

Lecture 18 – Type Systems

COSE212: Programming Languages

Jihyeok Park



2023 Fall

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 - **First-Class Continuations** (Lecture 16)
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 - Typing Recursive Functions
 - Type Inference
 - Algebraic Data Types
 - Parametric Polymorphism
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- In this lecture, we will focus on the **motivation** and **basic concepts** of type systems.

1. Motivation: Safe Language Systems

- Detecting Run-Time Errors

- Dynamic vs Static Analysis

- Soundness vs Completeness

2. Type Systems

- Types

- Type Errors

- Type Checking

- Type Soundness

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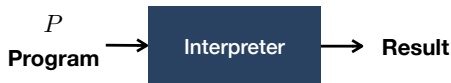
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- **Semantics**: a set of rules that defines the meaning of programs

and implemented their **interpreters** in Scala:



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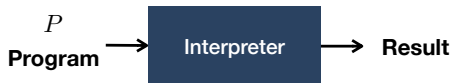


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However, we don't have any automatic system to **check** whether a program is evaluated without any **run-time errors**.

For example, following FAE expressions are syntactically correct, but they throw **run-time errors**:

```
/* FAE */  
x * 42           // error: free identifier  
0 + (x => x)     // error: cannot add a function  
1(2)            // error: cannot apply a number
```

Unexpected errors in **safety-critical software** cause serious problems:

<p>June 4, 1996: Ariane-5 explodes after lift off</p> <p>Today in History: June 4, 1996: Ariane-5 explodes after lift off</p> <p>Copyright: June 08, 2016 0:03 Abdellatif, David of Archibute</p> 	<p>Knight Capital Says Trading Glitch Cost It</p> <p>BY NATHANIEL POPPER AUGUST 3, 2013 9:07 AM 398</p> <p>Runaway Trades Spread Turmoil Across Wall St.</p> 	<p>Heathrow Airport apologises for IT failure disruption</p> <p>3:16 February 2020</p> 	<p>Cruise recalls all its driverless cars</p> <p>pedestrian hit and dragged</p> <p>In another setback, Cruise updates software on 950 driverless cars to fix its 'Collision Detection'</p> <p>By David Stewart</p> <p>Updated November 6, 2023 at 2:22 p.m. EST Published November 6, 2023 at 2:08 p.m. EST</p> 
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Then, how can we **prevent** such errors?

Can we **automatically** check whether a program does not have any **run-time errors**?

We can use various **analysis** techniques to detect run-time errors:



An **analyzer** is a program that takes a program as an input and determines whether the program has a certain property. In this case, the property is **run-time errors**.

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We can categorize them into two groups:

- **Dynamic Analysis**: analyze programs by **executing** them
- **Static Analysis**: analyze programs **without executing** them

Dynamic analysis is a program analysis technique by **executing** them.

Let's perform **dynamic analysis** for the following Scala program:

```
def abs(x: Int): Int = { /* L1 */  
  if (x < 0)             /* L2 */  
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  else                   /* L4 */  
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L1	-5
L2	-5
L3	5
L4	
L5	
L6	5

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L1	-5	42
L2	-5	
L3	5	
L4		42
L5		42
L6	5	42

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L1	-5	42	-7	99	0	...
L2	-5		-7			...
L3	5		7			...
L4		42		99	0	...
L5		42		99	0	...
L6	5	42	7	99	0	...

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We can easily get the **behavior** of the program for each **single input**.

However, it is **difficult** to get all the **possible behaviors** of the program for **all the inputs**.

Static analysis is a program analysis technique **without executing** them.

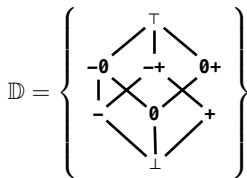
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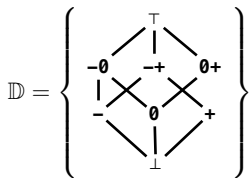


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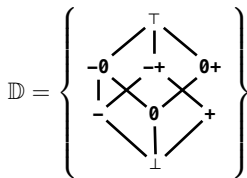
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L1	
L2	
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L5	
L6	

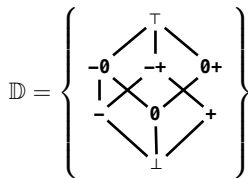
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L1	\top
L2	$-$
L3	$+$
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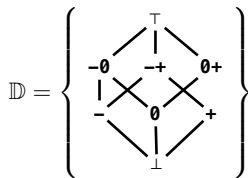
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We can prove that `abs` always returns a **non-negative** integer (i.e., $0+$).

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- $\vdash \psi$ denotes that a statement ψ is **true**.

In a **sound** proof system, all **provable** statements are **true**.

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Analysis techniques can be used to prove that a program is **error-free**.

- $\models P$ denotes that a program P is **analyzed** as error-free.
- $\vdash P$ denotes that a program P is truly **error-free**.

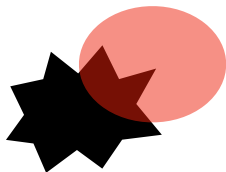
Then, is dynamic/static analysis **sound** or **complete**?

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 - All the detected errors are **true alarms** (**true positive (TP)**).
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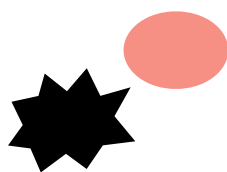
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P₁



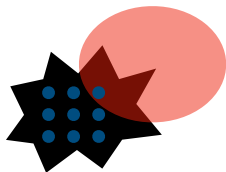
P₂



P₃

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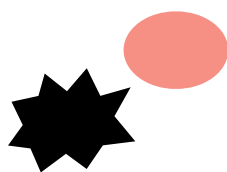
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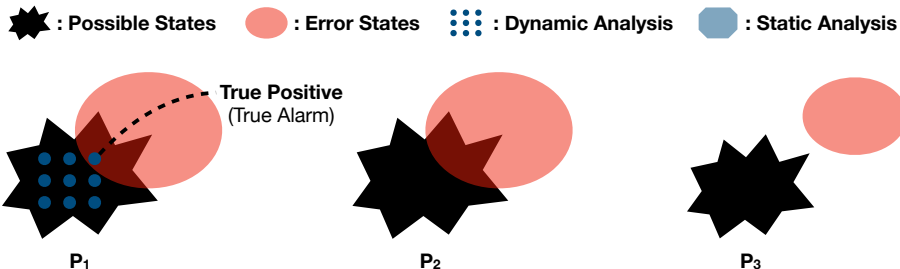


P₂

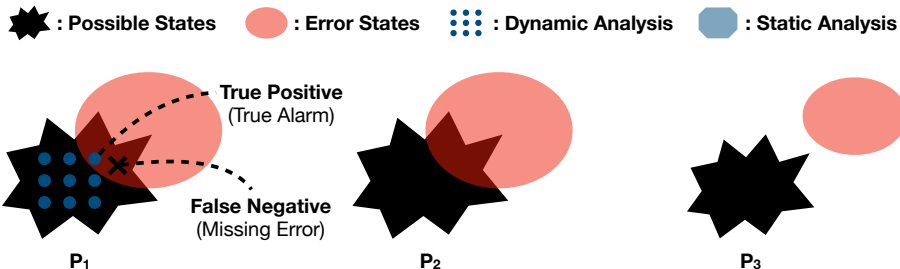


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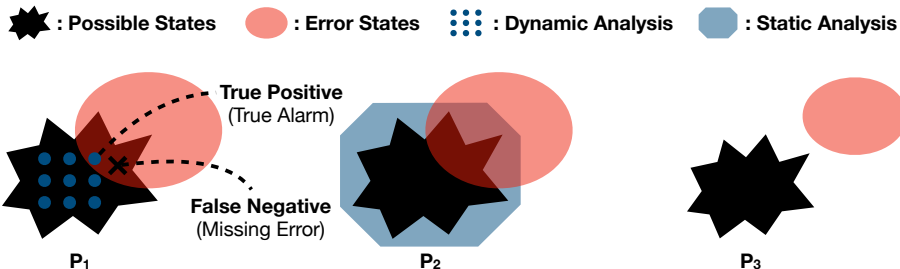
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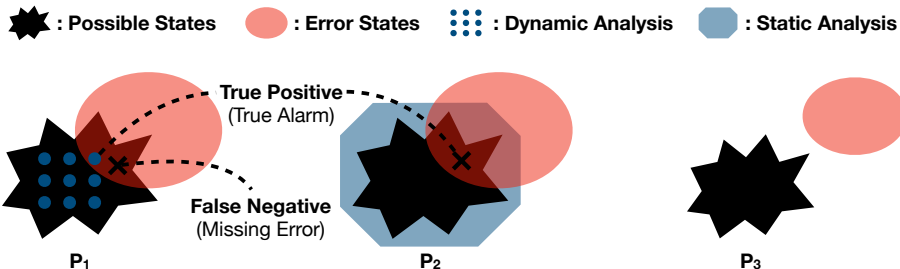
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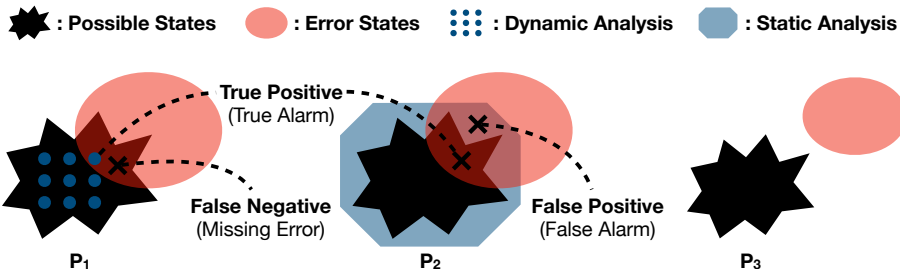
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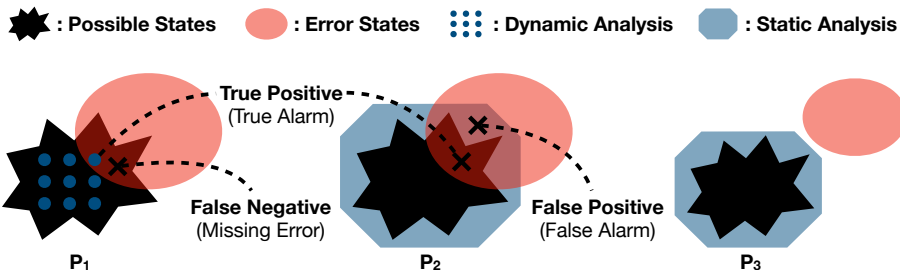
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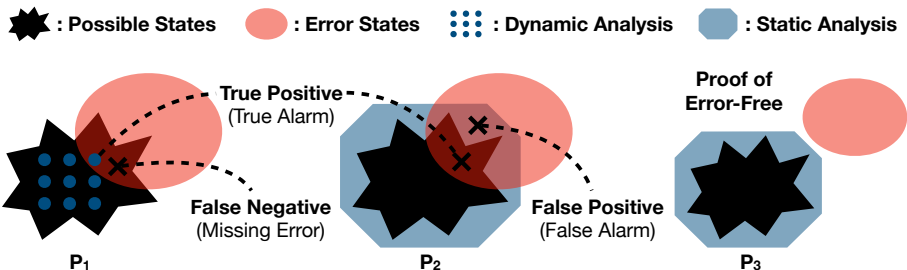
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Detecting Run-Time Errors

Dynamic vs Static Analysis

Soundness vs Completeness

2. Type Systems

Types

Type Errors

Type Checking

Type Soundness

Definition (Types)

A **type** is a set of values.

For example, the `Int`, `Boolean`, and `Int => Int` types are defined as the following sets of values in Scala.

$$\text{Int} = \{n \in \mathbb{Z} \mid -2^{31} \leq n < 2^{31}\}$$

$$\text{Boolean} = \{\text{true}, \text{false}\}$$

$$\text{Int} \Rightarrow \text{Int} = \{f \mid f \text{ is a function from Int to Int}\}$$

```
val n: Int = 42           // 42    : Int
n + 1                   // 43    : Int
val b: Boolean = n > 10   // true  : Boolean
def f(x: Int): Int = x + 1 // f     : Int => Int
f(42)                   // 43    : Int
```

Definition (Type Errors)

A **type error** occurs when a program tries to use a value having a type that is **incompatible** with the expected type.

For example, the following Scala program has type errors:

```
42 + true           // `Int` expected for `+`, but `Boolean` found
if (1) 2 else 3      // `Boolean` expected for `if`, but `Int` found
def f(x: Int): Int = x + 1
f(false)            // `Int` expected for `f`, but `Boolean` found
```

However, not all **run-time errors** are **type errors**:

```
42 / 0              // `ArithmeticException` at run-time
case class A(k: Int)
val x: A = null
x.k                 // `NullPointerException` at run-time
```

If the following conditions hold, we say “**the expression e has type τ** ”:

- e does not cause any type error, and
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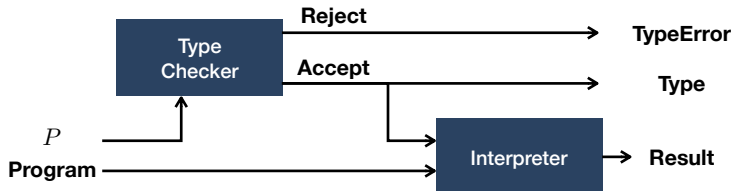
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Definition (Type Checking)

Type checking is a kind of static analysis checking whether a given expression e is **well-typed**. A **type checker** returns the **type** of e if it is well-typed, or rejects it and reports the detected **type error** otherwise.



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Type systems in most statically-typed languages are designed to be **sound**.

1. Motivation: Safe Language Systems

- Detecting Run-Time Errors
- Dynamic vs Static Analysis
- Soundness vs Completeness

2. Type Systems

- Types
- Type Errors
- Type Checking
- Type Soundness

- Please see this document¹ on GitHub.
- The due date is Nov. 27 (Mon.).
- Please only submit `Implementation.scala` file to **Blackboard**.

¹<https://github.com/ku-plrg-classroom/docs/tree/main/cose212/magnet>.

- Typed Languages

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