Junior problems

J103. The numbers $1, 2, \ldots, 9$ are randomly arranged on a circle. Prove that there are three adjacent numbers whose sum is at least 16.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J104. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a^2+b^2}{a^2+b^2+1}+\frac{b^2+c^2}{b^2+c^2+1}+\frac{c^2+a^2}{c^2+a^2+1} \geq \frac{a+b}{a^2+b^2+1}+\frac{b+c}{b^2+c^2+1}+\frac{c+a}{c^2+a^2+1}.$$

Proposed by Jingjun Han, Shanghai, China

J105. Let $A_1A_2\ldots A_n$ be a polygon that is inscribed in a circle $C(O,R)$ and at the same time circumscribed about a circle $\omega(I,r)$. The points of tangency of $A_1A_2\ldots A_n$ with ω form another polygon $B_1B_2\ldots B_n$. Prove that

$$\frac{P(A_1A_2\ldots A_n)}{P(B_1B_2\ldots B_n)} \leq \frac{R}{r},$$

where $P(S)$ stands for the perimeter of figure S.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J106. Prove that among any four positive real numbers there are two, say a and b, such that $ab + 1 \geq \frac{1}{\sqrt{3}}|a-b|$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J107. Find all quadruples (a, b, c, d) of positive integers such that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)\left(1+\frac{1}{d}\right) = 5.$$

Proposed by Shamil Asgarli, Burnaby, Canada

J108. Let n be a positive integer. Prove that the number of ordered pairs (a, b) of relatively prime positive divisors of n is equal to the number of divisors of n^2.

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

Senior problems

S103. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$x_1 + x_2 + \cdots + x_n + \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \geq (n+1) \sqrt[n]{x_1 x_2 \cdots x_n}.$$

Proposed by Nica Cristina-Paula and Nica Nicolae, Romania

S104. A set of four points in the plane is said to be "nice" if one can draw four circles centered at these points such that each circle is externally tangent to the other three. Given a triangle ABC with orthocenter H, incenter I, and excenters I_A, I_B, I_C, prove that $\{A, B, C, H\}$ and $\{I, I_a, I_b, I_c\}$ are nice if and only if triangle ABC is equilateral.

Proposed by Daniel Lasaosa, Universidad Publica de Navarra, Spain

S105. Let P be a point in the interior of a triangle ABC and let $d_a \geq d_b \geq d_c$ be distances from P to the triangle's sides. Prove that

$$\max(AP, BP, CP) \geq \sqrt{d_a^2 + d_b^2 + d_b d_c + d_c^2}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S106. Eight kids play two different games, A and B. At the beginning, they equally prefer the games. Each day starts with a random distribution of the kids in two groups of size 3 and 5. Every group plays the game preferred by the majority. However, each time a kid plays a game, he or she enjoys it so much, that it becomes his or her favorite game. Find the expected number of days after which all the kids will prefer the same game.

Proposed by Daniel Lasaosa, Universidad Publica de Navarra, Spain and Ivan Borsenco, MIT, USA

S107. Prove that the number of sets of integers of range n that also contain n is equal to the number of triangulations of a regular $(n+3)$-gon in which every triangle of the triangulation contains at least one side of the polygon. (Range of a set is the difference between the greatest and the least element in the set.)

Proposed by Zoran Sunic, Texas A&M University, USA

S108. In triangle ABC let D, E, F be the feet of the altitudes from vertices A, B, C. Denote by P and Q the feet of the perpendiculars from D onto AB and AC, respectively. Let $R = BE \cap DP$, $S = CF \cap DQ$, $M = BQ \cap CP$ and $N = RQ \cap PS$. Prove that M, N and H are collinear, where H is the orthocenter of triangle ABC.

Proposed by Gabriel Alexander Chicas Reyes, Tokyo, Japan

Undergraduate problems

U103. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n \leq n$. Prove that

$$a_1^{\frac{1}{a_1}} a_2^{\frac{1}{a_2}} \cdots a_n^{\frac{1}{a_n}} \leq 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U104. Let x_0 be a fixed real number and let $f : \mathbb{R} \to \mathbb{R}$ be a function such that f is a derivative on the intervals $(-\infty, x_0)$, (x_0, ∞) and continuous at x_0. Prove that f is a derivative on $\mathbb{R}$.

Proposed by Mihai Piticari, "Dragos Voda" National College, Romania

U105. Find $\min \left(\dfrac{\text{Im } z^5}{\text{Im}^5 z} \right)$ over all z in $\mathbb{C} \backslash \mathbb{R}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U106. Let x be a positive real number. Prove that

$$x^x - 1 \geq e^{x-1}(x - 1).$$

Proposed by Vasile Cartoaje, University of Ploiesti, Romania

U107. Let $f : [0, \infty) \to [0, \infty)$ be a continuous function for which there is a positive integer a such that $f(f(x)) = x^a$ for all x. Prove that

$$\int_0^1 (f(x))^2 \, dx \geq \frac{2a - 1}{a^2 + 6a - 3}.$$

Proposed by Mihai Piticari, "Dragos Voda" National College, Romania

U108. Find all $n \geq 3$ such that there is a surjective homomorphism $\phi : S_n \to S_{n-1}$, where S_n is the symmetric group on n elements.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Olympiad problems

O103. Let a, b, c, be positive real numbers such that $abc = 1$. Prove that

$$\sqrt[3]{(1+a)(1+b)(1+c)} \geq \sqrt[4]{4(1+a+b+c)}.$$

Proposed by Pham Huu Duc, Ballajura, Australia

O104. In a convex quadrilateral $ABCD$ let K, L, M, N be the midpoints of sides AB, BC, CD, DA, respectively. Line KM meets diagonals AC and BD at P and Q, respectively, and line LN meets diagonals AC and BD at R and S, respectively. Prove that if $AP \cdot PC = BQ \cdot QD$, then $AR \cdot RC = BS \cdot SD$.

Proposed by Nairi Sedrakian, Yerevan, Armenia

O105. Let $P(t)$ be a polynomial with integer coefficients such that $P(1) = P(-1)$. Prove that there is a polynomial $Q(x, y)$, with integer coefficients such that $P(t) = Q(t^2 - 1, t(t^2 - 1))$.

Proposed by Mircea Becheanu and Tiberiu Dumitrescu,
University of Bucharest, Romania

O106. A polynomial with integer coefficients is called *"good"* if it can be represented as a sum of cubes of several polynomials in x with integer coefficients. For example, $9x^3 - 3x^2 + 3x + 7 = (x-1)^3 + (2x)^3 + 2^3$ is good.

a) Is $3x^7 + 3x$ good?

b) Is $3x^{2008} + 3x^7 + 3x$ good?

Proposed by Nairi Sedrakian, Yerevan, Armenia

O107. Let p_1, p_2, p_3 be distinct primes and let n be a positive integer. Find the number of functions $f : \{1, 2, \ldots, 2n\} \to \{p_1, p_2, p_3\}$ for which $f(1)f(2)\cdots f(2n)$ is a perfect square.

Proposed by Dorin Andrica, "Babes-Bolyai" University and Mihai Piticari,
"Dragos Voda" National College, Romania

O108. Prove that the set of positive integers that cannot be written as a sum of four nonzero squares has density zero.

Proposed by Iurie Boreico, Harvard University, USA