On the area of a pedal triangle

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Geometry has been always the area of mathematics that attracted problem solvers with its exactness and intriguing results. The article presents one of such beautiful results - the Euler's Theorem for the pedal triangle and its applications. We start with the proof of this theorem and then we discuss Olympiad problems.

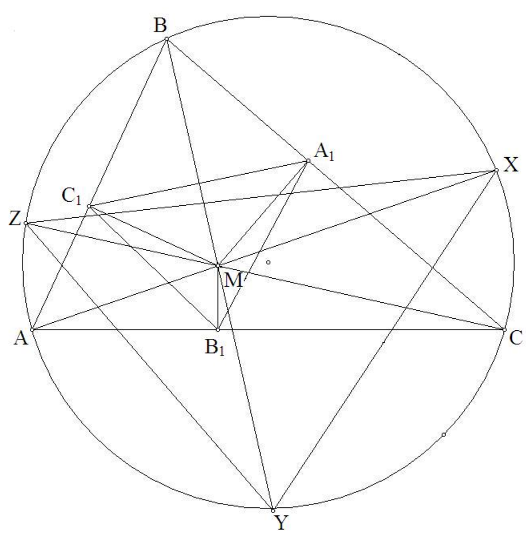
Theorem 1. Let $C(O, R)$ be the circumcircle of the triangle ABC. Consider a point M in the plane of the triangle. Denote by A_1, B_1, C_1 the projections of M on triangle's sides. The following relation holds

$$\frac{S_{A_1B_1C_1}}{S_{ABC}} = \frac{|R^2 - OM^2|}{4R^2}.$$

Proof: First of all note that quadrilaterals AB_1MC_1, BC_1MA_1, CA_1MB_1 are cyclic. Applying the extended Law of Cosines in triangle AB_1C_1 we get $B_1C_1 = AM\sin\alpha$. Analogously $A_1C_1 = BM\sin\beta$ and $B_1C_1 = CM\sin\gamma$. It follows that

$$\frac{B_1C_1}{BC} = \frac{AM}{2R},\ \frac{A_1C_1}{AC} = \frac{BM}{2R},\ \frac{A_1B_1}{BC} = \frac{CM}{2R}.$$

Suppose AM, BM, CM intersect the circumcircle $C(O, R)$ at points X, Y, Z.



Angle chasing yields

$$\angle A_1B_1C_1 = \angle A_1B_1M + \angle MB_1C_1 = \angle A_1CM + \angle MAC_1 = \angle ZYB + \angle BYX = \angle ZYX.$$

Similarly, $\angle B_1C_1A_1 = \angle YZX$ and $\angle B_1A_1C_1 = \angle YXZ$. Thus triangles $A_1B_1C_1$ and XYZ are similar and

$$\frac{A_1B_1}{XY} = \frac{R_{A_1B_1C_1}}{R}.$$

Because triangles MAB and MYX are also similar we have

$$\frac{XY}{AB} = \frac{MX}{MB}.$$

Combining the above results we obtain

$$\frac{S_{A_1B_1C_1}}{S_{ABC}} = \frac{R}{R_{A_1B_1C_1}} \cdot \frac{A_1B_1 \cdot B_1C_1 \cdot A_1C_1}{AB \cdot BC \cdot AC} = \frac{MX}{MB} \cdot \frac{MA}{2R} \cdot \frac{MB}{2R} = \frac{MA \cdot MX}{4R^2} = \frac{|R^2 - OM^2|}{4R^2}.$$

As we can see, the proof does not depend on the position of M (inside or outside the circle).

Corollary 1. If M lies on the circle, the projections of M onto triangle's sides are collinear. This fact is known as Simson's Theorem.

One more theorem we want to present without proof is the famous Lagrange Theorem.

Theorem 2. Let M be a point in the plane of triangle ABC with barycentric coordinates (u, v, w). For any point P in the plane of ABC the following relation holds

$$u \cdot PA^2 + v \cdot PB^2 + w \cdot PC^2 = (u + v + w)PM^2 + \frac{vwa^2 + uwb^2 + uvc^2}{u + v + w}.$$

The proof can be found after applying Stewart's Theorem a few times. The corollary of this theorem in which we are interested is the case when P coincides with the circumcenter O. We get

$$R^2 - OM^2 = \frac{vwa^2 + uwb^2 + uvc^2}{(u + v + w)^2}.$$

From this fact and the Euler's Theorem for pedal triangle we obtain the next result:

Theorem 3. Let M be a point in the plane of triangle ABC with barycentric coordinates (u, v, w). Denote by A_1, B_1, C_1 the projections of M onto triangle's sides. Then

$$\frac{S_{A_1B_1C_1}}{S_{ABC}} = \frac{vwa^2 + uwb^2 + uvc^2}{4R^2(u+v+w)^2}.$$

From the above results we can see that Theorem 1 and Theorem 3 give us insight on the area of pedal triangles. The Euler's Theorem for the area of a pedal triangle is a useful tool in solving geometry problems. The first one application we present is about Brocard points.

Let us introduce the definition of a Brocard point. In triangle ABC the circle that passes through A and is tangent to BC at B, circle that passes through B and is tangent to AC at C and the circle that passes through C and is tangent to AB at A are concurrent. The point of their concurrency is called Brocard point. In general, we have two Brocard points, the second one being obtained by reversing clockwise or counterclockwise the tangency of circles.

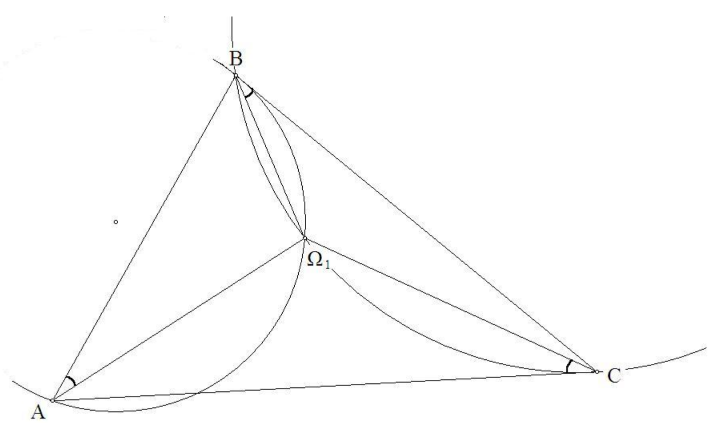
Problem 1. Let Ω_1 and Ω_2 be the two Brocard points of triangle ABC. Prove that $O\Omega_1 = O\Omega_2$, where O is the circumcenter of ABC.

Solution: From the definition of the Brocard points we see that

$\angle\Omega_1 AB = \Omega_1 BC = \Omega_1 CA = w_1$, and similarly $\angle\Omega_2 BA = \Omega_2 AC = \Omega_2 CB = w_2$. Let us prove that $w_1 = w_2$.

Observe that $\dfrac{S_{B\Omega_1 C}}{S_{ABC}} = \dfrac{B\Omega_1 \cdot BC \sin w_1}{AB \cdot BC \sin\beta} = \dfrac{\sin^2 w_1}{\sin^2\beta}$ and, analogously,

$$\frac{S_{C\Omega_1 A}}{S_{ABC}} = \frac{\sin^2 w_1}{\sin^2\gamma}, \quad \frac{S_{A\Omega_1 B}}{S_{ABC}} = \frac{\sin^2 w_1}{\sin^2\alpha}.$$



Summing up the areas we obtain

$$1 = \frac{S_{B\Omega_1 C} + S_{C\Omega_1 A} + S_{A\Omega_1 B}}{S_{ABC}} = \frac{\sin w_1^2}{\sin^2 \alpha} + \frac{\sin w_1^2}{\sin^2 \beta} + \frac{\sin w_1^2}{\sin^2 \gamma} \quad \text{or}$$

$$\frac{1}{\sin w_1^2} = \frac{1}{\sin^2 \alpha} + \frac{1}{\sin^2 \beta} + \frac{1}{\sin^2 \gamma}.$$

The same expression we get summing the areas for Ω_2:

$$\frac{1}{\sin w_2^2} = \frac{1}{\sin^2 \alpha} + \frac{1}{\sin^2 \beta} + \frac{1}{\sin^2 \gamma}$$

From two equalities we conclude that $w_1 = w_2 = w$.

The idea for the solution comes from Euler' Theorem for pedal triangles. Observe that Ω_1 and Ω_2 always lie inside triangle ABC, because every circle lies in that half of the plane that contains triangle ABC. It follows that the point of their intersection Ω_1 or Ω_2 lies inside the triangle. In order to prove that $O\Omega_1 = O\Omega_2$, we prove that their pedal areas are equal. If so, then we have $R^2 - O\Omega_1^2 = R^2 - O\Omega_2^2$ and therefore $O\Omega_1 = O\Omega_2$.

Denote by A_1, B_1, C_1 the projections from Ω_1 onto BC, AC, AB, respectively. Then using the extended Law of Sines we get $A_1C_1 = B\Omega_1 \sin b$, because $B\Omega_1$ is diameter in the cyclic quadrilateral $BA_1\Omega_1C_1$. Also from the Law of Sines in triangle $AB\Omega_1$ we have

$$\frac{B\Omega_1}{\sin w} = \frac{c}{\sin b}.$$

It follows that

$$A_1C_1 = B\Omega_1 \sin b = c \sin w.$$

Similarly we obtain $B_1C_1 = b \sin w$ and $A_1B_1 = a \sin w$. It is not difficult to see that triangle $A_1B_1C_1$ is similar to ABC with ratio of similarity $\sin w$. From the fact $w_1 = w_2 = w$ we conclude that pedal triangles of Ω_1 and Ω_2 have the same area. Thus $O\Omega_1 = O\Omega_2$, and we are done.

Remark: The intersection of symmedians in the triangle is denoted by K and is called Lemoine point. One can prove that $K\Omega_1 = K\Omega_2$. Moreover O, Ω_1, K, Ω_2 lie on a circle with diameter OK called Brocard circle.

We continue with an interesting approach to another classical problem.

Problem 2. Consider a triangle ABC and denote by O, I, H its circumcenter, incenter, and orthocenter, respectively. Prove that

$$OI \leq OH.$$

Solution: Many solvers will start to play with formulas for OI and OH. We choose another way, we consider the pedal areas for I and H. Clearly, the incenter is always inside the triangle ABC and if $OH \geq R$, then we are done. Therefore assume that $OH < R$. It follows that in this case we have an acute-angled triangle. In order to prove that $OI \leq OH$, we can use Euler's Theorem for pedal triangles and prove that the area of pedal triangle I, S_I is greater than the area of the pedal triangle of H, S_H. This is because if $S_I \geq S_H$ then $R^2 - OI^2 \geq R^2 - OH^2$ and therefore $OH \geq OI$.

Let us find both areas. Denote by A_1, B_1, C_1 the projections of H and A_2, B_2, C_2 the projections of I onto BC, CA, AB, respectively. Recalling the fact that A_1, B_1, C_1 lie on the Euler circle of radius $R/2$ we get

$$S_{A_1B_1C_1} = \frac{A_1B_1 \cdot B_1C_1 \cdot A_1C_1}{4 \cdot R/2}.$$

Using the Law of Sines it is not difficult to see that $B_1C_1 = a\cos\alpha, A_1C_1 = b\cos\beta, A_1B_1 = c\cos\gamma$ and therefore

$$S_{A_1B_1C_1} = \frac{abc \cdot \cos\alpha\cos\beta\cos\gamma}{2R} = 2S\cos\alpha\cos\beta\cos\gamma.$$

To calculate the area of triangle $A_2B_2C_2$, observe that

$$S_{A_2B_2C_2} = S_{A_2IB_2} + S_{A_2IC_2} + S_{B_2IC_2} = \frac{1}{2}r^2(\sin\alpha + \sin\beta + \sin\gamma) = \frac{r^2(a+b+c)}{4R}.$$

Recalling the formula $\dfrac{r}{4R} = \sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}$ we obtain

$$S_{A_2B_2C_2} = 2S \cdot \frac{r}{4R} = 2S\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.$$

It follows that it is enough to prove that

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \geq \cos\alpha\cos\beta\cos\gamma.$$

Now we use the classical Jensen Inequality. Consider the concave function $f : (0, \frac{\pi}{2}) \to R, f(x) = \ln\cos x$. Then

$$f(\frac{a+b}{2}) + f(\frac{b+c}{2}) + f(\frac{a+c}{2}) \geq f(a) + f(b) + f(c).$$

Thus $\ln\left(\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}\right) \geq \ln(\cos\alpha\cos\beta\cos\gamma)$, and the problem is solved.

The next two examples feature Olympiad problems where Euler's Theorem for pedal triangle is probably the shortest way to solve them.

Problem 3. (*Balkan Mathematical Olympiad*)

Let ABC be an acute triangle. Denote by A_1, B_1, C_1 the projections of centroid G onto triangle's sides. Prove that

$$\frac{2}{9} \leq \frac{S_{A_1B_1C_1}}{S_{ABC}} \leq \frac{1}{4}.$$

Solution: As we will see this problem requires a direct application of Euler's Theorem for pedal triangles. Applying it we have to prove

$$\frac{2}{9} \leq \frac{R^2 - OG^2}{4R^2} \leq \frac{1}{4}.$$

The right-hand side immediately follows. To prove the left-hand side recall the well known formula

$$9OG^2 = OH^2 = R^2(1 - 8\cos\alpha\cos\beta\cos\gamma).$$

Because triangle ABC is acute-angled we have $\cos\alpha, \cos\beta, \cos\gamma \geq 0$, hence $8\cos\alpha\cos\beta\cos\gamma \geq 0$, and thus $9OG^2 \leq R^2$ or $OG^2 \leq R^2/9$.

The conclusion now follows.

Problem 4. (*Mathematical Reflections, proposed by Ivan Borsenco*)

For every point M inside triangle ABC we define a triplet (d_1, d_2, d_3), where d_1, d_2, d_3 are distances from M to the sides BC, AC, AB. Prove that the set of points M, satisfing condition $d_1 \cdot d_2 \cdot d_3 \geq r^3$, where r is the incircle's radius, lies inside the circle with center O and radius OI.

Solution: Let A_1, B_1, C_1 be projections from M onto sides BC, AC, AB. Consider pedal triangle $A_1B_1C_1$ for point M, we have $\angle B_1MC_1 = 180 - \alpha$, $\angle A_1MC_1 = 180 - \beta$, $\angle A_1MB_1 = 180 - \gamma$. It follows that

$2 \cdot Area_{A_1B_1C_1} = 2S_1 = d_2 \cdot d_3 \cdot sin\alpha + d_1 \cdot d_3 \cdot sin\beta + d_1 \cdot d_2 \cdot sin\gamma$

and rewriting it we get $2S_1 = \dfrac{d_1 \cdot d_2 \cdot d_3}{2R} \cdot \left(\dfrac{a}{d_1} + \dfrac{b}{d_2} + \dfrac{c}{d_3}\right)$.

Also we know that $2 \cdot Area_{ABC} = 2S = a \cdot d_1 + b \cdot d_2 + c \cdot d_3$.

Applying the Cauchy-Schwarz inequality we obtain

$$4S{\cdot}S_1 = \frac{d_1 \cdot d_2 \cdot d_3}{2R}\left(ad_1 + bd_2 + cd_3\right)\left(\frac{a}{d_1} + \frac{b}{d_2} + \frac{c}{d_3}\right) \geq \frac{d_1 \cdot d_2 \cdot d_3(a+b+c)^2}{2R}.$$

Using Euler's Theorem for pedal triangles we have

$$\frac{4S^2(R^2 - OM^2)}{4R^2} \geq \frac{d_1 \cdot d_2 \cdot d_3(a+b+c)^2}{2R} \geq \frac{r^3(a+b+c)^2}{2R}.$$

From here it follows that $R^2 - OM^2 \geq 2Rr$ or $OI^2 = R^2 - 2Rr \geq OM^2$.

Thus for every point M in the set we proved that $OM \leq OI$, and we are done.

Finally, we want to direct the reader's attention to the result that finds unexpectedly simple approximation for the circumradius of the pedal triangle.

Problem 5. (*proposed by Ivan Borsenco*)

Let M be a point inside triangle ABC, which has barycentric coordinates (u, v, w). Denote by R_M the circumradius of the pedal triangle of M. Prove that

$$\sqrt{(u+v+w)\left(\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}\right)} \geq 6\sqrt{3} \cdot R_M.$$

Solution: Denote by N the isogonal point of M. We need the following two lemmas

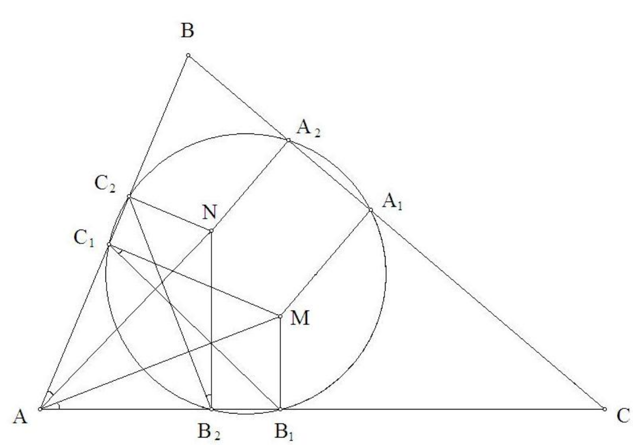
Lemma 1. If $A_1B_1C_1$ and $A_2B_2C_2$ are the pedal triangles of two isogonal points M and N then these six points lie on a circle.

Proof: Let us prove that $B_1B_2C_1C_2$ is a cyclic quadrilateral. Let $\angle BAM = \angle CAN = \phi$. Then because AB_1MC_1 is cyclic, we have

$$\angle AB_1C_1 = 90^\circ - \angle C_1B_1C_2 = 90^\circ - \angle C_1AM = 90^\circ - \phi.$$

Similarly, because AB_2NC_2 is cyclic, we have

$$\angle AC_2B_2 = 90^\circ - \angle B_1C_2B_2 = 90^\circ - \angle B_2AN = 90^\circ - \phi.$$



Hence $\angle AB_1C_1 = \angle AC_2B_2$, and so $B_1B_2C_2C_1$ is a cyclic quadrilateral. Analogously, we obtain that $A_1A_2B_2B_1$ and $A_1A_2C_2C_1$ are also cyclic. Consider the three circles that circumscribe our quadrilaterals. If they do not coincide on a common circle, they should have a radical center, which is the intersection of their radical axes. However, we can see that radical axes, that are A_1A_2, B_1B_2, C_1C_2, form a triangle, namely ABC, a contradiction. It follows that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on the same circle.

Lemma 2. If M and N are isogonal points, then the following equality holds
$$\frac{AM \cdot AN}{bc} + \frac{BM \cdot BN}{ac} + \frac{CM \cdot CN}{ab} = 1.$$

Proof: Let A_1, B_1, C_1 be the pedal triangle of M. It is not difficult to prove that $B_1C_1 \perp AN$. Thus the area of quadrilateral AB_1NC_1 is equal to $\frac{1}{2}B_1C_1 \cdot AN$. Because $B_1C_1 = AM \cdot \sin\alpha$, we get $S_{AB_1NC_1} = \frac{1}{2}AM \cdot AN \sin\alpha$. Analogously, we find that $S_{BC_1NA_1} = \frac{1}{2}BM \cdot BN \sin\beta$ and $S_{CA_1NB_1} = \frac{1}{2}CM \cdot CN \sin\gamma$.

Summing them up we obtain
$$S_{ABC} = S_{AB_1NC_1} + S_{BC_1NA_1} + S_{CA_1NB_1}$$
or
$$S_{ABC} = \frac{1}{2}\left(AM \cdot AN \sin\alpha + BM \cdot BN \sin\beta + CM \cdot CN \sin\gamma\right).$$
Using the Extended Law of Sines we obtain the desired result
$$\frac{AM \cdot AN}{bc} + \frac{BM \cdot BN}{ac} + \frac{CM \cdot CN}{ab} = 1.$$

Now let us return to the problem. Applying the AM-GM inequality in Lemma 2 we get
$$\frac{1}{27} = \frac{1}{27}\left(\frac{AM \cdot AN}{bc} + \frac{BM \cdot BN}{ac} + \frac{CM \cdot CN}{ab}\right)^3 \geq \frac{AM \cdot AN \cdot BM \cdot BN \cdot CM \cdot CN}{a^2b^2c^2}.$$
Using the fact that $AM_1 = \dfrac{B_1C_1}{sin\alpha},\ BM_1 = \dfrac{A_1C_1}{sin\beta},\ CM_1 = \dfrac{A_1B_1}{sin\gamma}$ and
$AN = \dfrac{B_2C_2}{sin\alpha},\ BN = \dfrac{A_2C_2}{sin\beta},\ CN = \dfrac{A_2B_2}{sin\gamma},$ we obtain
$$\frac{1}{27} \geq \frac{A_1B_1 \cdot B_1C_1 \cdot A_1C_1 \cdot A_2B_2 \cdot B_2C_2 \cdot A_2C_2}{a^2b^2c^2 \cdot \sin^2\alpha \cdot \sin^2\beta \cdot \sin^2\gamma}$$

As $a^2b^2c^2 \cdot \sin^2\alpha \cdot \sin^2\beta \cdot \sin^2\gamma = \dfrac{4S^4}{R^2}$ it follows that

$$\frac{1}{27} \geq \frac{R^2}{4S^4} \cdot A_1B_1 \cdot B_1C_1 \cdot A_1C_1 \cdot A_2B_2 \cdot B_2C_2 \cdot A_2C_2.$$

Recalling Lemma 1 and Euler's Theorem for areas of pedal triangles, we have

$$A_1B_1 \cdot B_1C_1 \cdot A_1C_1 = 4R_M \cdot \frac{R^2 - OM^2}{4R^2} \cdot S, \;\; A_2B_2 \cdot B_2C_2 \cdot A_2C_2 = 4R_M \cdot \frac{R^2 - ON^2}{4R^2} \cdot S.$$

Thus

$$\frac{1}{27} \geq \frac{R_M^2 (R^2 - OM^2)(R^2 - O^2N)}{4R^2 \cdot S^2}.$$

The next step is to use Theorem 3 for point M and its isogonal conjugate

$$R^2 - OM^2 = \frac{vwa^2 + uwb^2 + uvc^2}{(u+v+w)^2} = \frac{uvw(a^2/u + b^2/v + c^2/w)}{(u+v+w)^2}$$

The isogonal point of M, N has $\left(\dfrac{a^2}{u}, \dfrac{b^2}{v}, \dfrac{c^2}{w}\right)$ as barycentric coordinates and thus

$$R^2 - ON^2 = \frac{a^2b^2c^2 \cdot uvw(u+v+w)}{(vwa^2 + uwb^2 + uvc^2)^2} = \frac{a^2b^2c^2(u+v+w)}{uvw(a^2/u + b^2/v + c^2/w)^2}.$$

Combining these two results with the inequality we have

$$\frac{1}{27} \geq \frac{R_M^2 \cdot a^2b^2c^2}{4R^2 \cdot S^2 \cdot (u+v+w)(a^2/u + b^2/v + c^2/w)}.$$

Finally, we obtain

$$\sqrt{(u+v+w)\left(\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w}\right)} \geq 6\sqrt{3} \cdot R_M.$$

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