Lecture 14 – Continuations (1)

COSE212: Programming Languages

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PLRG

2023 Fall

Recall

- Lazy Evaluation
  - Call-by-Name (CBN)
  - Call-by-Need (CBN')
- LFAE – FAE with Lazy Evaluation

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- We will learn about **continuations** with the following topics:
  - **Continuations** (Lecture 14 & 15)
  - **First-Class Continuations** (Lecture 16)
  - **Compiling with continuations** (Lecture 17)

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- Lazy Evaluation
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- LFAE – FAE with Lazy Evaluation
- We will learn about **continuations** with the following topics:
  - **Continuations** (Lecture 14 & 15)
  - **First-Class Continuations** (Lecture 16)
  - **Compiling with continuations** (Lecture 17)
- In this lecture, we will focus on the meaning of **continuations**.

Contents

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Continuations

Many real-world programming languages support **control statements** to change the **control-flow** of a program.

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Many real-world programming languages support **control statements** to change the **control-flow** of a program.

For example, `C++` supports `break`, `continue`, and `return` statements:

```
int sumEvenUntilZero(int xs[], int len) {
  if (len <= 0) return 0;           // directly return 0 if len <= 0
  int sum = 0;
  for (int i = 0; i < len; i++) {
    if (xs[i] == 0) break;          // stop the loop if xs[i] == 0
    if (xs[i] % 2 == 1) continue;   // skip the rest if xs[i] is odd
    sum += xs[i];
  }
  return sum;                       // finally return the sum
}
int xs[] = {4, 1, 3, 2, 0, 6, 5, 8};
sumEvenUntilZero(xs, 8);            // 4 + 2 = 6
```

How can we represent them in functional languages?

Continuations

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int xs[] = {4, 1, 3, 2, 0, 6, 5, 8};
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How can we represent them in functional languages? **Continuations**!

Continuations

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/* FAE */
(1 + 3) * 5
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It **implicitly** represents the following computation:

1. Evaluate 1.
2. Evaluate 3.
3. Add 1 and 3 to get 4.
4. Evaluate 5.
5. Multiply 4 and 5 to get 20.

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The **continuation** of k-th step is the steps from $(k+1)$-th to the last one.

For instance, the **continuation** of the 3rd step is the 4th and 5th steps.

Continuations

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Yes! Let's represent the **continuation** of the k-th step as a **function** that

- **takes** the result of the k-th step as an argument and
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Continuations

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```
/* FAE */
{
  x1 => (x1 + 3) * 5      // step 2-5 (continuation of step 1)
}(1)                      // step 1
```

Continuations

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```
/* FAE */
{
  x1 => {
    x2 => (x1 + x2) * 5   // step 3-5 (continuation of step 2)
  }(3)                    // step 2
}(1)                      // step 1
```

Continuations

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```
/* FAE */
{
  x1 => {
    x2 => {
      x3 => x3 * 5          // step 4-5 (continuation of step 3)
    }(x1 + x2)              // step 3
  }(3)                      // step 2
}(1)                        // step 1
```

Continuations

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```
/* FAE */
{
  x1 => {
    x2 => {
      x3 => {
        x4 => x3 * x4     // step 5 (continuation of step 4)
      }(5)                // step 4
    }(x1 + x2)            // step 3
  }(3)                    // step 2
}(1)                      // step 1
```

Continuations

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```
/* FAE */
{
  x1 => {
    x2 => {
      x3 => {
        x4 => {
          x5 => x5          // no more steps (continuation of step 5)
        }(x3 * x4)          // step 5
      }(5)                  // step 4
    }(x1 + x2)              // step 3
  }(3)                      // step 2
}(1)                        // step 1
```

Continuations

1. Evaluate 1.
2. Evaluate 3.
3. Add 1 and 3 to get 4.
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```
/* FAE */
val x1 = 1                // step 1
val x2 = 3                // step 2
val x3 = x1 + x2          // step 3
val x4 = 5                // step 4
val x5 = x3 * x4          // step 5
x5                        // no more steps (continuation of step 5)
```

by using the syntactic sugar for variable declarations:

$$\mathcal{D}[\![\texttt{val}\ x\texttt{=}e\texttt{;}\ \ e']\!] = (\lambda x.\mathcal{D}[\![e']\!])(\mathcal{D}[\![e]\!])$$

Contents

Continuation-Passing Style (CPS)

Continuation-passing style (CPS) is a style of programming that passes the continuation as an explicit parameter to function calls.

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For example, consider the following Scala `sum` function:

```
def sum(n: Int): Int =
  if (n <= 1) 1
  else sum(n - 1) + n

sum(3) * 5              // (1 + 2 + 3) * 5 = 30
```

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```

Let's rewrite the `sum` function in CPS:

```scala
type Cont = Int => Int
def sumCPS(n: Int, k: Cont): Int = ???

sumCPS(3, x => x * 5)   // (1 + 2 + 3) * 5 = 30
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Continuation-Passing Style (CPS)

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def sum(n: Int): Int =
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```

Let's rewrite the `sum` function in CPS:

```
type Cont = Int => Int
def sumCPS(n: Int, k: Cont): Int = k(sum(n))
```

It is not the correct implementation of `sum` in CPS because it depends on the original `sum` function.

Let's replace `sum(n)` with the body of `sum`.

Continuation-Passing Style (CPS)

```
def sum(n: Int): Int =
  if (n <= 1) 1
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Let's rewrite the sum function in CPS:

```
type Cont = Int => Int
def sumCPS(n: Int, k: Cont): Int = k(
  if (n <= 1) 1
  else sum(n - 1) + n
)
```

Let's utilize the following equivalence:

```
e0(if (e1) e2 else e3)     ==     if (e1) e0(e2) else e0(e3)
```

Continuation-Passing Style (CPS)

```
def sum(n: Int): Int =
  if (n <= 1) 1
  else sum(n - 1) + n
```

Let's rewrite the `sum` function in CPS:

```
type Cont = Int => Int
def sumCPS(n: Int, k: Cont): Int =
  if (n <= 1) k(1)
  else k(sum(n - 1) + n)
```

But, it still depends on the original `sum` function.

Let's consider the continuation of `sum(n - 1)`:

```
k(sum(n - 1) + n)     ==     (x => k(x + n))(sum(n - 1))
                      ==     sumCPS(n - 1, x => k(x + n))
```

Continuation-Passing Style (CPS)

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```

If all functions are written in CPS, a program satisfies the properties:

- Every function takes a continuation as an explicit parameter.
- A continuation is used at most once in a function body.
- Every function call is in a tail position. (tail-call optimization)
- Every function ends with a function call.

Contents

Interpreter of FAE in CPS

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type Cont = Int => Int
def interpCPS(expr: Expr, env: Env, k: Cont): Value = ???
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```
type Cont = Int => Int
def interpCPS(expr: Expr, env: Env, k: Cont): Value =
  k(interp(expr, env))
```

Interpreter of FAE in CPS

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To **explicitly** represent continuations of the evaluation of each expression in the interpreter of FAE, we need to modify the interpreter in CPS:

```
type Cont = Int => Int
def interpCPS(expr: Expr, env: Env, k: Cont): Value = k(expr match
  case Num(n)    => NumV(n)
  case Add(l, r) => numAdd(interp(l, env), interp(r, env))
  case Mul(l, r) => numMul(interp(l, env), interp(r, env))
  case Id(x)     => env.getOrElse(x, error(s"free identifier: $x"))
  case Fun(p, b) => CloV(p, b, env)
  case App(f, a) => interp(f, env) match
    case CloV(p, b, fenv) => interp(b, fenv + (p -> interp(a, env)))
    case v                => error(s"not a function: ${v.str}")
)
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To **explicitly** represent continuations of the evaluation of each expression in the interpreter of FAE, we need to modify the interpreter in CPS:

```
type Cont = Int => Int
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  case Num(n)    => k(NumV(n))
  case Add(l, r) => k(numAdd(interp(l, env), interp(r, env)))
  case Mul(l, r) => k(numMul(interp(l, env), interp(r, env)))
  case Id(x)     => k(env.getOrElse(x, error(s"free identifier: $x")))
  case Fun(p, b) => k(CloV(p, b, env))
  case App(f, a) => k(interp(f, env) match
    case CloV(p, b, fenv) => interp(b, fenv + (p -> interp(a, env)))
    case v                => error(s"not a function: ${v.str}")
  )
```

Addition and Multiplication

```
type Cont = Int => Int
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Add(l, r) =>
    k(numAdd(interp(l, env), interp(r, env)))
  ...
```

The current evaluation part is `interp(l, env)`.

Its continuation is `lv => k(numAdd(lv, interp(r, env)))`.

Addition and Multiplication

```
type Cont = Int => Int
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Add(l, r) =>
    {
      lv => k(numAdd(lv, interp(r, env)))   // cont. of `interp(l, env)`
    }(interp(l, env))
  ...
```

Let's rewrite it by passing the continuation into `interpCPS`.

Addition and Multiplication

```scala
type Cont = Int => Int
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Add(l, r) =>
    (interpCPS(l, env, {
      lv => k(numAdd(lv, interp(r, env)))   // cont. of `interp(l, env)`
    })
  ...
```

Similarly, the current evaluation part is `interp(r, env)`.

Its continuation is `rv => k(numAdd(lv, rv))`.

Addition and Multiplication

```
type Cont = Int => Int
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Add(l, r) =>
    (interpCPS(l, env, {
      lv => {
        rv => k(numAdd(lv, rv))            // cont. of `interp(r, env)`
      }(interp(r, env))
    })
  ...
```

Let's rewrite it by passing the continuation into `interpCPS`.

Addition and Multiplication

```
type Cont = Int => Int
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
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      })
    })
  ...
```

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type Cont = Int => Int
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
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        rv => k(numAdd(lv, rv))
      })
    })
  case Mul(l, r) =>
    (interpCPS(l, env, {
      lv => interpCPS(r, env, {
        rv => k(numMul(lv, rv))
      })
    })
  ...
```

Function Application

```scala
type Cont = Int => Int
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case App(f, a) => k(interp(f, env) match
    case CloV(p, b, fenv) => interp(b, fenv + (p -> interp(a, env)))
    case v                => error(s"not a function: ${v.str}")
  )
  ...
```

In a similar way, we can rewrite function application case.

Function Application

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type Cont = Int => Int
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case App(f, a) => interpCPS(f, env, fv => k(fv match
    case CloV(p, b, fenv) => interp(b, fenv + (p -> interp(a, env)))
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    case CloV(p, b, fenv) =>
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        av => k(interp(b, fenv + (p -> av)))
      })
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  )
  ...
```

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  ...
  case App(f, a) => interpCPS(f, env, fv => fv match
    case CloV(p, b, fenv) =>
      interpCPS(a, env, {
        av => interpCPS(b, fenv + (p -> av), k)
      })
    case v => error(s"not a function: ${v.str}")
  )
  ...
```

Summary

1. Continuations

2. Continuation-Passing Style (CPS)

3. Interpreter of FAE in CPS
    - Addition and Multiplication
    - Function Application

Next Lecture

- Continuations (2)

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