A note on the Malfatti problem

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Introduction

In 1803 Malfatti[3] proposed the following problem:

Given a triangle find three non-intersecting circles inside of it so that the sum of their areas is maximized.

As noted in [1], Malfatti, and many others who consider the problem, assumed that the solution would be the three circles inscribed in the angles of the triangle and which are tangent to each other (Fig.1). These circles have become known as the Malfatti circles and we refer the reader to [2] for a history of the problem for derivation of their radii.

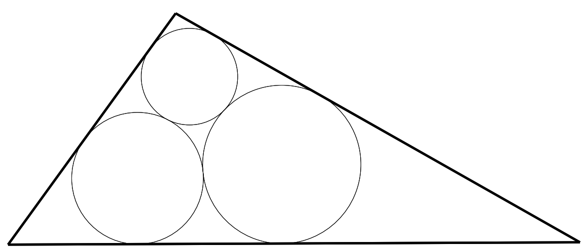
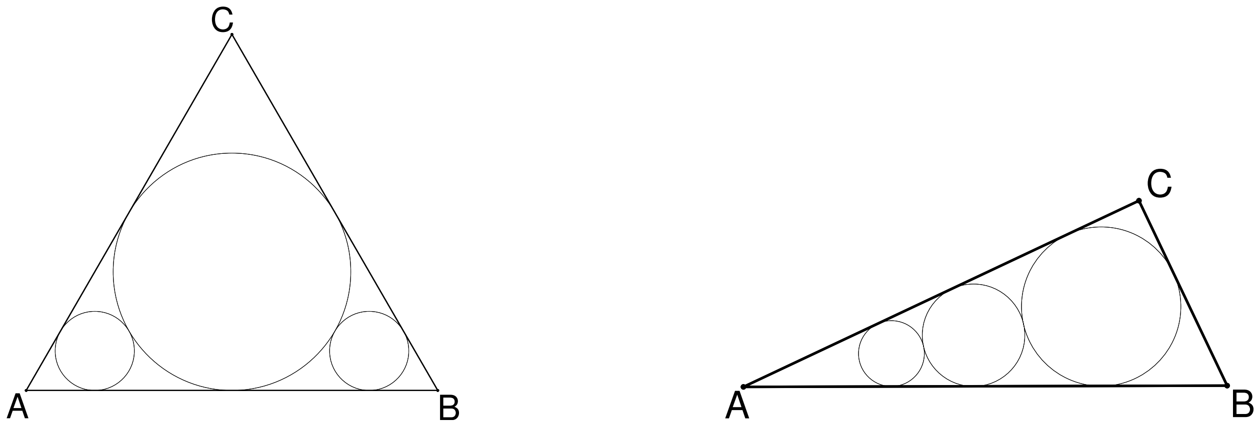


Fig. 1

In 1929, Lob and Richmond [2] noted that the Malfatti conjecture was not true. They remarked that for an equilateral triangle, the incircle with two little circles squeezed into the angles contain a greater area than the Malfatti circles. Moreover, Goldberg [1] proved in 1967 that the Malfatti circles never give a solution of the Malfatti problem.

Up to the authors knowledge, the Malfatti problem was first solved by Zalgaller and Loss [4] in 1991. They proved that for any triangle ABC with $\angle A \leq \angle B \leq \angle C$ the solution of the Malfatti problem is given by the following three circles K_1, K_2 and K_3, where K_1 is the incircle of the triangle; K_2 is the circle inscribed in angle A and tangent to K_1; K_3 is either the circle inscribed in angle B and tangent to K_1 (Fig.2, (a)), or the circle inscribed in angle A and tangent to K_2 (Fig.2, (b)), depending on whether $\sin\frac{A}{2} \geq \tan\frac{B}{4}$ or not.



(a) (b)

Fig. 2

The proof of Zalgaller and Loss uses different approaches, including computer calculations at an important step. Its main part is a case by case examination of fourteen specific arrangements of three non-intersecting circles in a triangle that are potential candidates for a solution of the problem.

The purpose of this note is to give a simple solution of the Malfatti problem for an equilateral triangle which is based on a dual approach to it.

2. A dual Malfatti problem

In this section we consider a problem which, in a sense, is dual to the Malfatti problem for an equilateral triangle.

Problem 1. Given two positive numbers a and b, find the side length of the smallest equilateral triangle, containing two non intersecting circles of radii a and b, respectively. **Solution.** we shall assume that $a \geq b$. Let ABC be an equilateral triangle of side length x, containing two non-intersecting circles $k_1(O_1, a)$ and $k_2(O_2, b)$, then the inradius of triangle ABC is not less than a and we get

$$x \geq 2a\sqrt{3}.$$

Note also that the center O_1 (resp. O_2) of the circles k_1 (resp k_2) lies in the equilateral triangle $A_1B_1C_1$ (resp. $A_2B_2C_2$) whose sides are at a distance a (resp. b) apart the respective sides of triangle ABC (Fig.3). then it is easy to see that

$$|O_1O_2| \leq |A_2C_1|.$$

To compute $|A_2C_1|$ denote by O the common center of triangles $ABC, A_1B_1C_1$ and $A_2B_2C_2$. Since the distance from O to AC and A_1C_1 are equal to $\frac{x}{2\sqrt{3}}$ and $\frac{x}{2\sqrt{3}} - a$ respectively, it follows that

$$\frac{|A_1C_1|}{AC} = \frac{\frac{x}{2\sqrt{3}} - a}{\frac{x}{2\sqrt{3}}}$$

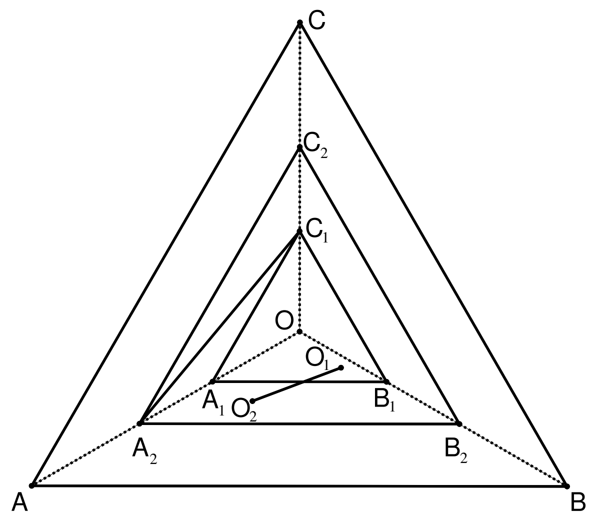


Fig. 3

Hence $|A_1C_1| = x - 2a\sqrt{3}$. Similarly $|A_2C_2| = x - 2b\sqrt{3}$. Since the height of the isosceles trapezoid $A_2A_1C_1C_2$ is $a - b$ we get that

$$|A_2C_1|^2 = (a-b)^2 + (x - \sqrt{3}(a+b))^2.$$

On the other hand $|O_1O_2|^2 \geq (a+b)^2$ (the circles k_1 and k_2 are non intersecting) and this together with (2) and (3) gives

$$x \geq \sqrt{3}(a+b) + 2\sqrt{ab}.$$

Set $t(a,b) = max\{2a\sqrt{3}, \sqrt{3}(a+b) + 2\sqrt{ab}\}$. Then (1) and (4) imply that $x \geq t(a,b)$. on the other hand it is easy to see that an equilateral triangle ABC of side length $t(a,b)$ contains two non-intersecting circles of radii a and b, respectively. Indeed, if $t(a,b) = 2a\sqrt{3}$ then $a \geq 3b$ and in this case k_1 is the incircle of triangle ABC and one can squeeze a circle k_2 of radius b into an angle of ABC. If $t(a,b) = \sqrt{3}(a+b) + 2\sqrt{ab}$ then $a \leq 3b$ and the circles $k_1(C_1, a)$ and $k_2(A_2, b)$ are non-intersecting and contained in triangle ABC.

Then the solution of Problem 1 is an equilateral triangle of side length

$$t(a,b) = \begin{cases} \sqrt{3}(a+b) + 2\sqrt{ab} & if \quad b \leq a \leq 3b \\ 2a\sqrt{3} & if \qquad a \geq 3b \end{cases}$$

Remark 1. the same argument as above show that the side length $s(a,b)$ of the smallest square, containing two non-intersecting circles of radii a and b, respectively is given by

$$s(a,b) = \begin{cases} (a+b)(1+\frac{1}{\sqrt{2}}) & if \quad b \leq a \leq b(3+2\sqrt{2}) \\ 2a & if \qquad a \geq b(3+2\sqrt{2}) \end{cases}$$

The Malfatti problem for an equilateral triangle

Now, we shall use problem 1 to solve the Malfatti problem for an equilateral triangle.

Theorem 1 the solution of the Malfatti problem for an equilateral triangle is given by the incircle and two circles which are inscribed in the angles of the triangle and are tangent to the incircle.

Proof. We shall assume that the side length of the equilateral triangle is 1. Suppose that it contains three non intersecting circles of radii a, b and c. Since we want to show that the solution to the Malfatti problem is given by the circles with radii $\frac{1}{2\sqrt{3}}, \frac{1}{6\sqrt{3}}$ and $\frac{1}{6\sqrt{3}}$, we have to prove that

$$a^2 + b^2 + c^2 \leq \frac{11}{108}$$

Assume that $a \geq b \geq c$. we shall consider two cases.

Case 1. let $a \geq 3b$. Since $a \leq \frac{1}{2\sqrt{3}}$, it follows that

$$a^2 + b^2 + c^2 \leq a^2 + 2b^2 \leq a^2 + \frac{2a^2}{9} \leq \frac{11}{108}$$

and the inequality (7) is proved, note that the inequality is attained only if $a = \frac{1}{2\sqrt{3}}, b = c = \frac{1}{6\sqrt{3}}$.

Case 2. Let $b \le a \le 3b$. Then it follows from Problem 1 that $\sqrt{3}(a+b) + 2\sqrt{ab} \le 1$.

Set $a = 3x^2 b$ where $x > 0$. Then the above inequalities are equivalent to $\frac{1}{\sqrt{3}} \le x \le 1$ and $b \le \frac{1}{\sqrt{3}(3x^2+2x+1)}$.

Hence

$$a^2 + b^2 + c^2 \le a^2 + 2b^2 = (9x^4 + 2)b^2 \le \frac{9x^4 + 2}{3(3x^2 + 2x + 1)^2}$$

and it is enough to prove that

$$\frac{9x^4 + 2}{(3x^2 + 2x + 1)^2} \le \frac{11}{36}$$

for $x \in [\frac{1}{\sqrt{3}}, 1]$. this inequality can be written as

$$(225x^3 + 93x^2 - 17x - 61)(x - 1) \le 0$$

which holds since $x - 1 \le 0$ and

$$\begin{aligned} 225x^3 + 93x^2 - 17x - 61 &= 51x\left(x^2 - \frac{1}{3}\right) + 174x^3 + 93x^2 - 61 \\ &\ge 174x^3 + 93x^2 - 61 \ge \frac{174}{3\sqrt{3}} + \frac{93}{3} - 61 > 0 \end{aligned}$$

Note that in this case the equality in (7) is attained only if $x = 1$, $b = c = \frac{1}{\sqrt{3}(3x^2+2x+1)}$ and we obtain again $a = \frac{1}{2\sqrt{3}}, b = c = \frac{1}{6\sqrt{3}}$.

Remark 2. Using similar arguments one can prove that the solution of the Malfatti problem for a square is given by the incircle and two circles which are inscribed on its angles and are tangent to the incircle.

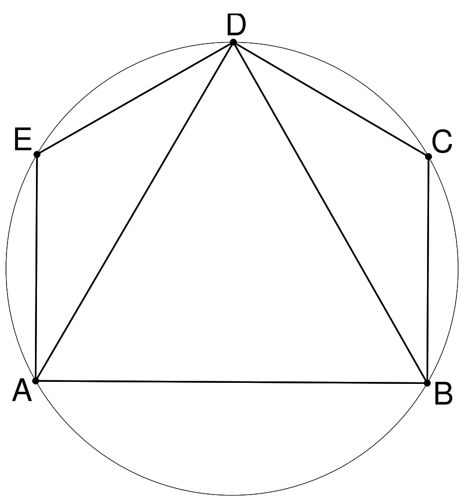
An open Malfatti type problem

The theorem of Zalgaller and Loss [4] shows that the solution of the Malfatti problem for a triangle is given according to a greedy algorithm: at any step we take a circle of greatest possible area. As noted in remark 2, the same holds for the Malfatti problem for a square. Of course, one can state many other Malfatti type problems and it is tempting to to conjecture that their solutions are also given according to the greedy algorithm. We next give an example of such a problem which shows that this is not always true. Consider the following problem:

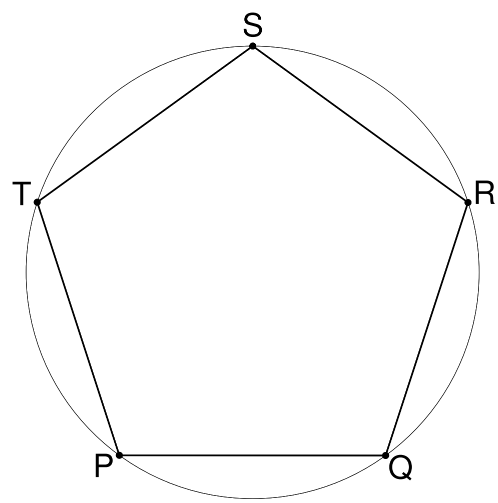
Cut three triangles from a given circle, so that the sum of their areas is maximized.

According to the greedy algorithm the solution of the problem would be an equilateral triangle ABD and two isosceles triangles BDC and ADE, inscribed in the circle (Fig.4(a)).

But this is not true since three isosceles triangles PQS, SQR and PST (Fig.4(b)), forming a regular pentagon inscribed in the circle, have a greater combined area. This follows from the well-known fact that among all pentagons inscribed on a circle the regular one has greatest area.



(a)



(b)

Fig. 4

it is easy to see that the solution of the same problem but for two triangles is given by two isosceles right triangles, forming a square. So it seems reasonable to conjecture that the solution of the above problem for n triangles is given by n triangles, forming a regular $(n+2)$-gon, inscribed in the circle.

References

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[3] G. Malfatti, Memoria sopra un problema sterotomico, Memorie di Matematica e di Fisica dalla Societa Italiana delle Scienze, No.10,1(1803),pp.235-244.

[4] V.A. Zalgaller and g.A. Loss, A solution of the Malfatti problem, Ukrainian Geometric Journal, 34(1991), pp. 14-33.