What is. . . type theory?

An introduction for mathematicians

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What is a type system?

A type system is a set of rules that assign a property called type to various constructs a computer program consists of.

The main purpose of a type system is to reduce possibilities for bugs in computer programs.

(paraphrased from Wikipedia)

Type theory is…

Surprisingly theoretical:

- deep links with category theory and homotopy theory
- alternative foundation of mathematics

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- deep links with category theory and homotopy theory
- alternative foundation of mathematics

Surprisingly practical:

- new programming languages
- proof assistants based on type theory

Goals of this talk

Introduce you to the world of type systems

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Show some beautiful ideas from type theory

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Introduce you to the world of type systems

Show some beautiful ideas from type theory

Explain what types can do for you as a mathematician

Type systems

Dependent types

Homotopy type theory

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Types in programming languages

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Goal: give meaning to computer programs without having to run them.

- Data types in C: `int x = 5;`
- Classes in C++ and Java:
  `class Triangle extends Shape`
- Polymorphic function types in Haskell:
  $\texttt{zip} : [a] \to [b] \to [(a, b)]$

Static versus dynamic typing

Statically typed languages guarantee no type errors will occur while running a program.

e.g. C, Java, Haskell, ...

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Statically typed languages guarantee no type errors will occur while running a program.

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Dynamically typed languages don't give any such guarantees.

e.g. JavaScript, Python, MATLAB, ...

Syntax of type theory

$$a : T$$

"The term a has type T."

Simply typed lambda calculus

Function types: $A \to B$

Alonzo
Church

Simply typed lambda calculus

Function types: $A \rightarrow B$

Function application:

$$\frac{f : A \rightarrow B \qquad x : A}{f\ x : B}$$

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Lambda abstraction:

$$\frac{x : A \vdash t : B}{\lambda x.\, t : A \to B}$$

Alonzo Church

The Curry-Howard correspondence

Haskell B. Curry

If we read a type $A \to B$ as an implication $A \Rightarrow B$, any given function corresponds to a proof of the implication and vice versa.

The Curry-Howard correspondence

Logic		Type system
proposition	A	type
proof	$a : A$	program
implication	$A \to B$	function type
conjunction	$A \times B$	pair type
disjunction	$A \uplus B$	sum type
true	$\top$	unit type
false	$\bot$	empty type

Constructive logic

Type theory is a *constructive* logic.

We can *run* a proof of "A or B" to get either a proof of A or a proof of B.

Luitzen E. J. Brouwer

Constructive logic

Type theory is a *constructive* logic.

We can *run* a proof of "A or B" to get either a proof of A or a proof of B.

In particular, the law of the excluded middle doesn't hold!

Luitzen E. J. Brouwer

The cake question

"Do you want the cake or can I have it?"

Classical mathematician: *"Yes!"*

Curry-Howard beyond propositional logic

Ideas from logic can be used in type systems, and vice versa:

- Modal logic / monads (Haskell)
- Linear logic / linear types (Rust)
- Classical logic / CPS (Scheme)
- Predicate logic / dependent types

Type systems

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Homotopy type theory

Dependent types

A dependent type $P\ x$ is a family of types indexed over terms $x : A$ of a base type.

E.g. `Vec` $A\ n$ is the type of vectors of length n.

Per Martin-Löf

Martin-Löf type theory

Logic		Type system
Universal quantification	$(x : A) \to B\ x$	Dependent function type
Existential quantification	$(x : A) \times B\ x$	Dependent pair type
Equality	$x \equiv_A y$	Identity type

Example: $(x : \mathbb{N}) \to (y : \mathbb{N}) \times (2 * x \equiv_{\mathbb{N}} y)$

Proof assistants

Thierry Coquand

Type theory provides a full alternative to both logic and set theory.

We can build proof assistants based on type theory! e.g. Coq, Agda, NuPRL

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- Check that your proofs are correct
- Look at the current assumptions and goal while writing a proof
- Write scripts that search a proof automatically (*tactics*)

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- Type membership is decidable, set membership is not.
- Terms in type theory can be *run* to simplify them.
- Set theory has only one type: *Set*.

Inductive families of datatypes

You can define new data types, for example:

$$
\begin{aligned}
&\textbf{data}\ \mathbb{N} : \texttt{Type}\ \textbf{where} \\
&\quad \texttt{z} : \mathbb{N} \\
&\quad \texttt{s} : \mathbb{N} \to \mathbb{N}
\end{aligned}
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You can also define predicates and relations:

$$
\begin{aligned}
&\textbf{data}\ \_ \leq \_ : \mathbb{N} \to \mathbb{N} \to \mathtt{Type}\ \textbf{where} \\
&\quad \mathtt{lz} : (n : \mathbb{N}) \to \mathtt{z} \leq n \\
&\quad \mathtt{ls} : (m\ n : \mathbb{N}) \to m \leq n \to \mathtt{s}\ m \leq \mathtt{s}\ n
\end{aligned}
$$

Dependent pattern matching

You can define functions by pattern matching:

$$
\begin{aligned}
&\_ + \_ : \mathbb{N} \to \mathbb{N} \to \mathbb{N} \\
&\mathtt{z} + y \qquad = y \\
&(\mathtt{s}\ x) + y = \mathtt{s}\ (x + y)
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$$

You can also prove things by pattern matching:

$$
\begin{aligned}
&\mathtt{assoc} : (x : \mathbb{N}) \to (y : \mathbb{N}) \to (z : \mathbb{N}) \to \\
&\qquad\qquad x + (y + z) \equiv_{\mathbb{N}} (x + y) + z \\
&\mathtt{assoc}\ \mathtt{z}\ y\ z \quad\quad\; = \mathtt{refl} \\
&\mathtt{assoc}\ (\mathtt{s}\ x)\ y\ z = \mathtt{cong}\ \mathtt{s}\ (\mathtt{assoc}\ x\ y\ z)
\end{aligned}
$$

Type systems

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Open questions about Martin-Löf's type theory

What is the structure of the identity type?

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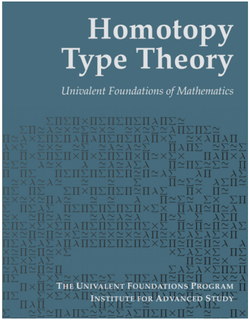
How to do reasoning 'up to isomorphism'?

Enter homotopy type theory



The HoTT book.

We can interpret types as topological spaces:

- Elements $x : A$ are points of the space A.
- Functions $A \to B$ are continuous maps from A to B.
- Proofs of $x \equiv_A y$ are *paths* from x to y.

Higher inductive types

You can define new types by giving their points *and paths between their points*:

$$
\begin{array}{l}
\textbf{data}\ \mathtt{I} : \mathtt{Type}\ \textbf{where} \\
\quad \mathtt{start} \quad\ \ : \mathtt{I} \\
\quad \mathtt{end} \qquad\ \ : \mathtt{I} \\
\quad \mathtt{segment} : \mathtt{start} \equiv_{\mathtt{I}} \mathtt{end}
\end{array}
$$

This way we can define quotients and subsets.

Synthetic homotopy theory

We can define topological spaces inductively:

$$
\begin{array}{l}
\textbf{data}\ \mathtt{S}^1 : \mathtt{Type}\ \textbf{where} \\
\quad \mathtt{base} : \mathtt{S}^1 \\
\quad \mathtt{loop} : \mathtt{base} \equiv_{\mathtt{S}^1} \mathtt{base}
\end{array}
$$

Synthetic homotopy theory

We can define topological spaces inductively:

```
data S¹ : Type where
  base : S¹
  loop : base ≡_S¹ base
```

Now we can do synthetic homotopy theory!

Synthetic homotopy theory

We can define topological spaces inductively:

data $\mathtt{S}^1$: `Type` **where**
  `base` : $\mathtt{S}^1$
  `loop` : `base` $\equiv_{\mathtt{S}^1}$ `base`

Now we can do synthetic homotopy theory!

We can prove that (`base` $\equiv_{\mathtt{S}^1}$ `base`) $\simeq \mathbb{Z}$.

Equivalence of types

A function $f : A \to B$ is an equivalence if it has a left and right inverse.

Two types are equivalent ($A \simeq B$) if there is an equivalence between them.

The univalence axiom

$$(A \equiv_{\texttt{Type}} B) \simeq (A \simeq B)$$

"Isomorphic structures can be identified"

Vladimir Voevodsky

Cubical type theory

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Very recently (2016), this has been solved by cubical type theory, based on ideas from cubical sets in category theory.

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Now we can actually run univalence to transport constructions between isomorphic structures!

Pattern matching in HoTT

Problem: With pattern matching, we can prove that any two paths are equal:

$$\mathtt{UIP} : (p\ q : x \equiv_A y) \to p \equiv_{x \equiv_A y} q$$
$$\mathtt{UIP}\ \mathtt{refl}\ \mathtt{refl} = \mathtt{refl}$$

Pattern matching in HoTT

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My PhD: how to make dependent pattern matching and HoTT work together?

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Lots of exciting things are happening right now, I've only scratched the surface!

References

Howard: *The formulae-as-types notion of construction* (1969).

Martin-Löf: *An intuitionistic theory of types* (1972).

Hofmann and Streicher: *The groupoid model refutes uniqueness of identity proofs* (1994).

Licata and Shulman: *Calculating the fundamental group of the circle in homotopy type theory* (2013).

Cohen, Coquand, Huber and Mörtberg: *Cubical type theory: a constructive interpretation of the univalence axiom* (2016).

Links

The HoTT book:

`homotopytypetheory.org/book`

The Coq proof assistant:

`coq.inria.fr`

The Agda proof assistant:

`wiki.portal.chalmers.se/agda`