Conjugate models 02

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1 The Poisson-Gamma model

Likelihood

The likelihood is:

$$(y|\lambda) \sim \text{Poisson}(\lambda),$$

with pmf:

$$f(y|\lambda) = e^{-\lambda}\,\frac{\lambda^y}{y!}, \quad y = 0, 1, 2, \ldots, \quad \lambda > 0,$$

Conjugate prior pdf

λ's prior is:

$$\text{Gamma}(\alpha, \beta), \quad \alpha, \beta > 0,$$

with pdf:

$$h(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\,\lambda^{\alpha-1}\,e^{-\beta\lambda}, \quad \alpha, \beta, \lambda > 0.$$

Joint pdf

The joint "density" of (y, λ) is:

$$f(y, \lambda) = f(y|\lambda) \cdot h(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\lambda^{\alpha+y-1}}{y!}\,\exp\{-(\beta+1)\lambda\},$$

for $\alpha, \beta, \lambda > 0$.

Marginal pmf of x (Prior predictive pmf)

To integrate with respect to λ we split $f(y, \lambda)$:

$$\begin{aligned} \frac{\beta^\alpha}{\Gamma(\alpha)} \quad &\cdot \frac{1}{y!} \cdot \frac{\Gamma(\alpha+y)}{(\beta+1)^{(\alpha+y)}} \\ &\times \frac{(\beta+1)^{(\alpha+y)}}{\Gamma(\alpha+y)} \cdot \lambda^{\alpha+y-1}\,\exp\{-(\beta+1)\lambda\}, \end{aligned}$$

The second line is a $\text{Gamma}(\alpha+y, \beta+1)$ pdf,
which integrates to 1.

Marginal pmf of x

After integrating out λ we get the marginal pmf of y:

$$f(y) = \frac{\Gamma(\alpha + y)}{\Gamma(\alpha)\, y!} \frac{\beta^{\alpha}}{(\beta + 1)^{\alpha + y}}, \quad y = 0, 1, \ldots, \quad \alpha, \beta > 0$$

Noting that:

$$\frac{\beta^{\alpha}}{(\beta + 1)^{\alpha + y}} = \left(\frac{1}{\beta + 1}\right)^{y} \cdot \left(\frac{\beta}{\beta + 1}\right)^{\alpha},$$

Marginal pmf of y

we see $f(y)$ is a $\mathrm{NB}(r, p)$ pmf, with parameters:

$$r = \alpha, \quad \text{and} \quad p = \frac{\beta}{\beta + 1}.$$

(see Section A in the Appendix).

This is the *prior predictive* distribution for y.

Posterior pdf

With Bayes' formula, dividing $f(y, \lambda)$ by the marginal pmf we obtain the posterior pdf of λ, given y (first line above):

$$h(\lambda \,|\, y) = \frac{(\beta + 1)^{\alpha + y}}{\Gamma(\alpha + y)} \lambda^{\alpha + y - 1} e^{-(\beta + 1)\lambda}, \quad \lambda > 0,$$

which is a $\mathrm{Gamma}(\alpha + y, \beta + 1)$ pdf.

Case of an n-sample: posterior pdf

For $y_1, \ldots, y_n$ i.i.d.$\sim \mathrm{Poisson}(\lambda)$, the sum:

$$y = \sum_{i=1}^{n} y_i \sim \mathrm{Poisson}(n\,\lambda)$$

(Additive property of Poisson r.v.)

Thus, for a prior $\lambda \sim \mathrm{Gamma}(\alpha, \beta)$ and n $\mathrm{Poisson}(\lambda)$ data, the posterior is a $\mathrm{Gamma}(\alpha + y, \beta + n)$.

Case of an n-sample: prior predictive pmf

Similarly, for n observed $\mathrm{Poisson}(\lambda)$ data, the prior predictive (marginal) distribution of the total count number $y = \sum_{i=1}^{n} y_i$ is a $\mathrm{NB}(r, p)$ r.v. , with parameters:

$$r = \alpha, \quad \text{and} \quad p = \frac{\beta}{\beta + n}.$$

An n-sample with different exposures

In many applications we find data of the form:

$$y_i \sim \mathrm{Poisson}(\lambda_i), \quad \text{where } \lambda_i = x_i \cdot \theta, \quad 1 \le i \le n.$$

The values x_i are known positive values of an explanatory variable, usually called *exposure*, and θ is the common *rate* parameter.

Posterior pdf for an n-sample with different exposures

With a $\theta \sim \mathrm{Gamma}(\alpha, \beta)$ prior,

observations $\boldsymbol{y} = (y_1, \ldots, y_n)$, exposures $\boldsymbol{x} = (x_1, \ldots, x_n)$,

the posterior pdf is:

$$\theta | \boldsymbol{y} \sim \mathrm{Gamma}(\alpha + y, \beta + x),$$

where $y = \sum_{i=1}^{n} y_i$ and $x = \sum_{i=1}^{n} x_i$.

2 The Dirichlet - Multinomial model

The Dirichlet - Multinomial model

Generalizes the Beta-Binomial conjugate pair.

The Dirichlet distribution is the multivariate Beta.

The Multinomial distribution is the multivariate Binomial.

Multivariate Bernoulli distribution

A partition $\Omega = A_1 \sqcup \cdots \sqcup A_m$, where the A_j are pairwise exclusive events whose union is the total space, and:

$$\theta = (\theta_1, \ldots, \theta_m), \quad \theta_j = \mathrm{P}(A_j), \quad 1 \le j \le m.$$

Each indicator

$$\mathbb{1}_{A_j} \sim \mathrm{Ber}(\theta_j) \quad 1 \le j \le m.$$

Multivariate Bernoulli distribution

The m-dimensional vector of indicators:

$$(\mathbb{1}_{A_1}, \ldots, \mathbb{1}_{A_m}),$$

follows an m-dimensional *multivariate Bernoulli* distribution with vector of probabilities:

$$\theta = (\theta_1, \ldots, \theta_m),$$

Multivariate Bernoulli distribution

The sum of the m probabilities is 1.

The sum of the m indicators is 1 **(Cannot be independent!!)**.

Each j-th marginal, $\mathbb{1}_{A_j} \sim \mathrm{Ber}(\theta_j)$.

$$\mathrm{E}(\mathbb{1}_{A_j}) = \theta_j, \qquad \mathrm{var}(\mathbb{1}_{A_j}) = \theta_j\,(1-\theta_j),$$

$$\mathrm{cov}(\mathbb{1}_{A_j}, \mathbb{1}_{A_k}) = -\theta_j\,\theta_k, \quad j \neq k.$$

The multinomial distribution

The m-dim. multinomial distribution of size n and probs. $\theta = (\theta_1, \ldots, \theta_m)$, $\theta_j \in [0,1]$, $\sum_{j=1}^m \theta_j = 1$, has joint pmf:

$$\frac{n!}{x_1! \cdot \cdots \cdot x_m!}\,\theta_1^{x_1} \cdot \cdots \cdot \theta_m^{x_m}.$$

for an m-dimensional vector $x = (x_1, \ldots, x_m)$, of integers $x_j \in [0, n]$ such that $\sum_{j=1}^m x_j = n$.

x is the sum of n m-dim. vectors i.i.d. $\sim \mathrm{Ber}(\theta)$.

The Dirichlet distribution

θ's joint pdf, with parameters $a = (a_1, \ldots, a_m)$, $a_i > 0$:

$$h(\theta_1, \ldots, \theta_m; a_1, \ldots, a_m) = \frac{1}{\mathrm{B}(a)} \prod_{i=1}^m \theta_i^{a_i - 1}, \quad \text{where}$$

$$\mathrm{B}(a) = \frac{\prod_{i=1}^{m} \Gamma(a_i)}{\Gamma\left(\sum_{i=1}^{m} a_i\right)} \quad \text{is the multivariate Beta function.}$$

Multinomial likelihood with Dirichlet prior

$$\text{If:} \quad (x|\theta) \equiv (x_1, \ldots, x_m|\theta_1, \ldots, \theta_m) \sim \text{Multinomial}(n, \theta),$$

with parameter vector:

$$\theta = (\theta_1, \ldots, \theta_m), \quad 0 < \theta_i < 1, \quad 1 \le i \le m, \quad \textstyle\sum_{i=1}^{m} \theta_i = 1,$$

and θ's joint prior is Dirichlet with parameters

$a = (a_1, \ldots, a_m)$, then θ's posterior is Dirichlet,

$$\text{with parameters: } a + x = (a_1 + x_1, \ldots, a_m + x_m).$$

3 Exponential-Gamma model

Exponential likelihood

Used to model *waiting times* or *insurance claims*.

The pdf of an outcome y, given θ, is:

$$f(y|\theta) = \theta \exp(-\theta \cdot y), \quad y > 0,$$

and $\theta = 1/\,\mathrm{E}(y|\theta)$ is called the *rate parameter.*

The exponential is a special case of a Gamma with $(\alpha, \beta) = (1, \theta)$.

Gamma prior for θ

The conjugate prior for θ is a Gamma(α, β), for $\alpha, \beta > 0$, with pdf:

$$h(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \theta^{\alpha-1} \cdot \exp\{-\beta\,\theta\}, \quad \theta > 0.$$

Likelihood for an n-sample

$y = (y_1, \ldots, y_n)$ i.i.d.$\sim$ Exp(θ).

$$f(y\,|\,\theta) = \prod_{i=1}^{n} \left(\theta\, e^{-\theta\, y_i}\right) = \theta^n \cdot \exp\{-n\,\theta\,\bar{y}\}.$$

Joint (y, θ) pdf

$$\begin{aligned} f(y, \theta) &= f(y\,|\,\theta) \cdot h(\theta) \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \theta^{\alpha+n-1} \cdot \exp\{-\theta \cdot (n\bar{y} + \beta)\} \end{aligned}$$

To get the $y = (y_1, \ldots, y_n)$ joint marginal (prior predictive pdf) we integrate $f(y, \theta)$ with respect to θ.

(Integral is a gamma function, after a variable change $t = \theta\,(n\,\bar{y} + \beta)$)

Prior predictive pdf

$$f(y_1, \dots, y_n) = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \cdot \frac{\beta^{\alpha}}{(n\,\bar{y} + \beta)^{\alpha+n}},$$

$$y_1 > 0, \dots, y_n > 0.$$

Caution! This is the joint pdf of the $y = (y_1, \dots, y_n)$ vector.
It is a function of the sum $n\,\bar{y}$ but it is NOT the pdf of $x = n\,\bar{y}$.
Obtaining it requires $n - 1$ further integrals, just like in the proof of Fisher's theorem in elementary statistics, to derive the distributions of $\bar{x}_n$ and s_n^2 in an n-sample of a Normal$(0, 1)$.

Prior predictive density of $x = n\,\bar{y}$ for small n

For $n = 1$,

$$f(x) = \frac{\alpha\,\beta^{\alpha}}{(x + \beta)^{\alpha+1}}, \qquad x > 0,$$

for $n = 2$,

$$f(x) = (\alpha + 1)\,\alpha\,\beta^{\alpha}\,\frac{x}{(x + \beta)^{\alpha+2}}, \qquad x > 0,$$

$\alpha > 0, \beta > 0$.

Prior predictive density of $x = n\,\bar{y}$ for small n

$$\begin{aligned}
\text{For } n &= 1, \\
\mathrm{E}(x) &= \frac{\beta}{\alpha - 1},\ \ \alpha > 1,\ \ \mathrm{var}(x) = \frac{\alpha\,\beta^2}{(\alpha - 2)\,(\alpha - 1)^2},\ \ \alpha > 2. \\
\text{For } n &= 2, \\
\mathrm{E}(x) &= \frac{2\,\beta}{\alpha - 1},\ \ \alpha > 1,\ \ \mathrm{var}(x) = \frac{2\,(\alpha + 1)\,\beta^2}{(\alpha - 2)\,(\alpha - 1)^2},\ \ \alpha > 2.
\end{aligned}$$

$(\beta > 0)$.

Posterior pdf of θ, given y

$$(\theta \,|\, y) \sim \mathrm{Gamma}(\alpha', \beta'),$$

where the updating formula is:

$$\begin{cases} \alpha' &= \alpha + n, \\ \beta' &= \beta + n\,\bar{y}. \end{cases}$$

4 Mixture priors: the spinning coin

Persi Diaconis

Stanford prof. Persi Diaconis, formerly a professional magician, famously found how many shuffles a deck of cards needs to give a truly random order (seven). He's also dabbled in coin games.

BTW: Persi Diaconis on randomness

See his 2013 video talk:

The Search for Randomness

The spinning coin Fact: if a coin is spinned on its edge instead of being flipped, proportion of heads or tails is not around 50% but rather such values as 25% or 75% are obtained.

Persi Diaconis on the spinning coin

According to Diaconis, *"the reasons for the bias are not hard to infer. The shape of the edge will be a strong determining factor – indeed, magicians have coins that are slightly shaved; the eye cannot detect the shaving,*
but the spun coin always comes up heads".

A prior for the spinning coin problem

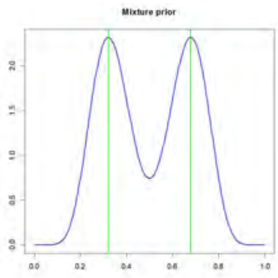
For n tosses of a spinning coin, the number x of heads up is a binomial $\mathrm{Binom}(n, \theta)$, and θ's prior will typically be bimodal (pdf with two local maxima). It cannot be a $\mathrm{Beta}(\alpha, \beta)$, which has a single mode at:

$$\frac{\alpha - 1}{\alpha + \beta - 2}$$

Diaconis, Persi and Donald Ylvisaker (1985) *Quantifying prior opinion.*
In:
J.M. Bernardo et al (eds), Bayesian Statistics 2, Elsevier, pp. 133-156.

A possible prior

$$0.50\ \mathrm{Beta}(10, 20) + 0.50\ \mathrm{Beta}(20, 10).$$



Interpretation of a mixture prior

The mixture prior can be thought of as a weighted combination of “beta populations”, the weights γ_i measuring the prior degree of belief that the actual coin was chosen from the i-th population.

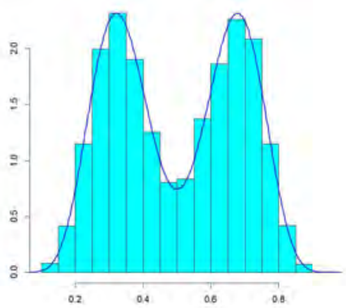
To generate random numbers from a mixture:

$$\gamma \cdot h_1 + (1-\gamma) \cdot h_2, \quad \gamma \in (0,1),$$

we generate a sequence of realizations of $I \sim \text{Ber}(\gamma)$ and:

- For each entry equal to 1, a realization of h_1,
- For each entry equal to 0, a realization of h_2.

Simulation of a mixture prior



Bayesian modelling with a mixture prior

Assume the prior pdf for θ is:

$$h(\theta) = \gamma \cdot h_1(\theta) + (1-\gamma) \cdot h_2(\theta),$$

and the likelihood is: $f(x|\theta)$. Then, the joint pdf is:

$$f(x,\theta) = \gamma \cdot f_1(x,\theta) + (1-\gamma) \cdot f_2(x,\theta).$$

where:

$$f_i(x,\theta) = f(x|\theta) \cdot h_i(\theta), \quad i = 1,2.$$

Prior predictive pdf from a mixture prior

Integrating out θ, the marginal for x is:

$$f(x) = \gamma \cdot f_1(x) + (1-\gamma) \cdot f_2(x),$$

where:

$$f_i(x) = \int f(x|\theta) \cdot h_i(\theta)\, d\theta, \quad i = 1,2.$$

Computing the posterior pdf from a mixture prior

From Bayes' formula:

$$h(\theta|x) = \frac{f(x,\theta)}{f(x)} = \frac{\gamma \cdot f_1(x,\theta) + (1-\gamma)\cdot f_2(x,\theta)}{\gamma \cdot f_1(x) + (1-\gamma)\cdot f_2(x)}.$$

With the obvious notation:

$$h_i(\theta|x) = \frac{f_i(x,\theta)}{f_i(x)} \qquad i = 1, 2,$$

Posterior pdf from a mixture prior

the posterior pdf is:

$$h(\theta|x) = \widehat{\gamma}(x) \cdot h_1(\theta|x) + (1-\widehat{\gamma}(x)) \cdot h_2(\theta|x),$$

where the *posterior mixture weights* are:

$$\widehat{\gamma}(x) = \frac{\gamma \cdot f_1(x)}{\gamma \cdot f_1(x) + (1-\gamma)\cdot f_2(x)}, \text{ and } 1 - \widehat{\gamma}(x).$$

A Appendix: The negative binomial distribution

First definition

A sequence of independent binary 0/1 experiments, with i.i.d.$\sim$ Ber(p) indicators, $p \in (0,1)$.

Number X of realizations to obtain a number $r \in \mathbb{N}$ of successes (1's), is a *negative binomial r.v.* with *size* r and probability p.

Alternative: $Y = X - r$ = number of failures (0's) before obtaining a number r of successes.

Probability mass function

$$\mathrm{P}(x) = \binom{x-1}{r-1} \cdot (1-p)^{x-r} \cdot p^r, \qquad\qquad x = 1, 2, \ldots,$$

$$\mathrm{P}(y) = \binom{y+r-1}{r-1} \cdot (1-p)^{y} \cdot p^r, \qquad\qquad y = 0, 1, 2, \ldots,$$

The second one is more usual (e.g., `dnbinom()` in `R`).

Alternative (and the reason for the name)

$$\mathrm{P}(y) = \binom{-r}{y} \cdot p^r \cdot (-q)^y, \quad \text{where } q = 1 - p.$$

Indeed:

$$\binom{-r}{y} = \frac{(-r)\cdot(-r-1)\cdot \cdots \cdot (-r-y+1)}{y!}$$

$$= (-1)^y \cdot \binom{y+r-1}{r-1}.$$

General definition

For an integer r,

$$\binom{y+r-1}{r-1} = \frac{(y+r-1)!}{(r-1)!\cdot y!} = \frac{\Gamma(y+r)}{\Gamma(r)\cdot y!}.$$

The right hand is valid for real $r > 0$. Thus the pmf:

$$\mathrm{P}(y) = \frac{\Gamma(y+r)}{\Gamma(r)\cdot y!}\cdot(1-p)^y\cdot p^r, \quad y = 0, 1, 2, \ldots,$$

defines the $\mathrm{NB}(r, p)$, for $r > 0$ and probability p.

Relation to the geometric distribution

For $r = 1$, the $\mathrm{NB}(1, p)$ is the $\mathrm{Geom}(p)$.

For integer r, a r.v. distributed as $\mathrm{NB}(r, p)$ can be considered as the sum of r i.i.d. copies of a $\mathrm{Geom}(p)$.

Expectation, variance of a negative binomial

For $Y \sim \mathrm{NB}(r, p)$,

$$\mathrm{E}(Y) = \mu \equiv r\cdot\frac{1-p}{p}$$

$$\mathrm{var}(Y) = \sigma^2 \equiv r\cdot\frac{1-p}{p^2} = \frac{\mu}{p} = \mu + \frac{\mu^2}{r}.$$
