Week 6 Graded Assignment Solution

1.

$$f(x) = x^2 + y^2$$
$$f_x = \frac{\partial(x,y)}{\partial x} = 2x$$
$$f_y = \frac{\partial(x,y)}{\partial y} = 2x$$

Put $f_x = 0$ and $f_y = 0$,

$$f_x = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$$

$$f_y = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$$

Thus, stationary points $= (0,0)$

2.

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Here, $a = 4 > 0$ and $ac - b^2 = 4(2) - 2^2 = 4 > 0$
So, A is positive definite.

Now,

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Here, $a = 1 > 0$ and $ac - b^2 = 1(2) - 1^1 = 1 > 0$
So, B is positive definite.

$A + B = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ $a = 5 > 0$ and $ac - b^2 = 5(4) - 3^2 = 11 > 0$
So, $A + B$ is also positive definite.

3.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

The characteristic polynomial is

$$\lambda^3 - 6\lambda^2 + \{3 + 3 + 3\}\lambda - 4 = 0$$

FORMULA:
$x^3 - [trace(A)]x^2 + \Sigma[\text{Minors of diagonal elements(A)}]x - det(A) = 0$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

Solving we get $\lambda = 4, 1, 1$ Since all eigenvalues are greater than zero, A is positive definite.

4.

$$f(x, y) = 2x^2 + 2xy + 2y^2 - 6x$$
$$f_x = 4x + 2y - 6$$
$$f_y = 2x + 4y$$

For stationary point, $f_x = 0$ and $f_y = 0$

$$4x + 2y - 6 = 0 \Rightarrow 2x + y = 3$$
$$2x + 4y = 0 \Rightarrow x = -2y$$

Solving we get $x = 2$ and $y = -1$

Hence, the stationary point $= (2, -1)$

5.

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} ax + dy + gz & bx + ey + hz & cx + fy + iz \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$ax^2 + ey^2 + iz^2 + (b + d)xy + (g + c)xz + (f + h)yz$$

Compare this with $x^2 + y^2 - z^2 - xy + yz + xz$ we get
$a = 1, e = 1, i = -1, b + d = -1, c + g = 1, f + h = 1$

Only in options A and C, the diagonal elements are $1, 1, -1$

- If we check in option C, then $f + h = -1$ which is not satisfied. So, this is incorrect.
- If we check in option A, then all $b + d = -1, c + g = 1, f + h = 1$ are satisfied.
So, option A is correct.

6.

$$f(x,y) = 3x^2 + 4xy + 2y^2$$
$$f_x = 6x + 4y \Rightarrow f_{xx} = 6$$
$$f_y = 4x + 4y \Rightarrow f_{yy} = 4$$

Since $f_{xx} > 0$ and $f_{yy} > 0$, the point $(0,0)$ is a minima.

7.

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Here, $a = 4 > 0$ and $ac - b^2 = 4(3) - 2^2 = 8 > 0$
So, A is positive definite.

8.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Here, $a = 1 > 0$ and $ac - b^2 = 1(1) - 2 * 2 = -1 < 0$
Since $a > 0$ but $ac - b^2 < 0$, A is NOT positive definite.

9.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Since A is a diagonal matrix, the eigenvalues are $3, 5$ and 7.

Now since all eigenvalues are greater than 0, A is positive definite.

10.

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

The characteristics polynomial of AA^T:

$$\lambda^3 - 6\lambda^2 + 6\lambda = 0$$

Solving we get

$$\lambda = 0, 3 \pm \sqrt{3}$$

Now, $\sigma = \sqrt{\lambda}$
So,

$$\sigma_1 = \sqrt{3+\sqrt{3}}\, and\, \sigma_2 = \sqrt{3-\sqrt{3}}$$

are the non-zero singular values.

11.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The characteristics polynomial of AA^T:

$$\lambda^2 - 4\lambda + 4 = 0$$

Solving we get

$$\lambda = 2, 2$$

$$\sigma = \sqrt{\lambda} = \sqrt{2}$$

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}$$

Now for $\lambda = 2$,

$$(AA^T - 2I)X = 0$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$v = k \times \begin{bmatrix} x \\ y \end{bmatrix}$$

for $x = 1$, $y = 0$; $v_1 = k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

for $x = 0$, $y = 1$; $v_2 = k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$y_1 = \frac{A^T x_1}{\sigma_1}$$

$$y_1 = \frac{1}{\sqrt{2}} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

Similarly,

$$y_2 = \frac{1}{\sqrt{2}} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

For the other two

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0$$

and

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

From above we have

$$a + c = 0 \, and \, b + d = 0$$

Let $c = k_1$ and $d = k_2$

$$k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} - \\ -1 \\ 0 \\ 1 \end{bmatrix} Normalizing \rightarrow \frac{1}{\sqrt{2}} k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} ; \frac{1}{\sqrt{2}} k_2 \begin{bmatrix} - \\ -1 \\ 0 \\ 1 \end{bmatrix} Now,$$

$$y_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} for \, k_1 = -1 \, and \, y_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} for \, k_2 = -1$$

So,

$$Q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$Q_2^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$