

# Lecture 24 – Subtype Polymorphism

## COSE212: Programming Languages

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2023 Fall

- Please see this document<sup>1</sup> on GitHub.
- The due date is Dec. 14 (Thu.).
- Please only submit `Implementation.scala` file to **Blackboard**.

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<sup>1</sup><https://github.com/ku-plrg-classroom/docs/tree/main/cose212/battery>.

- **Polymorphism** is to use a single entity as **multiple types**, and there are various kinds of polymorphism:
  - **Parametric polymorphism**
  - **Subtype polymorphism**
  - **Ad-hoc polymorphism**
  - ...
- **Parametric polymorphism** is a form of polymorphism by introducing **type variables** and instantiating them with **type arguments**.
- **PTFAE** – TFAE with **parametric polymorphism**.
- In this lecture, we will learn **subtype polymorphism**.
- **STFAE** – TFAE with **subtype polymorphism**.
  - **Interpreter** and **Natural Semantics**
  - **Type Checker** and **Typing Rules**

1. Subtype Polymorphism
2. STFAE – TFAE with Subtype Polymorphism
  - Concrete Syntax
  - Abstract Syntax
3. Interpreter and Natural Semantics for STFAE
4. Type Checker and Typing Rules
  - Records and Field Accesses
  - Exit
  - Immutable Variable Definition
5. Subtype Relation
  - Bottom Type and Top Type
  - Record Types
  - Function Types
  - Typing Rules with Subsumption
  - Algorithmic Typing Rules without Subsumption

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To easily explain **subtype polymorphism**, let's add new language syntax, **records** and **record types** to TFAE. (Also, type annotations for **val**.)

```
/* STFAE */  
// A record with two fields `a` and `b` whose types are `Number`  
val x: {a: Number, b: Number} = {a=1, b=2}  
x.a    // Access the field `a` of `x` and evaluate to `1`
```

Consider the following expression:

```
/* STFAE */  
val f = (x: ???) => x.a  
f({a=1}) + f({a=2, b=3}) + f({c=4, a=5})
```

Unfortunately, we cannot assign any type to `x` because the type of `x` should be ❶ `{a: Number}`, ❷ `{a: Number, b: Number}`, and ❸ `{c: Number, a: Number}`, simultaneously.

How can we resolve this problem? **Subtype Polymorphism!**

## Definition (Subtype Polymorphism)

**Subtype polymorphism** is a form of polymorphism by introducing **subtype relationships** between types.

```
/* STFAE */  
val f = (x: {a: Number}) => x.a      // Allow subtypes of `{a: Number}`  
f({a=1}) + f({a=2, b=3}) + f({c=4, a=5})
```

All the following types are **subtypes** of `{a: Number}`:

<code>{a: Number}</code>	<code>{a: Number, b: Number}</code>
<code>{c: Number, a: Number}</code>	<code>...</code>

It corresponds to the **subset relationship** between sets in mathematics, and most programming languages support **subtype polymorphism**.

**Subtype relationships** could be defined for other types (e.g., lists, pairs, datatypes, etc.) as well.

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Now, let's extend TFAE into STFAE to support **subtype polymorphism** with **records**:

```
/* STFAE */  
val x: {a: Number} = {a=1}           // A record with a field `a`  
x.a                                   // Access the field `a` of `x`  
val x: {a: Top, b: Top} = {a=1, b=x} // `Top` is the top type  
val x: Bot = exit                     // Exit the program  
...                                   // Not evaluated after `exit`
```

For STFAE, we need to extend **expressions** of TFAE with

- ① **Records**
- ② **Field Accesses**
- ③ **Exit** (to immediately exit the program)
- ④ **Record Types**
- ⑤ **Bottom Type** (corresponding to the empty set)
- ⑥ **Top Type** (corresponding to the universal set)

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We can extend the **concrete syntax** of TFAE as follows:

```
// expressions
<expr> ::= ...
        | "{" [<id> "=" <expr>]*{"", ""} "}"
        | <expr> "." <id>
        | "exit"

// types
<type> ::= ...
        | "{" [<id> ":" <type>]*{"", ""} "}"
        | "Bot"
        | "Top"
```

Expressions	$\mathbb{E} \ni e ::= \dots$	$e.x$ (Access)
	$\{[x = e]^*\}$ (Record)	$\text{exit}$ (Exit)

Types	$\mathbb{T} \ni \tau ::= \dots$	$\perp$ (BotT)
	$\{[x : \tau]^*\}$ (RecordT)	$\top$ (TopT)

```
enum Expr:
  ...
  case Record(fields: List[(String, Expr)])
  case Access(record: Expr, field: String)
  case Exit
```

```
enum Type:
  ...
  case RecordT(fields: Map[String, Type])
  case BotT
  case TopT
```

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For STFAE, we need to 1) implement the **interpreter** with environments:

```
def interp(expr: Expr, env: Env): Value = ???
```

and 2) define the **natural semantics** with environments:

$$\sigma \vdash e \Rightarrow v$$

with a new kind of values called **record values**:

Values  $\mathbb{V} \ni v ::= n$  (NumV)  
                  |  $\langle \lambda x. e, \sigma \rangle$  (CloV)  
                  |  $\{[x = v]^*\}$  (RecordV)

```
enum Value:  
  case NumV(number: BigInt)  
  case CloV(param: String, body: Expr, env: Env)  
  case RecordV(fields: Map[String, Value])
```

```
def interp(expr: Expr, env: Env): Value = expr match
  ...

  case Record(fs) =>
    RecordV(fs.map { case (f, e) => (f, interp(e, env)) }.toMap)

  case Access(r, f) => interp(r, env) match
    case RecordV(fs) => fs.getOrElse(f, error(s"no such field: $f"))
    case v             => error(s"not a record: ${v.str}")
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{Record} \frac{\sigma \vdash e_1 \Rightarrow v_1 \quad \dots \quad \sigma \vdash e_n \Rightarrow v_n}{\sigma \vdash \{x_1 = e_1, \dots, x_n = e_n\} \Rightarrow \{x_1 = v_1, \dots, x_n = v_n\}}$$

$$\text{Access} \frac{\sigma \vdash e \Rightarrow \{x_1 = v_1, \dots, x_n = v_n\} \quad 1 \leq i \leq n}{\sigma \vdash e.x_i \Rightarrow v_i}$$

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Exit => error("exit")
```

$$\sigma \vdash e \Rightarrow v$$

There is no rule for exit because it cannot produce any value.

We cannot draw the derivation tree for the following expression:

```
/* STFAE */ 1 + exit
```

However, We can draw the derivation tree for the following expression:

```
/* STFAE */ (x: Number) => 1 + exit
```

$$\text{Fun} \frac{}{\emptyset \vdash \lambda x:\text{num}.1 + \text{exit} \Rightarrow \langle \lambda x.1 + \text{exit}, \sigma \rangle}$$

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Let's ❶ design **typing rules** of STFAE to define when an expression is well-typed in the form of:

$$\boxed{\Gamma \vdash e : \tau}$$

and ❷ implement a **type checker** in Scala according to typing rules:

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = ???
```

The type checker returns the **type** of  $e$  if it is well-typed, or rejects it and throws a **type error** otherwise.

Similar to TFAE, we will keep track of the **variable types** using a **type environment**  $\Gamma$  as a mapping from variable names to their types.

$$\text{Type Environments} \quad \Gamma \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{T} \quad (\text{TypeEnv})$$

```
type TypeEnv = Map[String, Type]
```

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
...
case Record(fields) =>
  RecordT(fields.map { case (f, e) => (f, typeCheck(e, tenv)) }.toMap)
case Access(record, f) => typeCheck(record, tenv) match
  case RecordT(fs) => fs.getOrElse(f, error(s"no such field: $f"))
  case ty          => error(s"not a record type: ${ty.str}")
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-Record} \frac{\Gamma \vdash e_1 : \tau_1 \quad \dots \quad \Gamma \vdash e_n : \tau_n}{\Gamma \vdash \{x_1 = e_1, \dots, x_n = e_n\} : \{x_1 : \tau_1, \dots, x_n : \tau_n\}}$$

$$\tau\text{-Access} \frac{\Gamma \vdash e : \{x_1 : \tau_1, \dots, x_n : \tau_n\} \quad 1 \leq i \leq n}{\Gamma \vdash e.x_i : \tau_i}$$

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
  ...
  case Exit => BotT
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-Exit} \frac{}{\Gamma \vdash \text{exit} : \perp}$$

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
...
case Val(name, tyOpt, expr, body) =>
  val ty = typeCheck(expr, tenv)

  typeCheck(body, tenv + (name -> ty))
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-Val} \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma[x : \tau_1] \vdash e_2 : \tau_2}{\Gamma \vdash \text{val } x = e_1; e_2 : \tau_2}$$

```
def typeCheck(expr: Expr, tenv: TypeEnv): Type = expr match
...
case Val(name, tyOpt, expr, body) =>
  val ty = typeCheck(expr, tenv)
  tyOpt.map(givenTy => mustEqual(ty, givenTy))
  val nameTy = tyOpt.getOrElse(ty)
  typeCheck(body, tenv + (name -> nameTy))
```

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-Val} \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma[x : \tau_1] \vdash e_2 : \tau_2}{\Gamma \vdash \text{val } x = e_1; e_2 : \tau_2}$$

$$\tau\text{-Val}_\tau \frac{\Gamma \vdash e_1 : \tau_1 \quad \tau_1 = \tau_0 \quad \Gamma[x : \tau_0] \vdash e_2 : \tau_2}{\Gamma \vdash \text{val } x : \tau_0 = e_1; e_2 : \tau_2}$$

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-Val}_{\tau} \frac{\Gamma \vdash e_1 : \tau_1 \quad \tau_1 = \tau_0 \quad \Gamma[x : \tau_0] \vdash e_2 : \tau_2}{\Gamma \vdash \text{val } x : \tau_0 = e_1; e_2 : \tau_2}$$

Consider the following example:

```
/* STFAE */
val x: {a: Number} = {a=2, b=3}; x.a
```

It fails to type check because:

$$\{a: \text{Number}, b: \text{Number}\} \neq \{a: \text{Number}\}$$

Let's apply **subtype polymorphism** to fix this problem by introducing a **subtype relation** ( $<:$ ) between types.

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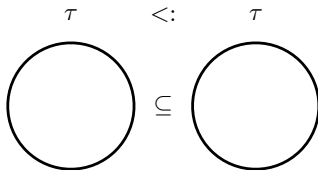
To support **subtype polymorphism**, we need to define a **subtype relation**  $<:$  between types.

$$\boxed{\tau <: \tau}$$

$\tau <: \tau'$  denotes  $\tau$  is a subtype of  $\tau'$  ( $\tau'$  is more general than  $\tau$ ).

First, any type is a **subtype** of itself:

$$\frac{}{\tau <: \tau}$$

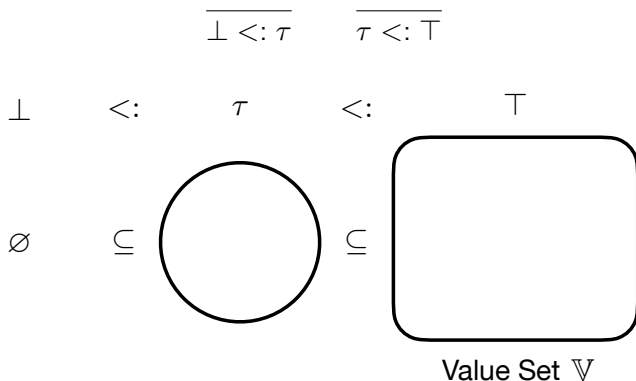




$$\boxed{\tau <: \tau}$$

The bottom type  $\perp$  and the top type  $\top$  represent the empty set of values and the universal set of values, respectively.

Thus,  $\perp$  is a subtype of any type, and any type is a subtype of  $\top$ :



$$\tau <: \tau$$

Let's consider the subtype relation between **record types**.

```
/* STFAE */  
val x: {a: Number, b: Number} = {a = 1, b = 2}  
val y: {a: Number} = x  
val z: Number = y.a  
...
```

If we **add** any new field to a record type, the resulting type should be a subtype of the original type.

$$\frac{}{\{x_1 : \tau_1, \dots, x_n : \tau_n, x : \tau\} <: \{x_1 : \tau_1, \dots, x_n : \tau_n\}}$$

$$\tau <: \tau$$

Let's consider the subtype relation between **record types**.

```
/* STFAE */  
val x: {a: Number, b: Number} = {a = 1, b = 2}  
val y: {a: Top, b: Top} = x  
val z: Top = y.a  
...
```

If all fields of a record type are **subtypes** of the corresponding fields of another record type, the resulting type should be a subtype of the other.

$$\frac{\tau_1 <: \tau'_1 \quad \dots \quad \tau_n <: \tau'_n}{\{x_1 : \tau_1, \dots, x_n : \tau_n\} <: \{x_1 : \tau'_1, \dots, x_n : \tau'_n\}}$$

$$\tau <: \tau$$

Let's consider the subtype relation between **record types**.

```
/* STFAE */  
val x: {a: Number, b: Number} = {a = 1, b = 2}  
val y: {b: Number, a: Number} = x  
val z: Number = y.a  
...
```

If the fields of a record type is a **permutation** of the fields of another record type, the resulting type should be a subtype of the other.

$$\frac{\{x_1 : \tau_1, \dots, x_n : \tau_n\} \text{ is a permutation of } \{x'_1 : \tau'_1, \dots, x'_n : \tau'_n\}}{\{x_1 : \tau_1, \dots, x_n : \tau_n\} <: \{x'_1 : \tau'_1, \dots, x'_n : \tau'_n\}}$$

$$\tau <: \tau$$

Let's consider the subtype relation between **function types**.

Consider the subtype relationship between **parameter types**.

```
val f3: Top => Number = f // Impossible: f cannot take Top </: Number
val f4: Bot => Number = f // Possible : f can take Bot <: Number
...
```

We need **super types** for the given **parameter types**.

Now, consider the subtype relationship between **return types**.

```
/* STFAE */
val f: Number => Number = (x: Number) => x
val f1: Number => Top = f // Possible : f returns Number <: Top
val f2: Number => Bot = f // Impossible: f returns Number </: Bot
...
```

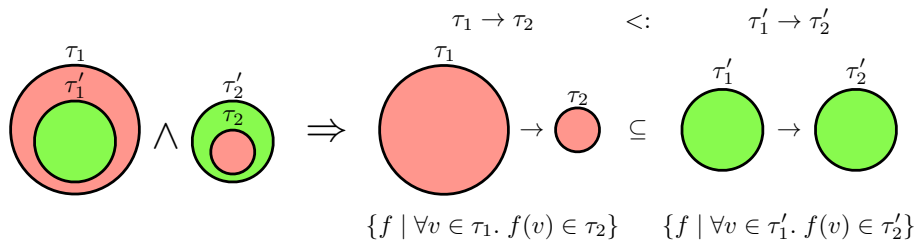
We need **subtypes** for the given **return types**.

$$\boxed{\tau <: \tau'}$$

We need **super types** for the given **parameter types**:  $\tau_1 :> \tau'_1$ .

We need **sub types** for the given **return types**:  $\tau_2 <: \tau'_2$ .

$$\frac{\tau_1 :> \tau'_1 \quad \tau_2 <: \tau'_2}{(\tau_1 \rightarrow \tau_2) <: (\tau'_1 \rightarrow \tau'_2)}$$



One possible way to support subtype polymorphism is to add a general **subsumption** rule to the typing rules:

$$\frac{\Gamma \vdash e : \tau \quad \tau <: \tau'}{\Gamma \vdash e : \tau'}$$

With this rule, we can type the following expression.

```
/* STFAE */
val x: {a: Number} = {a=2, b=3}; x.a
```

$$\frac{\begin{array}{c} \dots \\ \hline \emptyset \vdash \{a = 2, b = 3\} : \{a : \text{num}, b : \text{num}\} \end{array} \quad \begin{array}{c} \dots \\ \hline \{a : \text{num}, b : \text{num}\} <: \{a : \text{num}\} \end{array}}{\emptyset \vdash \{a = 2, b = 3\} : \{a : \text{num}\}} \quad \dots$$

$$\frac{\emptyset \vdash \{a = 2, b = 3\} : \{a : \text{num}\}}{\emptyset \vdash \text{val } x : \{a : \text{num}\} = \{a = 2, b = 3\}; x.a : \text{num}}$$

However, it is **not algorithmic** because we don't know which types are required as the result of subsumption.

Another way is to **directly apply** the subtype relation to the each typing rule without subsumption, and it is **algorithmic**.

$$\boxed{\Gamma \vdash e : \tau}$$

$$\tau\text{-Add} \frac{\Gamma \vdash e_1 : \tau_1 \quad \tau_1 <: \text{num} \quad \Gamma \vdash e_2 : \tau_2 \quad \tau_2 <: \text{num}}{\Gamma \vdash e_1 + e_2 : \text{num}}$$

$$\tau\text{-Mul} \frac{\Gamma \vdash e_1 : \tau_1 \quad \tau_1 <: \text{num} \quad \Gamma \vdash e_2 : \tau_2 \quad \tau_2 <: \text{num}}{\Gamma \vdash e_1 \times e_2 : \text{num}}$$

$$\tau\text{-Val}_\tau \frac{\Gamma \vdash e_1 : \tau_1 \quad \tau_1 <: \tau_0 \quad \Gamma[x : \tau_0] \vdash e_2 : \tau_2}{\Gamma \vdash \text{val } x : \tau_0 = e_1; e_2 : \tau_2}$$

$$\tau\text{-App} \frac{\Gamma \vdash e_0 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash e_1 : \tau_3 \quad \tau_3 <: \tau_1}{\Gamma \vdash e_0(e_1) : \tau_2}$$



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- Please see this document<sup>2</sup> on GitHub.
  - Implement `typeCheck` function.
  - Implement `interp` function.
- It is just an exercise, and you **don't need to submit** anything.
- However, some exam questions might be related to this exercise.

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<sup>2</sup><https://github.com/ku-plrg-classroom/docs/tree/main/cose212/stfae>.

- Type Inference (1)

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