Course: Machine Learning - Foundations
Week 2 - Graded assignment

---

1. Which of the following functions is/are continuous?

    A. $\frac{1}{x-1}$

    B. $\frac{x^2-1}{x-1}$

    C. $sign(x-2)$

    D. $sin(x)$

Answer: D

Explanation: Option A is not defined at $x = 1$ therefore, it'll have a breakpoint there. Hence, not continuous.
In option B, the function is again not continuous at $x = 1$. One may try to simplify the option as follows:

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1}$$

Please note that you cannot cancel out $(x-1)$ here because you would be assuming that $x-1$ is not equal to 0. But, we get $(x-1) = 0$ at $x = 1$. Here, limits exist but that doesn't necessarily mean that the function is continuous.
Option C is discontinuous at $x = 2$.
Option D is continuous at all points.

---

2. Regarding a d-dimensional vector $\mathbf{x}$, which of the following four options is not equivalent to the rest three options?

    A. $\mathbf{x}^T\mathbf{x}$

    B. $||\mathbf{x}||^2$

    C. $\sum_{i=1}^{d} x_i^2$

    D. $\mathbf{x}\mathbf{x}^T$

Answer: D

Explanation:

$$x \cdot x = x^T x = \sum_{i=1}^{d} x_i^2$$

$$||x|| = \sqrt{x_1^2 + x_2^2 + ... + x_d^2}$$

$$\implies ||x||^2 = x_1^2 + x_2^2 + ... + x_d^2 = \sum_{i=1}^{d} x_i^2$$

$$x^T x \neq x x^T$$

Therefore, options A, B, and C are equivalent but option D is different.

---

3. Consider the following function:

$$f(x) = \begin{cases} 3x+3, & \text{if } x \geq 3 \\ 2x+8, & \text{if } x<3 \end{cases}$$

Which of the following is/are true?

A. $f(x)$ is continuous at $x = 3$.

B. $f(x)$ is not continuous at $x = 3$.

C. $f(x)$ is differentiable at $x = 3$.

D. $f(x)$ is not differentiable at $x = 3$.

Answer: B, D

Explanation:

$f(x)$ is continuous at $x = 3$ if $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$

$$\lim_{x\to 3^-} (2x+8) = 2(3) + 8 = 14$$

$$\lim_{x\to 3^+} (3x+3) = 3(3) + 3 = 12$$

$LHL \neq RHL$

Therefore, the function is not continuous at $x = 3$

For a function to be differentiable, the minimum requirement for it is to be continuous at that point. As our function is not continuous, it cannot be differentiable.
Hence, options B and D are the correct options.

---

4. Approximate the value of $e^{0.011}$ by linearizing e^x around x=0.

Answer: 1.011

Explanation: To approximate the value of $e^{0.011}$ by linearizing e^x around $x = 0$, we can use the first-order Taylor expansion of e^x around the limit $x = a$, which is given by:

$$e^x \approx e^a + e^a(x - a)$$

where a is the point around which we are linearizing (in this case, $a = 0$).

Using this approximation, we have:

$$e^{0.011} \approx e^0 + e^0(0.011 - 0) = 1 + 1(0.011) = 1.011$$

Therefore, the approximate value of $e^{0.011}$ obtained by linearizing e^x around $x = 0$ is approximately 1.011.

---

5. Approximate $\sqrt{3.9}$ by linearizing $\sqrt{x}$ around $x = 4$.

   ____

   Answer: 1.975

   Explanation: To approximate the value of $\sqrt{3.9}$ by linearizing $\sqrt{x}$ around $x = 4$, we can use the first-order Taylor expansion of $\sqrt{x}$ around the limit $x = 0$, which is given by:

$$\sqrt{x} \approx \sqrt{a} + \frac{1}{2\sqrt{a}}(x - a)$$

   Using this approximation, we have:

$$\sqrt{3.9} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(3.9 - 4) = 2 + \frac{1}{4}(-0.1) = 2 - 0.025 = 1.975$$

   Therefore, the approximate value of $\sqrt{3.9}$ obtained by linearizing $\sqrt{x}$ around $x = 4$ is approximately 1.975.

---

6. Which of the following pairs of vectors are perpendicular to each other?

   A. [2, 3, 5] and [-2, 3, -1]

   B. [1, 0, 1] and [0, 1, 1]

   C. [1, 2, 0] and [0, 1, 2]

   D. [0, 1, 0] and [0, 0, 1]

   E. [2, -3, 5] and [-2, 3, -5]

   F. [1, 0, 0] and [0, 1, 0]

   Answer: A, D, E, F

   Explanation: If 2 vectors are perpendicular to each other, the 2 vectors must have the dot product equal to 0.

Only options A, D, E, and F result in a dot product = 0.

---

7. What is the linear approximation of $f(x, y) = x^3 + y^3$ around (2, 2)?
   A. $4x + 4y - 8$
   B. $12x + 12y - 32$
   C. $12x + 4y - 8$
   D. $12x + 12y + 32$

Answer: B

Explanation:

$$\nabla f(x, y) = \begin{bmatrix} 3x^2 \\ 3y^2 \end{bmatrix}$$
$$\implies \nabla f(2, 2) = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$$

$$\begin{aligned} L_{x*,y*}[f](x, y) =& f(x, y) + \nabla f(x^*, y^*)^T \cdot \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix} \\ =& 16 + \begin{bmatrix} 12 & 12 \end{bmatrix} \begin{bmatrix} x - 2 \\ y - 2 \end{bmatrix} \\ =& 16 + 12x - 24 + 12y - 24 \\ =& 12x + 12y - 32 \end{aligned}$$

---

8. What is the gradient of $f(x, y) = x^3y^2$ at (1, 2)?
   A. [12, 4]
   B. [4, 12]
   C. [1, 4]
   D. [4, 1]

Answer: A

Explanation:

$$\nabla f(x, y) = \begin{bmatrix} 3x^2y^2 \\ 2x^3y \end{bmatrix} \implies \nabla f(1, 2) = \begin{bmatrix} 3(1)^2(2)^2 \\ 2(1)^3(2) \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \end{bmatrix}$$

---

9. The gradient of $f = x^3 + y^2 + z^3$ at $x = 0$, $y = 1$ and $z = 1$ is given by,

A. [1, 2, 3]

B. [-1, 2, 3]

C. [0, 2, 3]

D. [2, 0, 3]

Answer: C

Explanation: The gradient of $f = x^3 + y^2 + z^3$ is given by:

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$

Taking the partial derivatives:

$\frac{\partial f}{\partial x} = 3x^2$, $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial z} = 3z^2$

Evaluating these partial derivatives at $x = 0$, $y = 1$, and $z = 1$:

$$\frac{\partial f}{\partial x}(0, 1, 1) = 3(0)^2 = 0$$
$$\frac{\partial f}{\partial y}(0, 1, 1) = 2(1) = 2$$
$$\frac{\partial f}{\partial z}(0, 1, 1) = 3(1)^2 = 3$$

Therefore, the gradient $\nabla f(0, 1, 1) = [0, 2, 3]$.

---

10. For two vectors **a** and **b**, which of the following is true as per Cauchy-Schwarz inequality?

(i) $\mathbf{a}^T\mathbf{b} \leq ||\mathbf{a}|| * ||\mathbf{b}||$
(ii) $\mathbf{a}^T\mathbf{b} \geq -||\mathbf{a}|| * ||\mathbf{b}||$
(iii) $\mathbf{a}^T\mathbf{b} \geq ||\mathbf{a}|| * ||\mathbf{b}||$
(iv) $\mathbf{a}^T\mathbf{b} \leq -||\mathbf{a}|| * ||\mathbf{b}||$

A. (i) only

B. (ii) only

C. (iii) only

D. (iv) only

E. (i) and (ii)

F. (iii) and (iv)

Answer: E ((i) and (ii))

Explanation: According to Cauchy-Schwarz inequality:

$$-||a|| \cdot ||b|| \leq a^T b \leq ||a|| \cdot ||b||$$

---

11. The directional derivative of $f(x, y, z) = x^3 + y^2 + z^3$ at (1, 1, 1) in the direction of unit vector along $\mathbf{v} = [1, -2, 1]$ is __________.

Answer: 0.816

Explanation: directional derivative is given by the dot product of gradient at a point with a unit vector along which the directional derivative is needed.

$$\nabla f(x, y, z) = \begin{bmatrix} 3x^2 \\ 2y \\ 3z^2 \end{bmatrix}$$

$$\implies \nabla f(1, 1, 1) = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

Next, let's find the unit vector along $[1, -2, 1]$. To do that, we divide the vector by its magnitude: $\mathbf{u} = \dfrac{[1, -2, 1]}{\|[1, -2, 1]\|}$

Calculating the magnitude: $\|[1, -2, 1]\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

$$\implies \mathbf{u} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$D_u[f](v) = \nabla f(1, 1, 1) \cdot \mathbf{u} = \begin{bmatrix} 3 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

Therefore, the directional derivative of $f(x, y, z)$ at $(1, 1, 1)$ in the direction of the unit vector along $[1, -2, 1]$ is $\dfrac{2}{\sqrt{6}}$.

---

12. The direction of steepest ascent for the function $2x + y^3 + 4z$ at the point $(1, 0, 1)$ is

A. $\left[\frac{2}{\sqrt{20}}, \quad 0 \quad \frac{4}{\sqrt{20}},\right]$

B. $\left[\frac{1}{\sqrt{29}}, \quad 0 \quad \frac{1}{\sqrt{29}},\right]$

C. $\left[\frac{-2}{\sqrt{29}}, \quad 0 \quad \frac{4}{\sqrt{29}},\right]$

D. $\left[\frac{2}{\sqrt{20}}, \quad 0 \quad \frac{-4}{\sqrt{20}},\right]$

Answer: A

Explanation:

Let $f(x, y, z) = 2x + y^3 + 4z$

$$\nabla f(x, y, z) = \begin{bmatrix} 2 \\ 3y^2 \\ 4 \end{bmatrix}$$

$$\implies \nabla f(1, 0, 1) = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

To obtain the direction of steepest ascent, we need to normalize the gradient vector.

The magnitude of the gradient vector is:

$$\|\nabla f(1, 0, 1)\| = \sqrt{2^2 + 0^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$$

Therefore, the direction of steepest ascent for the function $2x + y^3 + 4z$ at the point $(1, 0, 1)$ is $\left[\frac{2}{\sqrt{20}}, \quad 0 \quad \frac{4}{\sqrt{20}},\right]$

---

13. The directional derivative of $f(x, y, z) = x + y + z$ at $(-1, 1, 0)$ in the direction of unit vector along [1, -1, 1] is ________.

Answer: 0.577

Explanation: To find the directional derivative of $f(x, y, z) = x + y + z$ at $(-1, 1, 0)$ in the direction of the unit vector along $[1, -1, 1]$, we need to calculate the dot product of the gradient of f at that point with the unit vector.

$$\nabla f(x, y, z) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\implies \nabla f(-1, 1, 1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Next, let's find the unit vector along $[1, -1, 1]$. To do that, we divide the vector by its magnitude: $\mathbf{u} = \frac{[1, -1, 1]}{\|[1, -1, 1]\|}$

Calculating the magnitude: $\|[1,-1,1]\| = \sqrt{1^2+(-1)^2+1^2} = \sqrt{3}$

Therefore,

$$\mathbf{u} = \frac{1}{\sqrt{3}}[1,-1,1] = \left[\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$$

$D_u[f](v) = \nabla f(-1,1,0)\cdot\mathbf{u} = (1,1,1)\cdot\left[\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$

Therefore, the directional derivative of $f(x,y,z) = x+y+z$ at $(-1,1,0)$ in the direction of the unit vector along $[1,-1,1]$ is $\frac{1}{\sqrt{3}} \approx 0.577$.

---

14. Which of the following is the equation of the line passing through $(7,8,6)$ in the direction of vector $[1,2,3]$

    A. $[1,2,3]+\alpha[-6,-6,3]$

    B. $[7,8,9]+\alpha[-6,-6,3]$

    C. $[1,2,3]+\alpha[6,6,3]$

    D. $[7,8,6]+\alpha[6,6,3]$

    E. $[7,8,6]+\alpha[1,2,3]$

    F. $[1,2,3]+\alpha[7,8,6]$

Answer: E

Explanation: A line through the point $u \in \mathbb{R}^d$ along a vector $v \in \mathbb{R}^d$ is given by the equation

$$x = u + \alpha v$$

$$\implies x = [7,8,6] + \alpha[1,2,3]$$

So, option E is the answer.
