AwesomeMath Admission Test Cover Sheet

Your name (please print) _____ _____

Last   *First*

Admission Test   A/B/C (circle one)

Contact Information   _____ (phone number)

(please print)   _____ (email address)

Number of pages (not including this cover sheet)   _____

Admission Test C

- Don't be discouraged if you can't solve many of the questions: the test isn't made to be easy. We want to see the solutions you do come up with no matter how many you solve.

- You should include all significant steps in your reasoning and computation. We are interested in your ability to present your work, so unsupported answers will receive much less credit than well-reasoned progress towards a solution without a correct answer.

- In this document, you will find a cover sheet and an answer sheet. Print out each one, and make several copies of the blank answer sheet. Fill out the top of each answer sheet as you go, and then fill out the cover sheet when you're finished. **Start each problem on a new answer sheet.**

- **All the work you present must be your own**.

- **Don't be intimidated!** Some of the problems involve complex mathematical ideas, but all can be solved using only elementary techniques, admittedly combined in clever ways.

- **Be patient and persistent!** Learning comes more from struggling with problems than from solving them. Problem-solving becomes easier with experience. Success is not a function of cleverness alone.

- **You must postmark or submit your solutions by e-mail by Monday, May 15, 2006**. Make sure that the cover sheet is the first page of your submission, and that it is completely filled out.

  Solutions are to be mailed to the following address:

  Dr. Titu Andreescu
  School of Natural Sciences and Mathematics
  The University of Texas at Dallas
  2601 N. Floyd Road, FN 33
  Richardson, TX 75083

  If you e-mail your solutions, please send them to

  titu.andreescu@utdallas.edu

  E-mailed solutions may be written and scanned or typed in TeX. They should be sent as an attachment in either .doc or .pdf format. If you write and scan your solutions, insert the scans into a .doc or .pdf file, and send just the one file.

  Please go the next page for the problems.

Admission Test C

1. Find the least 20-digit perfect square.

2. In how many ways can a motorist pay the exact amount of a 75-cent toll by using some combination of nickels, dimes, and quarters?

3. Find all integers n such that $n - 260$ and $n + 260$ are both perfect cubes.

4. A class of 67 students takes a six-question exam. For the first question, a student receives 1 point for answering correctly and loses 1 point for answering incorrectly or not answering at all. For the second question, the student earns 2 points for a correct answer, and loses 2 points for an incorrect answer or no answer, and so on.
   a) Find the least possible positive difference between any two students' scores.
   b) Prove that some four students have the same final score.
   c) Prove that at least two students answered the same questions correctly and the same questions incorrectly.

5. Let a, b, and c be real numbers such that $abc = 1$. Prove that at least one of the numbers $2a - \frac{1}{b}$, $2b - \frac{1}{c}$, $2c - \frac{1}{a}$ does not exceed 1.

6. Find all triples (x, y, z) of positive integers satisfying the system of equations

$$\begin{aligned} xy + x - z &= 1 \\ xy + y + z &= 2006 \end{aligned}$$

7. Consider an isosceles triangle ABC such that $AB = AC$ and $\angle A = 20°$. Let M be the foot of the altitude from C and let N be the point on side AC such that $CN = \frac{1}{2}BC$. Find the measure of angle $\angle AMN$.

8. Prove that every nonnegative integer can be expressed in the form $a^2 + b^2 - c^2$, where a, b, and c are positive integers with $a < b < c$.

9. Let $x_n = \sqrt{n + \sqrt{n^2 - 1}}$ for all $n \geq 1$. Express $\dfrac{1}{x_1} + \dfrac{1}{x_2} + \cdots + \dfrac{1}{x_{49}}$ as $a + b\sqrt{2}$ for some integers a and b.

10. The result of the addition

```
    AWESOME
     SUMMER
+   PROGRAM
```

is a seven-digit number whose digits are all equal. Different letters stand for different decimal digits, and the solution is unique. Find the six-digit number that SUMMER represents.

Suggestions for Writing Proofs
 (By Tiankai Liu)

- All proofs should be written neatly and coherently in paragraphs of standard American English. Mathematical symbols like $\equiv$ and $\leq$ should be used only in equations, not as verbs or prepositions in a sentence. Do not write things like "all of the $\triangle$'s angles are $\leq 90°$"—this should be "$\angle A, \angle B, \angle C \leq 90°$" or "all of the triangle's angles are at most $90°$." Avoid the symbols $\wedge \vee \because \therefore \forall \exists$; instead, write out "and," "or," "because," "therefore," "for all," "there exists." Similarly, do not use $\Longrightarrow$ or $\Rightarrow\!\!\!\Leftarrow$ except as part of a sequence of equations.

- Write a statement using words rather than symbols unless this would be unnecessarily awkward. Do not invent more notation than is necessary to explain your solution.

- Write true statements. Do not write something that is only partially true, and then say how to fix it later. If you assume something in one of your statements, say clearly what you are assuming. Define all terms you make up. If you use figures, graphs, tables, etc., explain thoroughly what they represent.

- Use the following formats for common proof patterns:

  - *Proof by induction:* (1) state the claim, (2) check the base case, (3) prove the induction step, and (4) conclude with the words "induction is complete."
  - *Proof by contradiction:* (1) state the claim, (2) state that you are assuming the opposite, (3) derive a contradiction (and say why it is a contradiction), and (4) conclude that the claim follows.
  - *Proof by case analysis:* (1) state the claim, (2) state the various cases, (3) say why they exhaust all possibilities, (4) analyze the cases one by one, and (5) conclude that the claim follows.

- Draw accurate diagrams with compass and straightedge for geometry problems. This is for your own good as well as the graders'.

AwesomeMath Admission Test Answer Sheet

Your name (please print)

Admission Test A/B/C (circle one)   Problem Number ___   Page ___ of ___

Write neatly! Write all work inside the box. Do NOT write on the back of the page.