The Democratic Critiques and Populism

Problem Set - Game Theory

GV482*

This problem set will serve to illustrate many of the points we see in our lectures on the democratic critiques and on populism: the representation question and representation failure, how the latter opens the door to populists, when populists actually enter, how mainstream parties react. As such, it will be with you for a while (part of it will be solved in Week 2, other parts in Week 3, and you will finish the problem set in Week 4).

The basic idea is that political inequality (here, in term of attention to politics, but it could also be unequal turnout or resources) leads office-motivated candidates to cater to the needs of the elite rather than the common people. This very much opens the possibility for, what we will call, a representation failure. You will cover this aspect of the problem set (Part I) in the seminar on Thursday 27 January.

When the interests of the common people are not taken care of, populists may find it optimal to enter the electoral arena. The question is when will they do so and what the electoral consequences of their entry are. You will see that the populist's entry is intrinsically linked to the problem of representation failure. You will cover the voters electoral decisions with populist and (probably part of) the populist's entry decision in the seminar on Thursday 3 February.

The final part of this problem set regards the reaction of mainstream parties. Populist's entry, as you will see, has large electoral consequences. It seems that the mainstream parties should adjust their electoral strategy in anticipation to block the populists. You will study why this does not happen in the seminar on Thursday 10 February.

This problem set is based on Crutzen and co-authors' (2020) excellent paper (available here). The set-up you study contains a few differences with the original work, so it is not always helpful to refer

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to Crutzen et al. (2020). The problem set contains many questions as I try to help you navigate intermediary steps. But all questions should be quite short. If you get stuck answering one, do not hesitate to skip it since all results are given to you (you need to show the steps that lead from the premises to the provided results). I have also added comments on the set-up (in gray, italics, and sometimes between parentheses to highlight the important points, explain the set-up substantively, and offer some guidance on answering some harder questions).

Part I - The game without populist (to be solved on Thursday 26 January)

Consider an electorate of mass one, divided into two groups: Elite (E) and Common people or commoners (C). A proportion $\sigma < 1/2$ of the population belong to the Elite and a proportion $1-\sigma$ belong to the Common people.

(Comment: Think of it this way. There are millions of voters, a majority of them are common people and a minority are from the elite. For example, $\sigma = 0.2$ says there are 20% of voters from the elite and 80% belonging to the common people. Rather than working with millions, we say we have a continuum of voters, but the idea is the same.)

There are two mainstream candidates A and B who compete for an office they value. That is, they are office-motivated and gets a payoff of 1 when elected and 0 otherwise.

(Comment: This just says that they only care about winning the election. These candidates do not care about policies. When they win they get a positive utility/payoff/gain from it. When they lose, they get nothing.)

To win the election, each candidate commits to a policy choice x on a single-issue, with $x \in \{0, 1\}$.

Comment: The candidates can only do one of two policies: 0 or 1. Think about it as political orientations. The candidates can, for example, propose a open door platform ($x = 1$) or a protectionist platform ($x = 0$), both in terms of trade and immigration. Or it could be a minority rights platform or a traditionalist platform. Of course, it is a simplistic way to think about campaigns, but it is a way to think about political stances in a simple way.)

The impact of the policy for each group depends on an underlying state of the world, specific to each group $J \in \{C, E\}$: $\omega_J \in \{0, 1\}$. All citizens from group J gets a payoff of one if the policy matches their group-specific state and zero otherwise.

Comment: This says that the right policy for each group is uncertain. Continuing with our example *of open/protectionist policies. There are times when the elite benefits from open policies and times when they benefit from protectionist policies (e.g., the XIXth century). Same for the common voters. The 'state' ω_E captures what the best policy is for the members of the elite. That is, when $\omega_E = 1$, the best policy for the elite is open borders ($x = 1$). In contrast, when $\omega_E = 0$, the best policy for the elite is protectionist ($x = 0$). In turn, ω_C represents what the best policy is for the common voters. As above, when $\omega_C = 0$, then the best policy for a commoner voter is protectionist ($x = 0$), when $\omega_C = 1$, the optimal policy for a common person is open borders ($x = 1$). The important assumption is that when the elite benefits from protectionist policies, the common voters may, in turn, benefit from open policies, and vice versa (in formal language, the two states do not need to be equal, we can have $\omega_E = 1$ and $\omega_C = 0$, or vice versa). That is, the best policies for the elite and common voters are sometimes the same, sometimes different. We discuss this alignment/misalignment below in greater details.*

On top of it, each citizen receives a common valence shock δ, which is uniformly distributed over $[-\overline{\delta}, \overline{\delta}]$. We will impose some restriction on $\overline{\delta}$ below, but for the moment it is enough to add that it is relatively small. As will become clear, the valence shock serves to make sure citizens are almost never indifferent between the two mainstream candidates.

Comment: The valence shock captures the idea that other aspects, on top of the candidates' program, matter for voters' decisions. Here, we suppose that these other aspects are completely exogenous (out of the candidates' controls) as many before us. There are other models which make them endogenous. The valence shock serves to designate a winner when different candidates propose the same platform since one candidate is always favoured by the valence shock.

That is, we can write the players' payoffs as follows. For a citizen i from group $J \in \{C, E\}$, the payoff from voting for candidate A and B as a function of their policy choice x_A and x_B and the valence shock is, respectively:

$$U_{i,J}(A) = \begin{cases} 1 & \text{if } x_A = \omega_J \\ 0 & \text{if } x_A \neq \omega_J \end{cases} + \delta \quad \text{and} \quad U_{i,J}(B) = \begin{cases} 1 & \text{if } x_B = \omega_J \\ 0 & \text{if } x_B \neq \omega_J \end{cases} - \delta$$

An alternative notation for this sort of payoff is $\mathbb{I}_{\{x=\omega_J\}}$ an indicator function taking value one if the condition $x = \omega_J$ is satisfied and zero otherwise.

Comment: This is a representation of the idea discussed above. If the policy that maximizes the interest of the elite ($J = E$) is implemented ($x = \omega_E$), then a voter i from the elite ($J = E$) gets a positive payoff of 1. Otherwise, this voter gets a payoff of zero. So an elite voter ($J = E$) only gains from a protectionist policy ($x = 0$) if it is the best policy for its group ($\omega_E = 0$). On top of this, we have the valence shock which favours A when positive and B when negative.

For the mainstream candidates A and B, the payoff is a function of their electoral success:

$$U_K = \begin{cases} 1 & \text{if in office} \\ \\ 0 & \text{if not in office} \end{cases}, \quad K \in \{A, B\}$$

Comment: This is just a way to represent what we said before. If a mainstream candidate wins, he gets a payoff of one. He is happy to be in office. If a mainstream candidate loses, he gets a payoff of 0, he is unhappy. But the policy such candidate implements has no bearing on his payoff (on his utility).

Let us now turn to players' information. The commonly shared and common knowledge prior is that for both groups, each state is equally likely: $Pr(\omega_E = 1) = Pr(\omega_C = 1) = 1/2$.

Comment: This is to say (continuing our example) that there is half a chance that the elite benefits from an open policy ($x = 1$) and half a chance the elite benefits from a protectionist policy ($x = 0$). Same for the common voters.

We further assume that the states for both group are correlated. This does not matter much in this first part of the problem set, but will play an important part as we move forward. The correlation between the two states is represented in the next table.

ω_C/ω_E	0	1
0	$\frac{1}{2}\alpha$	$\frac{1}{2}(1-\alpha)$
1	$\frac{1}{2}(1-\alpha)$	$\frac{1}{2}\alpha$

Table 1: Joint distribution of the states

This table may be a little bit hard to read. It is telling you the realization of the possible states (i.e., optimal policies for both groups). So let's take the case of $\omega_C = 0$ and $\omega_E = 0$. Still with our example, this means that the optimal policies for both groups is the protectionist policy. In these cases, the interest of both groups are well aligned. When does it happen that both groups would benefit from the protectionist policies? Well, it happens with probability $\alpha/2$ (e.g., suppose $\alpha = 1/2$, then it happens with probability $1/4$). Now, take the case of $\omega_C = 0$ and $\omega_E = 1$. This means that for the common voters, the optimal policy is protectionist, but for the elite, it is the open policy. This case happens with probability $\frac{1-\alpha}{2}$. The rest of the table can be read in a similar way.

The parameter α measures the correlation between the state for the elite (which we also refer as their preferred policy) and the state of the commoners (which we also call commoners' preferred policy). When $\alpha \geq 1/2$, the interests of the elite and of the commoners are relatively well aligned (the two states are more likely than not to be the same). When $\alpha < 1/2$, the interests of the elite and of the commoners are poorly aligned (the two states are more likely than not to be different).

At the beginning of the game the states are unknown. Before choosing their respective platforms, Candidates A and B observe the states of the world specific to the elite and to the common people.

Comment: This means that the two mainstream candidates know what is the optimal policies for both groups. Whatever policies they choose, they choose it knowingly.

Before casting her ballot, citizen i in group J receives a signal about her group-specific state: ω_J, denoted $s_{i,J} \in \{0, 1\}$.

Comment: This is how we model the gathering of new information. Think about yourself. When you read a news story, you get some new pieces of information. This makes you adjust how you think about issues. Well, in a model, this process is represented as a signal. Here each voter from each group receives some pieces of information, some signals, about what the best policy for them is.

We assume that for the elite voters, the signal is perfectly informative: $Pr(s_{i,E} = 1|\omega_E = 1) = Pr(s_{i,E} = 0|\omega_E = 0) = 1$. So at the time of voting, elite members are perfectly informed of their interest.

Comment: Here, it tells you that the pieces of information for the elite perfectly reveal to them where their interest lies. That is, the elite voters are perfectly informed of what the optimal policy for the group is.

For the common people, the signal is noisy: $Pr(s_{i,C} = 1|\omega_C = 1) = Pr(s_{i,C} = 0|\omega_C = 0) = p$. Voters' signals are independent conditional on the state of the world.

Comment: In contrast to the elite voters, common voters do not learn for sure where their interest lies. They get some information about it, but are not certain of what is best for them. The value p denotes the quality of their information. Higher p means better information (indeed, for the elite voters, p is equal to one). The notion that signals are independent conditional on ω_C means that if one commoner receives signal $s_{i,C} = 1$, it does not mean that all commoners received the same signals (the same piece of information)!

The game, in turn, proceeds as follows:

0. Nature draws the states ω_E and ω_C.
1. Candidates A and B learn both states, each citizen i in group J receives one signal $s_{i,J}$.
2. Candidates A and B simultaneously choose their platform: $x_A \in \{0, 1\}$, $x_B \in \{0, 1\}$.
3. All citizens observe δ and simultaneously vote for A or B.
4. The candidate who receives the most votes is elected and implements his platform. Payoffs are realized, game ends.

The equilibrium concept is Perfect Bayesian Equilibrium. We assume throughout that citizens vote sincerely (i.e., for the candidate who maximises their expected payoff). To avoid a multiplication of cases, we impose several assumptions on the parameter values. We suppose that commoners' signal is informative: $p > \frac{1}{2}$.

Comment: The common voters learn something. They are not completely uninformed. They can make use of the piece of information they receive to adjust their opinion of what the optimal policy for them is.

Throughout, we assume that $\overline{\delta} < \max\left\{\frac{2\alpha-1}{2}, \frac{1-2\alpha}{2}\right\}$ (we have not made an assumption that $\alpha > 1/2$ or $\alpha < 1/2$, hence the need for a maximum in the assumption).

This assumption guarantees that platforms (x) are much more important than the valence shock (δ) in the voters' decisions.

Finally, we assume that $1 - p < \alpha < p$ so that an individual's signal is always his most informative piece of information.

Table 2 provides a point of reference by describing the main parameters and variables from Part 1.

Variables	Definition
σ	share of elite voters
$\omega_J \in \{0,1\}$	State of the world for group $J \in \{E, C\}$
α	Correlation between ω_E and ω_C
$s_{i,C} \in \{0,1\}$	Signal for a commoner
p	Probability commoners' signal is correct
δ	Valence shock
$x_K \in \{0,1\}$	Platform of a mainstream candidate $K \in \{A, B\}$

Table 2: Main variables and parameters (Part I)

To solve this type of games, we proceed backward (i.e., by backward induction). We first consider a voter's strategy as a function of her information and candidates' platforms. This allows us to determine the electoral chances of each candidate in each possible scenario.

Q1 In this question, we first show that there is no equilibrium in which the two candidates propose different platforms. To do so, we suppose that one candidate, here B, always proposes a policy, which matches the commoners' state of the world.

(a) Suppose that A and B propose different platforms. Explain briefly why all commoners vote for B under the scenario of this question.

(b) Suppose that the state of the world for the elite is $\omega_E = 0$ and the state of the world for the commoners is $\omega_C = 0$ (so candidate B proposes $x_B = 0$). Suppose candidate A proposes $x_A = 1$. Explain briefly why A receives no vote then.

(c) Suppose that the state of the world for the elite is $\omega_E = 0$ and the state of the world for the commoners is $\omega_C = 0$ (so candidate B proposes $x_B = 0$). Suppose candidate A proposes $x_A = 0$. Explain briefly why A wins with 50% chances.

The previous questions show that when the states are equal, all candidates converge to the policy matching ω_E and ω_C. We now turn to the case when both states are different.

(d) Suppose the voters anticipate that candidate A picks a policy that matches the state of the world for the elite (i.e., $x_A = \omega_E$). Explain briefly why when $\omega_E = 1$ and $\omega_C = 0$, A's vote share is σ and B's vote share is $1 - \sigma$.

(e) Suppose the voters anticipate that candidate A picks a policy that matches the state of the world for the elite (i.e., $x_A = \omega_E$). Explain briefly why A wins with probability 50% when he deviates and proposes $\hat{x}_A = x_B = 0$. Explain why this shows that there is no equilibrium in which one candidate picks a policy that matches the state of the world for the elite and the other picks a policy that matches the state of the world for the commoners.

From **Q1**, we know that candidates have to converge to one of the two states of the world (either the one for the elite or the one for the commoner).[1]

Q2 In this question, we look for conditions under which the two candidates find it optimal to match ω_C ($x_A = x_B = \omega_C$). To do so, assume in what follows that when a voter observes $x_A \neq x_B$, she votes according to which candidate provides the highest expected utility given her prior and posterior.

(a) Suppose that $\omega_C = 0$ and $\omega_E = 1$. Suppose candidate A deviates and proposes $\hat{x_A} = 1$ instead of $x_A = 0$ as prescribed. Explain briefly why a commoner who receives signal $s_{i,C} = 1$ believes that $\omega_C = 1$ with probability p given candidates' platforms ($Pr(\omega_C = 1|s_{i,C} = 1, x_A = 1, x_B = 0) = p$) and a commoner who receives signal $s_{i,C} = 0$ believes that $\omega_C = 1$ with probability $1 - p$ given candidates' platforms ($Pr(\omega_C = 1|s_{i,C} = 0, x_A = 1, x_B = 0) = 1 - p$).

Comment: Think whether the commoners can learn anything from the platforms. Further, use Bayes' rule to compute the posteriors.

(b) Suppose that $\omega_C = 0$ and $\omega_E = 1$. Suppose candidate A deviates and proposes $\hat{x_A} = 1$ instead of $x_A = 0$ as prescribed. Explain briefly why all common voters who receive signal $s_{i,C} = 1$ votes for A, whereas all common voters who receive signal $s_{i,C} = 0$ votes for B.

Comment: Compute the expected utility from voting for A and B given different signals. Then,

[1] To be complete, we would need to show that A or B also prefers to converge to the same policies (i) when one candidate is known to match the elite's preferred policy and (ii) when one always offers $x = 0$ and the other offers $x = 1$. This can be done adapting the steps from **Q1**.

recall our assumptions on $\overline{\delta}$ relative to α and on p relative to α.

(c) Suppose that $\omega_C = 0$ and $\omega_E = 0$. Suppose candidate A deviates and proposes $\hat{x}_A = 1$ instead of $x_A = 0$ as prescribed. Show that A's vote share is $(1-\sigma)(1-p)$.

Comment: Since we have a mass of citizens, the proportion of citizens who receive a certain signal is equal to the probability one citizen receives a particular signal.

(d) Suppose that $\omega_C = 0$ and $\omega_E = 1$. Suppose candidate A deviates and proposes $\hat{x}_A = 1$ instead of $x_A = 0$ as prescribed. Show that A's vote share is $\sigma + (1-\sigma)(1-p)$.

Comment: As before, since we have a mass of citizens, the proportion of citizens who receive a certain signal is equal to the probability one citizen receives a particular signal.

(e) Suppose that $\omega_C = 0$. Show that an equilibrium in which both candidates match the commoners' state of the world ($x_A = x_B = 0$) exist if and only if $\sigma + (1-\sigma)(1-p) \leq \frac{1}{2}$.

By symmetry of the model, when $\omega_C = 1$, both candidates match the commoners' state of the world ($x_A = x_B = 1$) exist if and only if $\sigma + (1-\sigma)(1-p) \leq \frac{1}{2}$. By following a reasoning similar to what you have done in **Q2**, we can also show that both candidates converge to the elite's state of the world when $\sigma + (1-\sigma)(1-p) \geq \frac{1}{2}$. We will not show this, but you are encouraged to do it to practice your game theory skills.

We label **'responsive equilibrium'** the equilibrium in which $x_A = x_B = \omega_C$ and **'elite-driven equilibrium'** the equilibrium in which $x_A = x_B = \omega_E$. We will use this term in the next parts. But before doing so, we amend the model above to incorporate a third type of candidate: a populist candidate P.

Amending the model for Parts II, III, and IV of the problem set

We now introduce a potential third candidate: P whom we refer to as a populist. The populist cares implementing the right policy for the common people.

(Comment: Unlike the mainstream candidates, P cares about policies when he implements them. He would like to implement a policy that benefits the common voters. In the language of formal

models, we say that the populist is legacy motivated.)

The populist also needs to pay a cost c to run for office.

(Comment: In this model, we consider the possibility of entry from scratch. (As we will see in the lecture, populists have always been around.) As the populist needs to set up the conditions to run, a structure, pay for campaigning, etc., we assume there is a cost of running for P which is not present for the mainstream candidates.)

His utility is then:

$$U_P(x_P) = \begin{cases} 1 & \text{if in office and picks } x_P = \omega_C \\ 0 & \text{otheriwse} \end{cases} - c \times \begin{cases} 1 & \text{if runs} \\ 0 & \text{if does not run} \end{cases}$$

Before deciding whether to run, the populist candidate observes a signal $s_{P,C} \in \{0,1\}$ of the state ω_C, which is as noisy as the a common voter's signal: $Pr(s_{P,C} = 1|\omega_C = 1) = Pr(s_{P,C} = 0|\omega_C = 0) = p$.

Comment: Basically, you can think of the populist as a representative of the pool of common voters. He has the same information as them. He has the same policy preferences, but he needs to pay the cost of running (which he does not necessarily wants to).

For a citizen i from group J, the payoff from voting for the populist is simply:

$$U_{i,J} = \begin{cases} 1 & \text{if } x_P = \omega_J \\ 0 & \text{if } x_P \neq \omega_J \end{cases}$$

Notice **importantly** that the valence shock does not affect the payoff from voting for the populist.

The modified game, in turn, proceeds as follows:

0. Nature draws the states ω_E and ω_C.

1. Candidates A and B learn both states, each citizen i in group J receives one signal $s_{i,J}$. The Populist P also receives his own signal $s_{P,C}$.

2. Candidates A and B simultaneously choose their platform: $x_A \in \{0,1\}$, $x_B \in \{0,1\}$.

3. The Populist P decides whether to enter. If he runs, he chooses a platform $x_P \in \{0,1\}$.

3. All citizens observe δ and simultaneously vote for A, B, or (when P enters) P.

4. The candidate who receives the most votes is elected and implements his platform. Payoffs are realized, game ends.

The equilibrium concept is Perfect Bayesian Equilibrium. On top of the assumptions above ($p > \frac{1}{2}$, $1-p < \alpha < p$, and $\overline{\delta} < \max\left\{\frac{2\alpha-1}{2}, \frac{1-2\alpha}{2}\right\}$), we also assume that running is costly, but not prohibitive for the populist: $1/2 < c < p$.

In the parts below, we will also make use of the following notation. We denote $\mu(s_{i,C}, x_A, x_B, x_P)$ a commoner's posterior that the state ω_C takes value 1 after observing her signal $s_{i,C}$, mainstream candidates' platforms x_A and x_B and (potentially) the populist's platform if he enters x_P (we will use $\emptyset$ if he does not enter). We denote $\rho(s_{P,C}, x_A, x_B)$ the posterior of the populist that the commoners' state is one after observing his signal and the two mainstream candidates' platforms.

Table 3 provides a point of reference by describing all the model's main parameters and variables.

Variables	Definition
σ	share of elite voters
$\omega_J \in \{0,1\}$	State of the world for group $J \in \{E, C\}$
α	Correlation between ω_E and ω_C
$s_{i,C} \in \{0,1\}$	Signal for a commoner
$s_{P,C} \in \{0,1\}$	Signal of the the populist
p	Probability commoners' and P's signal is correct
δ	Valence shock on voters' payoff from voting for *mainstream* candidates
$x_K \in \{0,1\}$	Platform of mainstream candidates
$x_P \in \{0,1\}$	Platform of populist if he runs
$\rho(s_{P,C}, x_A, x_B)$	P's posterior
$\mu(s_{i,C}, x_A, x_B, x_P)$	A common person i's posterior

Table 3: Main variables and parameters (with populist)

Part II - Voters' electoral decision with populist (to be solved on Thursday 2 February)

Q3 We first consider the voters' electoral decisions when the mainstream candidates find it optimal to converge to the commoners' preferred policy (i.e., citizens correctly anticipate that $x_A = x_B = \omega_C$).

(a) Suppose $x_A = x_B = 0$. Explain briefly why the populist gets no vote if $x_P = 0$.

Comment: Recall that the valence shock does not affects the payoff from voting for P.

(b) Suppose $x_A = x_B = 0$ and $\omega_E = 0$. Explain briefly why the populist gets no vote if $x_P = 1$.

(c) Suppose $x_A = x_B = 0$ and $\omega_E = 1$. Explain briefly why the populist gets $\sigma\%$ of the vote if $x_P = 1$.

Of course, a similar reasoning holds for $x_A = x_B = 1$. Hence, in a responsive equilibrium, if P enters, then P would get at most $\sigma\%$ of the vote (if he enters at $x_P \neq x_A = x_B$ and it happens that $\omega_E \neq \omega_C$).

For now and until noted otherwise, assume that both mainstream candidates propose the same platform and this platform is the elite's preferred policy. In other words, after learning that $\omega_E = 1$ (*in our example, the best policy for the elite is open borders*), candidates A and B both propose $x_A = x_B = 1$. After learning that $\omega_E = 0$ (*in our example, the best policy for the elite is protectionism*), candidates A and B both propose $x_A = x_B = 0$. A direct consequence is that the common voters learn ω_E from the platforms of the mainstream candidates. In part C, we will see that, indeed, matching the elite's optimal policy is a best response for the mainstream candidates (i.e., this is what gives them the highest expected payoff).

For now and until otherwise noted, assume as well that the populist, when he enters the electoral race, proposes a platform that matches his signal $s_{P,C}$. That is, if the information of the populist tells him $x = 1$ is the best policy for the voter (i.e., $s_{P,C} = 1$), then when the populist enters the race, he chooses platform $x_P = 1$. If the information of the populist tells him $x = 0$ is the best policy for the voter (i.e., $s_{P,C} = 0$), then when the populist enters the race, he chooses platform $x_P = 0$.

Q4: In this question, we are going to assume that $x_P \neq x_A = x_B$. That is the populist does not propose the same policy as the mainstream candidates. We are going to determine what a common voter i should do.

(a) Suppose a commoner has received signal $s_{i,C} = 1$. Further, the platforms satisfy $x_A = x_B = 1$, and $x_P = 0$. Explain carefully, but briefly why the expected payoff from voting for the populist voter is: $(1 - \mu(1, 1, 1, 0))$.

(b) Suppose a commoner has received signal $s_{i,C} = 1$. Further, the platforms satisfy $x_A = x_B = 1$, and $x_P = 0$. Explain carefully, but briefly why a commoner i votes for the populist if and only if

$\mu(1,1,1,0) < \frac{1-|\delta|}{2}$.

Comment: Recall that $|\cdot|$ is the absolute value function. Think about how the valence shock affects for which mainstream candidate voters cast a ballot.

(c) Suppose a commoner has received signal $s_{i,C} = 1$. Further, the platforms satisfy $x_A = x_B = 0$, and $x_P = 1$. Explain carefully, but briefly why the expected payoff from voting for the populist voter is: $\mu(1,0,0,1)$.

(d) Suppose a commoner has received signal $s_{i,C} = 1$. Further, the platform satisfies $x_A = x_B = 0$, and $x_P = 1$. Explain carefully, but briefly why a commoner i votes for the populist if and only if $\mu(1,0,0,1) > \frac{1+|\delta|}{2}$.

(e) More generally, show that after signal $s_{i,C}$, a common voter i casts a ballot for P if and only if:

- $\mu(s_{i,C}, x_A, x_B, x_P) < \frac{1-|\delta|}{2}$ if $x_A = x_B = 1$ and $x_P = 0$,
- $\mu(s_{i,C}, x_A, x_B, x_P > \frac{1+|\delta|}{2}$ if $x_A = x_B = 0$ and $x_P = 1$.

Q5: The previous question shows us the importance of looking at the common voter's belief. We are going to compute this belief in this question. The function $\mu(\cdot)$ represents a voter's posterior (belief) that the state ω_C is one given his information: $\mu(s_{i,C}, x_A, x_B, x_P) = Pr(\omega_C = 1|s_{i,C}, x_A, x_B, x_P)$. By Bayes' rule, we then have:

$$\mu(s_{i,C}, x_A, x_B, x_P) = \frac{Pr(s_{i,C}, x_A, x_B, x_P|\omega_C = 1)Pr(\omega_C = 1)}{Pr(s_{i,C}, x_A, x_B, x_P|\omega_C = 1)Pr(\omega_C = 1) + Pr(s_{i,C}, x_A, x_B, x_P|\omega_C = 0)Pr(\omega_C = 0)}$$

Comment: You need to know how Bayes' rule works. If the final exam requires the use of Bayes' rule, I won't give you the formula.

There are different terms we are going to consider in turn. To do so, we are going to work with the case when $x_A = x_B = 0$ and $x_P = 1$ in this question. We will consider the other case in **Q5**.

(a) The first term, we want to consider is the following: $Pr(s_{i,C}, x_A = 0, x_B = 0, x_P = 1|\omega_C = 1)$. Justify carefully why, under the assumptions, this probability satisfies:

$$Pr(s_{i,C}, x_A = 0, x_B = 0, x_P = 1|\omega_C = 1) = Pr(s_{i,C}, \omega_E = 0, s_{P,C} = 1|\omega_C = 1)$$

(Hint: Recall the strategies of A, B, and P. What can the voter learn from them?)

(b) Which assumption allows us to write:

$$Pr(s_{i,C}, \omega_E = 0, s_{P,C} = 1 | \omega_C = 1) = Pr(s_{i,C} | \omega_C = 1) Pr(\omega_E = 0 | \omega_C = 1) Pr(s_{P,C} = 1 | \omega_C = 1)?$$

Comment: Do not worry if you don't know the answer to this question. Just take the result as given.

(c) We are going to determine this quantity: $Pr(\omega_E = 0 | \omega_C = 1)$. Explain carefully why:

$$Pr(\omega_E = 0 | \omega_C = 1) = \frac{Pr(\omega_E = 0, \omega_C = 1)}{Pr(\omega_C = 1)} = \frac{\frac{1}{2}(1-\alpha)}{\frac{1}{2}}$$

Comment: Here you need to show why the two equalities are true. That is, why can we write $Pr(\omega_E = 0 | \omega_C = 1)$ *as* $\frac{Pr(\omega_E=0,\omega_C=1)}{Pr(\omega_C=1)}$ *and then why* $\frac{Pr(\omega_E=0,\omega_C=1)}{Pr(\omega_C=1)} = \frac{\frac{1}{2}(1-\alpha)}{\frac{1}{2}}$.

(d) Combining the results from (a) to (c), show that

$$Pr(s_{i,C}, x_A = 0, x_B = 0, x_P = 1 | \omega_C = 1) = Pr(s_{i,C} | \omega_C = 1) \times (1-\alpha) \times p$$

(Hint: Recall the distribution of signals for the populist. When does this piece of information correctly reveals the optimal policy for the common voters?)

(e) We are now going to work with $Pr(s_{i,C}, x_A = 0, x_B = 0, x_P = 1 | \omega_C = 0)$. Repeating steps (a) to (d) above, show that

$$Pr(s_{i,C}, x_A = 0, x_B = 0, x_P = 1 | \omega_C = 0) = Pr(s_{i,C} | \omega_C = 0) \times \alpha \times (1-p)$$

(f) Let's suppose that $s_{i,C} = 1$. Show the following step:

1. $Pr(s_{i,C} = 1, x_A = 0, x_B = 0, x_P = 1 | \omega_C = 1) = p \times (1-\alpha) \times p$

2. $Pr(s_{i,C} = 1, x_A = 0, x_B = 0, x_P = 1|\omega_C = 0) = (1-p) \times \alpha \times (1-p)$

3. $\mu(s_{i,C} = 1, x_A = 0, x_B = 0, x_P = 1) = \frac{(1-\alpha)p^2}{(1-\alpha)p^2 + \alpha(1-p)^2}$

(g) Let's suppose that $s_{i,C} = 0$. Proceeding with the same steps as in (f), show that

$$\mu(s_{i,C} = 0, x_A = 0, x_B = 0, x_P = 1) = \frac{(1-\alpha)p(1-p)}{(1-\alpha)p(1-p) + \alpha(1-p)p} = 1 - \alpha$$

We have now recovered the posterior in this case. Now, let's try to make sense of it. First, we are going to compare a common voter's posterior (our function μ) to her prior belief that the optimal policy for her group is one, which equals 1/2.

(h) Show that $\mu(s_{i,C} = 1, x_A = 0, x_B = 0, x_P = 1) > 1/2$. Provide some intuition why.

Comment: The first part ("Show that") is just a matter of algebra (good practice!). The second part is more important. Think about what the three different pieces of information tells the voter (what are these three pieces? Get back to (a) to find out). How does she wait them?

(i) In turn, $\mu(s_{i,C} = 0, x_A = 0, x_B = 0, x_P = 1) > 1/2$ if and only if $\alpha < 1/2$. Why is that?

(j) Show that all common citizens who receive a signal $s_{i,C} = 1$ vote for P for all values of α and common citizens who receive a signal $s_{i,C} = 0$ vote for P if and only if $\alpha < 1/2$.

*Comment: Combine your answers to **Q4**(e) and **Q5**(f) and (g). Recall our assumptions on $\overline{\delta}$, the upper bound of δ, and p.*

Q6: We are now considering the case when the mainstream parties are proposing policy 1 ($x_A = x_B = 1$) and the populist candidate is offering the policy 0 ($x_P = 0$).

(a) Show that:

$$
\begin{aligned}
\mu(s_{i,C} = 0, x_A = 1, x_B = 1, x_P = 0) =& \frac{(1-p)^2\alpha}{(1-p)^2\alpha + p^2(1-\alpha)} \\
\mu(s_{i,C} = 1, x_A = 1, x_B = 1, x_P = 0) =& \alpha
\end{aligned}
$$

*Comment: One way to go is to reproduce steps (a)-(g) of **Q5**. Another possibility is to make note of some specificities of the model. If you are not familiar with Bayes' rule, it is a better exercise to repeat **Q5**(a)-(g). If you are familiar with Bayes' rule, it is better to try the shortcut.*

(b) Show that $\mu(s_{i,C} = 0, x_A = 1, x_B = 1, x_P = 0) < 1/2$. Provide some intuition why.

*Comment: For (b) here and (c) below, your intuition should be the same as for **Q5**(h) and (i).*

(c) In turn, $\mu(s_{i,C} = 1, x_A = 1, x_B = 1, x_P = 0) > 1/2$ if and only if $\alpha > 1/2$. Why is that?

(d) Show that all common citizens who receive a signal $s_{i,C} = 0$ vote for P for all values of α and common citizens who receive a signal $s_{i,C} = 1$ vote for P if and only if $\alpha < 1/2$.

Part III - The Populist's entry decision (part of this will be solved on Thursday 2 February, the rest on Thursday 9 February)

In the previous part, we have computed the electoral decision of a voter from group C as a function of her information. We use our previous answers to compute P's electoral chances and entry decision under different scenarios.

Q7 In this question, we suppose that mainstream candidates find it optimal to the commoners' preferred policy.

(a) Using your answers to **Q3**, explain briefly why P never wins the election if he enters in this case.

Comment: Recall that the valence shock guarantees that one mainstream candidate is always preferred to the other.

(b) Explain why the populist never enters when $\sigma + (1-\sigma)(1-p) < \frac{1}{2}$.

Q8 In question **Q8** and **Q9**, we turn to the case when mainstream candidates converge to the elite citizens' preferred policy ($x_A = x_B = \omega_E$ and this is anticipated by commoners). In this question, we assume that $\alpha > 1/2$ and look at P's entry decision then.

(a) Assume that mainstream candidates propose $x_A = x_B = 0$ and the populist offers $x_P = 1$. Assume further than the state is $\omega_C = 0$ (i.e., that is the optimal policy happens to be zero for the common voters). Using your answer to **Q5**(i), show that the vote share of candidate P is: $(1-\sigma)(1-p)$. Explain why P does not win the election then.

(c) Assume that mainstream candidates propose $x_A = x_B = 0$ and the populist offers $x_P = 1$. Assume further than the state is $\omega_C = 1$ (i.e., that is the optimal policy happens to be zero for the common voters). Show that the vote share of candidate P is: $(1-\sigma)p$. Explain why P loses the election then.

Comment: Recall from Part I the condition for mainstream candidates to prefer converging to ω_E.

(d) Assume that mainstream candidates propose $x_A = x_B = 1$ and the populist offers $x_P = 0$. Explain why P never wins the election.

(e) Explain briefly why P never enters.

Q9 We now look at P's electoral chances and entry decision when $\alpha < 1/2$.

(a) Assume that mainstream candidates propose $x_A = x_B = 0$ and the populist offers $x_P = 1$. Using your answer to **Q5**(i), show that the vote share of candidate P is: $(1-\sigma)$. Explain why P wins the election then.

(b) Assume that mainstream candidates propose $x_A = x_B = 1$ and the populist offers $x_P = 0$. Explain why P always wins the election then.

We now need to compute P's expected payoff if P enters. We consider the populist's entry decision so we no longer assume that $x_P = s_{P,C}$ (this has to be a choice of the populist). However, we still assume below that citizens *anticipate* that $x_P = s_{P,C}$ if the populist enters (we will see that this anticipation is correct below). In turn, both the citizens and the populist anticipate that the mainstream candidates propose $x_A = x_B = \omega_E$ (again, we will see that this anticipation is correct in Part IV).

To calculate P's expected utility when he enters, we make use of $\rho(s_{P,C}, x_A, x_B)$ the populist's posterior that $\omega_C = 1$ given his signal and the mainstream candidates' platform choices.

(c) Show that the expected payoff of the populist candidate if he enters the race is:

- $0 - c$ if $x_P = x_A = x_B$
- $\rho(s_{P,C}, 0, 0) - c$ if $x_P = 1$ and $x_A = x_B = 0$
- $1 - \rho(s_{P,C}, 1, 1) - c$ if $x_P = 0$ and $x_A = x_B = 1$

Obviously, P does not enter at the same platform as the mainstream candidates. To see, whether P enters at a different platform, we need to compute P's posterior (belief) that the correct policy for group C is $x = 1$ after observing his own signal $s_{P,C}$ and the other candidates' platforms x_A and x_B. Recall as well that we still assume that mainstream candidates' platform is the optimal policy for the elite.

By Bayes' rule,

$$\rho(s_{P,C}, x_A, x_B) = \frac{Pr(s_{P,C}, x_A, x_B|\omega_C = 1)Pr(\omega_C = 1)}{Pr(s_{P,C}, x_A, x_B|\omega_C = 1)Pr(\omega_C = 1) + Pr(s_{P,C}, x_A, x_B|\omega_C = 0)Pr(\omega_C = 0)}$$

We again take these terms in turn.

(d) Suppose that $x_A = x_B = 0$ (at this point, it should be clear, I hope, that this means that the mainstream candidates propose policy 0). Explain briefly why $Pr(s_{P,C}, x_A = 0, x_B = 0|\omega_C = 1) = Pr(s_{P,C}, \omega_E = 0|\omega_C = 1)$.

(e) Suppose that $x_A = x_B = 0$. Explain why we can write:

$$Pr(s_{P,C}, \omega_E = 0|\omega_C = 1) = Pr(s_{P,C}|\omega_C = 1)Pr(\omega_E = 0|\omega_C = 1)$$

Comment: Again, if you cannot see why, just take this result as given.

(f) Suppose that $x_A = x_B = 0$. Show that

1. $Pr(s_{P,C}, \omega_E = 0|\omega_C = 1) = Pr(s_{P,C}|\omega_C = 1)(1 - \alpha)$
2. $Pr(s_{P,C}, \omega_E = 0|\omega_C = 0) = Pr(s_{P,C}|\omega_C = 1)\alpha$
3. $\rho(s_{P,C}, x_A = 0, x_B = 0) = \frac{Pr(s_{P,C}|\omega_C=1)(1-\alpha)}{Pr(s_{P,C}|\omega_C=1)(1-\alpha)+Pr(s_{P,C}|\omega_C=0)\alpha}$

(g) Using the previous result, show that

$$\rho(s_{P,C} = 1, x_A = 0, x_B = 0) = \frac{p(1-\alpha)}{p(1-\alpha) + (1-p)\alpha}$$
$$\rho(s_{P,C} = 0, x_A = 0, x_B = 0) = \frac{(1-p)(1-\alpha)}{(1-p)(1-\alpha) + p\alpha}$$

(h) Suppose now that $x_A = x_B = 1$. Show that

$$\rho(s_{P,C} = 1, x_A = 1, x_B = 1) = \frac{p\alpha}{p\alpha + (1-p)(1-\alpha)}$$
$$\rho(s_{P,C} = 0, x_A = 1, x_B = 1) = \frac{(1-p)\alpha}{(1-p)\alpha + p(1-\alpha)}$$

Now that we know P's belief about what the correct policy is, we can think about his entry decision.

(i) Recall that $1/2 < c < p$ and $1 - p < \alpha < p$. Show that when $x_A = x_B = 0$, the populist enters with platform $x_P = 1$ if and only if $s_{P,C} = 1$ and does not enter otherwise.

Comment: You should answer the following questions to get the result: Suppose P enters and proposes $x_P = 1$, what would be his payoff after signal $s_{P,C} = 1$? after signal $s_{P,C} = 0$? Is it bigger or smaller than zero (P's payoff from not running)?

By a similar reasoning as above, we can also show that when $x_A = x_B = 1$, the populist enters with platform $x_P = 0$ if and only if $s_{P,C} = 0$ and does not enter otherwise (you are encouraged to check this yourself). With this in mind, we turn to the mainstream parties' adaptation under the threat of populist entry.

Part IV - The mainstream parties' adaptation

In this last part, we consider how the mainstream parties react to the populist's threat of entry. As we have seen in part III, this threat materializes only if $\sigma + (1-\sigma)(1-p) > 1/2$ and $\alpha < 1/2$ so we will assume this holds in what follows (in all other cases, it is business as normal for the mainstream candidates). Further, we assume that the populist and citizens anticipate that candidates play an elite-driven strategy ($x_A = x_B = \omega_E$). When they observe diverging platform, all citizens follow their signal as we assumed in Part I. In the questions that follow, we consider A's incentive to deviate (keeping B's behavior constant, $x_B = \omega_E$). We ask ourselves: what should A do? Should he propose $x_A = \omega_E$ or should he deviate and go on proposing a different policy? We are going to answer these questions under different possible scenarios.

Q10: We consider the following case: $\omega_E = 0$ so B proposes policy $x_B = 0$, and $\omega_C = 0$ so the optimal policy for the common people is the same as for the elite. **(a)** Suppose A proposes $x_A = 1$. Show that the populist does not enter then.

(b) Suppose A proposes $x_A = 1$. Show that A's vote share is $(1-\sigma)(1-p)$. Explain briefly why A does not win the election then.

(c) Suppose now that $x_A = 0$. Show that if P enters at $x_P = 1$, then A loses the election for sure.

(d) Suppose now that $x_A = 0$. Explain carefully why A's chance of winning the election is $\frac{p}{2}$.
Comment: Here we take an ex-ante perspective. Prior to P's entry decision. So think what is A's winning chances if (i) P does not enter, (ii) if P enters. And then think about what the chances P enters are.

(e) Explain why A also has no incentive to deviate in this case (i.e., prefers to propose $x_A = 0$ over $x_A = 1$).

Q11 We now turn to a different case with $\omega_E = 0$, so B proposes policy $x_E = 0$, and $\omega_C = 1$ so the optimal policy for the common people is different than the optimal policy for the elite.

(a) Explain briefly why A's vote share is $(1-\sigma)p$ if he proposes $x_A = 1$.

(b) Suppose now that $x_A = 0$. Explain carefully why A's chance of winning the election is $\frac{1-p}{2}$.

(c) Explain why A also has no incentive to deviate in this case (i.e., prefers to propose $x_A = 0$ over $x_A = 1$).

Obviously, the same logic applies when $\omega_E = 1$. Further, since A and B face similar incentives, if A has no incentive to deviate so does B.

Q12: Given your answers above, can you provide some intuition why the two mainstream candidates do not react to the threat of entry of the populist candidate?

Congratulations. We have finally shown that there is an equilibrium in which the mainstream candidates cater to the elite ($x_K = \omega_E$, $K \in \{A, B\}$), the populist enters when $s_{P,C} \neq x_K$ and wins the election then. Notice that the populist's entry and success is, in this problem set, the result of a representation failure. This representation failure arises when two conditions are met:

1. Large political inequality (p is sufficiently low so $\sigma + (1-\sigma)(1-p) > (1-\sigma)p$), which leads to an elite-driven equilibrium from mainstream parties.
2. The interests of the elite and the commoners are misaligned ($\alpha < 1/2$), so that the populist can receive the vote of all commoners and has the electoral incentives to enter.

These are the themes we have been developing in the lectures.