Hello! Congratulations on getting to your last assignment to this class.

Notice: **make sure you use the commands figure, subplot, etc.**, so when I run your code I can see all the results you've gotten at once. Also, make sure to have a snapshot of all figures you have generated in this assignment submitted along with your results. I will take this into account when grading your assignment.

Since the problems you will find in this worksheet are classic, I suggest you to read about them and enjoy their beauty. However, **I strongly encourage you to be persistent and believe your calculations**. Many resources in the internet contain typos and might mislead you. Please ask me if you have any questions while you develop your answers. Your grade will mostly reflect the coherence of your work rather than rightness.

Problem 1. Let's play with chaos!

In this problem we will simulate the motion of a double pendulum. This system is composed by two pendulums attached to another. This is a simple example of a physical system which can exhibit chaotic behavior. These are their governing equations:

$$\begin{aligned}(m_1+m_2)l_1\ddot{\theta}_1 + m_2l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) + g(m_1+m_2)\sin(\theta_1) = 0\\ m_2l_2\ddot{\theta}_2 + m_2l_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2l_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) + m_2g\sin(\theta_2) = 0\end{aligned} \tag{1}$$

in which

$$\begin{aligned}x_1 &= l_1\sin(\theta_1);\\ y_1 &= -l_1\cos(\theta_1);\\ x_2 &= x_1 + l_2\sin(\theta_2);\\ y_2 &= y_1 - l_2\cos(\theta_2).\end{aligned} \tag{2}$$

a) (Paper and pencil) Express the equation above in terms of

$$\begin{aligned}A &= (m_1+m_2)l_1,\\ B &= m_2l_2\cos(\theta_1-\theta_2),\\ C &= m_2l_1\cos(\theta_1-\theta_2),\\ D &= m_2l_2,\\ E &= -m_2l_2\sin(\theta_1-\theta_2)\dot{\theta}_2^2 - g(m_1+m_2)\sin(\theta_1)\\ F &= m_2l_1\sin(\theta_1-\theta_2)\dot{\theta}_1^2 - m_2g\sin(\theta_2)\end{aligned}$$

parameters. Your result should only depend on those parameters and θ_1, θ_2 and their derivatives. In other words, you should not have m_1, m_2, l_1, l_2 on your final result.

b) (Paper and pencil) Now, create a vector s with four entries. The first entry should correspond to θ_1, the second to $\dot{\theta}_1$, the third to θ_2, the fourth to $\dot{\theta}_2$. Your result should not have any θ's anymore and you should have a system of four equations involving the entries of s.

c) (MATLAB) Set $m_1 = m_2 = 0.1$ kg, $l_1 = l_2 = 0.3$ m, $g = 9.8$ m/s^2. Create a variable $k = l1 + l2$. Make sure to put the units as a comment on the side of the variable. This helps you and other people reading your code when doing dimensional analysis.

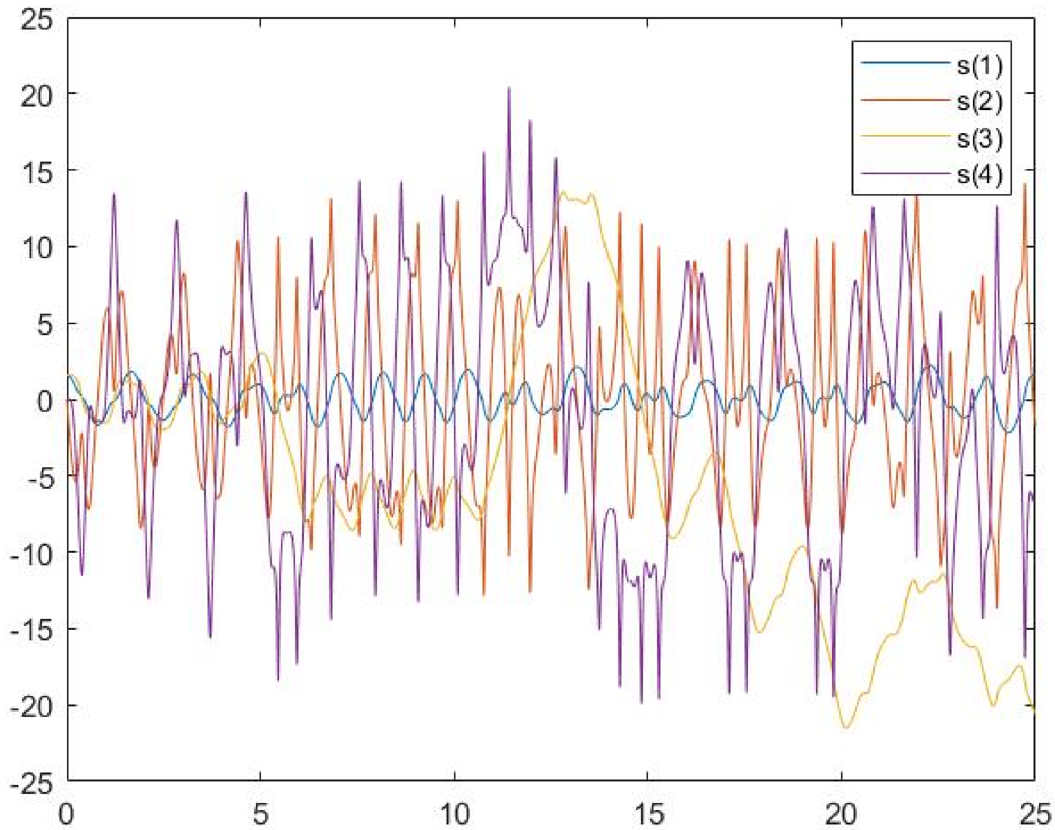
d) (MATLAB) Let's now express our governing equations in terms of a system of first order ODEs:

$$\frac{ds}{dt} = f(t, s)$$

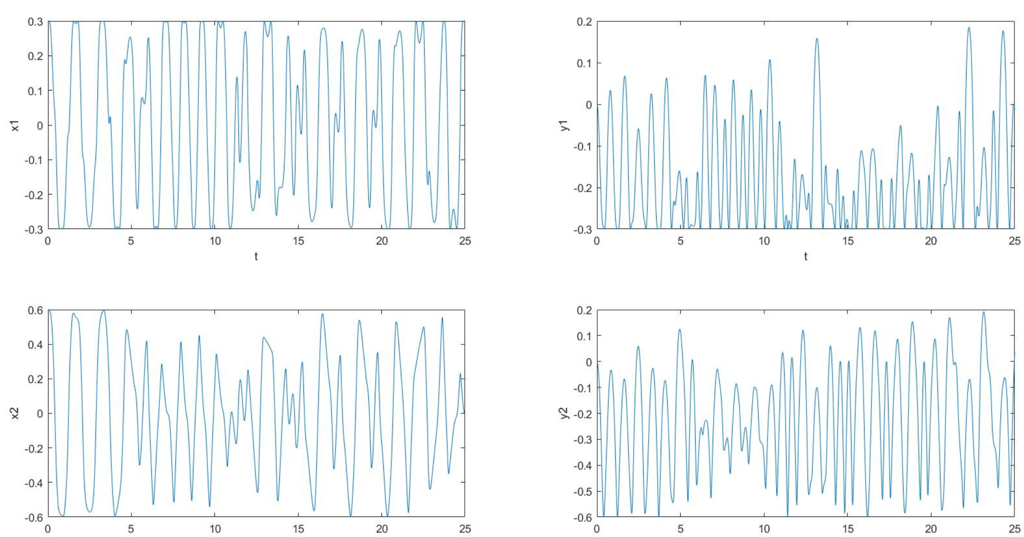
Create a "script" function and name it as `s_prime.m`. Its output should be f, and its input, (t, s). The vector f should have four components that reflect the calculations you did in part b). If you end up calling each entry of f as $f1, f2, f3, f4$, for example, make sure to output $f = [f1; f2; f3; f4]$. (It will give you an error if you do not follow this closely).

e) (MATLAB) Set up your initial conditions to be $\theta_1(0) = \frac{\pi}{2}$, $\dot{\theta}_1(0) = 0$, $\theta_2(0) = \frac{\pi}{2}$, $\dot{\theta}_2(0) = 0$. Remember to express this in terms of entries of s.

f) (MATLAB) Solve the problem using ode45, $t \in [0, T]$, $T = 25$. Plot the components of s against t. You should see something like this:



g) (MATLAB) Now, plot x_1, y_1, x_2, y_2 against t. You should see something like this:



h) (MATLAB) Make a movie using the `animate.m` available on Sakai! Use paramerts: $k = l1 + l2$; $v = 2$ (this is the velocity of the movie, feel free to change this to other natural numbers as you like);

i) Change the parameters so you simulate the simple pendulum from WS11;

j) Play with the parameters and find a situation where the pendulum moves in another interesting way.

k) Set up all the results above so that it is organized, commented, and nothing is over-written when I run your code.

l) Plot $\theta_1 \times \theta_1'$ and $\theta_2 \times \theta_2'$. If you are curious about phase planes, check the link to notes_phase_planes on Sakai.

Problem 2. Finally, we will consider the Lorenz Equations:

$$x' = \sigma(y - x)$$
$$y' = x(\rho - z) - y$$
$$z' = xy - \beta z$$

.

a) For this problem, we will consider $\sigma = 10$, $\rho = 28$, and $\beta = 8/3$. Use your favorite solver to solve these equations with initial conditions $x = 1$, $y = 1$, and $z = 1$ over the time interval $[0, 100]$ (with as many time-steps as your computer can comfortably handle).

b) Use `plot3` to plot your solution in 3d. This image is called the Lorenz Attractor! This system we solved is *chaotic*, meaning that the ultimate solution can vary drastically based on small variations at the beginning. This unpredictability and pretty butterfly shape, in part, inspired the popular concept of the "butterfly effect."

c) Let's add a splash of color! Use a `for` loop to solve these equations for many different initial conditions (think hundreds!) and plot them all on the same graph. You may want to shrink your time interval to $[0, 15]$ - feel free to experiment!

d) Once you have a picture you're happy with, screenshot it and include it in your `.zip` file when you upload your solutions.

I hope you had fun while learning in this class. I look forward to hearing your feedback on how to improve the experience for next students. Thank you again for your engagement and for sharing your presence and thoughts with me this Fall.