**Course: Machine Learning - Foundations**
**Week 4: Test questions**

1. (1 point) If $P$ is a projection matrix, then the eigenvalue corresponding to every nonzero vector orthogonal to the column space of $P$ is

   A. 0

   B. 1

   C. -1

   **Answer:** A

   **Solution:** If a vector (let us say $b$) is orthogonal to column space of $P$, then it means that $b$ vector is orthogonal to vector (let us say $a$) from which projection matrix $P$ is built.
   That is $b \perp a$. So, $a.b = 0$ and $Pb = 0$.
   $\Rightarrow Pb = 0b$
   Hence eigenvalue is zero.

2. (1 point) (Multiple select) Subtracting a multiple of one row from another

   A. changes the determinant of the matrix.

   B. changes the column space of the matrix.

   C. changes the rank of the matrix.

   D. changes the eigenvalues of the matrix.

   **Answer:** D

   **Solution**: By doing row operations, we already know that rank and columns space will not change.
   By doing row operations determinant too will not change. For example consider the following matrix and operations done.
   $A = \begin{bmatrix} R1 \\ R2 \\ R3 \end{bmatrix}$
   Where $R1, R2$, and $R3$ are row vectors. Let us form another matrix $B$ by doing row operations.
   Let $B = A - \begin{bmatrix} R1 \\ R1 \\ R1 \end{bmatrix}$
   Now$\Rightarrow$ $\det(B) = \det(A) - \det(\begin{bmatrix} R1 \\ R1 \\ R1 \end{bmatrix})$
   $\Rightarrow \det B = \det A$

Since determinant of matrix with same rows is zero.
Doing row operations will change eigenvalues because characteristic polynomial equation changes.

3. (1 point) Consider the following statements regarding a real symmetric matrix $A$

   1. The eigenvalues of $A$ are always real.
   2. The eigenvalues of $A$ may be imaginary.
   3. The eigenvectors corresponding to different eigenvalues of $A$ are linearly independent.
   4. The eigenvectors corresponding to different eigenvalues of $A$ are not linearly independent.
   5. $A$ is orthogonally diagonalizable.
   6. $A$ is not diagonalizable.

   Which of the above statements are true?

   A. 2, 3 and 5
   B. 1, 4 and 5
   C. 2, 4 and 6
   D. 1, 3 and 5

   **Answer:** D

   **Solution:** Eigenvalues of symmetric matrix are always real. Eigenvectors corresponding to two distinct eigen values of a matrix are independent. In specific if matrix is symmetric, then they are orthogonal too.
   Since, eigenvectors of symmetric matrix are orthogonal, the diagonlization of symmetric matrix will result in orthogonal diagonalization.

4. (1 point) The eigenvectors corresponding to distinct eigenvalues of a matrix

   A. are linearly independent
   B. are linearly dependent
   C. have no relation

   **Answer:** A

   **Solution:** As mentioned in the above question, eigen vectors corresponding to distinct eigenvalues of a matrix are independent.

5. (1 point) The determinant of a $3 \times 3$ matrix having eigenvalues 1, -2 and 3 is

   A. 2

B. 0

C. 6

D. -6

E. -2

**Answer:** D

**Solution:** The product of eigen values is equal to determinant of the matrix. So, determinant is $1 \times -2 \times 3$, which is $-6$.

6. (1 point) The trace of a $2 \times 2$ matrix is -1 and its determinant is -6. Its eigenvalues will be

A. -1, 3

B. 2, 3

C. 2, -3

D. -2, 3

**Answer:** C

**Solution:** Trace (sum of diagonal elements) of a matrix is equal to the sum of the eigenvalues.
Since the given matrix is of $2 \times 2$ dimension, it will have two eigen values.
$\lambda_1 + \lambda_2 = -1, \lambda_1\lambda_2 = -6$, by solving we get $\lambda_1 = 2$ and $\lambda_2 = -3$

7. (1 point) If the eigenvalues of a matrix are -1, 0 and 4, then its trace and determinant are
Trace:_____
Determinant:_____

**Answer:** 3, 0

**Solution:** By using same above relations, we can find trace as $3(-1 + 0 + 4)$ and determinant as $0(-1 \times 0 \times 4)$.

8. (1 point) The characteristic polynomial for the matrix $A=\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ is

A. $\lambda^2 - 4\lambda + 1$

B. $\lambda^2 - 4\lambda$

C. $\lambda^2 - 4\lambda - 2$

D. $\lambda^2 + 4\lambda + 2$

E. $\lambda^2 - 4\lambda + 2$

**Answer:** E

**Solution** For any matrix $A$, characteristic polynomial equation is formed by using relation $\det(A - \lambda I) = 0$.
$\Rightarrow \det(\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$
$\Rightarrow \det(\begin{bmatrix} 1-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}) = 0$
By simplifying we get $\lambda^2 - 4\lambda + 2 = 0$

9. (1 point) The eigenvalues of matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ are

 A. $2+\sqrt{3}, 2-\sqrt{3}$
 B. $\sqrt{3}, -\sqrt{3}$
 C. 0,1
 D. $\sqrt{5}, -\sqrt{5}$

**Answer:** A

**Solution:** To find eigenvalues, we need to solve characteristic polynomial equation just like we did for the above problem.
$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$.
$\det \begin{bmatrix} 1-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix} = 0$
$\Rightarrow (1-\lambda)(3-\lambda) - 2 * 1 = 0$
$\Rightarrow \lambda^2 - 4\lambda + 1 = 0$
$\Rightarrow \lambda = 2 \pm \sqrt{3}$.

10. (2 points) If the eigenvalues of a matrix $A$ are 0 -1 and 5, then the eigenvalues of $A^3$ are

 A. 0, -1 and 5
 B. 0, -1 and 125
 C. 0, 1 and -125
 D. 0, 1 and -5

**Answer:** B

**Solution:** $\lambda^k$ is eigenvalue of matrix $A^k$, if $\lambda$ is eigenvalue of matrix $A$.
Since eigen values of $A$ are $0, -1$, and 5, eigenvalues of $A^3$ are $0^3, -1^3$, and $5^3$.
Hence option B is correct.

11. (1 point) The 110th term of Fibonacci sequence is approximately given by

    A. $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{110}$

    B. $\frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{110}$

    C. $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{-110}$

    D. $\frac{-1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{110}$

    **Answer:** A

    **Solution:** By following the lectures you can see that
    $F_k = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^k + \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^k$
    $F_110 = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{110} + \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{110} \approx \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{110}$.
    Hence, option A is correct.

12. (1 point) (Multiple Select) Let $A$ be an $n \times n$ matrix. Which of the following statements is/are false?

    A. If A has $r$ non-zero eigenvalues, then rank of $A$ is at least $r$.

    B. If one of the eigenvalues of $A$ are zero, then $|A| \neq 0$.

    C. If $x$ is an eigenvector of $A$, then so is every vector on the line through $x$.

    D. If 0 is an eigenvalue of $A$, then $A$ cannot be invertible.

    **Answer:** B

    **Solution:**
    If a matrix has $r$ non-zero eigenvalues, then rank of the matrix is $r$, hence first option is true.
    If one of the eigenvalue is zero, then determinant is zero, since determinant is product of eigenvalues. So, option B is false.
    If $x$ is an eigenvector then $cx$ will also be an eigenvector, where $c$ is real constant value. So, option C is true.
    If 0 is eigenvalue, then determinant is zero, hence it is not invertible.

13. (2 points) The eigenvalues of the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ are

    **Answer:** 0, 5

    **Solution:** By using the characteristic polynomial equation, we get $(1-\lambda)(4-\lambda)-4=0$ $\Rightarrow \lambda^2 - 5\lambda = 0$. On solving them we get $\lambda$ eigenvalues as $0, 5$.

14. (2 points) (Multiple Select) For the matrix given in the previous question, which of the following vectors is/are its eigenvector(s)?

    A. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

B. $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

**Answer:** A, B

**Solution:** We will find eigenvector for each of the eigenvalue 0 and 5.
For $\lambda = 0$ case:
We know that $(A - \lambda I)x = 0$
$\Rightarrow (A - 0I)x = 0$
Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\Rightarrow \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
Let us take $x_1$ as 1, then we get $x_2$ as 0.5, so first eigen vector is $c \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$ Among the given options, b satifsfies when $c = -2$, that is $-2 \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

For $\lambda = 5$ case:
Here we will get equation as
$(A - 5I)x = 0$
$\Rightarrow \begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\Rightarrow -4x_1 + 2x_2 = 0$ and $\Rightarrow 2x_1 - x_2 = 0$
If $x_1 = 1$, then $x_2 = 2$
Hence, eigenvector is of the form $c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So, option A is correct.

15. (3 points) Suppose that A, P are 3 × 3 matrices, and P is an invertible matrix.
If $P^{-1}AP = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 3 & 8 \\ 0 & 0 & 4 \end{bmatrix}$, then the eigenvalues of the matrix $A^2$ are

**Answer:** 1,9,16

**Solution:** Here $P^{-1}AP$ gives upper triangular matrix where diagonal elements are eigenvalues. So, eigenvalues of $A$ are $-1, 3$, and 4. So, eigen values of $A^2$ are $(-1)^2, 3^2$, and $4^2$.

Eigenvalues of $A^2$ are 1,9, and 16.

16. (2 points) The best second order polynomial that fits the data set

| x | y |
|---|---|
| 0 | 0 |
| 1.3 | 1.5 |
| 4 | 1.2 |

is

A. $1.35x^2 + 0.3x$

B. $1.25x^2 + 0.45x$

C. $-0.316x^2 + 1.56x$

D. $-0.25x^2 + 0.5$

**Answer:** C

**Solution:** Here we are asked to find second order polynomial that fits the data. This is similar to linear regression of multiple features where we have one feature with order 2. The equation will be $y = \theta_0 + \theta_1 x + \theta_2 x^2$

$Y = \begin{bmatrix} 0 \\ 1.5 \\ 1.2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1.3 & 1.3^2 \\ 1 & 4 & 4^2 \end{bmatrix}$, and $\theta = \begin{bmatrix} \theta_0 \theta_1 \\ \theta_2 \end{bmatrix}$

In order to minimize error $\theta = (A^T A)^{-1} A^T Y$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1.3 & 4 \\ 0 & 1.69 & 16 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1.3 & 1.3^2 \\ 1 & 4 & 4^2 \end{bmatrix} = \begin{bmatrix} 1 & 5.3 & 17.69 \\ 5.3 & 17.69 & 66.19 \\ 17.69 & 66.19 & 258.8561 \end{bmatrix}$$

$$A^T Y = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1.3 & 4 \\ 0 & 1.69 & 16 \end{bmatrix} \begin{bmatrix} 0 \\ 1.5 \\ 1.2 \end{bmatrix} = \begin{bmatrix} 2.7 \\ 6.75 \\ 21.735 \end{bmatrix}$$

Yet to do

$$(A^T A)^{-1} \approx \begin{bmatrix} -1 & 1.017 & -0.197 \\ 1.017 & 0.273 & -0.14 \\ -0.197 & -0.14 & 0.05 \end{bmatrix}$$

$$\Rightarrow \theta = \begin{bmatrix} -1 & 1.0178 & -0.1917 \\ 1.0178 & 0.2738 & -0.1395 \\ -0.1917 & -0.1395 & 0.0526 \end{bmatrix} \begin{bmatrix} 2.7 \\ 6.75 \\ 21.735 \end{bmatrix} \approx \begin{bmatrix} -0.117045 & 1.545 & -0.315 \end{bmatrix}$$

So, option C is correct.