Course: Machine Learning - Foundations
Week 3: Test questions

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1. (1 point) The length of the vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is

   A. 2.342

   B. 2.308

   C. 2.440

   D. 2.449

Answer: D

Explanation:

$$v = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$||v|| = \sqrt{v \cdot v}$$

$$\implies ||v|| = \sqrt{\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}$$

$$\implies ||v|| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6} = 2.449$$

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2. (1 point) The inner product of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$ is

   A. 11

   B. 12

   C. 14

   D. 16

Answer: D

Explanation:

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$b = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$$

$$< a, b >= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$$

$$= -1 + 2 + 15 = 16$$

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3. (1 point) The rank of a 4×3 matrix is 1, what is the dimension of its null space?
   A. 3
   B. 1
   C. 2
   D. 4

Answer: C

Explanation:

$$\text{Rank} = dim(C(A)) = 1 = r \text{ (say)}$$

$$\text{According to Rank-Nullity theorem, if } n \text{ is the nullity, then:}$$

$$r + n = \text{number of variables}$$

$$\text{Number of variables is equal to the number of columns}$$

$$1 + n = 3$$

$$n = 2$$

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4. (1 point) Which of the following vector is orthogonal to $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$?
   A. [-1 1 -3]
   B. [1 2 1]

C. [-1 1 -3]

D. [-3 0 1]

Answer: D

Explanation: To see which vector is orthogonal to the given vector, find the dot product between the two. Orthogonal vectors have the dot product equal to 0.
For option A,

$$\begin{bmatrix}-1 & 1 & -3\end{bmatrix}\begin{bmatrix}1\\-1\\3\end{bmatrix} = -1-1-9 \neq 0$$

For option B,

$$\begin{bmatrix}1 & 2 & 1\end{bmatrix}\begin{bmatrix}1\\-1\\3\end{bmatrix} = 1-2+3 \neq 0$$

For option C,

$$\begin{bmatrix}-1 & 1 & -3\end{bmatrix}\begin{bmatrix}1\\-1\\3\end{bmatrix} = -1-1-9 \neq 0$$

For option D,

$$\begin{bmatrix}-3 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\-1\\3\end{bmatrix} = -3+0+3 = 0$$

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5. (1 point) The rank of the following matrix $\begin{bmatrix}1 & 2 & 3\\2 & 4 & 6\\3 & 6 & 9\end{bmatrix}$ is

A. 1

B. 2

C. 3

D. 4

Answer: A

Explanation:
The 2^{nd} row is twice the 1^{st} row.
The 3^{rd} row is thrice the 1^{st} row. So, the REF will have just 1 pivot element, which means that the rank is 1.

REF can be achieved by applying the following row operations:

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - 3R_1$$

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6. (1 point) Which of the following would be the smallest subspace containing the first quadrant of the space $\mathbb{R}^2$?

   A. The first quadrant

   B. The first and the third quadrant

   C. The first and second quadrant

   D. The whole space $\mathbb{R}^2$

   Answer: D

   Explanation: The smallest subspace containing the first quadrant of the space $\mathbb{R}^2$ would be the span of the vectors that define the first quadrant. The first quadrant in $\mathbb{R}^2$ consists of all points (x, y) where x $\geq$ 0 and y $\geq$ 0.

   To find the smallest subspace containing the first quadrant, we need to determine the vectors that span this region. Since the first quadrant is in the positive x and y directions, the vectors (1, 0) and (0, 1) would form a basis for this subspace.

   The span of (1, 0) and (0, 1) is all of $\mathbb{R}^2$.

   Therefore, the smallest subspace containing the first quadrant of $\mathbb{R}^2$ is $\mathbb{R}^2$ itself.

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7. (1 point) 5 peaches and 6 oranges cost 150 rupees. 10 peaches and 12 oranges cost 300 rupees. Form a matrix out of the given information and find its rank.

   A. Rank = 2

   B. Rank = 1

   C. Rank = 0

   D. Rank = 4

   Answer: B

   Explanation: Let number of peaches = x_1.
   Let number of oranges = x_2.

equations:

$$5x_1 + 6x_2 = 150$$
$$10x_1 + 12x_2 = 300$$

Given equations in matrix notation will be:

$$\begin{bmatrix} 5 & 6 \\ 10 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 150 \\ 300 \end{bmatrix}$$

It is clearly visible that the 2^{nd} row is twice of 1^{st} row (linear combination), so that rank will be 1.

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8. (1 point) Consider a set of 3 paired observations on $(x_i, b_i), i = 1, 2, 3$ as $((1, 6), (-1, 3), (3, 15))$. For the closest line b to go through these points, which of the following is the least squares solution $(\hat{\theta})$?
   - A. (3,4)
   - B. (4,3)
   - C. (5,3)
   - D. (3,5)

Answer: D

Explanation: To solve for the least square, we have to solve the equation

$$A^T A\hat{\theta} = A^T b$$

$$A^T A = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 3 \\ 3 & 3 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 15 \end{bmatrix} = \begin{bmatrix} 48 \\ 24 \end{bmatrix}$$

Now, we have to solve the above equation using the quantities we just calculated.

$$A^T A\hat{\theta} = A^T b$$

$$\implies \begin{bmatrix} 11 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 48 \\ 24 \end{bmatrix}$$

Upon solving, we get

$$\theta_1 = 3$$
$$\theta_2 = 5$$

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9. (1 point) Which of the following represents the null space of the matrix $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix}$

A. Span $\left\{ \begin{bmatrix} -9 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

B. Span $\left\{ \begin{bmatrix} 9 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

C. Span $\left\{ \begin{bmatrix} 9 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

D. Span $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Answer: A

Explanation: To get the null space of the given matrix(let's call it A), we will use the fact that the null space does not change when we apply row operations.

So, we can reduce the matrix A to REF to get the following matrix:

$$\begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From the equations to get:

$$x_1 + 9x_3 + 2x_4 = 0$$
$$x_2 - 3x_3 + x_4 = 0$$

$$\text{Let } x_3 = s; x_4 = t$$
$$\implies x_1 = -9s - 2t$$
$$x_2 = 3s - t$$

Now, we can form a vector out of it for our solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9s - 2t \\ 3s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} t$$

Take two cases, one where s=0, t=1 and another where t=0, s=1

You will get the following:

$$\begin{bmatrix} -9 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence, option A

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10. (1 point) Which of the two vectors are orthogonal to each other?
    A. [1 2 3],[-1 -2 3]
    B. [1 2 1],[0 -1 2]
    C. [1 2 5],[1 2 3]
    D. [1 2 3],[2 4 6]
    E. [1 2 1],[-1 0 -1]

Answer: B

Explanation: To check for orthogonality, find the dot product. If the dot product equals 0, they are orthogonal.

Only for option B, the dot product will be equal to 0.

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = 0 - 2 + 2 = 0$$

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11. (1 point) Find projection of [5,-4,1] along [3,-2,4]
    A. $\begin{bmatrix} \frac{27}{29} & \frac{-18}{29} & \frac{36}{29} \end{bmatrix}$
    B. $\begin{bmatrix} \frac{27}{29} & \frac{18}{29} & \frac{36}{29} \end{bmatrix}$
    C. $\begin{bmatrix} \frac{81}{29} & \frac{-54}{29} & \frac{108}{29} \end{bmatrix}$

D. $\begin{bmatrix} \frac{81}{29} & \frac{54}{29} & \frac{108}{29} \end{bmatrix}$

Answer: C

Explanation: Let a = [5, -4, 1] and b = [3, -2, 4]

$$\mathbf{P_b(a)} = \frac{<\mathbf{a}, \mathbf{b}>}{<\mathbf{b}, \mathbf{b}>}\mathbf{b}$$

$$\implies \left(\frac{\begin{bmatrix} 5 & -4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}}{\begin{bmatrix} 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}} \right) \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

$$= \frac{27}{29} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{81}{29} \\ \frac{-54}{29} \\ \frac{108}{29} \end{bmatrix}$$

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12. (1 point) The projection matrix for the matrix v= $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ is

A. $\frac{1}{14} \begin{bmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{bmatrix}$

B. $\frac{1}{14} \begin{bmatrix} 4 & 3 & 6 \\ 2 & 2 & 3 \\ 9 & 3 & 9 \end{bmatrix}$

C. $\frac{1}{14} \begin{bmatrix} 4 & -2 & 6 \\ 3 & 1 & 3 \\ 6 & 6 & 9 \end{bmatrix}$

D. $\frac{1}{14} \begin{bmatrix} 2 & 2 & -6 \\ 2 & -1 & 5 \\ 5 & 7 & 9 \end{bmatrix}$

Answer: A

Explanation: The projection matrix of a vector v is given by the formula:

$$P = \frac{vv^T}{v^Tv}$$

$$\implies \frac{\begin{bmatrix}2\\1\\3\end{bmatrix}\begin{bmatrix}2 & 1 & 3\end{bmatrix}}{\begin{bmatrix}2 & 1 & 3\end{bmatrix}\begin{bmatrix}2\\1\\3\end{bmatrix}}$$

$$= \frac{1}{14}\begin{bmatrix}4 & 2 & 6\\2 & 1 & 3\\6 & 3 & 9\end{bmatrix}$$

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13. (1 point) Find projection of [2,-4,4] along [2,-2,1]

A. $\begin{bmatrix}\frac{-32}{9} & \frac{-32}{9} & \frac{16}{9}\end{bmatrix}$

B. $\begin{bmatrix}\frac{32}{9} & \frac{-32}{9} & \frac{16}{9}\end{bmatrix}$

C. $\begin{bmatrix}\frac{32}{9} & \frac{16}{9} & \frac{-32}{9}\end{bmatrix}$

D. $\begin{bmatrix}\frac{32}{9} & \frac{-16}{9} & \frac{32}{9}\end{bmatrix}$

Answer: B

Explanation: Let a = [2, -4, 4] and b = [2, -2, 1]

$$\mathbf{P_b(a)} = \frac{< \mathbf{a}, \mathbf{b} >}{< \mathbf{b}, \mathbf{b} >}\mathbf{b}$$

$$\implies \left(\frac{\begin{bmatrix} 2 & -4 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}} \right) \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$= \frac{16}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{32}{9} \\ \frac{-32}{9} \\ \frac{16}{9} \end{bmatrix}$$