

# Lecture 22 – Algebraic Data Types (2)

## COSE212: Programming Languages

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- A way to define new types by combining existing types:
  - product type
  - union type
  - sum type (tagged union type)
  - **algebraic data type** (recursive sum type of product types)
- **ATFAE** – TRFAE with **ADTs** and **pattern matching**.
  - **Interpreter** and **Natural Semantics**
  - **Type Checker** and **Typing Rules**
- In this lecture, we will discuss on **Type Checker** and **Typing Rules**.

```
/* ATFAE */  
enum Tree {  
  case Leaf(Number)  
  case Node(Tree, Number, Tree)  
}  
Leaf(42) match {  
  case Leaf(v)      => v  
  case Node(l, v, r) => v  
}
```

The natural semantics of ATFAE ignores all the types.

Leaf and Node are not types but **variant names**.

Leaf and Node are **constructors** that take lists of values and produce **variant values** by adding their variant names as tags.

A **pattern matching** expression takes a **variant value** and finds the first match case whose name is equal to the variant name of the value.

## 1. Type Checker and Typing Rules

- Type Environment for ADTs

- Well-Formedness of Types

- (Recursive) Function Definition and Application

- Algebraic Data Types

- Pattern Matching

## 2. Type Soundness of ATFAE

- Recall: Type Soundness

- Algebraic Data Types - Revised (1)

- Algebraic Data Types - Revised (2)

## 1. Type Checker and Typing Rules

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Let's ① design **typing rules** of TRFAE to define when an expression is well-typed in the form of:

$$\boxed{\Gamma \vdash e : \tau}$$

and ② implement a **type checker** in Scala according to typing rules:

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = ???
```

The type checker returns the **type** of  $e$  if it is well-typed, or rejects it and throws a **type error** otherwise.

Similar to TRFAE, we will keep track of the **variable types** using a **type environment**  $\Gamma$  as a mapping from variable names to their types.

$$\text{Type Environments} \quad \Gamma \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{T} \quad (\text{TypeEnv})$$

```
type TypeEnv = Map[String, Type]
```

However, we need additional information about newly defined types by ADTs in type environments!

Type Environments  $\in (\mathbb{X} \xrightarrow{\text{fin}} \mathbb{T}) \times (\mathbb{X}_t \xrightarrow{\text{fin}} (\mathbb{X} \xrightarrow{\text{fin}} \mathbb{T}^*))$  (TypeEnv)

$$\Gamma(t) = x_1(\tau_{1,1}, \dots, \tau_{1,m_1}) + \dots + x_n(\tau_{n,1}, \dots, \tau_{n,m_n})$$

whose variants are **commutative**. For example,

$A = B(\text{bool}) + C(\text{num})$       equivalent to       $A = C(\text{num}) + B(\text{bool})$

```
case class TypeEnv(  
  vars: Map[String, Type] = Map(),  
  tys: Map[String, Map[String, List[Type]]] = Map()) {  
  def +(pair: (String, Type)): TypeEnv =  
    TypeEnv(vars + pair, tys)  
  def ++(pairs: Iterable[(String, Type)]): TypeEnv =  
    TypeEnv(vars ++ pairs, tys)  
  def addTy(tname: String, ws: Map[String, List[Type]]): TypeEnv =  
    TypeEnv(vars, tys + (tname -> ws))  
}
```

For example, consider the following an ADT for binary trees:

```
/* ATFAE */  
enum Tree {  
  case Leaf(Number)  
  case Node(Tree, Number, Tree)  
} ...
```

We can add the type information of the Tree ADT to an existing type environment  $\Gamma$  (or `tenv`) as follows:

$$\Gamma[\text{Tree} = \text{Leaf}(\text{num}) + \text{Node}(\text{Tree}, \text{num}, \text{Tree})]$$

```
val newTEnv = tenv.addTy(NameT("Tree"), Map(  
  "Leaf" -> List(NumT),  
  "Node" -> List(NameT("Tree"), NumT, NameT("Tree"))  
))
```



```
/* ATFAE */  
enum Tree {  
  case Leaf(Number)  
  case Node(Tree, Number, Tree)  
}  
def f(t: Tree): Int = t  
...
```

It is a well-typed ATFAE expression.

```
/* ATFAE */  
def f(t: Tree): Int = t  
...
```

How about this?

```
/* ATFAE */  
enum Tree {  
  case Leaf(Number)  
  case Node(Tree, Number, Tree)  
}  
def f(t: Tree): Int = t  
...
```

It is a well-typed ATFAE expression.

```
/* ATFAE */  
def f(t: Tree): Int = t  
...
```

How about this? **No!**

It is **syntactically correct** but the `Tree` type is **not defined**.

We need to check the **well-formedness** of types with **type environment**.

We need to check the **well-formedness** of types with **type environment**:

$$\boxed{\Gamma \vdash \tau}$$

$$\frac{}{\Gamma \vdash \text{num}} \qquad \frac{}{\Gamma \vdash \text{bool}} \qquad \frac{\Gamma \vdash \tau_1 \quad \dots \quad \Gamma \vdash \tau_n \quad \Gamma \vdash \tau}{\Gamma \vdash (\tau_1, \dots, \tau_n) \rightarrow \tau}$$

$$\frac{\Gamma(t) = x_1(\tau_{1,1}, \dots, \tau_{1,m_1}) + \dots + x_n(\tau_{n,1}, \dots, \tau_{n,m_n})}{\Gamma \vdash t}$$

```
def mustValid(ty: Type, tenv: TypeEnv): Type = ty match
  case NumT => NumT
  case BoolT => BoolT
  case ArrowT(ptys, rty) =>
    ArrowT(ptys.map(mustValid(_, tenv)), mustValid(rty, tenv))
  case NameT(x) =>
    if (!tenv.tys.contains(x)) error(s"free type variable: $x")
    NameT(x)
```

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
...
case Fun(params, body) =>
  val ptys = params.map(_.ty)
  for (pty <- ptys) mustValid(pty, tenv)
  val rty = typeCheck(body, tenv ++ params.map(p => p.name -> p.ty))
  ArrowT(ptys, rty)
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-Fun} \frac{\Gamma \vdash \tau_1 \quad \dots \quad \Gamma \vdash \tau_n \quad \Gamma[x_1 : \tau_1, \dots, x_n : \tau_n] \vdash e : \tau}{\Gamma \vdash \lambda(x_1 : \tau_1, \dots, x_n : \tau_n).e : (\tau_1, \dots, \tau_n) \rightarrow \tau}$$

We need to check the **well-formedness** of parameter types.

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
...
case Rec(f, params, rty, body, scope) =>
  val ptys = params.map(_.ty)
  for (pty <- ptys) mustValid(pty, tenv)
  mustValid(rty, tenv)
  val fty = ArrowT(ptys, rty)
  val newTEEnv = tenv + (f -> fty) ++ params.map(p => p.name -> p.ty)
  typeCheck(scope, tenv + (f -> fty))
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-Rec} \frac{\begin{array}{c} \Gamma \vdash \tau_1 \quad \dots \quad \Gamma \vdash \tau_n \quad \Gamma \vdash \tau \\ \Gamma[x_0 : (\tau_1, \dots, \tau_n) \rightarrow \tau, x_1 : \tau_1, \dots, x_n : \tau_n] \vdash e : \tau \\ \Gamma[x_0 : (\tau_1, \dots, \tau_n) \rightarrow \tau] \vdash e' : \tau' \end{array}}{\Gamma \vdash \text{def } x_0(x_1 : \tau_1, \dots, x_n : \tau_n) : \tau = e; e' : \tau'}$$

We need to check the **well-formedness** of parameter and return types.

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
...
case App(fun, args) => typeCheck(fun, tenv) match
  case ArrowT(ptys, retTy) =>
    if (ptys.length != args.length) error("arity mismatch")
    (ptys zip args).map((p, a) => mustSame(typeCheck(a, tenv), p))
    retTy
  case ty => error(s"not a function type: ${ty.str}")
```

$$\tau\text{-App} \frac{\Gamma \vdash e_0 : (\tau_1, \dots, \tau_n) \rightarrow \tau \quad \Gamma \vdash e_1 : \tau_1 \quad \dots \quad \Gamma \vdash e_n : \tau_n}{\Gamma \vdash e_0(e_1, \dots, e_n) : \tau}$$

**No change** in the type checking for **function application**.

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
...
case TypeDef(tname, ws, body) =>
  val newTEEnv = tenv.addTy(tname, ws.map(w => w.name -> w.ptys).toMap)
  for (w <- ws; pty <- w.ptys) mustValid(pty, newTEEnv)
  typeCheck(
    body,
    newTEEnv ++ ws.map(w => w.name -> ArrowT(w.ptys, NameT(tname)))
  )
```

$$\begin{array}{c}
 \Gamma' = \Gamma[t = x_1(\tau_{1,1}, \dots, \tau_{1,m_1}) + \dots + x_n(\tau_{n,1}, \dots, \tau_{n,m_n})] \\
 \Gamma' \vdash \tau_{1,1} \quad \dots \quad \Gamma' \vdash \tau_{n,m_n} \quad \Gamma' \left[ \begin{array}{l} x_1 : (\tau_{1,1}, \dots, \tau_{1,m_1}) \rightarrow t, \\ \dots, \\ x_n : (\tau_{n,1}, \dots, \tau_{n,m_n}) \rightarrow t \end{array} \right] \vdash e : \tau \\
 \hline
 \tau\text{-TypeDef} \quad \Gamma \vdash \text{enum } t \left\{ \begin{array}{l} \text{case } x_1(\tau_{1,1}, \dots, \tau_{1,m_1}) \\ \dots \\ \text{case } x_n(\tau_{n,1}, \dots, \tau_{n,m_n}) \end{array} \right\}; e : \tau
 \end{array}$$

It is indeed **type unsound**, and we will fix it later.

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
...
case Match(expr, cs) => typeCheck(expr, tenv) match
  case NameT(tname) =>
    val ts = tenv.tys.getOrElse(tname, error(s"unknown type: $tname"))
    val xs = cs.map(_.name).toSet
    if (ts.keySet != xs || xs.size != cs.length) error("invalid case")
    cs.map {
      case MatchCase(x, ps, b) => typeCheck(b, tenv ++ (ps zip ts(x)))
    }.reduce((lty, rty) => { mustSame(lty, rty); lty })
  case _ => error("not a variant")
```

$$\tau\text{-Match} \frac{\begin{array}{l} \Gamma \vdash e : t \quad \Gamma(t) = x_1(\tau_{1,1}, \dots, \tau_{1,m_1}) + \dots + x_n(\tau_{n,1}, \dots, \tau_{n,m_n}) \\ \forall 1 \leq i \leq n. \Gamma_i = \Gamma[x_{i,1} : \tau_{i,1}, \dots, x_{i,m_i} : \tau_{i,m_i}] \quad \Gamma_1 \vdash e_1 : \tau \quad \dots \quad \Gamma_n \vdash e_n : \tau \end{array}}{\Gamma \vdash e \text{ match } \left\{ \begin{array}{l} \text{case } x_1(x_{1,1}, \dots, x_{1,m_1}) \Rightarrow e_1 \\ \dots \\ \text{case } x_n(x_{n,1}, \dots, x_{n,m_n}) \Rightarrow e_n \end{array} \right\} : \tau}$$



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Algebraic Data Types - Revised (1)

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## Definition (Type Soundness)

A **type system** is **sound** if it guarantees that a **well-typed** program will **never** cause a **type error** at run-time.

Consider the following ATFAE expression:

```
/* ATFAE */
enum A { case X(Number) }           // X: Number => A
val f = (a: A) = a match { case X(n) => n } // f: A => Number
enum A { case X(Boolean) }          // X: Boolean => A
f(X(true)) + 1                      // TypeError: true + 1
```

It throws a type error when evaluating `true + 1` at run-time while this expression is **well-typed** according to the type system.

It happens because the **same type name** `A` is defined twice and **shadows** the previous one with **different types** for its **variants**.

Let's **forbid** the redefinition of **same type name** in the scope of **ADTs**!

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
...
case TypeDef(tname, ws, body) =>
  if (tenv.tys.contains(tname)) error(s"already defined type: $tname")
  val newTEnv = tenv.addTy(tname, ws.map(w => w.name -> w.ptys).toMap)
  for (w <- ws; pty <- w.ptys) mustValid(pty, newTEnv)
  typeCheck(
    body,
    newTEnv ++ ws.map(w => w.name -> ArrowT(w.ptys, NameT(tname)))
  )
```

$$\begin{array}{c}
 \Gamma' = \Gamma[t = x_1(\tau_{1,1}, \dots, \tau_{1,m_1}) + \dots + x_n(\tau_{n,1}, \dots, \tau_{n,m_n})] \\
 t \notin \text{Domain}(\Gamma) \quad \Gamma' \vdash \tau_{1,1} \quad \dots \quad \Gamma' \vdash \tau_{n,m_n} \\
 \Gamma' \left[ \begin{array}{l} x_1 : (\tau_{1,1}, \dots, \tau_{1,m_1}) \rightarrow t, \\ \dots, \\ x_n : (\tau_{n,1}, \dots, \tau_{n,m_n}) \rightarrow t \end{array} \right] \vdash e : \tau \\
 \hline
 \tau\text{-TypeDef} \quad \Gamma \vdash \text{enum } t \left\{ \begin{array}{l} \text{case } x_1(\tau_{1,1}, \dots, \tau_{1,m_1}) \\ \dots \\ \text{case } x_n(\tau_{n,1}, \dots, \tau_{n,m_n}) \end{array} \right\}; e : \tau
 \end{array}$$

Now, consider the following another ATFAE expression:

```
/* ATFAE */  
val f = {  
  enum A { case X(Number) }           // X: Number => A  
  (a: A) => a match { case X(n) => n }  
}  
enum A { case X(Boolean) }           // f: A => Number  
f(X(true)) + 1                       // X: Boolean => A  
                                     // TypeError: true + 1
```

It throws a type error when evaluating `true + 1` at run-time while this expression is **well-typed** according to the type system.

Since the second `A` type does not shadow the first one, the type system allows the definition of the second `A` type.

It happens because the first `A` type escapes its scope and is still visible in the scope of the second `A` type.

Let's **forbid** the escape of **ADTs** from their scope!

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
...
case TypeDef(tname, ws, body) =>
  if (tenv.tys.contains(tname)) error(s"already defined type: $tname")
  val newTEnv = tenv.addTy(tname, ws.map(w => w.name -> w.ptys).toMap)
  for (w <- ws; pty <- w.ptys) mustValid(pty, newTEnv)
  mustValid(typeCheck(
    body,
    newTEnv ++ ws.map(w => w.name -> ArrowT(w.ptys, NameT(tname)))
  ), tenv)
```

$$\begin{array}{c}
 \Gamma' = \Gamma[t = x_1(\tau_{1,1}, \dots, \tau_{1,m_1}) + \dots + x_n(\tau_{n,1}, \dots, \tau_{n,m_n})] \\
 t \notin \text{Domain}(\Gamma) \quad \Gamma' \vdash \tau_{1,1} \quad \dots \quad \Gamma' \vdash \tau_{n,m_n} \\
 \Gamma' \left[ \begin{array}{l} x_1 : (\tau_{1,1}, \dots, \tau_{1,m_1}) \rightarrow t, \\ \dots, \\ x_n : (\tau_{n,1}, \dots, \tau_{n,m_n}) \rightarrow t \end{array} \right] \vdash e : \tau \quad \Gamma \vdash \tau \\
 \tau\text{-TypeDef} \frac{}{\Gamma \vdash \text{enum } t \left\{ \begin{array}{l} \text{case } x_1(\tau_{1,1}, \dots, \tau_{1,m_1}) \\ \dots \\ \text{case } x_n(\tau_{n,1}, \dots, \tau_{n,m_n}) \end{array} \right\}; e : \tau}
 \end{array}$$

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- Parametric Polymorphism

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