

Overview of the “Barrier Approach”
to lower the upper bound of the
de Bruijn-Newman constant.

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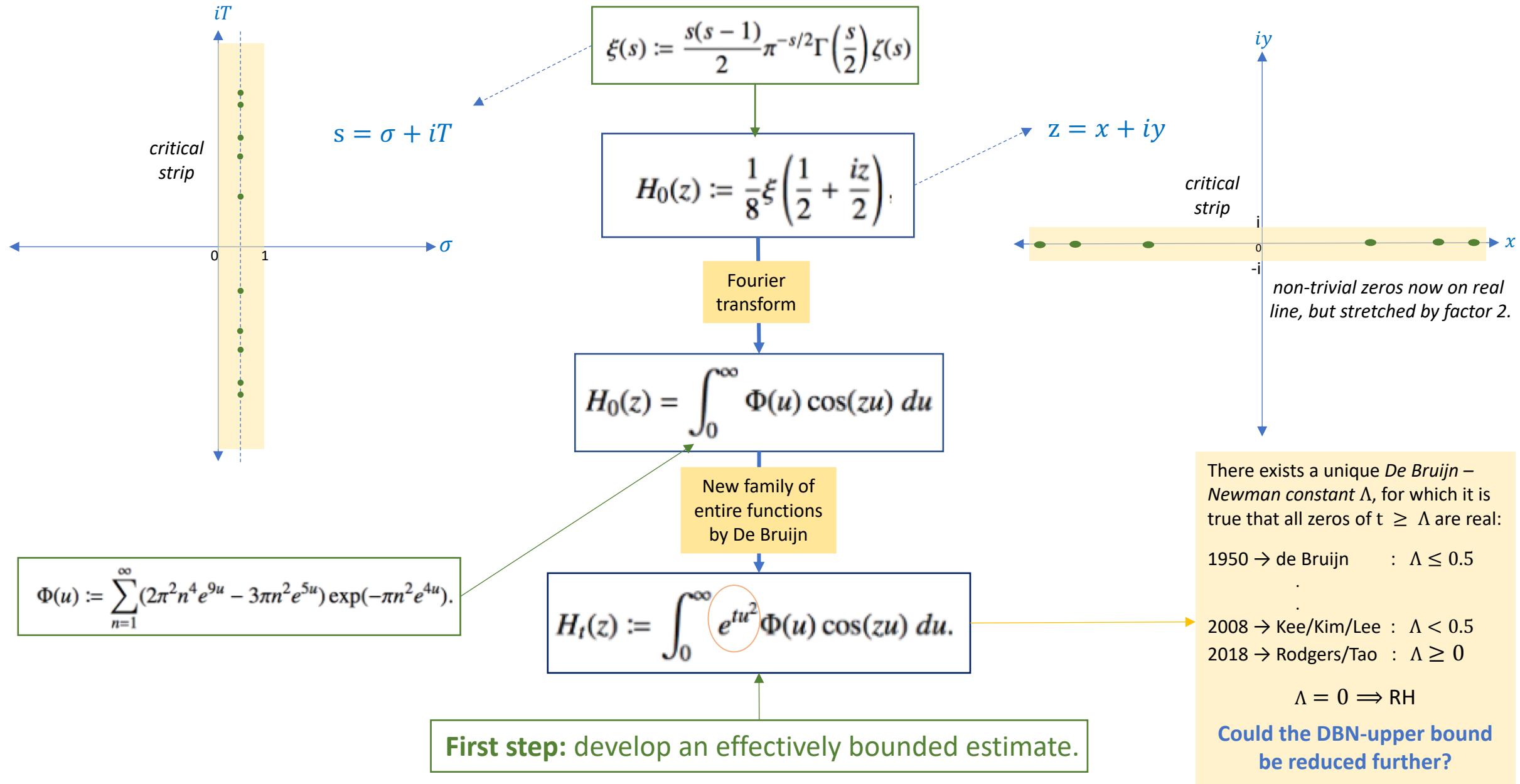
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v1.7

High level storyline

- The basic De Bruijn-Newman idea and the function $H_t(x+iy)$.
- How to effectively bound a good estimate for H_t ?
- Some observations on the zeros of H_t .
- How could zeros of H_t be 'blocked' to lower the Λ upper bound.
- The key ideas behind the “Barrier approach”.
- How to ensure no zeros have passed the Barrier?
- How to show that H_t doesn't vanish from the Barrier to N_b ?
- Optimizations used to improve computation speed.
- Numerical results showing $\Lambda < 0.22$ (and $\Lambda < 0.12$ conditionally on $\text{RH} < 10^{19}$).
- Software used, optimizations, and detailed results available.

The idea behind the De Bruijn-Newman constant



Estimating and effectively bounding $H_t(x+iy)$

Main estimate

$$A(x+iy) := M_t\left(\frac{1-y+ix}{2}\right) \sum_{n=1}^N \frac{b_n^t}{n^{\frac{1-y+ix}{2} + \frac{t}{2}\alpha(\frac{1-y+ix}{2})}}$$

$$B(x+iy) := M_t\left(\frac{1+y-ix}{2}\right) \sum_{n=1}^N \frac{b_n^t}{n^{\frac{1+y-ix}{2} + \frac{t}{2}\alpha(\frac{1+y-ix}{2})}}$$

Optionally: a more effective C-term is available



Designed for:

$$0 < t \leq \frac{1}{2}; \quad 0 \leq y \leq 1; \quad x \geq 200.$$

Error terms

$$E_A(x+iy) := |M_t\left(\frac{1-y+ix}{2}\right)| \sum_{n=1}^N \frac{b_n^t}{n^{\frac{1-y}{2} + \frac{t}{2}\operatorname{Re}\alpha(\frac{1-y+ix}{2})}} \varepsilon_{t,n}\left(\frac{1-y+ix}{2}\right)$$

$$E_B(x+iy) := |M_t\left(\frac{1+y-ix}{2}\right)| \sum_{n=1}^N \frac{b_n^t}{n^{\frac{1+y}{2} + \frac{t}{2}\operatorname{Re}\alpha(\frac{1+y-ix}{2})}} \varepsilon_{t,n}\left(\frac{1+y-ix}{2}\right)$$

$$E_{C,0}(x+iy) := \exp\left(\frac{t\pi^2}{64}\right) |M_0(it')| (1 + \tilde{\varepsilon}\left(\frac{1-y+ix}{2}\right) + \tilde{\varepsilon}\left(\frac{1+y-ix}{2}\right)).$$

$$H_t(x+iy) = A(x+iy) + B(x+iy) + O_{\leq}(E_A(x+iy) + E_B(x+iy) + E_{C,0}(x+iy))$$

Normalize by B_0 and bound effectively

$$\frac{H_t(x+iy)}{B_0(x+iy)} = \sum_{n=1}^N \frac{b_n^t}{n^{s_*}} + \gamma \sum_{n=1}^N n^y \frac{b_n^t}{n^{\bar{s}_*+\kappa}} + O_{\leq}(e_A + e_B + e_{C,0})$$

Main estimate lower bound

(triangle) $|f_t(x+iy)| \geq 1 - |\gamma| - \sum_{n=2}^N \frac{b_n^t}{n^\sigma} (1 + |\gamma| n^{y-\operatorname{Re}(\kappa)})$,

(lemma) $|f_t(x+iy)| \geq \left(\left| \sum_{n=1}^N \frac{b_n^t}{n^{s_*}} \right| - \left| \sum_{n=1}^N \frac{|\gamma| b_n^t n^y}{n^{s_*}} \right| \right)_+ - |\gamma| \sum_{n=1}^N \frac{b_n^t (n^{|\kappa|} - 1)}{n^{\sigma-y}}$.

Hence,

Error upper bounds

$$e_A + e_B \leq \sum_{n=1}^N (1 + |\gamma| N^{|\kappa|} n^y) \frac{b_n^t}{n^{\operatorname{Re}(s_*)}} \left(\exp\left(\frac{\frac{t^2}{16} \log^2 \frac{x}{4\pi n^2} + 0.626}{x - 6.66}\right) - 1 \right)$$

$$e_{C,0} \leq \left(\frac{x}{4\pi}\right)^{-\frac{1+y}{4}} \exp\left(-\frac{t}{16} \log^2 \frac{x}{4\pi} + \frac{1.24 \times (3^y + 3^{-y})}{N - 0.125} + \frac{3|\log \frac{x}{4\pi} + i\frac{\pi}{2}| + 10.44}{x - 8.52}\right)$$

$$\sum_{d=1}^D \frac{\lambda_d}{d^s} = \prod_{p \leq P} \left(1 - \frac{b_p^t}{p^s}\right)$$

Choice of 'Euler mollifiers'

If Lower bound \geq Upper bound then $H_t(x+iy) \neq 0$

Actual trajectories of real and complex zeros of $H_t(x+iy)$

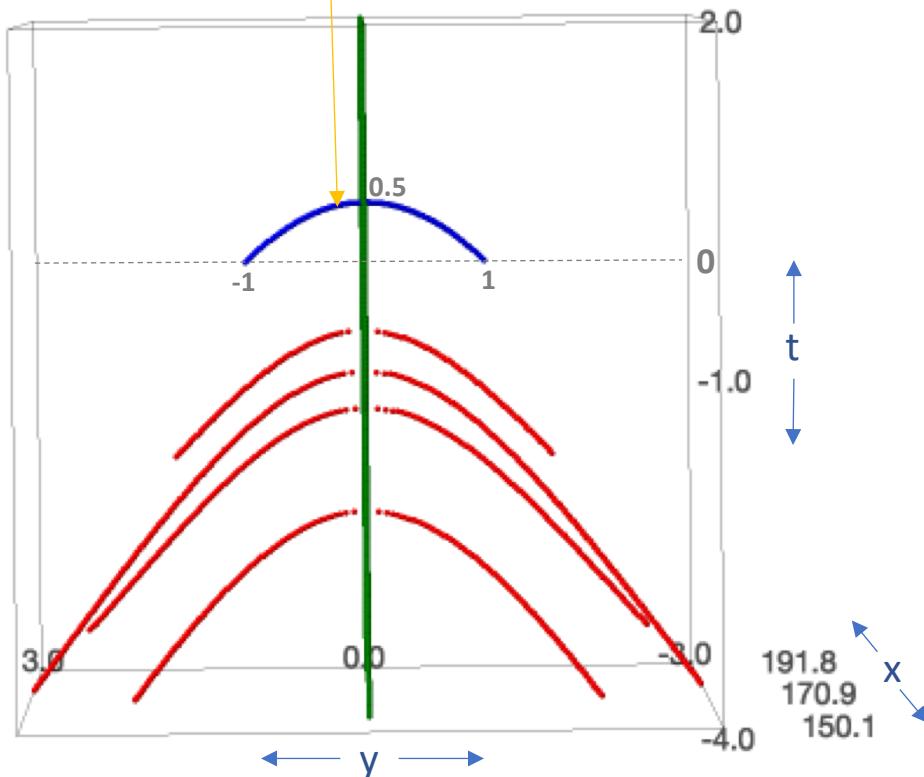
The complex parts of zeros attract each other and the real parts repel each other. From isolating the imaginary "force", it can be derived that all complex zeroes will be forced into the real axis in a finite time leading to the bound:

$$\Lambda \leq t + \frac{1}{2}\sigma_{\max}^2(t)$$

Zeroes get denser as one moves away from the origin, so there are more zeros to the right of x_n than to the left, hence their trajectories typically "lean" leftwards.

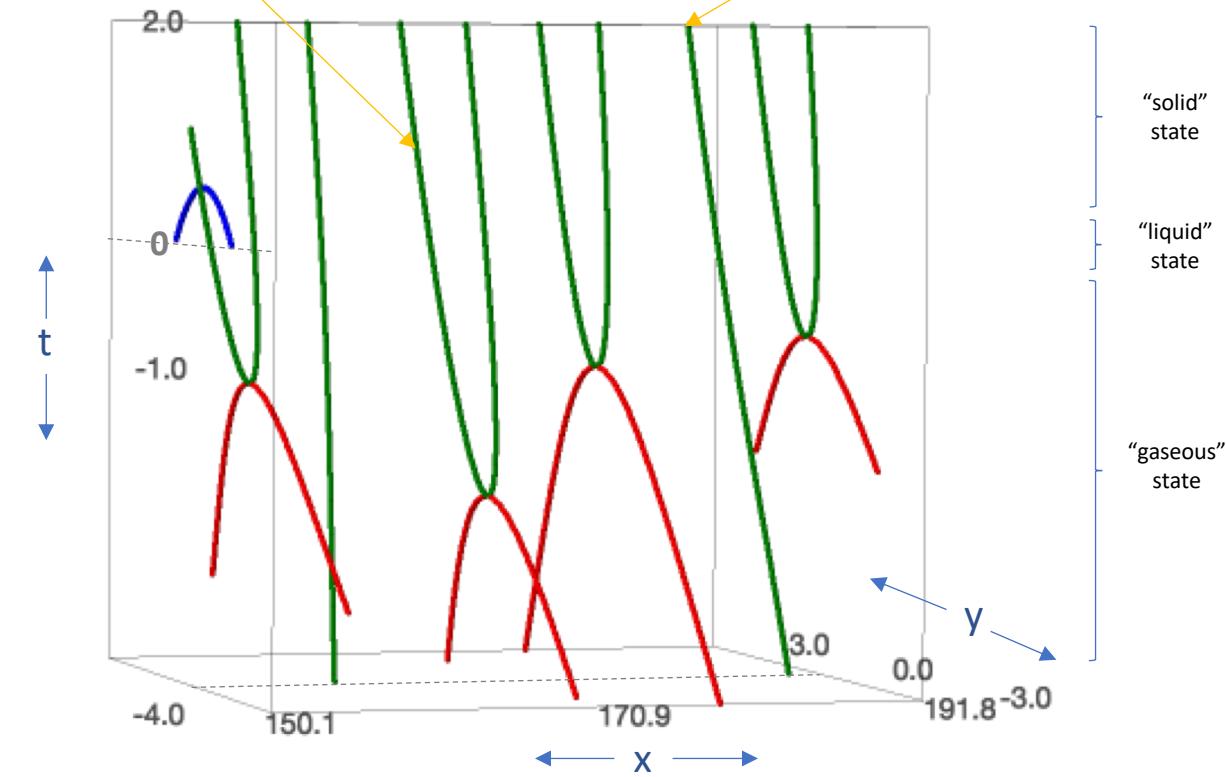
Once a zero becomes real, it stays real forever and ends up roughly equally spaced with:

$$z_{j+1}(t) - z_j(t) = (1 + o(1)) \frac{4\pi}{\log z_j(t)}$$



— trajectory of a complex zero

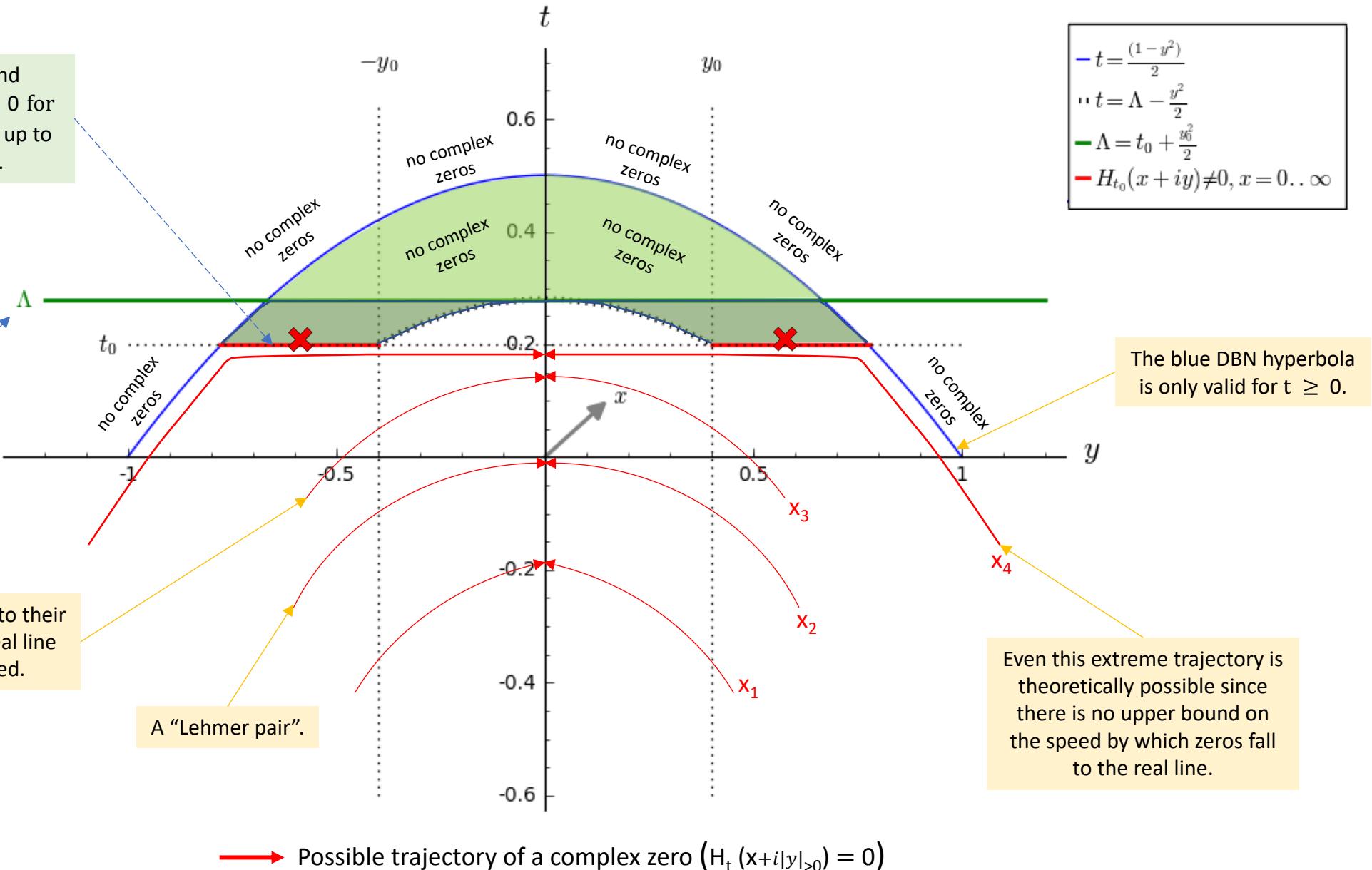
— trajectory of a real zero



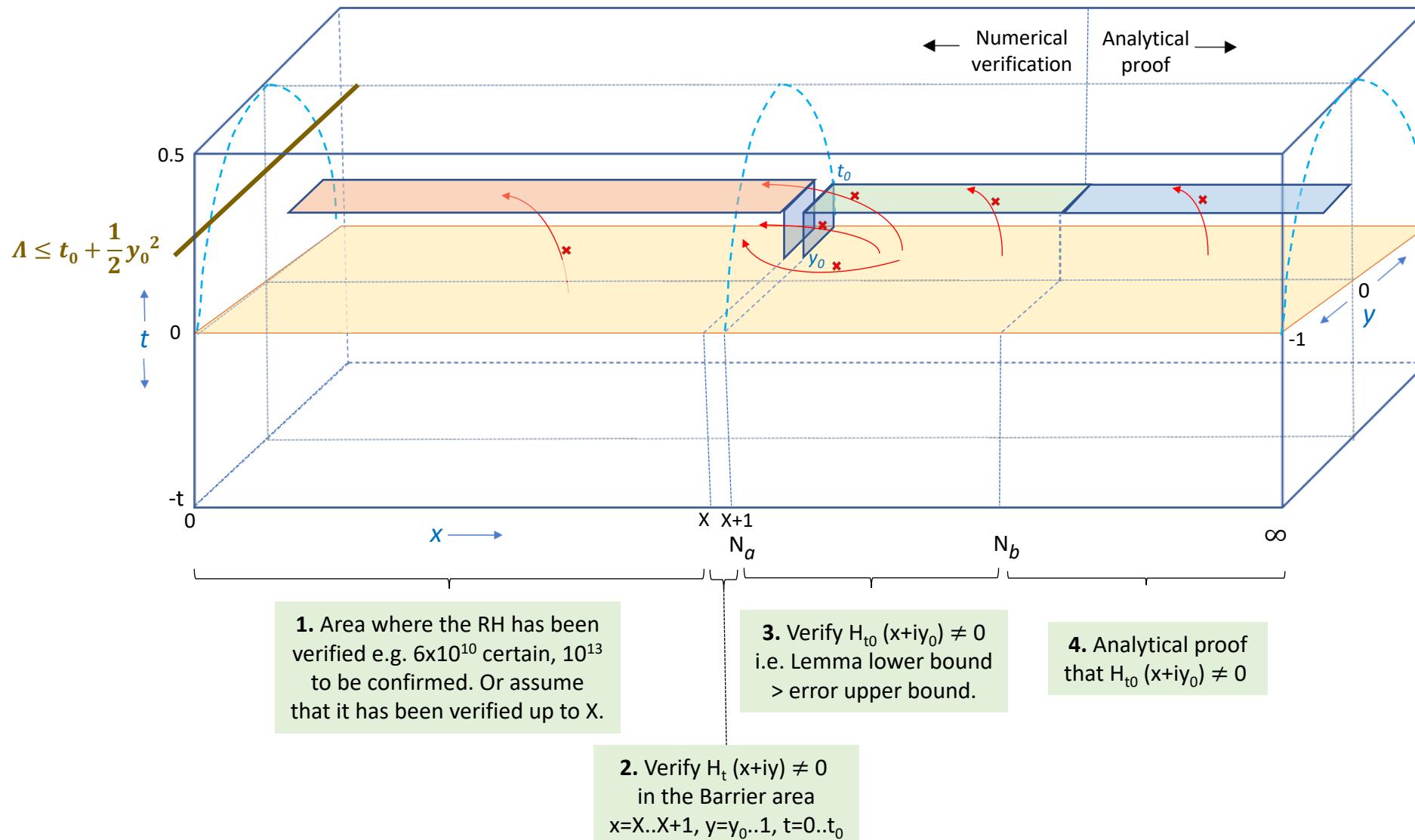
The De Bruijn – Newman Λ : introducing a ‘canopy’ the complex zeroes can’t cross

1. Introduce a ‘canopy’ and verify that $H_{t_0}(x+iy) \neq 0$ for $x = 0 \dots \infty, y = y_0 \dots 1$ (or up to the blue DBN-hyperbola).

2. If so, then the new upper bound:
 $\Lambda \leq t_0 + 0.5 y_0^2$
 has been established.

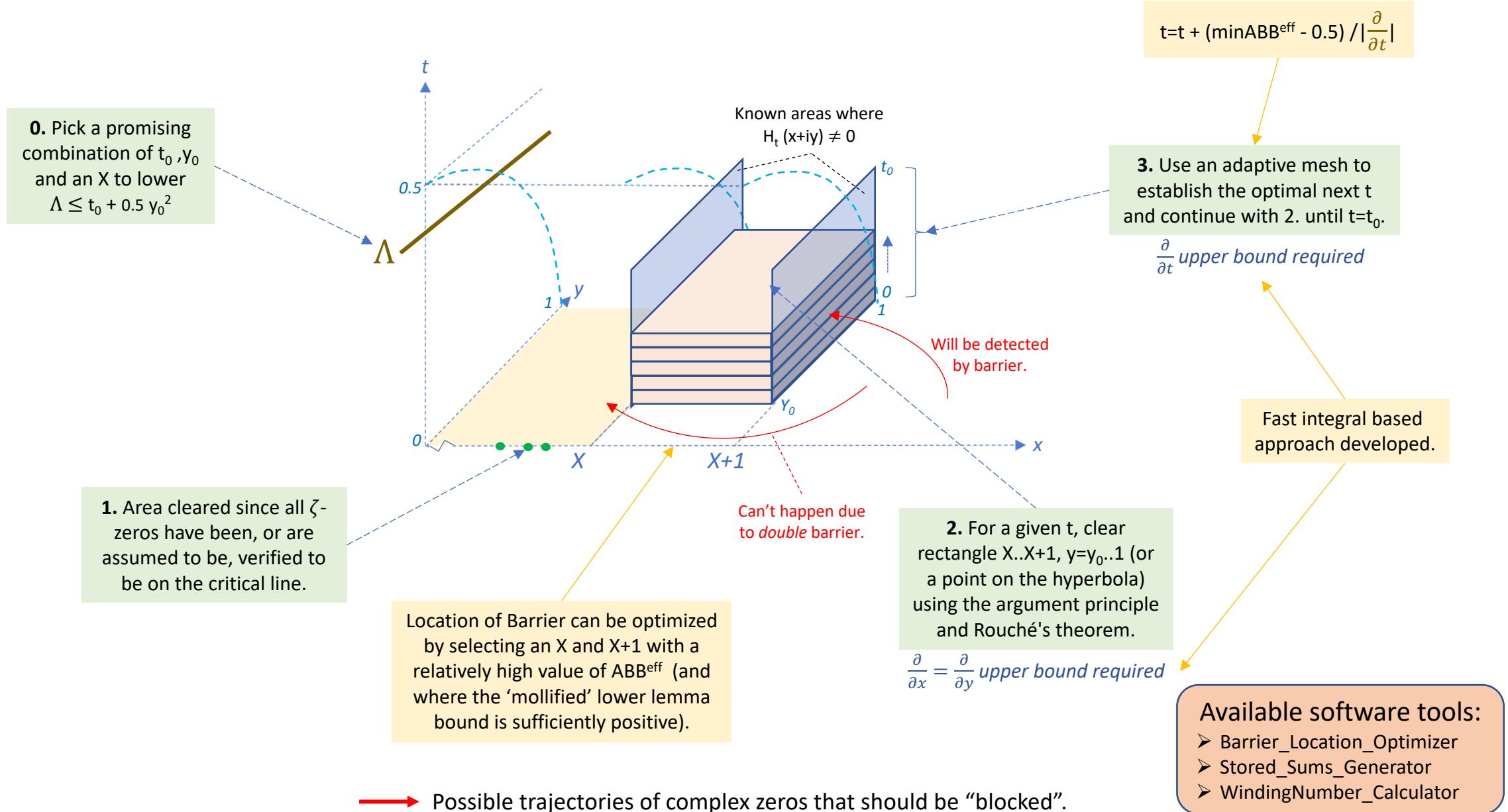


“Barrier” approach to assure $H_t(x+iy) \neq 0$ for a certain $y > y_0, t_0$.

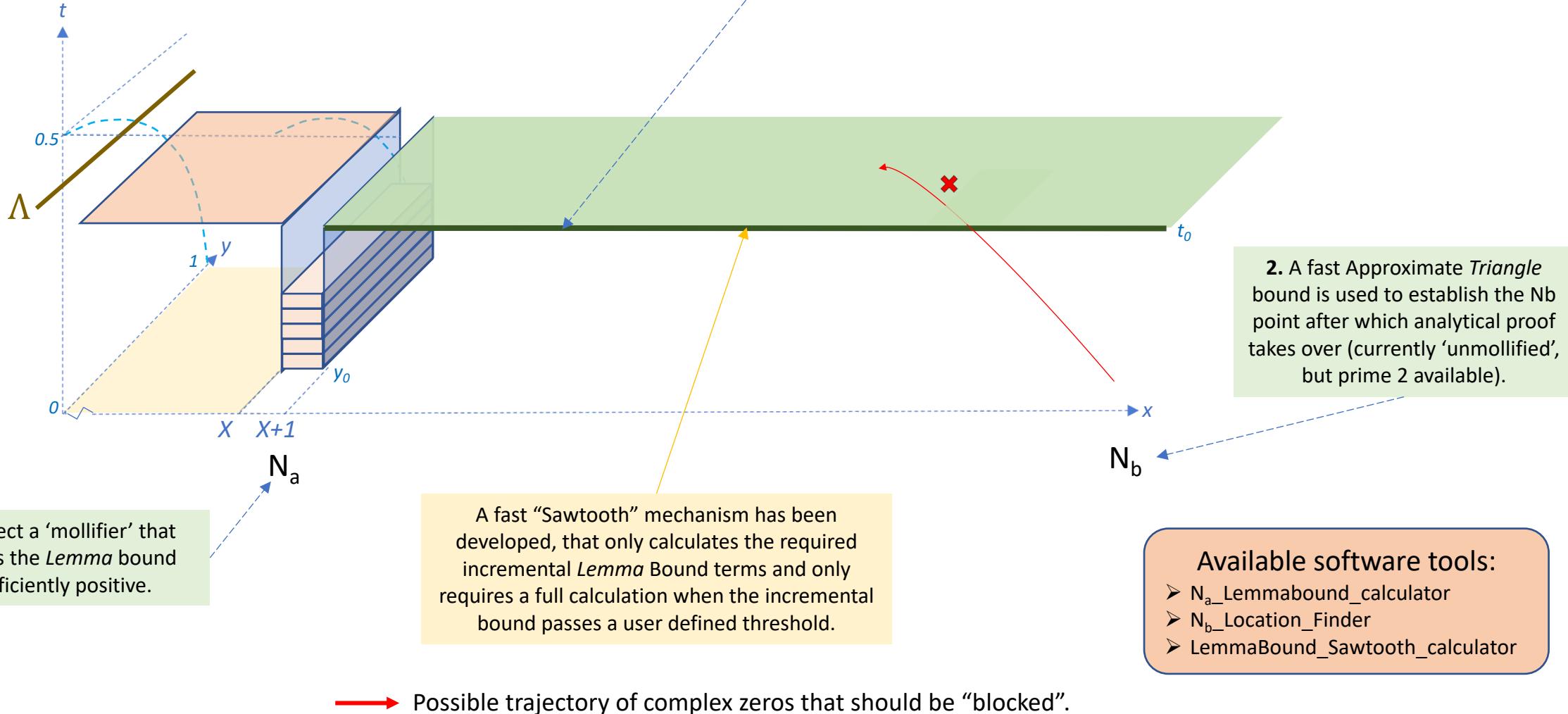


→ Possible trajectories of complex zeros that should be “blocked”.

“Barrier” approach: how to clear the barrier?

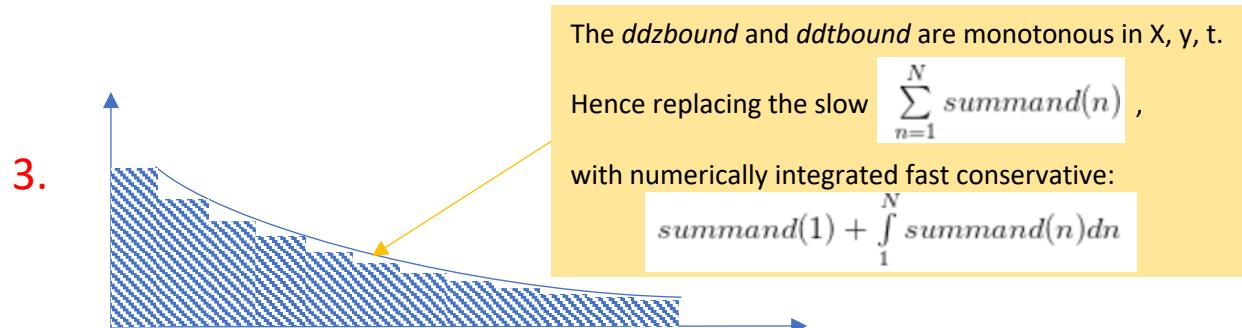
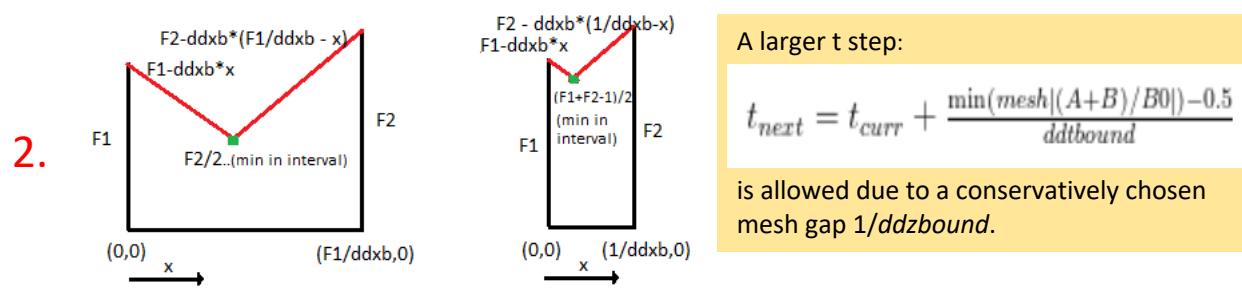
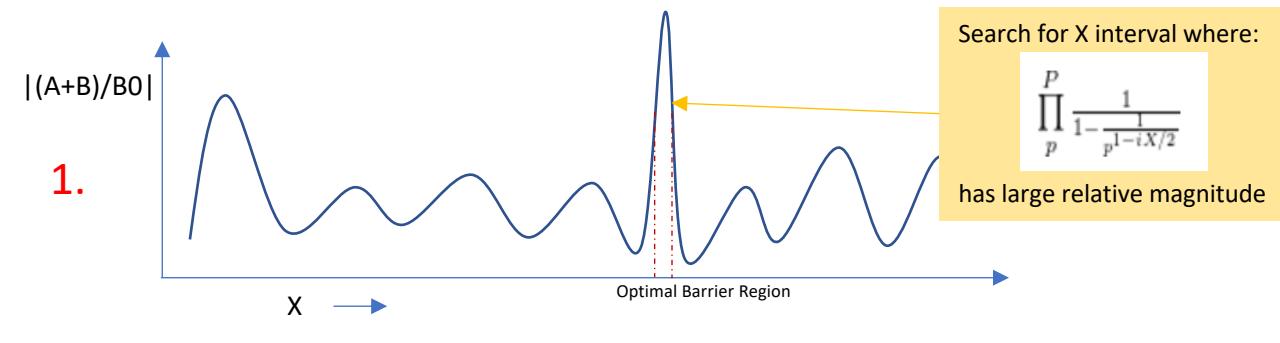


“Barrier” approach: how to verify the area from the barrier up to N_b ?

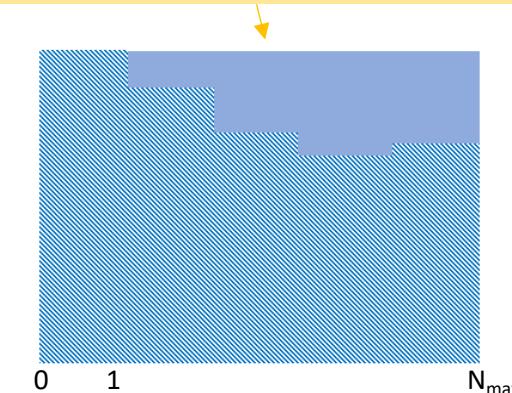
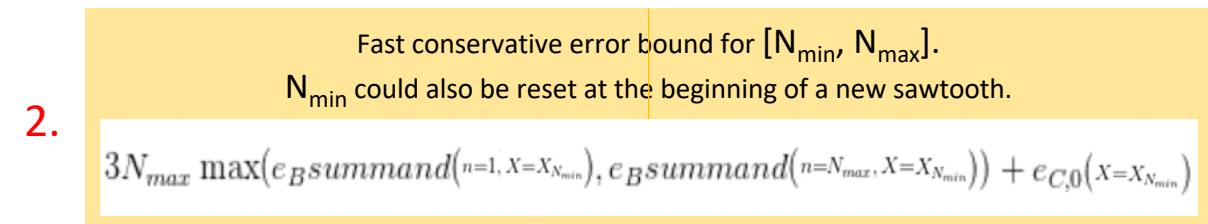
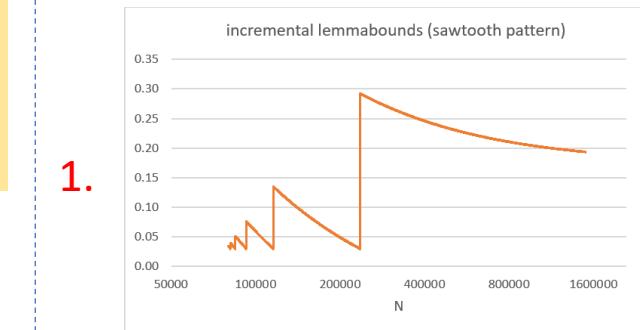


Optimizations made to improve computation speed

Barrier Optimizations



Euler Bounds Optimizations



The Barrier model - progress made

X-location where $\min_{\text{mesh}} |(A+B)/B_0|$
starts between 4 and 5 at $t=0$ and ends
near 1.5 at $t=t_0$.

Selected with the
LemmaBound utility

Selected with the
TriangleBound utility

x	Barrier offset	RH verified?	t_0	y_0	Λ	Winding number	mollifier primes	Lemma bound value	Na	mollifier primes	Triangle bound value	Nb	Status
6.00E+10	83951.5	yes ¹⁾	0.20	0.20	0.22	0	(2,3,5,7)	0.067	69098	(2)	0.098	2.5E+05	✓
6.00E+10	155019	yes	0.20	0.20	0.22	0	(2,3,5,7)	0.067	69098	(2)	0.098	2.5E+05	✓
2.00E+11	187700	tentative	0.19	0.20	0.21	0	(2,3,5,7)	0.095	126156	(0)	0.121	4.0E+06	✓
1.00E+12	46880	tentative	0.18	0.20	0.20	0	(2,3,5)	0.135	282094	(0)	0.112	7.0E+06	✓
5.00E+12	194858	tentative ²⁾	0.17	0.20	0.19	0	(2,3,5)	0.180	630783	(0)	0.116	1.5E+07	✓
1.00E+13	9995	not yet	0.16	0.20	0.18	0	(2,3,5)	0.109	892062	(0)	0.091	3.0E+07	✓
1.00E+14	2659	not yet	0.15	0.20	0.17	0	(2,3,5)	0.195	2820947	(0)	0.076	7.0E+07	✓
1.00E+15	21104	not yet	0.14	0.20	0.16	0	(2,3,5)	0.251	8920620	(0)	0.073	2.0E+08	✓
1.00E+16	172302	not yet	0.13	0.20	0.15	0	(2,3,5)	0.278	28209479	(0)	0.077	7.0E+08	✓
1.00E+17	31656	not yet	0.12	0.20	0.14	0	(2,3,5)	0.279	89206205	(0)	0.080	3.0E+09	✓
1.00E+18	44592	not yet	0.11	0.20	0.13	0	(2,3)	0.207	282094791	(0)	0.103	2.0E+10	✓
1.00E+19	12010	not yet	0.1135	0.1135	0.12	0	(2)	Not req.	892062059	(2)	0.036	=Na	✓
1.00E+19	12010	not yet	0.10	0.20	0.12	0	(2,3)	0.128	892062059	(2)	0.109	2.0E+09	✓
1.00E+20	37221	not yet	0.09	0.20	0.11	tbd	(2,3,5)	0.037	2820947918	(2)	0.094	1.3E+10	tbd

1) Platt et al, 2011

2) Gourdon-Demichel, 2004

Software used and useful links

All software was developed in two languages and results between them were reconciled:

- Symbolic math programming language - **pari/gp** (<https://pari.math.u-bordeaux.fr>)
 - Short development time
 - Relatively fast
- Arithmetic Balls C-based library - **Arb** (<http://arblib.org>)
 - Longer development time
 - Very fast (up to 20 x pari/gp)

All software and results are free to use (under the LGPL-terms) and can be found here:

https://github.com/km-git-acc/dbn_upper_bound