Questions 1-6 are based on common data

Consider these data points to answer the following questions: $x_1 = \begin{bmatrix}0\\1\\2\end{bmatrix}$, $x_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $x_3 = \begin{bmatrix}2\\1\\0\end{bmatrix}$

$$\bar{x} = \frac{1}{3}\begin{bmatrix}0+1+2\\1+1+1\\2+1+0\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

1) The mean vector of the data points x_1, x_2, x_3 is

- ( ) $\begin{bmatrix}0\\0\\0\end{bmatrix}$
- (●) $\begin{bmatrix}1\\1\\1\end{bmatrix}$
- ( ) $\begin{bmatrix}0.9\\0.6\\0.3\end{bmatrix}$
- ( ) $\begin{bmatrix}0.5\\0.5\\0.5\end{bmatrix}$

$$Z_1 = x_1 - \bar{x} = \begin{bmatrix}0-1\\1-1\\2-1\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$Z_2 = x_2 - \bar{x} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$Z_3 = x_3 - \bar{x} = \begin{bmatrix}2-1\\1-1\\0-1\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

2) The covariance matrix $C = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})^T$ of the data points x_1, x_2, x_3 is

- ( ) $\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$
- ( ) $\begin{bmatrix}0.33 & 0 & 0.33\\0 & 0 & 0\\0.33 & 0 & 0.33\end{bmatrix}$
- (●) $\begin{bmatrix}0.67 & 0 & -0.67\\0 & 0 & 0\\-0.67 & 0 & 0.67\end{bmatrix}$
- ( ) $\begin{bmatrix}2.88 & 0 & 1.44\\0 & 0 & 0\\1.44 & 0 & 2.88\end{bmatrix}$

Adding all $(x_i - \bar{x})$:

$$\frac{1}{3}\left(\begin{bmatrix}-1\\0\\1\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\end{bmatrix} + \begin{bmatrix}0\end{bmatrix} + \begin{bmatrix}1\\0\\-1\end{bmatrix}\begin{bmatrix}1 & 0 & -1\end{bmatrix}\right)$$

$$\begin{bmatrix}1 & 0 & -1\\0 & 0 & 0\\-1 & 0 & 1\end{bmatrix} + \begin{bmatrix}0\end{bmatrix} + \begin{bmatrix}1 & 0 & -1\\0 & 0 & 0\\-1 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}0.67 & 0 & -0.67\\0 & 0 & 0\\-0.67 & 0 & 0.67\end{bmatrix}$$

$(0.67)^2$
$= 0.4489$

$$\left|\begin{bmatrix} 0.67-\lambda & 0 & -0.67 \\ 0 & -\lambda & 0 \\ -0.67 & 0 & 0.67-\lambda \end{bmatrix}\right| = -\lambda\left((0.67-\lambda)^2 - (-0.67)^2\right)$$

$$= -\lambda\left(\lambda^2 + \cancel{0.4489} - 1.34\lambda - \cancel{0.4489}\right)$$

$$= -\lambda\left(\lambda^2 - 1.34\lambda\right)$$

$$= \lambda \cdot \lambda \cdot (\lambda - 1.34)$$

1.3, 0, 0

3) The eigenvalues of the covariance matrix $C = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})^T$ are

(Note: The eigenvalues should be arranged in the descending order from left to right.)

- ○ 2.66, 1.18, 0
- ○ 1.22, 0.74, 0.34
- ● 1.34, 0, 0
- ○ 0.56, 0.33, 0.14

4) The eigenvectors matrix of the covariance matrix $C = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})^T$ is

(Note: The eigenvectors should be arranged in the descending order of eigenvalues from left to right in the matrix.)

- ○ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
- ○ $\begin{bmatrix} 0.71 & 0 & 1 \\ 0 & 0.71 & 0 \\ 0.71 & 0.71 & 0 \end{bmatrix}$
- ● $\begin{bmatrix} -0.71 & 0 & 0.71 \\ 0 & 1 & 0 \\ 0.71 & 0 & 0.71 \end{bmatrix}$
- ○ $\begin{bmatrix} 0.33 & 0 & 0 \\ 0.33 & 1 & 0 \\ 0.34 & 0 & 1 \end{bmatrix}$

For 1.34;
$$\begin{bmatrix} 0.67 & 0 & -0.67 \\ 0 & 0 & 0 \\ -0.67 & 0 & 0.67 \end{bmatrix} - \begin{bmatrix} 1.34 & 0 & 0 \\ 0 & 1.34 & 0 \\ 0 & 0 & 1.34 \end{bmatrix} =$$

$$= \begin{bmatrix} -0.67 & 0 & -0.67 \\ 0 & -1.34 & 0 \\ -0.67 & 0 & -0.67 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

5) The data points x_1, x_2, x_3 are projected onto the one dimensional space using PCA as points z_1, z_2, z_3 respectively.

(Use eigenvector with the maximum eigenvalue for this projection.)

- ○ $z_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, z_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, z_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
- ○ $z_1 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, z_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, z_3 = \begin{bmatrix} -0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$
- ● $z_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, z_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, z_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$
- ○ $z_1 = \begin{bmatrix} 0 \\ 1 \\ 2.0164 \end{bmatrix}, z_2 = \begin{bmatrix} 1.0082 \\ 1 \\ 1.0082 \end{bmatrix}, z_3 = \begin{bmatrix} 2.0164 \\ 1 \\ 0 \end{bmatrix}$

Projections are just $x_i - \bar{x}$ vectors
So

6) The approximation error J on the given data set is given by $\Sigma_{i=1}^{n}||x_i - z_i||^2$. What is the reconstruction error?

- ○ $6.724 * 10^{-4}$
- ○ 5
- ● 10
- ○ 20

$$= \left(\left\| \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\|^2 \right)$$

$$= \left(\left\| \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\|^2 \right)$$

$$= \left(\sqrt{(-1)^2+1^2+1^2}\right)^2 + \left(\sqrt{1^2+1^2+1^2}\right)^2 + \left(\sqrt{1^2+1^2+(-1)^2}\right)^2$$

$$= 3+3+3 = 9$$

7) Which of the following is/are true about PCA?

- ■ PCA is an unsupervised method
- ■ All principal components are orthogonal to each other
- ■ It searches for the directions that data have the largest variance
- ■ PCA is a dimensional reduction technique.

8) Consider the following matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Check if $Av = \lambda v$ there exists a scalar λ

Which of the following is not the eigenvector of this matrix ?

- ○ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1+2 \\ 1+2+1 \\ 2+1+1 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \lambda = 4$ exists

- ○ $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - \cancel{2} + \cancel{2} \\ 1 - 4 + 1 \\ \cancel{2} - \cancel{2} + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ $\quad \lambda = 1$ exists

- ○ $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+0+2 \\ -1+0+1 \\ -2+0+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\quad \lambda = -1$ exists

- ● $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+1+0 \\ -1+2+0 \\ -2+1+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $\quad \lambda$ does not exist

9) Consider a square matrix A such that $A^T = A$. One learner told me that the following three vectors are the eigenvectors of this matrix A.

$$x = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, z = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Is learner telling the truth ?

- Yes
- No
- Can't say without knowing the element of A
- Yes, only if all diagonal elements of A are 1

Check orthogonality of vector

$x \cdot y = (-1 \times 1) + (1 \times 1) + (1 \times 1) = 1$

$x \cdot z = (-1 + 1 - 1) = -1$

$y \cdot z = (1 + 1 - 1) = 1$

None of them are orthogonal so they cannot be eigenvectors of a symmetric matrix