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READER REACTION

A Re-analysis of the Pump-Failure Data

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ABSTRACT. A fit of the Poisson-gamma model to the pump-failure data quoted by George *et al.* (*Scand. J. Statist.* 20, 147–156) using maximum conjugate likelihood gives pump effects that agree closely with the values obtained from a full Bayesian model. However, the model is a poor fit, because the estimates of the pump effects do not look like a sample from a gamma distribution. A split of the pumps into two subgroups does give an adequate fit.

Key words: Bayes inference, conjugate prior, EM algorithm, Gibbs sampling, maximum conjugate likelihood, maximum likelihood, model checking

1. Introduction

George *et al.* (*Scand. J. Statist.* **20,** 147–156) discuss conjugate likelihood distributions, and apply them to some data on pump-failure times, using Gibbs sampling to generate the relevant posterior means. They compare the values of the pump effects using four prior distributions with those derived from the simple maximum likelihood estimate, whereby each pump's failure rate is estimated as y_i/t_i. This, of course, is equivalent to making no assumptions about the failure rates when the pumps are considered as a sample from a population. George *et al.* note that there appears to be an appreciable difference between the Bayesian analyses and that using maximum likelihood. This does not mean, however, that there is necessarily a major difference between estimates obtained from Bayesian methods and those obtained from maximizing likelihood. A re-analysis of these data using the same Poisson-gamma model and maximizing the conjugate likelihood gives values for the pump effects that are very close to those from the Bayesian analyses. In section 2 we describe the method used, and in section 3 we use model-checking procedures to show that the model gives a poor fit, which can, however, be rectified by using prior information on the pumps to split them into two subgroups.

2. Maximum-conjugate-likelihood estimates (MCLE)

The Poisson-gamma model assumes that the data y have Poisson distributions with means μu_i, where the u_i form a sample from a gamma distribution with mean 1 and shape parameter ν. The restriction on the mean is for convenience and does not represent any loss of generality. The conjugate likelihood is derived from the product of the Poisson frequencies of y_i given u_i and the distribution of the u_i as independent (unobserved) variables from a gamma distribution. It thus differs from a "true" likelihood in that the second term refers to a distribution of unobservables. This conjugate likelihood is the analogue of the quantity used in Normal–Normal models to provide BLUP estimates of the random component $\mathbf{u}$ (Robinson, 1991). Furthermore, the estimates of the systematic components (here only the intercept) are the same as those obtained from the use of the marginal likelihood after integrating out u_i, provided only that the differential element in the distribution of u is written as $d(\log(u))$.

Table 1. *The pump data and various analyses*

Pump No.	1	2	3	4	5	6	7	8	9	10	
t	94.3	15.7	62.9	126	5.24	31.4	1.05	1.05	2.1	10.5	
y	5	1	5	14	3	19	1	1	4	22	
MLE	0.053	0.064	0.080	0.111	0.573	0.604	0.954	0.954	1.908	2.099	
π_1	0.060	0.102	0.089	0.116	0.602	0.609	0.901	0.894	1.588	1.994	
π_2	0.061	0.106	0.090	0.117	0.603	0.609	0.884	0.886	1.560	1.981	
π_3	0.061	0.107	0.091	0.117	0.584	0.605	0.806	0.808	1.453	1.936	
π_4	0.062	0.113	0.093	0.118	0.585	0.604	0.791	0.789	1.398	1.905	
MCLE	0.063	0.118	0.094	0.118	0.587	0.606	0.758	0.758	1.347	1.897	$\hat{v} = 1.042$

For given values of the shape parameter v, the estimates of $\boldsymbol{\beta}$, the systematic component given $\mathbf{u}$ can be obtained from the standard GLM estimating equations for Poisson errors with offset $\log(u)$, while the estimates of u_i, given μ_i, are given by the formula

$$\hat{u}_i = (y_i + v)/(\hat{\mu}_i + v).$$

These two steps are an instance of the EM algorithm, with u_i as the data to be completed. Finally the shape parameter v can be estimated by searching, using the conjugate likelihood as the quantity to be maximized.

Note that these data contain rate constants t_i, which are easily included by making $\log(t)$ an offset component in the E step. Finally, to match the estimates of u_i with the analysis of George *et al.* we must add on the estimate of the intercept from the E step.

The data, together with the analyses of George *et al.* and the current analysis, are shown in Table 1.

It is immediately obvious that the MCLEs are closer to those derived from the Bayesian analyses than to the simple MLE. This is not surprising since both MCLE and the Bayesian estimates make the same assumption that the failure rates can be modelled as a sample from the gamma distribution. There are two extreme cases: (i) as the shape parameter v tends to infinity the estimates of the pump effects are the same so that the MCLE is the same as that from a Poisson model only; (ii) as v tends to zero the estimates tend to those given above as the MLE.

3. Model checking

The above model requires that the u_i look like a sample from a gamma distribution. We can check this informally by fitting a GLM to the u_i, using a gamma error and a linear predictor consisting of an intercept; we then form the deviance residuals, which are approximately Normally distributed and make a half-Normal plot of them. The result is shown in Fig. 1; the plot is not linear, so that the model is contradicted. However, Gaver & O'Muircheartaigh (1987) mention that the pumps fall into two groups, in that Nos. 1, 3, 4, and 6 were operated continuously, whereas the rest were operated intermittently. If we set up a two-level factor to represent this distinction and repeat the analysis, we obtain values of the u equal to

$$0.340 \quad 0.160 \quad 0.486 \quad 0.597 \quad 0.661 \quad 2.575 \quad 0.963 \quad 0.963 \quad 1.422 \quad 1.830 \quad \hat{v} = 1.974$$

and Fig. 2 shows the half-Normal plot from this model. The linearity is quite satisfactory and the model is thus acceptable.

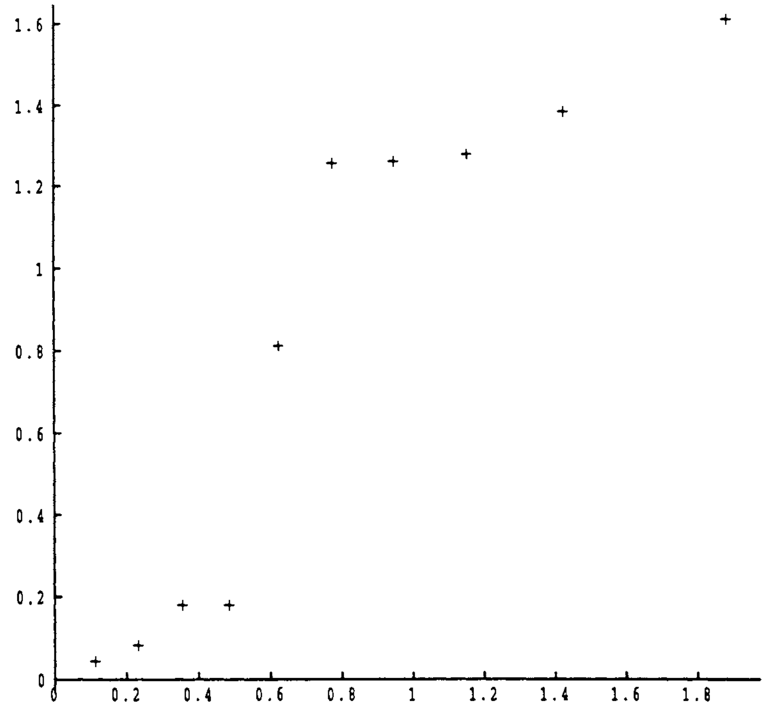

Fig. 1. Half-Normal plot of deviance residuals of $\hat{u}$ when pumps are treated as homogeneous.

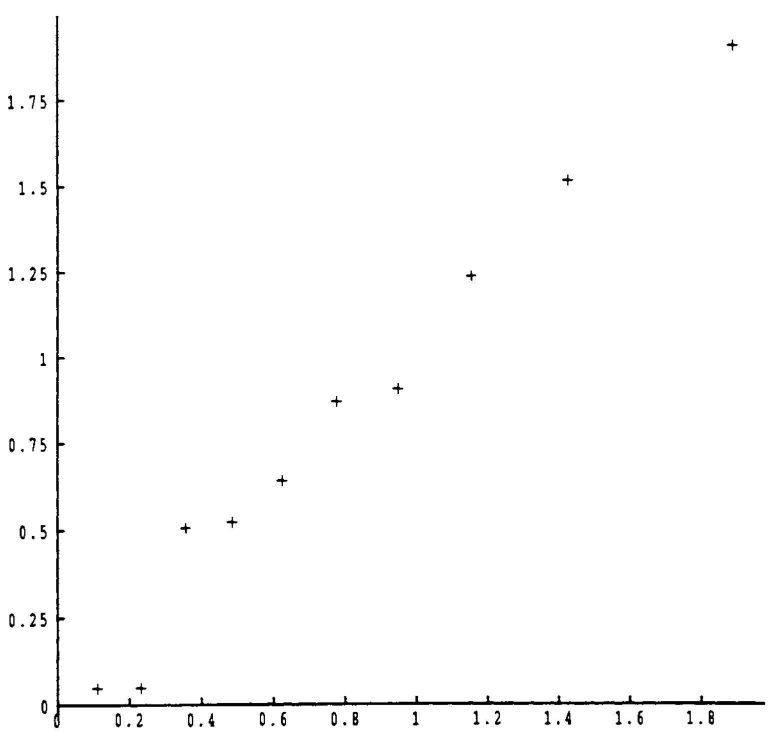

Fig. 2. Half-Normal plot of deviance residuals of $\hat{u}$ when pumps are split into two groups.

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RESPONSE

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We very much agree with the two main points of Professor Nelder's re-analyses. First, in the context of the gamma-Poisson hierarchical model, maximum likelihood estimation and Bayes estimation yield similar results, especially in the absence of strong prior information. What is much more important is the use of a probability distribution (here the gamma) to model the ensemble of Poisson rate parameters. This was the important innovation by Gaver & O'Muircheartaigh (1987).

Second, a hierarchical model which treats the two groups of pumps differently is indeed much more appropriate for these data. However, we did not mean to endorse the single group gamma-Poisson model as the best for this problem. Our purpose was simply to illustrate a fully Bayes gamma-Poisson analysis (which was facilitated by our conjugate likelihoods results) on a data set for which many competing gamma-Poisson analyses already appear in the literature. Admittedly, we assumed that others had checked the model, and

Table 2. Comparison of eB, fB and MCLE

	eB	fB	MCLE
Pump			
1	0.061	0.061	0.063
2	0.107	0.107	0.118
3	0.091	0.091	0.094
4	0.116	0.117	0.118
5	0.587	0.584	0.587
6	0.607	0.604	0.606
7	0.784	0.806	0.758
8	0.784	0.808	0.758
9	1.430	1.453	1.347
10	1.939	1.936	1.897
α	0.82	0.8	1.04
β	1.27	1.4	1.04

were astonished to see the serious lack of fit. Professor Nelder's observation highlights a crucial point, that the proper analysis of this and other hierarchical models must include a model checking component which gauges the appropriateness of parametric assumptions. Alternatively, if it is suspected *a priori* that there are two underlying populations (or a mainly homogeneous population with a few "outliers"), some form of mixture model can be specified for the second stage of the hierarchy.

Finally, it may be of interest to consider the following comparison of the original empirical Bayes (eB) estimates of Gaver & O'Muircheartaigh, our fully Bayes (fB) estimates using prior π_3, and the MCLE estimates of Professor Nelder. Notice how similar the eB and fB estimates are when compared to the MCLEs. These similarities and differences probably have to do with the treatment of the hyperparameters α and β. The eB estimates were obtained by maximizing the unconstrained likelihood under the gamma-Poisson model which occurs at $(\alpha, \beta) = (0.82, 1.27)$. (These numbers are incorrectly reversed in Table 2 of George *et al.*) These values are not very different from the fB posterior means of $(\alpha, \beta) = (0.8, 1.4)$. However, the MCLEs were obtained by maximizing the likelihood when the gamma is constrained to have mean 1, which is equivalent to contraining $\alpha = \beta$. Thus, the MCLEs here correspond to setting $(\alpha, \beta) = (1.04, 1.04)$.

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