

Lecture 6 – First-Order Functions

COSE212: Programming Languages

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- **VAE – AE with variables**
 - Evaluation with Environments
 - Interpreter and Natural Semantics

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- **F1VAE – VAE with first-order functions**
 - Concrete and Abstract Syntax
 - Interpreter and Natural Semantics

1. First-Order Functions
2. F1VAE – VAE with First-Order Functions
 - Concrete Syntax
 - Abstract Syntax
3. Interpreter and Natural Semantics for F1VAE
 - Evaluation with Function Environments
 - Function Application
4. Static Scoping vs Dynamic Scoping

1. First-Order Functions

2. F1VAE – VAE with First-Order Functions

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4. Static Scoping vs Dynamic Scoping

Let's calculate the square of several numbers in Scala.

```
1 * 1      // 1
2 * 2      // 4
3 * 3      // 9
42 * 42     // 1764
2434 * 2434 // 5925796
```

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```

With a **first-order function**, we can avoid the repetition of the code.

```
// A `square` function that takes an integer `n` and returns its square.
def square(n: Int): Int = n * n

square(1)      // 1
square(2)      // 4
square(3)      // 9
square(42)     // 1764
square(2434)   // 5924356
```


Most programming languages support **first-order functions**.

- Scala

```
def square(n: Int): Int = n * n
```

```
square(3) // 9
```

- Python

```
def square(n): return n * n
```

```
square(3) # 9
```

- C++

```
int square(int n) { return n * n; } square(3) // 9
```

- Rust

```
fn square(n: i32) -> i32 { return n * n; }
```

```
square(3) // 9
```

- ...

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Now, we want to extend VAE into F1VAE with **first-order functions**.

```
/* F1VAE */  
def square(n) = n * n;  
square(3) + 2 // 11
```

```
/* F1VAE */  
def add3(n) = n + 3;  
def mul2(m) = m * 2;  
mul2(add3(4)) // 14
```

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- An F1VAE **program** is a pair of
 - 1 a list of **function definitions**
 - 2 an **expression**
- We extend **expressions** with **function applications**.

Let's define the **concrete syntax** of F1VAE in BNF:

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Let's define the **concrete syntax** of F1VAE in BNF:

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```
// programs
<program> ::= <fdef>* <expr>

// function definitions
<fdef>      ::= "def" <id> "(" <id> ")" "=" <expr> ";"

// expressions
<expr>      ::= ...
               | "{" <expr> "}"
               | "val" <id> "=" <expr> ";" <expr>
               | <id>
               | <id> "(" <expr> ")"
```

Let's define the **abstract syntax** of F1VAE in BNF:

Programs	$\mathbb{P} \ni p ::= f^* e$	(Program)
Function Definitions	$\mathbb{F} \ni f ::= \text{def } x(x)=e$	(FunDef)
Expressions	$\mathbb{E} \ni e ::=$...
	val x=e; e	(Val)
	x	(Id)
	x(e)	(App)

Let's define the **abstract syntax** of F1VAE in BNF:

Programs	$\mathbb{P} \ni p$	$::= f^* e$	(Program)
Function Definitions	$\mathbb{F} \ni f$	$::= \text{def } x(x)=e$	(FunDef)
Expressions	$\mathbb{E} \ni e$	$::= \dots$	
		$\text{val } x=e; e$	(Val)
		x	(Id)
		$x(e)$	(App)

```
// programs
case class Program(fdefs: List[FunDef], expr: Expr)
// function definitions
case class FunDef(name: String, param: String, body: Expr)
enum Expr:
  ...
  case Val(name: String, init: Expr, body: Expr)
  case Id(name: String)
  // function application
  case App(fname: String, arg: Expr)
```


For example, let's **parse** the following F1VAE program:

```
/* F1VAE */  
def add3(n) = n + 3;  
def mul2(m) = m * 2;  
mul2(add3(4))
```

Then, the following **abstract syntax tree (AST)** is produced:

```
Program(  
  List(  
    FunDef("add3", "n", Add(Id("n"), Num(3))),  
    FunDef("mul2", "m", Mul(Id("m"), Num(2)))  
  ),  
  App("mul2", App("add3", Num(4)))  
)
```

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Let's evaluate the following F1VAE program:

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How to find the function definition of `add3` or `mul2`?

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/* F1VAE */  
def add3(n) = n + 3;  
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mul2(add3(4)) // 14
```

How to find the function definition of `add3` or `mul2`?

We need to construct a **function environment** that maps function names to function definitions from the **list of function definitions** in a program.

$$[\text{add3} \mapsto f_0, \text{mul2} \mapsto f_1]$$

where

$$\begin{aligned} f_0 &= \text{def add3}(n)=n+3 \\ f_1 &= \text{def mul2}(m)=m \times 2 \end{aligned}$$

For VAE, the interpreter takes an **expression** e with an **environment** σ and returns a number n as the result.

```
type Value = BigInt                                // values
type Env = Map[String, Value]                      // environments
def interp(expr: Expr, env: Env): Value = ... // interpreter
```

$$\sigma \vdash e \Rightarrow n$$

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def interp(expr: Expr, env: Env): Value = ... // interpreter
```

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Now, we extend it to take a **function environment** Λ , a mapping from function names to function definitions, as well:

```
type Value = BigInt                                // values
type Env = Map[String, Value]                      // environments
type FEnv = Map[String, FunDef]                    // function environments
def interp(expr: Expr, env: Env, fenv: FEnv): Value = ... // interpreter
```

$$\sigma, \Lambda \vdash e \Rightarrow n$$

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```
def createFEnv(fdefs: List[FunDef]): FEnv = fdefs.foldLeft(Map.empty) {  
  case (m: FEnv, fdef: FunDef) =>  
    val fname: String = fdef.name  
    // check if the function name is already in the function environment  
    if (m.contains(fname)) error(s"duplicate function: $fname")  
    else m + (fname -> fdef)  
}
```

However, an F1VAE program only contains a **list of function definitions**. Then, how to construct a **function environment** from a **list of function definitions**?

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    // check if the function name is already in the function environment  
    if (m.contains(fname)) error(s"duplicate function: $fname")  
    else m + (fname -> fdef)  
}
```

It will throw an error if there are **duplicate function names**:

```
createFEnv(List(  
  FunDef("add3", "n", Add(Id("n"), Num(3))),  
  FunDef("add3", "n", Add(Num(3), Id("n"))),  
)) // error: duplicate function: add3
```

For F1VAE, we need to 1) implement the **interpreter**:

```
def interp(expr: Expr, env: Env, fenv: FEnv): Value = ???
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def interp(expr: Expr, env: Env, fenv: FEnv): Value = ???
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and 2) define the **natural semantics** with environments and **function environments**:

$$\sigma, \Lambda \vdash e \Rightarrow n$$

Programs	$\mathbb{P} \ni p ::= f^* e$	(Program)
Function Definitions	$\mathbb{F} \ni f ::= \text{def } x(x)=e$	(FunDef)
Expressions	$\mathbb{E} \ni e ::= \dots$	
	$x(e)$	(App)

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where

Environments	$\sigma \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{Z}$	(Env)
Function Environments	$\Lambda \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{F}$	(FEnv)
Integers	$n \in \mathbb{Z}$	(BigInt)
Identifiers	$x \in \mathbb{X}$	(String)

```
def interp(expr: Expr, env: Env, fenv: FEnv): Value = expr match
  ...
  case App(f, e) => ???
```

$$\sigma, \Lambda \vdash e \Rightarrow n$$

$$\text{App} \frac{\text{???}}{\sigma, \Lambda \vdash x_0(e_1) \Rightarrow \text{???}}$$

```
def interp(expr: Expr, env: Env, fenv: FEnv): Value = expr match
...
case App(f, e) =>
  val fdef = fenv.getOrElse(f, error(s"unknown function: $f"))
  ...
```

$$\sigma, \Lambda \vdash e \Rightarrow n$$

$$\begin{array}{c} x_0 \in \text{Domain}(\Lambda) \quad \Lambda(x_0) = \text{def } x_0(x_1) = e_2 \\ \dots \\ \text{App} \frac{\quad}{\sigma, \Lambda \vdash x_0(e_1) \Rightarrow ???} \end{array}$$


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def interp(expr: Expr, env: Env, fenv: FEnv): Value = expr match
...
case App(f, e) =>
  val fdef = fenv.getOrElse(f, error(s"unknown function: $f"))
  ... interp(e, env, fenv) ...
```

$$\sigma, \Lambda \vdash e \Rightarrow n$$

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def interp(expr: Expr, env: Env, fenv: FEnv): Value = expr match
...
case App(f, e) =>
  val fdef = fenv.getOrElse(f, error(s"unknown function: $f"))
  ... Map(fdef.param -> interp(e, env, fenv)) ...
```

$$\boxed{\sigma, \Lambda \vdash e \Rightarrow n}$$

$$\text{App} \frac{\begin{array}{l} x_0 \in \text{Domain}(\Lambda) \quad \Lambda(x_0) = \text{def } x_0(x_1) = e_2 \\ \sigma, \Lambda \vdash e_1 \Rightarrow n_1 \quad \dots [x_1 \mapsto n_1] \dots \end{array}}{\sigma, \Lambda \vdash x_0(e_1) \Rightarrow n_2}$$

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def interp(expr: Expr, env: Env, fenv: FEnv): Value = expr match
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We skip the other cases because they are only augmented with passing function environments. If you are interested, please refer to this spec:

<https://github.com/ku-plrg-classroom/docs/blob/main/cose212/f1vae/f1vae-spec.pdf>

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4. Static Scoping vs Dynamic Scoping

The current semantics is called **static scoping (or lexical scoping)** because a binding occurrence is determined statically without considering the function application but only the function definition.

```
/* F1VAE */  
def f(x) = x + y;           // y is a free variable  
{ val y = 2; f(1) } + { val y = 4; f(3) }
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/* F1VAE */  
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{ val y = 2; f(1) } + { val y = 4; f(3) }
```

However, we can define the semantics of F1VAE in another way by using the **dynamic scoping** instead; a binding occurrence is determined dynamically when function application is executed:

```
/* F1VAE */  
def f(x) = x + y;           // y = 2 or y = 4 depending on the call-site  
{ val y = 2; f(1) } + { val y = 4; f(3) } // (1 + 2) + (3 + 4) = 10
```

We can design and implement the semantics of F1VAE with the **dynamic scoping** by changing the definition of the function application:

```
def interp(expr: Expr, env: Env, fenv: FEnv): Value = expr match
  ...
  case App(f, e) =>
    val fdef = fenv.getOrElse(f, error(s"unknown function: $f"))
    interp(fdef.body, env + (fdef.param -> interp(e, env, fenv)), fenv)
```

$$\sigma, \Lambda \vdash e \Rightarrow n$$

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We can design and implement the semantics of F1VAE with the **dynamic scoping** by changing the definition of the function application:

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def interp(expr: Expr, env: Env, fenv: FEnv): Value = expr match
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However, we will use the **static scoping** by default in this course.

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Expressions	$\mathbb{E} \ni e ::= \dots$ $x(e)$	(App)

```

type FEnv = Map[String, FunDef]
def interp(expr: Expr, env: Env, fenv: FEnv): Value = expr match
...
case App(f, e) =>
  val fdef = fenv.getOrElse(f, error(s"unknown function: $f"))
  interp(fdef.body, Map(fdef.param -> interp(e, env, fenv)), fenv)

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- Please see this document¹ on GitHub.
 - Implement `interp` function.
 - Implement `interpDS` function.
- It is just an exercise, and you **don't need to submit** anything.
- However, some exam questions might be related to this exercise.

¹<https://github.com/ku-plrg-classroom/docs/tree/main/cose212/f1vae>.

- First-Class Functions

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