ch5_solvedProblems

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P5.3 Which of the following sets of vectors are independent? Find the dimension of the vector space spanned by each set.

i. $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}$

Answer. Let

$$\mathbf{0} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3, \tag{1}$$

where the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ correspond with the above. For the set $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ to be linearly independent, $c_1 = c_2 = c_3 = 0$ must be the only solution satisfying (1).

Set up a matrix $[\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]$ via the three provided vectors, and augment with the zero vector $\mathbf{0}$ to create $[\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3\ \mathbf{0}]$, and row reduce. I.e.,

$$\begin{bmatrix}1 & 1 & 1 & 0\\1 & 0 & 2 & 0\\1 & 1 & 1 & 0\end{bmatrix} \sim \cdots \sim \begin{bmatrix}1 & 0 & 2 & 0\\0 & 1 & -1 & 0\\0 & 0 & 0 & 0\end{bmatrix}. \tag{2}$$

As the resulting matrix lacks a pivot in the third row, the system contains a free variable, meaning that a solution other than the trivial exists. Thus, the system is linearly dependent.

To demonstrate this explicitly, write out the solution suggested by the row-reduced matrix, using the free-variable row as a parameter.

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}-2c_3\\c_3\\c_3\end{bmatrix} = \begin{bmatrix}-2\\1\\1\end{bmatrix} c_3. \tag{3}$$

Let $c_3 = 1$, implying that $c_1 = -2$ and $c_2 = 1$. So then, $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = -2\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 = \mathbf{0}$, a non-trivial solution. So, again, the set of vectors is linearly dependent.

ii. $\sin t, \cos t, 2\cos\left(t + \frac{\pi}{4}\right)$

iii. $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\\1\end{bmatrix}$

Answer. Proceed similarly to part **i**. Let

$$\mathbf{0} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3, \tag{4}$$

where the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ correspond with the above. For the set $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ to be linearly independent, $c_1 = c_2 = c_3 = 0$ must be the only solution satisfying (4).

Set up a matrix $[\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]$ via the three provided vectors, and augment with the zero vector $\mathbf{0}$ to create $[\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3\ \mathbf{0}]$, and row reduce. I.e.,

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

As the resulting matrix lacks a pivot in the third row, the system contains a free variable, meaning that a solution other than the trivial exists. Thus, the system is linearly dependent.

To demonstrate this explicitly, write out the solution suggested by the row-reduced matrix, using the free-variable row as a parameter.

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2c_3 \\ c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} c_3. \tag{6}$$

Let $c_3 = 1$, implying that $c_1 = -2$ and $c_2 = 1$. So then, $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = -2\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 = \mathbf{0}$, a non-trivial solution. So, again, the set of vectors is linearly dependent.