

Lecture 16 – First-Class Continuations

COSE212: Programming Languages

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- We will learn about **continuations** with the following topics:
 - **Continuations** (Lecture 14 & 15)
 - **First-Class Continuations** (Lecture 16)
 - **Compiling with continuations** (Lecture 17)
- A **continuation** represents the **rest of the computation**.
 - Continuation Passing Style (CPS)
 - Interpreter of FAE in CPS
 - Small-step operational (reduction) semantics of FAE
- In this lecture, let's learn **first-class continuations**.
- **KFAE** – FAE with **first-class continuations**
 - Interpreter and Reduction semantics

1. First-Class Continuations

2. KFAE – FAE with First-Class Continuations

Concrete/Abstract Syntax

3. Interpreter and Reduction Semantics for KFAE

Recall: Interpreter and Reduction Semantics for FAE

Interpreter and Reduction Semantics for KFAE

First-Class Continuations

Function Application

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4. Control Statements

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4. Control Statements

In a programming language, an entity is said to be **first-class citizen** if it is treated as a **value**. In other words, it can be

- ① **assigned** to a **variable**,
- ② **passed** as an **argument** to a function, and
- ③ **returned** from a function.

For example, Scala supports **first-class functions**.

```
def inc(n: Int): Int = n + 1
// 1. We can assign a function to a variable.
val f: Int => Int =
// 2. We can pass a function as an argument to a function.
List(1, 2, 3).map(inc)           // List(2, 3, 4)
// 3. We can return a function from a function.
def addN(n: Int): Int => Int = m => n + m
val add3: Int => Int = addN(3)
add3(5)                        // 3 + 5 = 8
```

Similarly, a **first-class continuation** means that a continuation is treated as a **value**.

For example, **Racket** supports `let/cc` to create a first-class continuation.

```
; prefix notation  
(* 2 (+ 3 5))
```

```
; infix notation  
; 2 * (3 + 5) = 16
```

- ① Evaluate 2. (Result: 2)
- ② Evaluate 3. (Result: 3)
- ③ Evaluate 5. (Result: 5)
- ④ Add the results of step ② and ③. (Result: $3 + 5 = 8$)
- ⑤ Multiply the results of step ① and ④. (Result: $2 * 8 = 16$)

```
; first-class continuation with `let/cc`  
(* 2 (let/cc k (+ 3 (k 5))))
```

```
; 2 * 5 = 10
```

For example, Racket supports `let/cc` to create a first-class continuation.

```
; first-class continuation with `let/cc`  
(* 2 (let/cc k (+ 3 (k 5))))           ; 2 * 5 = 10
```

- ① Evaluate 2. (Result: 2)
 - ② Let k be the continuation of ② – ⑥. (k is $x \Rightarrow 2 * x$)
 - ③ Evaluate 3. (Result: 3)
 - ④ Evaluate 5. (Result: 5)
 - ⑤ Apply the result of step ④ to k . (Result of ② – ⑥: 5)
 - ⑥ Add the results of step ③ and ④ – ⑤. (Unreachable)
 - ⑦ Multiply the results of step ① and ② – ⑥. (Result: $2 * 5 = 10$)
- The step ② defines the continuation of ② – ⑥ as a value in k .
 - The step ⑤ changes the control flow using k with the result of ④.

Some functional languages support **first-class continuations**.

- Rakcet

```
(* 2 (let/cc k (+ 3 (k 5))))           ; 2 * 5 = 10
```

- Ruby

```
2 * (callcc { |k| 3 + k.call(5)})      # 2 * 5 = 10
```

- Haskell

```
do
  x <- callCC $ \k -> do
    y <- k 5
    return $ 3 + y
  return $ 2 * x                        -- 2 * 5 = 10
```

- ...

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Now, let's extend FAE into KFAE to support **first-class continuations**.
(Assume that `val` is supported in FAE as syntactic sugar.)

```
/* KFAE */  
2 * { vcc k; 3 + k(5) } // 2 * 5 = 10
```

```
/* KFAE */  
{  
  vcc done;  
  val f = {  
    vcc exit;  
    2 * done(1 + {  
      vcc k;  
      exit(k)  
    })  
  };  
  f(3) * 5  
}  
// 1 + 3
```

For KFAE, we need to extend **expressions** of FAE with

① first-class continuations (`vcc`)

We can extend the **concrete syntax** of FAE as follows:

```
// expressions  
<expr> ::= ... | "vcc" <id> ";" <expr>
```

and the **abstract syntax** of FAE as follows:

Expressions $\mathbb{E} \ni e ::= \dots \mid \text{vcc } x; e \quad (\text{Vcc})$

```
enum Expr:  
  ...  
  // first-class continuations  
  case Vcc(name: String, body: Expr)
```

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4. Control Statements

In the previous lecture, we have defined the **first-order representation** of **continuations** with **value stack**:

```
enum Cont:
  case EmptyK
  case EvalK(env: Env, expr: Expr, k: Cont)
  case AddK(k: Cont)
  case MulK(k: Cont)
  case AppK(k: Cont)

type Stack = List[Value]
```

Continuations	$\mathbb{K} \ni \kappa ::= \square$	(EmptyK)
	$(\sigma \vdash e) :: \kappa$	(EvalK)
	$(+)$:: κ	(AddK)
	(\times) :: κ	(MulK)
	$(@)$:: κ	(AppK)
Value Stacks	$\mathbb{S} \ni s ::= \blacksquare \mid v :: s$	(List[Value])

Then, we have defined the **reduction relation** $\rightarrow \in (\mathbb{K} \times \mathbb{S}) \times (\mathbb{K} \times \mathbb{S})$ between **states** consisting of pairs of **continuations** and **value stacks**:

```
def reduce(k: Cont, s: Stack): (Cont, Stack) = ???
```

$$\langle \kappa \parallel s \rangle \rightarrow \langle \kappa' \parallel s' \rangle$$

And the eval function **iteratively reduces** the state until it reaches the empty continuation \square and returns the single value in the value stack:

```
def eval(str: String): String =
  import Cont.*
  def aux(k: Cont, s: Stack): Value = reduce(k, s) match
    case (EmptyK, List(v)) => v
    case (k, s) => aux(k, s)
  aux(EvalK(Map.empty, Expr(str), EmptyK), List.empty).str
```

$$\langle (\emptyset \vdash e) :: \square \parallel \blacksquare \rangle \rightarrow^* \langle \square \parallel v :: \blacksquare \rangle$$

Now, let's extend the interpreter and reduction semantics for FAE to KFAE by adding the **first-class continuations**.

First, we need to extend the values of FAE with **continuation values** consisting of pairs of continuations and value stacks:

```
// values
enum Value:
  case NumV(number: BigInt)
  case CloV(param: String, body: Expr, env: Env)
  case ContV(cont: Cont, stack: Stack)
```

$$\begin{array}{ll} \text{Values } \mathbb{V} \ni v ::= n & (\text{NumV}) \\ & | \langle \lambda x.e, \sigma \rangle \quad (\text{CloV}) \\ & | \langle \kappa \parallel s \rangle \quad (\text{ContV}) \end{array}$$

Then, let's fill out the missing cases in the reduce function and reduction rules for \rightarrow in the reduction semantics of KFAE.

```
def reduce(k: Cont, s: Stack): (Cont, Stack) = (k, s) match
  case (EvalK(env, expr, k), s) => expr match
    ...
    case Vcc(x, b) => (EvalK(env + (x -> ContV(k, s)), b, k), s)
```

$$\langle \kappa \parallel s \rangle \rightarrow \langle \kappa \parallel s \rangle$$

$$\text{Vcc} \quad \langle (\sigma \vdash \text{vcc } x; e) :: \kappa \parallel s \rangle \rightarrow \langle (\sigma[x \mapsto \langle \kappa \parallel s \rangle] \vdash e) :: \kappa \parallel s \rangle$$

It defines a new immutable binding x in the environment σ that maps to a **continuation value** $\langle \kappa \parallel s \rangle$, and then evaluates the body expression e in the extended environment $\sigma[x \mapsto \langle \kappa \parallel s \rangle]$.


```
def reduce(k: Cont, s: Stack): (Cont, Stack) = (k, s) match
  case (EvalK(env, expr, k), s) => expr match
    ...
    case App(f, e) => (EvalK(env, f, EvalK(env, e, AppK(k))), s)
  ...
  case (AppK(k), a :: f :: s) => f match
    case CloV(p, b, fenv) => (EvalK(fenv + (p -> a), b, k), s)
    case ContV(k1, s1)    => (k1, a :: s1)
    case v                 => error(s"not a function: ${v.str}")
```

$$\langle \kappa \parallel s \rangle \rightarrow \langle \kappa \parallel s \rangle$$

$\text{App}_1 \quad \langle (\sigma \vdash e_1(e_2)) :: \kappa \parallel s \rangle \rightarrow \langle (\sigma \vdash e_1) :: (\sigma \vdash e_2) :: (@) :: \kappa \parallel s \rangle$
 $\text{App}_{2,\lambda} \quad \langle (@) :: \kappa \parallel v_2 :: \langle \lambda x.e, \sigma \rangle :: s \rangle \rightarrow \langle (\sigma[x \mapsto v_2] \vdash e) :: \kappa \parallel s \rangle$
 $\text{App}_{2,\kappa} \quad \langle (@) :: \kappa \parallel v_2 :: \langle \kappa' \parallel s' \rangle :: s \rangle \rightarrow \langle \kappa' \parallel v_2 :: s' \rangle$

The new $\text{App}_{2,\kappa}$ rule handles when the function expression evaluates to a continuation value $\langle \kappa' \parallel s' \rangle$. It changes the control flow to the continuation κ' with the given argument value v_2 and the value stack s' .

Example 1

Let's interpret the expression $2 \times (\text{vcc } k; (3 + k(5)))$:

	$\langle (\emptyset \vdash 2 \times (\text{vcc } k; (3 + k(5)))) :: \square \rangle$	$\parallel \blacksquare$	\rangle
(Mul_1)	$\rightarrow \langle (\emptyset \vdash 2) :: (\emptyset \vdash (\text{vcc } k; (3 + k(5)))) :: (\times) :: \square \rangle$	$\parallel \blacksquare$	\rangle
(Num)	$\rightarrow \langle (\emptyset \vdash (\text{vcc } k; (3 + k(5)))) :: (\times) :: \square \rangle$	$\parallel 2 :: \blacksquare$	\rangle
(Vcc)	$\rightarrow \langle (\sigma_0 \vdash (3 + k(5))) :: (\times) :: \square \rangle$	$\parallel 2 :: \blacksquare$	\rangle
(Add_1)	$\rightarrow \langle (\sigma_0 \vdash 3) :: (\sigma_0 \vdash k(5)) :: (+) :: (\times) :: \square \rangle$	$\parallel 2 :: \blacksquare$	\rangle
(Num)	$\rightarrow \langle (\sigma_0 \vdash k(5)) :: (+) :: (\times) :: \square \rangle$	$\parallel 3 :: 2 :: \blacksquare$	\rangle
(App_1)	$\rightarrow \langle (\sigma_0 \vdash k) :: (\sigma_0 \vdash 5) :: (@) :: (+) :: (\times) :: \square \rangle$	$\parallel 3 :: 2 :: \blacksquare$	\rangle
(Id)	$\rightarrow \langle (\sigma_0 \vdash 5) :: (@) :: (+) :: (\times) :: \square \rangle$	$\parallel \langle \kappa_0 \parallel s_1 \rangle :: 3 :: 2 :: \blacksquare$	\rangle
(Num)	$\rightarrow \langle (@) :: (+) :: (\times) :: \square \rangle$	$\parallel 5 :: \langle \kappa_0 \parallel s_1 \rangle :: 3 :: 2 :: \blacksquare$	\rangle
$(\text{App}_{2,\kappa})$	$\rightarrow \langle (\times) :: \square \rangle$	$\parallel 5 :: 2 :: \blacksquare$	\rangle
(Mul_2)	$\rightarrow \langle \square \rangle$	$\parallel 10 :: \blacksquare$	\rangle

$$\text{where } \begin{cases} \sigma_0 = [k \mapsto \langle \kappa_0 \parallel s_0 \rangle] \\ \kappa_0 = (\times) :: \square \\ s_0 = 2 :: \blacksquare \end{cases}$$

Example 2

Let's interpret the expression $(\lambda x. (\text{vcc } r; r(x + 1) \times 2))(3)$:

(App_1)	$\langle (\emptyset \vdash (\lambda x. (\text{vcc } r; r(x + 1) \times 2))(3)) :: \square \rangle$	$\parallel \blacksquare \rangle$
$\xrightarrow{\quad}$	$\langle (\emptyset \vdash (\lambda x. (\text{vcc } r; r(x + 1) \times 2))) :: (\emptyset \vdash 3) :: (@) :: \square \rangle$	$\parallel \blacksquare \rangle$
(Fun)	$\langle (\emptyset \vdash 3) :: (@) :: \square \rangle$	$\parallel \langle \lambda x. e_0, \emptyset \rangle :: \blacksquare \rangle$
$\xrightarrow{\quad}$	$\langle (@) :: \square \rangle$	$\parallel 3 :: \langle \lambda x. e_0, \emptyset \rangle :: \blacksquare \rangle$
(Num)	$\langle (@) :: \square \rangle$	$\parallel 3 :: \langle \lambda x. e_0, \emptyset \rangle :: \blacksquare \rangle$
$\xrightarrow{\quad}$	$\langle (@) :: \square \rangle$	$\parallel 3 :: \langle \lambda x. e_0, \emptyset \rangle :: \blacksquare \rangle$
$(\text{App}_{2,\lambda})$	$\langle (\sigma_0 \vdash \text{vcc } r; r(x + 1) \times 2) :: \square \rangle$	$\parallel \blacksquare \rangle$
$\xrightarrow{\quad}$	$\langle (\sigma_1 \vdash r(x + 1) \times 2) :: \square \rangle$	$\parallel \blacksquare \rangle$
(Vcc)	$\langle (\sigma_1 \vdash r(x + 1) \times 2) :: \square \rangle$	$\parallel \blacksquare \rangle$
$\xrightarrow{\quad}$	$\langle (\sigma_1 \vdash r(x + 1) \times 2) :: \square \rangle$	$\parallel \blacksquare \rangle$
(Mul_1)	$\langle (\sigma_1 \vdash r(x + 1)) :: (\sigma_1 \vdash 2) :: (@) :: (\times) :: \square \rangle$	$\parallel \blacksquare \rangle$
$\xrightarrow{\quad}$	$\langle (\sigma_1 \vdash r(x + 1)) :: (\sigma_1 \vdash 2) :: (@) :: (\times) :: \square \rangle$	$\parallel \blacksquare \rangle$
(App_1)	$\langle (\sigma_1 \vdash r) :: (\sigma_1 \vdash x + 1) :: (@) :: (\sigma_1 \vdash 2) :: (\times) :: \square \rangle$	$\parallel \blacksquare \rangle$
$\xrightarrow{\quad}$	$\langle (\sigma_1 \vdash r) :: (\sigma_1 \vdash x + 1) :: (@) :: (\sigma_1 \vdash 2) :: (\times) :: \square \rangle$	$\parallel \blacksquare \rangle$
\dots		
$\xrightarrow{*}$	$\langle (@) :: (\sigma_1 \vdash 2) :: (\times) :: \square \rangle$	$\parallel 4 :: \langle \square \parallel \blacksquare \rangle :: \blacksquare \rangle$
$(\text{App}_{2,\kappa})$	$\langle \square \rangle$	$\parallel 4 :: \blacksquare \rangle$

$$\text{where } \begin{cases} e_0 &= \text{vcc } r; r(x + 1) \times 2 \\ \sigma_0 &= [x \mapsto 3] \\ \sigma_1 &= \sigma_0[r \mapsto \langle \square \parallel \blacksquare \rangle] \end{cases}$$

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4. Control Statements

Many real-world programming languages support **control statements** to change the **control-flow** of a program.

For example, C++ supports **break**, **continue**, and **return** statements:

```
int sumEvenUntilZero(int xs[], int len) {  
    if (len <= 0) return 0;           // directly return 0 if len <= 0  
    int sum = 0;  
    for (int i = 0; i < len; i++) {  
        if (xs[i] == 0) break;        // stop the loop if xs[i] == 0  
        if (xs[i] % 2 == 1) continue; // skip the rest if xs[i] is odd  
        sum += xs[i];  
    }  
    return sum;                       // finally return the sum  
}  
  
int xs[] = {4, 1, 3, 2, 0, 6, 5, 8};  
sumEvenUntilZero(xs, 8);             // 4 + 2 = 6
```

Let's represent them using **first-class continuations**!

- **return** statement:

```
x => body
```

means

```
x => { vcc return;  
      body // return(e) directly returns e to the caller  
    }
```

- **break** and **continue** statements:

```
while (cond) body
```

means

```
{ vcc break;  
  while (cond) { vcc continue;  
    body // continue(e)/break(e) jumps to the next/end of the loop  
  }  
}
```

We can represent other control statements similarly, but think for yourself!

- exception in Python

```
try:
    x = y / z
except ZeroDivisionError:
    x = 0
```

- generator in JavaScript

```
const foo = function* () { yield 'a'; yield 'b'; yield 'c'; };
let str = '';
for (const c of foo()) { str = str + c; }
str // 'abc'
```

- coroutines in Kotlin
- async/await in C#
- ...

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4. Control Statements

- Please see this document¹ on GitHub.
 - Implement reduce function.
- It is just an exercise, and you **don't need to submit** anything.
- However, some exam questions might be related to this exercise.

¹<https://github.com/ku-plrg-classroom/docs/tree/main/cose212/kfae>.

- Compiling with Continuations

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