Junior problems

J1. Solve in real numbers the system of equations:

$$\begin{cases} x^4 + 2y^3 - x = -\frac{1}{4} + 3\sqrt{3} \\ y^4 + 2x^3 - y = -\frac{1}{4} - 3\sqrt{3}. \end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas

J2. Show that for any nonzero integer a one can find a nonzero integer b such that the equation

$$ax^2 - (a^2 + b)x + b = 0$$

has integral roots.

Proposed by Laurentiu Panaitopol, University of Bucharest, Romania

J3. Consider the sequence

$$a_n = \sqrt{1 + \left(1 + \frac{1}{n}\right)^2} + \sqrt{1 + \left(1 - \frac{1}{n}\right)^2}, \quad n \geq 1.$$

Prove that $\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots + \dfrac{1}{a_{20}}$ is an integer.

Proposed by Titu Andreescu, University of Texas at Dallas

J4. The unit cells of an $n \times n$ array are colored white or black in such a way that any 2×2 square contains either exactly one white cell or exactly one black cell. Find all possible values of n for wich such an array does not contain identical columns.

Proposed by Marius Ghergu, Romania

J5. Let x, y, z be positive real numbers such that $xyz = 1$. Show that the following inequality holds:

$$\frac{1}{(x+1)^2 + y^2 + 1} + \frac{1}{(y+1)^2 + z^2 + 1} + \frac{1}{(z+1)^2 + x^2 + 1} \leq \frac{1}{2}.$$

Proposed by Dr. Cristinel Mortici, Valahia University, Romania

J6. Let $ABCD$ be a convex quadrilateral such that the sides BC and CD have equal lengths and $2\angle A + \angle C = 180°$. Let M be the midpoint of the line segment BD. Prove that $\angle MAD = \angle BAC$

Proposed by Dinu Serbanescu, "Sf. Sava" National College, Romania

Senior problems

S1. Prove that the triangle ABC is right-angled if and only if

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}-\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{1}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas

S2. Circles with radii r_1, r_2, r_3 are externally tangent to each other. Two other circles, with radii R and r, are tangent to all previous circles. Prove that:

$$Rr \geq \frac{r_1 r_2 r_3}{r_1+r_2+r_3}.$$

Proposed by Ivan Borsenco, University of Texas at Dallas

S3. Let n be a positive integer such that $\gcd(n,6)=1$, and let k, l be positive integers. The entries of a $k \times l$ table are all positive numbers. One can simultaneously change the signs of any n consecutive horizontal, vertical, or diagonal entries. Prove that one can eventually make all entries negative numbers if and only if n divides k or l.

Proposed by Iurie Boreico, Harvard University

S4. Let $A_1A_2\ldots A_n$ be a convex polygon. Prove that the sum of distances from an interior point to its sides does not depend on the position of the point if and only if the following equalit holds:

$$\frac{1}{A_1A_2}\overrightarrow{A_1A_2}+\frac{1}{A_2A_3}\overrightarrow{A_2A_3}+\cdots+\frac{1}{A_nA_1}\overrightarrow{A_nA_1}=0.$$

Proposed by Bogdan Enescu,"B.P. Hasdeu" National College, Romania

S5. Let a and b be two real numbers such that a^p-b^p is a positive integer for each prime number p. Prove that a and b are integer numbers.

Proposed by Nairi Sedrakian, Yierevan, Armenia

S6. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

i. If $a \leq b \leq 1 \leq c$, then

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1};$$

ii. If $a \leq 1 \leq b \leq c$, then

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\leq\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1};$$

Proposed by Marian Tetiva, Romania

Undergraduate problems

U1. Evaluate

$$\int_0^1 \sqrt[3]{2x^3 - 3x^2 - x + 1}\,dx.$$

Proposed by Titu Andreescu, University of Texas at Dallas

U2. Solve in real numbers the equation

$$6^x + 1 = 8^x - 27^{x-1}.$$

Proposed by Titu Andreescu, University of Texas at Dallas

U3. Let $f : R \longrightarrow R$ be an indefinitely differentiable function and let a, b, c be distinct positive numbers such that:

$$g(x) = f(ax) + f(bx) + f(cx)$$

is a polynomial function. Prove that f is a polynomial function as well.

Proposed by Mihai Baluna and Mihai Piticari, Romania

U4. Let $\mathcal{A}$ be a commutative subalgebra of $M_n(\mathbb{R})$ such that $\det(X) \geq 0$, for all $X \in \mathcal{A}$.

Let k be a positive integer and $(X_{ij})_{1 \leq i,j \leq k}$ matricies in A. Prove that

$$\det \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix} \geq 0$$

Proposed by Gabriel Dospinescu, "Louis le Grand" College, France

U5. Let $s \geq 2$ be a positive integer. Prove that there is no rational function $R(x)$ such that

$$\frac{1}{2^s} + \frac{1}{3^s} + \ldots + \frac{1}{n^s} = R\left(n\right),$$

for all positive integers $n \geq 2$.

Proposed by Mihai Piticari, Dorin Andrica, Romania

U6. Find all positive integers a, b, c and all integers x, y, z satisfying the conditions:

$$\begin{cases} ax^2 + by^2 + cz^2 = abc + 2xyz - 1 \\ ab + bc + ca \geq x^2 + y^2 + z^2. \end{cases}$$

Proposed by Gabriel Dospinescu, "Louis le Grand" College, France

Olympiad problems

O1. A circle centered at O is tangent to all sides of the convex quadrilateral $ABCD$. The rays BA and CD intersect at K, the rays AD and BC intersect at L. The points X, Y are considered on the line segments OA, OC, respectively. Prove that $\angle XKY = \frac{1}{2}\angle AKC$ if and only if $\angle XLY = \frac{1}{2}\angle ALC$.

Proposed by Pavlo Pylyavskyy, MIT

O2. Find all positive integers n such that the set

$$A = \{1, 3, 5, \dots, 2n-1\}$$

can be partitioned into 12 subsets, the sum of elements in each subset being the same.

Proposed by Marian Tetiva, Romania

O3. Prove that there are infinitely many prime numbers p with the following property: in the main period of the decimal representation of $\frac{1}{p}$, the number of 1's plus the number of 3's equals the number of 6's plus the number of 8's.

Proposed by Adrian Zahariuc, Romania

O4. Let AB be a diameter of the circle Γ and let C be a point on the circle, different from A and B. Denote by D the projection of C on AB and let ω be a circle tangent to AD, CD, and Γ, touching Γ at X. Prove that the angle bisectors of $\angle AXB$ and $\angle ACD$ meet on AB.

Proposed by Liubomir Chiriac, Princeton

O5. Let p be a prime number of the form $4k+1$ such that $2^p \equiv 2 (\mod p^2)$. Prove that there exists a prime number q, divisor of $2^p - 1$, such that $2^q > (6p)^p$.

Proposed by Gabriel Dospinescu, "Louis le Grand" College, France

O6. Let x, y, z be nonnegative real numbers. Prove the inequality

$$x^4(y+z) + y^4(z+x) + z^4(x+y) \leq \frac{1}{12}(x+y+z)^5.$$

Proposed by Vasile Cartoaje, Romania