Assignment-1

Submit to Manohar at CSTAR on 27^{th} Jan, 2015 between 10AM and 4PM.
(No prior/post submissions allowed)

List of notations

1. $\mathbb{N}$ = Set of Natural Numbers.
2. $\mathbb{Z}$ = Set of Integers.
3. $\mathbb{Q}$ = Set of Rational Numbers.
4. $\mathbb{R}$ = Set of Real Numbers.
5. $M_{m\times n}(\mathbb{R})$ = Set of all $m \times n$ matrices over $\mathbb{R}$.
6. $Inv_n(\mathbb{R})$ = Set of all $n \times n$ invertible matrices over $\mathbb{R}$.

Definitions:

1. $\mathbb{R}^n(\mathbb{R}) = \{(x_1, x_2, x_3, \ldots, x_n)\ /\ x_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}$.
2. **Linear combination** : Let $v_i \in \mathbb{R}^n$ and $a_i \in \mathbb{R}$, for $i = 1, 2, \ldots, k$, then $\sum_{i=1}^{k} a_i v_i$ is called a linear combination of vectors $v_1, v_2, v_3, \ldots, v_k$.
3. **Zero linear combination**: Let $v_1, v_2, v_3, \ldots, v_k \in \mathbb{R}^n$,
   If suppose $\sum_{i=1}^{k} a_i v_i = 0$ then it must be true that $a_i = 0, \forall i = 1, 2, \ldots, k$.
4. Let $A = \{v_1, v_2, v_3, \ldots, v_k\}$ and $A \subseteq \mathbb{R}^n(\mathbb{R})$. A is said to be **independent** if $\exists$ no zero linear combination for any vectors of A. Otherwise we say that A is dependent.

Whenever required use the above notations and definitions to solve the following problems.

1. Which of the following sets form a group under the given operation, Justify your answers.
   (a) $(M_{n\times n}(\mathbb{R}), *)$; * is usual matrix multiplication.
   (b) $(Inv_n(\mathbb{R}), *)$; * is usual matrix multiplication.
   (c) $(\mathbb{N}, *)$; $a * b = a + b + k$, where k is a fixed and $k \in \mathbb{N}$.
   (d) $(\mathbb{Z}, *)$; $a * b = a + b - k$, where k is a fixed and $k \in \mathbb{Z}$.

(e) $(\mathbb{Z}, *)$; $a * b = a + b + ab$.

(f) $(\mathbb{Z} \setminus \{1\}, *)$; $a * b = a + b - ab$.

(g) $(\mathbb{Q} \setminus \{1\}, *)$; $a * b = a + b - ab$.

(h) $(\mathbb{Q}, *)$; $a * b = (a + b)/k$, where k is a fixed and $k \in \mathbb{N}$.

2. Let $M = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$. Find whether $(M, *)$ is group or not under usual matrix multiplication, if it is a group check whether this is abelian group or not.

3. Describe all possible groups with 4 elements.

4. Find out whether the following subsets of $\mathbb{R}^3(\mathbb{R})$ Independent or Dependent.
(a) $A_1 = \{(1, 0, 0), (0, 1, 0)\}$, $A_2 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
(b) $A_3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)(0, 0, 0)\}$, $A_4 = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$
(c) $A_5 = \{(1, 2, 3), (2, 3, 1), (3, 2, 1)\}$, $A_6 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 2, 3)\}$

Prove or Disprove the following statements:

5. Let $(G, *)$ be a group such that $a^2 = e, \forall a \in G$, where e is identity element of G, then $(G, *)$ is an abelian group.

6. If $(G, *)$ is a group such that $\forall a, b \in G, (a * b)^i = a^i * b^i$ for three consecutive integers then G is an abelian group.

7. If G is a finite group of even order, then there exists an element $a \neq e$ such that $a = a^{-1}$(i.e self inverse).

8. Let $(G, *)$ be a semi group(Closure and Associative) such that $\forall a, b, c \in G$ it satisfies the following two conditions then $(G, *)$ is a group.
(a) if $a * b = a * c$ then $b = c$ and
(b) if $b * a = c * a$ then $b = c$

9. Let $(G = \{g_1, g_2, \ldots, g_n\}, *)$ be a group and $x = g_1 * g_2 * \ldots * g_n$ then $x * x = e$, where e is the identity element of G.

10. Zero vector i.e $(0, 0, \ldots, 0)$ must belong to every independent set of $\mathbb{R}^n(\mathbb{R})$.

11. Subset of an independent set is always independent, i.e if $A \subseteq B$ and B is independent then A must be independent. Assume $A, B \subseteq \mathbb{R}^n(\mathbb{R})$.

12. Let H_1, H_2 are two subgroups of a group $(G, *)$ then $(H_1 \bigcup H_2, *)$ is a subgroup of G *if and only if* either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

13. Let $(G, *)$ be a group and for $a \in G$ define $N(a) = \{x \in G | xa = ax\}$ then $N(a)$ is a subgroup of G.

14. Let $(G, *)$ be a group and center of G is defined as
$Z = \{x \in G | xa = ax, \forall a \in G\}$ then Z is a subgroup of G.

15. Let $(G, *)$ be a group and $(H, *)$ is subgroup of G. Define
$\mathbf{C}(H) = \{x \in G | xh = hx, \forall h \in H\}$, then $\mathbf{C}(H)$ is a subgroup of G.

16. Let H, K are two subgroups of a group $(G, *)$.
Define $HK = \{hk | h \in H \text{ and } k \in K\}$ then HK is a subgroup of G.