Lecture 24 – Subtype Polymorphism

COSE212: Programming Languages

Jihyeok Park

PLRG

2023 Fall

Homework #4

- Please see this document[1] on GitHub.
- The due date is Dec. 14 (Thu.).
- Please only submit `Implementation.scala` file to **Blackboard**.

[1] https://github.com/ku-plrg-classroom/docs/tree/main/cose212/battery.

Recall

- **Polymorphism** is to use a single entity as **multiple types**, and there are various kinds of polymorphism:
  - **Parametric polymorphism**
  - **Subtype polymorphism**
  - **Ad-hoc polymorphism**
  - ...

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- **PTFAE** – TFAE with **parametric polymorphism**.
- In this lecture, we will learn **subtype polymorphism**.

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  - **Subtype polymorphism**
  - **Ad-hoc polymorphism**
  - ...
- **Parametric polymorphism** is a form of polymorphism by introducing **type variables** and instantiating them with **type arguments**.
- **PTFAE** – TFAE with **parametric polymorphism**.
- In this lecture, we will learn **subtype polymorphism**.
- **STFAE** – TFAE with **subtype polymorphism**.
  - **Interpreter** and **Natural Semantics**
  - **Type Checker** and **Typing Rules**

Contents

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Subtype Polymorphism

To easily explain **subtype polymorphism**, let's add new language syntax, **records** and **record types** to TFAE. (Also, type annotations for `val`.)

```
/* STFAE */
// A record with two fields `a` and `b` whose types are `Number`
val x: {a: Number, b: Number} = {a=1, b=2}
x.a    // Access the field `a` of `x` and evaluate to `1`
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Consider the following expression:

```
/* STFAE */
val f = (x: ???) => x.a
f({a=1}) + f({a=2, b=3}) + f({c=4, a=5})
```

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Unfortunately, we cannot assign any type to `x` because the type of `x` should be 1 `{a: Number}`, 2 `{a: Number, b: Number}`, and 3 `{c: Number, a: Number}`, simultaneously.

How can we resolve this problem?

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Subtype Polymorphism

Definition (Subtype Polymorphism)

Subtype polymorphism is a form of polymorphism by introducing **subtype relationships** between types.

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```
/* STFAE */
val f = (x: {a: Number}) => x.a      // Allow subtypes of `{a: Number}`
f({a=1}) + f({a=2, b=3}) + f({c=4, a=5})
```

All the following types are **subtypes** of `{a: Number}`:

`{a: Number}`   `{a: Number, b: Number}`

`{c: Number, a: Number}`   ...

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It corresponds to the **subset relationship** between sets in mathematics, and most programming languages support **subtype polymorphism**.

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`{a: Number}` `{a: Number, b: Number}`
`{c: Number, a: Number}` ...

It corresponds to the **subset relationship** between sets in mathematics, and most programming languages support **subtype polymorphism**.

Subtype relationships could be defined for other types (e.g., lists, pairs, datatypes, etc.) as well.

Contents

STFAE – TFAE with Subtype Polymorphism

Now, let's extend TFAE into STFAE to support **subtype polymorphism** with **records**:

```
/* STFAE */
val x: {a: Number} = {a=1}              // A record with a field `a`
x.a                                     // Access the field `a` of `x`
val x: {a: Top, b: Top} = {a=1, b=x}    // `Top` is the top type
val x: Bot = exit                       // Exit the program
...                                     // Not evaluated after `exit`
```

STFAE – TFAE with Subtype Polymorphism

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```
/* STFAE */
val x: {a: Number} = {a=1}                // A record with a field `a`
x.a                                       // Access the field `a` of `x`
val x: {a: Top, b: Top} = {a=1, b=x}      // `Top` is the top type
val x: Bot = exit                         // Exit the program
...                                       // Not evaluated after `exit`
```

For STFAE, we need to extend **expressions** of TFAE with

1. **Records**
2. **Field Accesses**
3. **Exit** (to immediately exit the program)
4. **Record Types**
5. **Bottom Type** (corresponding to the empty set)
6. **Top Type** (corresponding to the universal set)

Concrete Syntax

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Concrete Syntax

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We can extend the **concrete syntax** of TFAE as follows:

```
// expressions
<expr>     ::= ...
             | "{" [<id> "=" <expr>]*{","} "}"
             | <expr> "." <id>
             | "exit"
// types
<type>     ::= ...
             | "{" [<id> ":" <type>]*{","} "}"
             | "Bot"
             | "Top"
```

Abstract Syntax

$$
\begin{array}{llll}
\text{Expressions} & \mathbb{E} \ni e ::= \ldots & \mid e.x & (\texttt{Access}) \\
& \mid \{[x = e]^*\} \quad (\texttt{Record}) & \mid \texttt{exit} & (\texttt{Exit})
\end{array}
$$

$$
\begin{array}{llll}
\text{Types} & \mathbb{T} \ni \tau ::= \ldots & \mid \bot & (\texttt{BotT}) \\
& \mid \{[x : \tau]^*\} \quad (\texttt{RecordT}) & \mid \top & (\texttt{TopT})
\end{array}
$$

```
enum Expr:
  ...
  case Record(fields: List[(String, Expr)])
  case Access(record: Expr, field: String)
  case Exit

enum Type:
  ...
  case RecordT(fields: Map[String, Type])
  case BotT
  case TopT
```

Contents

Interpreter and Natural Semantics

For STFAE, we need to 1) implement the **interpreter** with environments:

```scala
def interp(expr: Expr, env: Env): Value = ???
```

and 2) define the **natural semantics** with environments:

$$\boxed{\sigma \vdash e \Rightarrow v}$$

Interpreter and Natural Semantics

For STFAE, we need to 1) implement the **interpreter** with environments:

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```

and 2) define the **natural semantics** with environments:

$$\boxed{\sigma \vdash e \Rightarrow v}$$

with a new kind of values called **record values**:

$$\begin{array}{llll} \text{Values} & \mathbb{V} \ni v ::= n & & (\texttt{NumV}) \\ & \quad\;\; \mid \langle \lambda x.e, \sigma \rangle & & (\texttt{CloV}) \\ & \quad\;\; \mid \{[x = v]^*\} & & (\texttt{RecordV}) \end{array}$$

```
enum Value:
  case NumV(number: BigInt)
  case CloV(param: String, body: Expr, env: Env)
  case RecordV(fields: Map[String, Value])
```

Records and Field Accesses

```
def interp(expr: Expr, env: Env): Value = expr match
  ...

  case Record(fs) =>
    RecordV(fs.map { case (f, e) => (f, interp(e, env)) }.toMap)

  case Access(r, f) => interp(r, env) match
    case RecordV(fs) => fs.getOrElse(f, error(s"no such field: $f"))
    case v           => error(s"not a record: ${v.str}")
```

$$\boxed{\sigma \vdash e \Rightarrow v}$$

$$\texttt{Record}\ \frac{\sigma \vdash e_1 \Rightarrow v_1 \qquad \ldots \qquad \sigma \vdash e_n \Rightarrow v_n}{\sigma \vdash \{x_1 = e_1, \ldots, x_n = e_n\} \Rightarrow \{x_1 = v_1, \ldots, x_n = v_n\}}$$

$$\texttt{Access}\ \frac{\sigma \vdash e \Rightarrow \{x_1 = v_1, \ldots, x_n = v_n\} \qquad 1 \leq i \leq n}{\sigma \vdash e.x_i \Rightarrow v_i}$$

Exit

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Exit => error("exit")
```

$$\boxed{\sigma \vdash e \Rightarrow v}$$

There is no rule for `exit` because it cannot produce any value.

Exit

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def interp(expr: Expr, env: Env): Value = expr match
  ...
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There is no rule for `exit` because it cannot produce any value.

We cannot draw the derivation tree for the following expression:

```
/* STFAE */ (x: Number) => 1 + exit
```

Exit

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def interp(expr: Expr, env: Env): Value = expr match
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/* STFAE */ (x: Number) => 1 + exit
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However, We can draw the derivation tree for the following expression:

```
/* STFAE */ (x: Number) => 1 + exit
```

$$\texttt{Fun}\ \frac{}{\varnothing \vdash \lambda x{:}\texttt{num}.1 + \texttt{exit} \Rightarrow \langle \lambda x.1 + \texttt{exit}, \sigma \rangle}$$

Contents

Type Checker and Typing Rules

Let's ❶ design **typing rules** of STFAE to define when an expression is well-typed in the form of:

$$\boxed{\Gamma \vdash e : \tau}$$

and ❷ implement a **type checker** in Scala according to typing rules:

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = ???
```

The type checker returns the **type** of e if it is well-typed, or rejects it and throws a **type error** otherwise.

Similar to TFAE, we will keep track of the **variable types** using a **type environment** Γ as a mapping from variable names to their types.

$$\text{Type Environments} \qquad \Gamma \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{T} \quad (\texttt{TypeEnv})$$

```
type TypeEnv = Map[String, Type]
```

Records and Field Accesses

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Record(fields) =>
    RecordT(fields.map { case (f, e) => (f, typeCheck(e, tenv)) }.toMap)
  case Access(record, f) => typeCheck(record, tenv) match
    case RecordT(fs) => fs.getOrElse(f, error(s"no such field: $f"))
    case ty          => error(s"not a record type: ${ty.str}")
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-}\mathtt{Record}\ \frac{\Gamma \vdash e_1 : \tau_1 \qquad \ldots \qquad \Gamma \vdash e_n : \tau_n}{\Gamma \vdash \{x_1 = e_1, \ldots, x_n = e_n\} : \{x_1 : \tau_1, \ldots, x_n : \tau_n\}}$$

$$\tau\text{-}\mathtt{Access}\ \frac{\Gamma \vdash e : \{x_1 : \tau_1, \ldots, x_n : \tau_n\} \qquad 1 \leq i \leq n}{\Gamma \vdash e.x_i : \tau_i}$$

Exit

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Exit => BotT
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{–}\texttt{Exit}\ \frac{}{\Gamma \vdash \texttt{exit} : \bot}$$

Immutable Variable Definition

```scala
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Val(name, tyOpt, expr, body) =>
    val ty = typeCheck(expr, tenv)


    typeCheck(body, tenv + (name -> ty))
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{–}\mathtt{Val}\ \frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma[x : \tau_1] \vdash e_2 : \tau_2}{\Gamma \vdash \mathtt{val}\ x = e_1;\ e_2 : \tau_2}$$

Immutable Variable Definition

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Val(name, tyOpt, expr, body) =>
    val ty = typeCheck(expr, tenv)
    tyOpt.map(givenTy => mustEqual(ty, givenTy))
    val nameTy = tyOpt.getOrElse(ty)
    typeCheck(body, tenv + (name -> nameTy))
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{–}\mathtt{Val}\ \frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma[x : \tau_1] \vdash e_2 : \tau_2}{\Gamma \vdash \mathtt{val}\ x\ \mathtt{=}\ e_1\mathtt{;}\ e_2 : \tau_2}$$

$$\tau\text{–}\mathtt{Val}_\tau\ \frac{\Gamma \vdash e_1 : \tau_1 \qquad \tau_1 = \tau_0 \qquad \Gamma[x : \tau_0] \vdash e_2 : \tau_2}{\Gamma \vdash \mathtt{val}\ x : \tau_0\ \mathtt{=}\ e_1\mathtt{;}\ e_2 : \tau_2}$$

Immutable Variable Definition

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$$\tau\text{−}\mathtt{Val}_\tau \; \frac{\Gamma \vdash e_1 : \tau_1 \qquad \tau_1 = \tau_0 \qquad \Gamma[x : \tau_0] \vdash e_2 : \tau_2}{\Gamma \vdash \mathtt{val}\ x : \tau_0 = e_1;\ e_2 : \tau_2}$$

Consider the following example:

```
/* STFAE */
val x: {a: Number} = {a=2, b=3}; x.a
```

It fails to type check because:

$$\mathtt{\{a:\ Number,\ b:\ Number\}} \neq \mathtt{\{a:\ Number\}}$$

Immutable Variable Definition

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$$\tau\text{–}\texttt{Val}_\tau \; \frac{\Gamma \vdash e_1 : \tau_1 \qquad \tau_1 = \tau_0 \qquad \Gamma[x : \tau_0] \vdash e_2 : \tau_2}{\Gamma \vdash \texttt{val } x : \tau_0 \texttt{ = } e_1\texttt{; } e_2 : \tau_2}$$

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It fails to type check because:

$$\texttt{\{a: Number, b: Number\}} \neq \texttt{\{a: Number\}}$$

Let's apply **subtype polymorphism** to fix this problem by introducing a **subtype relation** ($<:$) between types.

Contents

Subtype Relation

To support **subtype polymorphism**, we need to define a **subtype relation** $<:$ between types.

$$\boxed{\tau <: \tau}$$

$\tau <: \tau'$ denotes τ is a subtype of τ' (τ' is more general than τ).

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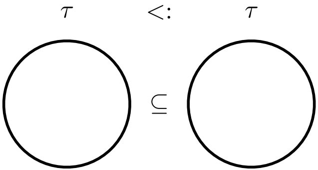
$\tau <: \tau'$ denotes τ is a subtype of τ' (τ' is more general than τ).

First, any type is a **subtype** of itself:

$$\frac{}{\tau <: \tau}$$



$$\boxed{\tau <: \tau}$$

The bottom type $\perp$ and the top type $\top$ represent the empty set of values and the universal set of values, respectively.

Subtype Relation – Bottom Type and Top Type

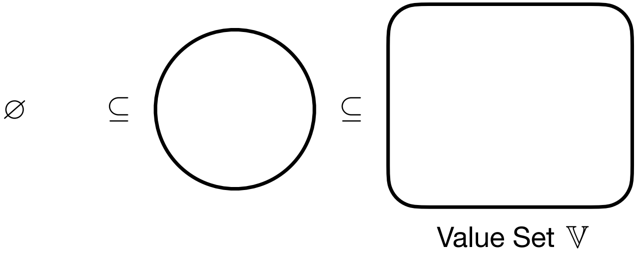
$$\boxed{\tau <: \tau}$$

The bottom type $\bot$ and the top type $\top$ represent the empty set of values and the universal set of values, respectively.

Thus, $\bot$ is a subtype of any type, and any type is a subtype of $\top$:

$$\frac{}{\bot <: \tau} \qquad \frac{}{\tau <: \top}$$

$$\bot \quad <: \quad \tau \quad <: \quad \top$$



Subtype Relation – Record Types (1)

$$\boxed{\tau <: \tau}$$

Let's consider the subtype relation between **record types**.

```
/* STFAE */
val x: {a: Number, b: Number} = {a = 1, b = 2}
val y: {a: Number} = x
val z: Number = y.a
...
```

If we **add** any new field to a record type, the resulting type should be a subtype of the original type.

$$\frac{}{\{x_1 : \tau_1, \ldots, x_n : \tau_n, x : \tau\} <: \{x_1 : \tau_1, \ldots, x_n : \tau_n\}}$$

Subtype Relation – Record Types (2)

$$\boxed{\tau <: \tau}$$

Let's consider the subtype relation between **record types**.

```
/* STFAE */
val x: {a: Number, b: Number} = {a = 1, b = 2}
val y: {a: Top, b: Top} = x
val z: Top = y.a
...
```

If all fields of a record type are **subtypes** of the corresponding fields of another record type, the resulting type should be a subtype of the other.

$$\frac{\tau_1 <: \tau_1' \qquad \ldots \qquad \tau_n <: \tau_n'}{\{x_1 : \tau_1, \ldots, x_n : \tau_n\} <: \{x_1 : \tau_1', \ldots, x_n : \tau_n'\}}$$

Subtype Relation – Record Types (3)

$$\boxed{\tau <: \tau}$$

Let's consider the subtype relation between **record types**.

```
/* STFAE */
val x: {a: Number, b: Number} = {a = 1, b = 2}
val y: {b: Number, a: Number} = x
val z: Number = y.a
...
```

If the fields of a record type is a **permutation** of the fields of another record type, the resulting type should be a subtype of the other.

$$\frac{\{x_1 : \tau_1, \ldots, x_n : \tau_n\} \text{ is a permutation of } \{x'_1 : \tau'_1, \ldots, x'_n : \tau'_n\}}{\{x_1 : \tau_1, \ldots, x_n : \tau_n\} <: \{x'_1 : \tau'_1, \ldots, x'_n : \tau'_n\}}$$

Subtype Relation – Function Types

$$\boxed{\tau <: \tau}$$

Let's consider the subtype relation between **function types**.

Subtype Relation – Function Types

$$\boxed{\tau <: \tau}$$

Let's consider the subtype relation between **function types**.

Consider the subtype relationship between **parameter types**.

```
val f3: Top => Number = f  // Impossible: f cannot take Top </: Number
val f4: Bot => Number = f  // Possible   : f can take Bot <: Number
...
```

We need **super types** for the given **parameter types**.

Subtype Relation – Function Types

$$\boxed{\tau <: \tau}$$

Let's consider the subtype relation between **function types**.

Consider the subtype relationship between **parameter types**.

```
val f3: Top => Number = f  // Impossible: f cannot take Top </: Number
val f4: Bot => Number = f  // Possible   : f can take Bot <: Number
...
```

We need **super types** for the given **parameter types**.

Now, consider the subtype relationship between **return types**.

```
/* STFAE */
val f: Number => Number = (x: Number) => x
val f1: Number => Top = f  // Possible   : f returns Number <: Top
val f2: Number => Bot = f  // Impossible: f returns Number </: Bot
 ...
```

We need **subtypes** for the given **return types**.

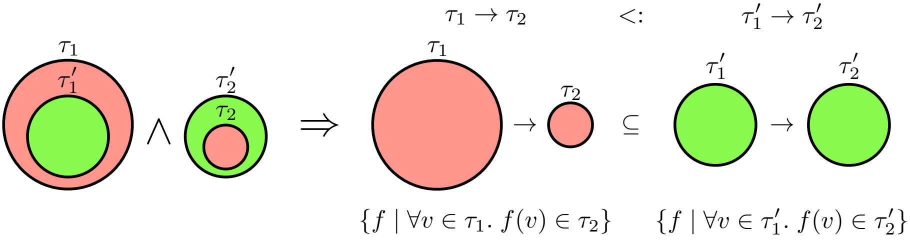
Subtype Relation – Function Types

$$\boxed{\tau <: \tau}$$

We need **super types** for the given **parameter types**: $\tau_1 :> \tau_1'$.

We need **sub types** for the given **return types**: $\tau_2 <: \tau_2'$.

$$\frac{\tau_1 :> \tau_1' \qquad \tau_2 <: \tau_2'}{(\tau_1 \to \tau_2) <: (\tau_1' \to \tau_2')}$$



Typing Rules with Subsumption

One possible way to support subtype polymorphism is to add a general **subsumption** rule to the typing rules:

$$\frac{\Gamma \vdash e : \tau \qquad \tau <: \tau'}{\Gamma \vdash e : \tau'}$$

With this rule, we can type the following expression.

```
/* STFAE */
val x: {a: Number} = {a=2, b=3}; x.a
```

$$\dfrac{\dfrac{\dfrac{\cdots}{\varnothing \vdash \{a = 2, b = 3\} : \{a : \mathtt{num}, b : \mathtt{num}\}} \qquad \dfrac{\cdots}{\{a : \mathtt{num}, b : \mathtt{num}\} <: \{a : \mathtt{num}\}}}{\varnothing \vdash \{a = 2, b = 3\} : \{a : \mathtt{num}\}} \qquad \cdots}{\varnothing \vdash \mathtt{val}\ x : \{a : \mathtt{num}\} = \{a = 2, b = 3\};\ x.a : \mathtt{num}}$$

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$$\dfrac{\dfrac{\dfrac{\cdots}{\varnothing \vdash \{a = 2, b = 3\} : \{a : \texttt{num}, b : \texttt{num}\}} \qquad \dfrac{\cdots}{\{a : \texttt{num}, b : \texttt{num}\} <: \{a : \texttt{num}\}}}{\varnothing \vdash \{a = 2, b = 3\} : \{a : \texttt{num}\}} \qquad \cdots}{\varnothing \vdash \texttt{val}\ x : \{a : \texttt{num}\} = \{a = 2, b = 3\};\ x.a : \texttt{num}}$$

However, it is **not algorithmic** because we don't know which types are required as the result of subsumption.

Algorithmic Typing Rules without Subsumption

Another way is to **directly apply** the subtype relation to the each typing rule without subsumption, and it is **algorithmic**.

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-}\mathtt{Add}\ \frac{\Gamma \vdash e_1 : \tau_1 \qquad \tau_1 <: \mathtt{num} \qquad \Gamma \vdash e_2 : \tau_2 \qquad \tau_2 <: \mathtt{num}}{\Gamma \vdash e_1 + e_2 : \mathtt{num}}$$

$$\tau\text{-}\mathtt{Mul}\ \frac{\Gamma \vdash e_1 : \tau_1 \qquad \tau_1 <: \mathtt{num} \qquad \Gamma \vdash e_2 : \tau_2 \qquad \tau_2 <: \mathtt{num}}{\Gamma \vdash e_1 \times e_2 : \mathtt{num}}$$

$$\tau\text{-}\mathtt{Val}_\tau\ \frac{\Gamma \vdash e_1 : \tau_1 \qquad \tau_1 <: \tau_0 \qquad \Gamma[x : \tau_0] \vdash e_2 : \tau_2}{\Gamma \vdash \mathtt{val}\ x : \tau_0 = e_1;\ e_2 : \tau_2}$$

$$\tau\text{-}\mathtt{App}\ \frac{\Gamma \vdash e_0 : \tau_1 \to \tau_2 \qquad \Gamma \vdash e_1 : \tau_3 \qquad \tau_3 <: \tau_1}{\Gamma \vdash e_0(e_1) : \tau_2}$$

Summary

1. Subtype Polymorphism
2. STFAE – TFAE with Subtype Polymorphism
   - Concrete Syntax
   - Abstract Syntax
3. Interpreter and Natural Semantics for STFAE
4. Type Checker and Typing Rules
   - Records and Field Accesses
   - Exit
   - Immutable Variable Definition
5. Subtype Relation
   - Bottom Type and Top Type
   - Record Types
   - Function Types
   - Typing Rules with Subsumption
   - Algorithmic Typing Rules without Subsumption

Exercise #14

- Please see this document[2] on GitHub.
  - Implement `typeCheck` function.
  - Implement `interp` function.
- It is just an exercise, and you **don't need to submit** anything.
- However, some exam questions might be related to this exercise.

[2] `https://github.com/ku-plrg-classroom/docs/tree/main/cose212/stfae`.

Next Lecture

- Type Inference (1)

Jihyeok Park
jihyeok_park@korea.ac.kr
https://plrg.korea.ac.kr