first order logic

- vocabulary
  - sentence letters: $p, q, \ldots, p_1, q_1, \ldots$ (0-place predicates)
  - individual vbls: $x, y, \ldots, x_1, y_1, \ldots$
  - individual constants: $a, b, \ldots, a_1, b_1, \ldots$ (0-place functions)
  - predicate constants: $F^1, G^1, \ldots, H^2, \ldots, F_2^1, \ldots.$
  - connectives: $\neg\ \wedge\ \vee\ \rightarrow\ \Leftrightarrow$ (truth functions)
  - quantifiers: $\forall\ \exists$
- wffs, sentences

1. all sentence letters are sentences (wffs)
2. if ϕ, ψ are sentences, then so are $\neg\phi$, $(\phi \vee \psi)$, $(\phi \wedge \psi)$, $(\phi \rightarrow \psi)$, $(\phi \Leftrightarrow \psi)$
3. if ϕ is a fmla (wff) , α a vbl, then $\forall\alpha\,\phi$ and $\exists\alpha\,\phi$ are fmlas
4. the result of writing an n-place predicate letter followed by n individual terms is a fmla

the heart of logic

$\vdash \;\Leftrightarrow\; \Vdash$

$P_1, \ldots, P_n \vdash Q$ vs. $P_1, \ldots, P_n \Vdash Q$

syntax (modus ponens etc.) vs. **semantics** (T, F, valid, satisfiable, cons)

an argument is **valid** iff, whenever all premisses are true, the conclusion is bound to be true: transmission of truth, retransmission of falsehood.

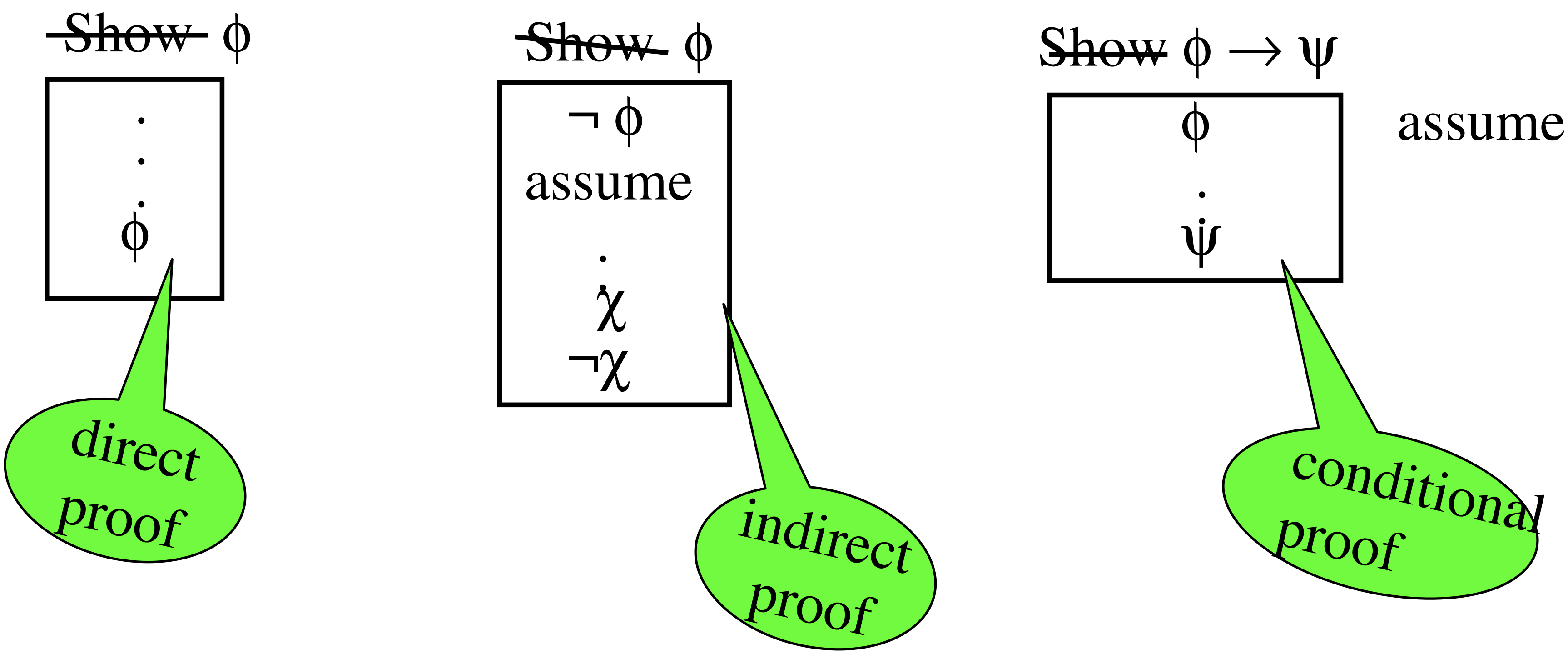
derivable ====> valid	consistent
valid ====> derivable	complete

$\vdash \;\Leftrightarrow\; \Vdash$

$P_1, \ldots, P_n \Vdash\ ?$ what do I believe?

$P_1, \ldots, P_n \Vdash Q, \neg Q$ which of my beliefs must I give up?

natural deduction



instead of resolution, use **natural** inference rules …

some inference rules

$\phi \wedge \psi \therefore \phi$	$\phi \wedge \psi \therefore \psi$	simplification
$\phi, \psi \therefore \phi \wedge \psi$		conjunction
$\phi \therefore \phi \vee \psi$		adjunction
$\phi, \phi \rightarrow \psi \therefore \psi$		modus ponens
$\neg\psi, \phi \rightarrow \psi \therefore \neg\phi$		modus tollens
$\neg\neg\phi \therefore \phi$		double negation
$\phi \vee \psi, \phi \rightarrow \zeta, \psi \rightarrow \chi \therefore \zeta \vee \chi$		constructive dilemma

also commutativity of $\vee$, $\wedge$; deMorgan's etc.

super rule: any tautology can be entered in any line of a proof.

example

S only if P

$S \rightarrow P$

Not Q given that U

$U \rightarrow \neg Q$

O only if R

$O \rightarrow R$

U or O, given that not S

$\neg S \rightarrow (U \vee O)$

$\therefore$ Not P only if either not Q or R

$\neg P \rightarrow (\neg Q \vee R)$

~~Show~~ $\neg P \rightarrow (\neg Q \vee R)$

$\neg P$	assume
$\neg S$	1.,MT
$U \vee O$	4., MP
$\neg Q \vee R$	2,3., dil

~~Show~~ $\neg P \rightarrow (\neg Q \vee R)$

$\neg P$ assume

~~Show~~ $\neg Q \vee R$

~~Show~~ $Q \rightarrow R$

Q assume

$\neg S$

$U \vee O$, $\neg U$, O, R

examples, cont.

Either it is not the case that Alfred pays attention (P) and does not lose track of the argument (T), or it is not the case that he does not take notes (N) and does not do well in the course (W). Alfred neither does well in the course nor loses track of the argument. If Alfred studies logic (S), then he does not do well in the course only if he does not take notes and pays attention.
Therefore, Alfred does not study logic.

$\neg(P \wedge \neg T) \vee \neg(\neg N \wedge \neg W)$

$\neg W \wedge \neg T$

$S \rightarrow (\neg W \rightarrow (\neg N \wedge P))$

$\therefore\ \neg S$

~~Show~~ $\neg S$	
S	ass.
$\neg W \rightarrow (\neg N \wedge P)$	
$\neg W$	simp
$\neg N \wedge P$	
$\neg N$	simp
$\neg N \wedge \neg W$	conj
$\neg(P \wedge \neg T)$	disj syll
$\neg P \vee T$	deMorgan
$\neg T$	simp
$\neg P$	disj syll
$P \wedge \neg P$	

try to prove this direct

predicate logic

- many arguments can only be analyzed by considering the internal structure of sentences
- **all F are G** i.e. only G are F: $\forall x(Fx \rightarrow Gx)$

  $\forall x(Fx \wedge Gx)$ too strong: it implies $\forall x Fx$!
- **some F are G:** $\exists x(Fx \wedge Gx)$

  $\exists x\,(Fx \rightarrow Gx)$ too weak: it is implied by $\exists x Gx$! e.g. $\exists x\,(\text{Cat } x \rightarrow \text{Dog } x)$ is true!
- **no F are G:** $\neg\exists x(Fx \wedge Gx)$ $\forall x(Fx \rightarrow \neg Gx)$
- **some F are not G:** $\exists x(Fx \wedge \neg Gx)$ $\neg\forall x(Fx \rightarrow Gx)$

$\forall x \Leftrightarrow \neg\exists x\neg$ $\neg\forall x \Leftrightarrow \exists x\neg$ $\neg\exists x \Leftrightarrow \forall x\neg$

semantics: interpretations and models

an **interpretation I** consists of

- a set of objects U
- a function mapping constant symbols to elements of U
- a function mapping function symbols to functions on U
- a function mapping predicate symbols to relations on U

suppose U = {■,▲,▼,◆,●,◗,☺,☻,☹,☀,☼,☁,✂}

- I(square) = ■, I(triangle) = ▲, I(face1) = ☺, etc.
- I(x) as yet undefined
- $I(f(t_1, ..., t_n)) = I(f)(I(t_1), ..., I(t_n))$ where f is many-one

- $F(t_1, ..., t_n)$ **holds in I** iff $\langle I(t_1), ..., I(t_n)\rangle \in I(F)$

semantics: interpretations and models

U = {■,▲,▼,◆,●,◗,☺,☻,☹,☀,☼,☁,✂}

I(square) = ■, I(triangle) = ▲, I(face1) = ☺, etc.

I(HasMoreSides) = { ⟨◗, ●⟩ , ⟨◆, ☺⟩ , ⟨▼, ☻⟩ , ⟨■, ▲⟩ ...}

I(IsFriendlier) = { ⟨●,◗⟩ , ⟨☺,◆⟩ , ⟨☻,▼⟩ , ⟨☼,▲⟩ , ⟨☺,☹⟩ ...}

remember: $F(t_1, ..., t_n)$ **holds in I** iff $\langle I(t_1), ..., I(t_n)\rangle \in I(F)$

does HasMoreSides(square, triangle) hold in I?
yes, because: ⟨■, ▲⟩ ∈ I(HasMoreSides)

does IsFriendlier(face1, triangle) hold in I?

$\vdash \;\Leftrightarrow\; \Vdash$

to see if a sentence with variables holds in an interpretation I, we temporarily bind an element $a \in U$ to x:

$$I_{x/a}$$

$\forall x\ \varphi$ holds in I iff φ holds in $I_{x/a}$, for **all** $a \in U$

$\exists x\ \varphi$ holds in I iff φ holds in $I_{x/a}$, for **some** $a \in U$

S is logically true ($\Vdash S$) iff S holds in **all** interpretations I.
A set of sentences S **implies** a sentence Q iff $\Vdash S \rightarrow Q$, i.e. **whenever S is true, Q is bound to be true as well.**

For fol, $\Vdash$ and $\vdash$ (syntactic derivability) coincide:
Gödel's completeness theorem!!

rules for quantifiers

UI: $\forall\alpha\ \phi$
$\therefore\ \phi'$

EG: ϕ'
$\therefore\ \exists\alpha\ \phi$

does not hold in intensional contexts!

EI: $\exists\alpha\ \phi$
$\therefore\ \phi'$
where the instantiating vbl must be new!

block: $\exists$x Fx, $\exists$x Gx, Fa, Ga $\therefore$ $\exists$x (Fx $\wedge$ Gx) !

UG: Show $\forall\alpha\ \phi$
χ_1
•
•
•
χ_m
where ϕ occurs unboxed among χ_i and α is not free in prior line!

pred logic, examples

$\forall x\,(Fx \rightarrow \forall y\,(Gy \rightarrow \neg Hxy))$

$\forall x\,(Fx \rightarrow \exists y\,(Jy \wedge Hxy))$

$\exists x\, Fx$

$\therefore \exists x\,(Jx \wedge \neg Gx)$

~~Show~~ $\exists x\,(Jx \wedge \neg Gx)$

Fa	EI
$\forall y\,(Gy \rightarrow \neg Hay)$	UI, MP
$\exists y\,(Jy \wedge Hay)$	UI, MP
$Ju \wedge Hau$	EI
$Gu \rightarrow \neg Hau$	UI
$\neg Gu$	MT
$Ju \wedge \neg Gu$	conj
$\exists x\,(Jx \wedge \neg Gx)$	EG

pred logic, examples

~~Show~~ $\neg\exists y\, \forall x\, (Fxy \Leftrightarrow \neg Fxx)$	remember Russell's paradox?
$\exists y\, \forall x\, (Fxy \Leftrightarrow \neg Fxx)$	ass.
$\forall x\, (Fxa \Leftrightarrow \neg Fxx)$	EI
$Faa \Leftrightarrow \neg Faa$	UI
$Faa \rightarrow \neg Faa$	taut
$\neg Faa \rightarrow Faa$	taut
$\neg Faa$	taut
Faa	MP

because

~~Show~~ $(p \rightarrow \neg p) \rightarrow \neg p$	
$p \rightarrow \neg p$	ass.
~~Show~~ $\neg p$	
p	ass.
$\neg p$	

pred logic, examples

Is $\exists y\ \forall x\ Fxy \rightarrow \forall x\ \exists y\ Fxy$ valid?

~~Show~~ $\exists y\ \forall x\ Fxy \rightarrow \forall x\ \exists y\ Fxy$	
$\exists y\ \forall x\ Fxy$	ass.
~~Show~~ $\forall x\ \exists y\ Fxy$	**UG!**
$\forall x\ Fxa$	EI
Fxa	UI
$\exists y\ Fxy$	EG

Is $\forall x\ \exists y\ Fxy \rightarrow \exists y\ \forall x\ Fxy$ valid? Can you prove it?

invalidity, consistency, independence of postulates, via interpretation

- if you can't prove something
  - you could be a poor logician (most unlikely!) or
  - it could be a very hard proof or
  - it's not valid after all
- to show non-validity, construct a counterexample
  - argument is invalid iff for <u>some</u> non-empty domain and for <u>some</u> extension assignment all premises are **clearly** true and conclusion is **clearly** false

$\forall x(Fx \rightarrow Gx); \exists x(Hx \wedge \neg Gx) \therefore \exists x(Fx \wedge \neg Hx)$. Valid?

domain: {pos ints}; $Fx \Leftrightarrow x > 10$; $Gx \Leftrightarrow x > 5$; $Hx \Leftrightarrow x > 0$;

$\forall x(x > 10 \rightarrow x > 5)$ (T); $\exists x(x > 0 \wedge \neg(x > 5))$ (T)

$\not\therefore \exists x(x > 10 \wedge \neg(x > 0))$ (F)

translating into fol: charity & don't scratch..

Everybody loves a lover. [Fx: x is a person; Lxy: x loves y]
$\forall x(Fx \rightarrow$ x loves every person who is a lover);
$\forall x(Fx \rightarrow \forall y((Fy \wedge \exists z(Fz \wedge Lyz)) \rightarrow Lxy))$

Somebody can beat everyone on a's team.
[Fx: x is a person; Ax: x is on a's team; Gxy: x can beat y]
$\exists x(Fx \wedge$ x can beat everyone on a's team)
$\exists x(Fx \wedge \forall y(Fy \wedge Ay \rightarrow Gxy))$

There's always a war somewhere. [Gxyz: x goes on at time y and place z; Wx: x is a war; Tx: x is a time; Px: x is a place]
$\forall x(Tx \rightarrow \exists y(Py \wedge \exists z(Wz \wedge Gzxy)))$
There's never a war everywhere.
$\forall x(Tx \rightarrow \exists y(Py \wedge \neg\exists z(Wz \wedge Gzxy)))$

a classical example...

All horses are animals. Therefore, all tails of horses are tails of animals. Valid?
[Hx: x is a horse; Ax: x is an animal; Txy: x is tail of y]

$\forall x(Hx \rightarrow Ax)$
$\therefore \forall y$(y is tail of a horse $\rightarrow$ y is tail of an animal)

~~Show~~ $\forall y(\exists x(Hx \wedge Tyx) \rightarrow \exists x(Ax \wedge Tyx))$

~~Show~~ $\exists x(Hx \wedge Tyx) \rightarrow \exists x(Ax \wedge Tyx)$	UG!
$\exists x(Hx \wedge Tyx)$	ass.
$Ha \wedge Tya$	EI
Aa	UI (prem), MP
$Aa \wedge Tya$	simpl, conj
$\exists x(Ax \wedge Tyx)$	EG

yet another example...

Only grown-ups (Gx) and children accompanied by their parents (Cx) were admitted (Ax).
$\forall x(Ax \rightarrow (Gx \vee Cx))$;
All grown-ups who stayed till the end (Sx) liked the show (Lx).
$\forall x(Gx \wedge Sx \rightarrow Lx)$;
Bob (b) was admitted. Bob didn't like the show. Ab; $\neg Lb$;
Therefore, some grown-ups didn't stay till the end or Bob was (a child) accompanied by his parents. $\exists x(Gx \wedge \neg Sx) \vee Cb$

~~Show~~ $\neg\exists x(Gx \wedge \neg Sx) \rightarrow Cb$

$\neg\exists x(Gx \wedge \neg Sx)$	ass.
$\neg(Gb \wedge \neg Sb)$, i.e. $Gb \rightarrow Sb$	UI, taut
$Gb \wedge Sb \rightarrow Lb$	UI
$\neg(Gb \wedge Sb)$, i.e. $Sb \rightarrow \neg Gb$	MT, taut
$Gb \rightarrow \neg Gb$	hyp syll
$\neg Gb$	taut
$Ab \rightarrow (Gb \vee Cb)$; $Gb \vee Cb$; Cb	UI, MP, disj syll

another example…

No one who is either an F or a G is an H.

$\forall x(Fx \vee Gx \rightarrow \neg Hx)$

Everyone is such that if he is a G only if he is not a J, then he is an F and an H.

$\forall x((Gx \rightarrow \neg Jx) \rightarrow Fx \wedge Hx)$

Therefore, everyone is a J, i.e. $\forall xJx$

~~Show~~ $\forall xJx$

$Fx \vee Gx \rightarrow \neg Hx$

$Fx \rightarrow \neg Hx$ because $p \vee q \rightarrow r$ implies $p \rightarrow r$

$\neg(Fx \wedge Hx)$

$\neg(Gx \rightarrow \neg Jx)$

$Gx \wedge Jx$

Jx