

# Dependent pattern matching and proof-relevant unification

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# Simple pattern matching

```
data N : Type where  
  zero : N  
  suc   : N → N
```

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**data**  $\mathbb{N}$  : **Type** **where**

**zero** :  $\mathbb{N}$

**suc** :  $\mathbb{N} \rightarrow \mathbb{N}$

**minimum** :  $\mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$

**minimum**  $x$   $y$  = { }

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# Why pattern matching?

- Intuitive way to write definitions by case analysis and recursion / induction.
- Computational meaning is obvious.
- Automates boring equational reasoning.

# Dependent pattern matching

```
data Vec (A : Type) : ℕ → Type where  
  nil   : Vec A zero  
  cons  : (n : ℕ) → A → Vec A n → Vec A (suc n)
```

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tail k xs = { }
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tail : (k : ℕ) → Vec A (suc k) → Vec A k
tail k nil = { } -- suc k = zero
tail k (cons n x xs) = { } -- suc k = suc n
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tail : (k : ℕ) → Vec A (suc k) → Vec A k
tail k nil = { } -- impossible
tail k (cons n x xs) = { } -- suc k = suc n
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tail k (cons n x xs) = { } --      k =      n
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# Matching on the identity type

**data**  $_ \equiv _ (x : A) : A \rightarrow \text{Type}$  **where**

**refl** :  $x \equiv x$

**trans** :  $(x\ y\ z : A) \rightarrow (x \equiv y) \rightarrow (y \equiv z) \rightarrow (x \equiv z)$

**trans**  $x\ y\ z\ p\ q = \{ \}$

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```

```
trans : (x y z : A)  $\rightarrow$  (x  $\equiv$  y)  $\rightarrow$  (y  $\equiv$  z)  $\rightarrow$  (x  $\equiv$  z)
```

```
trans x .x .x refl refl = refl
```

# Pattern matching implies uniqueness of identity proofs

$$\begin{aligned} \text{UIP} &: (x : A)(e : x \equiv_A x) \rightarrow e \equiv_{x \equiv_A x} \text{refl} \\ \text{UIP } x \ e &= \{ \} \end{aligned}$$



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- Univalence gives  $\text{swap} : \text{Bool} \equiv \text{Bool}$   
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such that  $\text{coerce swap true} = \text{false}$

hence  $\text{true} = \text{false}$ !

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How to avoid proofs of UIP in general?

**Answer:** by disabling the deletion rule of the unification algorithm.

This calls for a deeper investigation of unification in dependent type theory!

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# Specialization by unification

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The output of unification can change our notion of equality!

**Main question:** How to make sure the output of unification is correct?

# Proof-relevant unification

We want a unification algorithm that

- ... takes types of equations into account.
- ... represents input and output internally.
- ... produces *evidence* that its output is correct.

# Unification problems as telescopes

A **unification problem** consists of

1. Flexible variables  $x_1 : A_1, x_2 : A_2, \dots$
2. Equations  $u_1 = v_1 : B_1, \dots$

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This can be represented as a **telescope**:

$$(x_1 : A_1)(x_2 : A_2) \dots$$

$$(e_1 : u_1 \equiv_{B_1} v_1)(e_2 : u_2 \equiv_{B_2} v_2) \dots$$

e.g.  $(k : \mathbb{N})(n : \mathbb{N})(e : \text{succ } k \equiv_{\mathbb{N}} \text{succ } n)$



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A **unification problem** consists of

1. Flexible variables  $\Gamma$
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$$\Gamma$$
$$(e_1 : u_1 \equiv_{B_1} v_1)(e_2 : u_2 \equiv_{B_2} v_2) \dots$$

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# Unification problems as telescopes

A **unification problem** consists of

1. Flexible variables  $\Gamma$
2. Equations  $\bar{u} = \bar{v} : \Delta$

This can be represented as a **telescope**:

$$\Gamma(\bar{e} : \bar{u} \equiv_{\Delta} \bar{v})$$

e.g.  $(k : \mathbb{N})(n : \mathbb{N})(e : \text{succ } k \equiv_{\mathbb{N}} \text{succ } n)$

# Unifiers as telescope maps

A *unifier* of  $\bar{u}$  and  $\bar{v}$  is a substitution  $\sigma : \Gamma' \rightarrow \Gamma$  such that  $\bar{u}\sigma = \bar{v}\sigma$ .

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A *unifier* of  $\bar{u}$  and  $\bar{v}$  is a substitution  $\sigma : \Gamma' \rightarrow \Gamma$  such that  $\bar{u}\sigma = \bar{v}\sigma$ .

This can be represented as a *telescope map*:

$$f : \Gamma' \rightarrow \Gamma(\bar{e} : \bar{u} \equiv_{\Delta} \bar{v})$$

e.g.  $f : () \rightarrow (n : \mathbb{N})(e : n \equiv_{\mathbb{N}} \text{zero})$

# Evidence of unification

A map  $f : () \rightarrow (n : \mathbb{N})(e : n \equiv_{\mathbb{N}} \text{zero})$   
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gives us two things:

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2. Explicit **evidence**  $e$  of  $n \equiv_{\mathbb{N}} \text{zero}$

$\implies$  Unification is guaranteed to respect  $\equiv$ !



# Three valid unifiers

$f_1 : (k : \mathbb{N}) \rightarrow (k\ n : \mathbb{N})(e : k \equiv_{\mathbb{N}} n)$

$f_1\ k = k; k; \text{refl}$

$f_2 : () \rightarrow (k\ n : \mathbb{N})(e : k \equiv_{\mathbb{N}} n)$

$f_2\ () = \text{zero}; \text{zero}; \text{refl}$

$f_3 : (k\ n : \mathbb{N}) \rightarrow (k\ n : \mathbb{N})(e : k \equiv_{\mathbb{N}} n)$

$f_3\ k\ n = k; k; \text{refl}$

# What is a most general unifier?

A *most general unifier* of  $\bar{u}$  and  $\bar{v}$  is a unifier  $\sigma$  such that for any  $\sigma'$  with  $\bar{u}\sigma' = \bar{v}\sigma'$ , there is a  $\nu$  such that  $\sigma' = \sigma \circ \nu$ .

# What is a most general unifier?

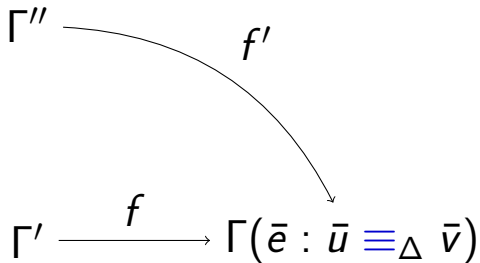
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To avoid ‘ghost variables’,  $\nu$  should be unique.

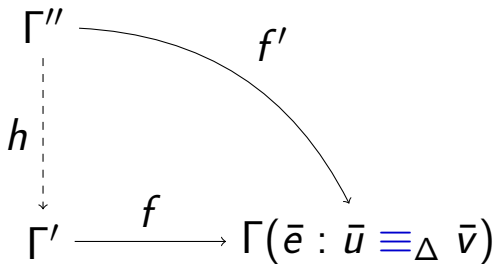
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$$\Gamma' \xrightarrow{f} \Gamma(\bar{e} : \bar{u} \equiv_{\Delta} \bar{v})$$

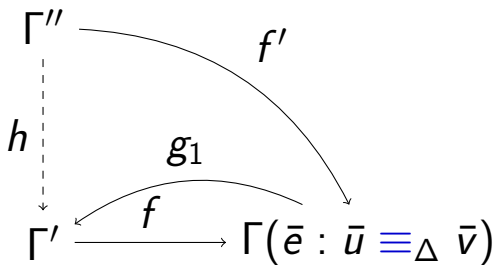
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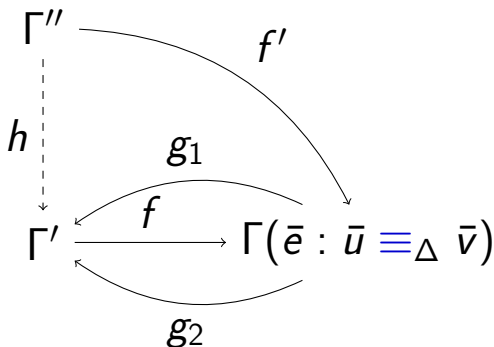


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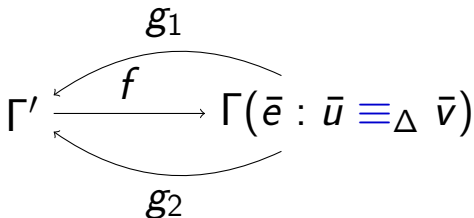


$(\forall f' : h \text{ exists}) \iff f \text{ has a right inverse } g_1$

$(\forall f' : h \text{ is unique}) \iff f \text{ has a left inverse } g_2$



# What is a most general unifier?



$f$  has a *right inverse*  $g_1$

$f$  has a *left inverse*  $g_2$

Most general unifiers  
are equivalences!

$$f : \Gamma(\bar{e} : \bar{u} \equiv_{\Delta} \bar{v}) \simeq \Gamma'$$

# Example of unification

$$(k\ n : \mathbb{N})(e : \text{succ } k \equiv_{\mathbb{N}} \text{succ } n)$$

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$$\begin{array}{c} (k \ n : \mathbb{N})(e : \text{succ } k \equiv_{\mathbb{N}} \text{succ } n) \\ \Downarrow \\ (k \ n : \mathbb{N})(e : k \equiv_{\mathbb{N}} n) \end{array}$$

# Example of unification

$$(k \ n : \mathbb{N})(e : \text{red suc } k \equiv_{\mathbb{N}} \text{red suc } n)$$

$$\Downarrow$$

$$(k \ \text{red } n : \mathbb{N})(\text{red } e : k \equiv_{\mathbb{N}} n)$$

$$\Downarrow$$

$$(k : \mathbb{N})$$

# Example of unification

$$\begin{aligned} & (k \ n : \mathbb{N})(e : \text{succ } k \equiv_{\mathbb{N}} \text{succ } n) \\ & \quad \Downarrow \\ & (k \ n : \mathbb{N})(e : k \equiv_{\mathbb{N}} n) \\ & \quad \Downarrow \\ & (k : \mathbb{N}) \end{aligned}$$

$$\begin{aligned} f & : (k : \mathbb{N}) \rightarrow (k \ n : \mathbb{N})(e : \text{succ } k \equiv_{\mathbb{N}} \text{succ } n) \\ f \ k & = k; k; \text{refl} \end{aligned}$$

# The solution rule

`solution` :  $(x : A)(e : x \equiv_A t) \simeq ()$

# The deletion rule

$$\text{deletion} : (e : t \equiv_A t) \simeq ()$$



# The injectivity rule

$$\text{injectivity}_{\text{suc}} : \\ (e : \text{suc } x \equiv_{\mathbb{N}} \text{suc } y) \simeq (e' : x \equiv_{\mathbb{N}} y)$$

# Negative unification rules

A *negative unification rule* applies to impossible equations, e.g. **suc**  $x$  = **zero**.

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A *negative unification rule* applies to impossible equations, e.g. **suc**  $x = \mathbf{zero}$ .

This can be represented by an equivalence:

$$(e : \mathbf{suc} \ x \equiv_{\mathbb{N}} \mathbf{zero}) \simeq \perp$$

where  $\perp$  is the **empty type**.

# The conflict rule

$$\text{conflict}_{\text{suc}, \text{zero}} : \\ (e : \text{suc } x \equiv_{\mathbb{N}} \text{zero}) \simeq \perp$$

# The cycle rule

$$\text{cycle}_{n, \text{suc } n} : (e : n \equiv_{\mathbb{N}} \text{suc } n) \simeq \perp$$

# Heterogeneous equations

Problem: What is the type of  $e_2$ ?

$$(e : (0, \text{nil}) \equiv_{\Sigma_{n:\mathbb{N}} \text{Vec } A \ n} (1, \text{cons } 0 \ x \ xs))$$

$\Downarrow$

$$(e_1 : 0 \equiv_{\mathbb{N}} 1)(e_2 : \text{nil} \equiv_{\text{Vec } A} ??? \text{cons } 0 \ x \ xs)$$

# Heterogeneous equations

**Problem:** What is the type of  $e_2$ ?

**Solution:** keep track of dependencies by introducing a new variable for each equation

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This is called a *telescopic equality*.



# Be careful with heterogeneous equations!

$$(e : (\text{Bool}, \text{true}) \equiv_{\Sigma_{A:\text{Type}} A} (\text{Bool}, \text{false}))$$

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$\perp$

The `conflict` rule does not apply!

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$\Downarrow$   
 $\perp$

Whether a unification rule can be applied  
depends on the **type** of the equation!

# Injectivity for indexed data

**Idea:** simplify equations between indices together with equation between constructors:

$$\begin{aligned} & (e_1 : \text{suc } k \equiv_{\mathbb{N}} \text{suc } n) \\ & (e_2 : \text{cons } k \ x \ xs \equiv_{\text{vec } A \ e_1} \text{cons } n \ y \ ys) \end{aligned}$$



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Length of the **Vec** must be *fully general*:  
must be an equation variable.

# Solving unsolvable equations

```
data Im ( $f : A \rightarrow B$ ) :  $B \rightarrow$  Type where  
  image : ( $x : A$ )  $\rightarrow$  Im  $f$  ( $f$   $x$ )
```

# Solving unsolvable equations

**data** `Im`  $(f : A \rightarrow B) : B \rightarrow \text{Type}$  **where**  
`image`  $: (x : A) \rightarrow \text{Im } f (f x)$

$(x_1 \ x_2 : A)(e_1 : f \ x_1 \equiv_B f \ x_2)$   
 $(e_2 : \text{image } x_1 \equiv_{\text{Im } f} \text{image } x_2)$

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$$\begin{array}{c} (x_1 \ x_2 : A)(e_1 : f \ x_1 \equiv_B f \ x_2) \\ (e_2 : \text{image } x_1 \equiv_{\text{Im } f} e_1 \ \text{image } x_2) \\ \Downarrow \\ (x_1 \ x_2 : A)(e : x_1 \equiv_A x_2) \end{array}$$

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**data**  $\text{Im}$   $(f : A \rightarrow B) : B \rightarrow \text{Type}$  **where**  
 $\text{image} : (x : A) \rightarrow \text{Im } f (f x)$

$$\begin{aligned} & (x_1 \ x_2 : A)(e_1 : f \ x_1 \equiv_B f \ x_2) \\ & (e_2 : \text{image } x_1 \equiv_{\text{Im } f} \text{image } x_2) \\ & \quad \Downarrow \\ & (x_1 \ x_2 : A)(e : x_1 \equiv_A x_2) \\ & \quad \Downarrow \\ & (x_1 : A) \end{aligned}$$

Dependent pattern matching

Proof-relevant unification

**Higher-dimensional unification**

Back to eliminators

What if the indices are not  
fully general?

$(e : \text{cons } n \ x \ xs \equiv_{\text{Vec } A (\text{suc } n)} \text{cons } n \ y \ ys)$



# What if the indices are not fully general?

$$\begin{aligned} & (e : \text{cons } n \ x \ xs \equiv_{\text{Vec } A} (\text{suc } n) \ \text{cons } n \ y \ ys) \\ & \quad \Downarrow \\ & \quad (e_1 : \text{suc } n \equiv_{\mathbb{N}} \text{suc } n) \\ & \quad (e_2 : \text{cons } n \ x \ xs \equiv_{\text{Vec } A} e_1 \ \text{cons } n \ y \ ys) \\ & \quad (p : e_1 \equiv_{\text{suc } n \equiv_{\mathbb{N}} \text{suc } n} \text{refl}) \end{aligned}$$

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$$(e_1 : \text{suc } n \equiv_{\mathbb{N}} \text{suc } n)$$

$$(e_2 : \text{cons } n \ x \ xs \equiv_{\text{Vec } A} e_1 \ \text{cons } n \ y \ ys)$$

$$(p : e_1 \equiv_{\text{suc } n \equiv_{\mathbb{N}} \text{suc } n} \text{refl})$$

$\Downarrow$

$$(e'_1 : n \equiv_{\mathbb{N}} n)(e'_2 : x \equiv_A y)(e'_3 : xs \equiv_{\text{Vec } A} e'_1 \ ys)$$

$$(p : \text{cong suc } e'_1 \equiv_{\text{suc } n \equiv_{\mathbb{N}} \text{suc } n} \text{refl})$$

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$$(e'_1 : n \equiv_{\mathbb{N}} n)(e'_2 : x \equiv_A y)(e'_3 : xs \equiv_{\text{Vec } A} e'_1 \ ys)$$

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# Higher-dimensional equations

$$(e'_1 : n \equiv_{\mathbb{N}} n)(e'_2 : x \equiv_A y)(e'_3 : xs \equiv_{\text{Vec } A \ e'_1} ys) \\ (p : \text{cong suc } e'_1 \equiv_{\text{suc } n \equiv_{\mathbb{N}} \text{suc } n} \text{refl})$$

We call an equation between equality proofs (e.g.  $p$ ) a **higher-dimensional equation**.

# How to solve higher-dimensional equations?

Existing unification rules do not apply...

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Existing unification rules do not apply...

We solve the problem in three steps:

1. lower the dimension of equations
2. solve lower-dimensional equations
3. lift unifier to higher dimension

# Step 1: lower the dimension of equations

We replace all equation variables  
by regular variables: instead of

$$(e_1 : n \equiv_{\mathbb{N}} n)(e_2 : x \equiv_A y)(e_3 : xs \equiv_{\text{Vec } A} e_1 \ ys) \\ (p : \text{cong suc } e_1 \equiv_{\text{suc } n \equiv_{\mathbb{N}} \text{suc } n} \text{refl})$$

let's first consider

$$(k : \mathbb{N})(u : A)(us : \text{Vec } A \ k) \\ (e : \text{suc } k \equiv_{\mathbb{N}} \text{suc } n)$$

## Step 2: solve lower-dimensional equations

This gives us an equivalence  $f$  of type

$$\begin{array}{c} (k : \mathbb{N})(u : A)(us : \text{Vec } A \ k) \\ (e : \text{succ } k \equiv_{\mathbb{N}} \text{succ } n) \\ \Downarrow \\ (u : A)(us : \text{Vec } A \ n) \end{array}$$



## Step 3: lift unifier to higher dimension

We lift  $f$  to an equivalence  $f^\uparrow$  of type

$$\begin{array}{c} (e_1 : n \equiv_{\mathbb{N}} n)(e_2 : x \equiv_A y) \\ (e_3 : xs \equiv_{\text{Vec } A \ e_1} ys) \\ (p : \text{cong suc } e_1 \equiv_{\text{suc } n \equiv_{\mathbb{N}} \text{suc } n} \text{refl}) \\ \Downarrow \\ (e_2 : x \equiv_A y)(e_3 : xs \equiv_{\text{Vec } A \ n} ys) \end{array}$$

# Final result of steps 1-3

$$\begin{array}{c} (e : \text{cons } n \ x \ xs \equiv_{\text{Vec } A} (\text{suc } n) \ \text{cons } n \ y \ ys) \\ \Downarrow \\ (e_2 : x \equiv_A y)(e_3 : xs \equiv_{\text{Vec } A \ n} ys) \end{array}$$

# Final result of steps 1-3

$$\begin{array}{c} (e : \text{cons } n \ x \ xs \equiv_{\text{Vec } A} (\text{suc } n) \ \text{cons } n \ y \ ys) \\ \Downarrow \\ (e_2 : x \equiv_A y)(e_3 : xs \equiv_{\text{Vec } A \ n} ys) \end{array}$$

This is the **forcing rule** for **cons**.

# Lifting equivalences: (mostly) general case

**Theorem.** If we have an equivalence  $f$  of type

$$(x : A)(e : b_1 x \equiv_{B x} b_2 x) \simeq C$$

we can construct  $f^\uparrow$  of type

$$\begin{aligned} (e : u \equiv_A v)(p : \text{cong } b_1 e \equiv_{r \equiv_{B e} s} \text{cong } b_2 e) \\ \quad \quad \quad \downarrow \\ (e' : f u r \equiv_C f v s) \end{aligned}$$

Dependent pattern matching

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# Desugaring pattern matching

1. Translate the definition to a case tree.

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4. Fill in recursive calls, using well-founded induction.

# From pattern matching ...

```
data _ ≤ _ : ℕ → ℕ → Type where  
  lz : (l : ℕ) → zero ≤ l  
  ls : (k l : ℕ) → k ≤ l → suc k ≤ suc l  
  
antisym : (x y : ℕ) → x ≤ y → y ≤ x → x ≡ y  
antisym .zero .zero (lz [zero]) (lz [zero]) = refl  
antisym .(suc k) .(suc l) (ls k l u) (ls .l .k v)  
    = cong suc (antisym k l u v)
```

... to a case tree ...

$$x \ y \ \underline{u} \ v \left\{ \begin{array}{l} \text{.zero } y \ (\text{lz } .y) \ \underline{u} \\ \quad \{ \text{.zero .zero } (\text{lz .zero}) (\text{lz .zero}) \mapsto \text{refl} \\ \text{.}(\text{suc } k) \text{.}(\text{suc } l) (\text{ls } k \ l \ u) \ \underline{v} \\ \quad \{ \text{.}(\text{suc } k) \text{.}(\text{suc } l) (\text{ls } k \ l \ u) (\text{ls } .l \ .k \ v) \\ \quad \quad \mapsto \text{cong suc } (\text{antisym } k \ l \ u \ v) \end{array} \right.$$

## ...to eliminators.

```
antisym : (x y : ℕ) → x ≤ y → y ≤ x → x ≡ℕ y
antisym = elim≤ (λx y u. y ≤ x → x ≡ℕ y)
  (λ/ v. elim≤ (λy x v. x ≡ℕ zero → x ≡ℕ y)
    (λx e. e)
    (λ/ k _ e. elim⊥ (suc k ≡ℕ suc l) (noConfℕ (suc k) zero e))
    / zero v refl)
(λk l _ H v. cong suc (H
  (elim≤ (λx y _ . x ≡ℕ suc l → u ≡ℕ suc k → l ≤ k)
    (λ/ e _ . elim⊥ (l ≤ k) (noConfℕ zero (suc l) e))
    (λk' l' v' _ e1 e2. subst (λn. n ≤ k)
      (noConfℕ (suc k') (suc l) e1)
      (subst (λm. k' ≤ m) (noConfℕ (suc l') (suc k) e2) v'))
    (suc l) (suc k) v refl refl))))
```