Course: Machine Learning - Foundations
Week 5: Test questions

1. (1 point) Consider two non-zero vectors $x \in \mathbb{C}$ and $y \in \mathbb{C}$. Suppose the inner product between x and y obeys commutative property (i.e., $x \cdot y = y \cdot x$), it implies that

   A. y must be a conjugate transpose of x

   B. y is equal to x

   C. y must be orthogonal to x

   D. y must be a scalar (possibly complex) multiple of x

   Explanation: Let us look at each option and see what makes the commutative property hold.

   Option A.
   Assuming $y = x^*$

   $$x \cdot y = x^* y = x^* x^*$$

   This multiplication is not defined, so option A is incorrect.

   Option B.
   Assuming $y = x$

   $$x \cdot y = y \cdot x$$

   This is trivially true, so option B is correct.

   Option C.
   Assuming $x^* y = 0$

   $$x \cdot y = x^* y = 0 = y^* x = y \cdot x$$

   So, option C is correct.

   Option D.
   Assuming $y = zx$ where $z \in \mathbb{C}$

   $$x \cdot y = x^* y = x^* z x = z x^* x \neq \overline{z} x^* x = (zx)^* x = y^* x = y \cdot x$$

   So, option D is incorrect.

   $\therefore$ Options B, C are correct.

2. (1 point) The inner product of two distinct vectors x and y that are drawn randomly from $\mathbb{C}^{100}$ is $0.8 - 0.37i$.The vector x is scaled by a scalar $1 - 2i$ to obtain a new vector z, then the inner product between z and y is

A. $0.06 - 1.97i$

B. $1.54 - 1.23i$

C. $1.54 + 1.23i$

D. $0.8 - 0.37i$

E. Not possible to calculate

Explanation: From the question, it is given that $x^*y = 0.8 - 0.37i$. Let us replace x with $(1 - 2i)x$ and compute -

$$[(1-2i)x]^*y = \overline{(1-2i)}\ x^*y = (1+2i)\ x^*y = (1+2i)(0.8-0.37i) = 1.54 + 1.23i$$

$\therefore$ Option C is correct.

3. (1 point) Select the correct statement(s).The Eigen-value decomposition for the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

A. doesn't exist over $\mathbb{R}$ but exists over $\mathbb{C}$

B. doesn't exist over $\mathbb{C}$ but exists over $\mathbb{R}$

C. neither exists over $\mathbb{R}$ nor exists over $\mathbb{C}$

D. exists over both $\mathbb{C}$ and $\mathbb{R}$

Explanation: Let us find the eigenvalues of the matrix A.

$$|A - \lambda I| = (-\lambda)^2 - (-1) = \lambda^2 + 1 \implies \lambda = i, -i$$

Here, we can see that the eigenvalues are complex. So, the decomposition doesn't exist over $\mathbb{R}$ but since there are two distinct eigenvalues, the decomposition will exist over $\mathbb{C}$. $\therefore$ Option A is correct.

4. (1 point) Consider the complex matrix $S = \begin{bmatrix} 1 & 1+i & -2-2i \\ 1-i & 1 & -i \\ -2+2i & i & 1 \end{bmatrix}$. The matrix is

A. Hermitian and Symmetric

B. Symmetric but not Hermitian

C. Neither Hermitian nor Symmetric

D. Hermitian but not Symmetric

Explanation: For this question, we can check both conditions. We can see that here $S^T \neq S$, so S is not symmetric. But taking the complex conjugate transpose, $S^* = S$, so S is hermitian.

$\therefore$ Option D is correct.

5. (1 point) Suppose that an unitary matrix U is multiplied by a diagonal matrix D with $d_{ii} \in \mathbb{R}$, then the resultant matrix will always be unitary. The statement is

   A. True

   B. False

   Explanation: The requirement for a matrix U to be unitary is that it's columns must be pairwise orthogonal and of unit length.

   Here, if we consider the diagonal matrix $D = 2I$, say. Then multiplying D with U will cause all the columns to double, which in turn will change the length of the columns. So, the resultant matrix will not be unitary.

   $\therefore$ Option B is correct.

6. (3 points) The eigenvectors of matrix $A = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$ are

   A. $\begin{bmatrix} -1 \\ 1+2i \\ 1 \end{bmatrix}, \begin{bmatrix} 1-21i \\ 6-9i \\ 13 \end{bmatrix}, \begin{bmatrix} 1+3i \\ -2-i \\ 5 \end{bmatrix}$

   B. $\begin{bmatrix} 1 \\ 1-2i \\ 1 \end{bmatrix}, \begin{bmatrix} 1-21i \\ 6-9i \\ 13 \end{bmatrix}, \begin{bmatrix} 1+3i \\ -2-i \\ 5 \end{bmatrix}$

   C. $\begin{bmatrix} -1 \\ 1-2i \\ -1 \end{bmatrix}, \begin{bmatrix} 1-21i \\ 6-9i \\ 13 \end{bmatrix}, \begin{bmatrix} 1+3i \\ -2-i \\ 5 \end{bmatrix}$

   D. $\begin{bmatrix} -1 \\ 1+2i \\ 1 \end{bmatrix}, \begin{bmatrix} 1-21i \\ 6-9i \\ 13 \end{bmatrix}, \begin{bmatrix} 1-3i \\ 2-i \\ -5 \end{bmatrix}$

   Explanation: To solve this question, we can use trial and error from the options and see which one of them is an eigenvector.

$$\begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1+2i \\ 1 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 1+2i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix} \begin{bmatrix} 1-21i \\ 6-9i \\ 13 \end{bmatrix} = 6 \begin{bmatrix} 1-21i \\ 6-9i \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix} \begin{bmatrix} 1+3i \\ -2-i \\ 5 \end{bmatrix} = -2 \begin{bmatrix} 1+3i \\ -2-i \\ 5 \end{bmatrix}$$

So, all the vectors from option A are eigenvectors as upon multiplication with the matrix A, the vectors get scaled.

$\therefore$ Option A is correct.

7. (1 point) A matrix $A = \frac{1}{2}\begin{bmatrix} k+i & \sqrt{2} \\ k-i & \sqrt{2}i \end{bmatrix}$ is unitary if k is

   A. $\frac{1}{2}$

   B. 1

   C. -$\frac{1}{2}$

   D. -1

   E. ± 1

   F. $\pm\frac{1}{2}$

   Explanation: To solve this question, we will use the fact that, for a unitary matrix A, its columns must be pairwise orthogonal. That is we need,

$$\begin{aligned} 0 &= < a_1, a_2 > \\ &= a_1^* \, a_2 \\ &= \frac{1}{2} \begin{bmatrix} k-i & k+i \end{bmatrix} \frac{1}{2} \begin{bmatrix} \sqrt{2} \\ \sqrt{2}i \end{bmatrix} \\ &= \frac{1}{4}\left(\sqrt{2}k - \sqrt{2}i + \sqrt{2}ki - \sqrt{2}\right) \\ &= \frac{(\sqrt{2}k - \sqrt{2}) + i(\sqrt{2}k - \sqrt{2})}{4} \end{aligned}$$

   So, we have, $\sqrt{2}k - \sqrt{2} = 0 \implies k = 1$

   $\therefore$ Option B is correct.

8. (3 points) A matrix $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$ can be written as $A = UDU^*$, where U is a unitary matrix and D is a diagonal matrix. Then, U and D, respectively, are

A. $U = \begin{bmatrix} 1+i & \sqrt{2} \\ \sqrt{2} & 1-i \end{bmatrix}$, $D = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$

B. $U = \begin{bmatrix} -1+i & \sqrt{2} \\ \sqrt{2} & -1-i \end{bmatrix}$, $D = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$

C. $U = \begin{bmatrix} 1+i & \sqrt{2} \\ \sqrt{2} & 1-i \end{bmatrix}$, $D = \begin{bmatrix} -1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$

D. $U = \begin{bmatrix} 1-i & \sqrt{2} \\ \sqrt{-2} & 1-i \end{bmatrix}$, $D = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$

Explanation: Since the matrix A can be written as UDU^*, A is unitarily diagonalizable. Let us start by finding the eigenvalues and eigenvectors. The characteristic polynomial is $c(\lambda)$.

$$c(\lambda) = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 1+i \\ 1-i & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 2 = \lambda^2 - 2\lambda - 1 \implies \lambda = 1+\sqrt{2},\ 1-\sqrt{2}$$

So, the eigenvalues are $1+\sqrt{2}$ and $1-\sqrt{2}$ $\implies D = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$

Let the eigenvectors be v_1 and v_2. Then,

$$E_{\lambda=1+\sqrt{2}} = null\left(\begin{bmatrix} -\sqrt{2} & 1+i \\ 1-i & -\sqrt{2} \end{bmatrix}\right) = null\left(\begin{bmatrix} 1 & \dfrac{-1-i}{\sqrt{2}} \\ 1-i & -\sqrt{2} \end{bmatrix}\right)$$
$$= null\left(\begin{bmatrix} 1 & \dfrac{-1-i}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}\right) \implies v_1 = \begin{bmatrix} 1+i \\ \sqrt{2} \end{bmatrix}$$

$$E_{\lambda=1-\sqrt{2}} = null\left(\begin{bmatrix} \sqrt{2} & 1+i \\ 1-i & \sqrt{2} \end{bmatrix}\right) = null\left(\begin{bmatrix} 1 & \dfrac{1+i}{\sqrt{2}} \\ 1-i & \sqrt{2} \end{bmatrix}\right)$$
$$= null\left(\begin{bmatrix} 1 & \dfrac{1+i}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}\right) \implies v_2 = \begin{bmatrix} -1-i \\ \sqrt{2} \end{bmatrix}$$

Converting v_1 and v_2 to unit vectors and putting them in a matrix we get $U = \begin{bmatrix} \dfrac{1+i}{2} & \dfrac{-1-i}{2} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix}$.

$\therefore$ Option A is correct.

9. (2 points) The matrix $Z = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has

   A. only real eigenvalues.

   B. one real and one complex eigenvalue.

   C. no real eigenvalues.

   Explanation: Let us compute the eigenvalues of Z.

$$|Z - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 \implies \lambda = \pm i$$

   $\therefore$ Option C is correct.

10. (1 point) (Multiple select) Which of the following matrices is/are unitary?

   A. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

   B. $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

   C. $\begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

   D. $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

   E. $\begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

   Explanation: The conditions for a particular matrix A to be unitary is that the columns need to be of unit length and pairwise orthogonal.

   We can see here that due to the property $\sin^2\theta + \cos^2\theta = 1$, all matrices here satisfy the first condition.
   For the second condition, we need the inner product of the columns to come out to 0. This will only be true for options C and D.

   $\therefore$ Option C, D are correct.

11. (2 points) Let U and V be two unitary matrices. Then

   1. UV is unitary.
   2. $U + V$ is unitary.

A. Both statements are true.

B. Both statements are false.

C. 1. is false.

D. 2. is false.

Explanation: The conditions for a particular matrix A to be unitary is that the columns need to be of unit length and pairwise orthogonal. These conditions are captured in the property that for a unitary matrix U, $\quad U^* = U^{-1}$. For the first statement, we have

$$(UV)^* = V^*U^* = V^{-1}U^{-1} = (UV)^{-1}$$

This shows that UV is a unitary matrix. However the same cannot be said for statement 2 because of the fact that $U^{-1} + V^{-1} \neq (U+V)^{-1}$
$\therefore$ Option D is correct.

12. (2 points) (Multiple select) Which of the following is/are eigenvectors of the Hermitian matrix $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$

A. $\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} -2-2i \\ 2 \end{bmatrix}$

C. $\begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

E. All of these.

Explanation: For this question, we can simply multiply the vectors in each of the options to the matrix A. If this multiplication has an effect of scaling the vectors, that implies it is an eigenvector.

Option A.

$$\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} \begin{bmatrix} -1-i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is an eigenvalue of 0, so option A is correct.

Option B.

This vector is simply a multiple of the vector in option A, so it is also an eigenvector.

Option C.

$$\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix} = 3 \begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix}$$

This is an eigenvalue of 3, so option C is correct.

Option D.
This vector is simply a multiple of the vector in option C, so it is also an eigenvector.

$\therefore$ All options are correct.