# Angle Inequalities in Tetrahedra

**Mark Chen**

Top performers in the International Mathematical Olympiad are undoubtedly familiar with the standard inequalities regarding angles in triangles. Several of these standard inequalities regarding the most basic of 2-D figures also have analogues in the most basic of 3-D figures, namely tetrahedra. While proving one of these analogues, we will also discover some interesting properties of tetrahedra.
We start with a basic theorem:

**Theorem 1.** *If $A$, $B$, and $C$ are the angles in an acute triangle $ABC$, then*

$$\cos A + \cos B + \cos C \leq \frac{3}{2}.$$

*Proof.* Because the cosine function is concave on the interval $\left[0, \frac{\pi}{2}\right]$, we have

$$\cos A + \cos B + \cos C \leq 3 \cdot \cos \frac{A+B+C}{3} = 3 \cdot \cos \frac{\pi}{3} = \frac{3}{2}$$

by Jensen's inequality with $f(x) = \cos x$. □

This leads to a main theorem regarding angles in tetrahedra:

**Theorem 2.** *Let $A_1A_2A_3A_4$ be a tetrahedron, and let $f_i$ be the face which opposes the vertex $A_i$. Furthermore, let $f_{ij} = f_{ji}$ be the angle between the faces $f_i$ and $f_j$. Then*

$$\sum_{1 \leq i < j \leq 4} \cos f_{ij} \leq 2.$$

From this point onwards, we will use the notation from Theorem 2. But before we can approach Theorem 2 itself, it is helpful to familiarize ourselves with some basic properties of tetrahedra. One of the these properties is the observation that every tetrahedron has a centroid.

**Definition 3.** *Let $A_1, A_1'$; $A_2, A_2'$; $A_3, A_3'$; $A_4, A_4'$ be the four pairs of diagonally opposite vertices of a parallelepiped, drawn so that face $A_1A_2'A_3A_4' \cong$ face $A_3'A_4A_1'A_2$ . Through one of its vertices, say $A_1$, draw the three diagonals $A_1A_2$, $A_1A_3$, and $A_1A_4$. We say that parallelepiped $A_1A_2'A_3A_4'A_3'A_4A_1'A_2$ is circumscribed about tetrahedron $A_1A_2A_3A_4$.*

**Definition 4.** *A median of a tetrahderon is a line segment connecting a vertex with the centroid of the opposite face.*

**Lemma 5.** *The medians of a tetrahedron lie along the diagonals of the tetraheron's circumscribed parallelepiped and are two-thrids the lengths of the respective diagonals.*

*Proof.* Let $A_1A_2A_3A_4$ be our tetrahdedron, and let its circumscibed parallelepiped be named as indicated in Definition 3. The line $A_3B$ joining the vertex $A_3$ to the midpoint $B$ of $A_1'A_3'$ lies in the plane of the parallelogram $A_1A_3'A_1'A_3$. Hence, $A_3B$ meets $A_1A_1'$ in the centroid $C$ of triangle $A_3A_3'A_1'$, and thus $\frac{A_3C}{CB} = 2$. Now $B$ is the common point to the diagonals $A_1'A_3'$ and $A_2A_4$ of face $A_1'A_2A_3'A_4$, so $A_3B$ is a median of the face $A_3A_2A_4$ of the tetrahedron, and $C$ is the centroid of this face. Therefore, the median $A_1C$ of the tetrahedron lies along the diagonal $A_1CA_1'$ of the parallelepiped. Let $O$ be the point of intersection of diagonals $A_1A_1'$ and $A_3A_3'$. The length of the segment $A_1'C$ is equal to two thirds of $A_1'O$, so it is equal to one third of $A_1A_1'$. Hence $A_1C$ is equal to two thirds of $A_1A_1'$. □

We will also need the following lemmas concerning tetrahedra:

**Lemma 6.** *Let the sides of a tetrahedron be $a_i$, $1 \le i \le 6$. If the radius of the circumsphere of the tetrahedron is R, then*

$$\sum_{i=1}^{6} a_i^2 \le 16R^2.$$

*Proof.* Let the coordinates of $A_i$ be $(x_i, y_i, z_i)$, $1 \le i \le 4$. It is a well-known fact that the centroid G of the tetrahedron is the point

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right).$$

Let W be a point $(x, y, z)$. Then

$$16WG^2 + \sum_{1\le i\le j\le 4} (A_iA_j)^2 = 16\sum_{cyc}\left[\left(x - \frac{\sum_{i=1}^4 x_i}{4}\right)^2 + \sum_{1\le i\le j\le 4}(x_i - x_j)^2\right]. \quad (1)$$

Because

$$\sum (x - x_i)^2 = x^2 - 2x \cdot \sum x_i + \sum x_i^2$$

and

$$\sum (x_i - x_j)^2 = 3 \cdot \sum x_i^2 - 2 \cdot \sum x_i x_j,$$

equation (1) reduces to:

$$\sum_{cyc}\left[16x^2 - 8x\cdot\sum_{i=1}^{4}x_i + \left(\sum_{i=1}^{4}x_i\right)^2 + 3\cdot\sum x_i^2 - 2\cdot\sum_{1\le i\le j\le 4}x_ix_j\right]$$
$$= \sum_{cyc}\left[16x^2 - 8x\cdot\sum_{i=1}^{4}x_i + 4\cdot\sum_{i=1}^{4}x_i^2\right]$$
$$= 4\cdot\sum_{cyc}(x-x_i)^2 = 4\cdot\sum_{i=1}^{4}(WA_i)^2$$

Letting point $W$ be equidistant from $A_1, A_2$ and $A_3, A_4$, we have $WA_i = R$ for all $i$. Since $WG \geq 0$, we have $\sum_{1\le i<j\le 4}(A_iA_j)^2 \leq 16R^2$, as desired. We immediately see that equality holds when $WG = 0$. $\square$

It is a well-known fact that every tetrahedron has a circumsphere (a shphere containing all four vertices of the tetrahedron) and an insphere (a sphere tangent to all four faces of the tetrahedron). We call the center of the circumsphere the circumcenter and the center of the insphere the incenter.
Now we find a necessary and sufficient condition for the coincidence of the circumcenter, incenter, and centroid of a tetrahedron.

**Lemma 7.** *A tetrahedron is isosceles (each pair of its opposite sides is congruent) if and only if its centroid, circumcenter, and incenter are coincident.*

*Proof.* We will first prove that a tetrahedron is isosceles if and only if its incenter coincides with its centroid. The "only if" direction is trivial. Now suppose that the incenter and the centroid of a tetrahedron are coincident at point $G$. Because $G$ is a centroid, the volumes of $GA_1A_2A_3$, $GA_1A_3A_4$, $GA_1A_4A_2$, and $GA_2A_3A_4$ are equal. Also, because $G$ is an incenter, $G$ is the same distance from all faces $f_i$ for $1 \leq i \leq 4$. Hence all four faces have equal area. Again, let us denote $f_{ij} = f_{ji}$ as the angle between the faces that oppose vertices $A_i$ and $A_j$. Projecting the areas of $A_1A_2A_3$, $A_1A_3A_4$, and $A_1A_4A_2$ onto $A_2A_3A_4$, we have the equation:

$$[A_1A_2A_3]\cos f_{14} + [A_1A_3A_4]\cos f_{12} + [A_1A_4A_2]\cos f_{13} = [A_2A_3A_4] \qquad (1)$$

Because all faces are of equal area, we have the system of equations:

$$\cos f_{12} + \cos f_{13} + \cos f_{14} = 1,$$

$$\cos f_{21} + \cos f_{23} + \cos f_{24} = 1,$$

$$\cos f_{31} + \cos f_{32} + \cos f_{34} = 1,$$

$$\cos f_{41} + \cos f_{42} + \cos f_{43} = 1.$$

After performing multiple substitutions, we find that

$$\cos f_{12} = \cos f_{34}, \cos f_{13} = \cos f_{24} \text{ and } \cos f_{14} = \cos f_{23}.$$

Also, because $0 < f_{ij} < \pi$ for $1 \le i < j \le 4$, we have $f_{12} = f_{34}$, $f_{13} = f_{24}$, and $f_{14} = f_{23}$. Now drop perpendicular $A_1B$ onto face $A_2A_3A_4$ and perpendicular $A_2C$ onto face $A_1A_3A_4$. Furthermore, drop perpendicular $A_1B'$ onto side $A_2A_3$ and perpendicular $A_2C'$ onto side $A_1A_4$. Calculating the volume of the tetrahdron in two ways, we have

$$\frac{1}{3}[A_2A_3A_4]A_1B = \frac{1}{3}[A_1A_3A_4]A_2C, . \tag{2}$$

from which we find that $A_1B = A_2C$. Noting that $BB' \perp A_2A_3$, and $CC' \perp A_1A_4$, we have $A_1BB' \cong A_2CC'$ by the AAS Postulate ($f_{23} = f_{14}$). Hence $A_1B' = A_2C'$. Because all faces are of equal area,

$$\frac{1}{2}A_1B' \times A_2A_3 = \frac{1}{2}A_2C' \times A_1A_4.$$

Therefore, $A_1A_4 = A_2A_3$, and similarly, other pairs of opposite sides are congruent. Hence our tetrahdron is isosceles. Now, it remains to prove that a tetrahedron is isosceles if and only if its centroid and circumcenter are coincident. Suppose we have an isosceles tetrahedron. It is easy to see that the circumscribed parallelepiped is rectangular. Hence the parallelepiped's diagonals are equal, which implies that the tetrahedron's medians are equal (median lengths equal $\frac{2}{3}$ corresponding diagonal lengths by Lemma 5). Therefore the centroid is equidistant from the vertices of the tetrahedron, and is also a circumcenter. Now suppose that the centroid is equidistant from the vertices. Then all medians of the the tetrahedron are equal, implying that all diagonals of the circumscribed parallelopiped are equal. Hence the circumscribed parallelepiped is rectangular and the tetrahedron is isosceles. □

Now, our final lemma:

**Lemma 8.** *Let the incenter of tetrahedron $A_1A_2A_3A_4$ meet face $f_i$ at $P_i$ for $1 \le i \le 4$. Then $A_1A_2A_3A_4$ is isosceles if and only if $P_1P_2P_3P_4$ is isosceles.*

*Proof.* ($\Rightarrow$) Let $I$ denote the incenter of tetrahedron $A_1A_2A_3A_4$. Because $IP_1 \perp \triangle A_2A_3A_4$ and $IP_4 \perp \triangle A_1A_2A_3$, it is easy to see that $\angle P_1IP_4 = \pi - f_{14}$, the supplement of the angle between faces $f_1$ and $f_4$. Similarly, $\angle P_2IP_3 = \pi - f_{23}$. In Lemma 4, we proved that $f_{14} = f_{23}$ for all isosceles tetrahedra, so we have the equivalence $\angle P_1IP_4 = \angle P_2IP_3$. Because $IP_1 = IP_2 = IP_3 = IP_4$, $\triangle P_1IP_4 \cong \triangle P_2IP_3$ by the SAS Postulate. Hence $P_1P_4 = P_2P_3$, and working similarly with other opposite edges, we find that $P_1P_2P_3P_4$ is isosceles.

($\Leftarrow$) As usual, we let $I$ denote the incenter of $A_1A_2A_3A_4$. Draw a plane through $I$, $P_1$, and $P_4$, intersecting $A_2A_3$ at point $B$. Also, draw a plane through $I$, $P_2$, and $P_3$, intersecting $A_1A_4$ at point $C$. Note that $IB \perp A_2A_3$ and that $IB$ is the perpendicular bisector of $P_1P_4$. Because $IP_1 = IP_2 = IP_3 = IP_4$, $I$ is the circumcenter of tetrahedron $P_1P_2P_3P_4$. But because tetrahedron $P_1P_2P_3P_4$ is isosceles, $I$ is also its incenter and centroid by Lemma 7. Let the intersection of $IB$ and $P_1P_4$ be $D$ and the intersection of $IC$ and $P_2P_3$ be $E$. Then $D$ and $E$ are the midpoints of $P_1P_4$ and $P_2P_3$ respectively. Because $I$ is the centroid of $P_1P_2P_3P_4$,

$$\vec{I} = \frac{\vec{P_1} + \vec{P_2} + \vec{P_3} + \vec{P_4}}{4} = \frac{1}{2}\left[\frac{\vec{P_1} + \vec{P_4}}{2} + \frac{\vec{P_2} + \vec{P_3}}{2}\right] = \frac{1}{2}[\vec{D} + \vec{E}].$$

Hence $D$, $I$, and $E$ are collinear (in particular, $B$, $D$, $I$, $E$, and $C$ are all collinear). Since $P_1P_4 = P_2P_3$, we have

$$\triangle IP_1P_4 \cong \triangle IP_2P_3 \Rightarrow \triangle IBP_1 \cong \triangle ICP_2 \Rightarrow I$$

so $I$ is the midpoint of segment $BC$. Now let the plane through $P_3$, $P_4$, and $I$ intersect $A_1A_2$ at $M$ and let the plane through $P_1$, $P_2$ and $I$ intersect $A_3A_4$ at $N$. Similarly, we find that $I$ is the midpoint of segment $MN$. Likewise, let the plane through $P_1$, $P_3$, and $I$ intersect $A_2A_4$ at $P$ and let the plane through $P_2$, $P_4$, and $I$ intersect $A_1A_3$ at $Q$. As before, we find that $I$ is the midpoint of segment $PQ$. It is not difficult to see that $BC$, $MN$, and $PQ$ perpendicularly bisect one another at point $I$. In particular, $PQ$ is perpendicular to the plane through $B, C, M, N$, and the lines $A_1A_3$ and $A_2A_4$. As $I$ is the midpoint of $PQ$, $B, C, M, N$ are the midpoints of $A_2A_3, A_1A_4, A_1A_2$ and $A_3A_4$ respectively. Hence $I$ is the centroid of $A_1A_2A_3A_4$, and by Lemma 7, $A_1A_2A_3A_4$ is isosceles. $\square$

Now we can finally prove Theorem 2!

*Proof.* As usual, let $I$ be the incenter of our tetrahedron. Let plane $S$ through points $I$, $P_1$, $P_4$ intersect $A_2A_3$ at point $X$. Now connect $AP_1$ and $AP_4$, and get $P_1XP_4 = (f_{14}) \Rightarrow P_1IP_4 = \pi - f_{14}$. If the inradius of the tetrahderon is $r$, then the law of cosines yields:

$$(P_1P_4)^2 = 2r^2 - 2r^2 \cos \angle P_1IP_4 = 2r^2(1 + \cos f_{14}).$$

Summing over all lengths $P_iP_j, 1 \leq i < j \leq 4$, we obtain:

$$\sum_{1 \leq i < j \leq 4} (P_iP_j)^2 = 2r^2 \left(6 + \sum_{1 \leq i < j \leq 4} \cos f_{ij}\right).$$

By Lemma 6, the above equation becomes

$$\sum_{1\le i<j\le 4}(P_iP_j)^2 \le 16r^2 \Leftrightarrow \sum_{1\le i<j\le 4}\cos f_{ij} \le 2.$$

By Lemmas 6, 7, and 8, we have equality if and only if $A_1A_2A_3A_4$ is isosceles. $\square$

We now present several extensions of Theorem 2. The analogue of the inequality

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le \frac{3\sqrt{3}}{8}$$

in a triangle $ABC$ is:

$$\prod_{1\le i<j\le 4}\cos\frac{f_{ij}}{2} \le \frac{8}{27}. \tag{3}$$

The analogue of the inequality $\cos A\cos B\cos C \le \frac{1}{8}$ in a triangle $ABC$ is:

$$\Pi_{1\le i<j\le 4}\cos f_{ij} \le \frac{1}{36}. \tag{4}$$

Finally, the analoge of the inequality $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2} \le \frac{9}{4}$ in a triangle ABC is:

$$\sum_{1\le i<j\le 4}\cos^2\frac{f_{ij}}{2} \le 4. \tag{5}$$

We leave the proofs of these simple extensions to the interested reader.

# 1 Bibliography


[1] Altshiller-Court, Nathan. Modern Pure Solid Geometry, Chelsea, 1964.

[2] Wang, Hsiang-Tung. Inequalities, He Nan Educational Press, 1994.