Problem 1.1.a

We would like to prove that $a\cos x + b\sin x = A\sin(x+\phi)$ for some amplitude A and phase shift ϕ. We first use the identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$:

$$\begin{aligned} A\sin(x+\phi) &= A(\sin x\cos\phi + \sin\phi\cos x) \\ a\cos x + b\sin x &= A\sin x\cos\phi + A\sin\phi\cos x \end{aligned}$$

By setting the components of $\cos x$ and $\sin x$ equal to each other,

$$\begin{aligned} a\cos x &= A\sin\phi\cos x \\ a &= A\sin\phi \\ b\sin x &= A\sin x\cos\phi \\ b &= A\cos\phi \end{aligned}$$

We can then determine A by,

$$\begin{aligned} (A\sin\phi)^2 + (A\cos\phi)^2 &= a^2 + b^2 \\ A^2(\sin^2\phi + \cos^2\phi) &= a^2 + b^2 \\ A &= \sqrt{a^2+b^2} \end{aligned}$$

We can also determine ϕ by,

$$\begin{aligned} \frac{A\sin\phi}{A\cos\phi} &= \frac{a}{b} \\ \phi &= \tanh(\frac{a}{b}) \end{aligned}$$

So we have that $a\cos x + b\sin x = A\sin(x+\phi)$ with amplitude $A = \sqrt{a^2+b^2}$ and phase shift $\phi = \tanh(\frac{a}{b})$.

Problem 1.1.b

The norm of $y = a\cos x + b\sin x$ is given by,

$$\begin{aligned}
||y||^2 &=< y, y > \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}(a\cos x + b\sin x)^2 dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}(a^2\cos^2 x + 2ab\cos x\sin x + b^2\sin^2 x)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}(a^2\cos^2 x)dx + \frac{1}{\pi}\int_{-\pi}^{\pi}(2ab\cos x\sin x)dx + \frac{1}{\pi}\int_{-\pi}^{\pi}(b^2\sin^2 x)dx
\end{aligned}$$

Solving each of the integrals separately,

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi}(\cos^2 x)dx &= \frac{1}{\pi}(\pi) = 1 \\
\frac{1}{\pi}\int_{-\pi}^{\pi}(\cos x\sin x)dx &= \frac{1}{\pi}(0) = 0 \\
\frac{1}{\pi}\int_{-\pi}^{\pi}(\sin^2 x)dx &= \frac{1}{\pi}(\pi) = 1
\end{aligned}$$

Giving us,

$$\begin{aligned}
||y||^2 &= a^2(1) + 2ab(0) + b^2(1) \\
&= a^2 + b^2
\end{aligned}$$

If $(a, b) \in S^1$, then $a^2 + b^2 = 1$. Therefore, $||y||^2 =$ and the resulting sine has unit norm.

Problem 2.3.b

The reason there is some distortion in the eigenvectors, when compared to the plots in Lecture 4, may be because the filters we used in the lab are subject to finite limits and constraints such as padding, boundary effects, and discretization of the space. Sorting the eigenvalues is helpful because the strongest signals come from eigenvectors with the largest eigenvalues, so plotting the strongest corresponding eigenvectors is best.

Problem 2.5.a

Convolution in the spatial domain corresponds to multiplication in the Fourier domain, meaning that the Fourier transform of a convolution operator should be diagonal. This is also desirable because it converts the expensive operation of convolution in the spatial domain into element-wise multiplication in the Fourier domain.

Problem 3.1

The response of $G^{(n)}$ to $f(x)$ for $n \geq 1$ is defined as $\int_{-\infty}^{\infty} G^{(n)}(u)f(x-u)du$. For $f(x) = 1$,

$$\begin{aligned}
(G^{(n)} * f)(x) &= \int_{-\infty}^{\infty} G^{(n)}(u)f(x-u)du \\
&= \int_{-\infty}^{\infty} G^{(n)}(u)du \\
&= 0
\end{aligned}$$

because $G(x)$ is a probability distribution, and the derivative of a probability distribution should integrate to 0. This implies that $G^{(n)}$ should be centered at 0/symmetric about 0 so that subsequent derivative filters also integrate to 0.

For $f(x) = x^n$, $(G^{(n)} * f)(x) = \int_{-\infty}^{\infty} G^{(n)}(u)(x-u)^n du$
. The Taylor expansion of $(x-u)^n$ gives us $(x-u)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}(-u)^k$. Inserting this into our integral, we get,

$$\begin{aligned}
(G^{(n)} * f)(x) &= \int_{-\infty}^{\infty} G^{(n)}(u) \sum_{k=0}^{n} \binom{n}{k} x^{n-k}(-u)^k du \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \int_{-\infty}^{\infty} G^{(n)}(u)(-u)^k du
\end{aligned}$$

For the inner integral $\int_{-\infty}^{\infty} G^{(n)}(u)(-u)^k du$, for $k < n$, the inner integral equals 0 because the symmetries of $G^{(n)}$ and $(-u)^k$ cancel out. For $k = n$, the inner integral is non-zero, and is in fact corresponds to the first non-zero moment of the Gaussian. Therefore, $(G^{(n)} * x^n)(x)$ is a constant, and $\sum_i G^{(n)}(x_i)x_i^n$ is also a constant. The weighted sum of $G^{(n)}$ must be a constant value.

Problem 3.4.b

The fact that both methods produced the same output demonstrates that steering and convolving the filter are commutable operations. Whether you steer the filter and convolve the steered filter with the image, or steer the image basis and convolve the steered image with the original filter, both produce the same output. This also indicates that the Fourier Transform handles rotations in a way that allows you to rotate both the image and the filter in the frequency domain using the same transformation, reinforcing the fact that rotations in the spatial domain can be represented in the frequency domain and vice versa.