Course: Machine Learning - Foundations
Test Questions
Lecture Details: Week 7

Questions 1-6 are based on common data
Consider these data points to answer the following questions:
$x_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

1. (1 point) The mean vector of the data points x_1, x_2, x_3 is

   A. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

   B. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   C. $\begin{bmatrix} 0.9 \\ 0.6 \\ 0.3 \end{bmatrix}$

   D. $\begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$

   Answer: B

   Explanation:

$$\overline{x} = \frac{1}{3} \sum_{i=1}^{3} = \frac{1}{3} \left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

   $\therefore$ Option B is correct.

2. (2 points) The covariance matrix $C = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})^T$ of the data points x_1, x_2, x_3 is

   A. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

   B. $\begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$

C. $\begin{bmatrix} 0.67 & 0 & -0.67 \\ 0 & 0 & 0 \\ -0.67 & 0 & 0.67 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Answer: C

Explanation: To solve this question we first take the data and center it by subtracting the mean. Doing this will give us the centered dataset.

$$x_1 - \overline{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ x_2 - \overline{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ x_3 - \overline{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Now, we can use the formula $C = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})^T$ to get the covariance matrix C.

$$\begin{aligned} C &= \frac{1}{3}\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}\right) \\ &= \frac{1}{3}\begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix} \\ &\approx \begin{bmatrix} 0.67 & 0 & -0.67 \\ 0 & 0 & 0 \\ -0.67 & 0 & 0.67 \end{bmatrix} \end{aligned}$$

$\therefore$ Option C is correct.

3. (2 points) The eigenvalues of the covariance matrix $C = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})^T$ are

   A. 2, 0, 0

   B. 1, 1, 1

   C. 1.34, 0, 0

   D. 0.5, 0, 0.5

Answer: C

Explanation: To find the eigenvalues, we find the characteristic polynomial and then find the roots.

$$
\begin{aligned}
|C - \lambda I| &= \begin{vmatrix} 0.67-\lambda & 0 & -0.67 \\ 0 & -\lambda & 0 \\ -0.67 & 0 & 0.67-\lambda \end{vmatrix} \\
&= (0.67-\lambda)(-\lambda)(0.67-\lambda) + (-0.67)(-0.67\lambda) \\
&= -\lambda^3 + 1.34\lambda^2 - 0.45\lambda + 0.45\lambda \\
&= \lambda^2(1.34 - \lambda)
\end{aligned}
$$

Solving for the roots, we get

$$|C - \lambda I| = 0 \implies \lambda = 1.34, 0, 0$$

$\therefore$ Option C is correct.

4. (2 points) The eigenvectors of the covariance matrix $C = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})^T$ are (Note: The eigenvectors should be arranged in the descending order of eigenvalues from left to right in the matrix.)

   A. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

   B. $\begin{bmatrix} 0.71 & 0 & 1 \\ 0 & 0.71 & 0 \\ 0.71 & 0.71 & 0 \end{bmatrix}$

   C. $\begin{bmatrix} -0.71 & 0 & 0.71 \\ 0 & 1 & 0 \\ 0.71 & 0 & 0.71 \end{bmatrix}$

   D. $\begin{bmatrix} 0.33 & 0 & 0 \\ 0.33 & 1 & 0 \\ 0.34 & 0 & 1 \end{bmatrix}$

Answer: C

Explanation: To solve this question, lets consider the eigenvalues one by one and find the eigenvectors.

Consider the eigenvalue $\lambda = 1.34$,

$$\begin{aligned} E_{1.34} &= null(C - 1.34I) \\ &= null\left(\begin{bmatrix} -0.67 & 0 & -0.67 \\ 0 & -1.34 & 0 \\ -0.67 & 0 & -0.67 \end{bmatrix}\right) \\ &= null\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \\ &= col\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) \end{aligned}$$

Now, $\lambda = 0$,

$$\begin{aligned} E_0 &= null(C) \\ &= null\left(\begin{bmatrix} 0.67 & 0 & -0.67 \\ 0 & 0 & 0 \\ -0.67 & 0 & 0.67 \end{bmatrix}\right) \\ &= null\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \\ &= col\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \end{aligned}$$

So, the eigenvectors in order of eigenvalue are -

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Keep in mind that the eigenvectors themselves can be scaled so, scaling it appropriately, we can see that the columns of option C match our eigenvectors.

$\therefore$ Option C is correct.

5. (2 points) The data points x_1, x_2, x_3 are projected onto the one dimensional space using PCA as points z_1, z_2, z_3 respectively. (Use eigenvector with the maximum eigenvalue for this projection.)

   A. $z_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $z_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $z_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

B. $z_1 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$, $z_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $z_3 = \begin{bmatrix} -0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$

C. $z_1 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$, $z_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $z_3 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

D. $z_1 = \begin{bmatrix} 0 \\ 1 \\ 2.0164 \end{bmatrix}$, $z_2 = \begin{bmatrix} 1.0082 \\ 1 \\ 1.0082 \end{bmatrix}$, $z_3 = \begin{bmatrix} 2.0164 \\ 1 \\ 0 \end{bmatrix}$

Answer: D

Explanation: To solve this question, we can calculate z_i to be the projection of x_i onto the principal eigenvector $v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \implies ||v||^2 = 2$. That is,

$$z_1 = proj_v x_1 = \frac{x_1 \cdot v}{||v||^2}\, v = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$z_2 = proj_v x_1 = \frac{x_2 \cdot v}{||v||^2}\, v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$z_3 = proj_v x_1 = \frac{x_3 \cdot v}{||v||^2}\, v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\therefore$ Option D is correct.

6. (1 point) The approximation error J on the given data set is given by $\frac{1}{n}\sum_{i=1}^{n} ||x_i - z_i||^2$. What is the reconstruction error?

A. 6.724×10^{-4}

B. 5

C. 10

D. 20

Answer: A

Explanation: Let us plugin the values of x_i and z_i into the formula and find the approximation error.

$$
\begin{aligned}
J &= \frac{1}{3} \sum_{i=1}^{3} ||x_i - z_1||^2 \\
&= \frac{1}{3} \left(|| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} ||^2 + || \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} ||^2 + || \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} ||^2 \right) \\
&= 3
\end{aligned}
$$

$\therefore$ Option A is correct.