Theoretical Guide
Humuhumunukunukuapua'a
UFMG

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1 Permutations

1.1 Cycles

Let $g_S(n)$ be the number of n-permutations whose cycle lengths all belong to the set S. Then

$$\sum_{n=0}^{\infty} g_S(n)\frac{x^n}{n!} = \exp\left(\sum_{n\in S}\frac{x^n}{n}\right)$$

1.2 Derangements - permutacao caotica

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1)+D(n-2)) = nD(n-1)+(-1)^n = \left\lfloor\frac{n!}{e}\right\rceil$$

$$D(n) = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$$

1.3 Burnside's lemma - contar com simetria

Given a group G of symmetries and a set X, the number of elements of X *up to symmetry* equals

$$\frac{1}{|G|}\sum_{g\in G}|X^g|,$$

where X^g are the elements fixed by g ($g.x = x$).
If $f(n)$ counts "configurations" (of some sort) of length n, we can ignore rotational symmetry using $G = \mathbb{Z}_n$ to get

$$g(n) = \frac{1}{n}\sum_{k=0}^{n-1} f(\gcd(n,k)) = \frac{1}{n}\sum_{k|n} f(k)\phi(n/k).$$

2 Partitions and subsets

2.1 Partition function

Number of ways of writing n as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1,\ p(n) = \sum_{k\in\mathbb{Z}\setminus\{0\}} (-1)^{k+1}p(n-k(3k-1)/2)$$

$$p(n) \sim 0.145/n\cdot\exp(2.56\sqrt{n})$$

n	0	1	2	3	4	5	6	7	8	9	20	50	100
$p(n)$	1	1	2	3	5	7	11	15	22	30	627	~2e5	~2e8

2.2 Lucas' Theorem

Let n, m be non-negative integers and p a prime. Write $n = n_kp^k + ... + n_1p + n_0$ and $m = m_kp^k + ... + m_1p + m_0$. Then $\binom{n}{m} \equiv \prod_{i=0}^{k}\binom{n_i}{m_i} \pmod{p}$.

3 General purpose numbers

3.1 Bernoulli numbers

EGF of Bernoulli numbers is $B(t) = \frac{t}{e^t-1}$ (FFT-able).
$B[0,\ldots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \ldots]$
Sums of powers:

$$\sum_{i=1}^{n} n^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k\cdot(n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\sum_{i=m}^{\infty} f(i) = \int_m^{\infty} f(x)dx - \sum_{k=1}^{\infty}\frac{B_k}{k!}f^{(k-1)}(m)$$

$$\approx \int_m^\infty f(x)dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m))$$

3.2 Stirling numbers of the first kind - permutacoes com K ciclos

Number of permutations on n items with k cycles. Pode usar FFT no polinomio ali.

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k),\ c(0,0) = 1$$
$$\textstyle\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\dots(x+n-1)$$

$c(8,k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$
$c(n,2) = 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \dots$

3.3 Eulerian numbers - permutacao com K subidas

Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j:s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j:s s.t. $\pi(j) \geq j$, k j:s s.t. $\pi(j) > j$.

$$E(n,k) = (n-k)E(n-1,k-1) + (k+1)E(n-1,k)$$

$$E(n,0) = E(n,n-1) = 1$$

$$E(n,k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n$$

3.4 Stirling numbers of the second kind - Particao de N itens em K grupos

Partitions of n distinct elements into exactly k groups.

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

$$S(n,1) = S(n,n) = 1$$

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} j^n$$

3.5 Bell numbers - numero total de particoes

Total number of partitions of n distinct elements. $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \dots$. For p prime,

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod{p}$$

3.6 Labeled unrooted trees and forests

- on n vertices: n^{n-2}
- on k existing trees of size n_i: $n_1 n_2 \cdots n_k n^{k-2}$
- with degrees d_i: $(n-2)!/((d_1-1)!\cdots(d_n-1)!)$
- According to one of generalizations of Cayley's formula, number of forests of x vertices, where vertices 1,2,...,y belong to different trees is $f(x,y) = y \cdot (x^(x-y-1))$

3.7 Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$
$$C_0 = 1,\ C_{n+1} = \frac{2(2n+1)}{n+2}C_n,\ C_{n+1} = \sum C_i C_{n-i}$$

$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$

- sub-diagonal monotone paths in an $n \times n$ grid.
- strings with n pairs of parenthesis, correctly nested.
- binary trees with with $n+1$ leaves (0 or 2 children).
- ordered trees with $n+1$ vertices.
- ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.
- permutations of $[n]$ with no 3-term increasing subseq.

Para o caso que algumas paranteses já foram contadas, podemos ver que se pegarmos os caminhos errados e invertemos sempre que eles passarem por cima da diagonal $y = x + 1$ temos uma bijeção e esses caminhos sempre param em (n-1,n+1). Dai C_n é o número de caminhos de (0,0) a (n,n) menos ate (n-1,n+1). Total $\binom{A+F}{F} - \binom{A+F}{F+1}$ onde A sao quantas "(" faltam e F quantas ")" faltam.

4 Game Theory

4.1 Nim-K : tirar de K pilhas

Nim podendo tirar de K heaps, aka Moore's Nimk Se soma xi mod (k+1) == 0 pra todo bit i, é uma P position.

4.2 Monotonic Nim : nao pode ficar decrescente

Se n é impar pega o xor de (a(2*i+1) - a(2*i)), se não insere um 0 no inicio e repete.

4.3 Misere Nim : se nao tem jogada ganha

É uma P position se: existe $a_i > 1$ e xor == 0 ou $a_i \leq 1$ e xor == 1.
P quer dizer que "previous ganha" (você perdeu)

5 Geometry

5.1 Formula de Euler - vertices arestas e faces

V - E + F = 2

5.2 Pick Theorem - pontos lattice plane

Para achar pontos em coords inteiras num poligono

$$Area = i + \frac{b}{2} - 1$$

onde i eh o o numero de pontos dentro do poligono e b de pontos no perimetro do poligono.

5.3 Two ears theorem

Todo poligono simples com mais de 3 vertices tem pelo menos 2 orelhas, vertices que podem ser removidos sem criar um crossing, remover orelhas repetidamente triangula o poligono

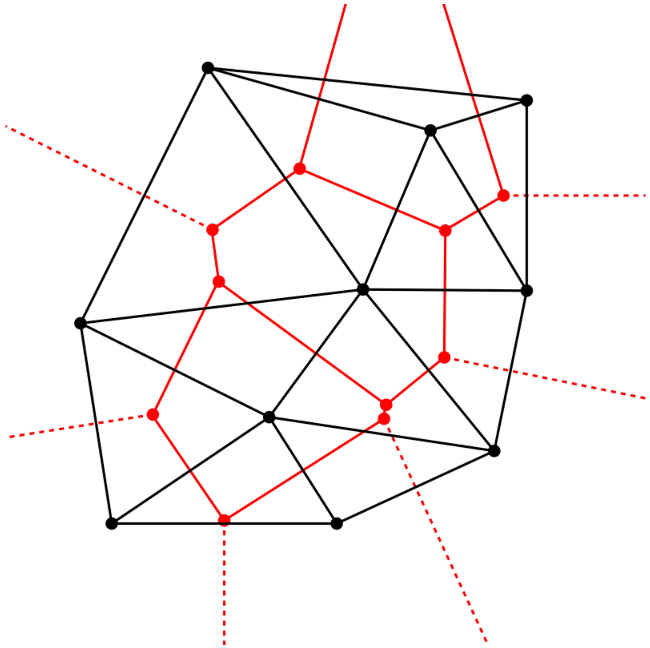
5.4 Incentro triangulo - bissetrizes - circ. inscrita

(a(Xa, Ya) + b(Xb, Yb) + c(Xc, Yc))/(a+b+c) onde a = lado oposto ao vertice a, incentro eh onde cruzam as bissetrizes, eh o centro da circunferencia inscrita e eh equidistante aos lados

5.5 Delaunay Triangulation

Triangulacao onde nenhum ponto esta dentro de nenhum circulo circunscrito nos triangulos.
Eh uma triangulacao que maximiza o menor angulo e a MST euclidiana de um conjunto de pontos eh um subconjunto da triangulacao.

5.6 voronoi diagram



5.7 Tangência

Dado um Circulo C na origem com raio R e um ponto P = (xp, yp) qualquer:
- Se P pertence a C, reta tangente que passa por P é da forma $x * (xp) + y * (yp) = r^2$

- Caso contrário, a interseção da reta r: $x \cdot (xp) + y \cdot (yp) = r^2$ com a circunferencia C são os dois pontos de tangencia

5.8 Brahmagupta's formula

Area cyclic quadrilateral s = (a+b+c+d)/2
area = $\sqrt((s-a)*(s-b)*(s-c)*(s-d))$
d = 0 (triangulo) a area = $\sqrt((s-a)*(s-b)*(s-c)*s)$

6 Graphs

6.1 Formula de Euler - vertices, arestas, faces e componentes

V - E + F = 2 (para grafo planar) / V - E + F = 1 + C (C sendo a qtd de componentes no grafo planar)

6.2 Handshaking

Numero par de vertices tem grau impar

6.3 Kirchhoff's Theorem

Monta matriz onde Mi,i = Grau[i] e Mi,j = -1 se houver aresta i-j ou 0 caso contrario, remove uma linha e uma coluna qualquer e o numero de spanning trees nesse grafo eh o det da matriz

6.4 Grafo contem caminho hamiltoniano se

Dirac's theorem: Se o grau de cada vertice for pelo menos n/2

6.5 Ore's theorem

Se a soma dos graus que cada par nao-adjacente de vertices for pelo menos n

6.6 Trees

Tem Catalan(N) Binary trees de N vertices
Tem Catalan(N-1) Arvores enraizadas com N vertices

6.7 Caley Formula

n^{n-2} arvores em N vertices com label
According to one of generalizations of Cayley's formula, number of forests of x vertices, where vertices 1,2,. . . ,y belong to different trees is
$f(x, y) = y \cdot (x^{x-y-1})$

6.8 Prufer code

Cada etapa voce remove a folha com menor label e o label do vizinho eh adicionado ao codigo ate ter 2 vertices. Prufer sequence tem tamanho n-2 e gera uma sequencia unica para cada arvore com label.

6.9 numero de arvores com sequencia de grau di

É multinomio de (n-2 , (d1-1 , .. , dn - 1))

6.10 Flow

- **Max Edge-disjoint paths**: Max flow com arestas com peso 1
- **Max Node-disjoint paths**: Faz a mesma coisa mas separa cada vertice em um com as arestas de chegadas e um com as arestas de saida e uma aresta de peso 1 conectando o vertice com aresta de chegada com ele mesmo com arestas de saida
- **Konig's Theorem**: minimum node cover = maximum matching se o grafo for bipartido, complemento eh o maximum independent set
- **Min Node disjoint path cover**: formar grafo bipartido de vertices duplicados, onde aresta sai do vertice tipo A e chega em tipo B, entao o path cover eh N - matching
- **Min General path cover**: Mesma coisa mas colocando arestas de A pra B sempre que houver caminho de A pra B
- **Dilworth's Theorem**: Min General Path cover = Max Antichain (set de vertices tal que nao existe caminho no grafo entre vertices desse set)
- **Hall's marriage**: um grafo tem um matching completo do lado X se para cada subconjunto W de X, |W| <= |vizinhosW| onde |W| eh quantos vertices tem em W

- **Weighted Independent set on bipartite graph**: Tudo menos mincut. Conecta da source com sink com capacidade igual a peso do vertice.

7 Math

7.1 Goldbach's

Todo numero par n > 2 pode ser representado com n = a + b onde a e b sao primos

7.2 Twin prime

Existem infinitos pares p, p + 2 onde ambos sao primos

7.3 Legendre's

Sempre tem um primo entre n^2 e $(n+1)^2$

7.4 Lagrange's

Todo numero inteiro pode ser inscrito como a soma de 4 quadrados

7.5 Zeckendorf's

Todo numero pode ser representado pela soma de dois numeros de fibonnacis diferentes e nao consecutivos

7.6 Euclid's - triplas pitagoricas

Toda tripla de pitagoras primitiva pode ser gerada com $(n^2 - m^2, 2nm, n^2 + m^2)$ onde n, m sao coprimos e um deles eh par

7.7 Wilson's

n eh primo quando $(n-1)!$ mod n = −1

7.8 Mcnugget - soma de coprimos

Para dois coprimos x, y o maior inteiro que nao pode ser escrito como $ax + by$ eh $(x-1)(y-1) - 1$ e tem $\frac{(x-1)(y-1)}{2}$ valores que é impossível escrever. Prova: usar teorema de pick, vendo os pontos embaixo da reta.

7.9 Fermat

Se p eh primo entao $a^{p-1}\ mod\ p = 1$
Se x e m tambem forem coprimos entao $x^k\ mod\ m = x^{(kmod(m-1))}\ mod\ m$

7.10 Euler's theorem

$x^{phi(m)}\ mod\ m = 1$ onde phi(m) eh o totiente de euler

7.11 Catalan number

Exemplo expressoes de parenteses bem formadas

- $C_0 = 1$, $C_n = \sum_{i=0}^{n-1} C_i \cdot C_{n-i+1}$
- $C_n = \frac{\binom{2n}{n}}{n+1}$
- Se ja tem alguns items voce tem $C(a+b, a) - C(a+b, b+1)$, com $a = n - openNoPrefix$ e $b = n - ClosedNoPrefix$.
- O número de caminhos de (0,0) até (n,n) que estão estritamente abaixo da diagonal y=x (mas podem tocar) em um grid é Catalan(n)

7.12 Bertrand's ballot theorem - sempre ganhar votacao

p votos tipo A e q votos tipo B com $p > q$, prob de em todo ponto ter mais As do que Bs antes dele =

$$\frac{(p-q)}{(p+q)}$$

Se puder empates entao prob = (p+1-q)/(p+1), para achar quantidade de possibilidades nos dois casos basta multiplicar por (p + q escolhe q)

7.13 Propriedades de Coeficientes Binomiais

$$\sum_{k=0} m(-1)^k * \binom{n}{k} = (-1)^m * \binom{n-1}{m}$$

$$\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{m=0}^{n}\binom{m}{k}=\binom{n+1}{k+1}$$

$$\sum_{k=0}^{m}\binom{n+k}{k}=\binom{n+m+1}{m}$$

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$$

$$\sum_{k=0}^{n}k\cdot\binom{n}{k}=n\cdot 2^(n-1)$$

$$\sum_{k=0}^{n}\binom{n-k}{k}=Fib(n+1)$$

7.14 Hockey-stick

$$\sum_{i=r}^{n}\binom{i}{r}=\binom{n+1}{r+1}$$

7.15 Vandermonde

$$\binom{m+n}{r}=\sum_{k=0}^{r}\binom{m}{k}\cdot\binom{n}{r-k}$$

7.16 Burnside lemma

colares diferentes nao contando rotacoes quando m = cores e n = comprimento

$$\frac{m^n+\sum_{i=1}^{n-1}m^{gcd(i,n)}}{n}$$

7.17 Distribuicao uniforme

$a, a+1, ..., b,$

$$Expected[X]=\frac{(a+b)}{2}$$

7.18 Distribuicao binomial

Com n tentativas de probabilidade p, X = sucessos:

$$P(X=x)=p^x\cdot(1-p)^{n-x}\cdot\binom{n}{x}$$

e

$$E[X]=p\cdot n$$

7.19 Distribuicao geometrica onde continuamos ate ter sucesso

$$P(X=x)=(1-p)^{x-1}\cdot p$$

$$E[X]=1/p$$

7.20 Linearity of expectation

Tendo duas variaveis X e Y e constantes a e b, o valor esperado de $aX+bY=a\cdot E[X]+b\cdot E[X]$

7.21 Variancia

$$var(x)=E[X^2]-E[X]^2$$

7.22 Higher order distributions of a coin

n trows and p chance of success in each.
$E[X^c]=\sum_{k=0}^{c}\left\{{c \atop k}\right\}n^{\underline{k}}p^k$ where $n^{\underline{k}}=n(n-1)\cdots(n-k+1)$ is the kth falling power of n.

7.23 funcao geradora

$(1+x)^{-n}=\sum_{k=0}^{\infty}((-1)^k\cdot\binom{n+k-1}{k}\cdot x^k)$

7.24 phi(m)

$$e >= log2(m)$$

,

$$n^e \ mod \ m=n^{(phi(m)+(e \ mod \ phi(m))} \ mod \ m$$

$phi(phi(...phi(m)))$ -> 1 em $O(logM)$ iterações

7.25 Number of times on k prefix sum's - Consertar

7.26 Multiplicative order

Smallest positive K such that $a^k https : //www.overleaf.com/project/63efa68f39aeeeef21218fb3 == 1 mod N$ -> $ord_n(a)$
As a consequence of Lagrange's theorem, $ord_n(a)$ always divides phi(n)
Se $gcd(a, n) \neq 1$ não existe $k > 0$

7.27 Pisano

k(m) = menor l tal que F[l] == 0 mod(m) F[l+1]==1 mod(m)
$k(a \cdot b) = lcm(k(a), k(b))$ se gcd(a,b)=1
$k(p^k)$ divide $p^{k-1} \cdot k(p)$
k(5)=20,k(2)=3,k(3)=8
Se $p > 5$:
k(p) divide (p-1) se p == +-1 mod 5
k(p) divide 2*(p+1) se p==+-2 mod 5
Tem que achar para cada p^k, testando todos os divisores com algum método de achar Fib rapido modulo.

7.28 Diofantinas

$$ax + by = c$$

divide tudo pelo $gcd(a, b)$; se $c\ mod\ gcd(a, b)\ neq 0$, não tem solução. Se não:
$a'x + b'y = c'$ tem solução dada pelo algoritmo de euclides
A resposta do problema original é:
$(X_g \cdot c/g, Y_g \cdot c/g)$, com X_g e Y_g achados com euclides.
OBS: passa $abs(a)$ e $abs(b)$ no euclides e depois inverter o sinal se era negativo.
Toda solução é na forma
$x = x0 + k * b'$
$y = y0 - k * a'$

7.29 Choromatic polynomial of a cycle - pintar ciclo com K cores

numero de modos de pintar um ciclo de tamanho n com K cores:

$$(k-1)^n + (-1)^n \cdot (k-1)$$

pode achar com expo tb se precisar...

7.30 Mersenne

: Primos de Mersenne $2^n - 1$
Lista de Ns que resultam nos primeiros 41 primos de Mersenne:
2; 3; 5; 7; 13; 17; 19; 31; 61; 89; 107; 127; 521; 607; 1.279; 2.203; 2.281; 3.217; 4.253; 4.423; 9.689; 9.941; 11.213; 19.937; 21.701; 23.209; 44.497; 86.243; 110.503; 132.049; 216.091; 756.839; 859.433; 1.257.787; 1.398.269; 2.976.221; 3.021.377; 6.972.593; 13.466.917; 20.996.011; 24.036.583;

8 Probability

8.1 Moment Generating Functions

Let X be a random variable. Define $M_X(t) = E[e^{tX}]$.

when X is Discrete

$$M_X(t) = \sum_{i=1}^{\infty} e^{tx_i} p_i$$

when X is Continuous

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Then we have:

$$M_X(0) = 0 \qquad M_X'(0) = E[x] \qquad \frac{d^k M_X(0)}{dt^k} = E[x^k]$$

8.2 Distributions

8.2.1 Binomial

- X is the number of successes in a sequence of n independent experiments.

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad E[X] = np \qquad Var(X) = np(1-p)$$

8.2.2 Geometric

- X is the number of failures in a sequence of independent experiment of Bernoulli until the first success.

$$P(X = k) = (1-p)^k p \qquad E[X] = \frac{1}{p} \qquad Var(X) = \frac{1-p}{p^2}$$

9 Graphs

9.1 Planar Graphs

1. If G has k connected components, then $n - m + f = k + 1$.
2. $m \leq 3n - 6$. If G has no triangles, $m \leq 2n - 4$.
3. The minimum degree is less or equal 5. And can be 6 colored in $\mathcal{O}(n + m)$.

9.2 Counting Minimum Spanning Trees - $\tau(G)$

- **Cayley's Formula**: $\tau(K_n) = n^{n-2}$.
- **Complete Bipartite Graphs**: $\tau(K_{p,q}) = p^{q-1}q^{p-1}$.
- **Kirchhoff's Theorem**: More generally, if we define the Laplacian matrix $\mathbf{L}(G) = \mathbf{D} - \mathbf{A}$, where $\mathbf{D}$ is the diagonal matrix with entries equal to the degree of vertices and $\mathbf{A}$ is the adjacency matrix. For $\mathbf{L}(G)_{ab}$ equal to $\mathbf{L}(G)$ without row a and column b, we have $\tau(G) = \det \mathbf{L}(G)_{ab}$, for any row a and column b.

9.3 Prüfer's Sequence

The Prüfer sequence is a bijection between labeled trees with n vertices and sequences with $n - 2$ numbers from 1 to n.
To get the sequence from the tree:

- While there are more than 2 vertices, remove the leaf with smallest label and append it's neighbour to the end of the sequence.

To get the tree from the sequence:

- The degree of each vertex is 1 more than the number of occurrences of that vertex in the sequence. Compute the degree d, then do the following: for every value x in the sequence (in order), find the vertex with smallest label y such that $d(y) = 1$ and add an edge between x and y, and also decrease their degrees by 1. At the end of this procedure, there will be two vertices left with degree 1; add an edge between them.

9.4 Erdős-Gallai Theorem

A sequence of non-negative integers $d_1 \geq ... \geq d_n$ can be represented as the degree sequence of a finite simple graph on n vertices if and only if $d_1 + ... + d_n$ is even and

$$\sum_{i=1}^{k} d_i \;\leq\; k(k-1) + \sum_{i=k+1}^{n} min(d_i, k)$$

holds for every k in $1 \leq k \leq n$.

9.5 Maximum Matching in Complete Multipartite graphs

The size of the maximum matching in a complete multipartite graph with n vertices and k vertices in its largest partition is (reference):

$$|M| = \min\left(\left\lfloor \frac{n}{2} \right\rfloor, n - k\right)$$

9.6 Dilworth's Theorem

9.6.1 Node-disjoint Path Cover

The node disjoint path cover in a `DAG` is equal to $|V| - |M|$, where M is the maximum matching in the bipartite flow network.

9.6.2 General Path Cover

The general path cover in a `DAG` is equal to $|V| - |M|$, where M is the maximum matching in the bipartite flow network of the transitive closure graph.

9.6.3 Dilworth's Theorem

The size of the maximum **antichain** in a `DAG`, that is, the maximum size of a set S of vertices such that no vertex in S can reach another vertiex in S, is equal to size of the minimum **general** path cover.

9.7 Sum of Subtrees of a Tree

For a rooted tree T with n vertices, let $sz(v)$ be the size of the subtree of v. Then the following holds:

$$\sum_{v \in V} \left[\mathrm{sz}(v) + \sum_{u \text{ child of } v} \mathrm{sz}(u)(\mathrm{sz}(v) - \mathrm{sz}(u)) \right] = n^2$$

10 Counting Problems

10.1 Stirling numbers of the first kind

These are the number of permutations of $[n]$ with exactly k disjoint cycles. They obey the recurrence:

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1, \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0$$

- The sum products of the $\binom{n}{k}$ subsets of size k of $\{0, 1, \ldots, n-1\}$ is $\left[{n \atop n-k}\right]$.
- $\sum_{k=0}^{n} \left[{n \atop k}\right] = n!$
- $\sum_{k=0}^{n} \left[{n \atop k}\right] x^k = x(x-1)(x-2)...(x-n+1)$

10.2 Stirling numbers of the second kind

These are the number of ways to partition $[n]$ into exactly k non-empty sets. They obey the recurrence:

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = k\begin{Bmatrix} n-1 \\ k \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = 1, \begin{Bmatrix} n \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ n \end{Bmatrix} = 0$$

A "closed" formula for it is:

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n$$

10.3 How many functions $f\colon [n] \to [k]$ are there?

$[n]$	$[k]$	Any f	Injective	Surjective
dist	dist	k^n	$\frac{k!}{(n-k)!}$	$k!\left\{{n \atop k}\right\}$
indist	dist	$\binom{k+n-1}{n}$	$\binom{k}{n}$	$\binom{n-1}{n-k}$
dist	indist	$\sum_{i=1}^{k}\left\{{n \atop i}\right\}$	$[n \le k]$	$\left\{{n \atop k}\right\}$
indist	indist	$\sum_{i=1}^{k} p_i(n)$	$[n \le k]$	$p_k(n)$

Where $p_k(n)$ is the number of ways to partition n into k terms.

10.4 Derangement

A derangement is a permutation that has no fixed points. Let d_n be the number of ways of derangement of a sequence of the sequence $1 \ldots n$. We have the recurrence $d_n = (n-1)(d_{n-1} + d_{n-2})$. Moreover, d_n is the closest integer to $\frac{n!}{e}$.

$$d_n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}$$

10.5 Bell numbers

These count the number of ways to partition $[n]$ into subsets. They obey the recurrence:

$$\mathcal{B}_{n+1} = \sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k$$

x	5	6	7	8	9	10	11	12
$\mathcal{B}_x$	52	203	877	4.140	21.147	115.975	678.570	4.213.597

10.6 Eulerian numbers

The Eulerian number $T(n, k)$ is the number of permutations of the numbers from 1 to n in which exactly k elements are greater than the previous element (permutations with k "ascents").

$$T(n, k) = \sum_{j=0}^{k}(-1)^j (k-j)^{(n+1)}\binom{n+1}{j}$$

10.7 Burside's Lemma

Let G be a group that acts on a set X. The Burnside Lemma states that the number of distinct orbits is equal to the average number of points fixed by an element of G.

$$T = \frac{1}{|G|}\sum_{g \in G}|\texttt{fix}(g)|$$

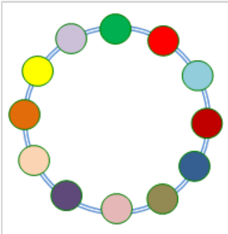
Where a orbit $\texttt{orb}(x)$ is defined as

$$\texttt{orb}(x) = \{y \in X : \exists g \in G \ gx = y\}$$

and $\texttt{fix}(g)$ is the set of elements in X fixed by g

$$\texttt{fix}(g) = \{x \in X : gx = x\}$$

Example: With k distinct types of beads how many distinct necklaces of size n can be made? Considering that two necklaces are equal if the rotation of one gives the other.



$$T = \frac{1}{n+1}\sum_{i=0}^{n} k^{gcd(i,n)}$$

10.8 Catalan Numbers

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452, 18367353072152, 69533550916004, 263747951750360, 1002242216651368.

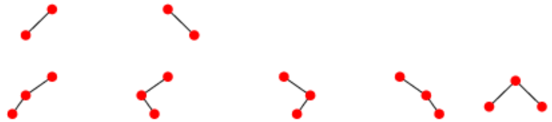
$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \prod_{k=2}^{n}\frac{n+k}{k}, \quad n \geq 0$$

Applications:

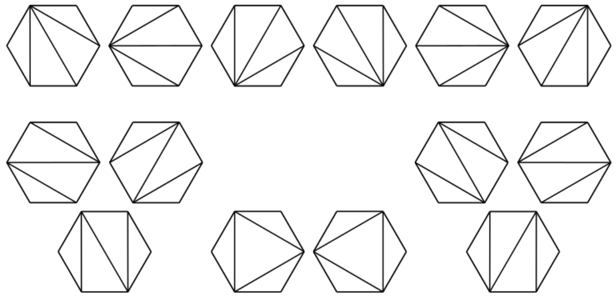
- C_n counts the number of expressions containing n pairs of parentheses which are correctly matched.

  ((())) ()(()) ()()() (())() ...

- Successive applications of a binary operator can be represented in terms of a full binary tree. (A rooted binary tree is full if every vertex has either two children or no children.) It follows that C_n is the number of full binary trees with n + 1 leaves:



- C_n is the number of different ways a convex polygon with n + 2 sides can be cut into triangles by connecting vertices with straight lines (a form of Polygon triangulation). The following hexagons illustrate the case n = 4:



10.9 Central Binomial Coefficient

To number of of subsets T of $S = \{\underbrace{1, 1, \ldots, 1}_{n}, \underbrace{-1, -1, \ldots, -1}_{n}\}$ that sum to 0 is

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n} = \frac{2n!}{(n!)^2} \approx \frac{2^{2n}}{\sqrt{n \cdot \pi}}$$

- The number of factors of 2 in $\binom{2n}{n}$ is equal to the number of 1's in the binary representation if n.
- $\binom{2n}{n}$ is never squarefree for $n > 4$.

11 Dynamic Programming Optimizations

11.1 Divide and Conquer

DP to compute the minimum cost to divide an array into k subarrays; the cost of a solution is equal to the sum of the costs of each subarray. The cost of a subarray $A[i..j]$ is $c(i, j)$.

$$dp[i][k] = \min_{j \geq i} (dp[j+1][k-1] + c(i,j))$$

- Define A to be the functions satisfying

$$dp[i][k] = dp[A(i,k)+1][k-1] + c(i, A(i,k)).$$

  If A also satisfy $A(i,k) \leq A(i+1,k)$, then the dp is optimizable.

- Another sufficient condition is, for every $a < b < c < d$:

$$c(a,d) + c(b,c) \geq c(a,c) + c(b,d)$$