Lecture 19 – Typed Languages

COSE212: Programming Languages

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Recall

- Safe Language Systems
  - Dynamic vs Static Analysis for Detecting Run-Time Errors
  - Soundness vs Completeness of Analysis
- Type Systems
  - Types
  - Type Errors
  - Type Checking
  - Type Soundness
- In this lecture, we will define our first **typed language**.
- **TFAE** – FAE with **type system**.
  - **Type Checking** and **Typing Rules**
  - Interpreter and Natural Semantics

Contents

1. TFAE – FAE with Type System
    - Concrete Syntax
    - Abstract Syntax
2. Type Checking and Typing Rules
    - Type Checking
    - Typing Rules
3. Interpreter and Natural Semantics

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   - Typing Rules
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TFAE – FAE with Type System

Before defining TFAE, consider the types of the following FAE expressions:

```
/* FAE */ 42
```

Since it produces a **number**, let's say it is a `Number` type expression.

```
/* FAE */ x => x + 1
```

It produces a function value, but can we say more about its type? **Yes!**

It should take a **number** type argument and return a **number**.

Let's say it is a `Number => Number` type expression.

```
/* FAE */ x => x
```

There is no information on the parameter x.

To handle such cases, let's explicitly add **type annotations** to the parameters of function definitions.

TFAE – FAE with Type System

Let's extend FAE into TFAE to support **type system** by adding **type annotations** to function definitions:

```
/* TFAE */
(x: Number) => x                    // x is `Number` type
(f: Number => Number) => f(42)      // f is `Number => Number` type
```

If we define immutable variable definitions as **syntactic sugar**, it requires the type annotations:

```
/* TFAE */
val x: Number = 42; x + 1           // == `((x: Number) => x + 1)(42)`
```

However, we we can infer variable types from their initial values if we **explicitly define** them rather than syntactic sugar:

```
/* TFAE */
val x = 42; x + 1                   // x is `Number` type because of `42`
```

Concrete Syntax

For TFAE, we need to extend **expressions** of FAE with

1. **function definitions** with **type annotations**
2. **immutable variable definitions** without **type annotations**
3. **types**

We can extend the **concrete syntax** of FAE as follows:

```
// expressions
<expr> ::= ...
         | "(" <id> ":" <type> ")" "=>" <expr>
         | "val" <id> "=" <expr> ";" <expr>

// types
<type> ::= "Number"                  // number type
         | <type> "=>" <type>        // arrow type
```

Since functions are first-class values, the parameter and return types could be recursively arrow types.

Abstract Syntax

We can extend the **abstract syntax** of FAE for TFAE as follows:

$$
\begin{array}{llll}
\text{Expressions} & \mathbb{E} \ni e ::= & \ldots & \\
 & & \mid \lambda x{:}\tau.e & (\texttt{Fun}) \\
 & & \mid \texttt{val}\ x\texttt{=}e;\ e & (\texttt{Val})
\end{array}
$$

$$
\begin{array}{llll}
\text{Types} & \mathbb{T} \ni \tau ::= & \texttt{Number} & (\texttt{NumT}) \\
 & & \mid \tau \rightarrow \tau & (\texttt{ArrowT})
\end{array}
$$

We can define the abstract syntax of TFAE in Scala as follows:

```
enum Expr:
  ...
  case Fun(param: String, ty: Type, body: Expr)
  case Val(name: String, init: Expr, body: Expr)

enum Type:
  case NumT
  case ArrowT(paramTy: Type, bodyTy: Type)
```

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Type Checking

If the following conditions hold, we say **"the expression e has type τ"**:

- e does not cause any type error, and
- e evaluates to a value of type τ or does not terminate.

If so, we use the following notation and say that e is **well-typed**:

$$\boxed{\vdash e : \tau}$$

It is defined by **typing rules** and implemented as the follows in Scala:

```scala
def typeCheck(expr: Expr): Type = ???
```

Type Checking

If the following conditions hold, we say **“the expression *e* has type τ”**:

- *e* does not cause any type error, and
- *e* evaluates to a value of type τ or does not terminate.

If so, we use the following notation and say that *e* is **well-typed**:

$$\boxed{\Gamma \vdash e : \tau}$$

It is defined by **typing rules** and implemented as the follows in Scala:

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = ???
```

We need a **type environment** Γ to keep track of the variable types in *e*:

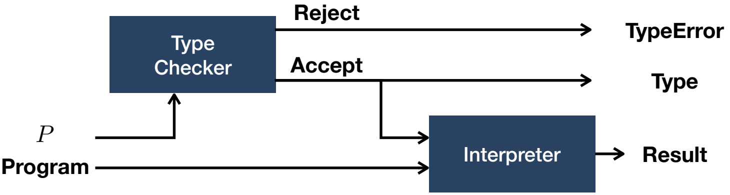
$$\text{Type Environments} \qquad \Gamma \;\in\; \mathbb{X} \xrightarrow{\text{fin}} \mathbb{T} \quad (\texttt{TypeEnv})$$

```
type TypeEnv = Map[String, Type]
```

Type Checking

Definition (Type Checking)

Type checking is a kind of static analysis checking whether a given expression e is **well-typed**. A **type checker** returns the **type** of e if it is well-typed, or rejects it and reports the detected **type error** otherwise.




```scala
def eval(str: String): String =
  val expr = Expr(str)
  val ty = typeCheck(expr, Map.empty)
  val v = interp(expr, Map.empty)
  s"${v.str}: ${ty.str}"
```

Typing Rules

Now, lets' define the typing rules for TFAE in the following form:

$$\boxed{\Gamma \vdash e : \tau}$$

and fill out the body of the `typeCheck` function:

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = ???
```

with type environments:

$$\text{Type Environments} \qquad \Gamma \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{T} \quad (\texttt{TypeEnv})$$

```
type TypeEnv = Map[String, Type]
```

Typing Rules – Numbers

```scala
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  case Num(_) => NumT
  ...
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{–}\texttt{Num}\ \frac{}{\Gamma \vdash n : \texttt{Number}}$$

The number literal n has `Number` type in any type environment Γ.

Typing Rules – Addition

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Add(left, right) =>
    ???
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\texttt{-Add}\ \frac{???}{\Gamma \vdash e_1 + e_2 : ???}$$

Typing Rules – Addition

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Add(left, right) =>
    typeCheck(left, tenv)
    ???
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{--}\mathtt{Add}\ \frac{\Gamma \vdash e_1 : \tau \qquad ???}{\Gamma \vdash e_1 + e_2 : ???}$$

Type checker should do

1. get the type of e_1 in Γ

Typing Rules – Addition

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Add(left, right) =>
    mustSame(typeCheck(left, tenv), NumT)
    ???


def mustSame(lty: Type, rty: Type): Unit =
  if (lty != rty) error(s"type error: ${lty.str} != ${rty.str}")
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{--}\mathtt{Add}\ \frac{\Gamma \vdash e_1 : \mathtt{Number} \qquad \mathtt{???}}{\Gamma \vdash e_1 + e_2 : \mathtt{???}}$$

Type checker should do

1. check the type of e_1 is $\mathtt{Number}$ in Γ

Typing Rules – Addition

```scala
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Add(left, right) =>
    mustSame(typeCheck(left, tenv), NumT)
    mustSame(typeCheck(right, tenv), NumT)
    ???

def mustSame(lty: Type, rty: Type): Unit =
  if (lty != rty) error(s"type error: ${lty.str} != ${rty.str}")
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{–}\mathtt{Add}\ \frac{\Gamma \vdash e_1 : \mathtt{Number} \qquad \Gamma \vdash e_2 : \mathtt{Number}}{\Gamma \vdash e_1 + e_2 : \color{red}{???}}$$

Type checker should do

1. check the types of e_1 and e_2 are `Number` in Γ

Typing Rules – Addition

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Add(left, right) =>
    mustSame(typeCheck(left, tenv), NumT)
    mustSame(typeCheck(right, tenv), NumT)
    NumT

def mustSame(lty: Type, rty: Type): Unit =
  if (lty != rty) error(s"type error: ${lty.str} != ${rty.str}")
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{–}\mathtt{Add}\ \frac{\Gamma \vdash e_1 : \mathtt{Number} \qquad \Gamma \vdash e_2 : \mathtt{Number}}{\Gamma \vdash e_1 + e_2 : \mathtt{Number}}$$

Type checker should do

1. check the types of e_1 and e_2 are `Number` in Γ
2. return `Number` as the type of $e_1 + e_2$

Typing Rules – Multiplication

```scala
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Mul(left, right) =>
    mustSame(typeCheck(left, tenv), NumT)
    mustSame(typeCheck(right, tenv), NumT)
    NumT

def mustSame(lty: Type, rty: Type): Unit =
  if (lty != rty) error(s"type error: ${lty.str} != ${rty.str}")
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{–}\texttt{Mul}\ \frac{\Gamma \vdash e_1 : \texttt{Number} \qquad \Gamma \vdash e_2 : \texttt{Number}}{\Gamma \vdash e_1 \times e_2 : \texttt{Number}}$$

Typing Rules – Immutable Variable Definition

```scala
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Val(x, init, body) =>
    val initTy = typeCheck(init, tenv)
    typeCheck(body, tenv + (x -> initTy))
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{–}\texttt{Val}\ \frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma[x \mapsto \tau_1] \vdash e_2 : \tau_2}{\Gamma \vdash \texttt{val}\ x\texttt{=}e_1\texttt{;}\ e_2 : \tau_2}$$

This rule stores the type of x in Γ inferred from the initial value.

```
/* TFAE */ val x = 1; x + 2      // `x: Number` in `tenv` <- `1: Number`
```

Typing Rules – Identifier Lookup

```scala
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Id(x) =>
    tenv.getOrElse(x, error(s"free identifier: $x"))
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{–}\mathtt{Id}\ \frac{x \in \mathsf{Domain}(\Gamma)}{\Gamma \vdash x : \Gamma(x)}$$

This rule looks up the type of x in Γ.

```
/* TFAE */ val x = 1; x + 2      // `x: Number` in `tenv` <- `1: Number`
```

Typing Rules – Function Definition

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Fun(param, paramTy, body) =>
    val bodyTy = typeCheck(body, tenv + (param -> paramTy))
    ArrowT(paramTy, bodyTy)
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{–}\texttt{Fun}\ \frac{\Gamma[x \mapsto \tau] \vdash e : \tau'}{\Gamma \vdash \lambda x{:}\tau.e : \tau \to \tau'}$$

We can check the body of a function with the its parameter type.

```
/* TFAE */ (x: Number) => x      // Number => Number
```

Typing Rules – Function Application

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case App(fun, arg) => typeCheck(fun, tenv) match
    case ArrowT(paramTy, bodyTy) =>
      mustSame(typeCheck(arg, tenv), paramTy)
      bodyTy
    case ty => error(s"not a function type: ${ty.str}")
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{–App}\ \frac{\Gamma \vdash e_0 : \tau_1 \to \tau_2 \qquad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash e_0(e_1) : \tau_2}$$

We don't have to check the type of the function body because it is already checked when the function is defined.

```
/* TFAE */ ((x: Number) => x)(1) // Number
```

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Interpreter and Natural Semantics

For interpreter and natural semantics for TFAE, it is just enough to extend the those for function definitions in FAE.

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Fun(p, t, b) => CloV(p, b, env)
```

$$\boxed{\sigma \vdash e \Rightarrow v}$$

$$\texttt{Fun}\ \frac{}{\sigma \vdash \lambda x{:}\tau.e \Rightarrow \langle \lambda x.e, \sigma \rangle}$$

The type annotation is ignored in the interpreter and natural semantics.

Summary

1. TFAE – FAE with Type System
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Exercise #10

- Please see this document[1] on GitHub.
  - Implement `typeCheck` function.
  - Implement `interp` function.
- It is just an exercise, and you **don't need to submit** anything.
- However, some exam questions might be related to this exercise.

[1] https://github.com/ku-plrg-classroom/docs/tree/main/cose212/tfae.

Next Lecture

- Typing Recursive Functions

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