

# Lecture 9 – Recursive Functions

## COSE212: Programming Languages

Jihyeok Park



2023 Fall

- Syntactic Sugar
  - FAE – Removing `val` from FVAE
  - Syntactic Sugar and Desugaring
- Lambda Calculus (LC)
  - Church Encodings
  - Church-Turing Thesis

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- In this lecture, we will learn **recursion** and **conditionals**.
- **RFAE – FAE with recursive functions**
  - Concrete and Abstract Syntax
  - Interpreter and Natural Semantics

## 1. Recursion

- Recursion in F1VAE

- Recursion in FVAE

- mkRec: Helper Function for Recursion

## 2. RFAE – FAE with Recursion and Conditionals

- Concrete Syntax

- Abstract Syntax

## 3. Interpreter and Natural Semantics for RFAE

- Definition with Desugaring

- Interpreter and Natural Semantics

- Arithmetic Comparison Operators

- Conditionals

- Recursive Function Definitions

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A **recursive function** is a function that calls itself, and it is useful for **iterative processes** on **inductive data structures**.

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Let's define a **recursive function** `sum` that computes the sum of integers from 1 to  $n$  in Scala:

```
def sum(n: Int): Int =  
  if (n < 1) 0           // base case  
  else n + sum(n - 1)    // recursive case  
  
sum(10) // 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 = 55
```



A **recursive function** is a function that calls itself, and it is useful for **iterative processes** on **inductive data structures**.

Let's define a **recursive function** `sum` that computes the sum of integers from 1 to  $n$  in Scala:

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```

For recursive functions, we need **conditionals** to define 1) **base cases** and 2) **recursive cases**.

Most programming languages support **recursive functions**:

- Scala

```
def sum(n: Int): Int = if (n < 1) 0 else n + sum(n - 1)
```

- C++

```
int sum(int n) { return n < 1 ? 0 : n + sum(n - 1); }
```

- Python

```
def sum(n): return 0 if n < 1 else n + sum(n - 1)
```

- Rust

```
fn sum(n: i32) -> i32 { if n < 1 {0} else {n + sum(n-1)} }
```

- ...

If we add **conditionals** to F1VAE, we can define recursive functions in F1VAE without any more extensions for recursion.

Programs  $\mathbb{P} \ni p ::= f^* e$  (Program)

Function Definitions  $\mathbb{F} \ni f ::= \text{def } x(x)=e$  (FunDef)

Expressions  $\mathbb{E} \ni e ::= \dots$

|  $e < e$  (Lt)

|  $\text{if } (e) e \text{ else } e$  (If)

Values  $\mathbb{V} \ni v ::= n \mid b \mid \langle \lambda x.e, \sigma \rangle$

Function Environments  $\Lambda \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{F}$  (FEnv)

Boolean  $b \in \mathbb{B} = \{\text{true}, \text{false}\}$  (Boolean)

```
/* F1VAE */
def sum(n) = if (n < 1) 0 else n + sum(n + -1)
```

$$\Lambda = [\text{sum} \mapsto f_0]$$

where  $f_0 = \text{def sum}(n)=\text{if } (n < 1) 0 \text{ else } n + \text{sum}(n + -1)$

However, the following FVAE expression is not a recursive function:

```
/* FVAE */  
val sum = n => {  
  if (n < 1) 0  
  else n + sum(n + -1)  
};  
sum(10)
```

Why?

However, the following FVAE expression is not a recursive function:

```
/* FVAE */  
val sum = n => {  
  if (n < 1) 0  
  else n + sum(n + -1)  
};  
sum(10)
```

Why?

`val` does not support recursive definitions. Thus, `sum` is **NOT** in the scope of the function body!

Let's pass the function as an argument to itself!

```
/* FVAE */  
val sumX = sumY => {  
  n => {  
    if (n < 1) 0  
    else n + sumY(sumY)(n + -1)  
  }  
};  
sumX(sumX)(10)
```

```
/* FVAE */  
val sumX = sumY => {  
  n => {  
    if (n < 1) 0  
    else n + sumY(sumY)(n + -1)  
  }  
};  
sumX(sumX)(10)
```

However, it is annoying to always pass the function as an argument to itself!

Let's wrap this to get sum back!

```
/* FVAE */  
val sum = n => {  
  val sumX = sumY => {  
    n => {  
      if (n < 1) 0  
      else n + sumY(sumY)(n + -1)  
    }  
  };  
  sumX(sumX)(n)  
};  
sum(10)
```



```
/* FVAE */  
val sum = n => {  
  val sumX = sumY => {  
    n => {  
      if (n < 1) 0  
      else n + sumY(sumY)(n + -1)  
    }  
  };  
  sumX(sumX)(n)  
};  
sum(10)
```

We can simplify this using  $\eta$ -**reduction**:

$$\lambda x.e(x) \quad \equiv \quad e \quad \text{only if } x \text{ is } \mathbf{NOT \ FREE} \text{ in } e.$$

```
/* FVAE */  
val sum = {  
  val sumX = sumY => {  
    n => { // ALMOST the same as the original body  
      if (n < 1) 0  
      else n + sumY(sumY)(n + -1)  
    }  
  };  
  sumX(sumX)  
};  
sum(10)
```

```
/* FVAE */  
val sum = {  
  val sumX = sumY => {  
    n => { // ALMOST the same as the original body  
      if (n < 1) 0  
      else n + sumY(sumY)(n + -1)  
    }  
  };  
  sumX(sumX)  
};  
sum(10)
```

The function body is almost the same as the original version except that we need to call the function as `sumY(sumY)` instead of `sum`.

Let's define a variable `sum` to be `sumY(sumY)`!

```
/* FVAE */  
val sum = {  
  val sumX = sumY => {  
    val sum = sumY(sumY); // INFINITE LOOP  
    n => { // EXACTLY the same as the original body  
      if (n < 1) 0  
      else n + sum(n + -1)  
    }  
  };  
  sumX(sumX)  
};  
sum(10)
```

```

/* FVAE */
val sum = {
  val sumX = sumY => {
    val sum = sumY(sumY); // INFINITE LOOP
    n => { // EXACTLY the same as the original body
      if (n < 1) 0
      else n + sum(n + -1)
    }
  };
  sumX(sumX)
};
sum(10)

```

Unfortunately, this is an **infinite loop**!

We need to **delay** the evaluation of `sum` using the  *$\eta$ -expansion*:

$$e \quad \equiv \quad \lambda x. e(x) \quad \text{only if } x \text{ is } \mathbf{NOT \text{ FREE}} \text{ in } e.$$

```
/* FVAE */
val sum = {
  val sumX = sumY => {
    val sum = x => sumY(sumY)(x);
    n => { // EXACTLY the same as the original body
      if (n < 1) 0
      else n + sum(n + -1)
    }
  };
  sumX(sumX)
};
sum(10)
```

```
/* FVAE */  
val sum = {  
  val sumX = sumY => {  
    val sum = x => sumY(sumY)(x);  
    n => { // EXACTLY the same as the original body  
      if (n < 1) 0  
      else n + sum(n + -1)  
    }  
  };  
  sumX(sumX)  
};  
sum(10)
```

Do we need to do this for every recursive function?

To avoid such boilerplate code, let's define a helper function `mkRec`!

```
/* FVAE */  
val mkRec = body => {  
  val fX = fY => {  
    val f = x => fY(fY)(x);  
    body(f)  
  };  
  fX(fX)  
};  
val sum = mkRec(sum => n => { // EXACTLY the same as the original body  
  if (n < 1) 0  
  else n + sum(n + -1)  
});  
sum(10)
```



```
/* FVAE */  
val mkRec = body => {  
  val fX = fY => {  
    val f = x => fY(fY)(x);  
    body(f)  
  };  
  fX(fX)  
};  
val sum = mkRec(sum => n => { // EXACTLY the same as the original body  
  if (n < 1) 0  
  else n + sum(n + -1)  
});  
sum(10)
```

For example, we can define factorial (fac) function using mkRec:

```
/* FVAE */  
val fac = mkRec(fac => n => if (n < 1) 1 else n * fac(n + -1));  
fac(5) // 5 * 4 * 3 * 2 * 1 = 120
```

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Now, let's extend FAE into RFAE with **recursion** and **conditionals**.

```
/* RFAE */  
def sum(n) = {  
  if (n < 1) 0  
  else n + sum(n + -1)  
};  
sum(10) // 55
```

```
/* RFAE */  
def fib(n) = {  
  if (n < 2) n  
  else fib(n + -1) + fib(n + -2)  
};  
fib(7) // 13
```

Now, let's extend FAE into RFAE with **recursion** and **conditionals**.

```
/* RFAE */  
def sum(n) = {  
  if (n < 1) 0  
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};  
sum(10) // 55
```

```
/* RFAE */  
def fib(n) = {  
  if (n < 2) n  
  else fib(n + -1) + fib(n + -2)  
};  
fib(7) // 13
```

For RFAE, we need to extend **expressions** of FAE with

- 1 **arithmetic comparison operators**
- 2 **conditionals**
- 3 **recursive function definitions**

```
// expressions
<expr> ::= ...
    | <expr> "<" <expr>
    | "if" "(" <expr> ")" <expr> "else" <expr>
    | "def" <id> "(" <id> ")" "=" <expr> ";" <expr>
```

For RFAE, we need to extend **expressions** of FAE with

- ① **arithmetic comparison operators**
- ② **conditionals**
- ③ **recursive function definitions**

Let's define the **abstract syntax** of FVAE in BNF:

Expressions  $\mathbb{E} \ni e ::= \dots$

$e < e$	(Lt)
$\text{if } (e) \ e \ \text{else } e$	(If)
$\text{def } x(x)=e; e$	(Rec)

Let's define the **abstract syntax** of FVAE in BNF:

Expressions  $\mathbb{E} \ni e ::= \dots$

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$\text{if } (e) \ e \ \text{else } e$	(If)
$\text{def } x(x)=e; e$	(Rec)

```
enum Expr:
    ...
    // less-than
    case Lt(left: Expr, right: Expr)
    // conditionals
    case If(cond: Expr, thenExpr: Expr, elseExpr: Expr)
    // recursive function definition
    case Rec(name: String, param: String, body: Expr, scope: Expr)
```

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One way to define the semantics of **recursive function definitions** by treating them as **syntactic sugar** and **desugar** them with `mkRec`:

$$\mathcal{D}[\text{def } x_0(x_1)=e_0; e_1] = \mathcal{D}[\text{val } x_0=\text{mkRec}(\lambda x_0.\lambda x_1.e_0); e_1]$$

One way to define the semantics of **recursive function definitions** by treating them as **syntactic sugar** and **desugar** them with `mkRec`:

$$\begin{aligned}\mathcal{D}[\text{def } x_0(x_1)=e_0; e_1] &= \mathcal{D}[\text{val } x_0=\text{mkRec}(\lambda x_0.\lambda x_1.e_0); e_1] \\ &= (\lambda x_0.\mathcal{D}[e_1])(\text{mkRec}(\lambda x_0.\lambda x_1.\mathcal{D}[e_0]))\end{aligned}$$

One way to define the semantics of **recursive function definitions** by treating them as **syntactic sugar** and **desugar** them with `mkRec`:

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For example,

```
/* RFAE */
def sum(n) = if (n<1) 0 else n+sum(n+1); sum(10)
// will be desugared into
(sum => sum(10))(mkRec(sum => (n => if (n<1) 0 else n+sum(n+1))))
```

```
/* RFAE */
def fib(n) = if(n<2) n else fib(n+1)+fib(n+2); fib(7)
// will be desugared into
(fib => fib(7))(mkRec(fib => (n => if(n<2) n else fib(n+1)+fib(n+2))))
```

Another way is to directly 1) implement the **interpreter**:

```
def interp(expr: Expr, env: Env): Value = ???
```

and 2) define the **natural semantics** for **recursive function definitions** and other new cases.

$$\sigma \vdash e \Rightarrow v$$

Expressions  $\mathbb{E} \ni e ::= \dots$

$e < e$	(Lt)
$\text{if } (e) \ e \text{ else } e$	(If)
$\text{def } x(x)=e; e$	(Rec)

Values  $\mathbb{V} \ni v ::= n \mid b \mid \langle \lambda x. e, \sigma \rangle$

```
enum Value:  
  case NumV(number: BigInt)  
  case BoolV(bool: Boolean)  
  case CloV(param: String, body: Expr, env: Env)
```

```

type NCOp = (BigInt, BigInt) => Boolean
def numCOp(x: String)(op: NCOp)(l: Value, r: Value): Value = (l, r)
  match
    case (NumV(l), NumV(r)) => BoolV(op(l, r))
    case (l, r) => error(s"invalid operation: ${l.str} $x ${r.str}")

val numLt: (Value, Value) => Value = numCOp("<")(_ < _)

def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Lt(l, r) => numLt(interp(l, env), interp(r, env))
    
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{Lt} \frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2}{\sigma \vdash e_1 < e_2 \Rightarrow n_1 < n_2}$$

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case If(c, t, e) => interp(c, env) match
    case BoolV(true)  => interp(t, env)
    case BoolV(false) => interp(e, env)
    case v             => error(s"not a boolean: ${v.str}")
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{If}_T \frac{\sigma \vdash e_0 \Rightarrow \text{true} \quad \sigma \vdash e_1 \Rightarrow v_1}{\sigma \vdash \text{if } (e_0) \text{ } e_1 \text{ else } e_2 \Rightarrow v_1}$$

$$\text{If}_F \frac{\sigma \vdash e_0 \Rightarrow \text{false} \quad \sigma \vdash e_2 \Rightarrow v_2}{\sigma \vdash \text{if } (e_0) \text{ } e_1 \text{ else } e_2 \Rightarrow v_2}$$

```
def interp(expr: Expr, env: Env): Value = expr match
...
case Rec(n, p, b, s) =>
  val newEnv: Env = ???
  interp(s, newEnv)
```

$$\boxed{\sigma \vdash e \Rightarrow v}$$

$$\text{Rec} \frac{\sigma' = \sigma[x_0 \mapsto \langle \lambda x_1. e_0, \sigma' \rangle] \quad \sigma' \vdash e_1 \Rightarrow v_1}{\sigma \vdash \text{def } x_0(x_1) = e_0; e_1 \Rightarrow v_1}$$

```
def interp(expr: Expr, env: Env): Value = expr match
...
case Rec(n, p, b, s) =>
  val newEnv: Env = env + (n -> CloV(p, b, newEnv)) // error
  interp(s, newEnv)
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{Rec} \frac{\sigma' = \sigma[x_0 \mapsto \langle \lambda x_1. e_0, \sigma' \rangle] \quad \sigma' \vdash e_1 \Rightarrow v_1}{\sigma \vdash \text{def } x_0(x_1) = e_0; e_1 \Rightarrow v_1}$$

Let's **delay** the evaluation of newEnv using the  $\eta$ -**expansion** again:

$$e \quad \equiv \quad \lambda x. e(x) \quad \text{only if } x \text{ is } \mathbf{NOT \text{ FREE}} \text{ in } e.$$



We augment the closure value with an **environment factory**  $() \Rightarrow \text{Env}$  rather than an **environment**  $(\text{Env})$ :

```
enum Value:
  ...
  case CloV(param: String, body: Expr, env: () => Env)

def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Func(p, b) => CloV(p, b, () => env)
  case App(f, e) => interp(f, env) match
    case CloV(p, b, fenv) => interp(b, fenv() + (p -> interp(e, env)))
    case v                => error(s"not a function: ${v.str}")
  case Rec(n, p, b, s) =>
    val newEnv: Env = env + (n -> CloV(p, b, () => newEnv)) // error
    interp(s, newEnv)
```

It still doesn't work because `newEnv` is not yet defined.

Let's use a **lazy value** (`lazy val`) to delay the evaluation of `newEnv`.

```
def interp(expr: Expr, env: Env): Value = expr match
...
case Rec(n, p, b, s) =>
  lazy val newEnv: Env = env + (n -> CloV(p, b, () => newEnv))
  interp(s, newEnv)
```

$$\boxed{\sigma \vdash e \Rightarrow v}$$

$$\text{Rec} \frac{\sigma' = \sigma[x_0 \mapsto \langle \lambda x_1. e_0, \sigma' \rangle] \quad \sigma' \vdash e_1 \Rightarrow v_1}{\sigma \vdash \text{def } x_0(x_1) = e_0; e_1 \Rightarrow v_1}$$

We will learn more about **lazy values** in the later lectures in this course.

- Please see this document<sup>1</sup> on GitHub.
  - Implement `interp` function.
- It is just an exercise, and you **don't need to submit** anything.
- However, some exam questions might be related to this exercise.

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<sup>1</sup><https://github.com/ku-plrg-classroom/docs/tree/main/cose212/rfae>.

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- Recursive Function Definitions

- Mutable Data Structures

Jihyeok Park

[jihyeok\\_park@korea.ac.kr](mailto:jihyeok_park@korea.ac.kr)

<https://plrg.korea.ac.kr>