OPTIMIZATION NIUB NUMBER: 21287070

PROBLEM SET 1: UNCONSTRAINED OPTIMIZATION

6) $I(x,y) = 2x+3y$

$C(x,y) = x^2-2xy+2y^2+6x-9y+5$

The profit function is $P(x,y) = I(x,y) - C(x,y) = 2x+3y-(x^2-2xy+2y^2+6x-9y+5)$

$P(x,y) = -x^2-2y^2+2xy-4x+12y-5$

The problem to solve is the following: $\begin{cases} \max\limits_{x,y} P(x,y) \\ x\geq 0,\ y\geq 0 \end{cases}$ $P: \mathbb{R}^2_+ \to \mathbb{R}$, $P\in C^2$

The maximum can be found where

$\nabla P(x,y) = 0$, $H_P(x,y)$ definite negative

$\frac{\partial P}{\partial x}(x,y) = -2x+2y-4$ $\quad \frac{\partial P}{\partial y}(x,y) = -4y+2x+12$ $\quad \nabla P(x,y) = \begin{bmatrix} \partial P/\partial x \\ \partial P/\partial y \end{bmatrix} = \begin{bmatrix} -2x+2y-4 \\ -4y+2x+12 \end{bmatrix}$

$\nabla P(x,y) = 0 \iff \begin{cases} -2x+2y-4=0 \longrightarrow y = x+2 \\ -4y+2x+12=0 \longrightarrow -4x-8+2x+12=0 \longrightarrow x=2,\ y=4 \end{cases}$

$H_P(x,y) = \begin{bmatrix} \partial^2P/\partial x^2 & \partial P/\partial x\partial y \\ \partial P/\partial y\partial x & \partial P/\partial y^2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}$ $\quad \left.\begin{array}{l} \det H_P = 8-4 = 4 = \lambda_1\lambda_2 > 0 \\ \operatorname{tr} H_P = -6 = \lambda_1+\lambda_2 < 0 \end{array}\right\} \lambda_1<0,\ \lambda_2<0$

$\Rightarrow H_P$ definite negative

The point $(2,4)$ satisfies $\nabla P(2,4)=0$ and $H_P(2,4)$ definite negative, which means it is a maximum for $P(x,y)$.

Furthermore, it satisfies $x\geq 0$ and $y\geq 0$, thus it is a solution for the given problem.

For the pair $(2,4)$ the profit is $P(2,4) = -4-32+16-8+48-5 = 15$ (millions of euros)

12) Let $x\in\mathbb{R}^2$, $A\in\mathbb{R}^{2\times 2}$ Symmetric, $f(x) = x^TAx$ and consider the problem $\begin{cases} \max f(x) \\ \text{s.t. } x^Tx = 1 \end{cases}$

Consider the Lagrangian associated to the problem given:

$L(x,\lambda) = f(x) - \lambda g(x) = x^TAx - \lambda(x^Tx-1)$

An optimal solution of the ploblem must satisfy the KKT conditions

$\nabla_x L(x^*,\lambda^*,\mu^*) = \nabla f(x^*) - \lambda\nabla g(x^*) = 0$

$= 2Ax^* - 2\lambda^* x^* = 0 \longrightarrow Ax^* = \lambda^* x^*$

Thus an optimal is of the form (x^*,λ^*): $\begin{cases} Ax^* = \lambda^* x^* \\ (x^*)^T x^* = 1 \end{cases}$

This means x^* is an eigenvector of A and $\lambda^*\in\mathbb{R}$, since A is symmetric, is the corresponding eigenvalue.

Since $x^Tx = 1$, we have $(x^*)^T\underbrace{Ax^*}_{=\lambda^* x^*} = (x^*)^T\lambda^* x^* = \lambda^* \underbrace{(x^*)^Tx^*}_{=1} = \lambda^*$

The given solution is a maximum of $f(x)$ if $H_f(x^*)$ is negative definite: $H_f(x) = 2A$.

$(x^*)^T H_f(x^*) x^* = (x^*)^T 2A x^* = 2\underbrace{(x^*)^TAx^*}_{=\lambda^*} = 2\lambda^* \longrightarrow H_f(x^*)$ neg. def. $\iff \lambda^* < 0$

Therefore, for (x^*,λ^*) to be a maximum λ^* needs to be a <u>negative</u> eigenvalue of A.

This reasoning can be extended to $n>2$ dimensions under the same conditions. ■

13) Given $y_1, \dots, y_m$, $y_j \in \mathbb{R}^2$ $j=1,\dots,m$, $w_1,\dots,w_m$, $w_j \in \mathbb{R}_+$, $j=1,\dots,m$

consider the problem $\begin{cases} \min \sum_{j=1}^{m} w_j \|x-y_j\| \\ \text{s.t. } x \in \mathbb{R}^2 \end{cases}$

(a) The function $f(x) = \sum_{j=1}^{m} w_j \|x-y_j\| = \sum_{j=1}^{m} w_j \sqrt{(x_1-y_{j1})^2+(x_2-y_{j2})^2}$

is minimized where the function $g(x) = \sum_{j=1}^{m} w_j \|x-y_j\|^2 = \sum_{j=1}^{m} w_j \left[(x_1-y_{j1})^2+(x_2-y_{j2})^2\right]$ is minimized.

Therefore, consider the equivalent problem $\begin{cases} \min g(x) \\ \text{s.t. } x \in \mathbb{R}^2 \end{cases}$

$$\partial g/\partial x_1 = \sum_{j=1}^{m} w_j \left[2(x_1-y_{j1})\right] = 2x_1 \sum_{j=1}^{m} w_j - 2\sum_{j=1}^{m} w_j y_{j1}$$

$$\partial g/\partial x_2 = \sum_{j=1}^{m} w_j \left[2(x_2-y_{j2})\right] = 2x_2 \sum_{j=1}^{m} w_j - 2\sum_{j=1}^{m} w_j y_{j2}$$

$$\Rightarrow \nabla g(x) = 2x\sum_{j=1}^{m} w_j - 2\sum_{j=1}^{m} w_j y_j = \begin{bmatrix} 2x_1\sum_{j=1}^{m} w_j - 2\sum_{j=1}^{m} w_j y_{j1} \\ 2x_2\sum_{j=1}^{m} w_j - 2\sum_{j=1}^{m} w_j y_{j2} \end{bmatrix}$$

$$\nabla g(x) = 0 \iff \begin{cases} x_1 \sum_{j=1}^{m} w_j - \sum_{j=1}^{m} w_j y_{j1} = 0 \\ x_2 \sum_{j=1}^{m} w_j - \sum_{j=1}^{m} w_j y_{j2} = 0 \end{cases} \qquad x^* = \begin{bmatrix} \sum_{j=1}^{m} w_j y_{j1} / \sum_{j=1}^{m} w_j \\ \sum_{j=1}^{m} w_j y_{j2} / \sum_{j=1}^{m} w_j \end{bmatrix}$$

Thus $x^* = \frac{1}{\sum_{j=1}^{m} w_j} \sum_{j=1}^{m} w_j y_j$ is such that $\nabla g(x^*) = 0$

Consider the Hessian: $H_g(x) = \begin{bmatrix} 2\sum_{j=1}^{m} w_j & 0 \\ 0 & 2\sum_{j=1}^{m} w_j \end{bmatrix}$, $\lambda = 2\sum_{j=1}^{m} w_j > 0$ $(w_j > 0,\ j=1,\dots,m)$

$\Rightarrow H_g(x^*)$ is positive definite

$\Rightarrow x^*$ is a minimum of $g(x)$ and an optimal solution for the given problem

Notice that x^* can be seen as the central point of the network with masses w_j in positions y_j since it has the same form of a center of mass.

(b) Since the Hessian $H_g(x)$ is positive definite $\forall x \in \mathbb{R}^2$, the function is convex. This implies that the optimal solution is unique.

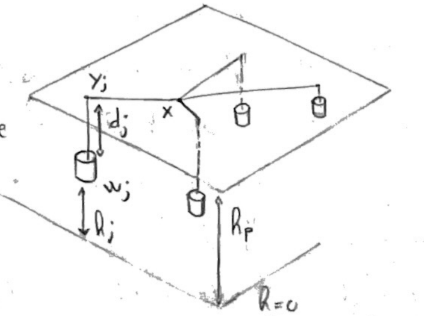
(c) Consider the potential energy of the mechanical system given $p = \sum_{j=1}^{m} w_j h_j$. The height h_j can be expressed in terms of x as follows:

$c = d_j + \|x-y_j\|$, c = constant $\rightarrow d_j = c - \|x-y_j\|$

$h_j = h_p - d_j \longrightarrow h_j = h_p - (c - \|x-y_j\|)$, h_p: height of $\mathbb{R}^2$ plane

$h_j = \underbrace{h_p - c}_{k} + \|x-y_j\|$

$h_j = k + \|x-y_j\|$, k = constant



Thus we have $p(x) = \sum_{j=1}^{m} w_j (k + \|x-y_j\|)$:

$$\Rightarrow p(x) = \underbrace{k\sum_{j=1}^{m} w_j}_{\text{constant}} + \underbrace{\sum_{j=1}^{m} w_j \|x-y_j\|}_{f(x)} = \text{const.} + f(x)$$

Therefore, $\min p(x) = \min[\text{const.} + f(x)] \Rightarrow (x^* \text{ min. for } f(x) \iff x^* \text{ min. for } p(x))$ ■