Junior problems

J115. Find all positive integers n for which $\sqrt{\sqrt{n}+\sqrt{n+2009}}$ is an integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania

J116. A bug is situated in one of the vertices of a cube. Each day it travels to another vertex of a cube. How many six day journeys end at the original vertex?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J117. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{2a^2+b^2+3}+\frac{b}{2b^2+c^2+3}+\frac{c}{2c^2+a^2+3}\leq\frac{1}{2}.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

J118. Prove that for each integer $n \geq 3$ there are n pairwise distinct positive integers such that each of them divides the sum of the remaining $n-1$.

Proposed by H. A. ShahAli, Tehran, Iran

J119. Let α, β, γ be angles of a triangle. Prove that

$$\cos^3\frac{\alpha}{2}\sin\frac{\beta-\gamma}{2}+\cos^3\frac{\beta}{2}\sin\frac{\gamma-\alpha}{2}+\cos^3\frac{\gamma}{2}\sin\frac{\alpha-\beta}{2}=0.$$

Proposed by Oleh Faynstein, Leipzig, Germany

J120. Let a, b, c be positive real numbers. Prove that

$$\frac{ab}{3a+4b+2c}+\frac{bc}{3b+4c+2a}+\frac{ca}{3c+4a+2b}\leq\frac{a+b+c}{9}.$$

Proposed by Baleanu Andrei Razvan, George Cosbuc Lyceum, Romania

Senior problems

S115. Prove that for each positive integer n, 2009^n can be written as sum of six nonzero perfect squares.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania

S116. Points P and Q lie on segment BC with P between B and Q. Suppose that BP, PQ, and QC form a geometric progression in some order. Prove that there is a point A in the plane such that AP and AQ are the trisectors of angle BAC if and only if PQ is less than BP and CQ.

Proposed by Daniel Campos Salas, Costa Rica

S117. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}+\frac{3abc}{2(ab+bc+ca)^2}\geq\frac{5}{a+b+c}.$$

Proposed by Shamil Asgarli, Burnaby, Canada

S118. An equilateral triangle is divided into n^2 congruent equilateral triangles. Let V be the set of all vertices and let E be the set of all edges of these triangles. Find all n for which we can paint all edges black or white such that for every vertex the number of incident edges of the black color is equal to the number of incident edges of the white color.

Proposed by Oles Dobosevych, Lviv National University, Ukraine

S119. Consider a point P inside a triangle ABC. Let AA_1, BB_1, CC_1 be cevians through P. The midpoint M of BC different from A_1, and T is the intersection of AA_1 and B_1C_1. Prove that if the circumcircle of triangle BTC is tangent to the line B_1C_1, then $\angle BTM = \angle A_1TC$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S120. Let P be a point interior to a triangle ABC and let d_a, d_b, d_c be the distances from P to the sides of the triangle. Prove that

$$\frac{4\cdot AP\cdot BP\cdot CP}{(d_a+d_b)(d_b+d_c)(d_a+d_c)}\geq\frac{AP}{d_b+d_c}+\frac{BP}{d_a+d_c}+\frac{CP}{d_a+d_b}+1.$$

Proposed by Oles Dobosevych, Lviv National University, Ukraine

Undergraduate problems

U115. Let $a_n = 2 - \dfrac{1}{n^2 + \sqrt{n^4 + \frac{1}{4}}}$, $n = 1, 2, \ldots$ Prove that $\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_{119}}$ is an integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U116. Let G be a K_4 complete graph without an edge. Find the number of closed walks of length n in G.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U117. Let n be an integer greater than 1 and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1 + x_2 + \cdots + x_n = n$. Prove that

$$\sum_{k=1}^{n} \frac{x_k}{n^2 - n + 1 - nx_k + (n-1)x_k^2} \le \frac{1}{n-1}$$

and find all equality cases.

Proposed by Iurie Boreico, Harvard University, USA

U118. Prove that there are infinitely many positive integers n such that $\phi(\sigma(n)) \mid n$, where ϕ denotes Euler's totient function and σ is the sum of divisors function.

Proposed by Cezar Lupu, University of Bucharest, Romania

U119. Let t be a real number greater than -1. Evaluate

$$\int_0^1 \int_0^1 x^t y^t \left\{\frac{x}{y}\right\} \left\{\frac{y}{x}\right\} dxdy,$$

where $\{a\} = a - \lfloor a \rfloor$ denotes the fractional part of a.

Proposed by Ovidiu Furdui, Cluj, Romania

U120. Let $x_n = \frac{1}{n+a_1} + \frac{1}{n+a_2} + \cdots + \frac{1}{n+a_k}$ and $y_n = \frac{\phi(n)}{n}$, where $a_1, a_2, \ldots, a_k$ are distinct positive integers less than n and relatively prime with n and ϕ is Euler's totient function. Prove that for all real numbers $a < 1$,

$$\lim_{n \to \infty} n^a (x_n - y_n \log 2) = 0.$$

Is this also true for $a = 1$?

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

Olympiad problems

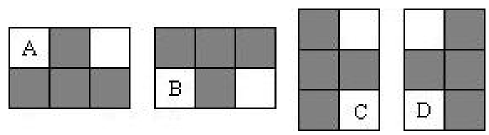
O115. Numbers 1 through 24 are written on a board. At any time, numbers a, b, c may be replaced by

$$\frac{2b+2c-a}{3}, \quad \frac{2c+2a-b}{3}, \quad \frac{2a+2b-c}{3}.$$

Can a number greater than 70 eventually appear on the board?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O116. Consider an $n \times n$ board tiled with T-tetraminos. Let a, b, c, d be the number of tetraminos of types A, B, C, D, respectively. Prove that $4 \mid a+b-c-d$.



Proposed by Oles Dobosevych, Lviv National University, Ukraine

O117. Consider a quadrilateral $ABCD$ with $\angle B = \angle D = 90^\circ$. Point M is chosen on segment AB such that $AD = AM$. Rays DM and CB intersect at point N. Let H and K be the feet of the perpendiculars from points D and C onto lines AC and AN, respectively. Prove that $\angle MHN = \angle MCK$.

Proposed by Nairi Sedrakian, Yerevan, Armenia

O118. Solve in positive integers the equation

$$x^2 + y^2 + z^2 - xy - yz - zx = w^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania

O119. Let a and b be nonzero integers with $|a| > 1$ and let P be a finite set of positive integers. Consider the sequence $x_n = m(a^n + b^n)$, where m is a positive integer. Prove that there are infinitely many integers n such that x_n is not a p_k-th power of an integer for each p_k in P.

Proposed by Tung Nguyen Tho, Hanoi University of Education, Vietnam

O120. Let $ABCDEF$ be a convex hexagon with area S. Prove that

$$AC(BD+BF-DF)+CE(BD+DF-BF)+AE(BF+DF-BD) \geq 2\sqrt{3}S.$$

Proposed by Nairi Sedrakian, Yerevan, Armenia