Lecture 21 – Algebraic Data Types (1)

COSE212: Programming Languages

Jihyeok Park

PLRG

2023 Fall

Recall

- **TFAE** – FAE with **type system**.
  - **Type Checker** and **Typing Rules**
  - Interpreter and Natural Semantics
- TRFAE – RFAE with **type system**.
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  - Interpreter and Natural Semantics

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- TAFAE – TRFAE with **ADTs** and **pattern matching**.
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- TAFAE – TRFAE with **ADTs** and **pattern matching**.
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- In this lecture, we will focus on **Interpreter** and **Natural Semantics**.

Contents

Contents

Recall: Types

Definition (Types)

A **type** is a set of values.

For example, the `Int`, `Boolean`, and `Int => Int` types are defined as the following sets of values in Scala.

$$
\begin{aligned}
\texttt{Int} &= \{n \in \mathbb{Z} \mid -2^{31} \leq n < 2^{31}\} \\
\texttt{Boolean} &= \{\texttt{true}, \texttt{false}\} \\
\texttt{Int => Int} &= \{f \mid f \text{ is a function from } \texttt{Int} \text{ to } \texttt{Int}\}
\end{aligned}
$$

```
val n: Int = 42              // 42   : Int
n + 1                        // 43   : Int
val b: Boolean = n > 10      // true : Boolean
def f(x: Int): Int = x + 1   // f    : Int => Int
f(42)                        // 43   : Int
```

Product Types

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Definition (Product Types)

A **product type** $(\tau_1, \ldots, \tau_n)$ is a set of values of the form $(v_1, \ldots, v_n)$ where τ_i is the type of v_i for $1 \leq i \leq n$.

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For example, we can define product types in Scala as follows:

```scala
// A product type consisting of three different types
val triple: (Int, Boolean, String) = (42, true, "abc")

// A rectangle type with its width and height
type Rectangle = (Int, Int)
val rectangle: Rectangle = (10, 20)
val (w, h) = rectangle
val perimeter: Int = 2 * (w + h)      // 2 * (10 + 20) = 60
```

Union Types

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For example, we can define union types in Scala as follows:

```scala
val a: Int | Boolean | String = 42
val b: Int | Boolean | String = true
val c: Int | Boolean | String = "abc"

type Square = Int                // A square type
type Triangle = Int              // A equilateral triangle type
val x: Square | Traingle = 42    // Is this a square or a triangle?
```

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How can we discriminate between a square and a triangle? **Sum types**!

Sum Types

Definition (Sum Types)

A **sum type** $x_1(\tau_1) + \ldots + x_n(\tau_n)$ consists of **variants** $x_i(\tau_i)$ for $1 \leq i \leq n$. For each variant $x_i(\tau_i)$, x_i is the **constructor**, a function that takes a value v of type τ_i and generates a value $x_i(v)$ of the sum type.

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It is corresponds to a **tagged union** of sets:

$$x_1(\tau_1) + \ldots + x_n(\tau_n) = \{x_i(v) \mid \exists 1 \leq i \leq n. \text{ s.t. } v \in \tau_i\}$$

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For example, we can define **sum types** in Scala as follows:

```scala
case class Square(side: Int)
case class Triangle(side: Int)
type Shape = Square | Triangle
val x: Shape = Square(42)      // It is a square
val y: Shape = Triangle(42)    // It is a triangle
```

Now, we can discriminate between a square and a triangle!

Sum Types

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A **sum type** $x_1(\tau_1) + \ldots + x_n(\tau_n)$ consists of **variants** $x_i(\tau_i)$ for $1 \leq i \leq n$. For each variant $x_i(\tau_i)$, x_i is the **constructor**, a function that takes a value v of type τ_i and generates a value $x_i(v)$ of the sum type.

```
case class Square(side: Int)               // A variant for squares
case class Triangle(side: Int)             // A variant for triangles
type Shape = Square | Triangle
val x: Square | Triangle = Square(42)      // It is a square
val y: Square | Triangle = Triangle(42)    // It is a triangle

// `Square` is a constructor that takes an `Int` and generates a `Shape`
Square: Int => Square

// `Square(42)` is a value of `Shape` by applying `42 to `Square`
Square(42): Shape
```

Algebraic Data Types (ADTs)

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An **algebraic data type** $x_1(\tau_{1,1}, \ldots, \tau_{1,m_1}) + \ldots + x_n(\tau_{n,1}, \ldots, \tau_{n,m_n})$ is a **recursive sum type** of **product types**.

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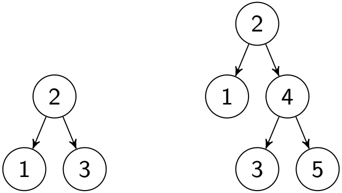
For example, we can define **algebraic data type** for trees in Scala:

```scala
enum Tree:
  case Leaf(v: Int)
  case Node(l: Tree, v: Int, r: Tree)

val t1: Tree = Node(Leaf(1), 2, Leaf(3))
val t2: Tree = Node(Leaf(1), 2, Node(Leaf(3), 4, Leaf(5)))
```



Pattern Matching

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Definition (Pattern matching)

We can use **pattern matching** for algebraic data types to identify which variant of the sum type a value belongs to and extract the data it contains.

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For example, we can define a function `sum` that sums all the values in a tree using pattern matching (`match`) on the `Tree` type in Scala:

```
enum Tree:
  case Leaf(v: Int)
  case Node(l: Tree, v: Int, r: Tree)

def sum(t: Tree): Int = t match
  case Leaf(v)       => v
  case Node(l, v, r) => sum(l) + v + sum(r)

sum(Node(Leaf(1), 2, Leaf(3)))                      // 6
sum(Node(Leaf(1), 2, Node(Leaf(3), 4, Leaf(5))))    // 15
```

Algebraic Data Types

Many functional languages support **algebraic data types**:

- Scala

```
enum Tree { Leaf(v: Int), Node(l: Tree, v: Int, r: Tree) }
```

- Haskell

```
data Tree = Leaf Int | Node Tree Int Tree
```

- Rust

```
enum Tree { Leaf(i32), Node(Tree, i32, Tree) }
```

- OCaml

```
type tree = Leaf of int | Node of tree * int * tree
```

- ...

Contents

TAFAE – TRFAE with ADTs and Pattern Matching

Now, let's extend TRFAE into TAFAE to support **algebraic data types** and **pattern matching**. (Assume that TRFAE supports multiple arguments for functions.)

```
/* TAFAE */
enum Tree {
  case Leaf(Number)
  case Node(Tree, Number, Tree)
}
Node(Leaf(1), 2, Leaf(3))
```

```
/* TAFAE */
enum NumList {
  case Nil()
  case Cons(Number, NumList)
}
Cons(1, Cons(2, Cons(3, Nil())))
```

For TAFAE, we need to extend **expressions** of TRFAE with

1. **algebraic data types (ADTs)**
2. **pattern matching**
3. **type variables**

Concrete Syntax

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Concrete Syntax

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We can extend the **concrete syntax** of FAE as follows:

```
// expressions
<expr> ::= ...
         | "enum" <id> "{" [ <variant> ";"? ]+ "}"
         | <expr> "match" "{" [ <mcase> ";"? ]+ "}"
// variants
<variant> ::= <id> "(" ")" | <id> "(" <type> [ "," <type> ]* ")"
// match cases
<mcase> ::= "case" <id> "(" ")" "=>" <expr>
          | "case" <id> "(" <id> [ "," <id> ]* ")" "=>" <expr>
// types
<type> ::= ...
         | <id>       // type variable
```

Abstract Syntax

$$
\begin{array}{llll}
\text{Expressions} & \mathbb{E} \ni e ::= & \ldots & \\
& & \mid \; \texttt{enum } t \; \{ \; [\texttt{case } x(\tau^*)]^* \; \}; \; e & (\texttt{TypeDef}) \\
& & \mid \; e \; \texttt{match} \; \{ \; [\texttt{case } x(x^*) \; \texttt{=>} \; e]^* \; \} & (\texttt{Match}) \\
\\
\text{Types} & \mathbb{T} \ni \tau ::= & \ldots & \\
& & \mid \; t & (\texttt{VarT})
\end{array}
$$

```
enum Expr:
  ...
  case TypeDef(name: String, varts: List[Variant], body: Expr)
  case Match(expr: Expr, mcases: List[MatchCase])

case class Variant(name: String, ptys: List[Type]):
case class MatchCase(name: String, params: List[String], body: Expr):

enum Type:
  ...
  case VarT(name: String)
```

Abstract Syntax

```
/* TAFAE */
enum Tree {
  case Leaf(Number)
  case Node(Tree, Number, Tree)
}
Node(Leaf(1), 2, Leaf(3))
```

will be parsed to the following abstract syntax tree (AST):

```
TypeDef("Tree", List(
    Variant("Leaf", List(NumT)),
    Variant("Node", List(VarT("Tree"), NumT, VarT("Tree")))
  ),
  App(Id("Node"), List(
    App(Id("Leaf"), List(Num(1))),
    Num(2),
    App(Id("Leaf"), List(Num(3)))
  ))
)
```

Contents

Interpreter and Natural Semantics for TAFAE

For TAFAE, we need to 1) implement the **interpreter** with environments:

```
def interp(expr: Expr, env: Env): Value = ???
```

and 2) define the **natural semantics** with environments:

$$\boxed{\sigma \vdash e \Rightarrow v}$$

Interpreter and Natural Semantics for TAFAE

For TAFAE, we need to 1) implement the **interpreter** with environments:

```scala
def interp(expr: Expr, env: Env): Value = ???
```

and 2) define the **natural semantics** with environments:

$$\boxed{\sigma \vdash e \Rightarrow v}$$

with a new kind of values called **constructor values** and **variant values**:

$$
\begin{array}{llll}
\text{Values} & \mathbb{V} \ni v ::= n & (\texttt{NumV}) & \mid \langle x \rangle \quad (\texttt{ConstrV}) \\
& \quad\mid b & (\texttt{BoolV}) & \mid x(v^*) \quad (\texttt{VariantV}) \\
& \quad\mid \langle \lambda x.(e, \ldots, e), \sigma \rangle & (\texttt{CloV}) &
\end{array}
$$

```scala
enum Value:
  ...
  case ConstrV(name: String)
  case VariantV(name: String, values: List[Value])
```

Algebraic Data Types

```scala
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case TypeDef(_, ws, body) =>
    interp(body, env ++ ws.map(w => w.name -> ConstrV(w.name)))
```

$$\boxed{\sigma \vdash e \Rightarrow v}$$

$$\texttt{TypeDef}\ \frac{\sigma[x_1 \mapsto \langle x_1 \rangle, \ldots, x_n \mapsto \langle x_n \rangle] \vdash e \Rightarrow v}{\sigma \vdash \texttt{enum}\ t \left\{ \begin{array}{l} \texttt{case}\ x_1(\tau_{1,1}, \ldots, \tau_{1,m_1}) \\ \ldots \\ \texttt{case}\ x_n(\tau_{n,1}, \ldots, \tau_{n,m_n}) \end{array} \right\};\ e \Rightarrow v}$$

Algebraic Data Types

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case App(f, es) => interp(f, env) match
    case CloV(ps, b, fenv) =>
      arityCheck(ps.length, es.length)
      interp(b, fenv() ++ (ps zip es.map(interp(_, env))))
    case ConstrV(name) => VariantV(name, es.map(interp(_, env)))
    case v             => error(s"not a function: ${v.str}")
```

$$\boxed{\sigma \vdash e \Rightarrow v}$$

$$\texttt{App}_{\langle - \rangle} \; \frac{\sigma \vdash e_0 \Rightarrow \langle x \rangle \qquad \sigma \vdash e_1 \Rightarrow v_1 \qquad \ldots \qquad \sigma \vdash e_n \Rightarrow v_n}{\sigma \vdash e_0(e_1, \ldots, e_n) \Rightarrow x(v_1, \ldots, v_n)}$$

Pattern Matching

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Match(expr, cases) => interp(expr, env) match
    case VariantV(wname, vs) => cases.find(_.name == wname) match
      case Some(MatchCase(_, ps, b)) =>
        arityCheck(ps.length, vs.length)
        interp(b, env ++ (ps zip vs))
      case None => error(s"no such case: $wname")
    case v => error(s"not a variant: ${v.str}")
```

$$\boxed{\sigma \vdash e \Rightarrow v}$$

$$\texttt{Match}\ \frac{1 \leq i \leq n \qquad \sigma \vdash e \Rightarrow x_i(v_i, \ldots, v_{m_i}) \qquad \sigma[x_{i,1} \mapsto v_1, \ldots, x_{i,m_i} \mapsto v_{m_i}] \vdash e_i \Rightarrow v}{\sigma \vdash e\ \texttt{match} \left\{ \begin{array}{l} \texttt{case}\ x_1(x_{1,1}, \ldots, x_{1,m_1})\ \texttt{=>}\ e_1 \\ \ldots \\ \texttt{case}\ x_n(x_{n,1}, \ldots, x_{n,m_n})\ \texttt{=>}\ e_n \end{array} \right\} \Rightarrow v}$$

Summary

1. Algebraic Data Types (ADTs) and Pattern Matching
   - Recall: Types
   - Product Types
   - Union Types
   - Sum Types
   - Algebraic Data Types (ADTs)
   - Pattern Matching
2. TAFAE – TRFAE with ADTs and Pattern Matching
   - Concrete Syntax
   - Abstract Syntax
3. Interpreter and Reduction Semantics for TAFAE
   - Algebraic Data Types
   - Function Application
   - Pattern Matching

Next Lecture

- Algebraic Data Types (2)

Jihyeok Park
jihyeok_park@korea.ac.kr
https://plrg.korea.ac.kr