On the extension of Carnot's Theorem

Tran Quang Hung

Abstract. In this article we present an extension of Carnot's identity and its applications.

1 Main Result

First of all we to introduce the concept of signed distance and signed area.

Consider the distances of two points to a line d. They have the same sign if the points are in the same half plane of d and have opposite signs if they are in different half planes of d. For an arbitrary triangle ABC, the signed distances with respect to the support lines of the sides are positive in the half plane that contains the triangle.

The same principle applies for signed area. Let XY be a segment and let P and Q be two arbitrary points. Then K_{PXY} and K_{QXY} are areas of the same sign if P and Q are in the same half plane of d, and have opposite signs if P and Q are in different half planes of d. The relation between signed area and signed distance is

$$\frac{d(P, XY)}{d(Q, XY)} = \frac{K_{PXY}}{K_{QXY}},$$

where $d(P, XY)$ and $d(Q, XY)$ are signed distances of P and Q with respect to XY.

Theorem 1. Let P be a point in the plane of triangle ABC with barycentric coordinates (u, v, w). Denote by d_a, d_b, d_c the distances from P to the sides BC, CA, AB, respectively. Then

$$xd_a + yd_b + zd_c = \frac{xubc + yvca + zwab}{2R(u + v + w)},$$

where R is the circumradius of triangle ABC.

Proof. Let h_a, h_b, h_c be the altitudes of triangle ABC. We have

$$\frac{d_a}{h_a} = \frac{K_{PBC}}{K_{ABC}} = \frac{u}{u + v + w}.$$

Note that $h_a = \dfrac{bc}{2R}$. Thus

$$d_a = \frac{ubc}{2R(u + v + w)},$$

and, analogously,

$$d_b = \frac{vca}{2R(u+v+w)}, \quad d_c = \frac{wab}{2R(u+v+w)}$$

Multiplying by x, y, z, respectively, and adding up we get

$$xd_a + yd_b + zd_c = \frac{xubc + yvca + zwab}{2R(u+v+w)},$$

as desired.

Corollary 1. Setting $x = y = z = 1$ we get

$$d_a + d_b + d_c = \frac{ubc + vca + wab}{2R(u+v+w)}.$$

Corollary 2. If P does not lie on the support lines of the sides of the triangle ABC, then

$$\frac{1}{d_a} + \frac{1}{d_b} + \frac{1}{d_c} = 2R(u+v+w)\left(\frac{1}{ubc} + \frac{1}{vca} + \frac{1}{wab}\right)$$

Proof. We just sum up the expressions for $\frac{1}{d_a}, \frac{1}{d_b}, \frac{1}{d_c}$.

Now we prove Carnot's Theorem.

Theorem 2. Let O be the circumcenter of triangle ABC and let d_a, d_b, d_c be the signed distances from O to sides BC, CA, AB, respectively. Then

$$d_a + d_b + d_c = R + r,$$

where R and r are the circumradius and the inradius, respectively.

Proof. The circumcenter of triangle ABC has barycentric coordinates $(\sin 2\alpha, \sin 2\beta, \sin 2\gamma)$. Using Theorem 1 for $P \equiv O$ we get

$$d_a + d_b + d_c = \frac{bc \sin 2\alpha + ca \sin 2\beta + ab \sin 2\gamma}{2R(\sin 2\alpha + \sin 2\beta + \sin 2\gamma)}.$$

Recall the well-known identities $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4 \sin\alpha \sin\beta \sin\gamma$ and

$$\cos\alpha + \cos\beta + \cos\gamma = 1 + \frac{r}{R}.$$

We have

$$\frac{bc \sin 2\alpha + ca \sin 2\beta + ab \sin 2\gamma}{2R(\sin 2\alpha + \sin 2\beta + \sin 2\gamma)} = \frac{8R^2 \sin\alpha \sin\beta \sin\gamma(\cos\alpha + \cos\beta + \cos\gamma)}{8R \sin\alpha \sin\beta \sin\gamma} = R+r.$$

Thus

$$d_a + d_b + d_c = R + r,$$

and we are done.

2 Applications

Problem 1. Let h_a, h_b, h_c be the length of the alitudes in triangle ABC. Prove that

$$\frac{2(5R-r)r}{R} \leq h_a + h_b + h_c \leq \frac{2(R+r)^2}{R}.$$

Solution. Let G be the centroid of triangle ABC. Then G has barycentric coordinates $(1,1,1)$ and the distances from G to the sides are $\frac{h_a}{3}, \frac{h_b}{3}, \frac{h_c}{3}$. Using Theorem 1 for $P \equiv G$ we have

$$\frac{h_a}{3} + \frac{h_b}{3} + \frac{h_c}{3} = \frac{ab+bc+ca}{6R}.$$

Recall that $ab+bc+ca = s^2 + (4R+r)r$ and $16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$. Thus

$$h_a + h_b + h_c = \frac{s^2 + (4R+r)r}{2R} \geq \frac{2(5R-r)r}{R}$$

and

$$h_a + h_b + h_c = \frac{s^2 + (4R+r)r}{2R} \leq \frac{2(R+r)^2}{R}.$$

Problem 2. Let t_a, t_b, t_c be the signed distances from the orthocenter H to the support lines of the sides of the triangle ABC. Prove that

$$\frac{t_a}{\cos\beta} + \frac{t_b}{\cos\gamma} + \frac{t_c}{\cos\alpha} = \frac{t_a}{\cos\gamma} + \frac{t_b}{\cos\alpha} + \frac{t_c}{\cos\beta},$$

where α, β, γ are the angles of triangle ABC.

Solution. We know that H has barycentric coordinates $(\tan\alpha, \tan\beta, \tan\gamma)$. Then using Theorem 1 for $P \equiv H$ with $x = \frac{1}{\cos\beta}, y = \frac{1}{\cos\gamma}, z = \frac{1}{\cos\alpha}$ and $x = \frac{1}{\cos\gamma}, y = \frac{1}{\cos\alpha}, z = \frac{1}{\cos\beta}$ we get

$$t_a + t_b + t_c = \frac{\frac{\tan\alpha}{\cos\beta}bc + \frac{\tan\beta}{\cos\gamma}ca + \frac{\tan\gamma}{\cos\alpha}ab}{2R(\tan\alpha + \tan\beta + \tan\gamma)}.$$

Recall the identity $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$ and from the Law of Sines we have

$$\begin{aligned}\frac{t_a}{\cos\beta} + \frac{t_b}{\cos\gamma} + \frac{t_c}{\cos\alpha} &= \frac{4R^2(\tan\alpha\tan\beta\sin\gamma + \tan\beta\tan\gamma\sin\alpha + \tan\gamma\tan\alpha\sin\beta)}{2R\tan\alpha\tan\beta\tan\gamma} \\ &= 2R(\cos\alpha + \cos\beta + \cos\gamma) \\ &= 2(R+r).\end{aligned}$$

Similarly,

$$\frac{t_a}{\cos\gamma} + \frac{t_b}{\cos\alpha} + \frac{t_c}{\cos\beta} = 2(R+r),$$

and we are done.

References

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[3] N. Dergiades, Signed Distances and the Erdos-Mordell Inequality, *Forum Geometricorum*, (2004), 66-68.