Problems for Mathematical Reflections 1(2007)

Juniors

J37. Let $a_1, a_2 ..., a_{2n+1}$ be distinct positive integers not exceeding $3n + 1$. Prove that among them there are two such that

$$a_i - a_j = m, \text{ for all } m \in \{1, 2, ..., n\}.$$

Proposed by Ivan Borsenco, University of Texas at Dallas

J38. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{a^2 + bc}{(a+b)(a+c)} + \frac{b^2 + ca}{(b+a)(b+c)} + \frac{c^2 + ab}{(c+a)(c+b)}.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

J39. Evaluate the product

$$(\sqrt{3} + \tan 1^\circ)(\sqrt{3} + \tan 2^\circ) \ldots (\sqrt{3} + \tan 29^\circ).$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

J40. A 5×6 rectangle is cut into eight rectangles with integral dimensions and whose sides are parallel to the ones of the initial rectangle. Prove that among them there are two congruent rectangles.

Proposed by Ivan Borsenco, University of Texas at Dallas

J41. Let a, b, c be positive real numbers such that $a + b + c + 1 = 4abc$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3 \geq \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}}.$$

Proposed by Daniel Campos Salas, Costa Rica

J42. Find all triples (m, n, p) of positive integers such that $m+n+p = 2008$ and the system of equations

$$\frac{x}{y} + \frac{y}{x} = m, \ \frac{y}{z} + \frac{z}{y} = n, \ \frac{z}{x} + \frac{x}{z} = p$$

has at least one solution in nonzero real numbers.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Seniors

S37. Let x, y, z be real numbers such that

$$\cos x + \cos y + \cos z = 0,$$

and

$$\cos 3x + \cos 3y + \cos 3z = 0.$$

Prove that

$$\cos 2x \cdot \cos 2y \cdot \cos 2z \leq 0.$$

Proposed by Bogdan Enescu, B.P. Hasdeu National College, Romania

S38. Prove that for each positive integer n, there is a positive integer m such that

$$(1+\sqrt{2})^n = \sqrt{m} + \sqrt{m+1}.$$

Proposed by Jean-Charles Mathieux, Dakar University, Sénégal

S39. Let a be a positive integer and let

$$A = \left\{\sqrt{a}, \sqrt[3]{a}, \sqrt[4]{a}, \ldots\right\}.$$

Prove that for every positive integer n the set A contains n consecutive terms of a geometric sequence, but it does not contain a geometric sequence with infinitely many terms.

Proposed by Bogdan Enescu, B.P. Hasdeu National College, Romania

S40. Let f and g be irreducible polynomials with rational coefficients and let a and b be complex numbers such that $f(a) = g(b) = 0$. Prove that if $a+b$ is a rational number, then f and g have the same degree.

Proposed by Bogdan Enescu, B.P. Hasdeu National College, Romania

S41. Prove that for any positive real numbers a, b and c,

$$\sqrt{\frac{b+c}{a}} + \sqrt{\frac{c+a}{b}} + \sqrt{\frac{a+b}{c}} \geq \sqrt{6 \cdot \frac{a+b+c}{\sqrt[3]{abc}}}.$$

Proposed by Pham Huu Duc, Ballajura, Australia

S42. Prove that in any triangle there exist a pair (M_1, M_2) of isogonal conjugates such that $OM_1 \cdot OM_2 > OI^2$, where O and I are the circumcenter and the incenter, respectively.

Proposed by Ivan Borsenco, University of Texas at Dallas

Undergraduate

U37. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with f' continuous, such that

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx = 1.$$

Prove that there exist $c \in (0,1)$ for which $f'(c) = 6$.

Proposed by Cezar Lupu, University of Bucharest, Romania

U38. Let $n > 1$ be an odd positive integer and let x be a real number. Set

$$t = \sum_{k=1}^{n-1} \arctan\left(\frac{\cos\left(\frac{2\pi k}{n}\right) - x}{\sin\left(\frac{2\pi k}{n}\right)}\right).$$

Compute the value of $\tan t$ in terms of n and x.

Proposed by Alex Anderson, New Trier High School, Winnetka, IL

U39. Prove that

$$\lim_{n\to\infty} \frac{1}{\ln n} \sum_{k=1}^{n} \frac{\{\ln k\}}{k} = \frac{1}{2},$$

where $\{x\}$ denotes the fractional part of x.

Proposed by Cristinel Mortici, Valahia University of Targoviste, Romania

U40. Show that $\mathrm{GL}_4(\mathbb{Q})$ has no element of order 7.

Proposed by Jean-Charles Mathieux, Dakar University, Sénégal

U41. Let k be a positive integer and let α be a real number greater than 1. A number is called a k-prime if it is the product of at most k (not necessarily distinct) primes. Let $p(r)$ be the probability that a random integer x contains r k-prime divisors $d_1 < d_2 < \ldots < d_r$ such that $d_r < \alpha d_1$. Prove that $\lim_{r\to\infty} p_r = 0$.

Proposed by Iurie Boreico, Moldova

U42. Let $A_1, ..., A_n, B_1, ..., B_n$ be points in a plane such that $B_iA_1 \cdot B_iA_2 \cdot ... \cdot B_iA_n \leq A_jB_i$ for all i and j. Prove that

$$\prod_{1\leq i<j\leq n} A_iA_j \cdot B_iB_j \leq n^{\frac{n}{2}},$$

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Olympiad

O37. Let a, b, c, d be nonnegative numbers such that $a^2+b^2+c^2+d^2=4$. Prove that

$$\sqrt{2}(4-ab-bc-cd-da) \geq (\sqrt{2}+1)(4-a-b-c-d).$$

Proposed by Vasile Cartoaje, University of Ploiesti, Romania

O38. Let w_1 be a circle smaller than and internally tangent to the circle w_2 at T. A tangent to w_1 (at T'), intersects w_2 at A and B. If A, T', and B are fixed, what is the locus of T.

Proposed by Alex Anderson, New Trier High School, Winnetka, IL

O39. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2+2bc}}+\frac{b}{\sqrt{b^2+2ca}}+\frac{c}{\sqrt{c^2+2ab}} \leq \frac{a+b+c}{\sqrt{ab+bc+ca}}.$$

Proposed by Ho Phu Thai, Da Nang, Vietnam

O40. In the AwesomeMath summer camp there are 80 boys and 40 girls. It has been noticed that any two boys have an even number of acquaintances among the girls and exactly 19 boys know an odd number of girls. Prove that one can choose a group of 50 boys such that any girl is acquainted to an even number of boys from this group.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

O41. Prove the following identity:

$$\sum_{0 \leq a \leq \sqrt{n}} \left[\sqrt{n-a^2}\right] = \sum_{0 \leq i \leq \frac{n}{2}} (-1)^i \left[\frac{n}{2i+1}\right].$$

Proposed by Ashay Burungale, India

O42. Let $a_1, a_2, ..., a_5$ be positive real numbers such that

$$a_1a_2...a_5 = a_1(1+a_2)+a_2(1+a_3)+...+a_5(1+a_1)+2.$$

Find the minimal value of $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{1}{a_5}$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris