A New Proof for Napoleon's Theorem

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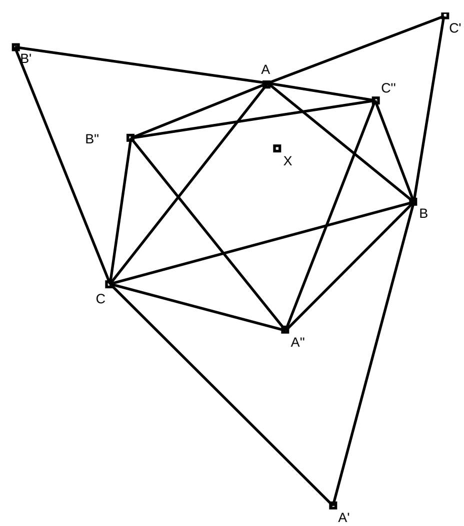
Introduction:

Napoleon's theorem is one of the most elegant statements of triangle geometry. In short, if one constructs equilateral triangles on the sides of any triangle, the centers of those 3 equilateral triangles form yet another equilateral triangle. There are many known methods of proving this statement that use relatively sophisticated geometric ideas (Ref. 1). This article offers a new proof that uses relatively intuitive geometric ideas like dissection and side equality. Perhaps Napoleon's alledged proof was more along these lines.

Diagram 1



Problem 1:

Given any triangle, ABC, construct equilateral triangles ABC', BCA', and CAB' (externally) with centers C'', A'', and B'', respectively. Prove that $A''B''C''$ is equilateral.

Proof:

Note: I will use the standard notation that $AB = c$, $BC = a$, $CA = b$, $\angle BAC = \angle A$, $\angle ABC = \angle B$, and $\angle BCA = \angle C$. We will consider when $\angle ABC$, $\angle BCA$, and $\angle CAB$ are less than $\frac{2\pi}{3}$. The other case creates a slight issue because $\triangle C''AB''$ lies outside $\triangle ABC$ instead of inside. Regardless, that case is sufficiently analogous.

I plan to show that the reflections of A, B, and C into $B''C''$, $C''A''$, and $A''B''$ are the same point. I define this point to be X.

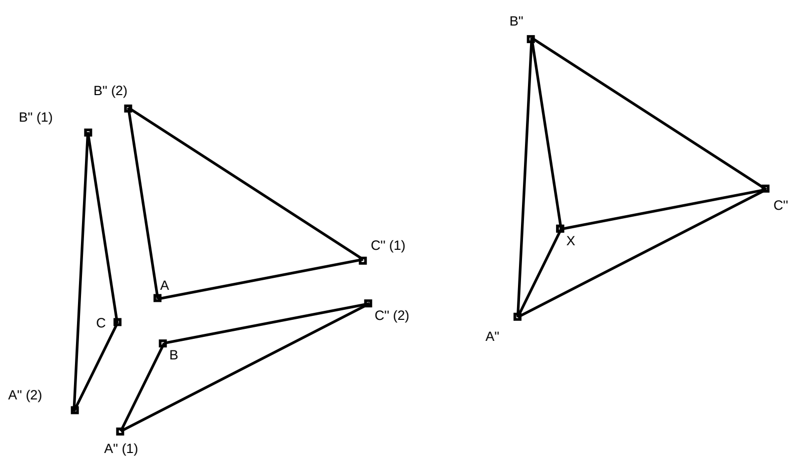
Consider triangle $AB''C''$. We have that

$$\angle B''AC'' = \angle B''AC + \angle CAB + \angle BAC'' = \frac{\pi}{6} + \angle C + \frac{\pi}{6} = \angle C + \frac{\pi}{3}$$

Using the 30-60-90 triangles, it follows that:

$$AB'' = B''C = \frac{b}{\sqrt{3}},\ BC'' = C''A = \frac{c}{\sqrt{3}},\ CA'' = A''B = \frac{a}{\sqrt{3}} \qquad (1)$$

Diagram 2



Now here is the crux move. On the left, we have the 3 triangles from Diagram 1 (just reflected). [see diagram 2 for appropriate labeling] On the right, we have another triangle that we are trying to construct; this triangle results from pushing the triangles on the left together. Push the upper two triangles together [so A and C coincide to point X and $B''(1)$

and $B''(2)$ coincide to point B'']. By (1), the sides AC'' and BC'' match up. Furthermore,

$$\angle A''XC'' = 2\pi - (\angle A''CB'' + \angle C''AB'') = 2\pi - \angle C - \frac{\pi}{3} - \angle B - \frac{pi}{3} = \angle A + \frac{\pi}{3} = \angle A''BC''$$

Now we show the lower triangle fits in. $A''X$ (right)$=A''(2)C$(left) (by construction) and $A''(2)C = A''(1)B$ by (1), so $A''X = A''(1)B$ and $C''X = C''(2)B$. It follows by SAS congruency that $\triangle XA''C'' \cong BA''(1)C''(2)$. So essentially, we took the triangles on the left, and pushed them together to form the triangle on the right. It directly follows that $A''B''$ on the right is congruent to $A''B''$ in the original diagram. Using SSS congruence, $A''B''C''$ is the same in diagram 1 and diagram 2, right. Now the important piece of information that we now have is that $\triangle XB''C'' \cong \triangle AB''C''$ and symmetrical statements. We have that $\angle XC''B'' = \angle AC''B''$ and $\angle XC''A'' = \angle BC''A''$. So:

$$\begin{aligned}
\angle B''C''A'' &= \tfrac{1}{2}(\angle XC''B'' + \angle XC''B'' + \angle XC''A'' + \angle XC''A'') \\
&= \tfrac{1}{2}(\angle AC''B'' + \angle XC''B'' + \angle XC''A'' + \angle BC''A'') \\
&= \tfrac{1}{2}\angle AC''B \\
&= \tfrac{\pi}{6}
\end{aligned}$$

By symmetry, all the angles of $A''B''C''$ are equal to $\frac{\pi}{3}$. It follows that $A''B''C''$ is equilateral; hence, the theorem is proved.

Problem 2:
X is the first fermat point of $\triangle ABC$. For $\angle A \leq \angle B \leq \angle C \leq \frac{2\pi}{3}$
Proof:

We note that $AB'' = XB'' = CB'' = B'B''$ hence X is the circumcenter of $AXCB'$. It follows that $\angle AXC = \pi - \angle AB'C = \frac{2\pi}{3}$. By symmetry, $\angle BXC = \angle CXA = \frac{2\pi}{3}$. This is a well-known definition for the first fermat point.

References: 1. "'Napoleon's Theorem"'
Available: http://www.mathpages.com/home/kmath270/kmath270.htm