fuzziness, vagueness ....

- most concepts are vague
  - not lists, **no necessary and sufficient criteria**

    for most concepts there are no precise criteria: what exactly is a furniture?
  - E. Rosch: subjects agree on how to classify instances of concepts according to **typicality**

    apples are more typical fruits than figs, sparrows are more typical birds than emus
  - x can be partly A and partly not-A

    a piano is somewhat a furniture but mainly a musical instrument
  - sorites 'paradox': remove one grain from a heap of sand…

    paradox disappears if one admits degrees of belonging to a concept: with every removed grain, the heap becomes less of a heap..
- with increasing complexity, precise statements lose meaning, and meaningful statements lose precision
  - see ecosystems, weather, stockmarket, brain, societies, etc.

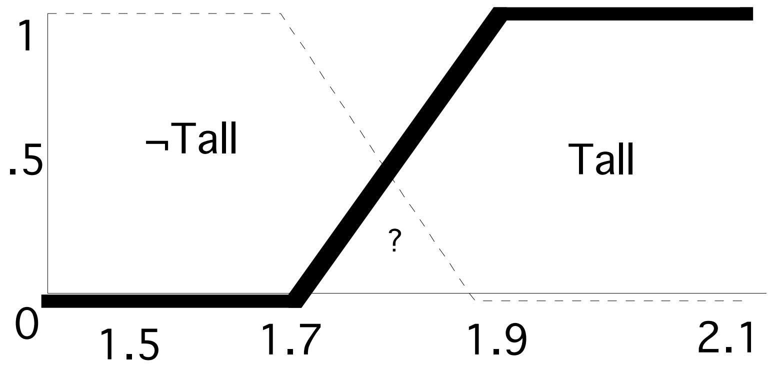
fuzziness and complexity

- **complexity is reduced by:**
  vagueness, imprecision, uncertainty
  - the car drives fast
  - the car is about 50 m away
  - there is weak evidence that …
- crisp sets
  - defined by enumeration
  - with a predicate or
  - with a **characteristic function**
    χ_Z: N $\rightarrow$ {0, 1} s.t. $\chi_Z(n) = 1$ if Pred(n), 0 otherwise.

Tall $=_{df}$ {x | x is a Person & x $\geq$ 180 cm}

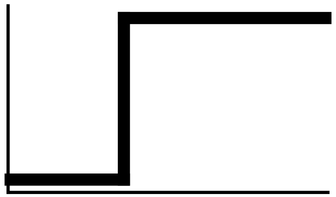
so if you are 179.99 you are small?

grey areas, cloud-like sets

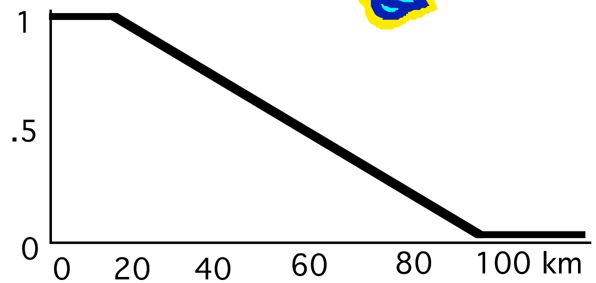


anybody under 1.7 is not tall…
zig-zag line is easy to compute.
Joe is 1.79 and thus tall to degree 0.45.

a fuzzy set **is** the degree of membership function!
¬Tall(x) = 1 - Tall(x)



ordinary, crisp set



I'm living near Ottawa

fuzzy sets

- a fuzzy set μ of X is a function from reference set X to [0, 1]:
  $\mu: X \to [0, 1]$ (F(X) = {all fuzzy sets of X})

  instead of [0,1], use a lattice $(l, \cup, \cap)$:
  $\mu: X \to L$. If L is linearly ordered, $l \cap l' = 1$ iff $l \leq l'$, $l \cup l' = 1$ iff $l \geq l'$

  a sample lattice:

  impossible<highly doubtful<slight chance<believe not<unlikely
  <probably not<chances slightly less than even<chances about even
  < chances slightly better than even<it's probable<chances good
  <we estimate<we believe<likely<highly likely<highly probable
  <we are convinced<virtually certain<certain

fuzzy set A in X

- is completely characterized by μ_A
- can be expressed as set of ordered pairs:

$$A = \{(x, \mu_A(x)) \mid x \in X\}$$

- can be based on discrete, unordered base set:

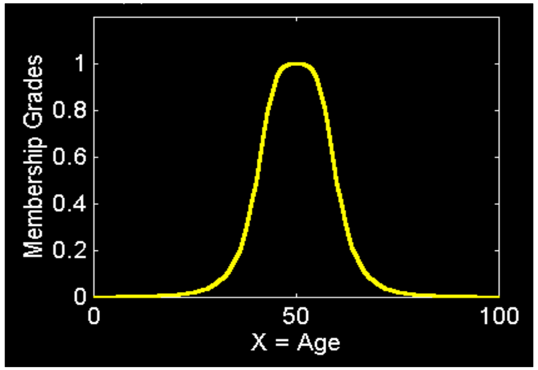
  A = "nice city to live in"; X = {Mtl, To, Van, Ott, Sask, MooseJaw…}

  A = {(Mtl,.6), (To,.7), (Van,.8), (Ott,7), (Sask,.2), (MJ,.1)}

- can be based on continuous base set:

  A = "about 50 years old"; X = [0,100]

$$\mu_A(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^2}$$



examples of fuzzy sets

tall $=_{df}$ $\mu_{a,b}(x) =_{df} \begin{cases} 0, & \text{if } x \le a \\ (x - a)/(b - a), & \text{if } a \le x \le b \\ 1, & \text{if } x \ge b \end{cases}$ $[a<b]$

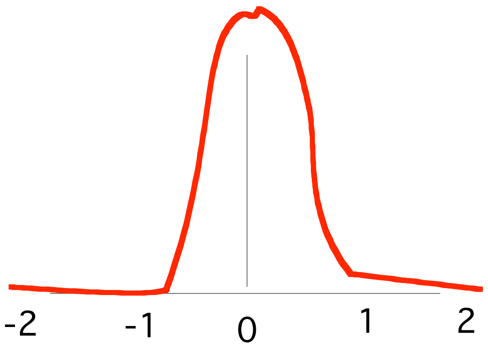
about 10 $=_{df}$ $\mu_{m,d}(x) =_{df} \begin{cases} 1 - |(m-x)/d|, & \text{if } m\text{-}d \le x \le m\text{+}d \\ 0, & \text{if } x < m\text{-}d \text{ or } x > m\text{+}d \end{cases}$

$[d>0, m \in R]$

approx.between b and c $=_{df}$ $\mu_{a,b,c,d}(x) =_{df} \begin{cases} (x-a)/(b-a), & \text{if } a \le x < b \\ 1, & \text{if } b \le x \le c \\ (x-d)/(c-d), & \text{if } c < x \le d \\ 0, & \text{if } x < a \text{ or } x > d \end{cases}$

$[a<b<c<d]$

examples of fuzzy sets, cont.



close to 0 $=_{df}$ $\mu_A(x) =_{df} 1 / (1 + 10x^2)$

$\mu(3) = .01$, $\mu(.25) = .62$, $\mu(0) = 1$.

close to a $=_{df}$ $1 / (1 + 10(x - a)^2)$

very close to 0: $(1 / (1 + 10x^2))^2$

μ's do not have to sum to 1!!

type 2 fuzzy sets: μ's are <u>themselves</u> type 1 fuzzy sets;

e.g. "intelligent" with μ's: average, below average, genius, etc.

level k fuzzy sets: fuzzy subsets of universal set whose elements are fuzzy sets.

e.g. { desired attributes} for a new house: cheap, good location, nice looking,...

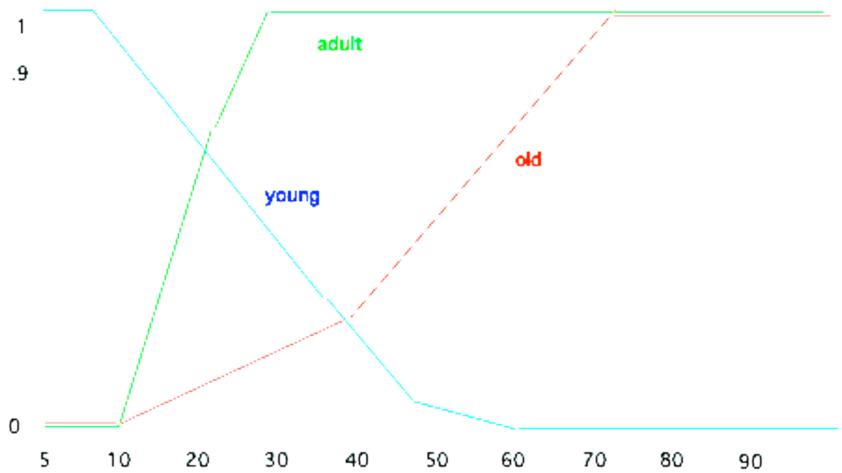
examples of fuzzy sets, cont.

- veryTall(x) = Tall(x -10 cm)

- 'he is heavy for his size' (HS)
  - suppose ideal weight: size in cm - 100, in kg
  - if you are more than 15% under the ideal weight, then HS = 0 …

$$
\begin{aligned}
HS(x) &= 0 \ \text{ if } \ weight(x) \ / \ (size(x) - 100) < 0.85 \\
&= 1 \ \text{ if } \ weight(x) \ / \ (size(x) - 100) > 1.15
\end{aligned}
$$

support, α-cuts



young, old, etc $\in \wp(X)$,
$X = \{5,10, 20,\ldots, 90\}$

$\text{support}_A = \{x \in X \mid \mu_A(x) > 0\}$
$\text{support}_{young} = \{5,10,20,30,40,50\}$

α-cut of A = $A_\alpha =_{df} \{x \in X \mid \mu_A(x)$! $\alpha\}$
$\text{young}_{.8} = \{5,10,20\}$, $\text{young}_1 = \{5,10\}$

scalar cardinality of A = $|A| = \sum_{x \in X} \mu_A(x)$; height of A = highest μ of any of its members; normalized fuzzy set A: at least 1 x s.t. $\mu_A(x) = 1$

vertical and horizontal representations

vertical: each $x \in X$ is assigned $\mu(x)$

horizontal (via α-cuts):

$$[\mu]_\alpha =_{df} \{x \in X \mid \mu(x)\ !\ \ \alpha\} \quad (\alpha\text{-cut})$$

- $[\mu]_0 = X$;
- $\alpha < \beta \Rightarrow [\mu]_\alpha \supseteq [\mu]_\beta$;
- $\bigcap_{\alpha:\alpha<\beta} [\mu]_\alpha = [\mu]_\beta$

any fuzzy set can be described by specifying its **α-cuts**!

- $$\mu(x) = \sup_{\alpha \in [0,1]} \{\min(\alpha, \chi_{[\mu]\alpha}(x))\}$$

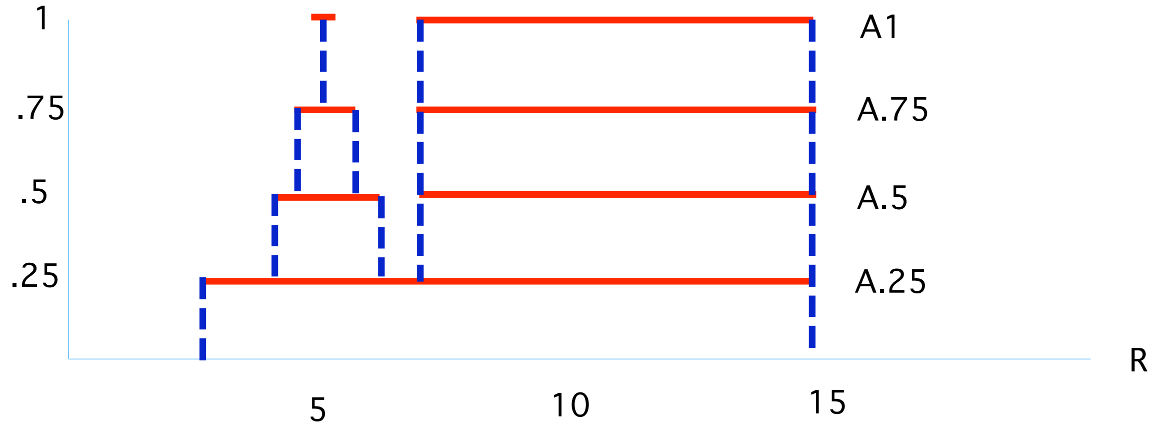
horizontal representation

e.g. $X = [0, 15]$;   vague datum: "ca. 5 or ! 7".

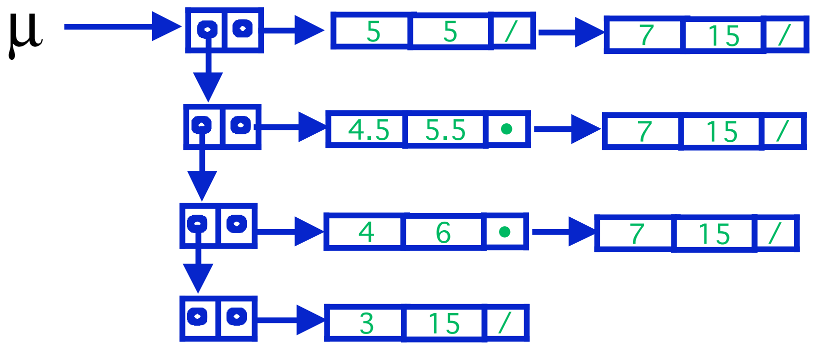
Expert chooses degrees L and α-cuts:

$L = \{0, .25, .5, .75, 1\}$; $A_0 = [0,15]$, $A_{.25} = [3, 15]$, $A_{.5} = [4, 6] \cup [7, 15]$,
$A_{.75} = [4.5, 5.5] \cup [7, 15]$, $A_1 = \{5\} \cup [7, 15]$;

$(A_\alpha)_{\alpha \in L}$ induces μ_α:



horizontal representation, cont.



1 linear list per α-level, (α " 0), storing finite union of closed intervals

horizontal view good for generalizing set operations...

basic fuzzy set operations

- $\forall x \in X, \mu_A(x) \leq \mu_B(x) \Rightarrow A \subseteq B$

  e.g. $\mu_{old}(x) \leq \mu_{adult}(x)$, so old $\subseteq$ adult

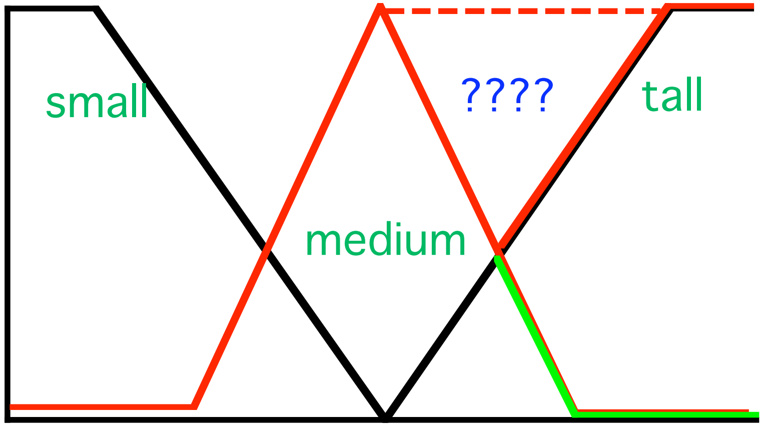
- $\forall x \in X, \mu_{\neg A}(x) = 1 - \mu_A(x)$

  but here, $\neg$old $\neq$ young

- $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$

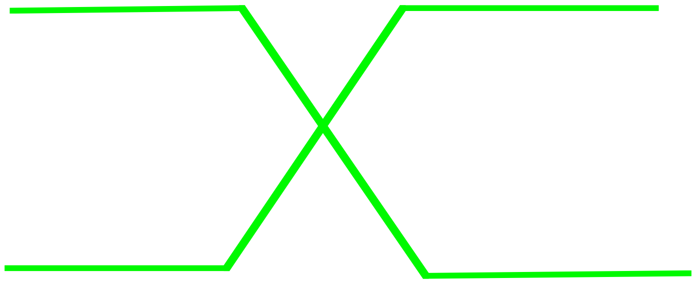
- $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$

$A \cup \neg A \neq U$!!! $A \cap \neg A \neq \varnothing$!!!



what is tall **or** medium?

Some prefer other definitions of $\cup$, $\cap$.



tall $\cup$ ¬tall !!

tall $\cap$ ¬tall !!

standard fuzzy $\cap$, $\cup$, complement ...

- form a pseudo-complemented distributive lattice on $\mathcal{P}(X)$, i.e. a boolean lattice without $A \cup \neg A = X$, $A \cap \neg A = \varnothing$

  Excluded Middle and NonContradiction can be preserved by giving up idempotency and distributivity

  fuzzy $\Rightarrow$:

  $$a \Rightarrow b \Leftrightarrow \neg a \vee b \Leftrightarrow \neg a \cup b$$
  $$\Leftrightarrow \neg(a \wedge \neg b) \Leftrightarrow \neg(a \cap \neg b)$$

  next: fuzzy relations (digraphs, equivalence relations etc)

standard membership functions

- standard mf:

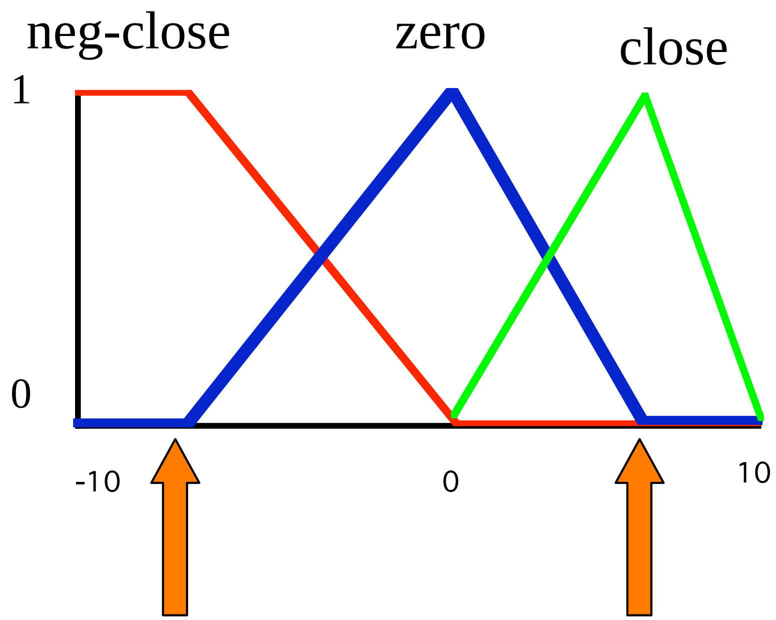
• The most typical value for each term gets $\mu=1$.

• For each term, set $\mu=0$ where neighboring terms have their most typical value.

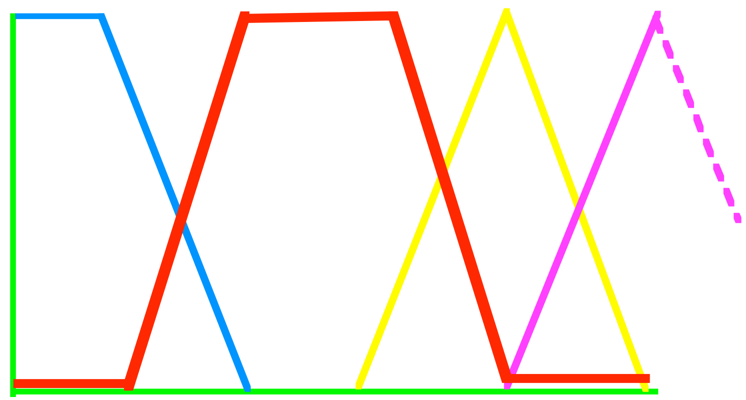
• Connect point $\mu=1$ with points $\mu=0$ by straight lines, so that mbf's for inner terms are of type Λ.

• Points left of leftmost and right of rightmost term: $\mu=1$.

membership functions Π and Λ



typical values for neg-close and close define $\mu = 0$ for zero.



basic types of mbf (x-coords a<b<c<d…)

Triangular MF:

$$
\text{trimf}(x; a,b,c) = \begin{cases} 0, & x \# a \text{ or } c \# x \\ (x-a)/(b-a), & a \# x \# b \\ (c-x)/(c-b), & b \# x \# c \end{cases}
$$

or:

$$
trimf(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)
$$

Trapezoidal MF:

$$
\text{trapmf}(x; a,b,c,d) = \begin{cases} 0, & x \# a \text{ or } d \# x \\ 1, & b \# x \# c \\ (x-a)/(b-a), & a \# x \# b \\ (d-x)/(d-c), & c \# x \# d \end{cases}
$$

or:

$$
trapmf(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)
$$

basic types of mbf, cont.

Gaussian MF:

c: center; σ: width

$$gaussmf(x;c,\sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

Generalized Bell MF:

$$gbellmf(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{b}\right|^{2b}}$$

other mf's

- **spline membership functions** are better for data analysis, decision support…

  • $\mu(x)$ is **continuous** over X, i.e. a small change in base variable does not result in a step in the evaluation!!

  • $d(\mu(x))/dx$ is continuous over X, i.e. a small change in base variable does not result in a step in the evaluation rate and

  • $\mu(x)$: $\min_{\mu}(\max_X(d^2(\mu(x))/dx^2))$; the change of slope is minimal.

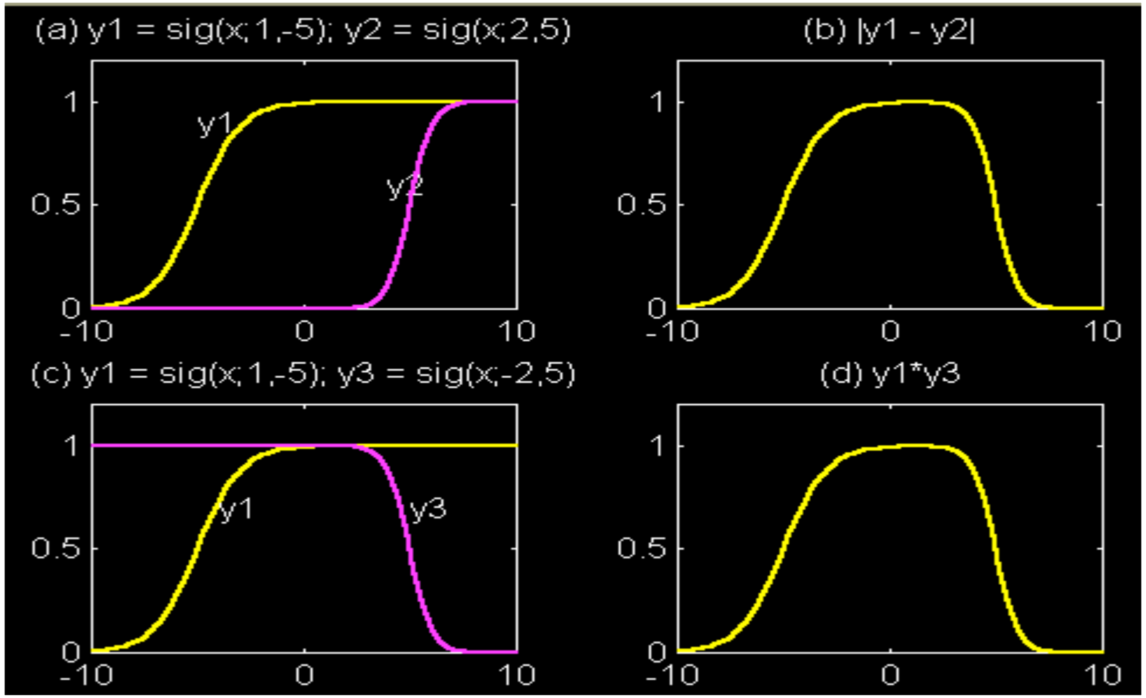
- To satisfy these assumptions, connect $\mu = 1$ and $\mu=0$ points with S-shaped interpolative cubic spline function instead of straight lines

sigmoidal mf's

$$sigmf(x;a,b,c) = \frac{1}{1+e^{-a(x-c)}}$$

here, *a* controls slope at xover point *x = c;*
sig(x;a,c) = 1 / [1 + exp[-a (x - c)]]

making a closed sigmoidal mf: $|y_1 - y_2|$ or $y_1 * y_2$



extension principle for fuzzification

- suppose f maps pairs from $X_1 = \{a, b, c\}$ and $X_2 = \{x, y\}$ to $Y = \{p, q, r\}$, i.e. f is:

	x	y
a	p	p
b	q	r
c	r	p

- $A_1 \in \wp(X_1)$ s.t. $A_1 = .3/a + .9/b + .5/c$; $A_2 \in \wp(X_2)$ s.t. $A_2 = .5/x + 1/y$;

$B = f(A_1, A_2) \in \wp(Y)$ via extension principle:

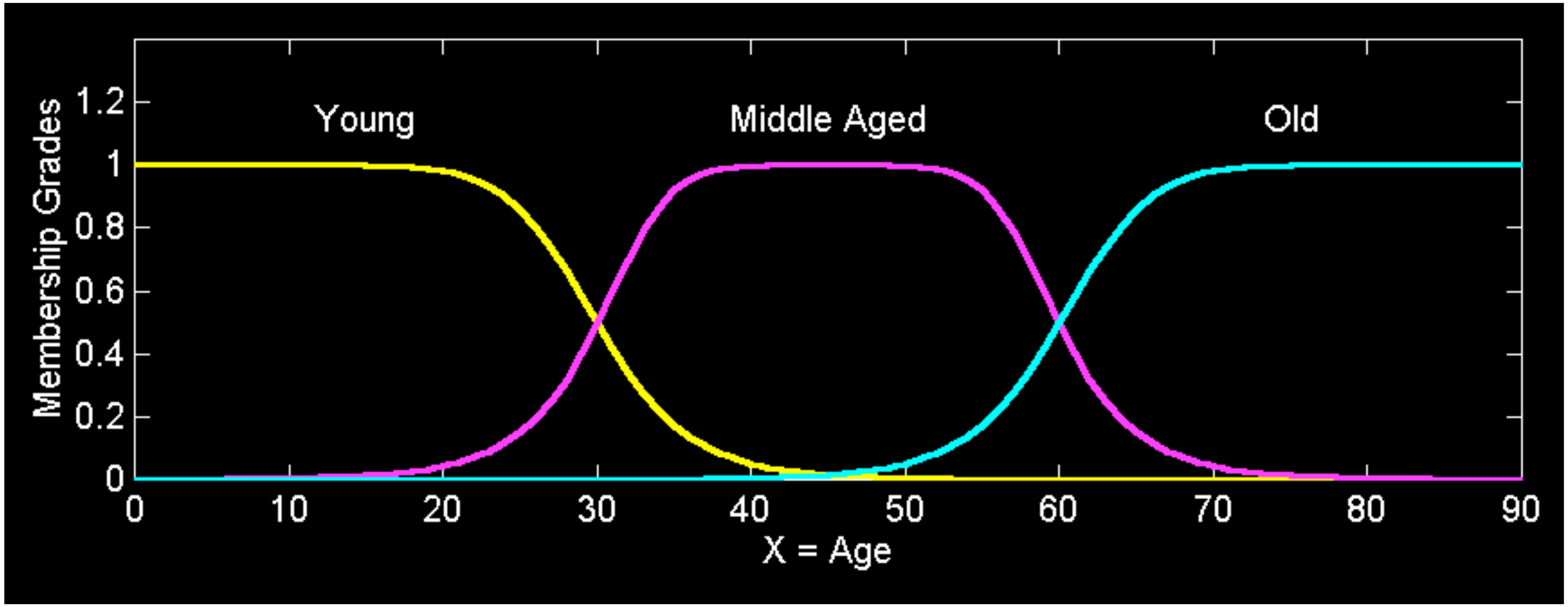
$\mu_B(p) = \max(\min(.3, .5), \min(.3, 1), \min(.5, 1)) = .5;$

$\mu_B(q) = \max(\min(.9, .5)) = .5;$

$\mu_B(r) = \max(\min(.5, .5), \min(.9, 1)) = .9;$

i.e. $f(A_1, A_2) = .5/p + .5/q + .9/r$

linguistic variables



linguistic variable (e.g. Age) has **linguistic values** like *Young*, *Old*, *very Old* etc. which form fuzzy partitions. **linguistic variable** =

< x,	T(x),	U,	G,	*M* > where
name of vbl	term set {names of ling.values}	universe of discourse	grammar rule to generate name	assigns fuzzy subsets of U as meanings

linguistic variables, cont.

- T(*Age*) = {old, very old, not so old, more or less young, quite young, very young}

- *M*(old) = {(u, μ_{old}(u)) | u ∈ [0, 100]} where
  $\mu_{old}(u) = 0$ if $u \in [0, 50]$; $= (1 + ((u-50)/5)^{-2})^{-1}$, if $u \in [50, 100]$;

- linguistic variables are good for approximate reasoning

- some linguistic terms are the result of applying fuzzy modifiers to others

- there are several styles of approximate reasoning ….
  via fuzzy modus ponens, fuzzy relations etc.

hedges, modifiers

concentration: $\mu_{con(A)}(u) = (\mu_A(u))^2$

dilation: $\mu_{dil(A)}(u) = (\mu_A(u))^{1/2}$

contrast intensification:

$$\mu_{int(A)}(u) = 2^*(\mu_A(u))^2, \text{ if } \mu_A(u) \in [0, .5],$$
$$= 1 - 2\,(1 - \mu_A(u))^2, \text{ otherwise.}$$

very A = con(A), more or less A = dil(A), plus A = $A^{1.25}$,

slightly A = int[plus A and not (very A)]

very very very … very old …. $T^{i+1} = \{old\} \cup \{very\ T^i\}$

fuzzy relations

x is little
x and y are approximately equal
thus, y is more or less little

X = {1,2,3,4}
little = {(1,1),(2,.6),(3,.2),(4,0)}

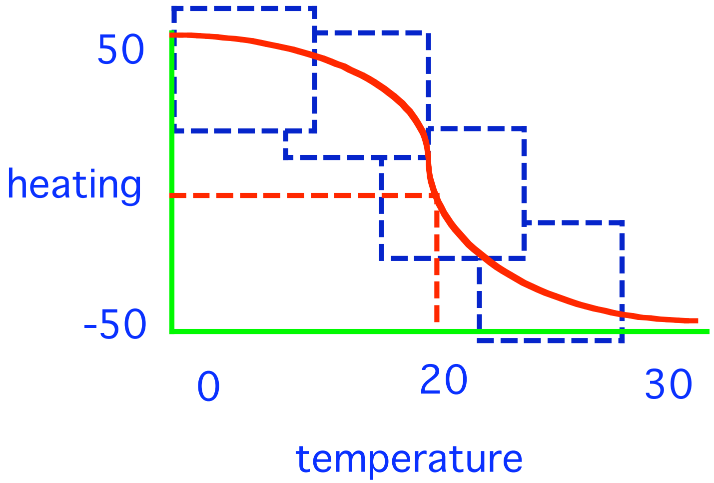
	1	2	3	4
1	1	.5	0	0
2	.5	1	.5	0
3	0	.5	1	.5
4	0	0	.5	1

R = approximately equal

max min $\{\mu_{little}(x), \mu_R(x,y)\}$ = {(1,1),(2,.6),(3,.5),(4,.2)}
max min composition for relations: similar to matrix multiplication.
$\mu_{R1 \circ R2}(x,z) = \max_y \min [\mu_{R1}(x, y), \mu_{R2}(y, z)]$

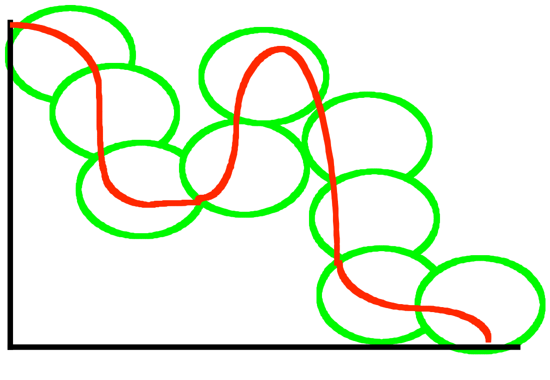
fuzzy control

control without models ….



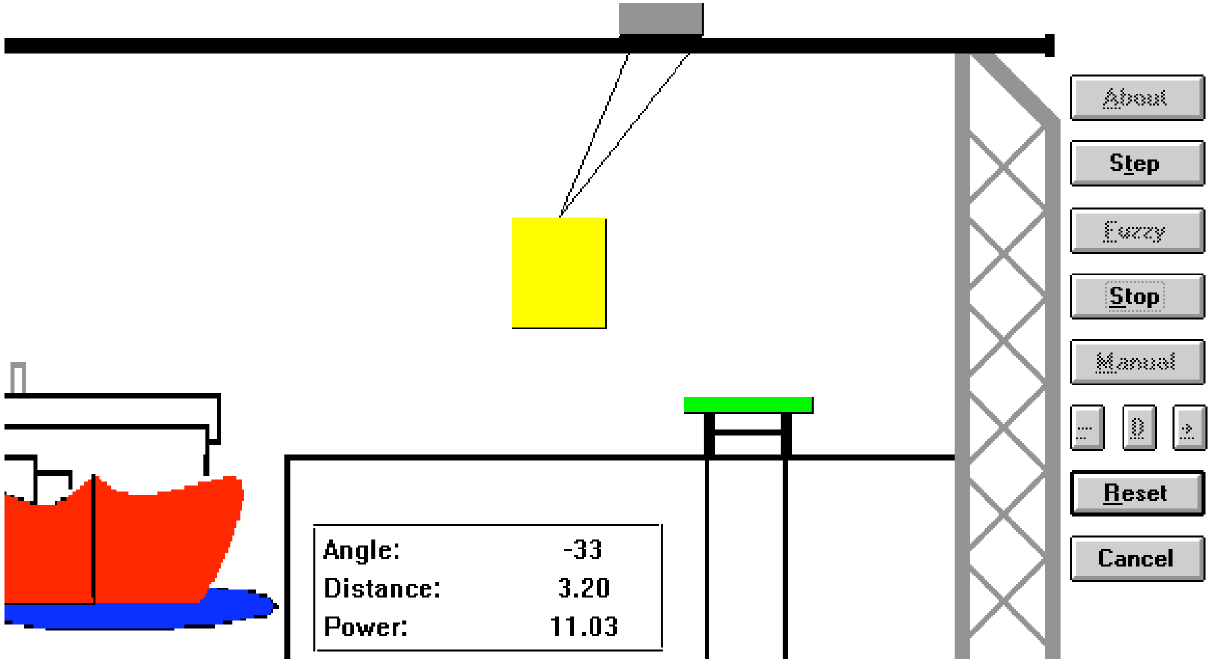
if it's cold, heat a lot;
if it's hot, cool a lot;
if temperature is about right, do nothing.

n-dimensional fuzzy clouds…



very small, quite small, small,
a little bit small….

fuzzy control, cont.



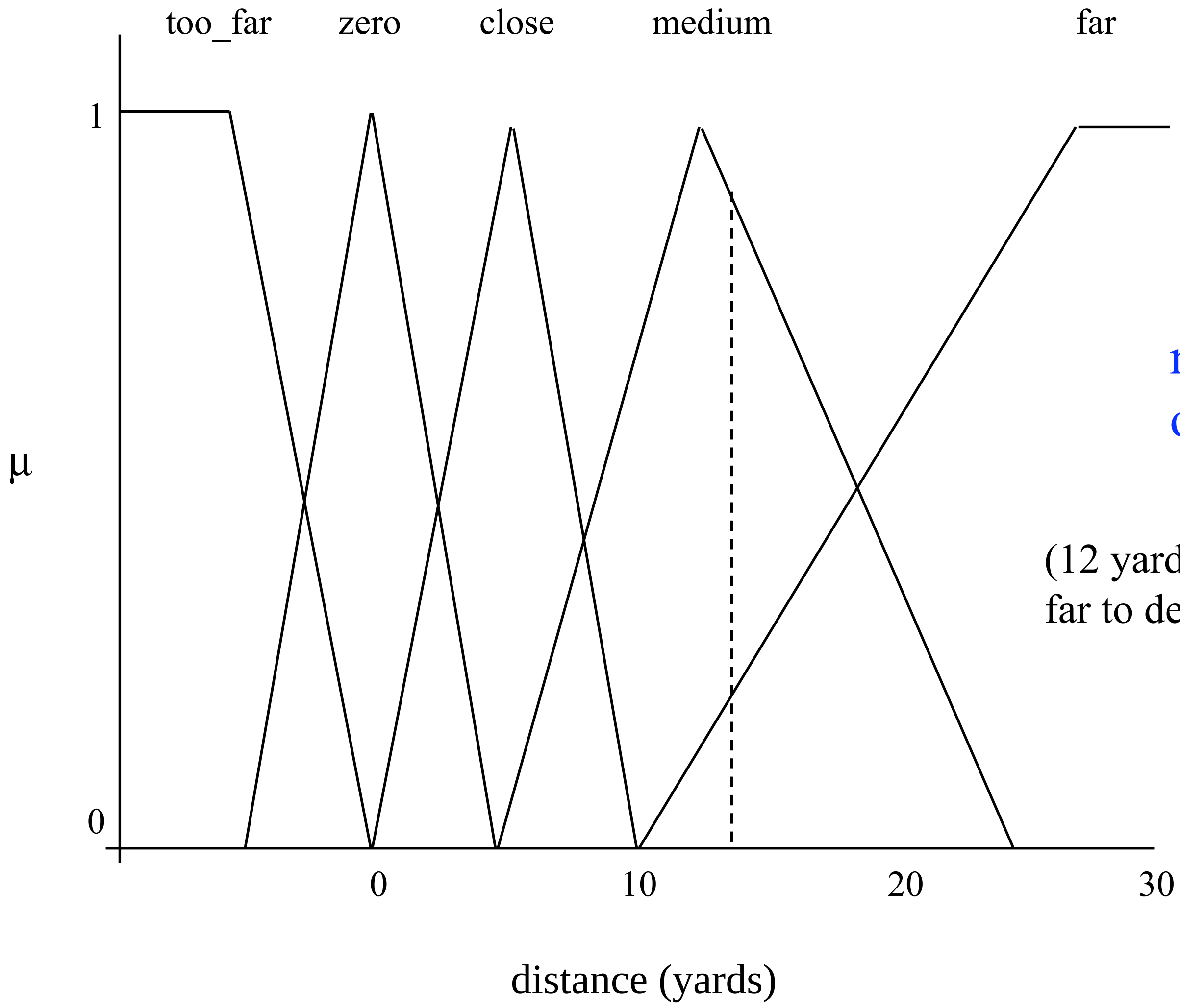
PID (proportional-integral-differential) control handles only linear tasks...
Mathematical models are hard to develop, ignore friction, wind, often too linear ...

thus, use **fuzzy logic**.

linguistic control strategy

1. start with medium power (**distance** = far & **angle** = 0 → **power** = pos_medium)
2. if started and still far from target, adjust motor power so that container gets a little behind crane head (distance = far & angle = neg_small → power = pos_big; distance = far & angle = neg_big → power = pos_medium)
3. if close to the target, reduce speed so the container gets a little ahead of crane head (distance = medium & angle = neg_small → power = neg_medium)
4. when container is very close to target, power up the motor (distance = close & angle = pos_small → power = pos_medium)
5. when container is over the target and the sway is zero, stop the motor (distance = 0 & angle = 0 → power = 0)

linguistic variables and terms



note the overlapping **terms** of variable **distance**!

(12 yards distance is medium to degree 0.9, far to degree 0.1, close to degree 0)

fuzzy (min/max) inference

suppose:

μ_{far}(12 yards) = 0.1, μ_{medium}(12 yards) = 0.9, μ_{zero}(+4°) = 0.2, μ_{pos_small}(+4°) = 0.8

rule 1: distance = medium & angle = pos_small → power = pos_medium

rule 2: distance = medium & angle = zero → power = zero

rule 3: distance = far & angle = zero → power = pos_medium

aggregate if-parts:

rule 1: min(0.9, 0.8) = 0.8, rule 2: min(0.9, 0.2) = 0.2, rule 3: min(0.1, 0.2) = 0.1.

and **compose** then-parts: Validity of action depends on adequacy of if-part to current situation, i.e.

rule 1 results in power = pos_medium to degree 0.8,

rule 2 results in power = zero to degree 0.2,

rule 3 results in power = pos_medium to degree 0.1.

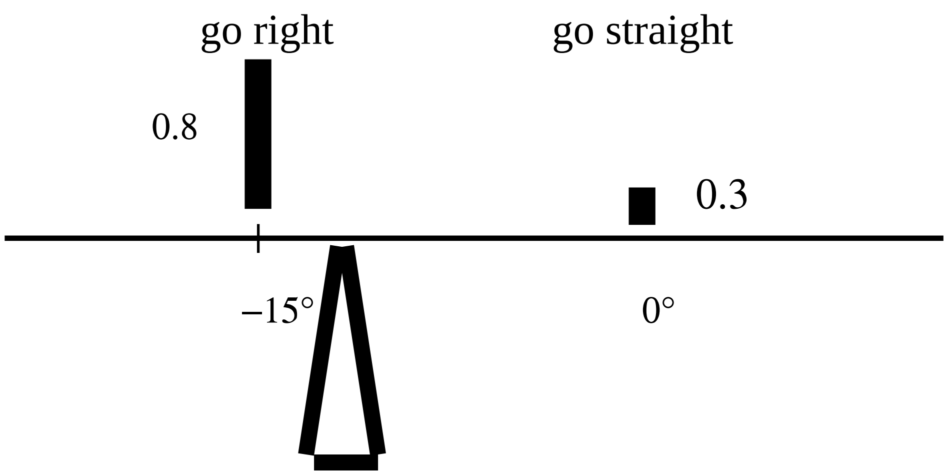
Thus, power = pos_medium to degree 0.8 = max(0.8, 0.1) and zero to degree 0.2.… now **defuzzify** for crisp motor power**...**

defuzzification methods

- defuzzification produces a crisp output value

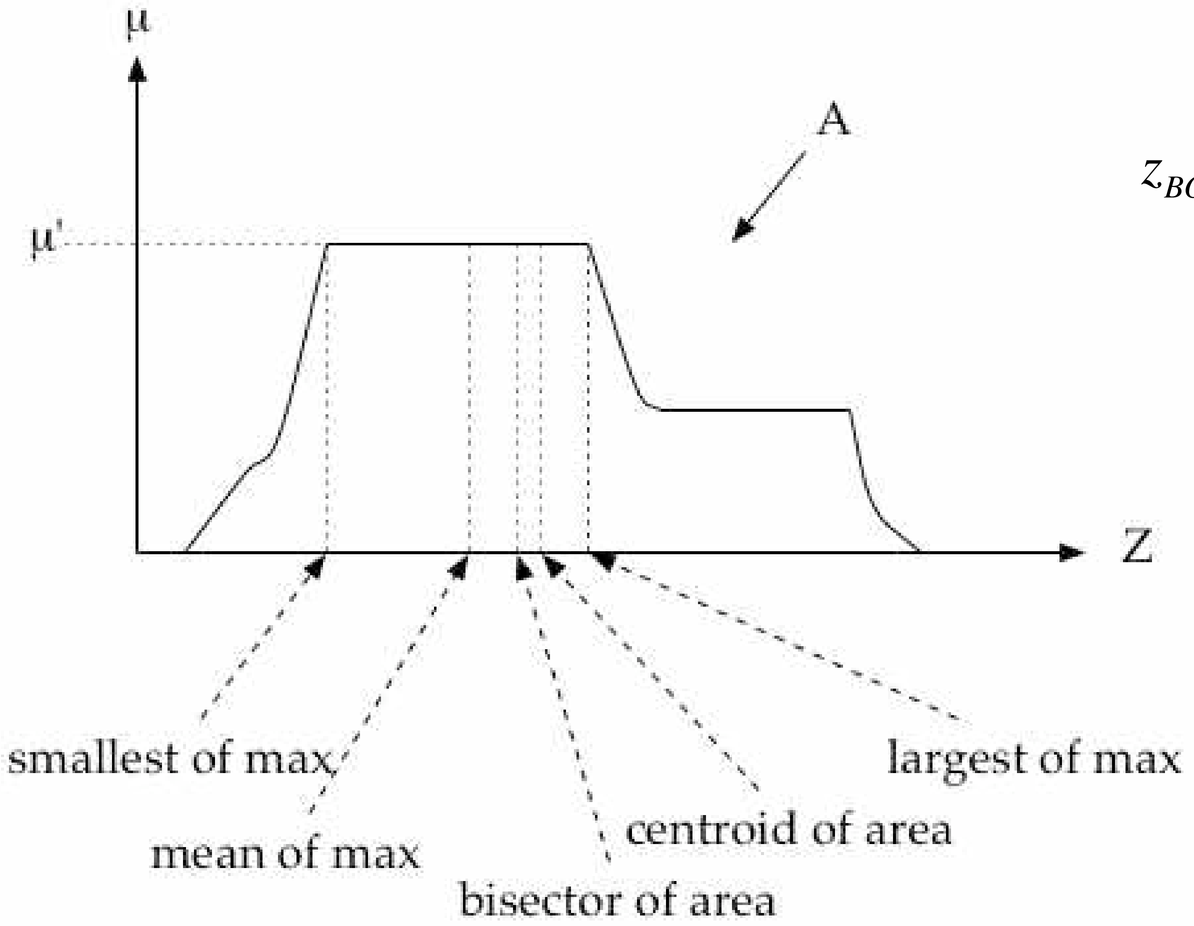
  '**best compromise**': center of maximum

  - find typical value of each term as max of its μ
  - balance out results of fuzzy inference at the locations of the typical values.



'**most plausible result**': typical value of most valid term

4 defuzzification methods



$$z_{BOA} = \int_{\min z}^{z_{BOA}} \mu_A(z)\partial z = \int_{z_{BOA}}^{\max z} \mu_A(z)\partial z$$

$$z_{COA} = \frac{\int_Z \mu_A(z) z \partial z}{\int_Z \mu_A(z)\partial z}$$

$$z_{MOM} = \frac{\int_{Z'} z \partial z}{\int_{Z'} \partial z}$$

where $Z' = \{z | \mu_A(z) = \mu'\}$

defuzzification

distance

deviation	far	near	goal	too far
left	fw	null	bw	bw
middle	fw	fw	null	bw
right	fw	fw	fw	bw

red entries deal with swinging: if deviation is left, we brake.

e.g. crane is 3m away, swings left 24 degr: near 0.6, goal 0.4, left 0.8, middle 0.2 ….

which rule should we use? Use **all** rules, weighted by their μ!!!

dist = near & dev = left ➞ motor = null: min(.6, .8) = .6,

i.e. motor = null to degree .6….

defuzzification, cont.

	far	near	goal	too far
left		null .6	bw .4	
middle		fw .2	null .2	
right				

4 **conflicting** fuzzy outputs!

center of gravity method combines everything, compromises. But compromise may be bad: left .9, right .8, so always go straight!!!

sometimes better: **Mean-of-Maximum method**

such outputs are computed several 1000 times/sec!

testing the rules

to prevent large swings, we might start the crane more softly:

1. **introduce finer fuzzy terms**:

forward slow, forward fast, forward very fast, etc.

2. **introduce conflicting rules**!

e.g. given Rule R: dist = far & dev = middle → motor = forward,

add R': dist = far & dev = middle → motor = null.

R conflicts with R'; result: 1/2 speed forward!!!

finally: GAs can be used to **learn** fuzzy rules...