On a class of sums involving the floor function

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For a real number x there is a unique integer n such that $n \le x < n+1$. We say that n is the *greatest integer less than or equal to* x or the *floor* of x. We denote $n = \lfloor x \rfloor$. The difference $x - \lfloor x \rfloor$ is called the *fractional part* of x and is denoted by $\{x\}$.

The integer $\lfloor x \rfloor + 1$ is called the *ceiling* of x and is denoted by $\lceil x \rceil$. For example,

$$\begin{array}{ccc} \lfloor 2.1 \rfloor = 2 & \{2.1\} = .1 & \lceil 2.1 \rceil = 3 \\ \lfloor -3.9 \rfloor = -4 & \{-3.9\} = .1 & \lceil -3.9 \rceil = -3 \end{array}$$

The following properties are useful:

1) If a and b are integers, $b > 0$, and q is the quotient when a is divided by b, then $q = \left\lfloor \frac{a}{b} \right\rfloor$.

2) For any real number x and any integer n, $\lfloor x+n \rfloor = \lfloor x \rfloor + n$ and $\lceil x+n \rceil = \lceil x \rceil + n$.

3) For any positive real number x and any positive integer n the number of positive multiples of n not exceeding x is $\left\lfloor \frac{x}{n} \right\rfloor$.

4) For any real number x and any positive integer n,

$$\left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor.$$

We will prove the last two properties. For 3) consider all multiples $1 \cdot n, 2 \cdot n, \ldots, k \cdot n$, where $k \cdot n \le x < (k+1)n$. That is $k \le \frac{x}{n} < k+1$ and the conclusion follows. For 4) denote $\lfloor x \rfloor = m$ and $\{x\} = \alpha$. From the Division Algorithm and property 1) above it follows that $m = n\left\lfloor \frac{m}{n} \right\rfloor + r$, where $0 \le r \le n-1$. We obtain $0 \le r + \alpha \le n - 1 + \alpha < n$, that is $\left\lfloor \frac{r+\alpha}{n} \right\rfloor = 0$ and

$$\left\lfloor \frac{x}{n} \right\rfloor = \left\lfloor \frac{m+\alpha}{n} \right\rfloor = \left\lfloor \left\lfloor \frac{m}{n} \right\rfloor + \frac{r+\alpha}{n} \right\rfloor = \left\lfloor \frac{m}{n} \right\rfloor + \left\lfloor \frac{r+\alpha}{n} \right\rfloor = \left\lfloor \frac{m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor.$$

The following result is helpful in proving many relations involving the floor function.

Theorem. *Let p be an odd prime and let q be an integer that is not divisible by p. If $f:\mathbb{Z}_+^* \to \mathbb{R}$ is a function such that:*

i) $\dfrac{f(k)}{p}$ *is not an integer,* $k=1,2,\ldots,p-1$*;*

ii) $f(k)+f(p-k)$ *is an integer divisible by p,* $k=1,2,\ldots,p-1$*, then*

$$\sum_{k=1}^{p-1}\left\lfloor f(k)\frac{q}{p}\right\rfloor = \frac{q}{p}\sum_{k=1}^{p-1} f(k) - \frac{p-1}{2}. \tag{1}$$

Proof. From ii) it follows that

$$\frac{qf(k)}{p}+\frac{qf(p-k)}{p}\in\mathbb{Z} \tag{2}$$

and from i) we obtain that $\dfrac{qf(k)}{p}\notin\mathbb{Z}$ and $\dfrac{qf(p-k)}{p}\notin\mathbb{Z}$, $k=1,\ldots,p-1$, hence

$$0<\left\{\frac{qf(k)}{p}\right\}+\left\{\frac{qf(p-k)}{p}\right\}<2.$$

But, from (1), $\left\{\dfrac{qf(k)}{p}\right\}+\left\{\dfrac{qf(p-k)}{p}\right\}\in\mathbb{Z}$, thus

$$\left\{\frac{qf(k)}{p}\right\}+\left\{\frac{qf(p-k)}{p}\right\}=1,\quad k=1,\ldots,p-1.$$

Summing up and dividing by 2 yields

$$\sum_{k=1}^{p-1}\left\{\frac{q}{p}f(k)\right\}=\frac{p-1}{2}.$$

It follows that

$$\sum_{k=1}^{p-1}\frac{q}{p}f(k)-\sum_{k=1}^{p-1}\left\lfloor\frac{q}{p}f(k)\right\rfloor=\frac{p-1}{2}$$

and the conclusion follows. $\square$

Application 1. The function $f(x)=x$ satisfies both i) and ii) in Theorem, hence

$$\sum_{k=1}^{p-1}\left\lfloor k\frac{q}{p}\right\rfloor=\frac{q}{p}\cdot\frac{(p-1)p}{2}-\frac{p-1}{2},$$

that is

$$\sum_{k=1}^{p-1} \left\lfloor k\frac{q}{p} \right\rfloor = \frac{(p-1)(q-1)}{2} \quad \text{(Gauss)}. \tag{3}$$

Remark. From the proof of our Theorem, it follows that the above formula holds for any relatively prime integers p and q.

Application 2. The function $f(x) = x^3$ also satisfies conditions i) and ii), hence

$$\sum_{k=1}^{p-1} \left\lfloor k^3\frac{q}{p} \right\rfloor = \frac{q}{p} \cdot \frac{(p-1)^2p^2}{4} - \frac{p-1}{2} = \frac{(p-1)(p^2q - pq - 2)}{4}. \tag{4}$$

For $q = 1$ we obtain the 2002 German Mathematical Olympiad problem:

$$\sum_{k=1}^{p-1} \left\lfloor \frac{k^3}{p} \right\rfloor = \frac{(p-2)(p-1)(p+1)}{4}. \tag{5}$$

Application 3. For $f : \mathbb{Z}_+^* \to \mathbb{R}$, $f(s) = (-1)^s s^2$, conditions i) and ii) in our Theorem are both satisfied. We obtain

$$\sum_{k=1}^{p-1} \left\lfloor (-1)^k k^2 \frac{q}{p} \right\rfloor = \frac{q}{p}(-1^2 + 2^2 - \ldots + (p-1)^2) - \frac{p-1}{2}$$

$$= \frac{q}{p} \cdot \frac{p(p-1)}{2} - \frac{p-1}{2},$$

hence

$$\sum_{k=1}^{p-1} \left\lfloor (-1)^k k^2 \frac{q}{p} \right\rfloor = \frac{(p-1)(q-1)}{2}. \tag{6}$$

Remark. By taking $q = 1$ we get

$$\sum_{k=1}^{p-1} \left\lfloor (-1)^k \frac{k^2}{p} \right\rfloor = 0.$$

Using now the identity $\lfloor -x \rfloor = -1 - \lfloor x \rfloor$, $x \in \mathbb{R} \setminus \mathbb{Z}$, the last display takes the form

$$\sum_{k=1}^{p-1} (-1)^k \left\lfloor \frac{k^2}{p} \right\rfloor = \frac{1-p}{2}. \tag{7}$$

Application 4. Similarly, applying our Theorem to $f : \mathbb{Z}_+^* \to \mathbb{R}$, $f(s) = (-1)^s s^4$ yields

$$\sum_{k=1}^{p-1} \left\lfloor (-1)^k k^4 \frac{q}{p} \right\rfloor = \frac{q(p-1)(p^2-p-1)}{2} - \frac{p-1}{2}. \tag{8}$$

Taking $q = 1$ gives

$$\sum_{k=1}^{p-1} \left\lfloor (-1)^k \frac{k^4}{p} \right\rfloor = \frac{(p-2)(p-1)(p+1)}{2}. \tag{9}$$

Application 5. For $f(s) = \dfrac{s^p}{p}$, conditions i) and ii) in our Theorem are also satisfied and for $q = 1$ we obtain

$$\sum_{k=1}^{p-1} \left\lfloor \frac{k^p}{p^2} \right\rfloor = \frac{1}{p} \sum_{k=1}^{p-1} \frac{k^p}{p} - \frac{p-1}{2} = \frac{1}{p^2} \left(\sum_{k=1}^{p-1} k^p - \frac{p(p-1)}{2} \right),$$

hence

$$\sum_{k=1}^{p-1} \left\lfloor \frac{k^p}{p^2} \right\rfloor = \frac{1}{2} \sum_{k=1}^{p-1} \frac{k^p - k}{p}. \tag{10}$$

Formula (10) shows that half of the sum of the quotients obtained when $k^p - k$ is divided by p (Fermat's Little Theorem) is equal to the sum of the quotients obtained when k^p is divided by p^2, $k = 1, 2, \ldots, p-1$.

References

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