

Overview of the “Barrier Approach”  
to lower the upper bound of the  
*de Bruijn-Newman* constant.

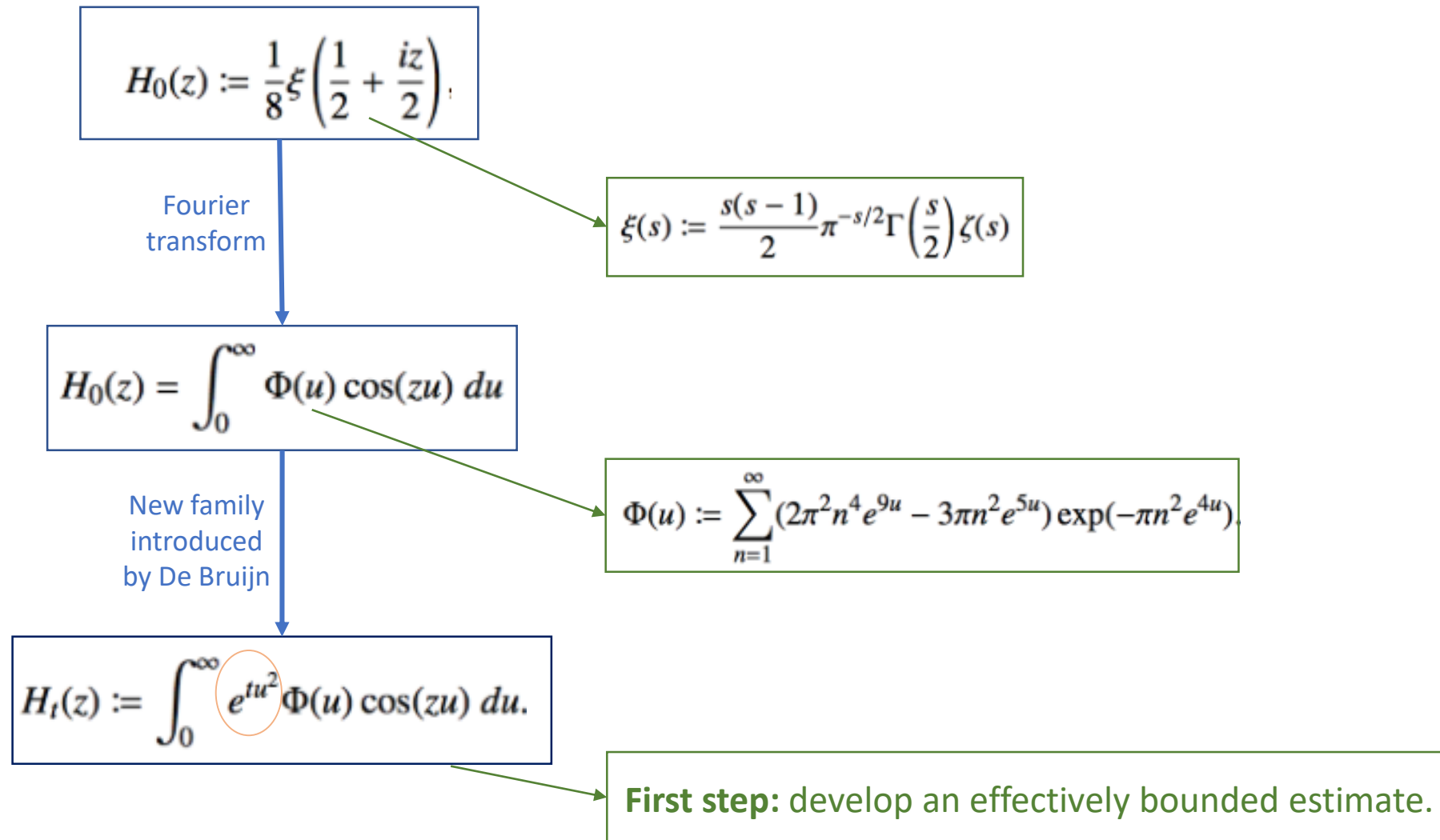
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## High level storyline

- The basic De Bruijn idea leading to the function  $H_t(x+iy)$ .
- How to effectively bound a good estimate for  $H_t$ ?
- Some observations on the zeros of  $H_t$ .
- How could zeros of  $H_t$  be 'blocked' to lower the  $\Lambda$  upper bound.
- The key ideas behind the “Barrier approach”.
- How to ensure no zeros have passed the Barrier?
- How to show that  $H_t$  doesn't vanish from the Barrier to  $N_b$  ?
- Optimizations used to improve computation speed.
- Numerical results showing  $\Lambda < 0.19$  (0.14 conditionally on  $RH < 10^{17}$ ).
- Software used, optimizations, and detailed results available.

# Basic idea by De Bruijn



# Estimating and effectively bounding $H_t(x+iy)$

Main estimate

$$A(x+iy) := M_t \left( \frac{1-y+ix}{2} \right) \sum_{n=1}^N \frac{b'_n}{n^{\frac{1-y+ix}{2} + \frac{t}{2} \alpha \left( \frac{1-y+ix}{2} \right)}}$$

$$B(x+iy) := M_t \left( \frac{1+y-ix}{2} \right) \sum_{n=1}^N \frac{b'_n}{n^{\frac{1+y-ix}{2} + \frac{t}{2} \alpha \left( \frac{1+y-ix}{2} \right)}}$$

Optionally: a more effective C-term is available

Designed for:  
 $0 < t \leq \frac{1}{2}; \quad 0 \leq y \leq 1; \quad x \geq 200.$

Error terms

$$E_A(x+iy) := \left| M_t \left( \frac{1-y+ix}{2} \right) \right| \sum_{n=1}^N \frac{b'_n}{n^{\frac{1-y}{2} + \frac{t}{2} \operatorname{Re} \alpha \left( \frac{1-y+ix}{2} \right)}} \varepsilon_{t,n} \left( \frac{1-y+ix}{2} \right)$$

$$E_B(x+iy) := \left| M_t \left( \frac{1+y+ix}{2} \right) \right| \sum_{n=1}^N \frac{b'_n}{n^{\frac{1+y}{2} + \frac{t}{2} \operatorname{Re} \alpha \left( \frac{1+y+ix}{2} \right)}} \varepsilon_{t,n} \left( \frac{1+y+ix}{2} \right)$$

$$E_{C,0}(x+iy) := \exp \left( \frac{t\pi^2}{64} \right) |M_0(iT')| \left( 1 + \tilde{\varepsilon} \left( \frac{1-y+ix}{2} \right) + \tilde{\varepsilon} \left( \frac{1+y+ix}{2} \right) \right).$$

$$H_t(x+iy) = A(x+iy) + B(x+iy) + O_{\leq}(E_A(x+iy) + E_B(x+iy) + E_{C,0}(x+iy))$$

Normalize by  $B_0$  and bound effectively

$$\frac{H_t(x+iy)}{B_0(x+iy)} = \sum_{n=1}^N \frac{b'_n}{n^{s_*}} + \gamma \sum_{n=1}^N n^y \frac{b'_n}{n^{\overline{s_*} + \kappa}} + O_{\leq}(e_A + e_B + e_{C,0})$$

Main estimate lower bound

(triangle)  $|f_t(x+iy)| \geq 1 - |\gamma| - \sum_{n=2}^N \frac{b'_n}{n^{\sigma}} (1 + |\gamma| n^{y - \operatorname{Re}(\kappa)}),$

(lemma)  $|f_t(x+iy)| \geq \left( \left| \sum_{n=1}^N \frac{b'_n}{n^{s_*}} \right| - \left| \sum_{n=1}^N \frac{|\gamma| b'_n n^y}{n^{s_*}} \right| \right)_+ - |\gamma| \sum_{n=1}^N \frac{b'_n (n^{|k|} - 1)}{n^{\sigma - y}}.$

Error upper bounds

$$e_A + e_B \leq \sum_{n=1}^N (1 + |\gamma| N^{|k|} n^y) \frac{b'_n}{n^{\operatorname{Re}(s_*)}} \left( \exp \left( \frac{\frac{t^2}{16} \log^2 \frac{x}{4\pi n^2} + 0.626}{x - 6.66} \right) - 1 \right)$$

$$e_{C,0} \leq \left( \frac{x}{4\pi} \right)^{-\frac{1+y}{4}} \exp \left( -\frac{t}{16} \log^2 \frac{x}{4\pi} + \frac{1.24 \times (3^y + 3^{-y})}{N - 0.125} + \frac{3 |\log \frac{x}{4\pi} + i \frac{\pi}{2}| + 10.44}{x - 8.52} \right)$$

Hence,

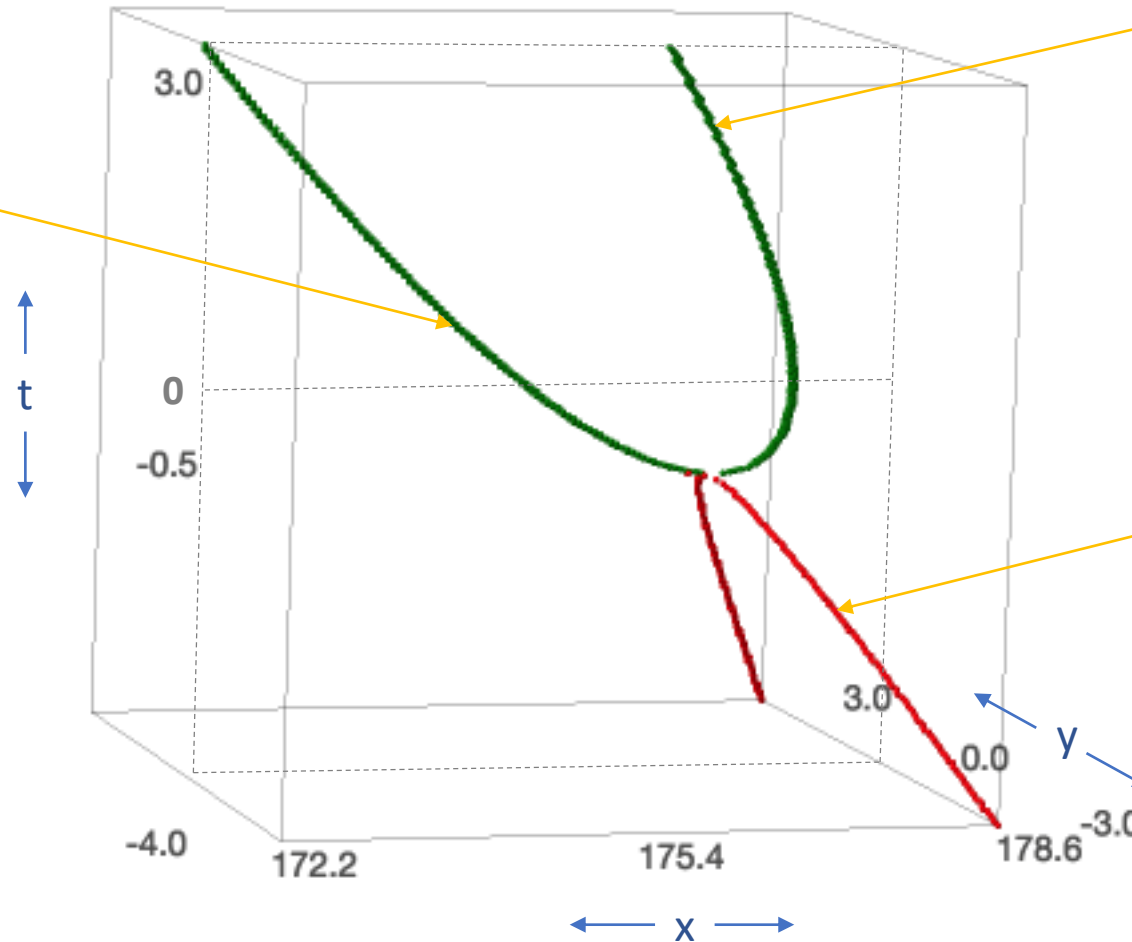
If Lower bound  $\geq$  Upper bound then  $H_t(x+iy) \neq 0$

$$\sum_{d=1}^D \frac{\lambda_d}{d^s} = \prod_{p \leq P} \left( 1 - \frac{b'_p}{p^s} \right)$$

Choice of 'Euler mollifiers'

# Real example of trajectories of real and complex zeros of $H_t(x+iy)$

Zeros get denser as one moves away from the origin, so there are more zeros to the right of  $x_n$  than to the left, hence their trajectories “lean” leftwards.



Once a zero becomes real, it stays real forever and ends up roughly equally spaced with:

$$z_{j+1}(t) - z_j(t) = (1 + o(1)) \frac{4\pi}{\log z_j(t)}$$

The complex parts of zeros attract each other and the real parts repel each other. From isolating the imaginary “force”, it can be derived that all complex zeroes will be forced into the real axis in a finite time leading to the bound:

$$\Lambda \leq t + \frac{1}{2} \sigma_{\max}^2(t)$$

— trajectory of a complex zero    — trajectory of a real zero

# The De Bruijn – Newman $\Lambda$ and a ‘ceiling’ the complex zeroes can’t cross

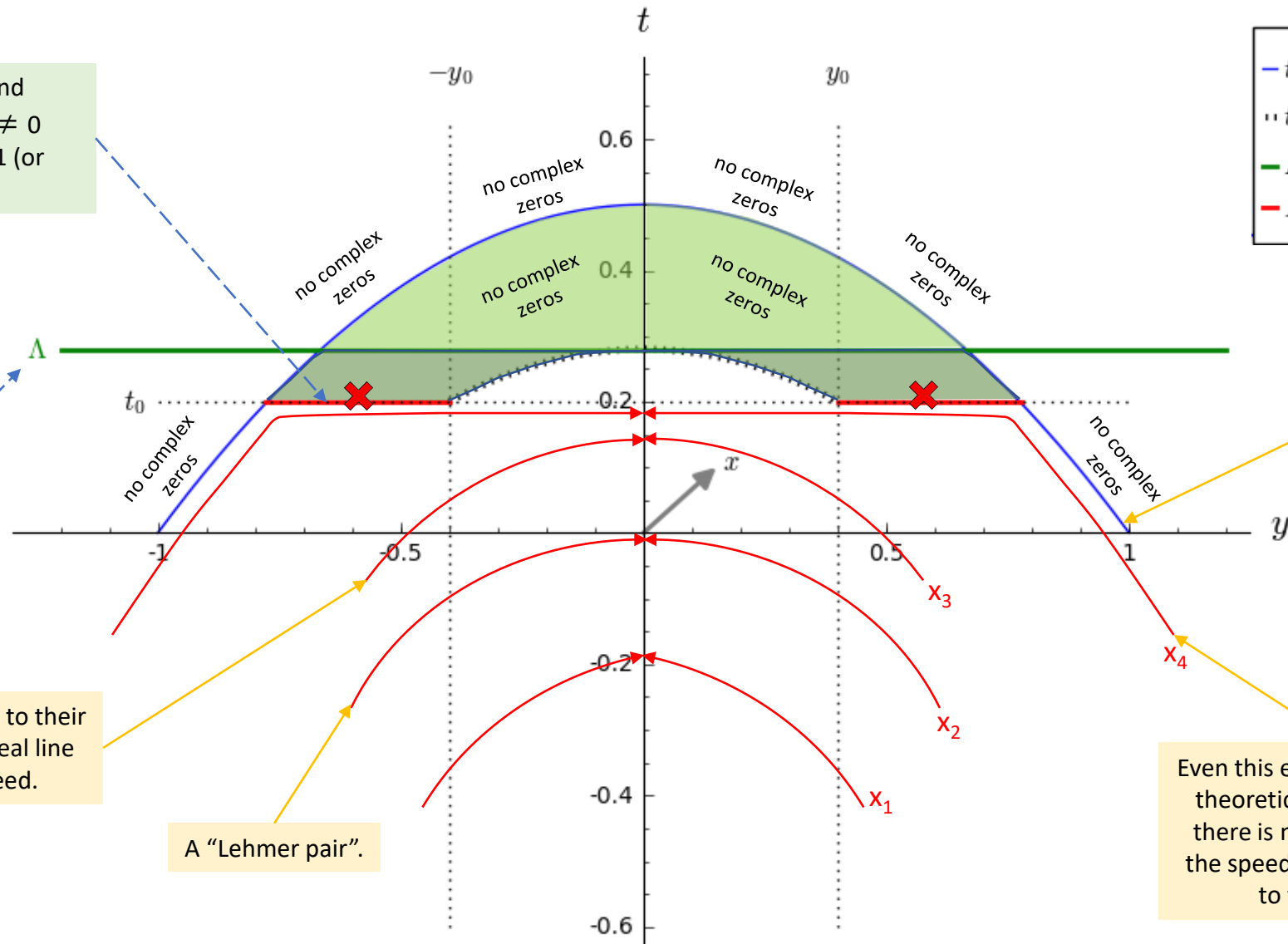
1. Introduce a ‘ceiling’ and verify that  $H_{t_0}(x+iy) \neq 0$  for  $x = 0 \dots \infty, y = y_0 \dots 1$  (or the blue hyperbola).

2. If so, then the new upper bound:  
 $\Lambda \leq t_0 + 0.5 y_0^2$   
 has been established.

Complex zeros are “attracted” to their conjugates and “fall” to the real line with a lower bounded speed.

A “Lehmer pair”.

Even this extreme trajectory is theoretically possible since there is no upper bound on the speed by which zeros fall to the real line.

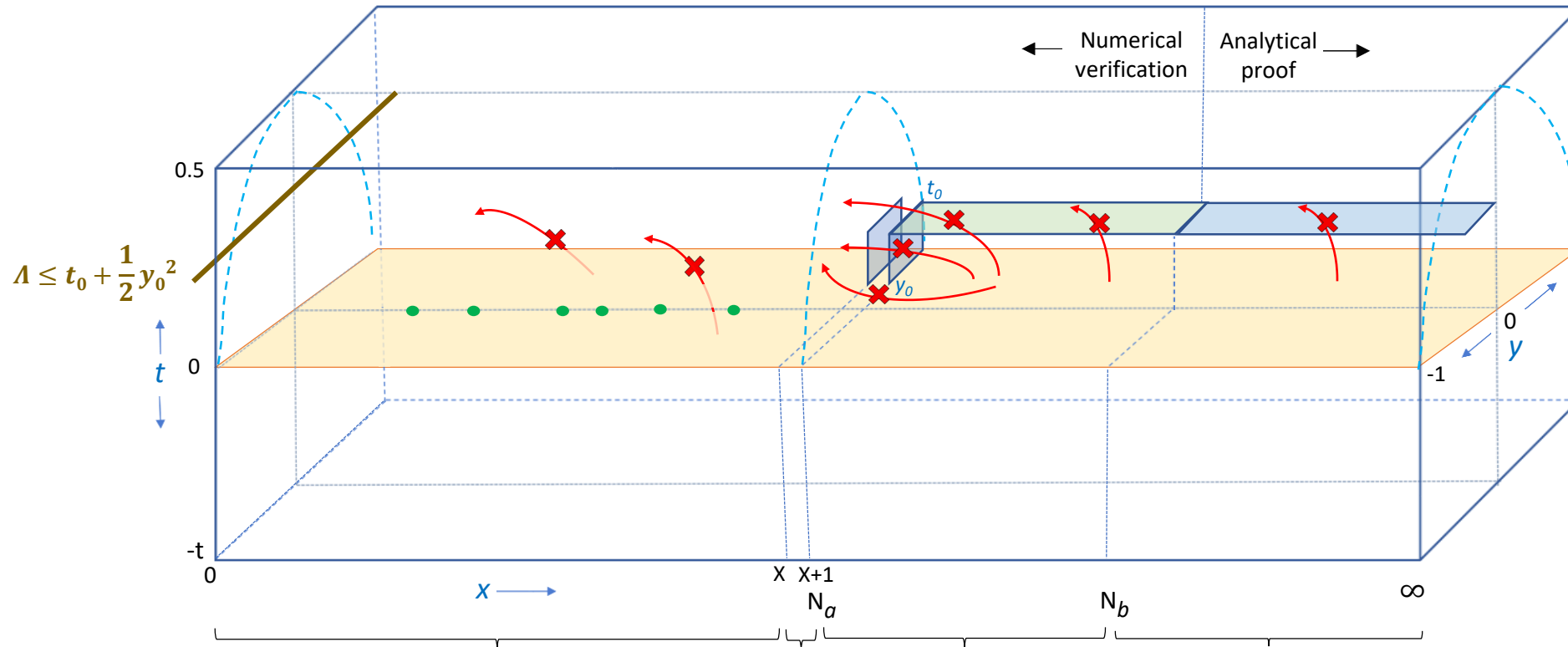


$$\begin{aligned} & \text{--- } t = \frac{(1-y^2)}{2} \\ & \text{--- } t = \Lambda - \frac{y^2}{2} \\ & \text{--- } \Lambda = t_0 + \frac{y_0^2}{2} \\ & \text{--- } H_{t_0}(x+iy) \neq 0, x = 0 \dots \infty \end{aligned}$$

The blue DbN hyperbola is only valid for  $t \geq 0$ .

→ Possible trajectory of a complex zero ( $H_t(x+iy) = 0$ )

“Barrier” approach to assure  $H_t(x+iy) \neq 0$  for a certain  $y > y_0$ ,  $t_0$ .



1. Area where the RH has been verified e.g.  $6 \times 10^{10}$  certain,  $10^{13}$  to be confirmed. Or assume that it has been verified up to X.

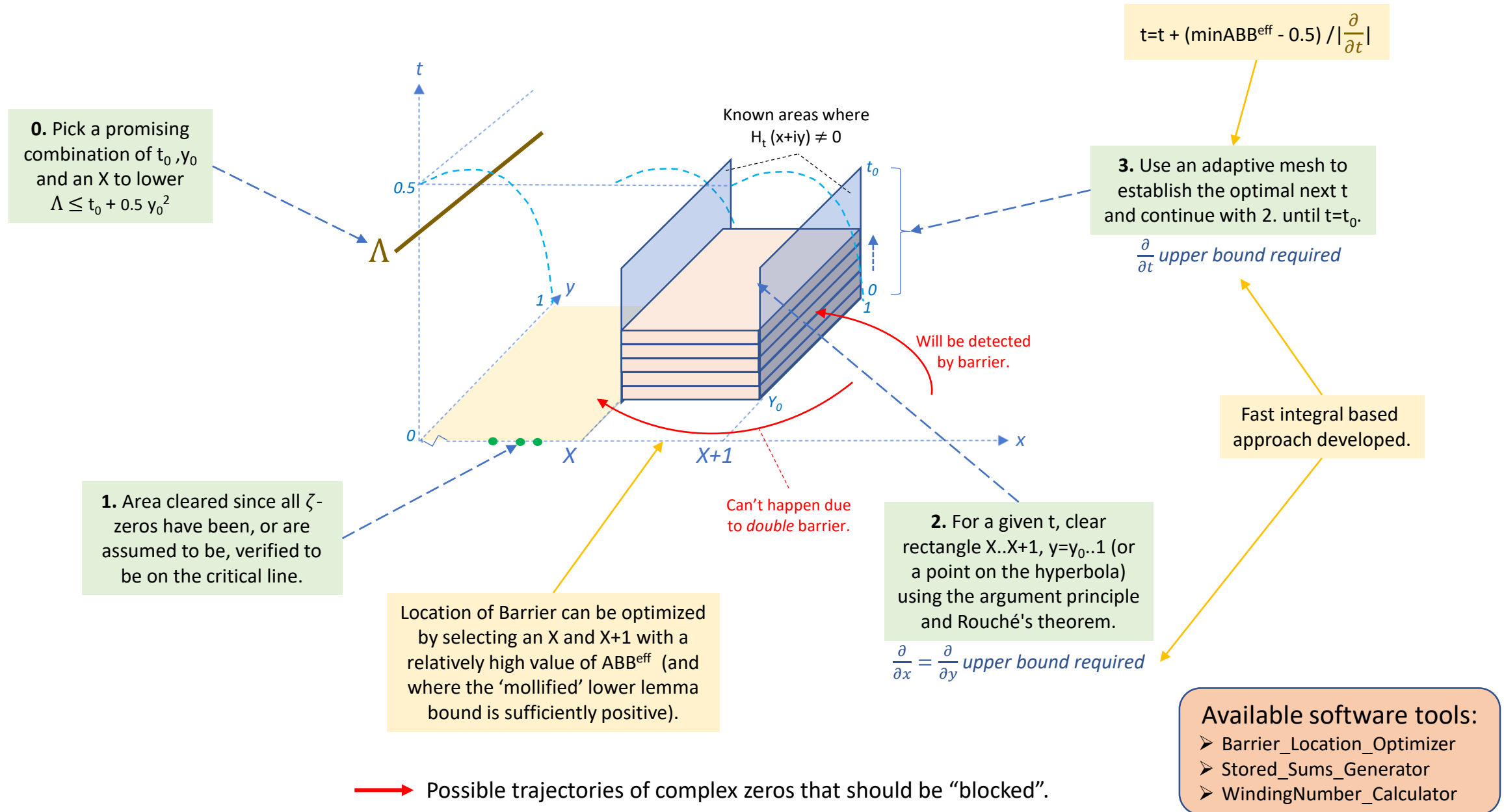
3. Verify  $H_{t_0}(x+iy_0) \neq 0$   
i.e. Lemma lower bound  
> Error upper bound.

4. Analytical proof that  $H_{t_0}(x+iy_0) \neq 0$

2. Verify  $H_t(x+iy) \neq 0$   
in the Barrier area  
 $x=X..X+1, y=y_0..1, t=0..t_0$

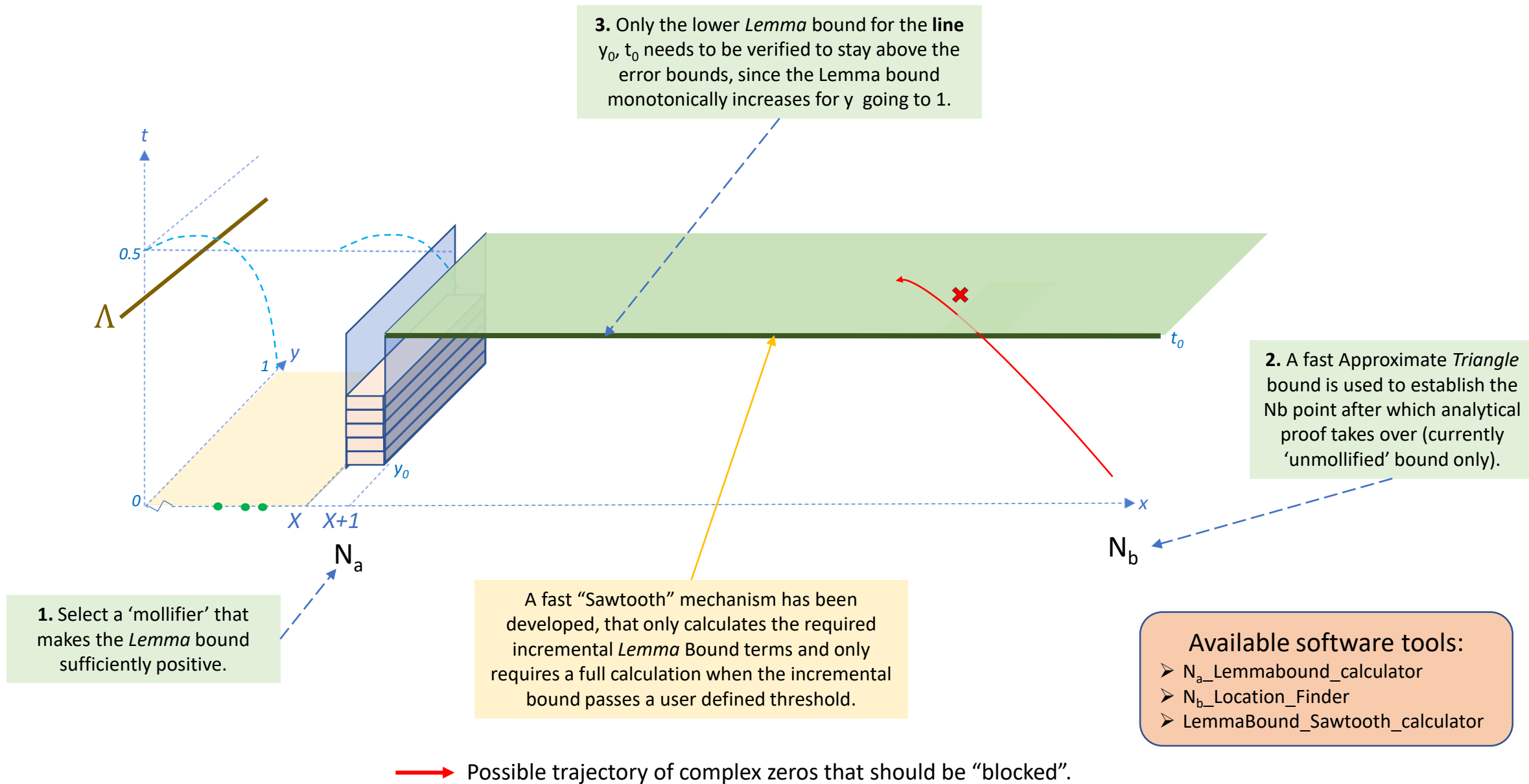
→ Possible trajectories of complex zeros that should be “blocked”.

# “Barrier” approach: how to clear the barrier?



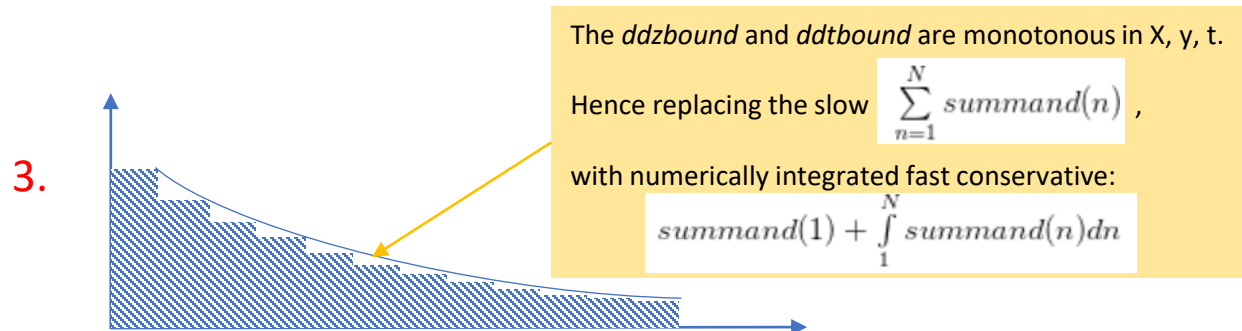
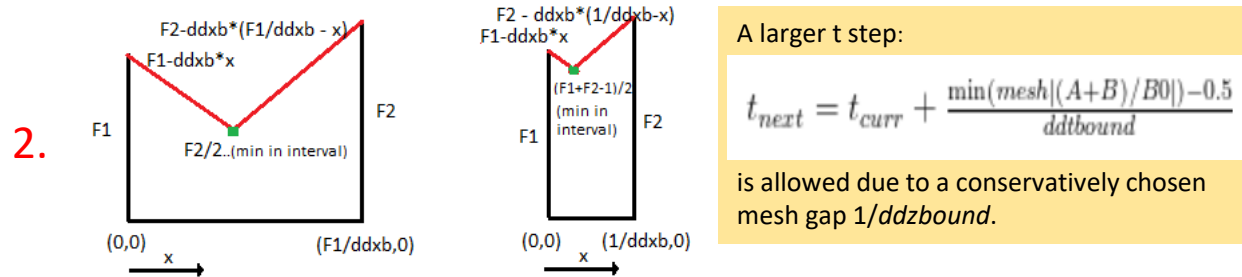
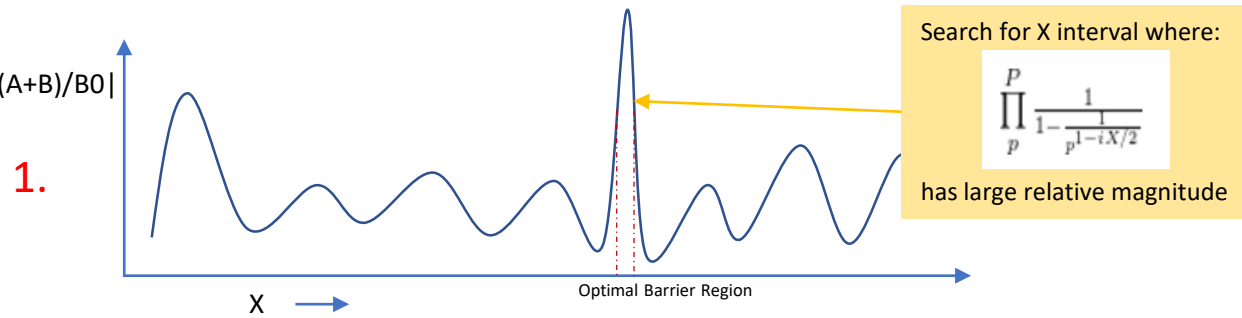


# “Barrier” approach: how to verify the area from the barrier up to $N_b$ ?

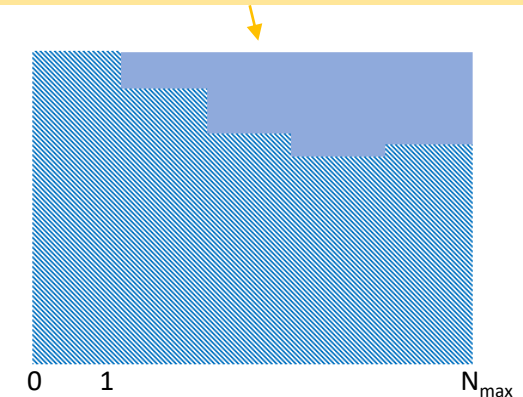
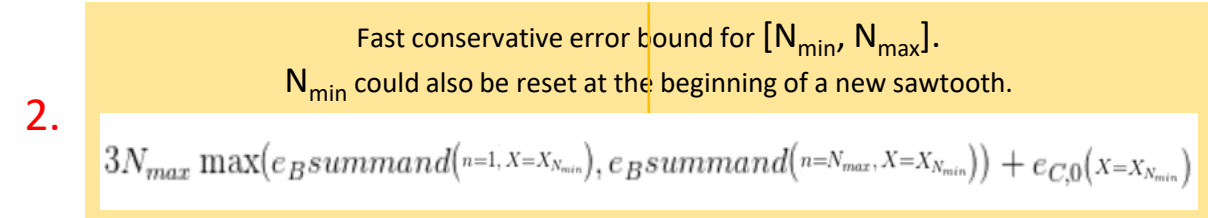
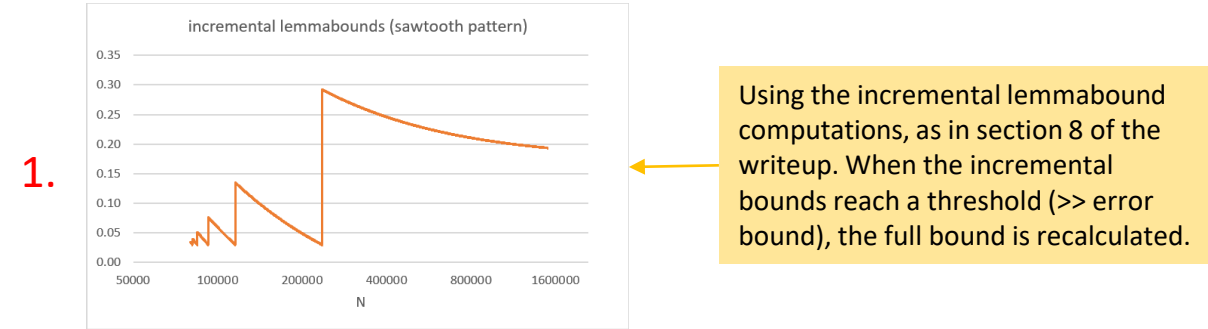


# Optimizations made to improve computation speed

## Barrier Optimizations



## Euler Bounds Optimizations



# The Barrier model in action: some real numbers (wip)

X-location where  $\min_{\text{mesh}} |(A+B)/B_0|$  starts between 4 and 5 at  $t=0$  and ends near 1.5 at  $t=t_0$ .

Selected with  
LemmaBound utility

Selected with  
 $N_b$  Location finder

x	Barrier offset	RH verified?	$t_0$	$y_0$	$\Lambda$	Winding number	mollifier primes	Lemma bound value	Na	Triangle bound value	Nb	
6.00E+10	155019	yes	0.20	0.20	0.22	0	(2,3,5,7)	0.067	69098	0.077	1.7E+06	✓
1.00E+11	78031	yes	0.19	0.20	0.21	0	(2,3,5,7)	0.067	89206	0.081	6.0E+06	✓
1.00E+12	46880	yes	0.18	0.20	0.20	0	(2,3,5)	0.135	282094	0.089	1.3E+07	✓
5.00E+12	194858	yes <sup>1)</sup>	0.17	0.20	0.19	0	(2,3,5)	0.180	630783	0.116	1.5E+07	✓
1.00E+13	9995	not yet	0.16	0.20	0.18	0	(2,3,5)	0.109	892062	0.091	3.0E+07	✓
1.00E+14	2659	not yet	0.15	0.20	0.17	0	(2,3,5)	0.195	2820947	0.076	7.0E+07	✓
1.00E+15	21104	not yet	0.14	0.20	0.16	0	(2,3,5)	0.251	8920620	0.073	2.0E+08	✓
1.00E+16	172302	not yet	0.13	0.20	0.15	0	(2,3,5)	0.278	28209479	0.077	7.0E+08	✓
1.00E+17	31656	not yet	0.12	0.20	0.14	0	(2,3,5)	0.279	89206205	0.080	3.0E+09	✓
1.00E+18	44592	not yet	0.11	0.20	0.13	0	(2,3)	0.207	282094791	0.103	2.0E+10	
1.00E+19	12010	not yet	0.10	0.20	0.12	tbd	(2,3)	0.128	892062059	0.097	1.5E+11	
1.00E+20	37221	not yet	0.09	0.20	0.11	tbd	(2,3,5)	0.037	2820947918	0.075	1.5E+12	

1) Gourdon-Demichel 2004

# Software used and useful links

All software was developed in two languages and all results were reconciled:

- Symbolic math programming language - **pari/gp** (<https://pari.math.u-bordeaux.fr>)
  - Short development time
  - Relatively fast
- Arithmetic Balls C-based library - **Arb** (<http://arblib.org>)
  - Longer development time
  - Very fast (up to 20 x pari/gp)

All software and results are free to use (under the LGPL-terms) and can be found here:

[https://github.com/km-git-acc/dbn\\_upper\\_bound](https://github.com/km-git-acc/dbn_upper_bound)