K_k versus $K_{k+1}\backslash\{e\}$

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Abstract. In this article we find the number of edges in a graph that ensures the existence of a $(k+1)$-clique without an edge as a subgraph.

A famous Turan's Theorem says that in the class of graphs G on n vertices with no k-clique we have

$$|E| \leq \frac{k-2}{k-1} \cdot \frac{n^2-r^2}{2} + \frac{(r-1)r}{2},$$

where $n \equiv r (\text{mod } k-1)$.

Let $f(S,n)$ be the greatest number of edges in a graph on n vertices with no particular structure S in it. From the above statement it follows that if $n \equiv r (\text{mod } k-1)$, then

$$f(K_k,n) = \frac{k-2}{k-1} \cdot \frac{n^2-r^2}{2} + \frac{(r-1)r}{2}.$$

Clearly, from here, the number $f(K_k,n)+1$ ensures the existence of a k-clique in the graph.

We would also like to introduce a complement of $f(S,n)$, a function $g(S,n)$ equal to the least number of edges we need to remove from a graph on n vertices to guarantee the nonexistence of a particular structure S in it. We have

$$g(K_k,n) = \frac{(n-1)n}{2} - f(K_k,n) = \frac{(n-r)(n+r-(k-1))}{2(k-1)},$$

where $n \equiv r (\text{mod } k-1)$.

We want to find the number of edges that ensures the existence of a $(k+1)$-clique without an edge in the graph. We claim that this number is the same as the number that ensures the exitence of a k-clique, that is

$$f(K_k,n) = f(K_{k+1}\backslash\{e\},n), \quad \text{for } n \geq k+1,\ k \geq 3.$$

Let us start with a simple observation. It is clear that

$$f(K_k,n) \leq f(K_{k+1}\backslash\{e\},n),$$

because $K_k \subset K_{k+1}\backslash\{e\}$. Thus $g(K_{k+1}\backslash\{e\},n) \leq g(K_k,n)$.

It is also clear that $g(S,n-1) < g(S,n)$. To prove this consider a graph on n vertices with $g(S,n)$ edges removed. The graph has no S in it. Delete one vertex that had an edge removed. We are left with $n-1$ vertices and at most $g(S,n)-1$ edges removed, with no S in the graph.

We use mathematical induction to prove our statement. The first and the most tedious thing we need to do is to prove the statement for $n \in \{k+1, k+2, \ldots, 2k\}$.

1^{st} case: $n = k - 1 + l$, $2 \leq l \leq k-1$. Using the formula above we have $g(K_k, n) = l$. Using induction on l we prove that $g(K_{k+1}\backslash\{e\}, n) = g(K_k, n) = l$. The base case $l = 2$: $g(K_{k+1}\backslash\{e\}, k+1) = g(K_k, k+1) = 2$ is true. Assume the statement holds for $m = 2, \ldots, l-1$. From the inequalities above we have

$$l - 1 = g(K_{k+1}\backslash\{e\}, n-1) < g(K_{k+1}\backslash\{e\}, n) \leq g(K_k, n) = l.$$

Thus $g(K_{k+1}\backslash\{e\}, n) = g(K_k, n) = l$, yielding $f(K_{k+1}\backslash\{e\}, n) = f(K_k, n)$.

2^{nd} case: $n = 2k-1$. We have $g(K_{k+1}\backslash\{e\}, 2k-1) \geq k$. Thus if we remove these edges there is a vertex v_0 which has at least two adjacent edges removed. Then $G\backslash\{v_0\}$ that does not contain $K_{k+1}\backslash\{e\}$ has at least $g(K_{k+1}\backslash\{e\}, 2k-2) = k-1$ edges removed. It follows that $g(K_{k+1}\backslash\{e\}, 2k-1) \geq k+1$, hence

$$k + 1 = g(K_k, 2k-1) \geq g(K_{k+1}\backslash\{e\}, 2k-1) \geq k - 1 + 2 = k + 1.$$

Thus $g(K_k, 2k-1) = g(K_{k+1}\backslash\{e\}, 2k-1)$ and we are done.

3^{rd} case: $n = 2k$. Then $g(K_{k+1}\backslash\{e\}, 2k-1) \geq k+2$. Using the same idea we get $g(K_{k+1}\backslash\{e\}, 2k-1) \geq k+3$. Therefore

$$k + 3 = g(K_k, 2k) \geq g(K_{k+1}\backslash\{e\}, 2k) \geq k + 1 + 2 = k + 3,$$

yielding $g(K_k, 2k) = g(K_{k+1}\backslash\{e\}, 2k)$ and the statement is proved.

Now, with the base cases for $n \in \{k+1, k+2, \ldots, 2k\}$ verified, we continue our mathematical induction. Assume the result holds for all positive integers less than n, $n \geq 2k$. We prove the statement for n.

Assume to the contrary that there is no subgraph $K_{k+1}\backslash\{e\}$ in the graph G on n vertices with $f(K_k, n) + 1$ edges. From Turan's Theorem, there is a subgraph K_k in G. Consider $G\backslash K_k$ that has $n-k$ vertices. Each of the vertices is connected to at most $k-2$ vertices of K_k. Also using the induction hypothesis, $G\backslash K_k$ has at most $f(K_k, n-k)$ edges. Thus

$$f(K_{k+1}\backslash\{e\}, n) \leq \frac{(k-1)k}{2} + (n-k)(k-2) + f(K_k, n-k).$$

We want to prove the following inequality

$$\frac{(k-1)k}{2} + (n-k)(k-2) + f(K_k, n-k) \leq f(K_k, n),$$

which will give us the desired contradiction.

1^{st} case: $n = (k-1)t$. Then $n - k = (k-1)(t-2) + k - 2$. We have $f(K_k, n) = \dfrac{(k-2)(k-1)t^2}{2}$ and $f(K_k, n-k) = \dfrac{(k-2)}{2} \cdot ((k-1)(t-2)^2 + k - 3)$. Then

$$f(K_k, n) - f(K_k, n-k) = 2(k-2)(k-1)(t-1) - \frac{(k-3)(k-2)}{2}.$$

Our inequality is equivalent to

$$\frac{(k-1)k}{2} + (k-2)(k-1)(t-1) - (k-2) \le 2(k-2)(k-1)(t-1) - \frac{(k-3)(k-2)}{2},$$

or

$$k^2 - 4k + 5 \le (k-2)(k-1)(t-1).$$

It clear that this inequality is true for all $k \ge 3$.

2^{nd} case: $n = (k-1)t + r$, $1 \le r \le k-2$. Then $n - k = (k-1)(t-1) + r - 1$.
We have

$$f(K_k, n) = \frac{k-2}{k-1} \cdot \frac{n^2 - r^2}{2} + \frac{(r-1)r}{2} = \frac{(k-2)t}{2}((k-1)t + 2r) + \frac{(r-1)r}{2}$$

and

$$\begin{aligned} f(K_k, n-k) &= \frac{k-2}{k-1} \cdot \frac{(n-k)^2 - (r-1)^2}{2} + \frac{(r-2)(r-1)}{2} \\ &= \frac{(k-2)(t-1)}{2}((k-1)(t-1) + 2r - 2) + \frac{(r-2)(r-1)}{2} \\ &= \frac{(k-2)(t-1)}{2}((k-1)t + 2r) - \frac{(k-2)(k+1)(t-1)}{2} + \frac{(r-2)(r-1)}{2}. \end{aligned}$$

It follows that

$$\begin{aligned} f(K_k, n) - f(K_k, n-k) &= \frac{(k-2)}{2}((k-1)t + 2r) + \frac{(k-2)(k+1)(t-1)}{2} + (r-1) \\ &= (k-2)k(t-1) + \frac{(k-2)(k-1+2r)}{2} + (r-1). \end{aligned}$$

Thus our inequality is equivalent to

$$\frac{(k-1)k}{2} + (k-2)((k-1)(t-1) + r - 1) \le (k-2)k(t-1) + \frac{(k-2)(k-1+2r)}{2} + r - 1,$$

or

$$(k-1) + (k-3)(r-1) \le (k-2)(t-1) + r(k-2),$$

or

$$0 \le (k-2)(t-1) + r - 2,$$

and we are done.

Thus we proved a stronger result than Turan's Theorem:
If the number of edges in the graph ensures the existence of a k-clique, it will ensure the existence of a $(k+1)$-clique without an edge.

References

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