

Lecture 26 – P, NP, and NP-Complete Problems

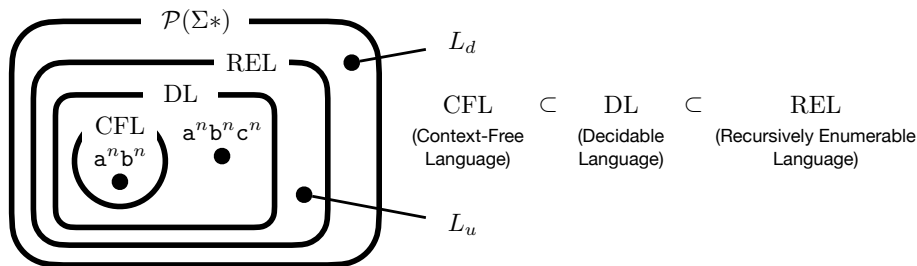
COSE215: Theory of Computation

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2023 Spring

- A language L is **recursively enumerable language (REL)** if there is a Turing machine (TM) M such that $L(M) = L$.
- A language L is **decidable language (DL)** if there is a TM M such that 1) $L(M) = L$ and 2) M **halts** on all inputs.



- What are **decision problems** and **time complexity**?
- Learn three classes of decision problems: **P**, **NP**, and **NP-Complete**.

1. Decision Problems

2. P

Time Complexity of TMs

P – Polynomial Time Complexity

3. NP

Time Complexity of NTMs

NP – Nondeterministic Polynomial Time Complexity

4. NP-complete

Polynomial Time Reduction (\leq_P)

NP-complete – Hardest Problems in **NP**

<SAT> – The First **NP-complete** Problem

Other **NP-complete** Problems

5. Major Unsolved Problem: **P = NP?**

Definition (Decision Problem)

A **decision problem** π is a computational problem whose answer is either **yes** or **no** for a given input.

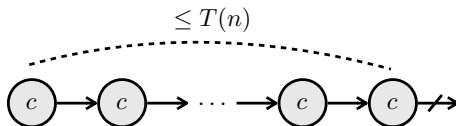
For example,

- Is word w an even-length palindrome?
- Is a natural number n a prime number?
- Is a graph G a tree?
- Is there a Hamiltonian path in a graph G ?
- Is a boolean formula ϕ satisfiable?
- ...

We say that a decision problem π is **decidable** (**solvable**) by a TM M if M halts on all inputs and $L(M) = \{w \mid \pi(w) = \text{yes}\}$.

Definition (Time Complexity of TMs)

We say a TM M has a **time complexity** $T : \mathbb{N} \rightarrow \mathbb{N}$ if M halts on w in at most $T(n)$ moves for all $w \in \Sigma^*$ whose length is n .



Definition (DTIME)

A decision problem π is in **DTIME**($T(n)$) if it is decidable by a TM M whose time complexity is $T(n)$.

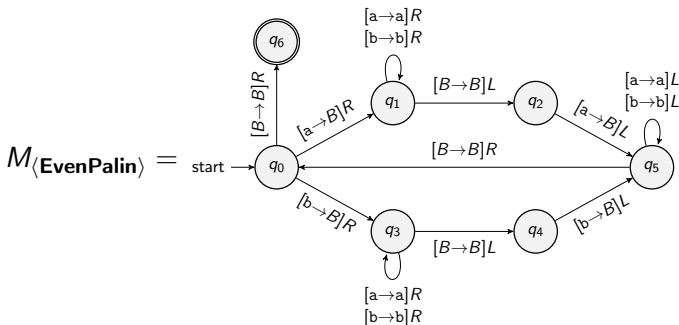
We often use a **big O notation** to describe the time complexity of a TM:

$$f(n) = O(g(n)) \iff \exists k \in \mathbb{N}, n_0 \in \mathbb{N}. \forall n \geq n_0. f(n) \leq k \cdot g(n)$$

$\langle \mathbf{EvenPalin} \rangle$ – Is a word $w \in \{a, b\}^*$ an even-length palindrome?

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...	B	a	b	a	a	b	a	B	...
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The decision problem $\langle \mathbf{EvenPalin} \rangle$ is decidable by a TM $M_{\langle \mathbf{EvenPalin} \rangle}$ whose time complexity is $T(n) = (n + 1)(n + 2)/2 = O(n^2)$.

$$\langle \mathbf{EvenPalin} \rangle \in \mathbf{DTIME}(O(n^2))$$

Definition (P – Polynomial Time Complexity)

A decision problem π is in **P** if it is decidable by a TM M whose time complexity is a **polynomial function** (i.e., $T(n) = O(n^k)$ for some $k \geq 0$).

$$P = \bigcup_{k \geq 0} \text{DTIME}(O(n^k))$$

For example, the decision problem $\langle \text{EvenPalin} \rangle$ is in **P**.

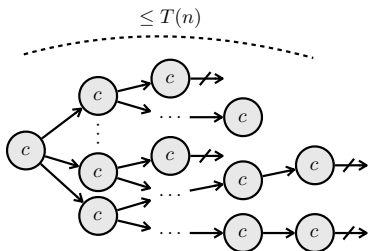
$$\langle \text{EvenPalin} \rangle \in \text{DTIME}(O(n^2)) \subseteq P$$

Definition (Tractable Problems)

A problem π is called a **tractable problem** if it is a **P** problem.

Definition (Time Complexity of NTMs)

We say a nondeterministic (NTM) M has a **time complexity** $T : \mathbb{N} \rightarrow \mathbb{N}$ if M halts on w in at most $T(n)$ moves for all $w \in \Sigma^*$ whose length is n .



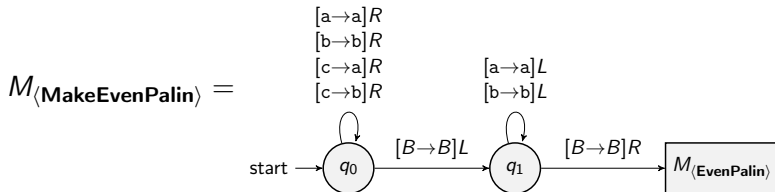
Definition (NTIME)

A decision problem π is in **NTIME**($T(n)$) if it is decidable by a NTM M whose time complexity is $T(n)$.

⟨MakeEvenPalin⟩ – Is a word $w \in \{a, b, c\}^*$ convertible to an even-length palindrome by replacing all c's with a's or b's?

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-----	---	---	---	---	---	---	---	---	-----



The decision problem $\langle \text{MakeEvenPalin} \rangle$ is decidable by a NTM whose time complexity is $T(n) = 2(n + 1) + O(n^2) = O(n^2)$.

$$\langle \text{MakeEvenPalin} \rangle \in \text{NTIME}(O(n^2))$$

Definition (NP – Nondeterministic Polynomial Time Complexity)

A decision problem π is in **NP** if it is decidable by a NTM M whose time complexity is a **polynomial function** (i.e., $T(n) = O(n^k)$ for some $k \geq 0$).

$$\text{NP} = \bigcup_{k \geq 0} \text{NTIME}(O(n^k))$$

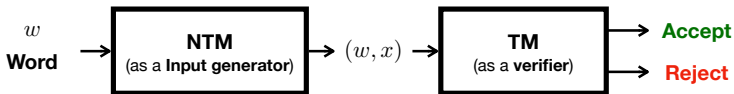
For example, the decision problem $\langle \text{MakeEvenPalin} \rangle$ is in **NP**.

$$\langle \text{MakeEvenPalin} \rangle \in \text{NTIME}(O(n^2)) \subseteq \text{NP}$$

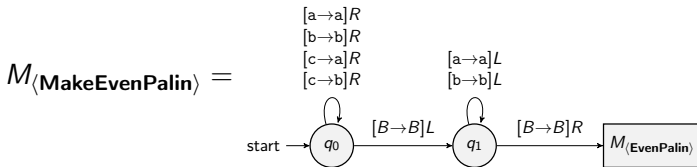
Definition (NP – Verifier-based Definition)

A decision problem π is in **NP** if there exists a TM M as a **verifier** whose time complexity is a polynomial function such that:

$$\forall w \in \Sigma^*. \pi(w) = \text{yes} \iff \exists x \in \Sigma^*. (w, x) \in L(M)$$



For example, the decision problem $\langle \text{MakeEvenPalin} \rangle$ is in **NP** because there exists a polynomial time verifier $M_{\langle \text{EvenPalin} \rangle}$.



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Yes! For example, $x_1 = \text{\#f}$, $x_2 = \text{\#f}$, and $x_3 = \text{\#t}$ is a satisfying assignment.

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Is it $\langle \text{SAT} \rangle$ in **NP**? Yes!

We can construct a polynomial time TM as a **verifier** for $\langle \text{SAT} \rangle$, which takes 1) a **Boolean formula** and 1) an **assignment** of Boolean variables, and checks whether the assignment satisfies the formula.

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We can construct a polynomial time TM as a **verifier** for $\langle \text{SAT} \rangle$, which takes 1) a **Boolean formula** and 1) an **assignment** of Boolean variables, and checks whether the assignment satisfies the formula.

In other words, we can construct a polynomial time NTM for $\langle \text{SAT} \rangle$ by 1) **generating all assignments** of Boolean variables and 2) **verifying** whether the assignment satisfies the formula using the verifier.

Definition (Polynomial Time Reduction (\leq_P))

A decision problem π_1 is **polynomial time reducible** to another decision problem π_2 (denoted by $\pi_1 \leq_P \pi_2$) if there exists a polynomial time computable function $f : \Sigma^* \rightarrow \Sigma^*$ such that:

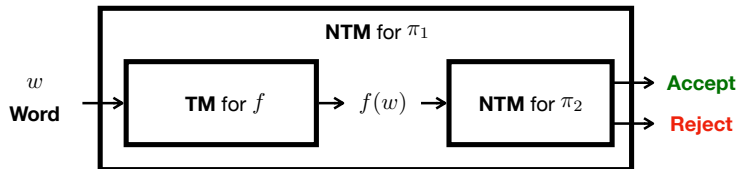
$$\forall w \in \Sigma^*. \pi_1(w) = \text{yes} \iff \pi_2(f(w)) = \text{yes}$$

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$$\forall w \in \Sigma^*. \pi_1(w) = \text{yes} \iff \pi_2(f(w)) = \text{yes}$$

We say that π_2 is **harder** than π_1 if $\pi_1 \leq_P \pi_2$ because we can solve π_1 in polynomial time if we can solve π_2 in polynomial time.

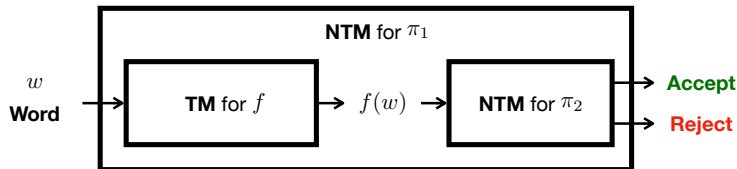


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We say that π_2 is **harder** than π_1 if $\pi_1 \leq_P \pi_2$ because we can solve π_1 in polynomial time if we can solve π_2 in polynomial time.



If a decision problem π_2 is in **NP** and $\pi_1 \leq_P \pi_2$, then π_1 is in **NP**.

Consider the following two decision problems:

- **⟨MakeEvenPalin⟩** – Is a word $w \in \{a, b, c\}^*$ convertible to an even-length palindrome by replacing all c's with a's or b's?
- **⟨SAT⟩** – Is a given Boolean formula satisfiable?

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We can show that **⟨MakeEvenPalin⟩** \leq_P **⟨SAT⟩** by the following polynomial time computable function f :

$$f(a_1 a_2 \cdots a_n) = \bigwedge_{i=1}^n ((x_i \wedge x_{n+1-i}) \vee (\neg x_i \wedge \neg x_{n+1-i})) \\ \wedge \bigwedge \{x_i \mid a_i = a\} \wedge \bigwedge \{\neg x_i \mid a_i = b\}$$

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For example,

$$f(acba) = ((x_1 \wedge x_4) \vee (\neg x_1 \wedge \neg x_4)) \wedge ((x_2 \wedge x_3) \vee (\neg x_2 \wedge \neg x_3)) \wedge x_1 \wedge \neg x_3 \wedge x_4$$

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Thus, we can solve **⟨MakeEvenPalin⟩** using an NTM for **⟨SAT⟩**, and it means that **⟨SAT⟩** is harder problem than **⟨MakeEvenPalin⟩**.

Definition (NP-hard – Harder Problems Than NP)

A decision problem π is in **NP-hard** if $\forall \pi' \in \text{NP}, \pi' \leq_P \pi$.

In other word, π is in **NP-hard** if π is **harder than all problems in NP**.

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Definition (NP-complete – Hardest Problems in NP)

A decision problem π is in **NP-complete** if

- 1 π is in **NP**, and
- 2 π is in **NP-hard** (i.e., $\forall \pi' \in \text{NP}, \pi' \leq_P \pi$).

In other word, π is in **NP-complete** if π is the **hardest problem in NP**.

Theorem (Cook–Levin theorem)

$\langle \text{SAT} \rangle$ *is in* NP-complete.

¹https://en.wikipedia.org/wiki/Cook-Levin_theorem

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We need to show that

- 1 $\langle \text{SAT} \rangle$ is in NP, and
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We already know that $\langle \text{SAT} \rangle$ is in NP.

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To show that $\langle \text{SAT} \rangle$ is in NP-hard, we need to show that $\forall \pi \in \text{NP}$, $\pi \leq_P \langle \text{SAT} \rangle$.

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We already know that $\langle \text{SAT} \rangle$ is in NP.

To show that $\langle \text{SAT} \rangle$ is in NP-hard, we need to show that $\forall \pi \in \text{NP}$, $\pi \leq_P \langle \text{SAT} \rangle$.

The core idea is to simulate an NTM M for π using a Boolean formula ϕ such that ϕ is satisfiable if and only if M accepts w . But, we skip the details of the proof. Please refer to the link¹ for the details.

¹https://en.wikipedia.org/wiki/Cook-Levin_theorem

Theorem (Lemma)

A decision problem π is in NP-hard if $\langle \text{SAT} \rangle \leq_P \pi$

²https://en.wikipedia.org/wiki/List_of_NP-complete_problems

Theorem (Lemma)

*A decision problem π is in **NP-hard** if $\langle \mathbf{SAT} \rangle \leq_P \pi$*

This lemma is very useful to show that a decision problem π is in **NP-complete** by showing that 1) π is in **NP** and 2) $\langle \mathbf{SAT} \rangle \leq_P \pi$.

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Theorem (Lemma)

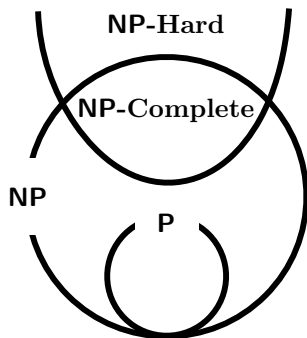
*A decision problem π is in **NP-hard** if $\langle \mathbf{SAT} \rangle \leq_P \pi$*

This lemma is very useful to show that a decision problem π is in **NP-complete** by showing that 1) π is in **NP** and 2) $\langle \mathbf{SAT} \rangle \leq_P \pi$. We can show that all of the following decision problems are in **NP-complete** by using this lemma:

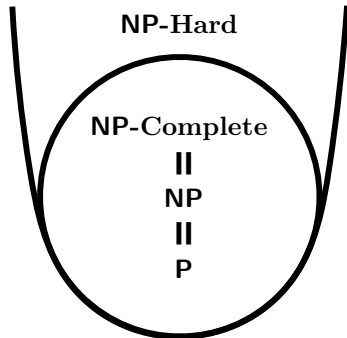
- $\langle \mathbf{SubsetSum} \rangle$ – Given a set of integers S and an integer t , is there a subset $S' \subseteq S$ such that $\sum S' = t$?
- $\langle \mathbf{Clique} \rangle$ – Given a graph G and an integer k , is there a clique of size k in G ?
- $\langle \mathbf{VertexCover} \rangle$ – Given a graph G and an integer k , is there a vertex cover of size k in G ?
- ...²

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Major Unsolved Problem: $P = NP?$



$P \neq NP$



$P = NP$

"If $P = NP$, then the world would be a profoundly different place than we usually assume it to be. There would be no special value in creative leaps, no fundamental gap between solving a problem and recognizing the solution once it's found."

— Scott Aaronson, *UT Austin*

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- Course Review

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