Optimization
Màster de Fonaments de Ciència de Dades

PART 2. Analysis

Chapter 3. Unconstrained and constrained optimization with equalities. Optimality conditions

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Chapter 5. Convex sets and functions

Chapter 4

Constrained optimization with inequalities.

Optimality conditions

The problem

Let $D \subset \mathbb{R}^n$ be an open set and let

$$\begin{aligned} & f : D \to \mathbb{R}, \\ & g_j : D \to \mathbb{R}, \; j = 1, \ldots, m, \text{ and} \\ & h_j : D \to \mathbb{R}, \; j = 1, \ldots, p, \end{aligned} \tag{1}$$

with $m << n$, be $\mathcal{C}^1$-functions defined in D.

Problem. The constrained optimization problem $(\mathcal{P})$ is defined by

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x})$$

$$\text{subject to:} \quad \begin{array}{ll} g_j(\boldsymbol{x}) = 0, & i = 1, \ldots, m \\ h_j(\boldsymbol{x}) \geq 0, & j = 1, \ldots, p. \end{array} \tag{2}$$

Constructing an equality constrained problem

Remark. Problem $\mathcal{P}$ may be written as an equality constrained problem by enlarging the number of variables.

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x})$$

$$\text{subject to:} \quad \begin{array}{ll} g_j(\boldsymbol{x}) = 0, & i = 1, \ldots, m \\ h_j(\boldsymbol{x}) - z_j^2 = 0, & j = 1, \ldots, p. \end{array} \tag{3}$$

Solutions of $\mathcal{P}$. Feasible set and points and directions

Definition. The set of points $\mathcal{X} \subset D$ satisfying conditions (12) are called feasible points and $\mathcal{X}$ is called the feasible set for the constrained optimization problem.

Definition. A point $\boldsymbol{x}^{\star} \in \mathcal{X}$ is called a local solution (minimum) of problem $\mathcal{P}$ if there exists ε such that $f(\boldsymbol{x}) \geq f(\boldsymbol{x}^{\star})$ for all $x \in \mathcal{X} \cap \boldsymbol{B}(\boldsymbol{x}^{\star}, \varepsilon)$.

Definition. A point $\boldsymbol{x}^{\star} \in \mathcal{X}$ is called a global solution (minimum) of problem $\mathcal{P}$ if $f(\boldsymbol{x}) \geq f(\boldsymbol{x}^{\star})$ for all $x \in \mathcal{X}$.

Definition. Let $\boldsymbol{x} \in \mathcal{X}$. A unitary vector z is called a feasible direction from x if for small enough $\delta > 0$ we have that if $|\theta| < \delta$ then

$$\{y \in \mathbb{R}^n \mid y = x + \theta z\} \subset \mathcal{X}$$

Active inequality constrains

Remark. The previous notion of local solution of $\mathcal{P}$ writes as

$$f(\boldsymbol{x}^\star + \theta z) \geq f(\boldsymbol{x}^\star), \text{ for } |\theta| < \delta,$$

with z being a feasible direction.

Definition. We introduce the following set.

$$\mathcal{I}(\boldsymbol{x}^\star) := \{j \ : \ h_j(\boldsymbol{x}^\star) = 0\}.$$

For those $j \in \mathcal{I}(\boldsymbol{x}^\star)$ we say that the inequality constrains h_j's are saturated or active at the solution $\boldsymbol{x}^\star$.

Feasible set and points and directions

Lemma. Let $\boldsymbol{x}^\star$ a local solution of $\mathcal{P}$. Suppose $k \in \mathcal{I}(\boldsymbol{x}^\star)$. Let z a feasible direction from $\boldsymbol{x}^\star$. Then $z^T \nabla h_k(\boldsymbol{x}^\star) \geq 0$.

Proof. Assume $z^T \nabla h_k(\boldsymbol{x}^\star) < 0$ We have that

$$h_k(\boldsymbol{x}^\star + \theta z) = h_k(\boldsymbol{x}^\star) + \theta \nabla h_k(\boldsymbol{x}^\star) + \varepsilon_k(\theta)$$

where $\varepsilon_k(\theta) \to 0$ as $\theta \to 0$. Hence for θ small enough $\theta \nabla h_k(\boldsymbol{x}^\star) + \varepsilon_k(\theta) < 0$ and so $h_k(\boldsymbol{x}^\star + \theta z) < 0$, a contradiction with z a feasible direction.

Lemma. Let $\boldsymbol{x}^\star$ a local solution of $\mathcal{P}$. Let z a feasible direction from $\boldsymbol{x}^\star$. Then $z^T \nabla g_j(\boldsymbol{x}^\star) = 0$ for all $j = 1, \ldots m$.

The linearizing cone $\mathcal{Z}^1(\mathbf{x}^\star)$

Definition. Assume previous notation. We define the linearizing cone of $\mathcal{X}$ at $\mathbf{x}^\star$ as

$$\mathcal{Z}^1(\mathbf{x}^\star) := \left\{ z \mid \begin{array}{l} z^T \nabla h_k(\mathbf{x}^\star) \geq 0 \text{ if } k \in \mathcal{I}(\mathbf{x}^\star), \text{ and} \\ \\ z^T \nabla g_j(\mathbf{x}^\star) = 0 \; j = 1, \dots m \end{array} \right\}$$

Lemma. If z is a feasible direction from $\mathbf{x}^\star \in \mathcal{X}$ (that is, $(\mathbf{x}^\star + \theta z) \in \mathcal{X}$ for θ small), then $z \in \mathcal{Z}^1(\mathbf{x}^\star)$.

Proof. We argue by contradiction. If $z \notin \mathcal{Z}^1(\mathbf{x}^\star)$ then either $z^T \nabla h_k(\mathbf{x}^\star) < 0$ for $k \in \mathcal{I}(\mathbf{x}^\star)$, or $z^T \nabla g_j(\mathbf{x}^\star) \neq 0$. Using linear expansion of h_k, $k \in \mathcal{I}(\mathbf{x}^\star)$ and g_j, $j = 1, \dots m$ these imply that either $h_k(\mathbf{x}^\star + \theta z) < 0$, $k \in \mathcal{I}(\mathbf{x}^\star)$ or $g_j(\mathbf{x}^\star + \theta z) \neq 0$, $j = 1, \dots m$, for θ small enough, respectively.

The set $\mathcal{Z}^2(\boldsymbol{x}^\star)$

Definition. Assume previous notation. We define the set

$$\mathcal{Z}^2(\boldsymbol{x}^\star) := \{z \mid z^T \nabla f(\boldsymbol{x}^\star) < 0\}$$

Lemma. If $z \in \mathcal{Z}^2(\boldsymbol{x}^\star)$ then $f(\boldsymbol{x}^\star + \theta z) < f(\boldsymbol{x}^\star)$, θ small enough.

The (generalized) Lagrangian associated to $\mathcal{P}$

Definition. Assume previous notation. We define the generalized Lagrangian associated to $\mathcal{P}$ as the function

$$L(x, \lambda, \mu) = f(x) - \sum_{j=1}^{m} \lambda_j g_j(\boldsymbol{x}) - \sum_{j=1}^{p} \mu_j h_j(\boldsymbol{x}).$$

Definition. A solution point $\boldsymbol{x}^\star$ is called regular if the equality constrains and the active inequality constrains at $\boldsymbol{x}^\star$ have linearly independent gradient vectors.

Remark. This definition generalize the previous technical condition of the Jacobian matrix $D(g)(\boldsymbol{x}^\star)$ having rank m.

Necessary conditions for minimum

Theorem (Karush-Kuhn-Tucker conditions). Assume previous notation. Let $\boldsymbol{x}^\star$ be a regular local minimum for $\mathcal{P}$. Then, there exist (unique) Lagrange multiplier vectors $\boldsymbol{\lambda}^\star = (\lambda_1^\star, \dots, \lambda_m^\star)$ and $\boldsymbol{\mu}^\star = \left(\mu_1^\star, \dots, \mu_p^\star\right)$ such that

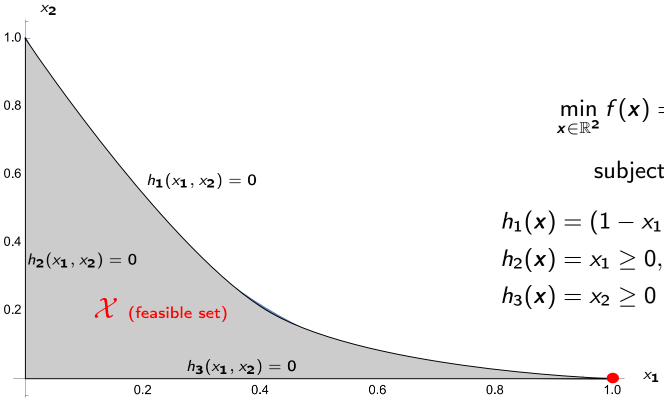
$$\nabla_x L\left(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star, \boldsymbol{\mu}^\star\right) = \nabla f\left(\boldsymbol{x}^\star\right) - \sum_{j=1}^{m} \lambda_j \nabla g_j(\boldsymbol{x}) - \sum_{j=1}^{p} \mu_j \nabla h_j(\boldsymbol{x}) = 0.$$

Moreover, $\mu_j \geq 0$ and $\mu_j h_j\left(\boldsymbol{x}^\star\right) = 0,\ j = 1, \dots m$. If f, g_j and h_j are $\mathcal{C}^2$-functions then

$$y^T H_x(L)\left(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star, \boldsymbol{\mu}^\star\right) y \geq 0$$

for all $y \in \mathbb{R}^n$ such that $\left(\nabla g_j\left(\boldsymbol{x}^\star\right)\right)^T y = 0,\ j = 1, \dots, m$ and $\left(\nabla h_k\left(\boldsymbol{x}^\star\right)\right)^T y = 0,\ k \in \mathcal{I}\left(\boldsymbol{x}^\star\right)$.

An exemple: non-regular local minimums



$$\min_{\boldsymbol{x}\in\mathbb{R}^2} f(\boldsymbol{x}) = -x_1$$

subject to

$$h_1(\boldsymbol{x}) = (1 - x_1)^3 - x_2 \geq 0$$
$$h_2(\boldsymbol{x}) = x_1 \geq 0,$$
$$h_3(\boldsymbol{x}) = x_2 \geq 0$$

An exemple: non-regular local minimums

Solution. Easily we can see that the point $\boldsymbol{x}^\star = (1, 0)$ is a local minimum of f under the constrains. However

$$\nabla h_1(\boldsymbol{x}) = \left(-3(1-x_1)^2, -1\right), \quad \nabla h_2(\boldsymbol{x}) = (1, 0), \nabla h_2(\boldsymbol{x}) = (0, 1),$$

and so, observe that $\nabla h_1(\boldsymbol{x}) = (0, -1)$ and $\nabla h_2(\boldsymbol{x}) = (0, 1)$ are not linearly independent. Moreover,

$$\nabla f(\boldsymbol{x}^\star) = (1, 0) \neq \mu_1(0, -1) + \mu_3(0, 1),$$

and so $\boldsymbol{x}^\star$ does not satisfies the necessary conditions.

Exercise. Prove that $\mathcal{Z}^1(\boldsymbol{x}^\star) \cup \mathcal{Z}^2(\boldsymbol{x}^\star) \neq \emptyset$. Indeed, this is the condition that characterizes non regular candidates.

Turning to sufficient conditions

Theorem. Assume previous notation and assume that all functions are of class $\mathcal{C}^2$. Assume that $\boldsymbol{x}^\star \in \mathbb{R}^n$, $\boldsymbol{\lambda}^\star \in \mathbb{R}^m$ and $\boldsymbol{\mu}^\star \in \mathbb{R}^p$ satisfy $g_j(\boldsymbol{x}^\star) = 0,\ j = 1, \dots, m,\ h_j(\boldsymbol{x}^\star) \geq 0,\ j = 1, \dots, p,$

$$\nabla_x L(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star, \boldsymbol{\mu}^\star) = 0, \quad \mu_j \geq 0, \quad \mu_j h_j(\boldsymbol{x}^\star) = 0,\ j = 1, \dots m,$$

and

$$y^T H_x(L)(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star, \boldsymbol{\mu}^\star)\, y \geq 0$$

for all $y \in \mathbb{R}^n$ such that $(\nabla g_j(\boldsymbol{x}^\star))^T y = 0,\ j = 1, \dots, m$ and $(\nabla h_k(\boldsymbol{x}^\star))^T y = 0,\ k \in \mathcal{I}(\boldsymbol{x}^\star)$. Assume also that $\mu_k^\star > 0$ for all $k \in \mathcal{I}(\boldsymbol{x}^\star)$.

Then, $\boldsymbol{x}^\star$ is a strict local minimum of f subject to the constrains given by $\mathcal{P}$.

An interesting example

Exercise. Discuss the following optimization problem in terms of the parameter $\beta > 0$.

$$\min_{\boldsymbol{x} \in \mathbb{R}^2} f(\boldsymbol{x}) = (x_1 - 1)^2 + x_2^2$$

subject to

$$h(x_1, x_2) = -x_1 + \beta x_2^2 \geq 0$$

Interpret the solutions geometrically in terms of the level curves and the restriction.

Saddlepoints of the Lagrangian

Definition. Let $\boldsymbol{x} \in E_x \subset \mathbb{R}^n$ and $\boldsymbol{y} \in E_y \subset \mathbb{R}^m$. Let φ a (continuous) function $\varphi : E_{\boldsymbol{x}} \times E_{\boldsymbol{y}} \to \mathbb{R}$. We say that a point $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \in E_{\boldsymbol{x}} \times E_{\boldsymbol{y}}$ is a saddlepoint of φ if

$$\varphi\left(\hat{\boldsymbol{x}}, \boldsymbol{y}\right) \leq \varphi\left(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}\right) \leq \varphi\left(\boldsymbol{x}, \hat{\boldsymbol{y}}\right).$$

Definition. We define the problem $(\mathcal{S})$ as follows. Find a saddlepoint $\hat{\boldsymbol{x}} \in \mathbb{R}^n$, $\hat{\boldsymbol{\lambda}} \in \mathbb{R}^m$ and $\hat{\boldsymbol{\mu}} \in \mathbb{R}^p$ with $\boldsymbol{\mu} \geq 0$ for the Lagrangian. That is

$$L\left(\hat{\boldsymbol{x}}, \boldsymbol{\lambda}, \boldsymbol{\mu}\right) \leq L\left(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\mu}}\right) \leq L\left(\boldsymbol{x}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\mu}}\right) \tag{4}$$

for every $\boldsymbol{x} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$ and $\boldsymbol{\mu} \in \mathbb{R}^p$ with $\boldsymbol{\mu} \geq 0$.

Connecting $(\mathcal{P})$ with $(\mathcal{S})$

Theorem. If $\left(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\mu}}\right)$ is a solution of $(\mathcal{S})$ then $\hat{x}$ is a solution of $(\mathcal{P})$.

Proof. Assume $\left(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\mu}}\right)$ is a solution of $(\mathcal{S})$. Then from (4) we have

$$\sum_{j=1}^{m}\left(\hat{\lambda}_j - \lambda_j\right) g_j\left(\hat{\boldsymbol{x}}\right) + \sum_{j=1}^{p}\left(\hat{\mu}_j - \mu_j\right) h_j\left(\hat{\boldsymbol{x}}\right) \leq 0 \qquad \text{(a)}$$

$$f\left(\hat{\boldsymbol{x}}\right) \leq f\left(\boldsymbol{x}\right) + \sum_{j=1}^{m} \hat{\lambda}_j \left(g_j\left(\hat{\boldsymbol{x}}\right) - g_j\left(\boldsymbol{x}\right)\right) + \sum_{j=1}^{p} \hat{\mu}_j \left(h_j\left(\hat{\boldsymbol{x}}\right) - h_j\left(\boldsymbol{x}\right)\right) \qquad \text{(b)}$$

After some computations we conclude that

$$g_j\left(\hat{\boldsymbol{x}}\right) = 0,\ j = 1, \ldots, m \quad \text{and} \quad \hat{\boldsymbol{\mu}}_j h_j\left(\hat{\boldsymbol{x}}\right) = 0,\ j = 1, \ldots p.$$

Hence

$$f\left(\hat{\boldsymbol{x}}\right) \leq f\left(\boldsymbol{x}\right) - \sum_{j=1}^{m} \hat{\lambda}_j g_j\left(\boldsymbol{x}\right) - \sum_{j=1}^{p} \hat{\mu}_j h_j\left(\boldsymbol{x}\right).$$

Connecting $(\mathcal{P})$ with $(\mathcal{S})$

Theorem. Suppose all functions are differentiable and suppose that $\left(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\mu}}\right)$ is a solution of $(\mathcal{S})$. Then $\mathcal{Z}^1(\hat{\boldsymbol{x}}) \cup \mathcal{Z}^2(\hat{\boldsymbol{x}}) \neq \emptyset$ and

$$\nabla_x L\left(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\mu}}\right) = 0 \quad \hat{\mu}_j h_j(\hat{\boldsymbol{x}}) = 0, \; j = 1, \ldots p,$$

with $\hat{\boldsymbol{\mu}} \geq 0$.

These were conditions for minimum of f under general inequality constrains.