Lecture 8 – Lambda Calculus

COSE212: Programming Languages

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PLRG

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Recall

- FVAE – VAE with First-Class Functions
  - First-Class Functions
  - Concrete and Abstract Syntax
  - Interpreter and Natural Semantics with Closures
  - Static and Dynamic Scoping

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- FVAE – VAE with First-Class Functions
  - First-Class Functions
  - Concrete and Abstract Syntax
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  - Static and Dynamic Scoping

- In this lecture, we will learn **syntactic sugar** and **lambda calculus**

Contents

1. Syntactic Sugar
   - No More `val`
   - FAE – Removing `val` from FVAE
   - Syntactic Sugar and Desugaring

2. Lambda Calculus
   - Definition
   - Church Encodings
   - Church-Turing Thesis

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No More `val`

```
/* FVAE */
val x = 1; x + 2
```

It assigns a **value** 1 to the **variable** x, and then evaluates the **body expression** $x + 2$ with the environment $[x \mapsto 1]$.

No More `val`

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/* FVAE */
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```

It assigns a **value** 1 to the **variable** x, and then evaluates the **body expression** $x + 2$ with the environment $[x \mapsto 1]$.

It is same as:

```
/* FVAE */
(x => x + 2)(1)
```

It assigns a **value** (argument) 1 to the **parameter** x, and then evaluates the **body expression** $x + 2$ with the environment $[x \mapsto 1]$.

No More `val`

In general, the following two expressions are equivalent:

$$\texttt{val}\ x\texttt{=}e;\ \ e' \qquad \text{is equivalent to} \qquad (\lambda x.e')(e)$$

Why?

No More `val`

In general, the following two expressions are equivalent:

$$\texttt{val}\ x\texttt{=}e\texttt{;}\ \ e' \qquad \text{is equivalent to} \qquad (\lambda x.e')(e)$$

Why?

The following inference rule for the semantics of `val` x=e; e':

$$\textsc{Val}\ \frac{\sigma \vdash e \Rightarrow v \qquad \sigma[x \mapsto v] \vdash e' \Rightarrow v'}{\sigma \vdash \texttt{val}\ x = e;\ e' \Rightarrow v'}$$

is equivalent to the following inference rule for the semantics of $(\lambda x.e')(e)$:

$$\texttt{App}\ \frac{\texttt{Fun}\ \dfrac{}{\sigma \vdash \lambda x.e' \Rightarrow \langle \lambda x.e', \sigma\rangle} \qquad \sigma \vdash e \Rightarrow v \qquad \sigma[x \mapsto v] \vdash e' \Rightarrow v'}{\sigma \vdash (\lambda x.e')(e) \Rightarrow v'}$$

FAE – Removing `val` from FVAE

Then, we can define a smaller language FAE

$$
\begin{array}{llll}
\text{Expressions} & \mathbb{E} \ni e ::= & n & (\texttt{Num}) \\
 & \mid & e + e & (\texttt{Add}) \\
 & \mid & e \times e & (\texttt{Mul}) \\
 & \mid & x & (\texttt{Id}) \\
 & \mid & \lambda x.e & (\texttt{Fun}) \\
 & \mid & e(e) & (\texttt{App})
\end{array}
$$

by removing `val` from FVAE using the following equivalence:

$$\texttt{val}\ x\texttt{=}e\texttt{;}\ \ e' \qquad \text{is equivalent to} \qquad (\lambda x.e')(e)$$

Syntactic Sugar and Desugaring

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Definition (Syntactic Sugar)

Syntactic elements that can be expressed in terms of other syntactic elements are called **syntactic sugar**.

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Definition (Desugaring)

Desugaring is a translation for removing syntactic sugar.

$$\mathcal{D}[\![-]\!] : \mathbb{E} \rightarrow \mathbb{E}$$

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$$\mathcal{D}[\![-]\!] : \mathbb{E} \to \mathbb{E}$$

$$\begin{array}{lcl}
\mathcal{D}[\![n]\!] & = & n \\
\mathcal{D}[\![e + e']\!] & = & \mathcal{D}[\![e]\!] + \mathcal{D}[\![e']\!] \\
\mathcal{D}[\![e \times e']\!] & = & \mathcal{D}[\![e]\!] \times \mathcal{D}[\![e']\!]
\end{array}
\qquad
\begin{array}{lcl}
\mathcal{D}[\![\texttt{val}\ x\texttt{=}e\texttt{;}\ e']\!] & = & (\lambda x.\mathcal{D}[\![e']\!])(\mathcal{D}[\![e]\!]) \\
\mathcal{D}[\![x]\!] & = & x \\
\mathcal{D}[\![\lambda x.e]\!] & = & \lambda x.\mathcal{D}[\![e]\!] \\
\mathcal{D}[\![e(e')]\!] & = & \mathcal{D}[\![e]\!](\mathcal{D}[\![e']\!])
\end{array}$$

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\mathcal{D}[\![\lambda x.e]\!] & = & \lambda x.\mathcal{D}[\![e]\!] \\
\mathcal{D}[\![e(e')]\!] & = & \mathcal{D}[\![e]\!](\mathcal{D}[\![e']\!])
\end{array}$$

For example,

$$\mathcal{D}[\![\texttt{val}\ x\texttt{=}42;\ \texttt{val}\ y\texttt{=}x + 1;\ y + 2]\!] = (\lambda x.(\lambda y.y + 2)(x + 1))(42)$$

Syntactic Sugar and Desugaring

We can also implement **desugaring** in Scala:

```
def desugar(expr: Expr): Expr = expr match
  case Num(n)       => Num(n)
  case Add(l, r)    => Add(desugar(l), desugar(r))
  case Mul(l, r)    => Mul(desugar(l), desugar(r))
  case Val(x, i, b) => App(Fun(x, desugar(b)), desugar(i))
  case Id(x)        => Id(x)
  case Fun(p, b)    => Fun(p, desugar(b))
  case App(f, e)    => App(desugar(f), desugar(e))
```

Note that we need to **recursively desugar** all sub-expressions of the given expression even if they are not syntactic sugars.

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  case Val(x, i, b) => App(Fun(x, desugar(b)), desugar(i))
  case Id(x)           => Id(x)
  case Fun(p, b)       => Fun(p, desugar(b))
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```

Note that we need to **recursively desugar** all sub-expressions of the given expression even if they are not syntactic sugars.

Then, we can desugar the example FVAE expression as follows:

```
val e1: Expr = Expr("val x = 42; val y = x + 1; y + 2")
val e2: Expr = Expr("(x => (y => y + 2)(x + 1))(42)")
desugar(e1) == e2
```

Syntactic Sugar and Desugaring

Most programming languages have **syntactic sugar**:

- Scala

```
for (x <- list) yield x * 2
```
≡
```
list.map(x => x * 2)
```

- C++

```
arr[i] + obj->field
```
≡
```
*(arr + i) + (*obj).field
```

- JavaScript

```
x += y; x *= y;
```
≡
```
x = x + y; x = x * y;
```

- Haskell

```
do x <- f; g x
```
≡
```
f >>= (\x -> g x)
```

- ...

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Lambda Calculus

What is the minimal language that can express all the syntactic elements of FVAE?

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The **lambda calculus (LC)** is a language only consisting of 1) **variables**, 2) **functions**, and 3) **applications**:

$$
\begin{array}{rrcl}
\text{Expressions} & \mathbb{E} \ni e & ::= & x \\
& & \mid & \lambda x.e \\
& & \mid & e(e)
\end{array}
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We already showed that the **variable definition** can be desugared to a combination of a **function definition** and an **application**:

$$\mathcal{D}[\![\texttt{val}\ x\texttt{=}e\texttt{;}\ \ e']\!] = (\lambda x.\mathcal{D}[\![e']\!])(\mathcal{D}[\![e]\!])$$

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Then, how can we desugar other syntactic elements of FVAE?

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Then, how can we desugar other syntactic elements of FVAE?

Let's learn the **Church encodings**!

Church Encodings – Church Numerals

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For example, **Church numerals** are a way to encode **natural numbers** in the **lambda calculus (LC)**.

The key idea is to encode a **natural number** n as a **function** that takes another function f and an argument x and applies f to x n times:

$$
\begin{array}{rcl}
\mathcal{D}[\![0]\!] & = & \lambda f.\lambda x.x \\
\mathcal{D}[\![1]\!] & = & \lambda f.\lambda x.f(x) \\
\mathcal{D}[\![2]\!] & = & \lambda f.\lambda x.f(f(x)) \\
\mathcal{D}[\![3]\!] & = & \lambda f.\lambda x.f(f(f(x))) \\
& \vdots &
\end{array}
$$

$$
\begin{array}{rcl}
\mathcal{D}[\![e_0 + e_1]\!] & = & \lambda f.\lambda x.\mathcal{D}[\![e_0]\!](f)(\mathcal{D}[\![e_1]\!](f)(x)) \\
\mathcal{D}[\![e_0 \times e_1]\!] & = & \lambda f.\lambda x.\mathcal{D}[\![e_0]\!](\mathcal{D}[\![e_1]\!](f))(x)
\end{array}
$$

Church Encodings – Church Numerals

For example,

$$
\begin{aligned}
\mathcal{D}[\![1+1]\!] &= \lambda f.\lambda x.\mathcal{D}[\![1]\!](f)(\mathcal{D}[\![1]\!](f)(x)) \\
&= \lambda f.\lambda x.(\lambda f.\lambda x.f(x))(f)((\lambda f.\lambda x.f(x))(f)(x)) \\
&= \lambda f.\lambda x.f((\lambda f.\lambda x.f(x))(f)(x)) \\
&= \lambda f.\lambda x.f(f(x)) \\
&= \mathcal{D}[\![2]\!]
\end{aligned}
$$

[1] https://en.wikipedia.org/wiki/Church_encoding

Church Encodings – Church Numerals

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$$
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\mathcal{D}[\![1+1]\!] & = & \lambda f.\lambda x.\mathcal{D}[\![1]\!](f)(\mathcal{D}[\![1]\!](f)(x)) \\
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& = & \lambda f.\lambda x.f(f(x)) \\
& = & \mathcal{D}[\![2]\!]
\end{array}
$$

We can represent other data or operations in the **LC** using **Church encodings**, such as **integers**, **booleans**, **pairs**, **lists**, and so on.[1]

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Church Encodings – Church Numerals

For example,

$$\begin{array}{rcl}\mathcal{D}[\![1+1]\!] & = & \lambda f.\lambda x.\mathcal{D}[\![1]\!](f)(\mathcal{D}[\![1]\!](f)(x)) \\ & = & \lambda f.\lambda x.(\lambda f.\lambda x.f(x))(f)((\lambda f.\lambda x.f(x))(f)(x)) \\ & = & \lambda f.\lambda x.f((\lambda f.\lambda x.f(x))(f)(x)) \\ & = & \lambda f.\lambda x.f(f(x)) \\ & = & \mathcal{D}[\![2]\!]\end{array}$$

We can represent other data or operations in the **LC** using **Church encodings**, such as **integers**, **booleans**, **pairs**, **lists**, and so on.[1]

Let's see one more example of **Church encoding** for **booleans** and **logical operations** (i.e., **Church booleans**).

[1] https://en.wikipedia.org/wiki/Church_encoding

Church Encodings – Church Booleans

The key idea is to encode a **boolean** b as a **function** that takes two arguments t and f and applies t if b is `true` or f if b is `false`:

$$\mathcal{D}[\![\texttt{true}]\!] = \lambda t.\lambda f.t \qquad \mathcal{D}[\![\texttt{false}]\!] = \lambda t.\lambda f.f$$

$$\begin{aligned}
\mathcal{D}[\![\texttt{if}(e_1)\ e_2\ \texttt{else}\ e_3]\!] &= \mathcal{D}[\![e_1]\!](\mathcal{D}[\![e_2]\!])(\mathcal{D}[\![e_3]\!]) \\
\mathcal{D}[\![e_1\ \texttt{\&\&}\ e_2]\!] &= \mathcal{D}[\![e_1]\!](\mathcal{D}[\![e_2]\!])(\mathcal{D}[\![e_1]\!]) \\
\mathcal{D}[\![e_1\ \texttt{||}\ e_2]\!] &= \mathcal{D}[\![e_1]\!](\mathcal{D}[\![e_1]\!])(\mathcal{D}[\![e_2]\!]) \\
\mathcal{D}[\![\texttt{!}\ e_0]\!] &= \lambda t.\lambda f.\mathcal{D}[\![e_0]\!](f)(t)
\end{aligned}$$

Church Encodings – Church Booleans

The key idea is to encode a **boolean** b as a **function** that takes two arguments t and f and applies t if b is `true` or f if b is `false`:

$$\begin{array}{rl} \mathcal{D}[\![\texttt{true}]\!] &= \lambda t.\lambda f.t \\ \mathcal{D}[\![\texttt{false}]\!] &= \lambda t.\lambda f.f \end{array} \qquad \begin{array}{ll} \mathcal{D}[\![\texttt{if}(e_1)\ e_2\ \texttt{else}\ e_3]\!] &= \mathcal{D}[\![e_1]\!](\mathcal{D}[\![e_2]\!])(\mathcal{D}[\![e_3]\!]) \\ \mathcal{D}[\![e_1\ \texttt{\&\&}\ e_2]\!] &= \mathcal{D}[\![e_1]\!](\mathcal{D}[\![e_2]\!])(\mathcal{D}[\![e_1]\!]) \\ \mathcal{D}[\![e_1\ \texttt{||}\ e_2]\!] &= \mathcal{D}[\![e_1]\!](\mathcal{D}[\![e_1]\!])(\mathcal{D}[\![e_2]\!]) \\ \mathcal{D}[\![!\ e_0]\!] &= \lambda t.\lambda f.\mathcal{D}[\![e_0]\!](f)(t) \end{array}$$

For example,

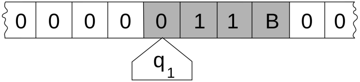
$$\begin{array}{rcl} \mathcal{D}[\![\texttt{true \&\& false}]\!] &=& \mathcal{D}[\![\texttt{true}]\!](\mathcal{D}[\![\texttt{false}]\!])(\mathcal{D}[\![\texttt{true}]\!]) \\ &=& (\lambda t.\lambda f.t)(\mathcal{D}[\![\texttt{false}]\!])(\mathcal{D}[\![\texttt{true}]\!]) \\ &=& \mathcal{D}[\![\texttt{false}]\!] \end{array}$$

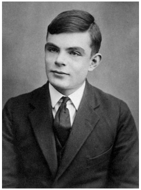
Church-Turing Thesis

Alonzo Church invented **lambda calculus (LC)** in 1930s, and it became the foundation of **programming languages**:

$$e ::= e \mid \lambda x.e \mid e(e)$$

Alan Turing invented **Turing machines (TM)** in 1936, and it became the foundation of **computers**:





Church-Turing Thesis: **LC** is **Turing complete**.

> *Any* **real-world computation** *can be translated into an equivalent computation involving a* **Turing machine** *or can be done using* **lambda calculus**.

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Next Lecture

- Recursive Functions

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