A Way to Prove the Inequality $R \geq 3r$

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Abstract

The radius R of the circumscribed sphere of a tetrahedron is greater than or equal to three times the radius r of the inscribed sphere of that tetrahedron. One can prove this by using homothety. In this paper we present another way to prove this inequality.

1 Introduction

Let κ denote a tetrahedron $A_1A_2A_3A_4$. The following theorem is a useful result in solid geometry.

Theorem 1. For every point M inside the tetrahedron κ,

$$V_1 \cdot \overrightarrow{MA_1} + V_2 \cdot \overrightarrow{MA_2} + V_3 \cdot \overrightarrow{MA_3} + V_4 \cdot \overrightarrow{MA_4} = 0, \tag{1}$$

where V_1, V_2, V_3, V_4 are the volumes of the tetrahedra

$$MA_2A_3A_4, MA_3A_4A_1, MA_4A_1A_2, MA_1A_2A_3,$$

respectively. For i$= 1, 2, 3, 4$; δ_i is the area of the face opposite to the vertex A_i. Let I denote the center of the incribed sphere of κ. Now, we will consider a corollary of Theorem 1.

Corollary 2. I is called the center of mass of the system of points A_1, A_2, A_3, A_4 with masses $\Delta_1, \Delta_2, \Delta_2, \Delta_2$ respectively, that is

$$\Delta_1 \cdot \overrightarrow{IA_1} + \Delta_2 \cdot \overrightarrow{IA_2} + \Delta_3 \cdot \overrightarrow{IA_3} + \delta_4 \cdot \overrightarrow{IA_4} = 0. \tag{2}$$

The proofs of Theorem 1 and Corollary 2 can be found in [1] and [2].

Denote by R and r the radius of the circumscribed sphere and the radius of inscribed sphere of κ, respectively.

Theorem 3.

$$R \geq 3r. \tag{3}$$

2 Two Propositions

For $i = 1, 2, 3, 4$, denote by h_i the length of the altitude of κ from vertex A_i.

Proposition 4. Let O be the center of the circumscribed sphere of κ. Then

$$OI^2 = R^2 - r^2 \sum_{1 \le i < j \le 4} \frac{A_i A_j^2}{h_i h_j}. \tag{4}$$

Proof. Using Theorem 1 for $M \equiv I$, we deduce

$$\overrightarrow{OI} = \frac{V_1 \cdot \overrightarrow{OA_1} + V_2 \cdot \overrightarrow{OA_2} + V_3 \cdot \overrightarrow{OA_3} + V_4 \cdot \overrightarrow{OA_4}}{V},$$

where V_1, V_2, V_3, V_4, and V are the volumes of the tetrahedra

$$IA_2A_3A_4, IA_3A_4A_1, IA_4A_1A_2, IA_1A_2A_3, \kappa,$$

respectively. Thus

$$OI^2 = \frac{1}{V^2}\left[(V_1 + V_2 + V_3 + V_4)^2 \cdot R^2 - \sum_{1 \le i < j \le 4} V_i V_j \overrightarrow{OA_i} \cdot \overrightarrow{OA_j}\right].$$

Note that $2\overrightarrow{OA_i} \cdot \overrightarrow{OA_j} = OA_i^2 + OA_j^2 - A_iA_j^2 = 2R^2 - A_iA_j^2$, so

$$OI^2 = \frac{1}{V^2}\left[(V_1 + V_2 + V_3 + V_4)^2 \cdot R^2 - \sum_{1 \le i < j \le 4} V_i V_j A_i A_j^2\right].$$

It is easy to see that $V_1 + V_2 + V_3 + V_4 = V$ and $\dfrac{V_i}{V} = \dfrac{r}{h_i}$, hence

$$OI^2 = R^2 - r^2 \sum_{1 \le i < j \le 4} \frac{A_i A_j^2}{h_i h_j}.$$

Proposition 4.

$$\sum_{1 \le i < j \le 4} \frac{A_i A_j^2}{h_i h_j} \ge 9. \tag{5}$$

Proof. Inequality (5) is equivalent to

$$\sum_{1 \le i < j \le 4} \Delta_i \Delta_j A_i A_j^2 \ge 9V^2. \tag{6}$$

Squaring both sides of (2) and observing that

$$2\overrightarrow{IA_i}\cdot\overrightarrow{IA_j} = IA_i^2 + IA_j^2 - A_iA_j^2,$$

we get

$$\sum_{1\le i<j\le 4} \Delta_i\Delta_j A_iA_j^2 = \sum_{i=1}^{4}\Delta_i^2 IA_i^2 + \sum_{1\le i<j\le 4}(IA_i^2 + IA_j^2).$$

Thus

$$\sum_{1\le i<j\le 4} \Delta_i\Delta_j A_iA_j^2 = \left(\sum_{i=1}^{4}\Delta_i IA_i\right)^2 + \sum_{1\le i<j\le 4}\Delta_i\Delta_j(IA_i - IA_j)^2.$$

It follows that

$$\sum_{1\le i<j\le 4} \Delta_i\Delta_j A_iA_j^2 \ge \left(\sum_{i=1}^{4}\Delta_i IA_i\right)^2.$$

It is not difficult to see that $IA_i \ge h_i - r = \dfrac{3V}{\Delta_i} - r$. Hence

$$\Delta_i IA_i \ge 3V - \Delta_i r = 3(V - V_i)$$

for all i. From this and $V_1 + V_2 + V_3 + V_4 = V$ we deduce that

$$\sum_{i=1}^{4}\Delta_i IA_i \ge 3V.$$

Therefore

$$\sum_{1\le i<j\le 4} \Delta_i\Delta_j A_iA_j^2 \ge \left(\sum_{i=1}^{4}\Delta_i IA_i\right)^2 \ge (3V)^2 = 9V^2.$$

and the proof is complete.

Corollary 6.

$$\max_{1\le i<j\le 4}\{A_iA_j\} \ge 2\sqrt{6}r. \tag{7}$$

Proof. Using the equality $\dfrac{1}{h_1} + \dfrac{1}{h_2} + \dfrac{1}{h_3} + \dfrac{1}{h_4} = \dfrac{1}{r}$ and the inequality

$$ab + ac + ad + bc + bd + cd \le \frac{3}{8}(a+b+c+d)^2$$

for all real numbers a, b, c, and d we have

$$\sum_{1\le i<j\le 4}\frac{1}{h_ih_j} \le \frac{3}{8}\left(\sum_{i=1}^{4}\frac{1}{h_i}\right) = \frac{3}{8r^2}.$$

Furthermore,

$$\sum_{1\le i<j\le 4} \frac{A_iA_j^2}{h_ih_j} \le \max_{1\le i<j\le 4}\{A_iA_j^2\} \cdot \sum_{1\le i<j\le 4} \frac{1}{h_ih_j}.$$

Hence

$$\sum_{1\le i<j\le 4} \frac{A_iA_j^2}{h_ih_j} \le \frac{3}{8r^2} \cdot \max_{1\le i<j\le 4}\{A_iA_j^2\}.$$

From this and Proposition 5, we obtain

$$\frac{3}{8r^2} \cdot \max_{1\le i<j\le 4}\{A_iA_j^2\} \ge 9,$$

which implies

$$\max_{1\le i<j\le 4}\{A_iA_j\} \ge 2\sqrt{6}r.$$

The equality occurs if and only if κ is a regular tetrahedron.

3 Proof of Theorem 3

From Proposition 4 and Proposition 5 it follows that

$$\left(\frac{R}{r}\right) \ge \sum_{1\le i<j\le 4} \frac{A_iA_j^2}{h_ih_j}.$$

Thus $R \ge 3r$.

4 A Corollary and a Result

Corollary 7.

$$\sum_{1\le i<j\le 4} \Delta_i\Delta_jA_iA_j \le \frac{\sqrt{6}}{4}\Delta^2R, \tag{8}$$

where $\Delta = \Delta_1 + \Delta_2 + \Delta_2 + \Delta_3.$, is the total area of the tetrahedron.

Proof. According to Cauchy-Schwarz inequality

$$\left(\sum_{1\le i<j\le 4} \frac{A_iA_j}{h_ih_j}\right)^2 \le \left(\sum_{1\le i<j\le 4} \frac{1}{h_ih_j}\right)\left(\sum_{1\le i<j\le 4} \frac{A_iA_j^2}{h_ih_j}\right).$$

Because $\displaystyle\sum_{1\le i<j\le 4} \frac{1}{h_ih_j} \le \frac{3}{8r^2}$ and $\displaystyle\sum_{1\le i<j\le 4} \frac{A_iA_j^2}{h_ih_j} \le \left(\frac{R}{r}\right)^2$, we deduce

$$\sum_{1\le i<j\le 4} \frac{A_iA_j^2}{h_ih_j} \le \sqrt{\frac{3}{8}}\frac{R}{r^2} = \frac{\sqrt{6}}{4}\frac{R}{r^2}. \tag{9}$$

Using $h_i = \dfrac{3V}{\Delta}, i = 1, 2, 3, 4$ and $3V = \Delta r$, (9) implies

$$\sum_{1\le i<j\le 4} \Delta_i \Delta_j A_i A_j \le \frac{\sqrt{6}}{4}\Delta^2 R.$$

The equality occurs if and only if κ is a regular tetrahedron.

Proposition 8. Let the incribed sphere of κ be tangent to the face opposite to vertex A_i at B_i $(i = 1, 2, 3, 4)$. Then,

$$\sum_{1\le i<j\le 4} \frac{B_i B_j^2}{h_i h_j} = 1.$$

Proof. Because $\dfrac{\overrightarrow{IB_i}}{IB_i}$ is a unit vector perpendicular to the face opposite to the vertex A_i and its direction is out of the tetrahedron, we have

$$\Delta_1 \cdot \frac{\overrightarrow{IB_1}}{IB_1} + \Delta_2 \cdot \frac{\overrightarrow{IB_2}}{IB_2} + \Delta_3 \cdot \frac{\overrightarrow{IB_3}}{IB_3} + \Delta_1 \cdot \frac{\overrightarrow{IB_i}}{4} IB_4 = \overrightarrow{0}.$$

Furthermore $IB_1 = IB_2 = IB_3 = IB_4 = r$. Consequently

$$\sum_{i=1}^{4} \Delta_i \cdot \overrightarrow{IB_i} = \overrightarrow{0}.$$

Squaring both sides of the above equality and noting that $\displaystyle\sum_{i=1}^{4} \frac{1}{h_i} = \frac{1}{r}$ and that

$$2\overrightarrow{IB_i} \cdot \overrightarrow{IB_j} = IB_i^2 + IB_j^2 - B_i B_j^2 = 2r^2 - B_i B_j^2,$$

we get

$$\left(\sum_{i=1}^{4} \frac{1}{h_i^2} + 2 \sum_{1\le i<j\le 4} \frac{1}{h_i h_j} \right) r^2 - \sum_{1\le i<j\le 4} \frac{B_i B_j^2}{h_i h_j} = 0.$$

Then

$$\sum_{1\le i<j\le 4} \frac{B_i B_j^2}{h_i h_j} = \left(\sum_{i=1}^{4} \frac{1}{h_i} \right)^2 r^2 = 1.$$

References

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[3] Kiran S. Kedlaya, 2006, *Geometry unbound*.

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