OPTIMIZATION $\qquad$ NIUB NUMBER: 21287070

PROBLEM SET 2

4) Let $\{S_i\}_{i\in I}$ be an arbitrary family of convex sets, thus satisfying

$$\left.\begin{array}{l} x\in S_i \\ y\in S_i \\ \lambda\in[0,1] \end{array}\right\} \Rightarrow \lambda x+(1-\lambda)y\in S_i \quad \forall i\in I$$

and consider the intersection $\bigcap_{i\in I} S_i$

If $\bigcap_{i\in I} S_i$ is empty or contains only one point, then convexity follows by definition

Otherwise, consider $x\in\bigcap_{i\in I} S_i$, $y\in\bigcap_{i\in I} S_i$, $\lambda\in[0,1]$, then we also have $x\in S_i\ \forall i\in I$, $y\in S_i\ \forall i\in I$ and for the convexity of S_i we have $\lambda x+(1-\lambda)y\in S_i\ \forall i\in I$, which implies $\lambda x+(1-\lambda)y\in\bigcap_{i\in I} S_i$.

Therefore, $\bigcap_{i\in I} S_i$ is a convex set. (Notice that the intersection does not need to be finite for this to hold) ■

9) Let $f:\mathbb{R}^n\to\mathbb{R}$, $x_0,z\in\mathbb{R}^n$, $\theta\in\mathbb{R}$, $\phi(\theta)=f(x_0+\theta z)$ and consider the quadratic approximation of ϕ, $\hat\phi(\theta)=a+b\theta+c\theta^2$, $a,b,c\in\mathbb{R}$

Applying the quadratic approximation method we have that the minimum of ϕ is θ^*: $\hat\phi'(\theta^*)=0$, $\hat\phi''(\theta^*)>0$

$\hat\phi'(\theta)=b+2c\theta$, $\hat\phi''(\theta)=2c$

$\hookrightarrow \theta^*: \hat\phi'(\theta^*)=0 \longrightarrow \theta^*=-\dfrac{b}{2c}$

Since f is evaluated on three points, $\theta_1,\theta_2,\theta_3\in\mathbb{R}$, we have

$$\left.\begin{array}{l} \phi(\theta_1)\simeq\hat\phi(\theta_1)=a+b\theta_1+c\theta_1^2 \\ \phi(\theta_2)\simeq\hat\phi(\theta_2)=a+b\theta_2+c\theta_2^2 \\ \phi(\theta_3)\simeq\hat\phi(\theta_3)=a+b\theta_3+c\theta_3^2 \end{array}\right\}$$

solving the system we obtain the values of a,b,c in terms of $\theta_1,\theta_2,\theta_3,\phi(\theta_1),\phi(\theta_2),\phi(\theta_3)$

$$a=\frac{\phi(\theta_1)(\theta_2^2\theta_3-\theta_3^2\theta_2)+\phi(\theta_2)(\theta_3^2\theta_1-\theta_1^2\theta_3)+\phi(\theta_3)(\theta_1^2\theta_2-\theta_2^2\theta_1)}{-(\theta_1-\theta_2)(\theta_2-\theta_3)(\theta_3-\theta_1)}$$

$$b=-\frac{\phi(\theta_1)(\theta_2^2-\theta_3^2)+\phi(\theta_2)(\theta_3^2-\theta_1^2)+\phi(\theta_3)(\theta_1^2-\theta_2^2)}{-(\theta_1-\theta_2)(\theta_2-\theta_3)(\theta_3-\theta_1)}$$

$$c=\frac{\phi(\theta_1)(\theta_2-\theta_3)+\phi(\theta_2)(\theta_3-\theta_1)+\phi(\theta_3)(\theta_1-\theta_2)}{-(\theta_1-\theta_2)(\theta_2-\theta_3)(\theta_3-\theta_1)}$$

Thus we have $\theta^*=-\dfrac{b}{2c}=\dfrac{(\theta_2^2-\theta_3^2)\phi(\theta_1)+(\theta_3^2-\theta_1^2)\phi(\theta_2)+(\theta_1^2-\theta_2^2)\phi(\theta_3)}{2\left[(\theta_2-\theta_3)\phi(\theta_1)+(\theta_3-\theta_1)\phi(\theta_2)+(\theta_1-\theta_2)\phi(\theta_3)\right]}$

which is a minimum if $\hat\phi''(\theta^*)=2c>0$, thus

$$2\,\frac{\phi(\theta_1)(\theta_2-\theta_3)+\phi(\theta_2)(\theta_3-\theta_1)+\phi(\theta_3)(\theta_1-\theta_2)}{-(\theta_1-\theta_2)(\theta_2-\theta_3)(\theta_3-\theta_1)}>0 \iff \frac{(\theta_2-\theta_3)\phi(\theta_1)+(\theta_3-\theta_1)\phi(\theta_2)+(\theta_1-\theta_2)\phi(\theta_3)}{2[(\theta_1-\theta_3)(\theta_3-\theta_1)(\theta_1-\theta_2)]}<0$$

■

Exercise 15

(a) Consider the function $f(x) = x^2 + e^x - 3$, this function is C^∞ and its derivative is $f'(x) = 2x + e^x$. Consider now two intervals, $[-2, -1]$ and $[\frac{1}{2}, 0.99]$; the function evaluated in those points gives the following values
$f(-2) = e^{-2} + 1 = 1.135... > 0$ and $f(-1) = e^{-1} - 2 = -1.632... < 0$,
$f(\frac{1}{2}) = e^{\frac{1}{2}} - \frac{11}{4} = -1.101... < 0$ and $f(0.99) = e^{0.99} - 2.0199 = 0.671... > 0$
So we have $f(a)f(b) < 0$ for both intervals, thus the hypothesis of Bolzano's theorem are satisfied. This shows that in those intervals there is a zero. However, this does not imply unicity. In order for those zeros to be unique it would be ideal to use the corollary of Rolle's theorem. However, in order to solve $f'(x) = 0$ and make sure no root falls in those intervals we would have to solve $f'(x) = 2x + e^x$, which presents the same problems as the problem we are tackling.
However, this is not needed in this case, since the function f can be seen as the sum of a two-degree polynomial $x^2 - 3$ and e^x. Since e^x is a monotonous and positive function, f is bound to have the same behavior of a polynomial of degree two, thus allowing for a maximum of two zeros (and one extreme point). Since we have that in both the considered intervals there must be a zero, then f has exactly two zeros, which fall in those intervals.

(b) Consider the interval $[-2, -1]$, in order to find the zero we can use a fixed point method using $g(x) = -\sqrt{3 - e^x}$, obtained from $f(x) = 0$ assuming $3 - e^x > 0$, i.e. $x \leq ln3 = 1.099...$, which is satisfied in this interval. The assumption for the convergence of the method is satisfied since $|g'(x)| = |\frac{e^x}{2\sqrt{3-e^x}}| = \frac{e^x}{2\sqrt{3-e^x}} < |g'(-1)| = \frac{e^{-1}}{2\sqrt{3-e^{-1}}} = 0.113...$. (This is true because $|g'(x)|$ is an increasing function)
So we have that $|g'(x)| \leq k < 1$, with $k = 0.113...$ on the whole interval.
Consider the interval $[\frac{1}{2}, 0.99]$, in order to find the zero we can use a fixed point method using $g(x) = ln(3 - x^2)$, obtained from $f(x) = 0$ assuming $-\sqrt{3} < x < \sqrt{3}$, i.e. $-1.732... < x < 1.732...$, which is satisfied in this interval. The assumption for the convergence of the method is satisfied since $|g'(x)| = |\frac{-2x}{3-x^2}| = \frac{2x}{3-x^2} < |g'(0.99)| = 0.980....$ (This is true because $|g'(x)|$ is an increasing function)
So we have that $|g'(x)| \leq k < 1$, with $k = 0.980...$ on the whole interval.

(c) Using the a priori estimate of the number of iterations needed, $10^{-6} < \frac{k^n}{1-k}|x_0 - x_1|$, we obtain the following
First zero: $k = 0.113...$, initial point $x_0 = -1.5$, $x_1 = g(-1.5) = -1.666...$, we obtain $n > 5.568$, thus a priori the estimate is $n = 6$
Second zero: $k = 0.980...$, initial point $x_0 = 0.75$, $x_1 = g(0.75) = 0.891...$, we obtain $n > 780.516$, thus a priori the estimate is $n = 781$ (This large value is due to the derivative being close to 1)

(d) Here it follows a program written to compute the two zeros and the obtained results

```
import numpy as np

def f(x):
  return x**2+np.exp(x)-3
def g_1(x):
  return -np.sqrt(3-np.exp(x))
def g_2(x):
  return np.log(3-(x**2))

def fixed_point_method(x0, k, tol=1e-6):
  # Initialization ------------------------------------------------------------
  niter = 1
  # First zero ----------------------------------------------------------------
  if k==0.113:
    x1 = g_1(x0)
    while k/(1-k)*np.abs(x1-x0)>tol:
      x0 = x1
      x1 = g_1(x0)
      niter = niter+1
  # Second zero ---------------------------------------------------------------
  if k==0.980:
    x1 = g_2(x0)
    while k/(1-k)*np.abs(x1-x0)>tol:
      x0 = x1
      x1 = g_2(x0)
      niter = niter+1
  return x1, niter
```

```
x1, niter = fixed_point_method(-1.5, 0.113)
print("The first zero is ", x1)
print("The method terminated in ", niter, " iterations")
```

```
The obtained solution is  -1.6772326034814298
The method terminated in  5  iterations
```

```
x1, niter = fixed_point_method(0.75, 0.980)
print("The second zero is ", x1)
print("The method terminated in ", niter, " iterations")
```

```
The obtained solution is  0.8344868572425469
The method terminated in  50  iterations
```

The obtained number of iterations is lower than the a priori boundary; for the first zero only 5 iterations are necessary, while for the second 50 iterations are required.