Problem 1.1.a

We would like to prove that $a \cos x + b \sin x = A \sin (x + \phi)$ for some amplitude A and phase shift ϕ. We first use the identity $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$:

$$\begin{aligned} A \sin(x + \phi) &= A(\sin x \cos \phi + \sin \phi \cos x) \\ a \cos x + b \sin x &= A \sin x \cos \phi + A \sin \phi \cos x \end{aligned}$$

By setting the components of $\cos x$ and $\sin x$ equal to each other,

$$\begin{aligned} a \cos x &= A \sin \phi \cos x \\ a &= A \sin \phi \\ b \sin x &= A \sin x \cos \phi \\ b &= A \cos \phi \end{aligned}$$

We can then determine A by,

$$\begin{aligned} (A \sin \phi)^2 + (A \cos \phi)^2 &= a^2 + b^2 \\ A^2(\sin^2 \phi + \cos^2 \phi) &= a^2 + b^2 \\ A &= \sqrt{a^2 + b^2} \end{aligned}$$

We can also determine ϕ by,

$$\begin{aligned} \frac{A \sin \phi}{A \cos \phi} &= \frac{a}{b} \\ \phi &= \tanh(\frac{a}{b}) \end{aligned}$$

So we have that $a \cos x + b \sin x = A \sin (x + \phi)$ with amplitude $A = \sqrt{a^2 + b^2}$ and phase shift $\phi = \tanh(\frac{a}{b})$.

Problem 1.1.b

The norm of $y = a\cos x + b\sin x$ is given by,

$$\begin{aligned}
||y||^2 &=< y, y > \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}(a\cos x + b\sin x)^2 dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}(a^2\cos^2 x + 2ab\cos x\sin x + b^2\sin^2 x)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}(a^2\cos^2 x)dx + \frac{1}{\pi}\int_{-\pi}^{\pi}(2ab\cos x\sin x)dx + \frac{1}{\pi}\int_{-\pi}^{\pi}(b^2\sin^2 x)dx
\end{aligned}$$

Solving each of the integrals separately,

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi}(\cos^2 x)dx &= \frac{1}{\pi}(\pi) = 1 \\
\frac{1}{\pi}\int_{-\pi}^{\pi}(\cos x\sin x)dx &= \frac{1}{\pi}(0) = 0 \\
\frac{1}{\pi}\int_{-\pi}^{\pi}(\sin^2 x)dx &= \frac{1}{\pi}(\pi) = 1
\end{aligned}$$

Giving us,

$$\begin{aligned}
||y||^2 &= a^2(1) + 2ab(0) + b^2(1) \\
&= a^2 + b^2
\end{aligned}$$

If $(a, b) \in S^1$, then $a^2 + b^2 = 1$. Therefore, $||y||^2 =$ and the resulting sine has unit norm.