The first few values of the $\{\lambda_n\}_{n\ge 0}$ are 1, 1, 1, 0, −3, −9, −18, ….

To complete this example we want to prove the helpful property (2.4.5) of the cube roots of unity. But for every $r > 1$, the rth roots of unity do the same sort of thing, namely

$$\frac{1}{r}\sum_{\omega^r=1}\omega^n = \begin{cases} 1 & \text{if } r\backslash n \\ 0 & \text{else.}\end{cases} \tag{2.4.9}$$

Indeed, the left side is

$$\frac{1}{r}\sum_{j=0}^{r-1} e^{(2\pi ijn)/r},$$

which is a finite geometric series whose sum is easy to find, and is as stated in (2.4.9). So, with more or less difficulty, it is always possible to select a subset of the terms of a convergent series in which the exponents form an arithmetic progression. See exercise 25. ■

2.5 Some useful power series

Generatingfunctionologists need reference lists of known power series and other series that occur frequently in applications of the theory. Here is such a list. For each series we show the series and its sum. The radius of the largest open disk, centered at the origin, in which convergence takes place will be, of course, the modulus of the singularity of the function that is nearest to the origin. Considering the relatively simple forms of the functions, the locations of those singularities will be sufficiently obvious that the radii of convergence are not explicitly shown in the table below.

$$\frac{1}{1-x} = \sum_{n\ge 0} x^n \tag{2.5.1}$$

$$\log\frac{1}{1-x} = \sum_{n\ge 1}\frac{x^n}{n} \tag{2.5.2}$$

$$e^x = \sum_{n\ge 0}\frac{x^n}{n!} \tag{2.5.3}$$

$$\sin x = \sum_{n\ge 0}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \tag{2.5.4}$$

$$\cos x = \sum_{n\ge 0}(-1)^n\frac{x^{2n}}{(2n)!} \tag{2.5.5}$$

$$(1+x)^\alpha = \sum_k \binom{\alpha}{k} x^k \tag{2.5.6}$$

$$\frac{1}{(1-x)^{k+1}} = \sum_n \binom{n+k}{n} x^n \tag{2.5.7}$$

$$\frac{x}{e^x - 1} = \sum_{n \ge 0} \frac{B_n x^n}{n!} \tag{2.5.8}$$

$$\tan^{-1} x = \sum_{n \ge 0} (-1)^n \frac{x^{2n+1}}{2n+1} \tag{2.5.9}$$

$$\begin{aligned} \frac{1}{2x}(1 - \sqrt{1-4x}) &= \sum_n \frac{1}{n+1} \binom{2n}{n} x^n \\ &= 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 \\ &\quad + 429x^7 + 1430x^8 + 4862x^9 + \cdots \end{aligned} \tag{2.5.10}$$

$$\begin{aligned} \frac{1}{\sqrt{1-4x}} &= \sum_k \binom{2k}{k} x^k \\ &= 1 + 2x + 6x^2 + 20x^3 + 70x^4 + 252x^5 + 924x^6 \\ &\quad + 3432x^7 + 12870x^8 + 48620x^9 + \cdots \end{aligned} \tag{2.5.11}$$

$$\begin{aligned} x \cot x &= \sum_{k \ge 0} \frac{(-4)^k B_{2k}}{(2k)!} x^{2k} \\ &= 1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2\,x^6}{945} - \frac{x^8}{4725} - \frac{2\,x^{10}}{93555} - \cdots \end{aligned} \tag{2.5.12}$$

$$\begin{aligned} \tan x &= \sum_{r \ge 1} (-1)^{r-1} \frac{2^{2r}(2^{2r}-1) B_{2r}}{(2r)!} x^{2r-1} \\ &= x + \frac{x^3}{3} + \frac{2\,x^5}{15} + \frac{17\,x^7}{315} + \frac{62\,x^9}{2835} + \frac{1382\,x^{11}}{155925} + \cdots \\ &\quad + \frac{21844\,x^{13}}{6081075} + \frac{929569\,x^{15}}{638512875} + \cdots \end{aligned} \tag{2.5.13}$$

$$\frac{x}{\sin x} = \sum_{r\geq 0}(-1)^{r-1}\frac{(4^r-2)B_{2r}}{(2r)!}x^{2r}$$
$$= 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \frac{31x^6}{15120} + \cdots \tag{2.5.14}$$

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^k = \sum_n \binom{2n+k}{n}x^n \tag{2.5.15}$$

$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^k = \sum_{n\geq 0}\frac{k(2n+k-1)!}{n!(n+k)!}x^n \qquad (k\geq 1) \tag{2.5.16}$$

$$\sin^{-1}(x) = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots \tag{2.5.17}$$

$$e^x\sin x = \sum_{n\geq 1}\frac{2^{\frac{n}{2}}\sin\frac{n\pi}{4}}{n!}x^n \tag{2.5.18}$$
$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} - \frac{x^7}{630} + \cdots$$

$$\frac{1}{2}\tan^{-1}(x)\log(1+x^2) = \sum_{r\geq 1}(-1)^{r-1}H_{2r}\frac{x^{2r+1}}{2r+1} \tag{2.5.19}$$
$$= \frac{x^3}{2} - \frac{5x^5}{12} + \frac{7x^7}{20} - \frac{761x^9}{2520} + \cdots$$

$$\frac{1}{4}\tan^{-1}(x)\log\frac{1+x}{1-x} = \sum_{r\geq 0}\frac{x^{4r+2}}{4r+2}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{1}{4r+1}\right)$$
$$= \frac{x^2}{2} + \frac{13x^6}{90} + \frac{263x^{10}}{3150} + \cdots \tag{2.5.20}$$

$$\frac{1}{2}\left\{\log\frac{1}{1-x}\right\}^2 = \sum_{r\geq 2}\frac{H_{r-1}}{r}x^r \tag{2.5.21}$$

$$= \frac{x^2}{2} + \frac{x^3}{2} + \frac{11x^4}{24} + \frac{5x^5}{12} + \frac{137x^6}{360} + \frac{7x^7}{20} + \cdots$$

$$\begin{aligned}\sqrt{\frac{1-\sqrt{1-x}}{x}} &= \sum_{k=0}^{\infty} \frac{(4k)!}{16^k\sqrt{2}(2k)!(2k+1)!}x^k \qquad (2.5.22)\\ &= \frac{1}{\sqrt{2}}\left(1 + \frac{x}{8} + \frac{7\,x^2}{128} + \frac{33\,x^3}{1024} + \frac{715\,x^4}{32768} + \frac{4199\,x^5}{262144}\right.\\ &\quad + \frac{52003\,x^6}{4194304} + \frac{334305\,x^7}{33554432} + \frac{17678835\,x^8}{2147483648}\\ &\quad\left. + \frac{119409675\,x^9}{17179869184} + \frac{1641030105\,x^{10}}{274877906944} + \cdots\right)\end{aligned}$$

$$\begin{aligned}e^{\arcsin x} &= \sum_{k=0}^{\infty} \frac{\prod_{j=0}^{k-1}(4j^2+1)}{(2k)!}x^{2k} + \sum_{k=0}^{\infty} \frac{4^k\prod_{j=1}^{k}(\frac{1}{2}-j+j^2)}{(2k+1)!}x^{2k+1}\\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{5\,x^4}{24} + \frac{x^5}{6} + \frac{17\,x^6}{144} + \frac{13\,x^7}{126}\\ &\quad + \frac{629\,x^8}{8064} + \frac{325\,x^9}{4536} + \frac{8177\,x^{10}}{145152} + \cdots \qquad (2.5.23)\end{aligned}$$

$$\begin{aligned}\left(\frac{\arcsin x}{x}\right)^2 &= \sum_{k=0}^{\infty} \frac{4^k k!^2}{(k+1)(2k+1)!}x^{2k} \qquad (2.5.24)\\ &= 1 + \frac{x^2}{3} + \frac{8\,x^4}{45} + \frac{4\,x^6}{35} + \frac{128\,x^8}{1575} + \frac{128\,x^{10}}{2079} + \cdots\end{aligned}$$

$$\begin{aligned}(x+\sqrt{1+x^2})^a &= \sum_{k=0}^{\infty} \frac{2^k\cdot(\frac{a}{2}-\frac{k}{2}+1)^{\overline{k}}}{(1+k/a)k!}x^k \qquad (2.5.25)\\ &= 1 + a\,x + \frac{a^2\,x^2}{2} + \left(\frac{-a}{6}+\frac{a^3}{6}\right)x^3 + \left(\frac{-a^2}{6}+\frac{a^4}{24}\right)x^4\\ &\quad + \frac{a\left(9-10\,a^2+a^4\right)\,x^5}{120} + \frac{a^2\left(64-20\,a^2+a^4\right)\,x^6}{720}\\ &\quad + \frac{a\left(-225+259\,a^2-35\,a^4+a^6\right)\,x^7}{5040}\\ &\quad + \frac{a^2\left(-2304+784\,a^2-56\,a^4+a^6\right)\,x^8}{40320} + \cdots\end{aligned}$$