Some Remarks on a Multiplicative Function

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Abstract

The main purpose of this paper is to define and study the arithmetic function S_k, the number of representations of the positive integer n as a product of k positive integers. The Dirichlet series of S_k is obtained in Theorem 3.1.

1 Introduction

In Problem O124 (Mathematical Reflections **3**(2009)) is defined the following interesting multiplicative function S, where for any positive integer n, $S(n)$ is the number of pairs of positive integers (x, y) such that $xy = n$ and $gcd(x, y) = 1$. The problem asks to prove the relation

$$\sum_{d|n} S(d) = \tau(n^2), \tag{1.1}$$

where $\tau(s)$ is the number of divisors of the positive integer s. A simple argument in order to prove relation (1.1) is to note that the function S is multiplicative, that is, for any relatively prime integers m and n we have $S(mn) = S(m)S(n)$. After that it is sufficient to note that for any prime p and any positive integer α, $S(p^\alpha) = 2$, hence we get $S(n) = 2^s$, where $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ is the prime factorization of n.

Another example connected to this topic is contained in Problem J108 (Mathematical Reflections **1**(2009)), which asks to show that the number of ordered pairs (a, b) of relatively prime positive divisors of n is equal to $\tau(n^2)$, the number of divisors of n^2.

The main purpose of this paper is to define and study a family of arithmetic multiplicative functions S_k, $k \geq 1$, that extend the first above mentioned example. These functions and some of their properties are presented in the book [2]. For details concerning the general theory of multiplicative functions we refer to the book [1].

2 The functions S_k

Denote by $S_k(n)$ the number of representations of the positive integer n as a product of k positive integers, that is, the number of solutions in positive integers of the equation

$$x_1 x_2 \cdots x_k = n. \tag{2.1}$$

In this way, for a fixed positive integer k, we define the arithmetic function $n \mapsto S_k(n)$. It is clear that $S_1 = \mathbf{1}$, the constant function 1.

A first result concerning the function S_k is the following.

Theorem 2.1. *The function S_k is multiplicative.*

Proof. Let m and n be two relatively prime integers. Consider $(x_1, \cdots, x_k)$ and $(y_1, \cdots, y_k)$ solutions in positive integers of the corresponding equations to m and n, that is, we have the relations $x_1x_2 \cdots x_k = m$ and $y_1y_2 \cdots y_k = n$. Then by multiplication we get $(x_1y_1)(x_2y_2) \cdots (x_ky_k) = mn$, that is the product of two solutions (component by component) gives a solution to the corresponding equation to mn. Conversely, let $(z_1, \cdots, z_k)$ be any solution to the equation $z_1z_2 \cdots z_k = mn$. Define $x_i = gcd(z_i, m)$ and $y_i = gcd(z_i, n)$, $i = 1, \cdots, k$. It is clear that $x_1x_2 \cdots x_k = m$, $y_1y_2 \cdots y_k = n$ and $(x_1y_1)(x_2y_2) \cdots (x_ky_k) = mn$, hence $S_k(mn) = S_k(m)S_k(n)$. □

Theorem 2.2. *S_k is the summation function of S_{k-1}, that is for any positive integer n the following relation holds:*

$$S_k(n) = \sum_{d|n} S_{k-1}(d). \tag{2.2}$$

Proof. For a fixed divisor d of n consider all solutions $(x_1, \cdots, x_k)$ to equation (2.1) such that $x_1 = d$. The number of such solutions is $S_{k-1}(\frac{n}{d})$. It follows that

$$S_k(n) = \sum_{d|n} S_{k-1}(\frac{n}{d}) = \sum_{d|n} S_{k-1}(d),$$

and we are done. □

From Theorem 2.2 it follows that $S_2(n) = \sum_{d|n} S_1(d) = \sum_{d|n} \mathbf{1}(d) = \sum_{d|n} 1 = \tau(n)$, hence we obtain $S_2 = \tau$, the number of divisors function.

Theorem 2.3. *If p is a prime and α is a positive integer, then*

$$S_k(p^\alpha) = \binom{\alpha + k - 1}{k - 1}. \tag{2.3}$$

Proof. We proceed by induction on k. Clearly, we have $S_1(p^\alpha) = 1$. According to relation (2.3) we get $S_2(p^\alpha) = \sum_{d|p^\alpha} S_1(d) = 1 + \cdots + 1 = \alpha + 1 = \binom{\alpha+1}{1}$, and the property holds. Assume that $S_k(p^\alpha) = \binom{\alpha+k-1}{k-1}$. Using formula (2.2) it follows that $S_{k+1}(p^\alpha) = \sum_{d|p^\alpha} S_k(d) = \sum_{j=0}^{\alpha} S_k(p^j) = \binom{k-1}{k-1} + \binom{k}{k-1} + \cdots + \binom{\alpha+k-1}{k} = \binom{\alpha+k}{k}$, where we have used the well-known combinatorial identity $\binom{s}{s} + \binom{s+1}{s} + \cdots + \binom{s+l}{s} = \binom{s+l+1}{s+1}$ □

Corollary 2.4. *Assume that $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ is the prime factorization of the positive integer n. Then*

$$S_k(n) = \binom{\alpha_1 + k - 1}{k - 1} \cdots \binom{\alpha_s + k - 1}{k - 1} \tag{2.4}$$

Proof 1. Taking into account that the function S_k is multiplicative it follows $S_k(n) = S_k(p_1^{\alpha_1} \cdots p_s^{\alpha_s}) = S_k(p_1^{\alpha_1}) \cdots S_k(p_s^{\alpha_s}) = \binom{\alpha_1+k-1}{k-1} \cdots \binom{\alpha_s+k-1}{k-1}$, and we are done.

Proof 2. For another proof we can use the summation formula in Theorem 2.2 and the Euler's product formula. We have

$$S_k(n) = \sum_{d|n} S_{k-1}(d) = \prod_{i=1}^{s}(1 + S_{k-1}(p_i) + \cdots + S_{k-1}(p_i^{\alpha_i})) =$$

$$\prod_{i=1}^{s}(\binom{k-1}{0} + \binom{k-1}{1} + \binom{k}{2} \cdots + \binom{\alpha_i + k - 3}{\alpha_i - 1} + \binom{\alpha_i + k - 2}{\alpha_i}) =$$

$$\prod_{i=1}^{s}(\binom{k}{1}+\binom{k}{2}+\cdots+\binom{\alpha_i+k-3}{\alpha_i-1}+\binom{\alpha_i+k-2}{\alpha_i})=$$

$$\prod_{i=1}^{s}(\binom{k+1}{2}+\cdots+\binom{\alpha_i+k-3}{\alpha_i-1}+\binom{\alpha_i+k-2}{\alpha_i})=\cdots=$$

$$\prod_{i=1}^{s}(\binom{\alpha_i+k-2}{\alpha_i-1}+\binom{\alpha_i+k-2}{\alpha_i})=\prod_{i=1}^{s}\binom{\alpha_i+k-2}{\alpha_i}.$$

$\square$

Remark. Assume that $n=p_1^{\alpha_1}\cdots p_s^{\alpha_s}$. From formulas (2.2) and (2.4) we have $S_{k+1}(n)=\sum_{d|n}S_k(d)=\sum_{0\le r_i\le\alpha_i}S_k(p_1^{r_1}\cdots p_s^{r_s})=\sum_{0\le r_i\le\alpha_i}\binom{r_1+k-1}{k-1}\cdots\binom{r_s+k-1}{k-1}$, hence we have derived the following combinatorial identity involving the decomposition of a product of binomial coefficients as a sum of terms of the same form:

$$\binom{\alpha_1+k-1}{k-1}\cdots\binom{\alpha_s+k-1}{k-1}=\sum_{0\le r_i\le\alpha_i}\binom{r_1+k-1}{k-1}\cdots\binom{r_s+k-1}{k-1} \tag{2.5}$$

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3 The Dirichlet series of S_k

Let f and g be two arithmetic functions. Define their *convolution product* $f*g$ by

$$(f*g)(n)=\sum_{d|n}f(d)g(\frac{n}{d}). \tag{3.1}$$

The convolution product has nice algebraic properties, for instance it is commutative and associative (see [1, pp.108-111]).

Given an arithmetic function f, the series

$$F(z)=\sum_{n=1}^{\infty}\frac{f(n)}{n^z} \tag{3.2}$$

is called the *Dirichlet series* associate with f. A Dirichlet series can be regarded as a purely formal infinite series, or as a function of the complex variable z, defined in the region in which the series converges.

Let f and g be arithmetic functions with associated Dirichlet series $F(z)$ and $G(z)$. Let $h=f*g$ be the convolution product of f and g, and let $H(z)$ be its associated Dirichlet series. If $F(z)$ and $G(z)$ converge absolutely at some point z, then so does $H(z)$, and we have $H(z)=F(z)G(z)$. Indeed, we have

$$F(z)G(z)=(\sum_{l=1}^{\infty}\frac{f(l)}{l^z})(\sum_{m=1}^{\infty}\frac{g(m)}{m^z})=$$

$$\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{f(l)g(m)}{l^z m^z}=\sum_{n=1}^{\infty}\frac{1}{n^z}(\sum_{lm=n}f(l)g(m))=\sum_{n=1}^{\infty}\frac{(f*g)(n)}{n^z},$$

where the rearranging of the terms in the double sum is justified by the absolute convergence of the series $F(z)$ and $G(z)$.

The most famous Dirichlet series is the *Riemann zeta function* $\zeta(z)$, defined as the Dirichlet series associated with constant function $\mathbf{1}$, that is,

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \tag{3.3}$$

and defined for $Re(z) > 1$.

Theorem 3.1. *The following relations hold:*

1. $S_k = \mathbf{1} * \mathbf{1} * \cdots * \mathbf{1}$, *where there are k factors appearing in the convolution product.*
2. $\sum_{n=1}^{\infty} \frac{S_k(n)}{n^z} = (\zeta(z))^k, Re(z) > 1$, *where ζ is the Riemann zeta function.*

Proof. 1. Using the result in Theorem 2.2 we get

$$S_k(n) = \sum_{d|n} S_{k-1}(d) = \sum_{d|n} S_{k-1}(d)\mathbf{1}(\frac{n}{d}) = (S_{k-1} * \mathbf{1})(n),$$

hence $S_k = S_{k-1} * \mathbf{1}$. Since $S_1 = \mathbf{1}$, from the associativity property of the convolution product, it follows $S_k = \mathbf{1} * \mathbf{1} * \cdots * \mathbf{1}$, where in the convolution product there are k factors, and we are done.

2. According to the above presented general result about Dirichlet series, we have

$$\sum_{n=1}^{\infty} \frac{S_k(n)}{n^z} = \sum_{n=1}^{\infty} \frac{(\mathbf{1} * \mathbf{1} * \cdots * \mathbf{1})(n)}{n^z} = (\zeta(z))^k.$$

$\square$

Remark. From the first relation in Theorem 3.1 it follows the relation $S_{k+l} = S_k * S_l$.

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References

[1] T.Andreescu, D.Andrica, *Number Theory. Structures, Examples, and Problems*, Birkhauser Boston, 2009.

[2] S. Dodunenkov, K.Chakvrian, *Problems in Number Theory* (Bulgarian), Regalia,1999.

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