Lecture 18 – Type Systems

COSE212: Programming Languages

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PLRG

2023 Fall

Recall

- We learned about **continuations** with the following topics:
  - **Continuations** (Lecture 14 & 15)
  - **First-Class Continuations** (Lecture 16)
  - **Compiling with continuations** (Lecture 17)

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  - Typed Languages
  - Typing Recursive Functions
  - Algebraic Data Types
  - Parametric Polymorphism
  - Subtype Polymorphism
  - Type Inference

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- In this lecture, we will focus on the **motivation** and **basic concepts** of type systems.

Contents

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Run-Time Errors

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- **Syntax**: a grammar that defines the structure of programs
- **Semantics**: a set of rules that defines the meaning of programs

and implemented their **interpreters** in Scala:

Run-Time Errors

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However, we don't have any automatic system to **check** whether a program is evaluated without any **run-time errors**.

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and implemented their **interpreters** in Scala:

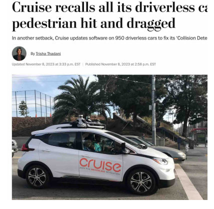
However, we don't have any automatic system to **check** whether a program is evaluated without any **run-time errors**.

For example, following FAE expressions are syntactically correct, but they throw **run-time errors**:

```
/* FAE */
x * 42                // error: free identifier
0 + (x => x)          // error: cannot add a function
1(2)                  // error: cannot apply a number
```

Errors in Saftety-Critical Software

Unexpected errors in **safety-critical software** cause serious problems:



Rocket	Financial	Airport	Auto. Vehicle
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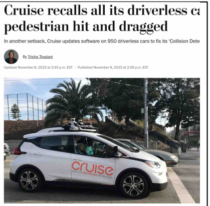

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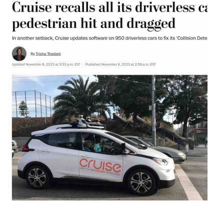

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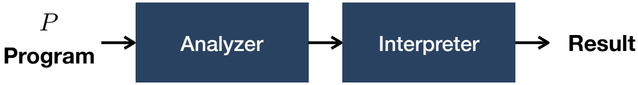
Then, how can we **prevent** such errors?

Can we **automatically** check whether a program does not have any **run-time errors**?

Detecting Run-Time Errors

We can use various **analysis** techniques to detect run-time errors:



An **analyzer** is a program that takes a program as an input and determines whether the program has a certain property. In this case, the property is **run-time errors**.

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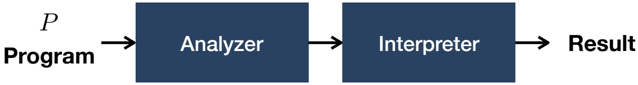

An **analyzer** is a program that takes a program as an input and determines whether the program has a certain property. In this case, the property is **run-time errors**.

We can categorize them into two groups:

- **Dynamic Analysis**: analyze programs by **executing** them
- **Static Analysis**: analyze programs **without executing** them

Dynamic Analysis

Dynamic analysis is a program analysis technique by **executing** them.

Let's perform **dynamic analysis** for the following Scala program:

```
def abs(x: Int): Int = { /* L1 */
  if (x < 0)             /* L2 */
    -x                   /* L3 */
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L1	-5
L2	-5
L3	5
L4	
L5	
L6	5

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L1	-5	42
L2	-5	
L3	5	
L4		42
L5		42
L6	5	42

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L1	-5	42	-7	99	0	...
L2	-5		-7			...
L3	5		7			...
L4		42		99	0	...
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We can easily get the **behavior** of the program for each **single input**.

However, it is **difficult** to get all the **possible behaviors** of the program for **all the inputs**.

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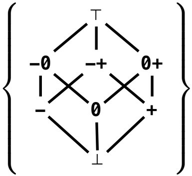
$\mathbb{D} =$



Let's define an **abstract domain** $\mathbb{D}$ for integers to analyze the program.

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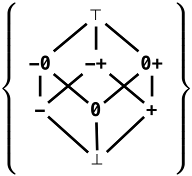

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$$
\begin{array}{lll}
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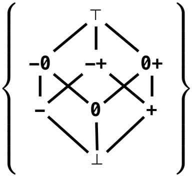

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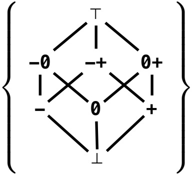
$\mathbb{D} =$ 

L1	$\top$
L2	-
L3	+
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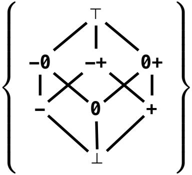

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We can prove that `abs` always returns a **non-negative** integer (i.e., 0+).

Soundness vs Completeness

- $\vDash \psi$ denotes that a statement ψ is **provable**.
- $\vdash \psi$ denotes that a statement ψ is **true**.

In a **sound** proof system, all **provable** statements are **true**.

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In a **complete** proof system, all **true** statements are **provable**.

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Analysis techniques can be used to prove that a program is **error-free**.

- $\vDash P$ denotes that a program P is **analyzed** as error-free.
- $\vdash P$ denotes that a program P is truly **error-free**.

Then, is dynamic/static analysis **sound** or **complete**?

Soundness vs Completeness

- **Dynamic analysis** is **complete** but **unsound** in general.
  - All the detected errors are **true alarms** (**true positive (TP)**).
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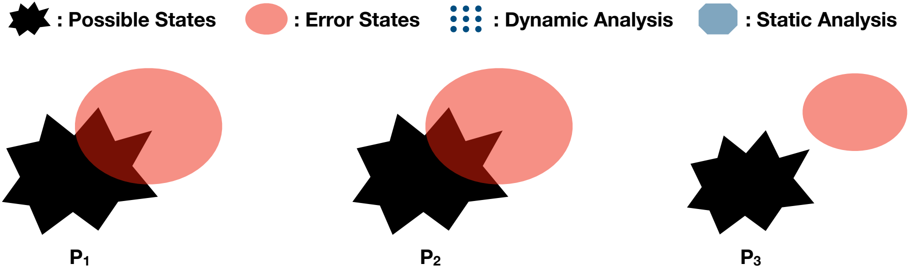

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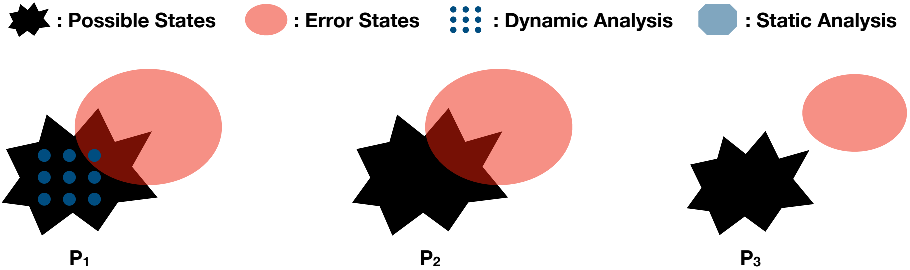

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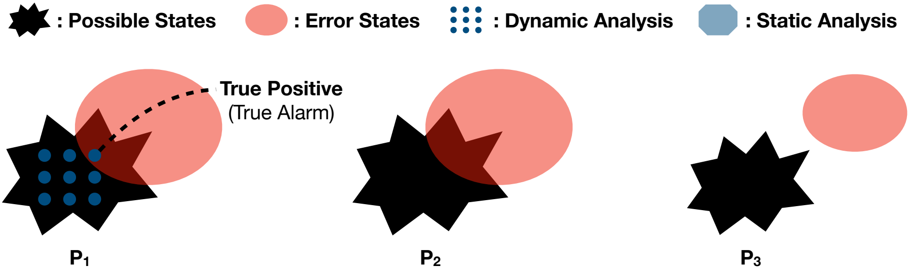

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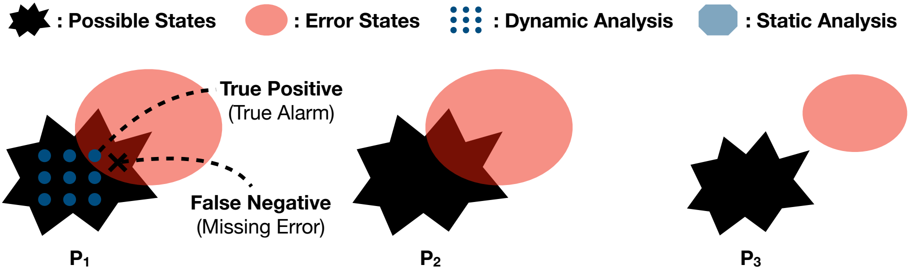

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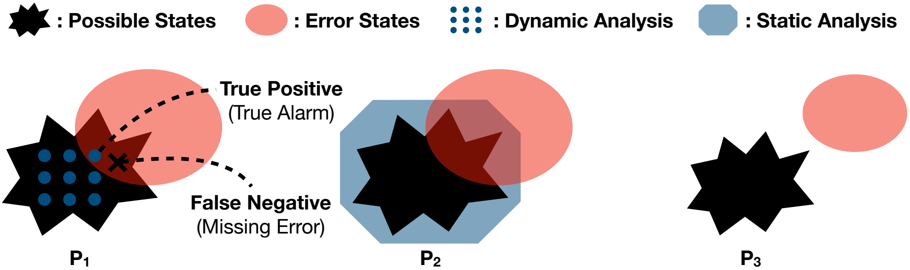

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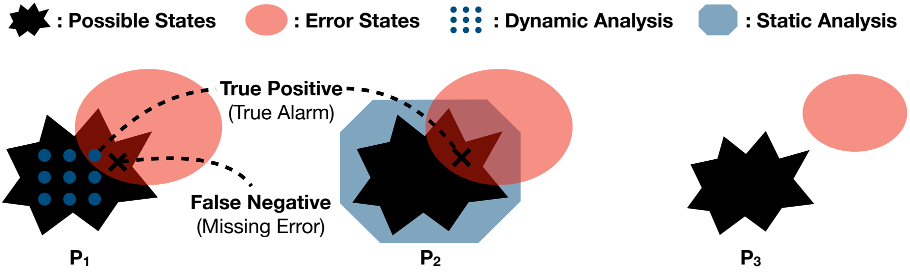

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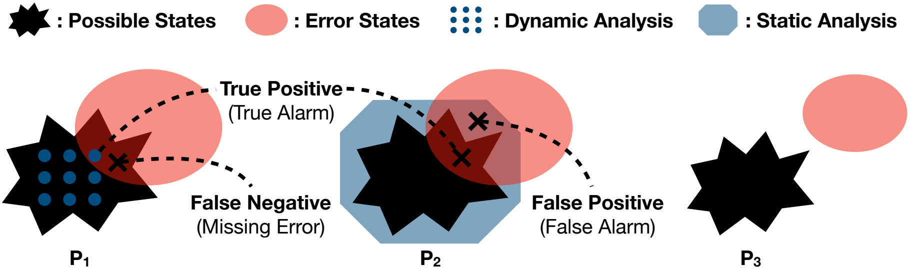

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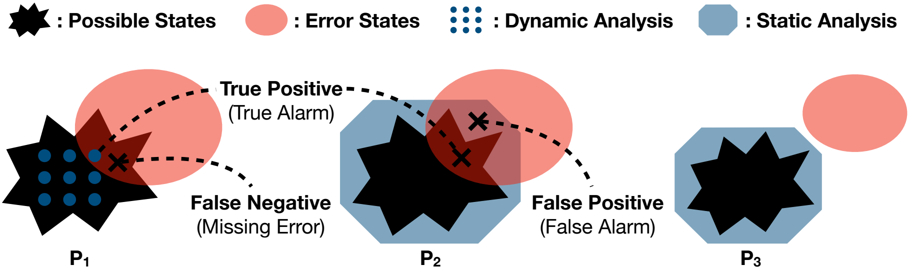

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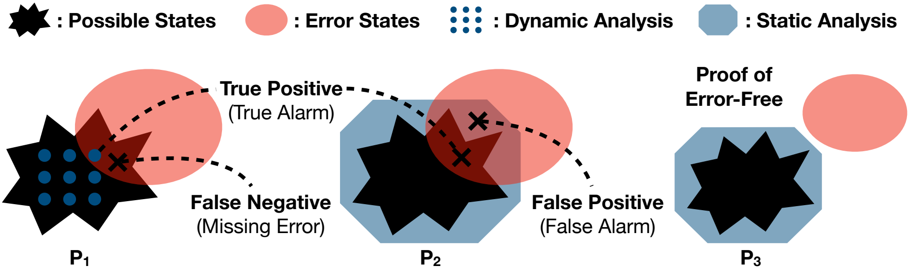

Contents

Types

Definition (Types)

A **type** is a set of values.

For example, the `Int`, `Boolean`, and `Int => Int` types are defined as the following sets of values in Scala.

$$
\begin{aligned}
\texttt{Int} &= \{n \in \mathbb{Z} \mid -2^{31} \leq n < 2^{31}\} \\
\texttt{Boolean} &= \{\texttt{true}, \texttt{false}\} \\
\texttt{Int => Int} &= \{f \mid f \text{ is a function from } \texttt{Int} \text{ to } \texttt{Int}\}
\end{aligned}
$$

```
val n: Int = 42              // 42   : Int
n + 1                        // 43   : Int
val b: Boolean = n > 10      // true : Boolean
def f(x: Int): Int = x + 1   // f    : Int => Int
f(42)                        // 43   : Int
```

Type Errors

PLRG

Definition (Type Errors)

A **type error** occurs when a program tries to use a value having a type that is **incompatible** with the expected type.

For example, the following Scala program has type errors:

```
42 + true              // `Int` expected for `+`, but `Boolean` found
if (1) 2 else 3        // `Boolean` expected for `if`, but `Int` found
def f(x: Int): Int = x + 1
f(false)               // `Int` expected for `f`, but `Boolean` found
```

However, not all **run-time errors** are **type errors**:

```
42 / 0                 // `ArithmeticException` at run-time
case class A(k: Int)
val x: A = null
x.k                    // `NullPointerException` at run-time
```

Type Checking

If the following conditions hold, we say **"the expression e has type τ"**:

- e does not cause any type error, and
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If so, we use the following notation and say that e is **well-typed**:

$$\boxed{\vdash e : \tau}$$

Type Checking

If the following conditions hold, we say **"the expression** e **has type** τ**"**:

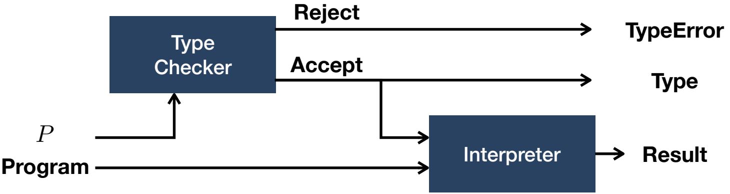
- e does not cause any type error, and
- e evaluates to a value of type τ or does not terminate.

If so, we use the following notation and say that e is **well-typed**:

$$\boxed{\vdash e : \tau}$$

Definition (Type Checking)

Type checking is a kind of static analysis checking whether a given expression e is **well-typed**. A **type checker** returns the **type** of e if it is well-typed, or rejects it and reports the detected **type error** otherwise.



Type Soundness

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There are two categories of languages in the context of type system:

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Type systems in most statically-typed languages are designed to be **sound**.

Summary

1. Motivation: Safe Language Systems
    - Detecting Run-Time Errors
    - Dynamic vs Static Analysis
    - Soundness vs Completeness

2. Type Systems
    - Types
    - Type Errors
    - Type Checking
    - Type Soundness

Homework #3

- Please see this document[1] on GitHub.
- The due date is Nov. 27 (Mon.).
- Please only submit `Implementation.scala` file to **Blackboard**.

[1] https://github.com/ku-plrg-classroom/docs/tree/main/cose212/magnet.

Next Lecture

- Typed Languages

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