Junior problems

J115. Find all positive integers n for which $\sqrt{\sqrt{n}+\sqrt{n+2009}}$ is an integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania

First solution by Brian Bradie, Christopher Newport University, USA

Suppose $\sqrt{\sqrt{n}+\sqrt{n+2009}} = m$ for some integer m. Then

$$\sqrt{n} = \frac{m^4 - 2009}{2m^2}.$$

For n to be an integer, m must be odd with

$$m^2 \big| 2009.$$

Because $2009 = 7^2 \cdot 41$, it follows that $m = 7$ and $n = 16$. Thus $n = 16$ is the only integer n for which $\sqrt{\sqrt{n}+\sqrt{n+2009}}$ is an integer.

Second solution by Magkos Athanasios, Kozani, Greece

Suppose there is a positive integer k such that

$$\sqrt{n}+\sqrt{n+2009} = k^2.$$

Taking the conjugate we obtain

$$\sqrt{n+2002} - \sqrt{n} = \frac{2009}{k^2}.$$

Subtracting the above equations we get

$$2\sqrt{n} = k^2 - \frac{2009}{k^2}.$$

From this equation we infer that $\sqrt{n}$ must be rational and hence integer, since n is integer. Thus, k^2 is a divisor of 2009. However, the only square numbers that divide 2009 are 1^2 and 7^2. It is clear that $k \neq 1$. If $k = 7$ we get $n = 16$ and it is easily verified that $\sqrt{\sqrt{16}+\sqrt{16+2009}} = 7$ is an integer.

Third solution by Dimitar Trenevski, Yahya Kemal College, Macedonia

For $\sqrt{\sqrt{n}+\sqrt{n+2009}}$ to be an integer $n, n+2009, \sqrt{n}+\sqrt{n+2009}$ must be perfect squares. So let $n=a^2, n+2009=b^2$ with a, b positive and $a+b$ a perfect square. We have $b^2-a^2=2009=7^2\cdot 41$ giving the systems

$$\begin{aligned} b-a&=41\\ b+a&=49 \end{aligned}$$

or

$$\begin{aligned} b-a&=2009\\ b+a&=1 \end{aligned}$$

First system gives the solution $n=16$. Second system implies $a=-1004$, contradiction. Therefore, $n=16$ is the only solution.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Elmir Sukanovic, Bosnia and Herzegovina; Ercole Suppa, "Liceo Scientifico Statale E.Einstein, Teramo", Italy; John T. Robinson, Yorktown Heights, NY, USA; Michel Bataille, France;

J116. A bug is situated in one of the vertices of a cube. Each day it travels to another vertex of a cube. How many six day journeys end at the original vertex?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Ercole Suppa, "Liceo Scientifico Statale E.Einstein, Teramo", Italy

Denote the vertices of the cube with the numbers $1, 2, \ldots, 8$. Consider the graph $G = (V, E)$ obtained from the vertices and the edges of the cube and let $A = (a_{ij})$ be its adjacency matrix:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Writing the (i, j)-entry $a_{ij}^{(2)}$ of the matrix A^2 in the form

$$a_{ij}^{(2)} = \sum_{k=1}^{n} a_{ik} a_{kj}$$

we see that $a_{ij}^{(2)}$ counts the number of intermediate nodes that are connected to both i and j. In other words, $a_{ij}^{(2)}$ gives the number of different walks of length two from i to j. More in general it is not difficult to show that the (i, j)-entry of the matrix A^n is equal to the number of walks of length n that originate at vertex i and terminate at vertex j.
Therefore the number of six day journeys is given by the $(1, 1)$ entry of the matrix A^6:

$$A^6 = \begin{pmatrix} 183 & 0 & 0 & 182 & 0 & 182 & 182 & 0 \\ 0 & 183 & 182 & 0 & 182 & 0 & 0 & 182 \\ 0 & 182 & 183 & 0 & 182 & 0 & 0 & 182 \\ 182 & 0 & 0 & 183 & 0 & 182 & 182 & 0 \\ 0 & 182 & 182 & 0 & 183 & 0 & 0 & 182 \\ 182 & 0 & 0 & 182 & 0 & 183 & 182 & 0 \\ 182 & 0 & 0 & 182 & 0 & 182 & 183 & 0 \\ 0 & 182 & 182 & 0 & 182 & 0 & 0 & 183 \end{pmatrix}$$

i.e. there are 183 six day journeys which start from the vertex 1 and end in vertex 1.

Second solution by John T. Robinson, Yorktown Heights, NY, USA

Suppose the vertices of the cube are placed at lattice points (x, y, z) where each of x, y, z is 0 or 1; then one day's movement of the bug corresponds to flipping a single coordinate (that is 0 goes to 1 or 1 goes to 0 for exactly one of x, y, or z). Starting at (0,0,0), suppose x is flipped a times, y is flipped b times, and z is flipped c times. Since it is a six days journey we must have $a + b + c = 6$. Finally, each of a, b, and c must be even in order to return to (0,0,0). With these constraints, the possibilities for (a, b, c) are: (6,0,0), (0,6,0), or (0,0,6); (4,2,0), (4,0,2), (2,4,0), (2,0,4), (0,4,2), or (0,2,4); or (2,2,2). Each flipping operation can take place at six possible times, so it follows that the total number of distinct six day journeys starting and ending at (0,0,0) is

$$\binom{6}{2\ 2\ 2} + 6\binom{6}{4\ 2\ 0} + 3\binom{6}{6\ 0\ 0} = 90 + 6 \cdot 15 + 3 = 183\,.$$

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Elmir Sukanovic, Bosnia and Herzegovina.

J117. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{2a^2+b^2+3}+\frac{b}{2b^2+c^2+3}+\frac{c}{2c^2+a^2+3}\leq\frac{1}{2}.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

Solution by Ivanov Andrei, Moldova

We have

$$2a^2+b^2+3=2(a^2+1)+(b^2+1)\geq 4a+2b$$

By replacing in our inequality we get

$$\sum_{cyclic}\frac{a}{2a^2+b^2+3}\leq\frac{1}{2}\Longleftrightarrow\sum_{cyclic}\frac{a}{2a+b}\leq 1$$

Which is equivalent to

$$ab^2+bc^2+ca^2\geq 3abc,$$

clearly true by the $AM-GM$ inequality.

Also solved by Arkady Alt, San Jose, California, USA; Brian Bradie, Newport News, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Elmir Sukanovic, Bosnia and Herzegovina; Ercole Suppa, "Liceo Scientifico Statale E.Einstein, Teramo", Italy; John T. Robinson, Yorktown Heights, NY, USA; Magkos Athanasios, Kozani, Greece; Manh Dung Nguyen, Hanoi University of Science, Vietnam; Michel Bataille, France;Oleh Faynshteyn, Leipzig, Germany; Pedrag Gruevski, Yahya Kemal College, Macedonia; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy.

J118. Prove that for each integer $n \geq 3$ there are n pairwise distinct positive integers such that each of them divides the sum of the remaining $n-1$.

Proposed by H. A. ShahAli, Tehran, Iran

First solution by Andrea Cameli, Universit degli Studi di L'Aquila, Italy

By induction:

- The statement holds for $n = 3$, take $(3, 6, 9)$ as an example;
- let $(x_1, x_2, \ldots, x_n)$ be n pairwise distinct positive integers such that the statement holds, and let s be their sum: then the statement also holds for $(x_1, x_2, \ldots, x_n, s)$, in fact

$$x_i \mid \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} x_j + s \right) = \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} x_j + x_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} x_j \right) \quad , \quad \forall i \in \{1, \ldots, n\}$$

  for inductive hypothesis; the check for s instead is trivial.

Then the statement holds for all $n \geq 3$.

Second solution by Michel Bataille, France

For each $n \geq 3$, the following pairwise distinct positive integers $x_1 = 1, x_2 = 2$ and $x_k = 3 \cdot 2^{k-3}$, $\quad (k = 3, 4, \ldots, n)$ answer the question. Note that $x_{k+1} = 2x_k$ for $3 \leq k \leq n-1$. Also, an easy induction shows that $x_k = x_1 + x_2 + \cdots + x_{k-1}$ for $k = 3, 4, \ldots, n$. Now, let $S_k = \sum_{j=1, j \neq k}^{n} x_j, \quad k = 1, 2, \ldots, n$. Certainly $x_1 = 1$ divides S_1, x_n divides $S_n = x_1 + x_2 + \cdots + x_{n-1} = x_n$ and

$$S_2 = x_1 + (x_3 + \cdots x_n) = 1 + 3 + 3 \cdot 2 + \cdots + 3 \cdot 2^{n-3}$$

is divisible by $x_2 = 2$. Finally, for $3 \leq k \leq n-1$,

$$S_k = \sum_{j=1}^{k-1} x_j + \sum_{j=k+1}^{n} x_j = x_k + 2x_k + 2^2 x_k + \cdots 2^{n-k} x_k$$

is divisible by x_k. This completes the proof.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Elmir Sukanovic, Bosnia and Herzegovina; John T. Robinson, Yorktown Heights, NY, USA.

J119. Let α, β, γ be angles of a triangle. Prove that

$$\cos^3\frac{\alpha}{2}\sin\frac{\beta-\gamma}{2}+\cos^3\frac{\beta}{2}\sin\frac{\gamma-\alpha}{2}+\cos^3\frac{\gamma}{2}\sin\frac{\alpha-\beta}{2}=0.$$

Proposed by Oleh Faynstein, Leipzig, Germany

First solution by Michel Bataille, France

Let a, b, c be the sides opposite α, β, γ, respectively. Then, from the law of sines

$$\frac{a-b}{c}=\frac{\sin\alpha-\sin\beta}{\sin\gamma}=\frac{2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}}{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}=\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\gamma}{2}}$$

(since $\frac{\alpha+\beta}{2}=\frac{\pi}{2}-\frac{\gamma}{2}$).
Also, using the law of cosines and denoting by s the semiperimeter $\frac{a+b+c}{2}$, we have

$$2\cos^2\frac{\gamma}{2}=1+\cos\gamma=1+\frac{a^2+b^2-c^2}{2ab}=\frac{(a+b)^2-c^2}{2ab}=\frac{2s(s-c)}{ab}.$$

It follows that

$$\sin\frac{\alpha-\beta}{2}\cos^3\frac{\gamma}{2}=\frac{a-b}{c}\cos^4\frac{\gamma}{2}=\frac{s^2}{abc}\left[(s-c)^2\left(\frac{1}{b}-\frac{1}{a}\right)\right].$$

Thus the proposed equality holds if and only if

$$(s-c)^2\left(\frac{1}{b}-\frac{1}{a}\right)+(s-b)^2\left(\frac{1}{a}-\frac{1}{c}\right)+(s-a)^2\left(\frac{1}{c}-\frac{1}{b}\right)=0.$$

Now, the left-hand side rewrites as

$$\frac{(s-b)^2-(s-c)^2}{a}+\frac{(s-c)^2-(s-a)^2}{b}+\frac{(s-a)^2-(s-b)^2}{c}=\frac{a(c-b)}{a}+\frac{b(a-c)}{b}+\frac{c(b-a)}{c}$$

which obviously vanishes. This completes the proof.

Second solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

We have

$$\begin{aligned}\cos^3\tfrac{\alpha}{2}\sin\tfrac{\beta-\gamma}{2} &= \cos^2\tfrac{\alpha}{2}\cdot\left(\cos\tfrac{\alpha}{2}\sin\tfrac{\beta-\gamma}{2}\right)\\ &= \tfrac{1+\cos\alpha}{2}\cdot\left(\sin\tfrac{\beta+\gamma}{2}\sin\tfrac{\beta-\gamma}{2}\right) \qquad \text{since} \quad \tfrac{\alpha}{2}=90^\circ-\tfrac{\beta+\gamma}{2}\\ &= \tfrac{1+\cos\alpha}{2}\cdot\tfrac{1}{2}\left(\cos\gamma-\cos\beta\right)\end{aligned}$$

and cyclically.

Therefore

$$\begin{aligned}\sum_{\text{cyclic}} \cos^3 \tfrac{\alpha}{2} \sin \tfrac{\beta-\gamma}{2} &= \tfrac{1}{4} \sum_{\text{cyclic}} (1+\cos\alpha)(\cos\gamma - \cos\beta) \\ &= 0\end{aligned}$$

as desired.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Ercole Suppa, "Liceo Scientifico Statale E.Einstein, Teramo", Italy; Ivanov Andrei, Moldova; Magkos Athanasios, Kozani, Greece; Manh Dung Nguyen, Hanoi University of Science, Vietnam; Vicente Vicario Garcia, Huelva, Spain.

J120. Let a, b, c be positive real numbers. Prove that

$$\frac{ab}{3a+4b+2c}+\frac{bc}{3b+4c+2a}+\frac{ca}{3c+4a+2b} \le \frac{a+b+c}{9}.$$

Proposed by Baleanu Andrei Razvan, George Cosbuc Lyceum, Romania

First solution by Gheorghe Pupazan, Moldova

The inequality is equivalent to

$$\frac{9ab}{3a+4b+2c}+\frac{9bc}{3b+4c+2a}+\frac{9ca}{3c+4a+2b} \le a+b+c$$

From Cauchy-Schwartz inequality it is known that

$$\frac{9ab}{3a+4b+2c} \le \frac{ab}{a+b+c}+\frac{ab}{a+b+c}+\frac{ab}{a+2b}$$

Adding the other 2 similar inequalities we get that it suffices to show that

$$\frac{2(ab+bc+ca)}{a+b+c}+\frac{ab}{a+2b}+\frac{bc}{b+2c}+\frac{ca}{c+2a} \le a+b+c$$

Because from the AM-GM inequality $(a+b+c)^2 \ge 3(ab+bc+ca)$, we conclude that

$$\frac{2(ab+bc+ca)}{a+b+c} \le \frac{2(a+b+c)}{3}$$

So it is enough to prove that

$$\frac{ab}{a+2b}+\frac{bc}{b+2c}+\frac{ca}{c+2a} \le \frac{a+b+c}{3}$$

But from AM-GM inequality

$$(a+2b)(b+2a) \ge 9ab \iff \frac{ab}{a+2b} \le \frac{b+2a}{9}$$

Adding the other two similar inequalities we get that

$$\frac{ab}{a+2b}+\frac{bc}{b+2c}+\frac{ca}{c+2a} \le \frac{b+2a}{9}+\frac{c+2b}{9}+\frac{a+2c}{9}=\frac{a+b+c}{3}$$

The proof is complete.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Elmir Sukanovic, Bosnia and Herzegovina; Ercole Suppa, "Liceo Scientifico Statale E.Einstein, Teramo", Italy; Manh Dung Nguyen, HUS, Vietnam; Oleh Faynshteyn, Leipzig, Germany; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy.

Senior problems

S115. Prove that for each positive integer n, 2009^n can be written as a sum of six nonzero perfect squares.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania

Solution by Ivanov Andrei, Moldova

Let $a_n \geq b_n \geq c_n \geq d_n \geq e_n \geq f_n \in \mathbb{N}$, such that

$$a_n^2 + b_n^2 + c_n^2 + d_n^2 + e_n^2 + f_n^2 = 2009^n = (7^2)^n \cdot 41^n. \tag{1}$$

If we denote $a_n = x_n \cdot 7$ and analogues, then (1) is equivalent to

$$x_n^2 + y_n^2 + z_n^2 + u_n^2 + v_n^2 + w_n^2 = 41^n.$$

We have

$$(x_1, y_1, z_1, u_1, v_1, w_1) \in \{(3,3,3,3,2,1); (4,5,2,2,2,2); (5,3,2,1,1,1); (6,1,1,1,1,1)\}$$

$$(x_2, y_2, z_2, u_2, v_2, w_2) \in \{(31,12,12,12,12,12); ...\}$$

For $n \geq 3$ we choose $x_n = 41 \cdot x_{n-2}$ and analogues. Then

$$x_n^2 + y_n^2 + z_n^2 + u_n^2 + v_n^2 + w_n^2$$

$$= 41^2(x_{n-2}^2 + y_{n-2}^2 + z_{n-2}^2 + u_{n-2}^2 + v_{n-2}^2 + w_{n-2}^2) = 41^2 \cdot 41^{n-2} = 41^n.$$

We proved that for every n we can find numbers $a_n, b_n, c_n, d_n, e_n, f_n$.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Elmir Sukanovic, Bosnia and Herzegovina; John T. Robinson, Yorktown Heights, NY, USA.

S116. Points P and Q lie on segment BC with P between B and Q. Suppose that BP, PQ, and QC form a geometric progression in some order. Prove that there is a point A in the plane such that AP and AQ are the trisectors of angle BAC if and only if PQ is less then BP and CQ.

Proposed by Daniel Campos Salas, Costa Rica

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Denote $\rho_B = \frac{PB}{PQ}$ and $\rho_C = \frac{QC}{QP}$. If AP, AQ trisect $\angle BAC$, then AP bisects $\angle BAQ$ and AQ bisects $\angle PAC$, or $\frac{AB}{AQ} = \frac{PB}{PQ} = \rho_B$ and $\frac{AC}{AP} = \frac{QC}{QP} = \rho_C$. Clearly, A is then defined by the intersection of two Apollonius' circles, Γ_P with diameter PP' and described by $\frac{XB}{XQ} = \rho_B$ for each $X \in \Gamma_P$, and Γ_Q with diameter QQ' and described by $\frac{YC}{YP} = \rho_C$ for each $Y \in \Gamma_Q$.

Now, if $\rho_B < 1$, then P' is on ray PB, and $1 - \rho_B = \frac{QB}{QP'}$, yielding $QP' = \frac{1+\rho_B}{1-\rho_B}PQ$ and $PP' = \frac{2\rho_B}{1-\rho_B}PQ$. Similarly, if $\rho_B > 1$, then P' is on ray PQ, and $\rho_B - 1 = \frac{BQ}{QP'}$, yielding $QP' = \frac{\rho_B+1}{\rho_B-1}PQ$ and $PP' = \frac{2\rho_B}{\rho_B-1}PQ$. Analogously, $QQ' = \frac{2\rho_C}{\rho_C-1}PQ$ if $\rho_C > 1$ with Q' on ray QC, and $QQ' = \frac{2\rho_C}{1-\rho_C}PQ$ if $\rho_C < 1$ with Q' on ray QP. Now, if $\rho_B, \rho_C < 1$, then diameters PP' and QQ' are disjoint, circles Γ_P and Γ_Q are exterior, they do not intersect, and A cannot exist. If wlog $\rho_B > 1 > \rho_C$, note that $\rho_C = \frac{1}{\rho_B}$, or $QQ' = \frac{2}{\rho_B-1}PQ < QP'$, and diameter QQ' is entirely contained inside diameter PP', or circle Γ_Q is interior to circle Γ_P, and they do not intersect either. Finally, if $\rho_B, \rho_C > 1$, then PQ is the common segment of diameters PP' and QQ', and circles Γ_P and Γ_Q intersect at some point A such that AP, AQ trisect $\angle BAC$. Note that if $BP = PQ = QC$, Γ_P and Γ_Q become the perpendiculars to BC through P and Q respectively, and they do not intersect, or A cannot exist either. The conclusion follows.

S117. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}+\frac{3abc}{2(ab+bc+ca)^2}\geq\frac{5}{a+b+c}.$$

Proposed by Shamil Asgarli, Burnaby, Canada

Solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

Clearing the denominators, the inequality is equivalent to

$$\sum_{\text{sym}} 2a^5b^2+a^4b^2c+2a^5bc\geq\sum_{\text{sym}} 2a^4b^3+2a^3b^3c+a^3b^2c^2$$

and the conclusion follows by Muirhead's theorem since $[5,2,0]\succ[4,3,0]$, $[4,2,1]\succ[3,3,1]$, $[5,1,1]\succ[3,2,2]$. The underlying AGM's are:

$[5,2,0]\succ[4,3,0]$. $(a^5b^2+a^5b^2+a^2b^5)/3\geq a^4b^3$, $(a^2b^5+a^2b^5+a^5b^2)/3\geq a^3b^4$ and cyclic.

$[4,2,1]\succ[3,3,1]$. $a^4b^2c+b^4a^2c\geq 2a^3b^3c$ and cyclic

$[5,1,1]\succ[3,2,2]$. $(2a^5bc+b^5ac+c^5ab)/4\geq a^3b^2c^2$ and cyclic. The proof is complete.

Also solved by Arkady Alt, San Jose, California, USA; Brian Bradie, Newport News, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Elmir Sukanovic, Bosnia and Herzegovina; Ercole Suppa, "Liceo Scientifico Statale E.Einstein, Teramo", Italy; Ivanov Andrei, Moldova; Manh Dung Nguyen, Hanoi University of Science, Vietnam; Michel Bataille, France;Oleh Faynshteyn, Leipzig, Germany.

S118. An equilateral triangle is divided into n^2 congruent equilateral triangles. Let V be the set of all vertices and let E be the set of all edges of these triangles. Find all n for which we can paint all edges black or white such that for every vertex the number of incident edges of black color is equal to the number if incident edges of the white color.

Proposed by Oles Dobosevych, Lviv National University, Ukraine

First solution by John T. Robinson, Yorktown Heights, NY, USA

Since there are n^2 triangles and each has three edges, with each interior edge shared by two triangles and each of the $3n$ exterior edges part of only one triangle, there are $\dfrac{3n^2+3n}{2}$ edges (we add $3n$ in order to “double count” all edges including the exterior edges and then divide by 2). Next, given a vertex v, let $B(v)$ and $W(v)$ be the number of black and white edges incident to v respectively. If for all $v \in V$ we have $B(v) = W(v)$, then $\sum_{v\in V} B(v) = \sum_{v\in V} W(v)$. Since each edge is incident to two vertices, these two sums are twice number of black and white edges respectively. If follows that if for all $v \in V$ $B(v) = W(v)$, then the number of black edges is equal to the number of white edges. Therefore $\dfrac{3n^2+3n}{2} = \dfrac{3n(n+1)}{2}$ must be even. By considering the cases where n is even or odd, it is easily seen that n must be of the form $4i+3$ or $4i+4$ (that is congruent to 3 or 0 modulo 4) for $i \geq 0$. The first few such n are 3, 4, 7, 8, 11, 12, etc.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Out of the $3n^2$ edges of the n^2 triangles, all are shared by exactly two triangles except for the $3n$ edges that form the edges of the large initial triangle, or the total number of edges is $\frac{3n(n+1)}{2}$. Note that the total number of edges must be even, since adding for all vertices the number of black and white edges that concur in it, each edge is counted exactly twice, and the total number of white edges and of black edges must be equal. Therefore, the total number of edges must be even, and 4 must divide either n or $n+1$. This condition is necessary, let us also show that it is sufficient by showing, for any n such that either n or $n+1$ is divisible by 4, how the edges may be painted for the condition of the problem to be fulfilled.

For $n = 3$ and $n = 4$, figure 1 shows how the coloring may be assigned (for clarity white edges have been represented by thin red segments).

For $n = 4m$ with $m \geq 2$, divide the equilateral triangle into m^2 congruent equilateral triangles with sidelength $4u$, where u is the sidelength of one of the n^2 triangles. Color each one of these m^2 triangles like the triangle for $n = 4$. Note that the colors ”fit”, ie, when the sides of two triangles for $n = 4$ are joined, after inverting one of them with respect to one of its sides, the 4 segments of alternating color into which each side of each triangle is divided, are color-matched when both sides are joined. It is easy to see that, when the coloring is performed in this way, each vertex of the n^2 triangles will have as many black as white edges incident, because each vertex on the side of the $n = 4$ triangle has exactly one black edge and one white edge toward the interior or the triangle, and exactly one black edge and one white edge along the side of the triangle. For $n = 7$, the triangle may be colored. Note that if we eliminate the first row of triangles on each side, we obtain a triangle with $n = 4$ and exactly the same coloring. Note now that, for $n = 4m+3$ with $m \geq 2$, eliminating the first row of triangles on each side, we would obtain a triangle with $n = 4m$. The eliminated rows of triangles may be then added back, and colored as the in the case $n = 7$, ie, with a zigzag pattern of black edges all along the first row, and with the same color configuration in each vertex of the original triangle. The fact that colors alternate along the sides of the initial triangle in the $n = 4m$ case guarantees that the task is possible. Therefore, the proposed task is possible if and only if n or $n + 1$ are multiples of 4.

S119. Consider a point P inside a triangle ABC. Let AA_1, BB_1, CC_1 be cevians through P. The midpoint M of BC different from A_1, and T is the intersection of AA_1 and B_1C_1. Prove that if the circumcircle of triangle BTC is tangent to the line B_1C_1, then $\angle BTM = \angle A_1TC$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Note first that $\angle BTM = \angle A_1TC$ iff $\angle CTM = \angle A_1TB$ since $\angle BTM + \angle CTM = \angle A_1TB + \angle A_1TC = \angle BTC$, or we may exchange B and C at will without altering the problem. Call S the point where the tangent to the circumcircle of BTC meets BC. By Pappus' harmonic theorem, $\frac{CA_1}{A_1B} = \frac{CS}{SB}$, or since $SA_1 = SB + BA_1 = SC - CA_1$ wlog, then $SA_1 \cdot SM = SA_1 \frac{SB+SC}{2} = SB \cdot SC$. But the power of S with respect to the circumcircle of BTC is $ST^2 = SB \cdot SC$, or the power of S with respect to the circumcircle of A_1TM is $SA_1 \cdot SM = ST^2$, and ST is the tangent at T to the circumcircle of A_1TM. Therefore,

$$\angle BTM = \angle STM - \angle STB = \angle SA_1T - \angle TCB = \pi - \angle CA_1T - \angle TCA_1 = \angle A_1TC.$$

The conclusion follows.

Also solved by Ercole Suppa, "Liceo Scientifico Statale E.Einstein, Teramo", Italy; Ivanov Andrei.

S120. Let P be a point interior to a triangle ABC and let d_a, d_b, d_c be the distances from P to the sides of the triangle. Prove that

$$\frac{4 \cdot AP \cdot BP \cdot CP}{(d_a+d_b)(d_b+d_c)(d_c+d_a)} \geq \frac{AP}{d_b+d_c} + \frac{BP}{d_a+d_c} + \frac{CP}{d_a+d_b}.$$

Proposed by Oles Dobosevych, Lviv National University, Ukraine

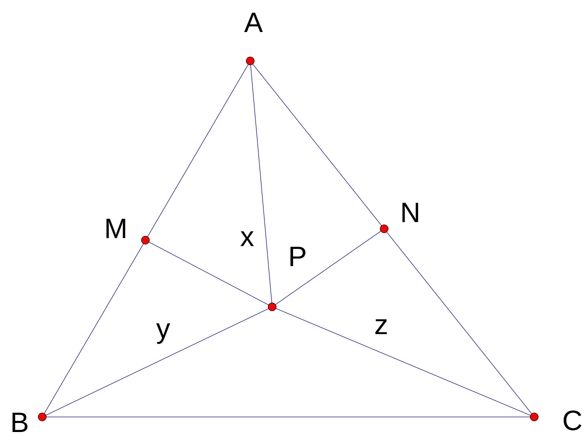
Solution by Magkos Athanasios, Kozani, Greece

For convenience set

$$AP = x, BP = y, CP = z, d_a = p, d_b = q, d_c = r.$$

First we prove the following

Lemma: On the sides AB, AC of triangle ABC take the points M, N respectively.



Then, the following inequality holds:

$$xMN \geq rAM + qAN.$$

We argue as follows: Using brackets to denote area, we have

$$[AMPN] = \frac{1}{2}AP \cdot MN \cdot \sin\phi \leq \frac{1}{2}x \cdot MN,$$

where ϕ is the angle between AP and MN. Since

$$[AMPN] = [AMP] + [ANP] = \frac{1}{2}rAM + \frac{1}{2}qAN,$$

we are done.

Consider now the points M, N such that $AM = AN = k$. An easy computation yields $MN = 2k\sin\frac{A}{2}$ and the above lemma implies $2x\sin\frac{A}{2} \geq q + r$. In the same way we have

$$r + p \leq 2y\sin\frac{B}{2}, p + q \leq 2z\sin\frac{C}{2}.$$

We can now prove the initial inequality. Rewrite it as

$$4xyz \geq x(p+q)(p+r) + y(q+r)(q+p) + z(r+p)(r+q).$$

By the above inequalities it suffices to prove that

$$4xyz \geq 4xyz\sum\sin\frac{B}{2}\sin\frac{C}{2} + 8xyz\prod\sin\frac{A}{2},$$

or

$$1 \geq \sum\sin\frac{B}{2}\sin\frac{C}{2} + 2\prod\sin\frac{A}{2}.$$

Using the well known identity

$$\sum\sin^2\frac{A}{2} + 2\prod\sin\frac{A}{2} = 1,$$

it reduces to

$$\sum sin^2\frac{A}{2} \geq \sum sin\frac{B}{2}\sin\frac{C}{2},$$

which clearly holds.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain.

Undergraduate problems

U115. Let $a_n = 2 - \frac{1}{n^2+\sqrt{n^4+\frac{1}{4}}}, n = 1, 2, \ldots$ Prove that $\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_{119}}$ is an integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Americo Tavares, Queluz, Portugal

First note that

$$\sqrt{a_n} = \sqrt{2 - \frac{1}{n^2 + \sqrt{n^4 + \frac{1}{4}}}} = \sqrt{2 + 4n^2 - \sqrt{16n^4 + 4}}.$$

Now we will use the following algebraic identity. Let A and B be two positive real numbers. Assume that $A^2 - B \geq 0$ and $C = \sqrt{A^2 - B}$. Then

$$\sqrt{A - \sqrt{B}} = \sqrt{\frac{A+C}{2}} - \sqrt{\frac{A-C}{2}}.$$

If $A = 2 + 4n^2$ and $B = 16n^4 + 4$, then $A^2 - B = 16n^2 \geq 0$ and $C = 4n$. Thus

$$\sqrt{2 + 4n^2 - \sqrt{16n^4 + 4}} = \sqrt{2n^2 + 2n + 1} - \sqrt{2n^2 - 2n + 1}.$$

Set $A_n = \sqrt{2n^2 + 2n + 1}$. Then $A_{n-1} = \sqrt{2n^2 - 2n + 1}$and

$$\sqrt{a_n} = A_n - A_{n-1}.$$

Therefore we need only to evaluate the telescoping sum

$$\sum_{n=1}^{119} (A_n - A_{n-1}) = A_{119} - A_0.$$

Since $A_{119} = \sqrt{2\,(119) + 2\,(119)^2 + 1} = 169$ and $A_0 = \sqrt{2\,(0) + 2\,(0)^2 + 1} = 1$, the given sum is equal to the integer 168.

Also solved by Arkady Alt, San Jose, California, USA; Brian Bradie, Newport News, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Ercole Suppa, "Liceo Scientifico Statale E.Einstein, Teramo", Italy; John T. Robinson, Yorktown Heights, NY, USA; Magkos Athanasios, Kozani, Greece; Oleh Faynshteyn, Leipzig, Germany; Vicente Vicario Garcia, Huelva, Spain.

U116. Let G be a K_4 complete graph without an edge. Find the number of closed walks of length n in G.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Label the vertices of the graph $\{1, 2, 3, 4\}$ such that 1 and 4 are not connected, and call $N_{i,j}(n)$ the number of paths of length n that begin at vertex i and end at vertex j, where $n \geq 0$ and $i, j \in \{1, 2, 3, 4\}$. By symmetry, we may exchange vertices 1 and 4, or exchange vertices 2 and 3, or walk backwards from j to i for a given path from i to j, leading to $N_{1,2}(n) = N_{1,3}(n) = N_{4,2}(n) = N_{4,3}(n) = N_{2,1}(n) = N_{2,4}(n) = N_{3,1}(n) = N_{3,4}(n)$, $N_{2,3}(n) = N_{3,2}(n)$, $N_{1,4}(n) = N_{4,1}(n)$, $N_{1,1}(n) = N_{4,4}(n)$ and $N_{2,2}(n) = N_{3,3}(n)$, which we will denote respectively by a_n, b_n, c_n, x_n and y_n. Note finally that, since 1 and 4 may be reached only from 2 or 3, then $c_n = x_n$ for $n \geq 1$, but $c_0 = 0$ but $x_0 = 1$. Clearly, the problem statement asks for $2x_n + 2y_n$.

Vertex 1 can be reached in one step only from 2 or 3, or a closed path from 1 to 1 of length n may be written only as a path from 1 to 2 of length $n-1$ and a step from 2 back to 1, or from 1 to 3 with length $n-1$ and a step from 3 back to 1, ie, $x_n = 2a_{n-1}$. Since 2 may be reached in one step from any other vertex except for itself, one can find the number of paths with length n from 1 to 2 as the sum of number of paths with lengths $n-1$ from 1 to 1, 3 and 4, ie, $a_n = a_{n-1} + 2x_{n-1}$. Note also that the number of paths from 2 to 1 with length n equals the sum of the number of paths from 2 to itself and to 3, followed by one step to 1, ie, $a_n = y_{n-1} + b_{n-1}$. Therefore, $a_n + 2x_n = b_n + y_n$. Similarly, we conclude that $y_n = b_{n-1} + 2a_{n-1} = 3a_{n-1} + 2x_{n-1} - y_{n-1}$. Writing the recursive relations in matrix form,

$$\begin{pmatrix} a_n \\ x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ x_{n-1} \\ y_{n-1} \end{pmatrix},$$

we find the characteristic polynomial of this matrix to be $(\rho + 1)(\rho^2 - \rho - 4)$, with roots $-1, \frac{1 \pm \sqrt{17}}{4}$. The initial conditions are clearly $a_1 = 1$, $x_1 = y_1 = 0$, from where we may deduce that $a_2 = 1$, $x_2 = 2$, $y_2 = 3$ and $x_3 = 2$, $y_3 = 4$; the number of closed paths with length n, $z_n = 2(x_n + y_n)$, satisfies $z_1 = 0$, $z_2 = 10$, $z_3 = 12$, and given the roots of the characteristic polynomial, in general

$$z_n = A(-1)^n + B\left(\frac{1 + \sqrt{17}}{2}\right)^n + C\left(\frac{1 - \sqrt{17}}{2}\right)^n.$$

Substitution of $n = 1, 2, 3$ yields a linear system of 3 equations with 3 unknowns A, B, C, easily solved to produce $A = B = C = 1$, or the number of closed paths with length n is

$$(-1)^n + \left(\frac{1+\sqrt{17}}{2}\right)^n + \left(\frac{1-\sqrt{17}}{2}\right)^n.$$

Since we have used $c_n = x_n$ in order to simplify the problem, note that this result is not valid for $n = 0$; clearly $z_0 = 4$, the closed paths corresponding to each vertex of the graph.

Also solved by Ercole Suppa, "Liceo Scientifico Statale E.Einstein, Teramo", Italy; John T. Robinson, Yorktown Heights, NY, USA.

U117. Let n be an integer greater than 1 and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1 + x_2 + \cdots + x_n = n$. Prove that

$$\sum_{k=1}^{n} \frac{x_k}{n^2 - n + 1 - nx_k + (n-1)x_k^2} \leq \frac{1}{n-1}$$

and find all equality cases.

Proposed by Iurie Boreico, Harvard University, USA

Solution by Arkady Alt, San Jose, California, USA

Lemma.

For any $a, b, c > 0$ such that $c \geq b,\ b^2 < 4ac,$ and any $t \geq -1$ holds inequality

$$\frac{c^2(t+1)}{at^2 + bt + c} \leq (c-b)\,t + c$$

with equality condition $t = 0$.

Proof.

Note that $at^2 + bt + c > 0$ for any t and in particular $a + c - b > 0,$ since $b^2 < 4ac$ and $a > 0$.Then we have

$$\frac{c^2(t+1)}{at^2 + bt + c} \leq (c-b)\,t + c \iff 0 \leq ((c-b)\,t + c)\left(at^2 + bt + c\right) - c^2(t+1) \iff$$

$t^2\left(a(c-b)\,t + bc + ca - b^2\right) \geq 0$ where latter inequality holds because

$a(c-b)\,t + bc + ca - b^2 = a(c-b)(t+1) + bc + ab - b^2 =$

$a(c-b)(t+1) + b(a + c - b)$ and $c - b \geq 0, t + 1 \geq 0, a + c - b > 0.$

Using the Lemma for $a = n-1, b = n-2, c = n^2 - n$ for which obviously holds $a, b, c > 0$ and $c \geq b,\ b^2 < 4ac$ gives us the following inequality

$$(1) \qquad \frac{t+1}{(n-1)\,t^2 + (n-2)\,t + n^2 - n} \leq \frac{\left(n^2 - 2n + 2\right)t + n(n-1)}{n^2(n-1)^2}.$$

Substitution $t = x - 1$ in (1) yields

$$(2) \qquad \frac{x}{n^2 - n + 1 - nx + (n-1)\,x^2} \leq \frac{\left(n^2 - 2n + 2\right)x + n - 2}{n^2(n-1)^2}$$

which becomes equality if and only if $x = 1$.

Using (2) we obtain

$$\sum_{k=1}^{n} \frac{x_k}{n^2 - n + 1 - nx_k + (n-1)\, x_k^2} \le \sum_{k=1}^{n} \frac{\left(n^2 - 2n + 2\right) x_k + n - 2}{n^2 \left(n-1\right)^2} =$$

$$\frac{\left(n^2 - 2n + 2\right) \sum_{k=1}^{n} x_k + (n-2)\, n}{n^2 \left(n-1\right)^2} = \frac{\left(n^2 - 2n + 2\right) n + (n-2)\, n}{n^2 \left(n-1\right)^2} = \frac{1}{n-1}.$$

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain

U118. Prove that there are infinitely many positive integers n such that $\phi(\sigma(n)) \mid n$, where ϕ denotes Euler's totient function and σ is the sum of divisors function.

Proposed by Cezar Lupu, University of Bucharest, Romania

Solution by Cezar Lupu, University of Bucharest, Romania

We will start by proving the following lemma.

Lemma. For any increasing sequence of positive integers $(x_n)_{n\geq 1}$ such that $\lim\limits_{n\to\infty}\dfrac{x_n}{n}=0$, the sequence $\left(\frac{n}{x_n}\right)_{n\geq 1}$ contains all positive integers. In particular, x_n divides n for infinitely many n.

Proof. We consider a positive integer m and the set

$$X=\left\{n\geq 1,\, \frac{a_{mn}}{mn}\geq \frac{1}{mn}\right\}.$$

Clearly, our set contains 1 and it is bounded, since $\lim\limits_{n\to\infty}\dfrac{x_{mn}}{mn}=0$. Thus, we have a maximal element, namely l. If $\frac{x_{ml}}{ml}=\frac{1}{ml}$, then it follows that m is in the sequence $\left(\frac{n}{x_n}\right)_{n\geq 1}$. Contrary, we would have that $x_{m(k+1)}\geq x_{mk}\geq k+1$ and we have that $k+1$ is also in the set in contradiction with the maximality of k. Now, we get back to our problem. Let $x_n=\phi(\sigma(n))$. We need to show that

$$\lim_{n\to\infty}\frac{\phi(\sigma(n))}{n}=0.$$

But, this was prove in [1] by Alaoglu and Erdos. In fact, the showed that for every $\epsilon>0$ we have $\phi(\sigma(n))<\epsilon n$, except a set of density 0. The proof follows from the following two observations. The first one, for a given prime p, the set of all n such that $\sigma(n)\equiv 0 \pmod p$ is of density 1. The second observation is that except for ϵx integers n less than x we have that $\sigma(n)<c(e)n$. As remarked in [1], much more can be show, i.e. except for a set of density 0, we have

$$\epsilon^{\gamma}\phi(\sigma(n))\log\log\log n\approx \sigma(n)$$

which is equivalent to

$$\frac{\phi(\sigma(n))}{n}\approx\frac{\sigma(n)}{\epsilon^{\gamma}n\log\log\log n}.$$

[1] Leon Alaoglu, Paul Erdos. *A conjecture in elementary number theory.* Bull. of AMS 50(1944), 881882.

U119. Let t be a real number greater than -1. Evaluate

$$\int_0^1 \int_0^1 x^t y^t \left\{\frac{x}{y}\right\} \left\{\frac{y}{x}\right\} dxdy,$$

where $\{a\} = a - \lfloor a \rfloor$ denotes the fractional part of a.

Proposed by Ovidiu Furdui, Cluj, Romania

Solution by Brian Bradie, Newport News, USA

Using symmetry and the facts

$$\left\{\frac{y}{x}\right\} = \frac{y}{x} \text{ when } y < x \quad \text{and} \quad \left\{\frac{x}{y}\right\} = \frac{x}{y} - n \text{ when } \frac{x}{n+1} < y < \frac{x}{n},$$

it follows that

$$\begin{aligned}
\int_0^1 \int_0^1 x^t y^t \left\{\frac{x}{y}\right\} \left\{\frac{y}{x}\right\} dxdy &= 2\int_0^1 \int_0^x x^t y^t \left\{\frac{x}{y}\right\} \left\{\frac{y}{x}\right\} dydx \\
&= 2\sum_{n=1}^{\infty} \int_0^1 \int_{x/(n+1)}^{x/n} x^{t-1} y^{t+1} \left(\frac{x}{y} - n\right) dydx \\
&= \int_0^1 \int_0^1 x^t y^t \, dydx \\
&\qquad -2\sum_{n=1}^{\infty} n \int_0^1 \int_{x/(n+1)}^{x/n} x^{t-1} y^{t+1} \, dydx \quad (1)
\end{aligned}$$

Now,

$$\int_0^1 \int_0^1 x^t y^t \, dydx = \frac{1}{(t+1)^2}. \qquad (2)$$

Moreover,

$$n\int_0^1 \int_{x/(n+1)}^{x/n} x^{t-1} y^{t+1} \, dydx = \frac{n}{2(t+1)(t+2)} \left(\frac{1}{n^{t+2}} - \frac{1}{(n+1)^{t+2}}\right),$$

so

$$2\sum_{n=1}^{\infty} n \int_0^1 \int_{x/(n+1)}^{x/n} x^{t-1} y^{t+1} \, dydx = \frac{1}{(t+1)(t+2)} \zeta(t+2). \qquad (3)$$

Combining (1), (2) and (3), we find

$$\int_0^1 \int_0^1 x^t y^t \left\{\frac{x}{y}\right\} \left\{\frac{y}{x}\right\} dxdy = \frac{1}{(t+1)^2} - \frac{1}{(t+1)(t+2)} \zeta(t+2).$$

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy.

U120. Let $x_n = \frac{1}{n+a_1} + \frac{1}{n+a_2} + \cdots + \frac{1}{n+a_k}$ and $y_n = \frac{\phi(n)}{n}$, where $a_1, a_2, \ldots, a_k$ are distinct positive integers less than n and relatively prime with n and ϕ is Euler's totient function. Prove that for all real numbers $a < 1$

$$\lim_{n\to\infty} n^a (x_n - y_n \log 2) = 0.$$

Is this also true for $a = 1$?

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Let $d \neq 1, n$ be any divisor of n, and let $b_1, b_2, \ldots$ be the positive integers less than n that are divisible by d. Call $q = \frac{n}{d}$, $\sigma_d = \frac{1}{n+b_1} + \frac{1}{n+b_2} + \ldots$, and $\Delta_d = \sigma_d - \frac{\log 2}{d}$. Then,

$$\Delta_d = \frac{1}{n+b_1} + \frac{1}{n+b_2} + \cdots - \frac{\log 2}{d} = \frac{1}{d}\left(\frac{1}{q+1} + \frac{1}{q+2} + \cdots + \frac{1}{2q-1} - \log 2\right) =$$

$$= \frac{H_{2q-1} - H_q - \log 2}{d} = \frac{1}{d}\log\frac{2q-1}{2q} + \frac{1}{d}O\left(\frac{1}{q}\right) = O\left(\frac{1}{n}\right),$$

since $\log\frac{2q-1}{2q} = -\frac{1}{2q} + O\left(\frac{1}{q}\right)$, and where Landau notation has been used. Call now D_k the set of divisors of n that consist on the product of exactly k distinct primes (e.g., if $n = 72$, then $D_1 = \{2, 3\}$, $D_2 = \{6\}$, and $D_k = \emptyset$ for $k \geq 3$). Clearly,

$$\frac{1}{n+a_1} + \frac{1}{n+a_2} + \cdots + \frac{1}{n+a_k} = H_{2n-1} - H_n - \sum_{d\in D_1}\sigma_d + \sum_{d\in D_2}\sigma_d - \cdots =$$

$$= \log 2\left(1 - \sum_{d\in D_1}\frac{1}{d} + \sum_{d\in D_2}\frac{1}{d} - \ldots\right) + O\left(\frac{1}{n}\right) + \sum_{d\in\bigcup D_k}(-1)^k\Delta_d =$$

$$= \log 2\frac{\phi(n)}{n} + O\left(\frac{1}{n}\right) + \sum_{d\in\bigcup D_k}(-1)^k\Delta_d,$$

and calling finally $a = 1 - \delta$, where $\delta > 0$ when $a < 1$, we conclude that

$$n^a\left(x_n - \log 2 y_n\right) = O\left(\frac{1}{n^\delta}\right) + n^a\sum_{d\in\bigcup D_k}(-1)^k\Delta_d.$$

The limit of the first term in the right hand side is clearly 0 when $n \to \infty$. Since when $n \to \infty$, the number of distinct prime divisors of n is much smaller than $\log n$, then the second term in the right hand side has an upper bound less than $\frac{o(\ln n)}{n^\delta}$, which clearly tends to 0 when $n \to \infty$ for any $\delta > 0$, i.e., for any $a < 1$. The conclusion follows.

For $a = 1$, the proposed result is not true. Consider any prime p. Clearly,

$$x_p = H_{2p-1} - H_p = \log \frac{2p-1}{p} + \frac{1}{2(2p-1)} - \frac{1}{2p} + O\left(\frac{1}{n^2}\right), \qquad y_p = \frac{p-1}{p},$$

$$p\left(x_p - y_p \log 2\right) = \log 2 + \frac{p}{2(2p-1)} - \frac{1}{2} + p \log \frac{2p-1}{2p} + O\left(\frac{1}{p}\right).$$

Clearly, when $p \to \infty$, the right hand side tends to $\log 2 - \frac{3}{4}$, which is nonzero. The limit when $n \to \infty$ cannot therefore be 0, since it is not 0 at least for all the infinite primes.

Olympiad problems

O115. Numbers 1 through 24 are written on a board. At any time, numbers a, b, c may be replaced by

$$\frac{2b+2c-a}{3},\ \frac{2c+2a-b}{3},\ \frac{2a+2b-c}{3}.$$

Can a number greater than 70 appear on the board?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ercole Suppa, "Liceo Scientifico Statale E.Einstein, Teramo", Italy

Let $S_n = \left\{x_1^{(n)}, \dots, x_{24}^{(n)}\right\}$ be the numbers written on the board after n iterations. For any three real numbers a, b, c we have

$$\left(\frac{2b+2c-a}{3}\right)^2 + \left(\frac{2c+2a-b}{3}\right)^2 + \left(\frac{2a+2b-c}{3}\right)^2 = a^2 + b^2 + c^2$$

Therefore the function

$$I(n) = \max\left\{a^2 + b^2 + c^2 \mid a, b, c \in S_n\right\}$$

is an invariant, i.e. $I(n)$ does not change during the whole process. Since

$$I(1) = 22^2 + 23^2 + 24^2 = 1589$$

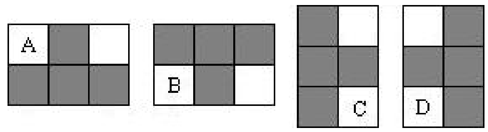
a number greater than 70 can not appear on the board (after n steps) because we would have

$$I(n) \geq 70^2 = 4900 > 1589$$

This is a contradiction and the result follows.

Also solved by Baleanu Andrei Razvan, "George Cosbuc" National College, Romania; Daniel Lasaosa, Universidad Publica de Navarra, Spain; John T. Robinson, Yorktown Heights, NY, USA.

O116. Consider an $n \times n$ board tiled with $T-$tetraminos. Let a, b, c, d be the number of tetraminos of types A, B, C, D, respectively. Prove that $4 \mid a+b-c-d$.



Proposed by Oles Dobosevych, Lviv National University, Ukraine

Solution by John T. Robinson, Yorktown Heights, NY, USA

We make use of a result due to Walkup (D. W. Walkup, Covering a rectangle with T-tetrominoes, *American Mathematical Monthly 72*, 9, November 1965, 986-988): an M by N rectangle can be tiled by T-tetrominoes if and only if M and N are both divisible by 4. Therefore for the current problem we have $n = 4m$. Next, suppose we color alternating columns of the board black and white. Each tetromino of type A or B will then always cover two black and two white squares, whereas each tetromino of type C or D will cover either 3 black squares and 1 white square, or 1 black square and 3 white squares. Since there is an equal number of black and white squares, and if we remove the tetrominoes of types A and B from the board then there will still be an equal number, this means that there must be equal numbers of type C or D tetrominoes that cover either 3 black squares and 1 white square or 3 white squares and 1 black square. It follows that $c+d$ is even, say $c+d = 2k$. Therefore

$$4(a+b+c+d) = n^2 = 16m^2$$

$$a+b+c+d = 4m^2$$

$$a+b-c-d = 4m^2 - 2(c+d) = 4m^2 - 4k$$

showing that $a+b-c-d$ is divisible by 4.

O117. Consider a quadrilateral $ABCD$ with $\angle B = \angle D = 90^\circ$. Point M is chosen on segment AB such that $AD = AM$. Rays DM and CB intersect at point N. Let H and K be the feet of the perpendiculars from points D and C onto lines AC and AN, respectively. Prove that $\angle MHN = \angle MCK$.

Proposed by Nairi Sedrakian, Yerevan, Armenia

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Clearly $ABCD$ is cyclic and AC is a diameter, its midpoint O being the circumcenter of $ABCD$. Choose then a system of orthonormal coordinates with origin at O such that $A \equiv (1,0)$ and $C \equiv (-1,0)$. Angles $0 < \beta, \delta < \pi$ exist such that $B \equiv (\cos\beta, \sin\beta)$ and $D \equiv (\cos\delta, -\sin\delta)$. Point M is clearly the intersection of circle $(x-1)^2 + y^2 = 4\sin^2\frac{\delta}{2}$ with center A and radius $AD = 2\sin\frac{\delta}{2}$, and line $AB \equiv y = \cot\frac{\beta}{2}(1-x)$, yielding $M \equiv \left(1 - 2\sin\frac{\beta}{2}\sin\frac{\delta}{2}, 2\cos\frac{\beta}{2}\sin\frac{\delta}{2}\right)$. Lines DM and BC have respective equations

$$y + \sin\delta = \frac{\cos\frac{\beta}{2} + \cos\frac{\delta}{2}}{\sin\frac{\beta}{2} - \sin\frac{\delta}{2}}(\cos\delta - x), \qquad y = \tan\frac{\beta}{2}(x+1),$$

yielding $N \equiv \left(2\cos\frac{\beta}{2}\cos\frac{\delta}{2} - 1, 2\sin\frac{\beta}{2}\cos\frac{\delta}{2}\right)$. The projection of D on the horizontal axis AC clearly is $H \equiv (\cos\delta, 0)$, producing

$$\vec{HM} \equiv 2\sin\frac{\delta}{2}\left(\sin\frac{\delta}{2} - \sin\frac{\beta}{2}, \cos\frac{\beta}{2}\right), \qquad \vec{HN} \equiv 2\cos\frac{\delta}{2}\left(\cos\frac{\beta}{2} - \cos\frac{\delta}{2}, \sin\frac{\beta}{2}\right),$$

and by the definition of scalar product,

$$\cos\angle MHN = \frac{\sin\frac{\beta+\delta}{2} - \frac{1}{2}\sin\delta}{\sqrt{\left(1 + \sin^2\frac{\delta}{2} - 2\sin\frac{\beta}{2}\sin\frac{\delta}{2}\right)\left(1 + \cos^2\frac{\delta}{2} - 2\cos\frac{\beta}{2}\cos\frac{\delta}{2}\right)}}.$$

Note that K will be on the circumcircle of $ABCD$, since $\angle AKC = 90^\circ$, ie, it will satisfy equation $x^2 + y^2 = 1$. Calling m the slope of AN, which will clearly have equation $AN \equiv y = m(x-1)$, and since the intersections of AN with the circumcircle of $ABCD$ are A and K, satisfying $(1+x)(1-x) = y^2 = m^2(1-x)^2$, then $K \equiv \left(\frac{m^2-1}{m^2+1}, -\frac{2m}{m^2+1}\right)$ since $x = 1$ produces A. Now, $m = \frac{\sin\frac{\beta}{2}\cos\frac{\delta}{2}}{\cos\frac{\beta}{2}\cos\frac{\delta}{2} - 1}$ is negative, or

$$\vec{CK} \equiv 2m\left(\frac{m}{m^2+1}, -\frac{1}{m^2+1}\right), \qquad \vec{CM} \equiv 2\left(1 - \sin\frac{\beta}{2}\sin\frac{\delta}{2}, \cos\frac{\beta}{2}\sin\frac{\delta}{2}\right),$$

$$\cos \angle MCK = \frac{\cos\frac{\beta}{2}\sin\frac{\delta}{2} - m\left(1 - \sin\frac{\beta}{2}\sin\frac{\delta}{2}\right)}{\sqrt{(m^2+1)\left(1 + \sin^2\frac{\delta}{2} - 2\sin\frac{\beta}{2}\sin\frac{\delta}{2}\right)}},$$

and after inserting the value of m and some algebra, this result is found to be identical to $\cos \angle MHN$. The conclusion follows.

O118. Solve in positive integers the equation

$$x^2+y^2+z^2-xy-yz-zx=w^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania

Solution by Ercole Suppa, "Liceo Scientifico Statale E.Einstein, Teramo", Italy

We use the following

LEMMA. All integral solutions of the Diophantine equation

$$x^2+xy+y^2=z^2$$

are given by

$$\begin{cases} x=k\left(n^2+2mn\right) \\ y=k\left(m^2-n^2\right) \\ z=k\left(m^2+n^2+mn\right) \end{cases} \quad , \quad \begin{cases} x=k\left(m^2-n^2\right) \\ y=k\left(n^2+2mn\right) \\ z=k\left(m^2+n^2+mn\right) \end{cases}$$

where $k,m,n\in\mathbb{Z}$.

Proof. The proof can be found in the book: *Titu Andreescu and Dorin Andrica, An Introduction to Diophantine Equations , GILL Publishing House, 2002.* ■

Coming back to the problem, the given equation can be written in the form:

$$(x-y)^2+(y-z)^2+(z-x)^2=2w^2 \tag{1}$$

By setting $X=x-z$, $Y=y-z$ the equation (1) rewrites as

$$X^2+Y^2+(X+Y)^2=2w^2$$

$$X^2+XY+Y^2=w^2$$

By the LEMMA there are $k,m,n\in\mathbb{Z}$ such that

$$\begin{cases} X=k\left(n^2+2mn\right) \\ Y=k\left(m^2-n^2\right) \\ w=k\left(m^2+n^2+mn\right) \end{cases} \quad , \quad \begin{cases} X=k\left(m^2-n^2\right) \\ Y=k\left(n^2+2mn\right) \\ w=k\left(m^2+n^2+mn\right) \end{cases}$$

By solving the system

$$\begin{cases} x-y=X \\ y-z=Y \\ z-x=-X-Y \end{cases}$$

we get $x = X + Y + h$, $y = Y + h$, $z = h$, where $h \in \mathbb{Z}$. Thus the positive integers solutions of the our equation, up to permutations of x, y, z, are given by:

$$\begin{cases} x = k\left(m^2 + 2mn\right) + h \\ y = k\left(m^2 - n^2\right) + h \\ z = h \\ w = k\left(m^2 + n^2 + mn\right) \end{cases}, \qquad \begin{cases} x = k\left(m^2 - n^2\right) + h \\ y = k\left(n^2 + 2mn\right) + h \\ z = h \\ w = k\left(m^2 + n^2 + mn\right) \end{cases}$$

where $h, k, m, n \in \mathbb{N}_0$, $m > n$.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Raul A. Simon, Chile.

O119. Let a and b be nonzero integers with $|a| > 1$ and let P be a finite set of positive integers. Consider the sequence $x_n = m(a^n + b^n)$, where m is a positive integer. Prove that there are infinitely many integers n such that x_n is not a p_k-th power of an integer for each p_k in P.

Proposed by Tung Nguyen Tho, Hanoi University of Education, Vietnam

Solution by Gabriel Dospinescu, Ecole Normal Superieur, France

If $d = (a, b)$ we can write $a = du, b = dv$ with $(u, v) = 1$. We may suppose $|u| > 1$ (everything is clear if $|u| \leq 1$ and $|v| \leq 1$). Note that any two of the numbers $u^{2^n} + v^{2^n}$ have greatest common divisor at most 2 (the only prime that can divide $u^{2^n} + v^{2^n}$ and $u^{2^m} + v^{2^m}$ with $m > n$ is 2 and 4 cannot divide $x^2 + y^2$ unless x, y are even). Therefore, since $|u| > 1$ there is a prime $p > m + d$ that divides some number $u^{2^N} + v^{2^N}$. We recall the following standard result: if $p > 2$ and $a \equiv b \pmod p$, but $(p, ab) = 1$, then $v_p(a^n - b^n) = v_p(a - b) + v_p(n)$. Applying this we obtain that $v_p(u^{2^N \cdot 2^M} + v^{2^N \cdot 2^M}) = v_p(u^{2^N} + v^{2^N}) + M$. Therefore, if we chose M such that the last quantity is 1 modulo the product of elements in P (and obviously there are infinitely many such M), $m \cdot (a^{2^N \cdot 2^M} + b^{2^N \cdot 2^M})$ cannot be a power of exponent p for any $p \in P$.

O120. Let $ABCDEF$ be a convex hexagon with area S. Prove that

$$AC(BD+BF-DF)+CE(BD+DF-BF)+AE(BF+DF-BD)\geq 2\sqrt{3}S.$$

Proposed by Nairi Sedrakian, Yerevan, Armenia

Solution by Holden Lee

Lemma: Let the incircle of BDF touch DF, FB, BD at X, Y, Z respectively. Then there exists a point O such that $OB \leq \frac{2}{\sqrt{3}}BY$, $OD \leq \frac{2}{\sqrt{3}}DZ$, $OF \leq \frac{2}{\sqrt{3}}FX$.
Proof: Let $u = DZ, v = FX, t = BY$. Consider circles centered at B, D, F with radii rt, ru, and rv, respectively. As r increases from 1, the circles expand so eventually there will be a point in XYZ on the boundary or in the interior of all three circles. Take the least r such that this is true. Then if all three circles do not have a point in common, there is a point P strictly inside all three circles. After decreasing r by a sufficiently small amount, P will still be inside all three circles, contradicting the minimality of r. Thus, the circles centered at B, D, F with radii rt, ru, rv have a point in common. Call this point O. We claim $r \leq \frac{2}{\sqrt{3}}$. Since $\angle BOD + \angle DOF + \angle FOB = 360°$ one of these angles is at least $120°$, say without loss of generality $\angle FOD$. Then using the Law of Cosines on ΔFOD,

$$\begin{aligned}-\frac{1}{2} = \cos 120° \geq \cos\angle FOD &= \frac{OF^2+OD^2-FD^2}{2OF\cdot OD}\\ &= \frac{r^2v^2+r^2u^2-(u+v)^2}{2uvr^2}\end{aligned}$$

Multiplying both sides by $2uvr^2$,

$$\begin{aligned}-uvr^2 &\geq r^2(v^2+u^2)-(u+v)^2\\ u^2+2uv+v^2 &\geq r^2(u^2+uv+v^2)\end{aligned}$$

Now if $r > \frac{2}{\sqrt{3}}$ then $r^2 > \frac{4}{3}$ and

$$\begin{aligned}u^2+2uv+v^2 &> \frac{4}{3}(u^2+uv+v^2)\\ 2uv &> u^2+v^2\end{aligned}$$

contradicting the Arithmetic Mean- Geometric Mean inequality. So $r \leq \frac{2}{\sqrt{3}}$ and O satisfies the conditions.

Let P be the intersection of BO and AC. Then using the formulas $BZ = \frac{BD+BF-DF}{2}$ and $[ABCO] = \frac{1}{2} \cdot AC \cdot BO \cdot \sin \angle BPC \leq \frac{1}{2} \cdot AC \cdot BO$,

$$\begin{aligned} AC \cdot (BD + BF - DF) &= 2AC \cdot BZ \\ &\geq 2AC \cdot \frac{\sqrt{3}}{2} BO \\ &\geq 2\sqrt{3}\,[ABCO] \end{aligned}$$

Similarly,

$$\begin{aligned} CE \cdot (BD + DF - BF) &\geq 2\sqrt{3}\,[CDEO] \\ AE \cdot (BF + DF - BD) &\geq 2\sqrt{3}\,[EFAO] \end{aligned}$$

Adding up these inequalities we obtain

$$AC \cdot (BD + BF - DF) + CE \cdot (BD + DF - BF) + AE \cdot (BF + DF - BD) \geq 2\sqrt{3}S$$