Lecture 26 – Type Inference (2)

COSE212: Programming Languages

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PLRG

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Recall

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- We have seen three examples to learn how the type inference works.

```
/* RFAE */ def sum(x) = if (x < 1) 0 else x + sum(x - 1); sum
```

```
/* RFAE */ val app = n => f => f(n); app(42)(x => x)
```

```
/* RFAE */ val id = x => x; val n = id(42); val b = id(true); b
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/* RFAE */ val id = x => x; val n = id(42); val b = id(true); b
```

- In this lecture, let's learn the details of the type inference algorithm.
- **TIFAE** – TRFAE with **type inference**.
  - **Type Checker** and **Typing Rules** with **Type Inference**
  - Interpreter and Natural Semantics

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Type Checker and Typing Rules

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Let's ❶ design **typing rules** of TIFAE to define when an expression is well-typed in the form of:

$$\boxed{\Gamma \vdash e : \tau}$$

and ❷ implement a **type checker** in Scala according to typing rules:

```scala
def typeCheck(expr: Expr, tenv: TypeEnv): Type = ???
```

The type checker returns the **type** of e if it is well-typed, or rejects it and throws a **type error** otherwise.

We will keep track of the **variable types** using a **type environment** Γ as a mapping from variable names to their types.

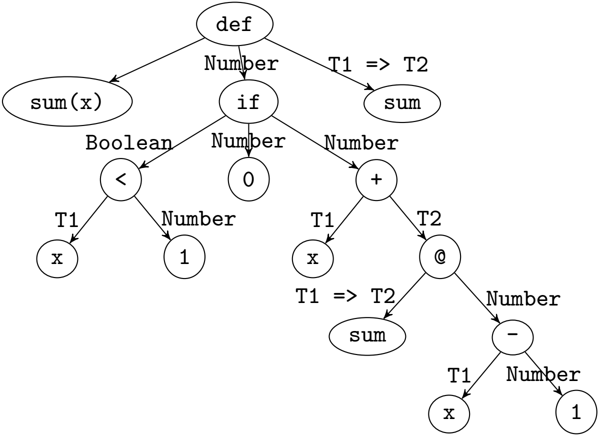
$$\text{Type Environments} \qquad \Gamma \quad \in \mathbb{\Gamma} = \mathbb{X} \xrightarrow{\text{fin}} \mathbb{T} \quad (\texttt{TypeEnv})$$

```scala
type TypeEnv = Map[String, Type]
```

Recall: Example 1 – `sum`

In addition, we need to keep track of the **solution** for **type constraints** over **type variables** to infer the types of expressions.

```
/* RFAE */ def sum(x) = if (x < 1) 0 else x + sum(x - 1); sum
```



Type Environment

$\mathbb{X}$	$\mathbb{T}$
`x`	`T1`
`sum`	`T1 => T2`

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
`T1`	`Number`
`T2`	`Number`

Solutions for Type Constraints

A **solution** is a mapping from **type variables** to **types**.

$$\begin{array}{llll} \text{Solutions} & \psi \in \Psi = \mathbb{X}_\alpha \xrightarrow{\text{fin}} (\mathbb{T} \uplus \{\bot\}) & (\texttt{Solution}) \\ \text{Type Variables} & \alpha \in \mathbb{X}_\alpha & (\texttt{Int}) \end{array}$$

```
type Solution = Map[Int, Option[Type]]
```

Note that $\bot$ (`None`) represents a **not yet solved** (**free**) type variable.

Solutions for Type Constraints

A **solution** is a mapping from **type variables** to **types**.

Solutions	$\psi \in \Psi = \mathbb{X}_\alpha \xrightarrow{\text{fin}} (\mathbb{T} \uplus \{\bot\})$	(`Solution`)
Type Variables	$\alpha \in \mathbb{X}_\alpha$	(`Int`)

```
type Solution = Map[Int, Option[Type]]
```

Note that $\bot$ (`None`) represents a **not yet solved** (**free**) type variable.

Now, we have new forms of **type checker** and **typing rules**.

```
def typeCheck(expr: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = ???
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

Similar to the memory passing in MFAE for mutation, we will pass the solution ψ and update it during type checking.

Numbers

```
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...
  case Num(n) => (NumT, sol)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{–}\texttt{Num}\ \frac{}{\Gamma, \psi \vdash n : \texttt{num}, \psi}$$

Additions

```
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...
  case Add(l, r) =>
    val (lty, sol1) = typeCheck(l, tenv, sol)
    val (rty, sol2) = typeCheck(r, tenv, sol1)
    val sol3 = unify(lty, NumT, sol2)
    val sol4 = unify(rty, NumT, sol3)
    (ty, sol4)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\!-\!\texttt{Add}\ \frac{\begin{array}{cc}\Gamma, \psi_0 \vdash e_1 : \tau_1, \psi_1 & \Gamma, \psi_1 \vdash e_2 : \tau_2, \psi_2 \\ \texttt{unify}(\tau_1, \texttt{num}, \psi_2) = \psi_3 & \texttt{unify}(\tau_2, \texttt{num}, \psi_3) = \psi_4\end{array}}{\Gamma, \psi_0 \vdash e_1 \texttt{ + } e_2 : \texttt{num}, \psi_4}$$

The `unify` function that takes two types must be the same and updates the solution. We will see how it works later.

Conditionals

```
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...
  case If(c, t, e) =>
    val (cty, sol1) = typeCheck(c, tenv, sol)
    val (tty, sol2) = typeCheck(t, tenv, sol1)
    val (ety, sol3) = typeCheck(e, tenv, sol2)
    val sol4 = unify(cty, BoolT, sol3)
    val sol5 = unify(tty, ety, sol4)
    (tty, sol5)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{-}\mathtt{If}\ \frac{\begin{array}{c}\Gamma, \psi \vdash e_c : \mathtt{bool}, \psi_c \qquad \Gamma, \psi_c \vdash e_t : \tau_t, \psi_t \qquad \Gamma, \psi_t \vdash e_e : \tau_e, \psi_e \\ \mathtt{unify}(\tau_c, \mathtt{bool}, \psi_e) = \psi' \qquad \mathtt{unify}(\tau_t, \tau_e, \psi') = \psi''\end{array}}{\Gamma, \psi \vdash \mathtt{if}\ (e_c)\ e_t\ \mathtt{else}\ e_e : \tau_t, \psi''}$$

Immutable Variable Defs. and Identifier Lookup

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```
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...

  case Val(x, e, b) =>
    val (ety, sol1) = typeCheck(e, tenv, sol)
    typeCheck(b, tenv + (x -> ety), sol1)

  case Id(x) => tenv.getOrElse(x, error(s"free identifier: $x"))
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{–}\mathtt{Val}\ \frac{\Gamma, \psi_0 \vdash e_1 : \tau_1, \psi_1 \qquad \Gamma[x : \tau_1] \vdash e_2 : \tau_2, \psi_2}{\Gamma, \psi_0 \vdash \mathtt{val}\ x = e_1;\ e_2 : \tau_2, \psi_2}$$

$$\tau\text{–}\mathtt{Id}\ \frac{x \in \mathsf{Domain}(\Gamma)}{\Gamma, \psi \vdash x : \Gamma(x), \psi}$$

Function Definitions

```
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...
  case Fun(p, b) =>
    val (pty, sol1) = newTypeVar(sol)
    val (rty, sol2) = typeCheck(b, tenv + (p -> pty), sol1)
    (ArrowT(pty, rty), sol2)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{–}\mathtt{Fun}\ \frac{\alpha_p \notin \psi \qquad \Gamma[x : \alpha_p], \psi[\alpha_p \mapsto \bot] \vdash e : \tau, \psi'}{\Gamma, \psi \vdash \lambda x.e : \alpha_p \to \tau, \psi'}$$

Recursive Function Definitions

```scala
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...
  case Rec(f, p, b, s) =>
    val (pty, sol1) = newTypeVar(sol)
    val (rty, sol2) = newTypeVar(sol1)
    val fty = ArrowT(pty, rty)
    val tenv1 = tenv + (f -> fty)
    val tenv2 = tenv1 + (p -> pty)
    val (bty, sol3) = typeCheck(b, tenv2, sol2)
    val sol4 = unify(bty, rty, sol3)
    typeCheck(s, tenv1, sol4)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\texttt{-Rec}\ \frac{\begin{array}{c}\alpha_p, \alpha_r \notin \psi \qquad \alpha_p \neq \alpha_r \qquad \Gamma_1 = \Gamma[x_f \mapsto (\alpha_p \to \alpha_r)] \\ \Gamma_2 = \Gamma_1[x_p \mapsto \alpha_p] \qquad \Gamma_2, \psi[\alpha \mapsto \bot, \alpha' \mapsto \bot] \vdash e_b : \tau_b, \psi_b \\ \texttt{unify}(\tau_b, \alpha_r, \psi_b) = \psi_r \qquad \Gamma_1, \psi_r \vdash e_s : \tau_s, \psi_s\end{array}}{\Gamma, \psi \vdash \texttt{def}\ x_f(x_p) \texttt{ = } e_b;\ \ e_s : \tau_s, \psi_s}$$

Function Applications

```scala
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...
  case App(f, a) =>
    val (fty, sol1) = typeCheck(f, tenv, sol)
    val (aty, sol2) = typeCheck(a, tenv, sol1)
    val (rty, sol3) = newTypeVar(sol2)
    val sol4 = unify(ArrowT(aty, rty), fty, sol3)
    (rty, sol4)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{-App}\ \frac{\Gamma, \psi \vdash e_f : \tau_f, \psi_f \qquad \Gamma, \psi_f \vdash e_a : \tau_a, \psi_a \qquad \alpha_r \notin \psi_a \qquad \texttt{unify}(\tau_a \to \alpha_r, \tau_f, \psi_a[\alpha_r \mapsto \bot]) = \psi'}{\Gamma, \psi \vdash e_1(e_2) : \alpha_r, \psi'}$$

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Type Unification

Definition (Type Unification)

Type unification is the process of updating a solution to make two types equal. If the types are not unifiable, then this process fails and throws an exception.

$$\boxed{\texttt{unify} : (\mathbb{T} \times \mathbb{T} \times \Psi) \rightharpoonup \Psi}$$

For example, if we unify a type variable α and the number type $\texttt{num}$, the empty solution $\varnothing$ is updated to $[\alpha \mapsto \texttt{num}]$.

$$\texttt{unify}(\alpha, \texttt{num}, \varnothing) = [\alpha \mapsto \texttt{num}]$$

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$$\texttt{unify}(\alpha, \texttt{num}, \varnothing) = [\alpha \mapsto \texttt{num}]$$

Before, we define the type unification, we need to define the **type resolving** and **occurrence checking** functions.

1. **Type resolving** is the process of recursively resolving a type variable to its representative type to deal with the **type aliasing**.
2. **Occurrence checking** is the process of checking whether a type variable occurs in a type to detect **cyclic types**.

Type Resolving

To understand why we need the **type resolving** function, let's consider the following example:

$$\texttt{unify}(\alpha_1, \alpha_2, \psi_1) = \psi_2 \qquad \texttt{unify}(\alpha_1, \texttt{num}, \psi_2) = \psi_3$$

$\psi_1 =$ Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
α_1	$\perp$
α_2	$\perp$

$\psi_2 =$ Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
α_1	α_2
α_2	$\perp$

$\psi_3 =$ Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
α_1	$\texttt{num}$
α_2	$\perp$

Type Resolving

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α_1	$\perp$
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$\mathbb{X}_\alpha$	$\mathbb{T}$
α_1	α_2
α_2	$\perp$

$\psi_3 =$ Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
α_1	$\texttt{num}$
α_2	$\perp$

Unfortunately, we cannot know that α_2 are `num` with the solution ψ_3.

Type Resolving

To understand why we need the **type resolving** function, let's consider the following example:

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$\psi_1 =$ Solution

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α_1	$\perp$
α_2	$\perp$

$\psi_2 =$ Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
α_1	α_2
α_2	$\perp$

$\psi_3 =$ Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
α_1	`num`
α_2	$\perp$

Unfortunately, we cannot know that α_2 are `num` with the solution ψ_3.

We need to **resolve** the type variable α_1 to find its **representative type** and update its solution to `num` to deal with the **type aliasing**.

$$\texttt{unify}(\texttt{resolve}(\alpha_1, \psi_2), \texttt{num}, \psi_2) = \psi_3' =$$

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
α_1	α_2
α_2	`num`

Type Resolving

We can define the **type resolving** function as follows:

$$\boxed{\texttt{resolve} : (\mathbb{T} \times \Psi) \rightarrow \mathbb{T}}$$

$$\texttt{resolve}(\tau, \psi) = \begin{cases} \texttt{resolve}(\tau', \psi) & \text{if } \tau = \alpha \wedge \psi(\alpha) = \tau' \\ \tau & \text{otherwise} \end{cases}$$

and implement it in Scala as follows:

```scala
def resolve(ty: Type, sol: Solution): Type = ty match
  case VarT(k) => sol(k) match
    case Some(ty) => resolve(ty, sol)
    case None => ty
  case _ => ty
```

Occurrence Checking

Let's understand why we need the **occurrence checking** function:

$$\texttt{unify}(\alpha_1, \texttt{num} \rightarrow \alpha_1, \psi) = \psi'$$

It actually fails because the type variable α_1 occurs in the type $\texttt{num} \rightarrow \alpha_1$, which means it requires **cyclic types** not supported in our type system.

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Let's define the **occurrence checking** function as follows:

$$\boxed{\texttt{occur} : (\mathbb{X}_\alpha \times \mathbb{T} \times \Psi) \to \texttt{bool}}$$

$$\texttt{occur}(\alpha, \tau, \psi) = \begin{cases} \texttt{true} & \text{if } \tau = \alpha \\ \texttt{occur}(\alpha, \tau_p, \psi) \lor \texttt{occur}(\alpha, \tau_r, \psi) & \text{if } \tau = (\tau_p \to \tau_r) \\ \texttt{false} & \text{otherwise} \end{cases}$$

and implement it in Scala as follows:

```scala
def occurs(k: Int, ty: Type, sol: Solution): Boolean = resolve(ty, sol) match
  case VarT(l) => k == l
  case ArrowT(pty, rty) => occurs(k, pty, sol) || occurs(k, rty, sol)
  case _ => false
```

Type Unification

Using the **type resolving** and **occurrence checking** functions, we could define the **type unification** as a partial function:

$$\boxed{\texttt{unify} : (\mathbb{T} \times \mathbb{T} \times \Psi) \rightharpoonup \Psi}$$

$$\texttt{unify}(\tau_1, \tau_2, \psi) = \begin{cases} \psi & \text{if } \tau_1' = \texttt{num} \land \tau_2' = \texttt{num} \\ \psi & \text{if } \tau_1' = \texttt{bool} \land \tau_2' = \texttt{bool} \\ \texttt{unify}(\tau_{1,r}, \tau_{2,r}, \texttt{unify}(\tau_{1,p}, \tau_{2,p}, \psi)) & \text{if } \tau_1' = (\tau_{1,p} \to \tau_{1,r}) \land \tau_2' = (\tau_{2,p} \to \tau_{2,r}) \\ \psi & \text{if } \tau_1' = \alpha = \tau_2' \\ \psi[\alpha \mapsto \tau_2'] & \text{if } \tau_1' = \alpha \land \neg\texttt{occur}(\alpha, \tau_2') \\ \psi[\alpha \mapsto \tau_1'] & \text{if } \tau_2' = \alpha \land \neg\texttt{occur}(\alpha, \tau_1') \end{cases}$$

where $\tau_1' = \texttt{resolve}(\tau_1, \psi)$ and $\tau_2' = \texttt{resolve}(\tau_2, \psi)$.

1. First, it resolves the types τ_1 and τ_2 with the current solution ψ into τ_1' and τ_2' using the **type resolving** function `resolve`.
2. If only one of them (τ_1' or τ_2') is a type variable, it checks cyclic types using the **occurrence checking** function `occur`.
3. Then, it unifies types τ_1' and τ_2' and updates the solution ψ.

Type Unification

And, we can implement the **type unification** function in Scala as follows:

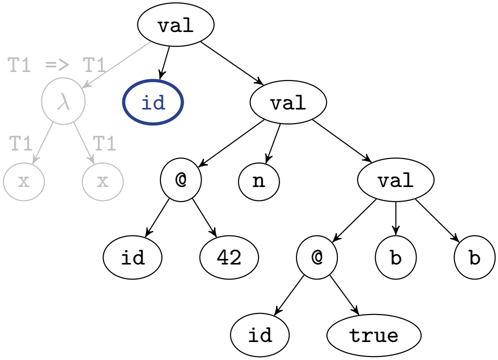
```
def unify(lty: Type, rty: Type, sol: Solution): Solution =
  (resolve(lty, sol), resolve(rty, sol)) match
    case (NumT, NumT) => sol
    case (BoolT, BoolT) => sol
    case (ArrowT(lpty, lrty), ArrowT(rpty, rrty)) =>
      unify(lrty, rrty, unify(lpty, rpty, sol))
    case (VarT(k), VarT(l)) if k == l => sol
    case (VarT(k), rty) if !occurs(k, rty, sol) => sol + (k -> Some(rty))
    case (lty, VarT(k)) if !occurs(k, lty, sol) => sol + (k -> Some(lty))
    case _ => error(s"Cannot unify ${lty.str} and ${rty.str}")
```

$$\texttt{unify}(\tau_1, \tau_2, \psi) = \begin{cases} \psi & \text{if } \tau_1' = \mathbf{num} \land \tau_2' = \mathbf{num} \\ \psi & \text{if } \tau_1' = \mathbf{bool} \land \tau_2' = \mathbf{bool} \\ \texttt{unify}(\tau_{1,r}, \tau_{2,r}, \texttt{unify}(\tau_{1,p}, \tau_{2,p}, \psi)) & \text{if } \tau_1' = (\tau_{1,p} \to \tau_{1,r}) \land \tau_2' = (\tau_{2,p} \to \tau_{2,r}) \\ \psi & \text{if } \tau_1' = \alpha = \tau_2' \\ \psi[\alpha \mapsto \tau_2'] & \text{if } \tau_1' = \alpha \land \neg\texttt{occur}(\alpha, \tau_2') \\ \psi[\alpha \mapsto \tau_1'] & \text{if } \tau_2' = \alpha \land \neg\texttt{occur}(\alpha, \tau_1') \end{cases}$$

where $\tau_1' = \texttt{resolve}(\tau_1, \psi)$ and $\tau_2' = \texttt{resolve}(\tau_2, \psi)$.

Contents

Recall: Example 3 – `id`



Type Environment

$\mathbb{X}$	$\mathbb{T}$
`id`	`[T1]` $\{$ `T1 => T1` $\}$

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
`T1`	-

Let's **generalize** the type `T1 => T1` into a **polymorphic type** for `id` with **type variable** `T1` as a **type parameter**.

We call this **let-polymorphism** because it only introduces polymorphism for the let-binding (e.g., `val`).

Type Generalization

We can define the **type generalization** function $\texttt{gen}$ as follows:

$$\boxed{\texttt{gen} : (\mathbb{T} \times \mathbb{\Gamma} \times \Psi) \to \mathbb{T}^{\forall}}$$

$$\texttt{gen}(\tau, \Gamma, \psi) = \forall \alpha_1, \ldots, \alpha_m . \tau \qquad \text{where} \qquad \texttt{free}_{\tau}(\tau, \psi) \setminus \texttt{free}_{\Gamma}(\Gamma, \psi) = \{\alpha_1, \ldots, \alpha_m\}$$

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and the **free type variables** in each component as follows:

$$\boxed{\texttt{free}_{\tau} : (\mathbb{T} \times \Psi) \rightarrow \mathcal{P}(\mathbb{X}_{\alpha})}$$

$$\texttt{free}_{\tau}(\tau, \psi) = \begin{cases} \texttt{free}_{\tau'}(\tau', \psi) & \text{if } \tau = \alpha \wedge \psi(\alpha) = \tau' \\ \{\alpha\} & \text{if } \tau = \alpha \wedge \psi(\alpha) = \bot \\ \texttt{free}_{\tau_p}(\tau_p, \psi) \cup \texttt{free}_{\tau_r}(\tau_r, \psi) & \text{if } \tau = (\tau_p \rightarrow \tau_r) \\ \varnothing & \text{otherwise} \end{cases}$$

$$\boxed{\texttt{free}_{\tau^{\forall}} : (\mathbb{T}^{\forall} \times \Psi) \rightarrow \mathcal{P}(\mathbb{X}_{\alpha})}$$

$$\texttt{free}_{\tau^{\forall}}(\forall \alpha_1, \ldots, \alpha_m.\tau, \psi) = \texttt{free}_{\tau}(\tau, \psi) \setminus \{\alpha_1, \ldots, \alpha_m\}$$

$$\boxed{\texttt{free}_{\Gamma} : (\mathbb{\Gamma} \times \Psi) \rightarrow \mathcal{P}(\mathbb{X}_{\alpha})}$$

$$\texttt{free}_{\Gamma}([x_1 : \tau_1^{\forall}, \ldots, x_n : \tau_n^{\forall}], \psi) = \texttt{free}_{\tau_1^{\forall}}(\tau_1^{\forall}, \psi) \cup \ldots \cup \texttt{free}_{\tau_n^{\forall}}(\tau_n^{\forall}, \psi)$$

Immutable Variable Defs. with Type Generalization

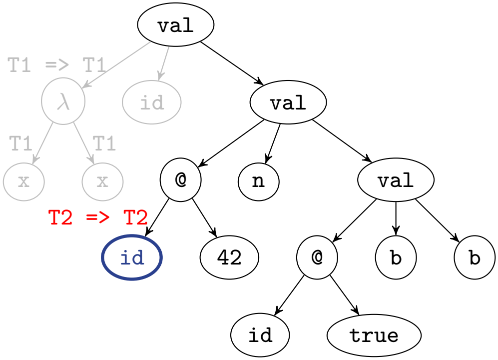
```
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...

  case Val(x, e, b) =>
    val (ety, sol1) = typeCheck(e, tenv, sol)
    val polyty = gen(ety, tenv, sol1)
    typeCheck(b, tenv + (x -> polyty), sol1)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{−}\texttt{Val}\ \frac{\Gamma, \psi_0 \vdash e_1 : \tau_1, \psi_1 \qquad \texttt{gen}(\tau_1, \Gamma, \psi_1) = \tau_1^{\forall} \qquad \Gamma[x : \tau_1^{\forall}] \vdash e_2 : \tau_2, \psi_2}{\Gamma, \psi_0 \vdash \texttt{val}\ x = e_1;\ e_2 : \tau_2, \psi_2}$$

Recall: Example 3 – `id`



Type Environment

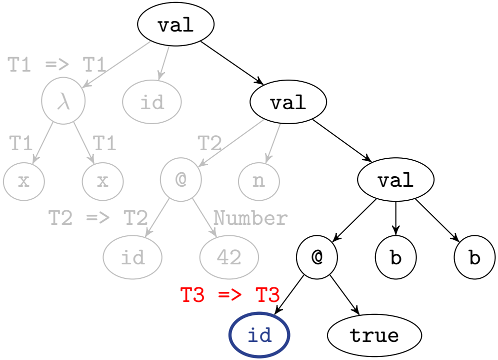
$\mathbb{X}$	$\mathbb{T}$
id	[T1] { T1 => T1 }

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-
T2	-

Let's define a new **type variable `T2`** to **instantiate** the **type variable `T1`**. And, **substitute** `T1` with `T2`.

Recall: Example 3 – `id`



Type Environment

$\mathbb{X}$	$\mathbb{T}$
id	[T1] { T1 => T1 }
n	T2

Solution

$\mathbb{X}_\alpha$	$\mathbb{T}$
T1	-
T2	Number
T3	-

Let's define a new **type variable `T3`** to **instantiate** the **type variable `T1`**. And, **substitute** `T1` with `T3`.

Type Instantiation

We can define the **type instantiation** function `inst` as follows:

$$\boxed{\texttt{inst} : (\mathbb{T}^{\forall} \times \Psi) \to (\mathbb{T} \times \Psi)}$$

$$\begin{array}{l}
\texttt{inst}(\forall \alpha_1, \ldots, \alpha_m.\tau, \psi) = (\\
\qquad \texttt{subst}(\tau, \psi[\alpha_1 \mapsto \alpha'_1, \ldots, \alpha_m \mapsto \alpha'_m]), \\
\qquad \psi[\alpha'_1 \mapsto \bot, \ldots, \alpha'_m \mapsto \bot] \\
)
\end{array}$$

where $\qquad \alpha'_1, \ldots, \alpha'_m \notin \psi \land \forall 1 \leq i < j \leq m.\ \alpha'_i \neq \alpha'_j$

Type Instantiation

We can define the **type instantiation** function `inst` as follows:

$$\boxed{\texttt{inst} : (\mathbb{T}^{\forall} \times \Psi) \rightarrow (\mathbb{T} \times \Psi)}$$

$$\begin{array}{l}
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\qquad \psi[\alpha'_1 \mapsto \bot, \ldots, \alpha'_m \mapsto \bot] \\
)
\end{array}$$

$$\text{where} \qquad \alpha'_1, \ldots, \alpha'_m \notin \psi \wedge \forall 1 \leq i < j \leq m.\ \alpha'_i \neq \alpha'_j$$

and the **type substitution** function `subst` as follows:

$$\boxed{\texttt{subst} : (\mathbb{T} \times \Psi) \rightarrow \mathbb{T}}$$

$$\texttt{subst}(\tau, \psi) = \begin{cases} \texttt{subst}(\tau', \psi) & \text{if } \tau = \alpha \wedge \psi(\alpha) = \tau' \\ \texttt{subst}(\tau_p, \psi) \rightarrow \texttt{subst}(\tau_r, \psi) & \text{if } \tau = (\tau_p \rightarrow \tau_r) \\ \tau & \text{otherwise} \end{cases}$$

Identifier Lookup with Type Instantiation

```
def typeCheck(e: Expr, tenv: TypeEnv, sol: Solution): (Type, Solution) = e match
  ...

  case Id(x) =>
    val ty = tenv.getOrElse(x, error(s"free identifier: $x"))
    inst(ty, sol)
```

$$\boxed{\Gamma, \psi \vdash e : \tau, \psi}$$

$$\tau\text{–}\mathtt{Id}\ \frac{\Gamma(x) = \tau^{\forall} \qquad \mathtt{inst}(\tau^{\forall}, \psi) = (\tau, \psi')}{\Gamma, \psi \vdash x : \tau, \psi'}$$

Summary

1. Type Checker and Typing Rules with Type Inference
   - Solutions for Type Constraints
   - Numbers
   - Additions
   - Conditionals
   - Immutable Variable Definitions and Identifier Lookup
   - Function Definitions
   - Recursive Function Definitions
   - Function Applications
2. Type Unification
   - Type Resolving
   - Occurrence Checking
   - Type Unification
3. Type Inference with Let-Polymorphism
   - Type Generalization
   - Type Instantiation

Exercise #15

- Please see this document[1] on GitHub.
  - Implement `typeCheck` function.
  - Implement `interp` function.
- It is just an exercise, and you **don't need to submit** anything.
- However, some exam questions might be related to this exercise.

[1] https://github.com/ku-plrg-classroom/docs/tree/main/cose212/tifae.

Next Lecture

- Course Review

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