Theoretical Guide
Humuhumunukunukuapua'a
UFMG

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1 Permutations

1.1 Cycles

Let $g_S(n)$ be the number of n-permutations whose cycle lengths all belong to the set S. Then

$$\sum_{n=0}^{\infty} g_S(n)\frac{x^n}{n!} = \exp\left(\sum_{n\in S}\frac{x^n}{n}\right)$$

1.2 Derangements - permutacao caotica

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1)+D(n-2)) = nD(n-1)+(-1)^n = \left\lfloor\frac{n!}{e}\right\rceil$$

$$D(n) = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$$

1.3 Burnside's lemma - contar com simetria

Given a group G of symmetries and a set X, the number of elements of X *up to symmetry* equals

$$\frac{1}{|G|}\sum_{g\in G}|X^g|,$$

where X^g are the elements fixed by g ($g.x = x$).
If $f(n)$ counts "configurations" (of some sort) of length n, we can ignore rotational symmetry using $G = \mathbb{Z}_n$ to get

$$g(n) = \frac{1}{n}\sum_{k=0}^{n-1} f(\gcd(n,k)) = \frac{1}{n}\sum_{k|n} f(k)\phi(n/k).$$

2 Partitions and subsets

2.1 Partition function

Number of ways of writing n as a sum of positive integers, disregarding the order of the summands.

$$p(0)=1,\ p(n) = \sum_{k\in\mathbb{Z}\setminus\{0\}}(-1)^{k+1}p(n-k(3k-1)/2)$$

$$p(n) \sim 0.145/n\cdot\exp(2.56\sqrt{n})$$

n	0	1	2	3	4	5	6	7	8	9	20	50	100
$p(n)$	1	1	2	3	5	7	11	15	22	30	627	~2e5	~2e8

2.2 Lucas' Theorem

Let n, m be non-negative integers and p a prime. Write $n = n_kp^k + ... + n_1p + n_0$ and $m = m_kp^k + ... + m_1p + m_0$. Then $\binom{n}{m} \equiv \prod_{i=0}^{k}\binom{n_i}{m_i} \pmod{p}$.

3 General purpose numbers

3.1 Bernoulli numbers

EGF of Bernoulli numbers is $B(t) = \frac{t}{e^t-1}$ (FFT-able).
$B[0,\ldots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \ldots]$
Sums of powers:

$$\sum_{i=1}^{n} n^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k\cdot(n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\sum_{i=m}^{\infty} f(i) = \int_m^{\infty} f(x)dx - \sum_{k=1}^{\infty}\frac{B_k}{k!}f^{(k-1)}(m)$$

$$\approx \int_m^\infty f(x)dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m))$$

3.2 Stirling numbers of the first kind - permutacoes com K ciclos

Number of permutations on n items with k cycles. Pode usar FFT no polinomio ali.

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k),\ c(0,0) = 1$$
$$\textstyle\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\dots(x+n-1)$$

$c(8,k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$
$c(n,2) = 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \dots$
EGF para coluna m: $(-log(1-x))^k/k!, k >= 0$

3.3 Eulerian numbers - permutacao com K subidas

Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j:s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j:s s.t. $\pi(j) \geq j$, k j:s s.t. $\pi(j) > j$.

$$E(n,k) = (n-k)E(n-1,k-1) + (k+1)E(n-1,k)$$

$$E(n,0) = E(n,n-1) = 1$$

$$E(n,k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n$$

3.4 Stirling numbers of the second kind - Particao de N itens em K grupos

Partitions of n distinct elements into exactly k groups.

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

$$S(n,1) = S(n,n) = 1$$

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

Or a FFT-table formula with $p_i = \frac{(-1)^i}{i!}$ and $q_j = \frac{j^n}{j!}$.

$$S(n,k) = \sum_{i=0}^{k} \frac{(-1)^i}{i!} \cdot \frac{(k-i)^n}{(k-i)!}$$

3.5 Bell numbers - numero total de particoes

Total number of partitions of n distinct elements. $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \dots$. For p prime,

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod p$$

3.6 Labeled unrooted trees and forests

- on n vertices: n^{n-2}
- on k existing trees of size n_i: $n_1 n_2 \cdots n_k n^{k-2}$
- with degrees d_i: $(n-2)!/((d_1-1)! \cdots (d_n-1)!)$
- According to one of generalizations of Cayley's formula, number of forests of x vertices, where vertices 1,2,. . . ,y belong to different trees is $f(x,y) = y \cdot (x^(x-y-1))$

3.7 Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1,\ C_{n+1} = \frac{2(2n+1)}{n+2} C_n,\ C_{n+1} = \sum C_i C_{n-i}$$

$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$

- sub-diagonal monotone paths in an $n \times n$ grid.
- strings with n pairs of parenthesis, correctly nested.
- binary trees with with $n+1$ leaves (0 or 2 children).
- ordered trees with $n+1$ vertices.
- ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.

- permutations of $[n]$ with no 3-term increasing subseq.

Para o caso que algumas paranteses já foram contadas, podemos ver que se pegarmos os caminhos errados e invertemos sempre que eles passarem por cima da diagonal $y = x + 1$ temos uma bijeção e esses caminhos sempre param em (n-1,n+1). Dai C_n é o número de caminhos de (0,0) a (n,n) menos ate (n-1,n+1). Total $\binom{A+F}{F} - \binom{A+F}{F+1}$ onde A sao quantas "(" faltam e F quantas ")" faltam.

4 Game Theory

4.1 Nim-K : tirar de K pilhas

Nim podendo tirar de K heaps, aka Moore's Nimk Se soma xi mod (k+1) == 0 pra todo bit i, é uma P position.

4.2 Monotonic Nim : nao pode ficar decrescente

Se n é impar pega o xor de (a(2*i+1) - a(2*i)), se não insere um 0 no inicio e repete.

4.3 Misere Nim : se nao tem jogada ganha

É uma P position se: existe $a_i > 1$ e xor == 0 ou $a_i \leq 1$ e xor == 1.
P quer dizer que "previous ganha" (você perdeu)

5 Geometry

5.1 Formula de Euler - vertices arestas e faces

V - E + F = 2

5.2 Pick Theorem - pontos lattice plane

Para achar pontos em coords inteiras num poligono

$$Area = i + \frac{b}{2} - 1$$

onde i eh o o numero de pontos dentro do poligono e b de pontos no perimetro do poligono.

5.3 Two ears theorem

Todo poligono simples com mais de 3 vertices tem pelo menos 2 orelhas, vertices que podem ser removidos sem criar um crossing, remover orelhas repetidamente triangula o poligono

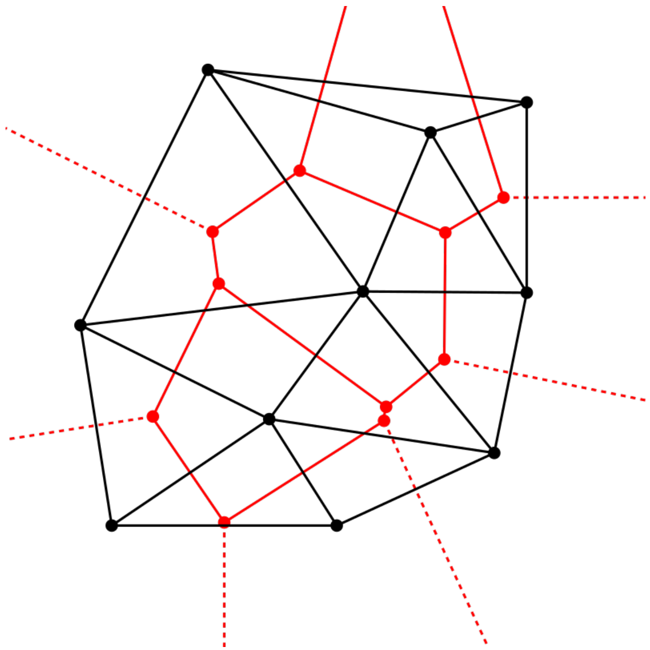
5.4 Incentro triangulo - bissetrizes - circ. inscrita

(a(Xa, Ya) + b(Xb, Yb) + c(Xc, Yc))/(a+b+c) onde a = lado oposto ao vertice a, incentro eh onde cruzam as bissetrizes, eh o centro da circunferencia inscrita e eh equidistante aos lados

5.5 Delaunay Triangulation

Triangulacao onde nenhum ponto esta dentro de nenhum circulo circunscrito nos triangulos.
Eh uma triangulacao que maximiza o menor angulo e a MST euclidiana de um conjunto de pontos eh um subconjunto da triangulacao.

5.6 voronoi diagram



5.7 Tangência

Dado um Circulo C na origem com raio R e um ponto P = (xp, yp) qualquer:
- Se P pertence a C, reta tangente que passa por P é da forma
$x * (xp) + y * (yp) = r^2$
- Caso contrário, a interseção da reta r: $x \cdot (xp) + y \cdot (yp) = r^2$ com a circunferencia C são os dois pontos de tangencia

5.8 Brahmagupta's formula

Area cyclic quadrilateral s = (a+b+c+d)/2
area = $\sqrt((s-a)*(s-b)*(s-c)*(s-d))$
d = 0 (triangulo) a area = $\sqrt((s-a)*(s-b)*(s-c)*s)$

6 Graphs

6.1 Formula de Euler - vertices, arestas, faces e componentes

V - E + F = 2 (para grafo planar) / V - E + F = 1 + C (C sendo a qtd de componentes no grafo planar)

6.2 Handshaking

Numero par de vertices tem grau impar

6.3 Kirchhoff's Theorem

Monta matriz onde Mi,i = Grau[i] e Mi,j = -1 se houver aresta i-j ou 0 caso contrario, remove uma linha e uma coluna qualquer e o numero de spanning trees nesse grafo eh o det da matriz

6.4 Grafo contem caminho hamiltoniano se

Dirac's theorem: Se o grau de cada vertice for pelo menos n/2

6.5 Ore's theorem

Se a soma dos graus que cada par nao-adjacente de vertices for pelo menos n

6.6 Trees

Tem Catalan(N) Binary trees de N vertices
Tem Catalan(N-1) Arvores enraizadas com N vertices

6.7 Caley Formula

n^{n-2} arvores em N vertices com label
According to one of generalizations of Cayley's formula, number of forests of x vertices, where vertices 1,2,. . . ,y belong to different trees is
$f(x, y) = y \cdot (x^{x-y-1})$

6.8 Prufer code

Cada etapa voce remove a folha com menor label e o label do vizinho eh adicionado ao codigo ate ter 2 vertices. Prufer sequence tem tamanho n-2 e gera uma sequencia unica para cada arvore com label.

6.9 numero de arvores com sequencia de grau di

É multinomio de (n-2 , (d1-1 , .. , dn - 1))

6.10 Flow

- **Max Edge-disjoint paths**: Max flow com arestas com peso 1
- **Max Node-disjoint paths**: Faz a mesma coisa mas separa cada vertice em um com as arestas de chegadas e um com as arestas de saida e uma aresta de peso 1 conectando o vertice com aresta de chegada com ele mesmo com arestas de saida
- **Konig's Theorem**: minimum node cover = maximum matching se o grafo for bipartido, complemento eh o maximum independent set
- **Min Node disjoint path cover**: formar grafo bipartido de vertices duplicados, onde aresta sai do vertice tipo A e chega em tipo B, entao o path cover eh N - matching
- **Min General path cover**: Mesma coisa mas colocando arestas de A pra B sempre que houver caminho de A pra B
- **Dilworth's Theorem**: Min General Path cover = Max Antichain (set de vertices tal que nao existe caminho no grafo entre vertices desse set)

- **Hall's marriage**: um grafo tem um matching completo do lado X se para cada subconjunto W de X, |W| <= |vizinhosW| onde |W| eh quantos vertices tem em W
- **Weighted Independent set on bipartite graph**: Tudo menos mincut. Conecta da source com sink com capacidade igual a peso do vertice.

7 Math

7.1 Goldbach's

Todo numero par n > 2 pode ser representado com n = a + b onde a e b sao primos

7.2 Twin prime

Existem infinitos pares p, p + 2 onde ambos sao primos

7.3 Legendre's

Sempre tem um primo entre n^2 e $(n+1)^2$

7.4 Lagrange's

Todo numero inteiro pode ser inscrito como a soma de 4 quadrados

7.5 Zeckendorf's

Todo numero pode ser representado pela soma de dois numeros de fibonnacis diferentes e nao consecutivos

7.6 Euclid's - triplas pitagoricas

Toda tripla de pitagoras primitiva pode ser gerada com $(n^2 - m^2, 2nm, n^2 + m^2)$ onde n, m sao coprimos e um deles eh par

7.7 Wilson's

n eh primo quando $(n-1)!$ mod n $= -1$

7.8 Mcnugget - soma de coprimos

Para dois coprimos x, y o maior inteiro que nao pode ser escrito como $ax + by$ eh $(x-1)(y-1) - 1$ e tem $\frac{(x-1)(y-1)}{2}$ valores que é impossível escrever. Prova: usar teorema de pick, vendo os pontos embaixo da reta.

7.9 Fermat

Se p eh primo entao a^{p-1} mod $p = 1$
Se x e m tambem forem coprimos entao x^k mod $m = x^{(kmod(m-1))}$ mod m

7.10 Euler's theorem

$x^{phi(m)}$ mod $m = 1$ onde phi(m) eh o totiente de euler

7.11 Catalan number

Exemplo expressoes de parenteses bem formadas

- $C_0 = 1$, $C_n = \sum_{i=0}^{n-1} C_i \cdot C_{n-i+1}$
- $C_n = \frac{\binom{2n}{n}}{n+1}$
- Se ja tem alguns items voce tem $C(a+b, a) - C(a+b, b+1)$, com $a = n - openNoPrefix$ e $b = n - ClosedNoPrefix$.
- O número de caminhos de (0,0) até (n,n) que estão estritamente abaixo da diagonal y=x (mas podem tocar) em um grid é Catalan(n)

7.12 Bertrand's ballot theorem - sempre ganhar votacao

p votos tipo A e q votos tipo B com $p > q$, prob de em todo ponto ter mais As do que Bs antes dele =

$$\frac{(p-q)}{(p+q)}$$

Se puder empates entao prob = (p+1-q)/(p+1), para achar quantidade de possibilidades nos dois casos basta multiplicar por (p + q escolhe q)

7.13 Propriedades de Coeficientes Binomiais

$$\sum_{k=0} m(-1)^k * \binom{n}{k} = (-1)^m * \binom{n-1}{m}$$

$$\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{m=0}^{n} \binom{m}{k} = \binom{n+1}{k+1}$$

$$\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot 2^(n-1)$$

$$\sum_{k=0}^{n} \binom{n-k}{k} = Fib(n+1)$$

7.14 Hockey-stick

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$

7.15 Vandermonde

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \cdot \binom{n}{r-k}$$

7.16 Burnside lemma

colares diferentes nao contando rotacoes quando m = cores e n = comprimento

$$\frac{m^n + \sum_{i=1}^{n-1} m^{gcd(i,n)}}{n}$$

7.17 Distribuicao uniforme

$a, a+1, ..., b,$

$$Expected[X] = \frac{(a+b)}{2}$$

7.18 Distribuicao binomial

Com n tentativas de probabilidade p, X = sucessos:

$$P(X = x) = p^x \cdot (1-p)^{n-x} \cdot \binom{n}{x}$$

e

$$E[X] = p \cdot n$$

7.19 Distribuicao geometrica onde continuamos ate ter sucesso

$$P(X = x) = (1-p)^{x-1} \cdot p$$
$$E[X] = 1/p$$

7.20 Linearity of expectation

Tendo duas variaveis X e Y e constantes a e b, o valor esperado de $aX + bY = a \cdot E[X] + b \cdot E[X]$

7.21 Variancia

$$var(x) = E[X^2] - E[X]^2$$

7.22 Higher order distributions of a coin

n trows and p chance of success in each.
$E[X^c] = \sum_{k=0}^{c} \left\{ {c \atop k} \right\} n^{\underline{k}} p^k$ where $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$ is the kth falling power of n.

7.23 funcao geradora

$(1+x)^{-n} = \sum_{k=0}^{\infty}((-1)^k \cdot \binom{n+k-1}{k} \cdot x^k)$

7.24 phi(m)

$$e >= log2(m)$$

,

$$n^e \ mod \ m = n^{(phi(m)+(e \ mod \ phi(m))} \ mod \ m$$

$phi(phi(...phi(m)))$ -> 1 em $O(logM)$ iterações

7.25 Number of times on k prefix sum's - Consertar

7.26 Multiplicative order

Smallest positive K such that $a^k https : //www.overleaf.com/project/63efa68f39aeeeef21218fb3 == 1 modN$ -> $ord_n(a)$
As a consequence of Lagrange's theorem, $ord_n(a)$ always divides phi(n)
Se $gcd(a,n) \neq 1$ não existe $k > 0$

7.27 Pisano

k(m) = menor l tal que F[l] == 0 mod(m) F[l+1]==1 mod(m)
$k(a \cdot b) = lcm(k(a), k(b))$ se gcd(a,b)=1
$k(p^k)$ divide $p^{k-1} \cdot k(p)$
k(5)=20,k(2)=3,k(3)=8
Se $p > 5$:
k(p) divide (p-1) se p == +-1 mod 5
k(p) divide 2*(p+1) se p==+-2 mod 5
Tem que achar para cada p^k, testando todos os divisores com algum método de achar Fib rapido modulo.

7.28 Diofantinas

$$ax + by = c$$

divide tudo pelo $gcd(a,b)$; se $c \ mod \ gcd(a,b) \ neq 0$, não tem solução. Se não:
$a'x + b'y = c'$ tem solução dada pelo algoritmo de euclides
A resposta do problema original é:
$(X_g \cdot c/g, Y_g \cdot c/g)$, com X_g e Y_g achados com euclides.

OBS: passa $abs(a)$ e $abs(b)$ no euclides e depois inverter o sinal se era negativo.
Toda solução é na forma
$x = x0 + k * b'$
$y = y0 - k * a'$

7.29 Choromatic polynomial of a cycle - pintar ciclo com K cores

numero de modos de pintar um ciclo de tamanho n com K cores:

$$(k-1)^n + (-1)^n \cdot (k-1)$$

pode achar com expo tb se precisar...

7.30 Mersenne

: Primos de Mersenne $2^n - 1$
Lista de Ns que resultam nos primeiros 41 primos de Mersenne:
2; 3; 5; 7; 13; 17; 19; 31; 61; 89; 107; 127; 521; 607; 1.279; 2.203; 2.281; 3.217; 4.253; 4.423; 9.689; 9.941; 11.213; 19.937; 21.701; 23.209; 44.497; 86.243; 110.503; 132.049; 216.091; 756.839; 859.433; 1.257.787; 1.398.269; 2.976.221; 3.021.377; 6.972.593; 13.466.917; 20.996.011; 24.036.583;

8 Probability

8.1 Moment Generating Functions

Let X be a random variable. Define $M_X(t) = E[e^{tX}]$.

when X is Discrete

$$M_X(t) = \sum_{i=1}^{\infty} e^{tx_i} p_i$$

when X is Continuous

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Then we have:

$$M_X(0) = 0 \qquad M_X'(0) = E[x] \qquad \frac{d^k M_X(0)}{dt^k} = E[x^k]$$

8.2 Distributions

8.2.1 Binomial

- X is the number of successes in a sequence of n independent experiments.

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad E[X] = np \qquad Var(X) = np(1-p)$$

8.2.2 Geometric

- X is the number of failures in a sequence of independent experiment of Bernoulli until the first success.

$$P(X=k) = (1-p)^k p \qquad E[X] = \frac{1}{p} \qquad Var(X) = \frac{1-p}{p^2}$$

9 Graphs

9.1 Planar Graphs

1. If G has k connected components, then $n - m + f = k + 1$.
2. $m \leq 3n - 6$. If G has no triangles, $m \leq 2n - 4$.
3. The minimum degree is less or equal 5. And can be 6 colored in $\mathcal{O}(n+m)$.

9.2 Counting Minimum Spanning Trees - $\tau(G)$

- **Cayley's Formula**: $\tau(K_n) = n^{n-2}$.
- **Complete Bipartite Graphs**: $\tau(K_{p,q}) = p^{q-1}q^{p-1}$.
- **Kirchhoff's Theorem**: More generally, if we define the Laplacian matrix $\mathbf{L}(G) = \mathbf{D} - \mathbf{A}$, where $\mathbf{D}$ is the diagonal matrix with entries equal to the degree of vertices and $\mathbf{A}$ is the adjacency matrix. For $\mathbf{L}(G)_{ab}$ equal to $\mathbf{L}(G)$ without row a and column b, we have $\tau(G) = \det \mathbf{L}(G)_{ab}$, for any row a and column b.

9.3 Prüfer's Sequence

The Prüfer sequence is a bijection between labeled trees with n vertices and sequences with $n-2$ numbers from 1 to n.
To get the sequence from the tree:

- While there are more than 2 vertices, remove the leaf with smallest label and append it's neighbour to the end of the sequence.

To get the tree from the sequence:

- The degree of each vertex is 1 more than the number of occurrences of that vertex in the sequence. Compute the degree d, then do the following: for every value x in the sequence (in order), find the vertex with smallest label y such that $d(y) = 1$ and add an edge between x and y, and also decrease their degrees by 1. At the end of this procedure, there will be two vertices left with degree 1; add an edge between them.

9.4 Erdős-Gallai Theorem

A sequence of non-negative integers $d_1 \geq ... \geq d_n$ can be represented as the degree sequence of a finite simple graph on n vertices if and only if $d_1 + ... + d_n$ is even and

$$\sum_{i=1}^{k} d_i \;\leq\; k(k-1) + \sum_{i=k+1}^{n} min(d_i, k)$$

holds for every k in $1 \leq k \leq n$.

9.5 Maximum Matching in Complete Multipartite graphs

The size of the maximum matching in a complete multipartite graph with n vertices and k vertices in its largest partition is (reference):

$$|M| = \min\left(\left\lfloor \frac{n}{2} \right\rfloor, n-k\right)$$

9.6 Dilworth's Theorem

9.6.1 Node-disjoint Path Cover

The node disjoint path cover in a `DAG` is equal to $|V| - |M|$, where M is the maximum matching in the bipartite flow network.

9.6.2 General Path Cover

The general path cover in a `DAG` is equal to $|V| - |M|$, where M is the maximum matching in the bipartite flow network of the transitive closure graph.

9.6.3 Dilworth's Theorem

The size of the maximum **antichain** in a `DAG`, that is, the maximum size of a set S of vertices such that no vertex in S can reach another vertiex in S, is equal to size of the minimum **general** path cover.

9.7 Sum of Subtrees of a Tree

For a rooted tree T with n vertices, let $sz(v)$ be the size of the subtree of v. Then the following holds:

$$\sum_{v \in V} \left[\mathrm{sz}(v) + \sum_{u \text{ child of } v} \mathrm{sz}(u)(\mathrm{sz}(v) - \mathrm{sz}(u)) \right] = n^2$$

10 Dynamic Programming Optimizations

10.1 Divide and Conquer

DP to compute the minimum cost to divide an array into k subarrays; the cost of a solution is equal to the sum of the costs of each subarray. The cost of a subarray $A[i..j]$ is $c(i, j)$.

$$dp[i][k] = \min_{j \geq i} \left(dp[j+1][k-1] + c(i, j) \right)$$

- Define A to be the functions satisfying

  $$dp[i][k] = dp[A(i,k)+1][k-1] + c(i, A(i,k)).$$

  If A also satisfy $A(i,k) \leq A(i+1,k)$, then the dp is optimizable.

- Another sufficient condition is, for every $a < b < c < d$:

  $$c(a,d) + c(b,c) \geq c(a,c) + c(b,d)$$