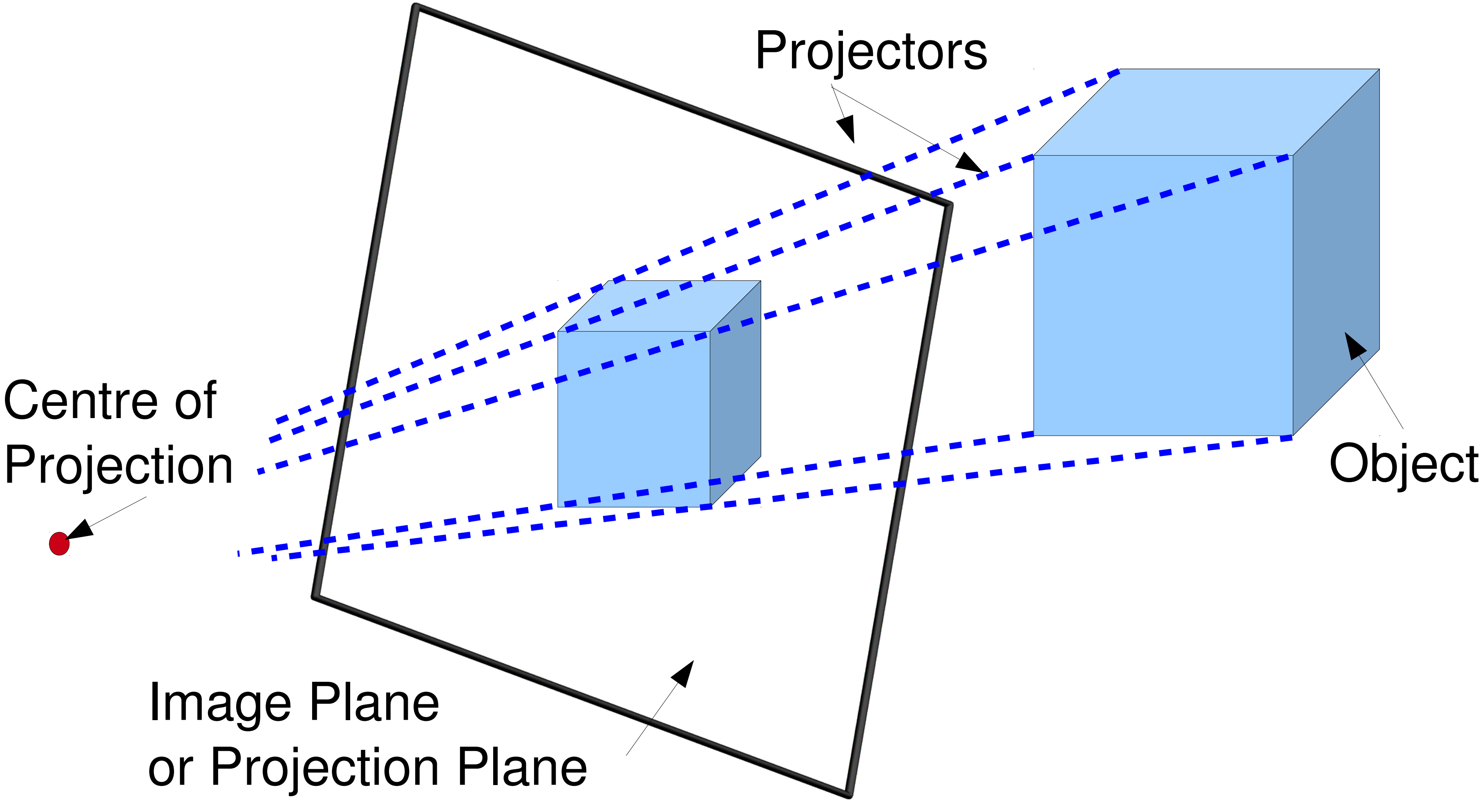
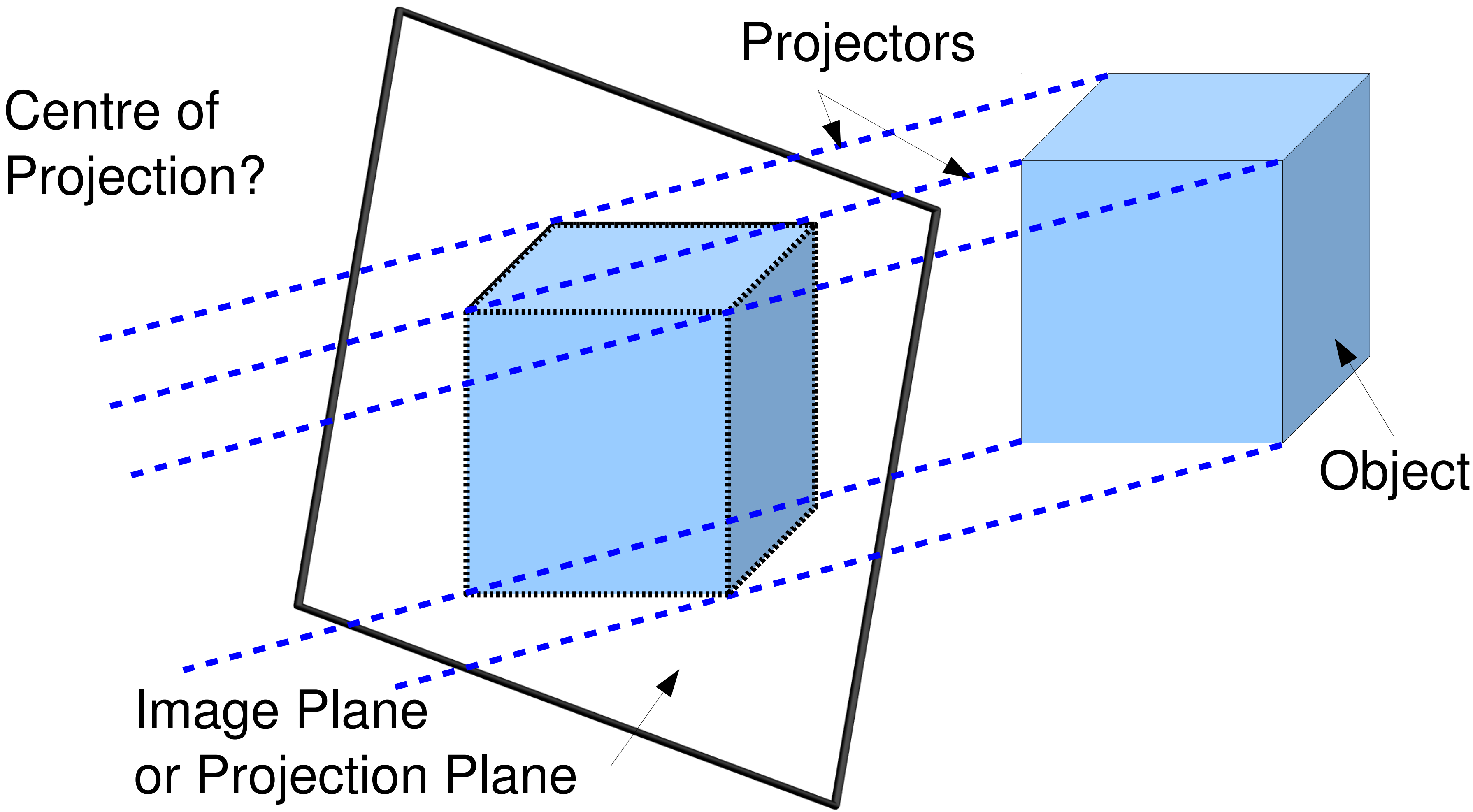
CS475m - Computer Graphics

Lecture 6 : Viewing

Perspective Projection



Parallel Projection

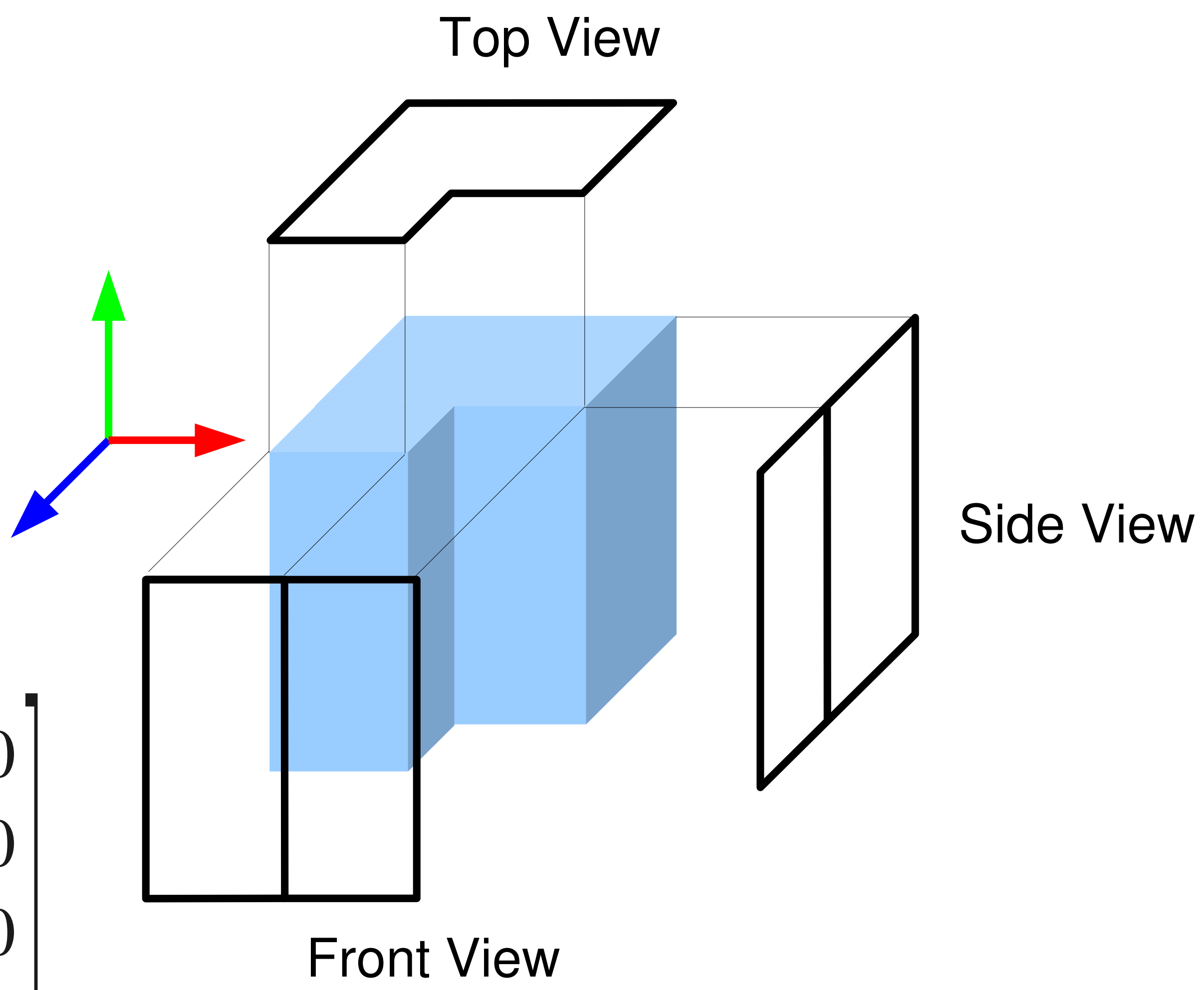


Parallel Projection

Orthographic Projection

- Multiviews (x=0, y=0 or z=0 or principal planes).
- True size or shape for lines.
- For projection on the z=0 plane we get the projection matrix as

$$P(z=0)=\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$



Parallel Projection

Axonometric Projection

- Transform and then project using an orthographic projection such that at multiple adjacent faces are visible – better representation of a 3D object using 1 view. Face parallel to projection plane shows true shape and size.

- If U be the matrix formed by stacking up the unit vectors along the three axes, and T be the axonometric projection, then

$$T.U = T.\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} x_x' & x_y' & x_z' \\ y_x' & y_y' & y_z' \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Parallel Projection

Axonometric Projection

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- The foreshortening ratios for each projected principal axes are then given by:

$$f_x = \sqrt{x_x'^2 + y_x'^2} \qquad f_y = \sqrt{x_y'^2 + y_y'^2} \qquad f_z = \sqrt{x_z'^2 + y_z'^2}$$

Parallel Projection

Axonometric Projection

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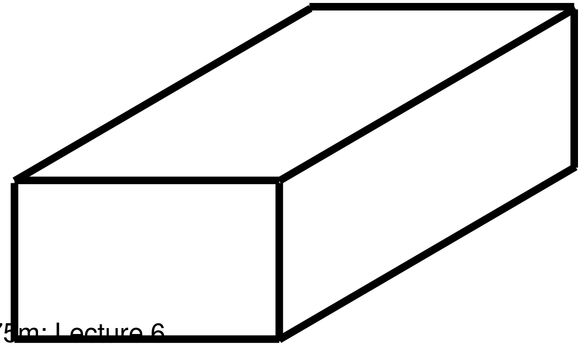
Parallel Projection

Axonometric Projection

- Depending on the kind of foreshortening they cause we can have three types of axonometric projections
  - Trimetric (all foreshortenings are different)

$$f_x \neq f_y \neq f_z$$

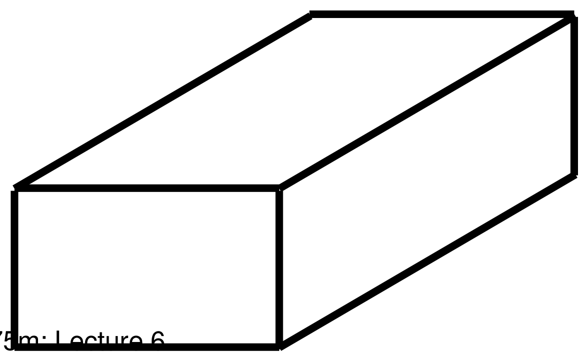


Parallel Projection

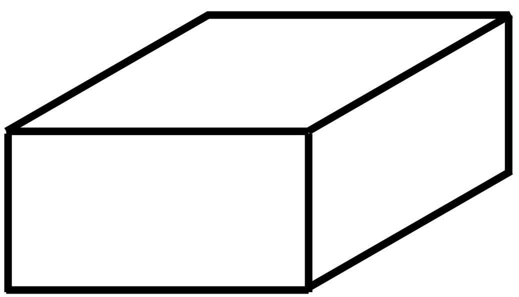
Axonometric Projection

- Depending on the kind of foreshortening they cause we can have three types of axonometric projections
  - Trimetric (all foreshortenings are different)
  - Dimetric (two foreshortenings are the same)

$f_x \neq f_y \neq f_z$



$f_x = f_z$

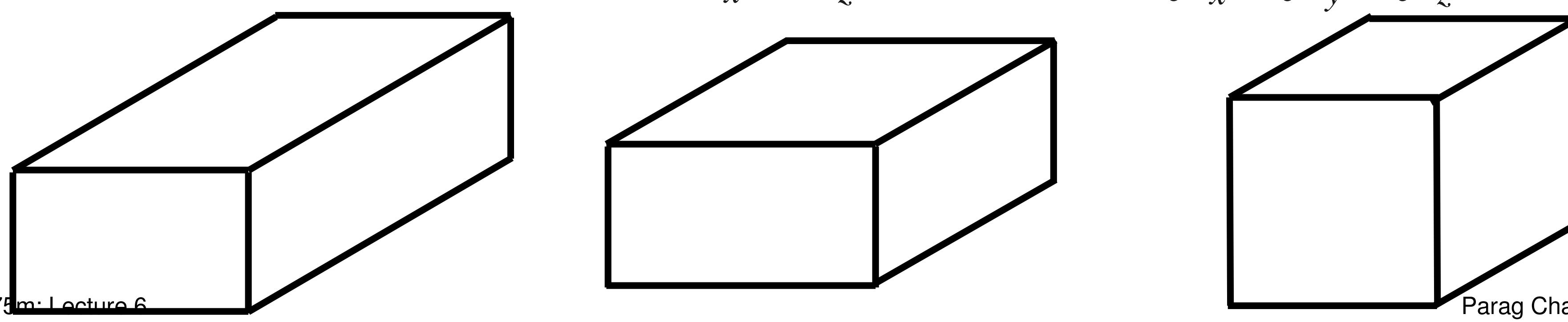


Parallel Projection

Axonometric Projection

- Depending on the kind of foreshortening they cause we can have three types of axonometric projections
  - Trimetric (all foreshortenings are different)
  - Dimetric (two foreshortenings are the same)
  - Isometric (all foreshortenings are the same)

$f_x \neq f_y \neq f_z$ $\qquad$ $f_x = f_z$ $\qquad$ $f_x = f_y = f_z$



Parallel Projection

Axonometric Projection

- Assuming we rotate by $R_y(\phi)$ and $R_x(\theta)$ before we do the projection on the z=0 plane.

$$T = P(z=0) . R_x(\theta) . R_y(\phi)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ \sin\theta\sin\phi & \cos\theta & -\sin\theta\cos\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Parallel Projection

Axonometric Projection

- Now we apply this axonometric projection T to U

$$T.U = \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ \sin\theta\sin\phi & \cos\theta & -\sin\theta\cos\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ \sin\theta\sin\phi & \cos\theta & -\sin\theta\cos\phi \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Parallel Projection

Axonometric Projection

- So the foreshortening ratios become

$$f_x^2 = \cos^2\phi + \sin^2\theta \sin^2\phi$$

$$f_y^2 = \cos^2\theta$$

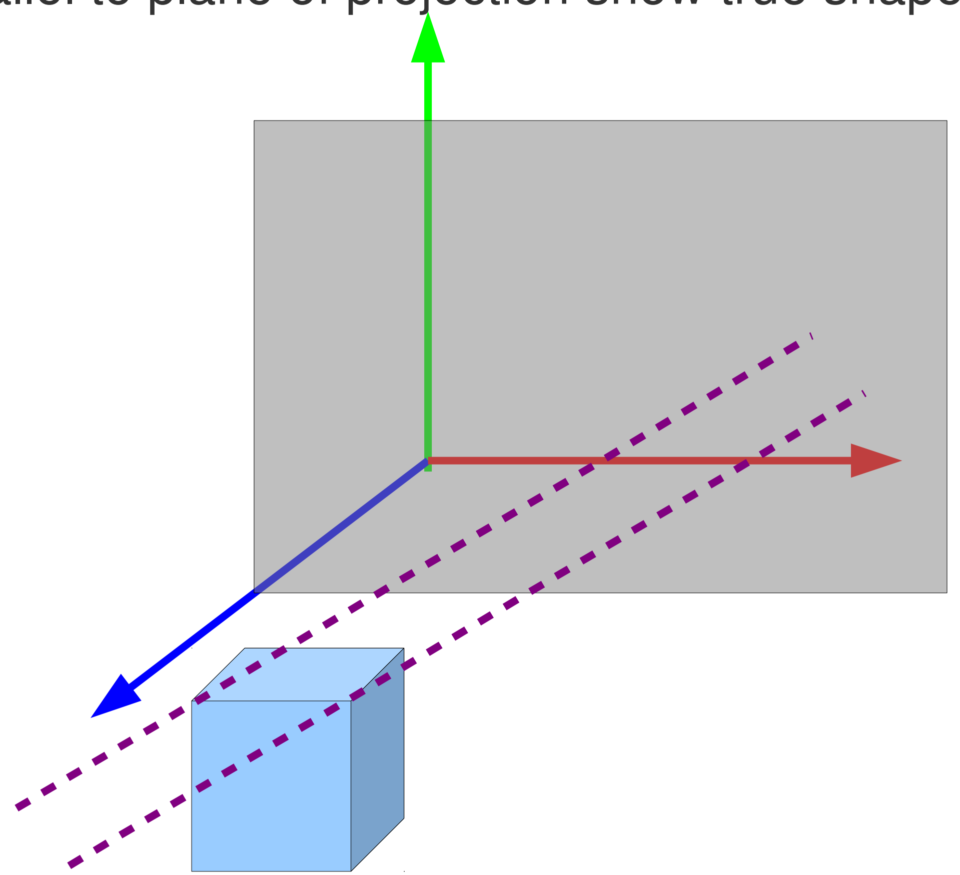
$$f_z^2 = \sin^2\phi + \sin^2\theta \cos^2\phi$$

- For Isometric projections, if we solve for $f_x = f_y = f_z$ then we get $\theta = 35.26^{\circ}$ and $\phi = \pm 45^{\circ}$

Parallel Projection

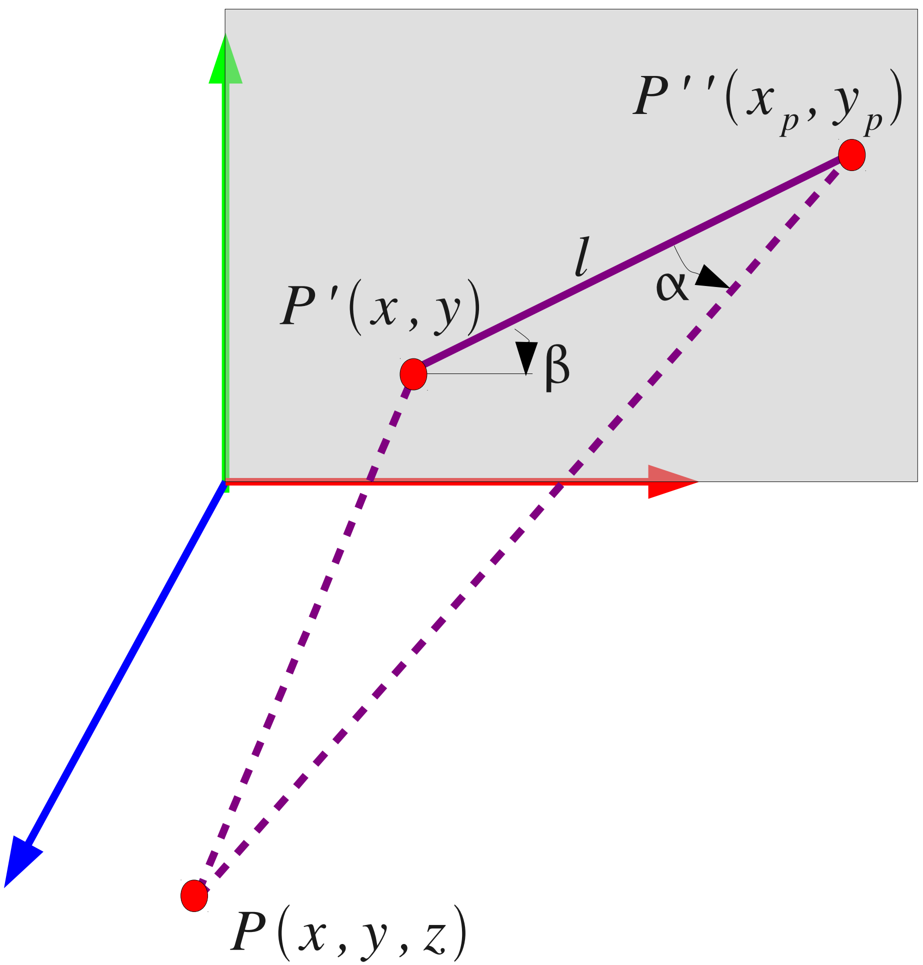
Oblique Projection

- The projectors are parallel to each other but they are not perpendicular to the plane of projection.
- Only planes parallel to plane of projection show true shape and size.



Parallel Projection

Oblique Projection



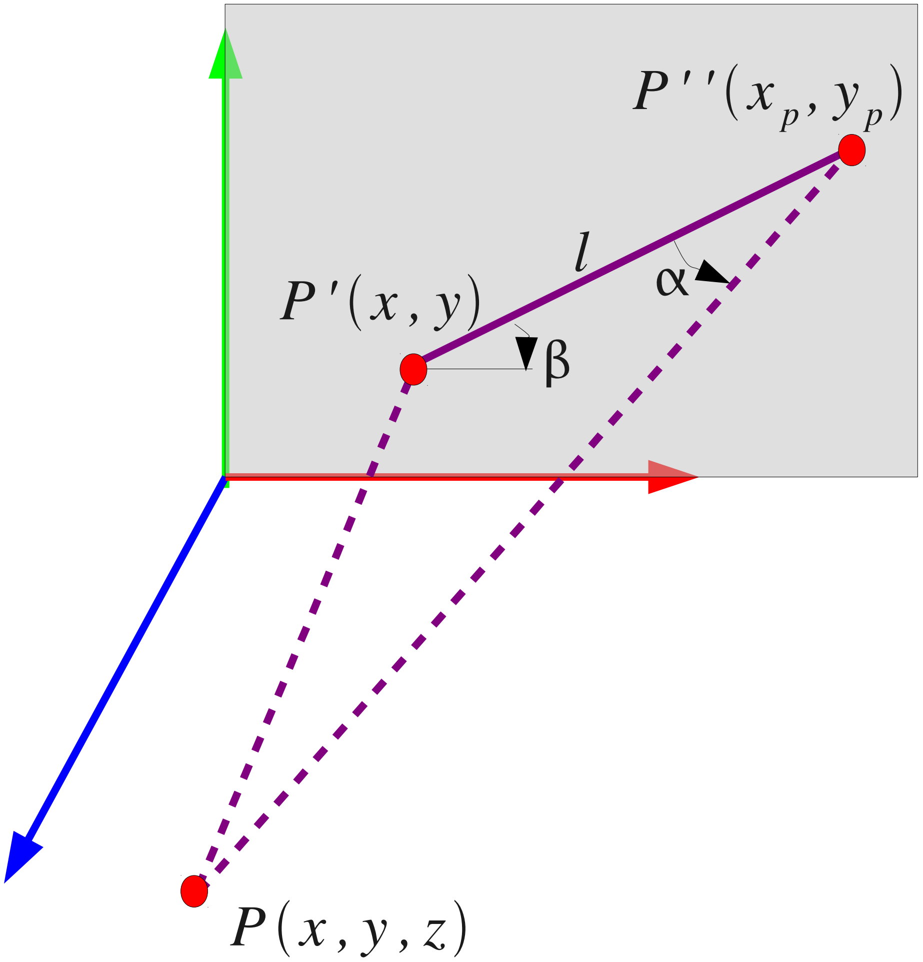
$$x_p = x + l\cos\beta$$

$$y_p = y + l\sin\beta$$

$$\tan\alpha = \frac{z}{l} \quad \text{or} \quad l = z\cot\alpha$$

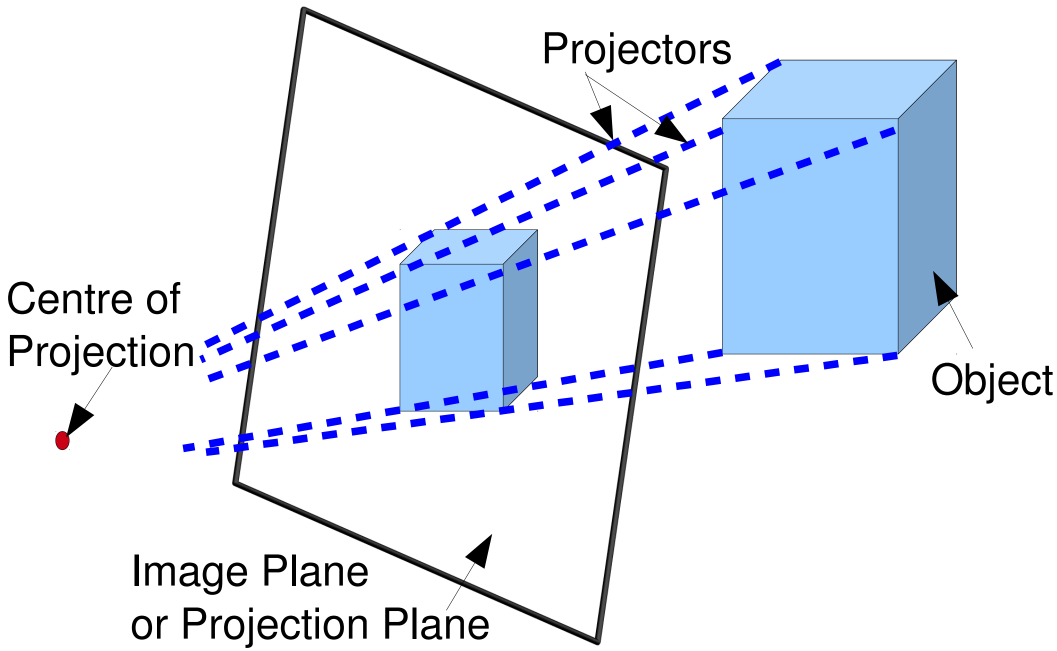
Parallel Projection

Oblique Projection



- When $\alpha = 45^{\circ}$ we get a Cavalier projection. Lines perpendicular to the projection plane are not foreshortened.
- When $\cot \alpha = 1/2$ we get Cabinet projections. Lines perpendicular to the projection plane are foreshortened by half.
- β is typically 30° or 45°.

Perspective Projection



- Projectors converge at a finite centre of projection.
- Parallel lines converge.
- We get non-uniform foreshortening.
- Shape is not preserved.
- We *see* in perspective – so perspective viewing seems natural and helps in depth perception.

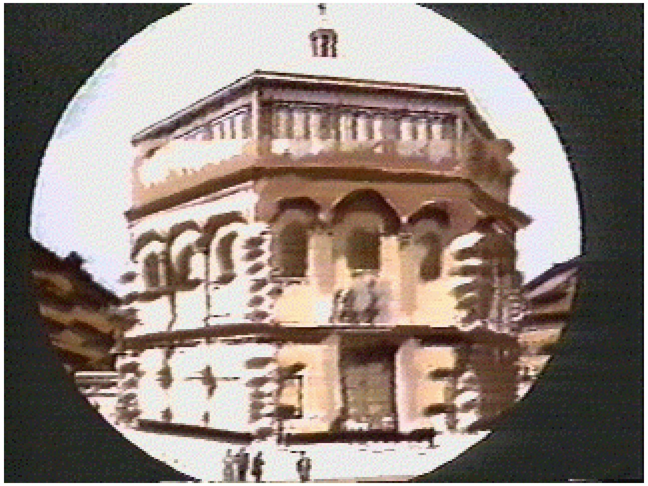
A digression into art

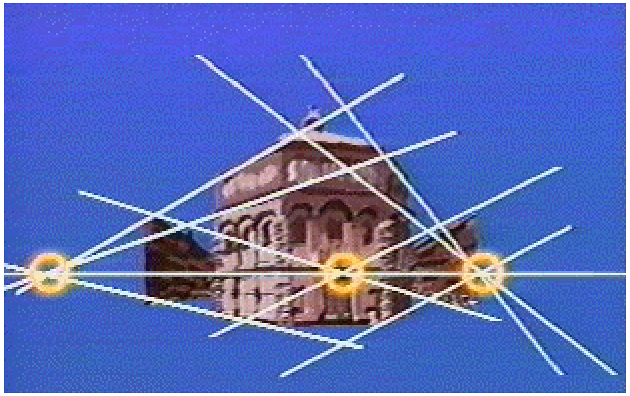
13^{th} century , Arezzo by Giotto

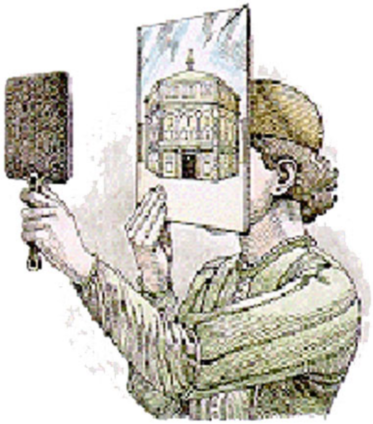
A digression into art

Early 15th century , The Little Garden of Paradise

A digression into art

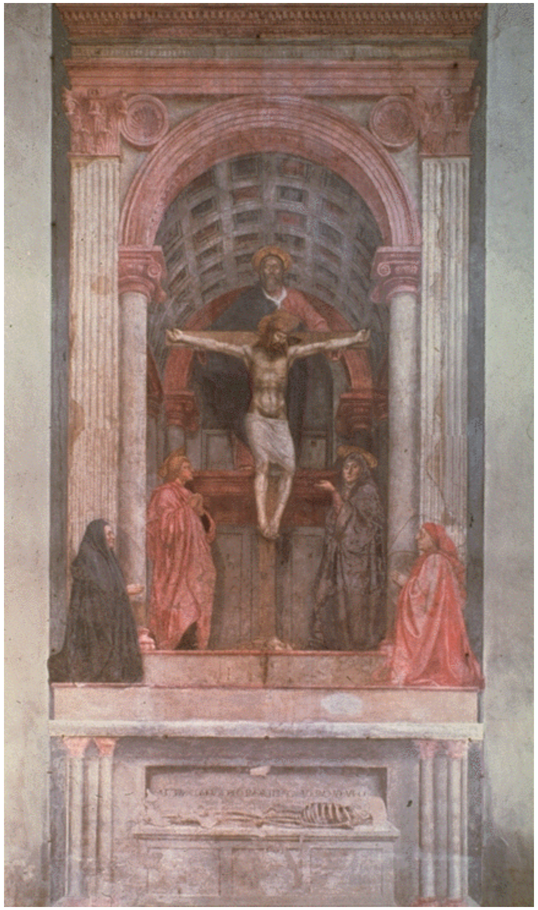


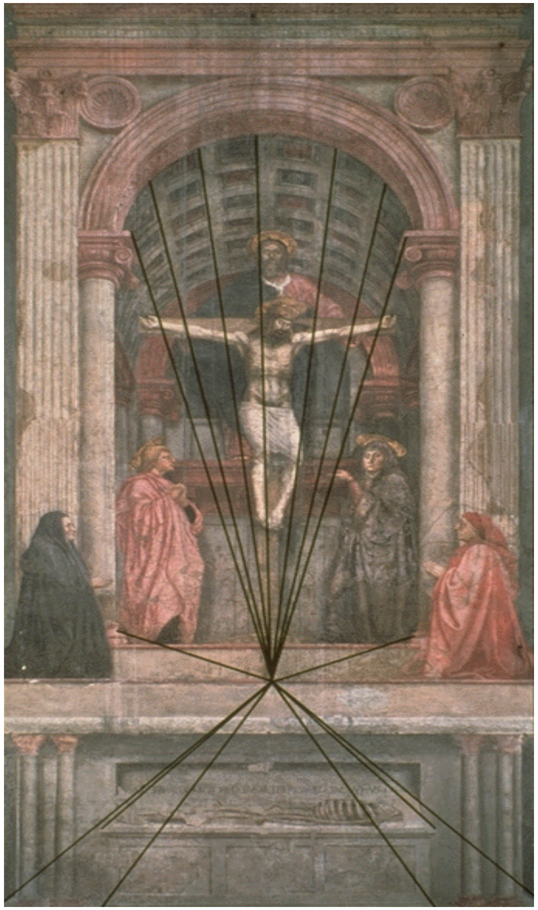




15th century , The Baptistry in Florence, Filippo Brunelleschi

A digression into art



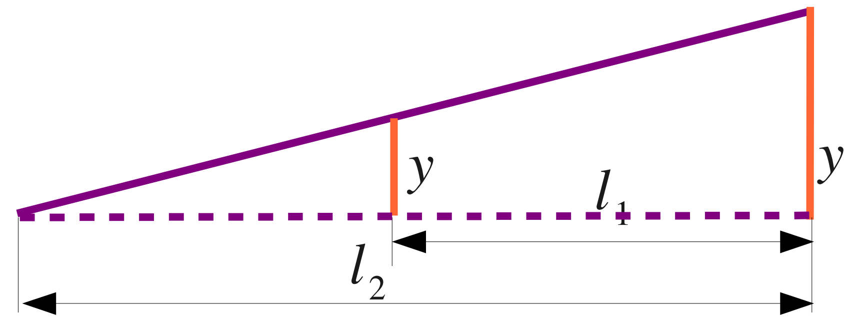


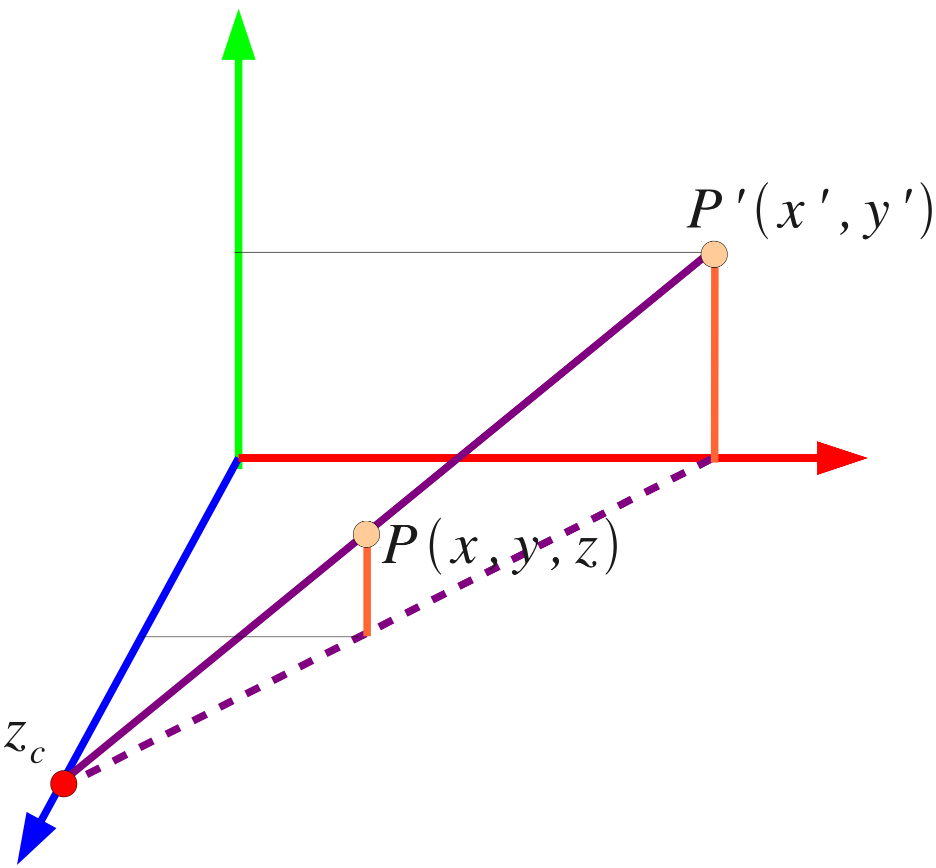
15th century , Fresco of Holy Trinity, Masaccio

A digression into art

15th century , School of Athens, Raphael

Perspective Projection

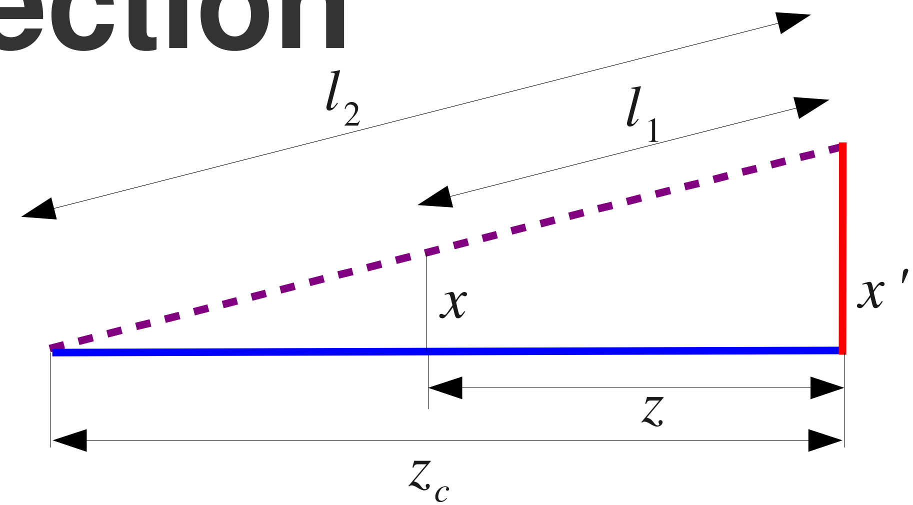


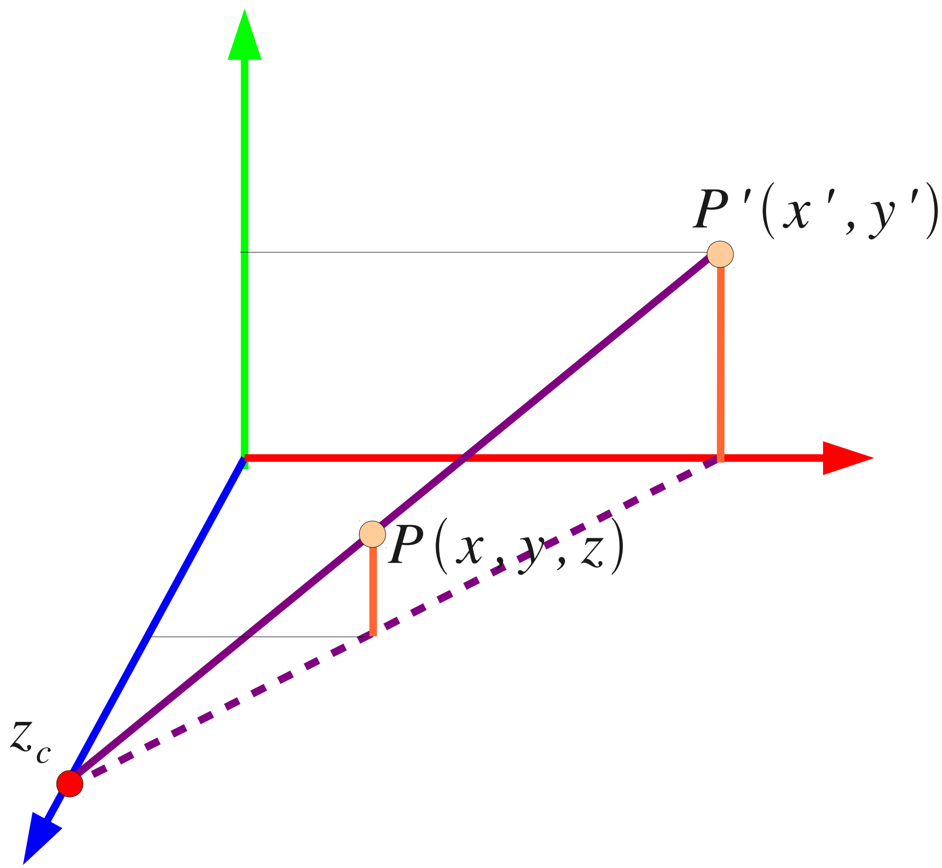


$$\frac{y'}{l_2}=\frac{y}{l_2-l_1} \qquad \frac{z_c}{l_2}=\frac{z_c-z}{l_2-l_1}$$

$$\Rightarrow y'=\frac{y}{1-\frac{z}{z_c}}$$

Perspective Projection





$$\frac{x'}{l_2}=\frac{x}{l_2-l_1} \qquad \frac{z_c}{l_2}=\frac{z_c-z}{l_2-l_1}$$

$$\Rightarrow x'=\frac{x}{1-\frac{z}{z_c}}$$

Perspective Projection

- First we apply a perspective transform to a point X that takes it to X'

$$X' = P_r . X$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ rz+1 \end{bmatrix}$$

$$x' = \frac{x}{rz+1}, \quad y' = \frac{y}{rz+1}, \quad z' = \frac{z}{rz+1}$$

Perspective Projection

- First we apply a perspective transform to a point X that takes it to X'
- Now we add projection on the $z=0$ plane.

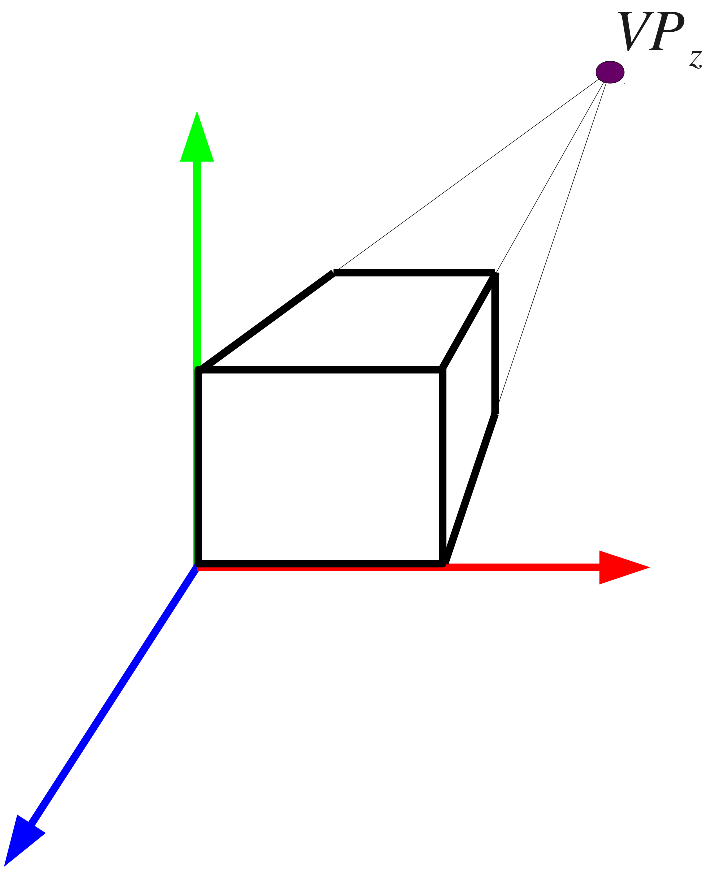
$$X' = P(z=0).P_r.X$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ rz+1 \end{bmatrix}$$

$$x' = \frac{x}{rz+1}, \quad y' = \frac{y}{rz+1}, \quad z' = 0$$

- If $r = \frac{-1}{z_c}$ then we get $x' = \frac{x}{1-\frac{z}{z_c}}, \quad y' = \frac{y}{1-\frac{z}{z_c}}$

Perspective Projection



- Vanishing point in the z direction.
- Set of lines not parallel to the projection plane converge at a vanishing point.

Perspective Projection

- To find the vanishing point along the z direction we apply the perspective transformation to the point at infinity along the z direction.

$$X' = P_r . X$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} . \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ r \end{bmatrix} \qquad x' = 0, \quad y' = 0, \quad z' = \frac{1}{r}$$

- If $r = -1/z_c$ then we get $z' = -z_c$, i.e., the vanishing point lies an equal distance on the opposite side of the projection plane as the center of projection.

Perspective Projection

Single point perspective

- Centre of projection (CoP) on x axis

$$X' = P_p . X$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ px+1 \end{bmatrix}$$

$$x' = \frac{x}{px+1}, \quad y' = \frac{y}{px+1}, \quad z' = \frac{z}{px+1}$$

- CoP is at $(-1/p, 0, 0, 1)$, VP is at $(1/p, 0, 0, 1)$

Perspective Projection

Single point perspective

- Centre of projection (CoP) on y axis

$$X' = P_q . X$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & q & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ qy+1 \end{bmatrix}$$

$$x' = \frac{x}{qy+1}, \quad y' = \frac{y}{qy+1}, \quad z' = \frac{z}{qy+1}$$

- CoP is at $(0, -1/q, 0, 1)$, VP is at $(0, 1/q, 0, 1)$

Perspective Projection

Two point perspective

$$P_{pq} = P_p . P_q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & q & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & q & 0 & 1 \end{bmatrix}$$

$$x' = \frac{x}{px + qy + 1}, \quad y' = \frac{y}{px + qy + 1}, \quad z' = \frac{z}{px + qy + 1}$$

- Two vanishing points
- Two CoPs?

From http://gatewayhsart.wordpress.com

Perspective Projection

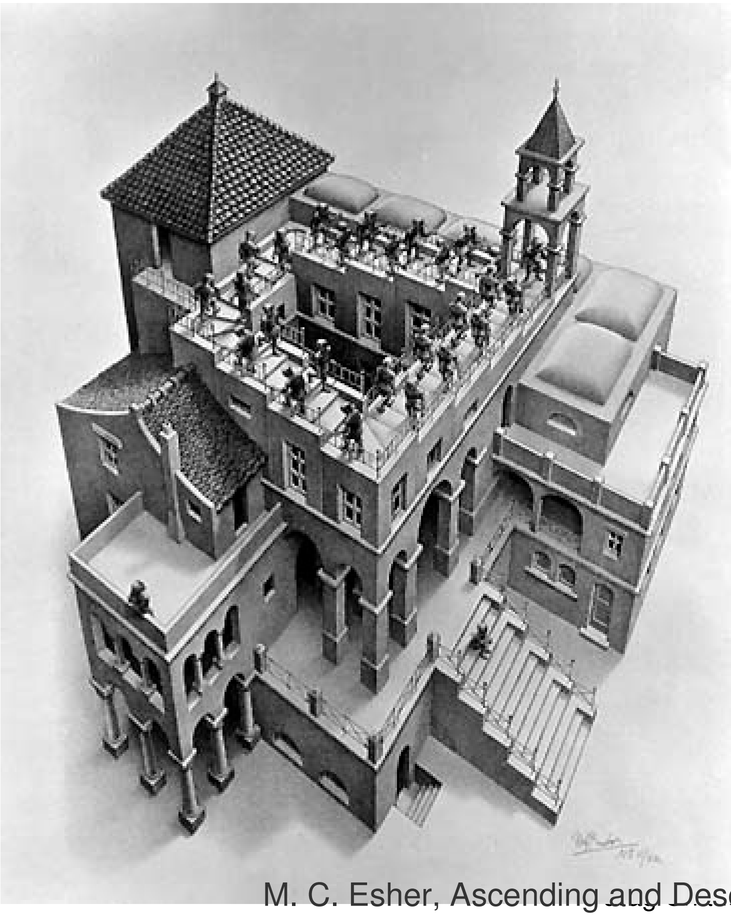
Three point perspective

$$P_{pqr} = P_p . P_q . P_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & q & r & 1 \end{bmatrix}$$

$$x' = \frac{x}{px+qy+rz+1}, \quad y' = \frac{y}{px+qy+rz+1}$$

$$z' = \frac{z}{px+qy+rz+1}$$

- Three vanishing points
- Three CoPs?



M. C. Esher, Ascending and Descending

Perspective Projection

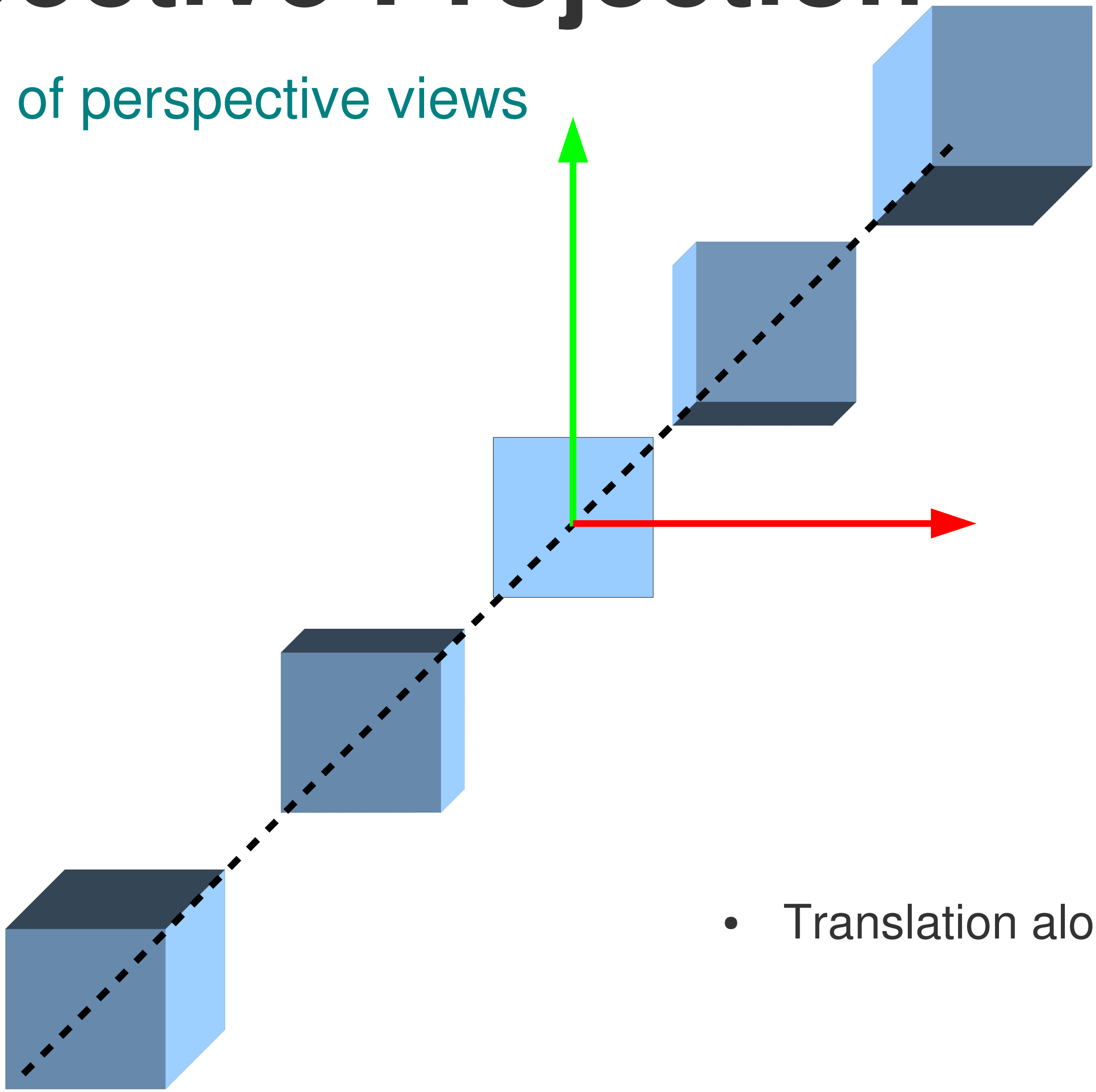
Generation of perspective views

- Transform and then apply single point persepective.
- Let us try to translate, apply a perspective and project to z=0

$$T = P_r(z=0).T(l,m,n) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & rn+1 \end{bmatrix}$$

Perspective Projection

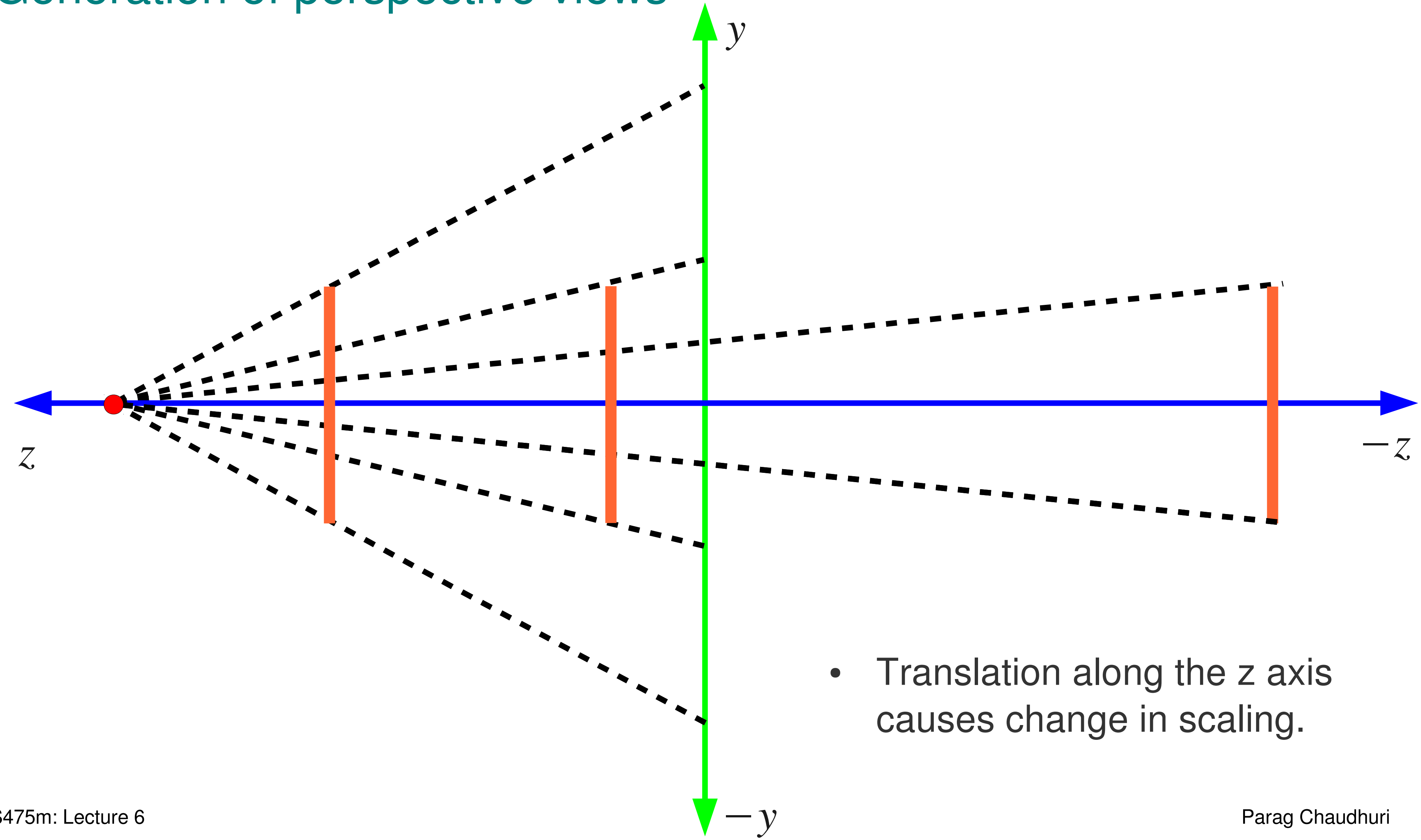
Generation of perspective views



- Translation along x=y line.

Perspective Projection

Generation of perspective views



- Translation along the z axis causes change in scaling.

Perspective Projection

Generation of perspective views

- Rotate about y axis and then apply single point persepective projection.

$$T = P_r(z=0) \,.\, R_y(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} . \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Perspective Projection

Generation of perspective views

- Rotate about y axis and then apply single point persepective projection.

$$T = P_r(z=0).\,R_y(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} . \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -r\sin\phi & 0 & r\cos\phi & 1 \end{bmatrix}$$

- We get a two point perspective.

Perspective Projection

Generation of perspective views

- Rotate about y axis, x axis and then apply single point persepective projection.

$$T = P_r(z=0) \,.\, R_x(\theta) \,.\, R_y(\phi)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Perspective Projection

Generation of perspective views

- Rotate about x axis, y axis and then apply single point persepective projection.

$$T = P_r(z=0).\, R_y(\phi), R_x(\theta)$$

$$= \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ \sin\theta\sin\phi & \cos\theta & -\sin\theta\cos\phi & 0 \\ 0 & 0 & 0 & 0 \\ -r\cos\theta\sin\phi & r\sin\theta & r\cos\theta\cos\phi & 1 \end{bmatrix}$$

- We get a three point perspective.

Taxonomy

- Planar Projections
  - Parallel
    - Orthographic
      - Front
      - Top
      - Side
    - Axonometric
      - Trimetric
      - Dimetric
      - Isometric
    - Oblique
      - Cavalier
      - Cabinet
  - Perspective
    - One Point
    - Two Point
    - Three Point