Math 319: Introduction to Real Analysis[1]

Arpit Kumar

Fall 2023/24

[1] Taught by Prof. Andrew Rechnitzer

These are my class notes, further research and a possibly random collection of facts from this course.

Digit Strings

There are several ways to define and construct the set of real numbers. We follow the approach largely attributed to Weierstrass, of using **digit strings**. Thinking of real numbers as strings of digits has the benefit of being very explicit, but comes at the expense of the following annoying technicality.

$$1.000... = 0.999...$$

Proof. Let $x = 0.999...$, then we can say that $10x = 9 + x$, which implies $9x = 9 \implies x = 1$ □

Definition 0.1. Digit Strings are defined as

$$< +a_1\, a_2 \,...\, a_m \,.\, b_1\, b_2 \,...> \text{ or } < -a_1\, a_2 \,...\, a_m \,.\, b_1\, b_2 \,... >$$

where

- each digit a_i, b_j lies in the set $\{0, 1, ..., 9\}$
- the leading digit a_1 cannot be zero unless $m = 1$
- the string $< -0.000... >$ is forbidden (only one definition of zero is allowed, i.e $< +0.000... >$)
- we denote the set of all digit strings by $\mathbb{D}$

To connect this construct to real numbers, we need a mapping

$$f : \mathbb{D} \to \mathbb{R}$$

Consider

$$f(d \in \mathbb{D}) = \underbrace{a_1\, a_2 \,...\, a_m}_{\in \mathbb{Z}} + \sum_{k=1}^{\infty} \frac{b_k}{10^k}$$

This seems to serve our purpose - but - we haven't defined the meaning of convergence yet, and as such should not be using an infinite sum in our mapping.

Definition 0.2. Terminating and deceptive digit strings. Let $x =< \pm a_1\, a_2 \,...\, a_m \,.\, b_1\, b_2 \,... >$. Then

- If there exists some k so that $b_k = b_{k+1} = b_{k+2} = ... = 9$ then we say that x is **deceptive**.
- If there exists some k so that $b_k = b_{k+1} = b_{k+2} = ... = 0$ then we say that x is **terminating**.

We use $\mathbb{D}_0$ to denote the set of all terminating digit strings. Similarly, we use $\mathbb{D}_9$ to denote the set of all deceptive digit strings. [2]

[2] Note that it is easy to map $\mathbb{D}_0$ to $\mathbb{Q}$ since we no longer need an infinite sum. Consider $x =< \pm a_1\, a_2\, ...\, a_m\ .\ b_1\, b_2\, ... b_l >$. As a rational, x = $\underbrace{a_1\, a_2\, ...\, a_m}_{\in \mathbb{Z}} + \underbrace{\sum_{k=1}^{l} \frac{b_k}{10^k}}_{\in \mathbb{Q}}$.

Truncations and Orders

Warning 0.1. To simplify technicalities in our proofs, we will only be considering the positive digit strings in our argument. Arguing for the other cases follows closely from the positive case.

Definition 0.3. Given a digit string $x =< a_1\, a_2\, ...\, a_m\ .\ b_1\, b_2\, ... >$ we define the truncation of x to k places to be $< +a_1\, a_2\, ...\, a_m\ .\ b_1\ b_2\ \underbrace{0\ 0\ 0}_{\textit{append zeros}}\ ... >$

$t_k : \mathbb{D} \to \mathbb{D}_0$

Definition 0.4. We define rational truncation $T_k : \mathbb{D} \to \mathbb{Q}$ by

$$T_k(x) = a_1\, a_2\, ...\, a_m + \sum_{j=1}^{k} \frac{b_j}{10^j}$$

Observe that $10^k \cdot T_k(x) \in \mathbb{Z}$

Also notice that $T_k(x)$ does not change much with k. Consider $x =< 1.732150808... >$, then

$$\begin{aligned} T_2(x) &= \frac{173}{100} \\ T_3(x) &= \frac{1732}{1000} \\ T_4(x) &= \frac{1732}{1000} + \frac{1}{10^4} \\ &\vdots \\ T_{k+1} &= T_k(x) + \frac{b_k}{10^k} \end{aligned}$$

We can see that the sequence of $T_k(x)$ is weakly monotonically increasing.

Lemma 0.1. Let $x \in \mathbb{D}$, then $T_k(x) \leq T_{k+1}(x) < T_k(x) + 10^{-k}$

Proof. Let $x =< a_1\, a_2\, ...\, a_m\ .\ b_1\, b_2\, ... >$. Then

$$T_k(x) = a_1\, a_2\, ...\, a_m + \sum_{j=1}^{k} \frac{b_j}{10^j}$$

and

$$T_{k+1}(x) = T_k(x) + b_{k+1} \cdot 10^{-k-1}$$

Since $b_{k+1} \geq 0$ and ≤ 9, the inequality follows. $\square$

Truncations gives us a way to order digit strings. Let $x, y \in \mathbb{D}$, then $x < y$ when there is some k such that $T_k(x) < T_k(y)$.

Warning 0.2. Note that in the reals, $1 = 0.999...$, BUT in digit strings, we have that $1.000... > 0.999...$[3]

[3] Ommitted the angular brackets wrapping digits strings here since that just looked ugly

Definition 0.5. Let $x, y \in \mathbb{D}$. We say that $x < y$ when there exists some k such that $T_k(x) < T_k(y)$. We say $x \leq y$ when $x < y$ or $\underbrace{x = y}_{\text{every single digit is equal}}$

Lemma 0.2. Let $x, y \in \mathbb{D}$. Then,

$$T_k(x) < T_k(y) \implies \forall l \geq k, T_l(x) < T_l(y)$$

Proof. @TODO $\square$

We are working towards trichotomy for $\mathbb{D}$. That is, we want $x, y \in \mathbb{D}$ such that exactly one of

$$\begin{aligned} x &< y \\ x &= y \\ x &> y \end{aligned}$$

is true.

Lemma 0.3. Let $x, y \in \mathbb{D}$, then it cannot be that $x < y$ and $y > x$

Proof. Assume the contrary that $x < y$ and $y < x$. Then we have k, l such that $T_k(x) < T_k(y)$ and $T_l(y) > T_l(x)$. Note that $k \neq l$ since $T_k(...)$ is a rational, i.e are trichotomous. So we get two cases, $k < l$ and $l > k$.

If $k < l$, then $T_k(x) < T_k(y)$ and $T_l(x) > T_l(y)$ which contradicts **lemma 0.2** (2.2.6 in text). A similar argument can be made for the remaining case. Hence the result holds. $\square$

Therefore, we have shown that $\mathbb{D}$ is trichotomous. Or have we? We still need to consider the case of equality and inequality.

Proof. @TODO

$\square$

Definition 0.6. [4] Let $\mathcal{R}$ be a relation on a set $\mathcal{A}$. We say that $\mathcal{R}$ is an order when

[4] We can say that trichotomy + transitivity = order. Also note that trichotomy does not imply transitivity.

- If $x, y \in \mathcal{A}$ then exactly one of $x \mathrel{\mathcal{R}} y$ or $x = y$ or $y \mathrel{\mathcal{R}} x$ is true.
- $\mathcal{R}$ is transitive.

Not quite dense...

@TODO

Bounds and supremum

@TODO

Defining real numbers

Informally, we define a mapping from $\mathbb{D} \to \mathbb{R}$ as follows

- We transform a digit string $x =< a_1\, a_2 \ldots a_m \,.\, b_1\, b_2 \ldots >$ to a decimal expansion $a_1 \cdots a_m \,.\, b_1 b_2 \cdots$
- **except** when x is deceptive, in which case we map it to it's terminating pair. This ensures a single expansion for each digit string.

  More formally, we use equivalence classes to define this mapping.

Definition 0.7. Let $x, y \in \mathbb{D}$. We say that $x \equiv y$ when

- $x = y$ or
- x, y are a terminating-deceptive sibling pair (i.e $y = \psi(x)$)

and otherwise we write $x \not\equiv y$. This defines a equivalence relation on $\mathbb{D}$.

We can define the real numbers as equivalence classes of this relation.

Definition 0.8. The real numbers $\mathbb{R}$ are the set of equivalence classes of digit strings under the equivalence relation '$\equiv$'.

@TODO

Arithmetic

We know use our work so far to define arithmetic on the reals. We will prove that $\mathbb{R}$ is ordered field (like $\mathbb{Q}$) with LUB property. We use the supremum as a sneaky way to calculate limits.

Addition, subtraction, multiplication, division

@TODO

Homework hint: ($\sqrt{2}$ proof question) Let $a \in \mathbb{R}$, such that $a^2 < 2$. Show that we can construct $a + \frac{1}{n}, n \in \mathbb{N}$ such that $(a + \frac{1}{n})^2 < 2$. We want to say that n is really, really big.

$a^2 + \frac{2a}{n} + \frac{1}{n^2} < 2$
$\frac{2a}{n} + \frac{1}{n^2} < 2 - a^2$
Factoring and some tricks we get

$$\frac{1}{n}\left(2a + \frac{1}{n}\right) \leq \frac{1}{n}(2a+1) \overbrace{<}^{\text{require}} 2 - a^2$$

The whole point of digit strings was to prove LUB on the reals.
We have $\mathbb{R}$, it has LUB $\to$ so what? What can we do with this fact?

Theorem. Let $x, y \in \mathbb{R}$, then

$$|x+y| \leq |x| + |y| \text{triangle inequality}$$

$$|x-y| \geq ||x| - |y|| \text{ reverse triangle inequality}$$

Proof of reverse triangle inequality. Start with the TI with x, y

$$\begin{aligned} |x+y| &\leq |x| + |y| && \text{set } y = b - a \text{ and } x = a \\ |b| &\leq |a| + |b-a| \\ -|b-a| &\leq |a| - |b| \end{aligned}$$

Now set $x = a - b$, $y = b$

$$\begin{aligned} |a| &\leq |a-b| + |b| \\ |a| - |b| &\leq |a-b| \end{aligned}$$

Hence $-|b-a| \leq |a| - |b| \leq |b-a|$ so

$$||a| - |b|| \leq |b-a|$$

$\square$

Archimedes

5

[5] Archimedes basically wrote down that we can measure things with a ruler

Theorem. Let $x, y \in \mathbb{R}$ with $x, y > 0$. Then there is $n \in \mathbb{N}$ such that $nx > y$

Proof. Assume to the contrary that there are $x, y \in \mathbb{R}$, $x, y > 0$ so that for all $n \in \mathbb{N}, nx \leq y$
Consider the set $A = \{nx \text{st} n \in \mathbb{N}\}$
It has an upper bound (by assumption) of y
Since $A \neq \emptyset$, the LUB tells us $u = supA$ exists and is in $\mathbb{R}$.

- $u =$ upperboud so $nx \leq u$ for all $n \in \mathbb{N}$.
- Since $u = supA$, $u - x$ is not an upperbound.

Hence there is $k \in \mathbb{N}$ such that $u - x < kx \leq u$. But then

$$u < (k+1)x \leq u + x$$

However, $(k+1)x \in A$ which contradicts the fact that $u = supA$. Hence there is no such x, y. Thus, the result holds. □

Corollary 0.1. The following are logically equivalent [6]

- For all $x, y \in \mathbb{R}^+$, there is $n \in \mathbb{N}$, such that $nx > y$.
- For all $x \in \mathbb{R}^+$, there is $n \in \mathbb{N}$ such that $n > x$ (set x = 1 and y = x)
- For all $x \in \mathbb{R}^+$, there is $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$

[6] Instead of proving all pairs, you prove a cycle - $A \implies B \implies C \implies A$, this is a very standard proof structure to use

Proof. We show that $(1) \implies (2) \implies (3) \implies (1)$

- $(1) \implies (2)$ Assume (1), then set $x = 1, y = x$
- $(2) \implies (3)$ Assume (2). Let $x \in \mathbb{R}^+$ and set $y = \frac{1}{x} > 0$
  Then (2) tells us there is $n \in \mathbb{N}$ such that $n > y > 0$. So multiply by x divide by n to get
  $$0 < \frac{1}{n} < x$$
  as required
- $(3) \implies (1)$ Assume (3). Let $x, y \in \mathbb{R}^+$. (3) tells us there is $n \in \mathbb{N}$ such that ... just multiply by ny to get $y < n < x$ as required.

□

Want to show that $\mathbb{Q}$ is dense inside $\mathbb{R}$

Corollary 0.2. Let $x, y \in \mathbb{R}$ with $x < y$. Then there is $z \in \mathbb{Q}$ such that $x < z < y$

Proof. @TODO homework □

Lemma 0.4. Let $x, y \in \mathbb{R}$ such that $1 < y - x$. Then there is $n \in \mathbb{Z}$ such that $x < n < y$

Proof. @TODO also homework □

Nested intervals (equivalent to LUB)

Rough idea: Take a sequence of intervals $I_1 \supset I_2 \supset I_3 \supset \cdots I_n \supset \cdots$, then (with some conditions), there is a unique real that sits inside all I_s.

Definition 0.9. Let $a, b \in \mathbb{R}$.

- The closed interval $[a, b] = \{x \in \mathbb{R} \text{ such that } a \leq x \leq b\}$

- The open interval $(a, b) = \{x \in \mathbb{R} \text{ such that } a < x < b\}$
- Length of interval $|I| = |b - a|$

  A motivating example

$$\begin{aligned}\sqrt{17} &\in I_1 = [1, 17] \\ &\in I_2 = [1, 9] \\ &\in I_3 = [1, 5] \\ &\in I_4 = [3, 5] \\ &\in I_5 = [4, 5]\end{aligned}$$

We need to prove that there is something unique that lies in all these intervals.

@TODO Lot's of holes

20 October

Last time

- Finished IVT
- Started EVT, boundedness

  Lemma 0.5. Let $g : [a, b] \to \mathbb{R}$ if g is continuous then g is bounded

Theorem. Let $f : [a, b] \to \mathbb{R}$ be continuous then there exists $c, d \in [a, b]$ such that

$$f(c) \leq f(x) \leq f(d)$$

for all $x \in [a, b]$

Essentially telling us that $f([a, b]) = [f(c), f(d)]$

Not necessarily takes open intervals to open intervals!

Example 1. $f : (-1, 1) \to [0, 1)$ by $f(x) = x^2$

But it does take closed intervals to closed intervals.

Proof. Let f be as given and form

$$Y = \{f(x) \text{ such that } x \in [a, b]\}$$

$Y \neq \emptyset$ and from the lemma above, we know that a continous function on a closed interval is bounded. Therefore, $M = supY$ exists.

If there is $d \in [a, b]$ such that $f(d) = M$ we are done. [7]

So now, to the contrary, assume that there is no such $d \in [a, b]$. Form

$$g : [a, b] \to \mathbb{R} = \frac{1}{M - f(x)}$$

[7] is this cheating? feels like it, very direct

Since $f(x) < M$ (by assumption) and $M - f(x) \neq 0$ for all $x \in [a,b]$ so g is continuous on $[a,b]$ and g is also bounded. [8]

Since $M = supY$, for any $\epsilon > 0$, we know $M - \epsilon$ is not an upper-bound. Pick $\epsilon = 1/n$ then there is $Y_n \in Y$ such that $M - \epsilon < Y_n < M$. Hence there is $x_n \in [a,b]$ such that $f(x_n) > M - \epsilon$. Thus $g(x_n) = \frac{1}{M - f(x_n)} > \frac{1}{\epsilon} = n$.[9] So g is unbounded and cannot be continuous, giving our contradiction. Thus, there exists $d \in [a,b]$ such that $f(d) = M$. The argument for the minimum is similar. □

[8] notice how a contradiction is going to show up - g is going to blow up since $f(x)$ comes arbitrarily close to M

[9] you could also use this to show that g is not continuous at that point

Derivatives

Definition 0.10. Let f be a function defined on a neighbourhood of c. We say f is differentiable at c when

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = f'(c)$$

exists, we call $f'(c)$ the derivative of f at c. If $f'(x)$ exists for all $x \in (a,b)$ we say f is differentiable on (a,b).

Example 2. $g(x) = |x|$ then

$$g'(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \\ DNE & x = 0 \end{cases}$$

Lemma 0.6. If f is differentiable at c, then f is continuous at c. Converse is false. If f is differentiable on $\mathbb{D}$, then f is continuous on $\mathbb{D}$. At the same time, the derivative function f' need not be continuous on $\mathbb{D}$.

Proof. @Homework □

Definition 0.11. Let $f : \mathbb{D} \to \mathbb{R}$ be a function. Then derivative function $f'(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The domain of f' is the subset of $\mathbb{D}$ where $f'(x)$ exists.

Definition 0.12. Let $f : [a,b] \to \mathbb{R}$ be differentiable at $c \in [a,b]$. The tangent line to f at c is given by

$$T(x) = f(c) + f'(c) \cdot (x - c)$$

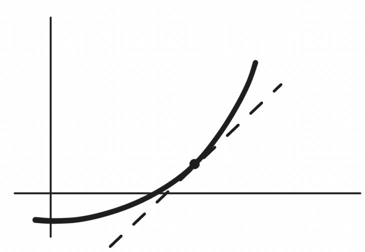

Figure 1: Tangent line to f

Lemma 0.7. Let f be differentiable at c and T be the tangent line at c. Then,

$$\lim x \to c \frac{f(x) - T(c)}{x - c} = 0$$

Further, T is the unique linear function with this property.

Proof. Let f, T be as stated. Then

$$\frac{f(x) - T(x)}{x - c} = \frac{f(x) - (f(c) + (x - c) \cdot f'(c))}{x - c} = \frac{f(x) - f(c)}{x - c} - f'(c)$$

So limit goes to 0 as required. Uniqueness = @Homework. $\square$

To make proofs of chain rule, etc work, it helps to be able to say $f(x)$ "look like" $T(x)$ at $x = c$

$$f(x) = f(c) + (x - c)\phi(x)$$

$\phi(x)$ looks like $f'(c)$ and

$$\lim_{x \to c} \phi(x) = f'(c)$$

Theorem. If g is differentiable at $x = c$ then there exists ϕ such that $g(x) - g(c) = \phi(x)(x - c)$ and ϕ is continuous with $\lim_{x \to c} \phi(x) = \phi(c) = g'(c)$.

The converse also holds. If you can find such a ϕ then g is differentiable at c with $g'(c) = \phi(c)$.

Proof. Let

$$\phi(x) = \begin{cases} g'(c) & x = c \\ \frac{g(x) - g(c)}{x - c} & x \neq c \end{cases}$$

Since g is differentiable at c, we know

$$\lim_{x \to c} \phi(x) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = g'(c)$$

So ϕ is continuous at $x = c$.

Conversely, if such a continuous ϕ exists then

$$\phi(x) = \frac{g(x) - g(c)}{x - c}$$

So $\lim_{x \to c} \phi(x)$ exists since ϕ is continuous at c and so $\phi(c) = g'(c)$ and so g is differntiable at $x = c$. $\square$

Proof of the Product Rule!

Proof. Assume f, g is differentiable at $x = c$, then by the theorem above we have that

$$f(x) = f(c) + \phi(x) \cdot (x - c)\phi(x) = f'(c)$$
$$g(x) = g(c) + \gamma(x) \cdot (x - c)\gamma(x) = g'(c)$$

$$\frac{d}{dx}(f \cdot g)\,|_{x=c} = \lim_{x \to c} \left(\frac{f(x) \cdot g(x) - f(c) \cdot g(c)}{x - c} \right)$$

Which boils down to

$$
\begin{aligned}
&= \lim_{x \to c} \left[(c) + \phi(x)(x-c)\right]\left[g(c) + \gamma(x)(x-c)\right] - f(c)g(c) \\
&= \lim_{x \to c} \left[\frac{g(c)\phi(x)(x-c) + f(c)\gamma(x)(x-c) + \phi \cdot \gamma(x-c)^2}{x-c}\right] \\
&= \lim_{x \to c} \left[g(c)\phi(x) + f(c)\gamma(x) + (x-c)\phi(x)\gamma(x)\right] \\
&= g(c)f'(c) + f(c)g'(c) + 0
\end{aligned}
$$

□

Proof of Chain Rule

Proof. f is defined around c and differentiable at c
g is defined around $f(c)$ and differentiable at $f(c)$
By the theorem above, we have that

$$
\begin{aligned}
&f(x) = f(c) + \phi(x)(x-c) && \phi(c) = f'(c) \\
&g(t) = g(d) + \gamma(t)(t-d) && d = f(c), \gamma(d) = g'(d)
\end{aligned}
$$

And so

$$
\begin{aligned}
g(f(x)) &= g(d) + \gamma(f(x))(f(x) - d) \\
&= g(f(c)) + \gamma(f(x)) \cdot (f(x) - f(c)) \\
\frac{g(f(x)) - g(f(c))}{x-c} &= \frac{\gamma(f(x))(f(x) - f(c))}{x-c}
\end{aligned}
$$

Which gives

$$
\lim_{x \to c} = \frac{d}{dx}(g(f(x)))|_{x=c} = \gamma(f(c)) \cdot f'(c) = g'(f(c))f'(c)
$$

□

Theorem. Arithmetic of derivatives

Proof. Consider

$$
g(x) = \begin{cases} x^2 sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}
$$

If $x \neq 0$ then $g'(x) = 2xsin(1/x) - cos(1/x)$

$$
g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x} = \lim_{x \to 0} xsin(1/x)
$$

Since $-x \leq xsin(1/x) \leq x$ and $\lim_{x \to 0} \pm x = 0$, by sandwhich theorem, $g'(0) = 0$.

So

$$g'(x) = \begin{cases} xsin(1/x) - cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This is not continuous at $x = 0$ so g is differentiable everywhere but its derivative is not continuous. □

Now consider

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

is continuous at 0 but nowhere else.

$$g(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & \text{elsewhere} \end{cases}$$

is differentiable at 0 but nowhere else.

Proof.

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x} = \lim_{x \to 0} \frac{g(x)}{x} = \lim_{x \to 0} f(x)$$

@TODO □

Increasing, Decreasing and Derivative

Lemma 0.8. Let

- $f : \mathbb{D} \to \mathbb{R}$
- $c \in \mathbb{D}$ and $w > 0$ such that $(c - w, c + w) \subseteq \mathbb{D}$
- f is differentiable at c with $f'(c) > 0$

Then, there is $\delta > 0$, so that

- if $x \in (c, c + \delta)$ then $f(x) > f(c)$
- if $x \in (c - \delta, c)$ then $f(x) < f(c)$

Similar statement for $f'(c) < 0$