Computer Exploration of Thrackable Graphs

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Motivation

Let $G = (V, E)$ be a simple connected graph. We will learn about thrackle drawings G.
A *drawing* of G is a representation on the plane, where each edge corresponds to a Jordan arcs joining its endpoints.
A thrackle of $G = (V, E)$ is a drawing that satisfies the following property: Every pair of edges meets exactly once, either:

1. At a common end vertex, or
2. At a proper crossing,

but not both.
Our aim is to generate and classify the set of thrackle drawings of a path of length n. Our goal is to make possible progress on a long-standing conjecture of J.H. Conway regarding thrackles.

Introduction

Definitions

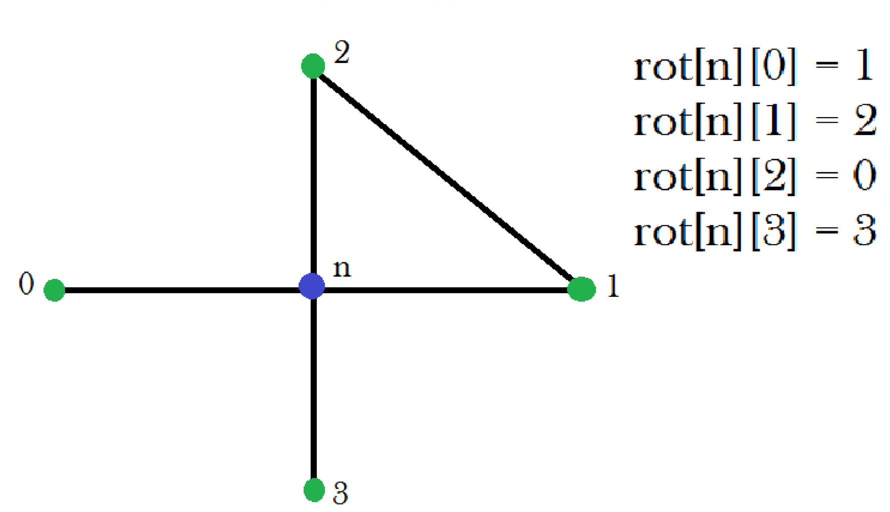
Proper Crossing A crossing p of two curves is *proper* if at p one curve passes from one side to the other of the other curve.

Thrackle A drawing of G is called a *thrackle* if every pair of edges meets exactly once: either at an end vertex or at a proper crossing.

Thrackable We say that G is *thrackable* if it has a thrackle drawing.

Rotation Scheme We represent a thrackle by introducing a new vertex of degree 4 at each crossing and recording, for each crossing and spot, the clockwise circular ordering of its neighbors, called the rotation rotation scheme.

Two thrackle drawings are *combinatorially equivalent* if they can be vertex-labeled so that their rotation schemes are the same.



Conway's Thrackle Conjecture

Over forty years ago, John H. Conway conjectured: In any given thrackle G, the following inequality holds:

$$|E(G)| \leq |V(G)| \tag{1}$$

Observe that the converse does not hold as a 4-cycle is not thrackable.
A Dumbbell consists of two cycles either joined by a path or whose intersection is a path.

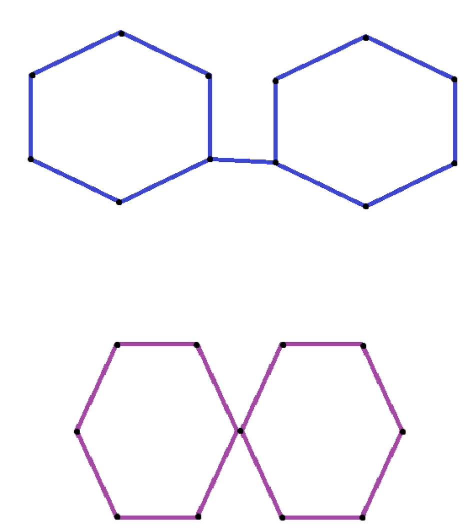


Figure 2: From top to bottom: $D(6,6,1)$ and $D(6,6,0)$

Conway's Conjecture is equivalent to the claim that no Dumbbell is thrackable.

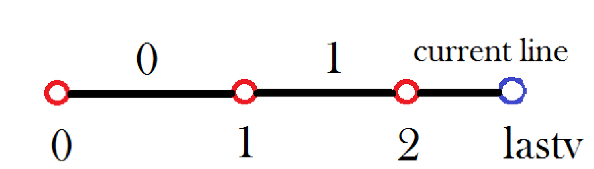
Our Strategy

We design an algorithm that performs an exhaustive Depth-First-Search of all the combinatorially inequivalent thrackle drawings for a path of length n.
Our goal is to try to classify the thrackles of a path of length n, and which pairs of these can occur in a path of length $n+1$.
Fulek and Pach did this by hand for $n = 4$, and thereby proved that $D(6,6,-3)$ is not thrackable. We hope to extend their work with computer assistance.

Algorithm

A Depth-First-Search is conducted. At any point we maintain the rotation scheme of a partially drawn path and attempt to extend the drawing in all possible ways.
INPUT Length of path.

- Step 1. Initialize data.



Challenge

Try to find a thrackle drawing of C_4. If you are unable to find one, try to give a rigorous explanation.

Two Thrackle Drawings of C_6

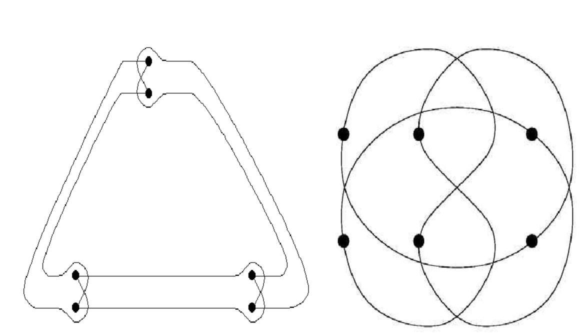
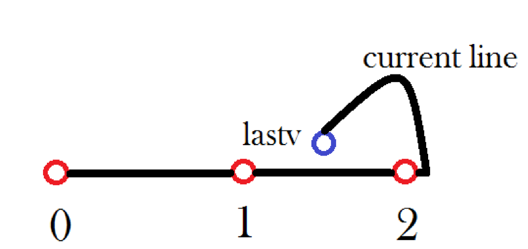


Figure 3: Two drawings of C_6 that are not combinatorially equivalent

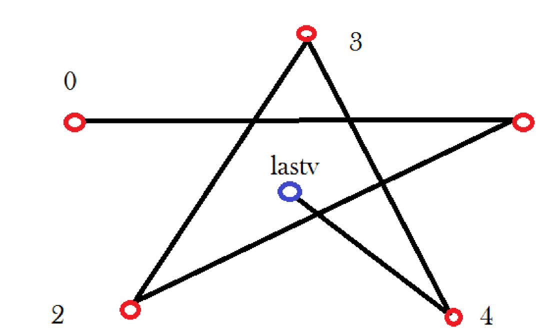
Algorithm (continued)

- Step 2. We examine, in sequence, the segments bounding the face containing the end vertex, lastv. Determine whether lastv needs to cross the segment.



Algorithm (continued)

- Step 2. Case 1: if the two lines still need to cross: cross and update data. Add a new spot if necessary. If a solution has been found, then record data.



- Step 2. Case 2: if the two lines should not be crossed, then examine the next segment along the face of lastv. If the face has been completely scanned, then backtrack lastv to the previous face. Remove a spot if necessary.

On Results and Open Questions

What pairs of 5-path thrackles can occur in a 6-path thrackle?
Once we have computed this, the challenge will be to use restrictions to prove that certain Dumbbells are not thrackable. This was done using 4-paths by Fulek and Pach to show that $D(6,6,-3)$ is not trackable.

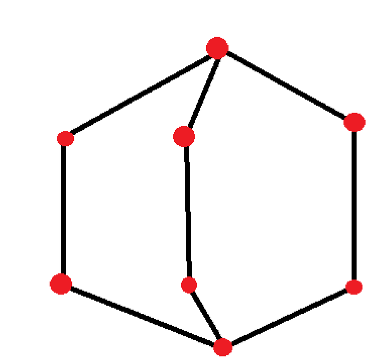


Figure 4: $D(6,6,-3)$

References

- Lovasz, L.; Pach, J.; Szegedy, M. (1997), "On Conway's Thrackle Conjecture".
- Fulek, R.; Pach, J. (2011), "A Computational approach to Conway's Thrackle Conjecture".
- http://www.thrackle.org/.