Junior problems

J145. Find all nine-digit numbers *aaaabbbb* that can be written as a sum of fifth powers of two positive integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

By Newton's binomial formula, $(5u+v)^5 \equiv v^5 \pmod{25}$ for any integers u, v, or any fifth power n^5 leaves remainders $-7, -1, 0, 1, 7$ modulus 25, when n is respectively congruent to $3, 4, 0, 1, 2$ modulus 5. Since $aaaabbbbb = m^5 + n^5$ for integers m, n, and $m^5 + n^5$ leaves remainders $0, \pm 1, \pm 2, \pm 6, \pm 7, \pm 8, \pm 14$, then $b \in \{0, 1, 3, 4, 7, 9\}$ for the last two digits of $m^5 + n^5$ to be equal.

Since $1^5 \equiv 1 \pmod{11}$, $2^5 = 32 \equiv -1 \pmod{11}$, $3^5 = 243 \equiv 1 \pmod{11}$, $4^5 = 1024 \equiv 1 \pmod{11}$ and $5^5 = 3125 \equiv 1 \pmod{11}$, then $m^5 + n^5$ leaves remainders $-2, -1, 0, 1, 2$ modulus 11. Moreover, $aaaabbbb0$ is clearly a multiple of 11, and since no digit may be congruent to -1 modulus 11, then $b \in \{9, 0, 1, 2\}$. Together with the previous result, we may conclude that the possible values for b are $b = 9$ with wlog $m \equiv 0 \pmod{5}$ and $n \equiv 4 \pmod{5}$ and exactly one of m, n is a multiple of 11, or $b = 0$ with $m + n$ a multiple of 5, or $b = 1$ with $m, n \equiv 3 \pmod{5}$ and exactly one of m, n a multiple of 11.

If $b = 1$, clearly m, n cannot finish in the same digit or otherwise $m^5 + n^5$ would finish in 8, hence wlog m ends in 3 and n ends in 8. Since $88^5 > 999999999$, $m = 33$ must be a multiple of 11, and $n = 10u + 8$ for some digit u. Now, 8^5 ends in 68 and 33^3 ends in 93, but $(10u+8)^5 - 8^5$ is a multiple of 100, so $m^5 + n^5$ formed in this way ends in 61, not in 11. No solution is possible in this case.

If $b = 9$, either m is a multiple of 10 and n^5 finishes in 99999, or m is odd and finishes in 5 and n^5 must finish in 4. In the first case, a digit u must exist such that $(10u + 9)^5$ must end in 99999, or since $9^5 = 59049$, $50 \cdot 9^4 u = 328050u$, hence $50u$ must end in 950, yielding $u \equiv -1 \pmod{20}$, impossible for a digit. In the second case, $m = 10u + 5$ and $n = 10v + 4$, and $m^5 + n^5$ ends in the same two digits as $5^5 + 4^5 + 5^5 10u + 4^4 50v$, which are also the same three digits as $125 + 24 + 250u + 800v = 800v + 250u + 149$. Clearly u must be odd, and $250u$ ends in 750 or 250, and $800v$ must finish respectively in 100 (impossible) or in 600, hence $v = 2$ or $v = 7$. But $74^5 > 999999999$, or $n = 24$ not a multiple of 11, yielding $m = 55$. Now, $55^5 + 24^5 = 511246999$, and no solution exists in this case.

Finally, if $b = 0$, then $m + n$ is a multiple of 5 with multiplicity $2 \geq \alpha \geq 1$, because if $m + n$ is a multiple of 5^3, either $m + n \geq 250$ and either m^5 or n^5 exceeds 10^{10}, or $m + n = 125$, and wlog m^5 ends in 0 and n^5 ends in 5, absurd. Note that 5^5 divides $m^5+n^5 = (m+n)^5-5mn(m+n)^3+5m^2n^2(m+n)$, yielding either $\alpha = 1$ and m^2n^2 is a multiple of 5^4, or $\alpha = 2$ and m^2n^2 is a multiple of 5^2. In either case, either m or n is a multiple of 5, hence so is the other, and clearly one ends in 0 iff the other one ends in 0, hence either $m = 10u + 5$ and $n = 10v + 5$ or $m = 10u$ and $n = 10v$. In the first case, the last four digits of $m^5 + n^5$ are also the last four digits of $5^3 1000(u^2 + v^2) + 5^5 10(u + v) + 2 \cdot 5^5$, hence the last four digits of $5000(u^2 + v^2 - 1) + 1250(u + v + 1)$. Now, $u + v + 1$ must be a multiple of 4 for the last three digits to be 0, hence $u^2 + v^2 - 1$ is even, and $1250(u+v+1)$ must end in 0000, yielding $u+v+1 = 8$, since $u+v+1 = 16$ produces either u or $v \geq 8$, and $85^5 > 999999999$. Trying all possible alternatives with $u + v = 7$, we find that $u \geq 6$ results in $m^5 > 999999999$, $u = 5$ and $v = 2$ results in $m^5 + n^5 = 513050000$, and $u = 4$ and $v = 3$ results in 237050000, so no solution exists in this case. Finally, when $m = 10u$ and $n = 10v$, the problem is equivalent to finding all digits u, v such that $u^5 + v^5 = aaaa$. Combining the numbers whose fifth powers do not exceed 10^4, we find that this result is only true when $6^5 + 1^5 = 7777$.

We conclude that the only number of the form *aaaabbbbb* that is the sum of two fifth powers is $777700000 = 60^5 + 10^5$.

J146. Let $A_1A_2A_3A_4A_5$ be a convex pentagon and let $X \in A_1A_2$, $Y \in A_2A_3$, $Z \in A_3A_4$, $U \in A_4A_5$, $V \in A_5A_1$ be points such that A_1Z, A_2U, A_3V, A_4X, A_5Y intersect at P. Prove that

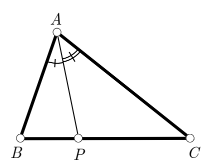
$$\frac{A_1X}{A_2X} \cdot \frac{A_2Y}{A_3Y} \cdot \frac{A_3Z}{A_4Z} \cdot \frac{A_4U}{A_5U} \cdot \frac{A_5V}{A_1V} = 1.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

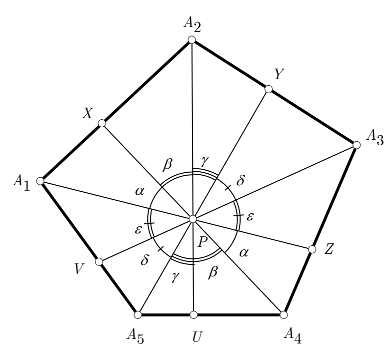
Solution by Ercole Suppa, Teramo, Italy

Lemma. *If P is a point on the side BC of a triangle $\triangle ABC$,*

$$\frac{PB}{PC} = \frac{AB}{AC} \cdot \frac{\sin \angle PAB}{\sin \angle PAC}$$



Coming back to the problem, let us denote $\angle A_1PX = \angle A_4PZ = \alpha$, $\angle XPA_2 = \angle UPA_4 = \beta$, $\angle A_2PY = \angle A_5PU = \gamma$, $\angle YPA_3 = \angle VPA_5 = \delta$, $\angle A_3PZ = \angle A_1PV = \epsilon$, as shown in figure. Applying the above lemma to the triangles $\triangle A_1PA_2$, $\triangle A_2PA_3$, $\triangle A_3PA_4$, $\triangle A_4PA_5$, $\triangle A_5PA_1$ we get



$$\frac{A_1X}{A_2X} \cdot \frac{A_2Y}{A_3Y} \cdot \frac{A_3Z}{A_4Z} \cdot \frac{A_4U}{A_5U} \cdot \frac{A_5V}{A_1V} = \frac{\sin\alpha}{\sin\beta} \cdot \frac{\sin\gamma}{\sin\delta} \cdot \frac{\sin\alpha}{\sin\beta} \cdot \frac{\sin\epsilon}{\sin\alpha} \cdot \frac{\sin\beta}{\sin\gamma} \cdot \frac{\sin\delta}{\sin\epsilon} = 1.$$

Also solved by Aravind Srinivas L, Chennai, India; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Michel Bataille, France; Pedro Henrique O. Pantoja, Natal, Brazil; Roberto Bosch Cabrera, Havana, Cuba.

J147. Let $a_0 = a_1 = 1$ and

$$a_{n+1} = 1 + \frac{a_1^2}{a_0} + \cdots + \frac{a_n^2}{a_{n-1}}$$

for $n \geq 1$. Find a_n in closed form.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Aravind Srinivas L, Chennai, India

Let us prove by induction that $a_n = n!$. The result is not difficult to verify for $n = 0, 1$. Assume that $a_k = k!$ for $k \leq n - 1$. Now

$$a_{k+1} = \left(1 + \frac{a_1^2}{a_0} + \cdots + \frac{a_{k-1}^2}{a_k}\right) + \frac{a_k^2}{a_{k-1}}.$$

By the induction hypothesis,

$$a_{k+1} = a_k + \frac{a_k^2}{a_{k-1}} = a_k\left(1 + \frac{a_k}{a_{k-1}}\right) = \frac{a_k}{a_{k-1}}(a_k + a_{k-1})$$

which is equivalent to

$$\frac{k!}{(k-1)!}((k-1)! + k!) = k(k-1)!(k+1) = (k+1)!.$$

Thus $a_n = n!$ for $n \geq 1$.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; G. C. Greubel, Newport News, USA; Irfan Besic, Sarajevo, Bosnia and Herzegovina; Michel Bataille, France; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Roberto Bosch Cabrera, Havana, Cuba; Samuel Gómez Moreno, Departamento de Matemáticas, Universidad de Jaén, Spain; Shai Covo, Kiryat-Ono, Israel.

J148. Find all n such that for each $\alpha_1, \ldots, \alpha_n \in (0, \pi)$ with $\alpha_1 + \cdots + \alpha_n = \pi$ the following equality holds

$$\sum_{i=1}^{n} \tan \alpha_i = \frac{\sum_{i=1}^{n} \cot \alpha_i}{\prod_{i=1}^{n} \cot \alpha_i}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Roberto Bosch Cabrera, Havana, Cuba;

If $n = 1$ we have $\tan \alpha_1 = \tan \pi = 0 = \frac{\cot \alpha_1}{\cot \alpha_1} = 1$ contradiction. If $n = 2$ we obtain

$$\tan \alpha_1 + \tan \alpha_2 = \frac{\tan \alpha_1 + \tan \alpha_2}{\tan \alpha_1 \tan \alpha_2} \cdot \tan \alpha_1 \tan \alpha_2$$

so the equality holds. Now suppose $n \geq 3$. Letting $\alpha_1 = \cdots = \alpha_n = \frac{\pi}{n}$ yields $(\tan \frac{\pi}{n})^{n-2} = 1$; hence $\tan \frac{\pi}{n} = \pm 1$, so $\frac{\pi}{n} = \pm(k\pi + \frac{\pi}{4})$, therefore $n = 4$. To verify that equality holds for $n = 4$ we have to prove that $\tan \alpha_1 + \tan \alpha_2 + \tan \alpha_3 + \tan \alpha_4$ is equal to

$$\begin{aligned} \tan \alpha_1 \tan \alpha_2 \tan \alpha_3 + {} & \tan \alpha_1 \tan \alpha_2 \tan \alpha_4 \\ & + \tan \alpha_1 \tan \alpha_3 \tan \alpha_4 \\ & + \tan \alpha_2 \tan \alpha_3 \tan \alpha_4 \end{aligned}$$

with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \pi$, expanding $\tan(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 0$ we obtain the desired conclusion. The values of n are 2 and 4.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Shai Covo, Kiryat-Ono, Israel; icente Vicario Garcia, Huelva, Spain.

J149. Let $ABCD$ be a quadrilateral with $\angle A \geq 60°$. Prove that

$$AC^2 < 2\left(BC^2 + CD^2\right).$$

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

According to Ptolemy's inequality, $AC \cdot BD \leq AB \cdot CD + BC \cdot DA$, with equality iff $ABCD$ is cyclic. Moreover, since $\angle A > 60°$, then $\cos A < \frac{1}{2}$, and the Cosine Law guarantees that $BD^2 > AB^2 + AD^2 - AB \cdot AD$. Note therefore that

$$AC < \frac{AB \cdot CD + BC \cdot DA}{\sqrt{AB^2 + AD^2 - AB \cdot AD}},$$

and it suffices to show that

$$\frac{(AB \cdot CD + BC \cdot DA)^2}{AB^2 + AD^2 - AB \cdot AD} \leq 2(BC^2 + CD^2).$$

This inequality rewrites as

$$(BC^2 + CD^2)(AB - AD)^2 + (AB \cdot BC - CD \cdot DA)^2 \geq 0,$$

clearly true, with equality iff $AB = AD$ and $BC = CD$. The conclusion follows.

Note that equality in the proposed inequality could be reached when $\angle A = 60°$, if and only if $ABCD$ is a cyclic kite with $AB = AD$ and $BC = CD$.

Second solution by Roberto Bosch Cabrera, Havana, Cuba

Let $\angle DAC = x$ and $\angle BAC = y$. We apply Cosines Law to triangles ADC and ABC, let see

$$\begin{aligned} CD^2 &= AD^2 + AC^2 - 2AD \cdot AC \cos x \\ BC^2 &= AB^2 + AC^2 - 2AB \cdot AC \cos y \end{aligned}$$

summing up

$$CD^2 + BC^2 = AD^2 + AB^2 + 2AC^2 - 2AC(AD \cos x + AB \cos y)$$

and hence we need to prove the right side is greater than $\frac{AC^2}{2}$ with $x + y \geq 60°$. That is to say

$$3AC^2 - 4(AD \cos x + AB \cos y)AC + 2(AD^2 + AB^2) > 0$$

this expression can be considered as an quadratic in AC, since $3 > 0$ it suffices show that the discriminant Δ_1 is negative.

$$\begin{aligned} \Delta_1 &= 16(AD \cos x + AB \cos y)^2 - 24(AD^2 + AB^2) < 0 \\ &\Leftrightarrow 2(AD \cos x + AB \cos y)^2 - 3(AD^2 + AB^2) < 0 \\ &\Leftrightarrow (2\cos^2 x - 3)AD^2 + 4\cos x \cos y AB \cdot AD + (2\cos^2 y - 3)AB^2 < 0 \end{aligned}$$

but this expression is an quadratic in AD, and since $2\cos^2 x - 3 < 0$ it suffices show that $\Delta_2 = AB^2\left(16\cos^2 x\cos^2 y - 4(2\cos^2 x - 3)(2\cos^2 y - 3)\right)$ is negative too. Note that $\Delta_2 < 0 \Rightarrow \Delta_1 < 0$. Hence just rest prove $2\cos^2 x + 2\cos^2 y - 3 < 0$. But the left side is equal to $\cos 2x + \cos 2y - 1 = 2\cos(x+y)\cos(x-y) - 1$. We suppose without loss of generality $x \geq y$. Note that we have three cases:

- $60^\circ \leq x + y \leq 90^\circ$
  $\Rightarrow 0 \leq x - y < x + y \leq 90^\circ$ hence
  $0 \leq 2\cos(x+y) \leq 1$ and $0 \leq \cos(x-y) \leq 1$, multiplying we obtain our inequality. Note that is strict because we have equality if and only if $90^\circ = x - y < x + y = 60^\circ$ contradiction.
- $90^\circ \leq x + y < 180^\circ$ and $0 \leq x - y \leq 90^\circ$
  $\Rightarrow \cos(x+y) \leq 0$ and $\cos(x-y) \geq 0$, then our inequality yields.
- $90^\circ \leq x + y < 180^\circ$ and $90^\circ \leq x - y$
  $\Rightarrow x > 90^\circ$ hence $CD^2 > AD^2 + AC^2$ and so $2(BC^2 + CD^2) > 2AC^2 + 2AD^2 + 2BC^2 > AC^2$.

Also solved by Arkady Alt, San Jose, California, USA

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J150. Let n be an integer greater than 2. Find all real numbers x such that $\{x\} \leq \{nx\}$, where $\{a\}$ denotes the fractional part of a.

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania and Mihai Piticari, "Dragos-Voda" National College, Romania

Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

Let's suppose $k \leq x < k+1$ and then $nk \leq nx < nk+n$. We write $[k,k+1) = \bigcup_{r=0}^{n-1}[k+\frac{r}{n}, k+\frac{r+1}{n}) \doteq \bigcup_{r=0}^{n-1} I_r$. For $x \in I_r$ the inequality $\{x\} \leq \{nx\}$ reads as

$$x - k \leq nx - nk - r \qquad \text{that is} \qquad x \geq k + \frac{r}{n-1}$$

The compatibility conditions are

$$x \in I_r \quad \Longleftrightarrow \quad k + \frac{r}{n} \leq k + \frac{r}{n-1} \leq k + \frac{r+1}{n}$$

and we conclude that:

1) if $x \in I_0$, it satisfies $\{x\} \leq \{nx\}$,

2) if $x \in I_r$, $r \geq 1$, only those x in the subinterval $J_r \subset I_r$, $J_r = [k+\frac{r}{n-1}, k+\frac{r+1}{n})$ satisfy the inequality.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Michel Bataille, France; Samuel Gómez Moreno, Departamento de Matemáticas, Universidad de Jaén, Spain.

Senior problems

S145. Let k be a nonzero real number. Find all functions $f : R \longrightarrow \mathbb{R}$ such that

$$f(xy) + f(yz) + f(zx) - k\left[f(x)f(yz) + f(y)f(zx) + f(z)f(xy)\right] \geq \frac{3}{4k},$$

for all $x, y, z \in R$.

Proposed by Marin Bancos, North University of Baia Mare, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Taking $x = y = z = 0$, the condition becomes $(2kf(0) - 1)^2 \leq 0$, or $f(0) = \frac{1}{2k}$. Take $x = y = z = 1$, the condition becomes again $(2kf(1) - 1)^2 \leq 0$, or $f(1) = \frac{1}{2k}$.

Take now $y = z = 0$, the condition becomes $f(x) \leq \frac{1}{2k}$.

Take finally $x = y$ and $z = 1$, the condition becomes $2kf(x^2) + 8kf(x) \geq 3 + 8k^2 f^2(x)$. But $2kf(x^2) \leq 1$, or $(2kf(x) - 1)^2 \leq 0$, yielding $f(x) = \frac{1}{2k}$ for all real x.

Also solved by Arkady Alt, San Jose, California, USA; Mohammed Kharbach, Dolphin Energy, UAE; Roberto Bosch Cabrera, Havana, Cuba.

S146. Let m_a, m_b, m_c be the medians, k_a, k_b, k_c the symmedians, r the inradius, and R the circumradius of a triangle ABC. Prove that

$$\frac{3R}{2r} \geq \frac{m_a}{k_a} + \frac{m_b}{k_b} + \frac{m_c}{k_c} \geq 3.$$

Proposed by Pangiote Ligouras, Bari, Italy

Solution by Michel Bataille, France

Let K be the symmedian point of ΔABC and let $BC = a, CA = b, AB = c$. It is well known that K is the barycentre of A, B, C with masses a^2, b^2, c^2, respectively. This gives

$$(b^2 + c^2)A_1 = b^2 B + c^2 C$$

where A_1 is the point of intersection of the lines AK and BC and it follows that

$$\begin{aligned}(b^2+c^2)^2 k_a^2 = (b^2+c^2)^2 AA_1^2 &= (b^2\overrightarrow{AB} + c^2\overrightarrow{AC})^2 \\ &= b^4c^2 + b^2c^4 + b^2c^2(b^2+c^2-a^2) \\ &= b^2c^2(2b^2+2c^2-a^2)\end{aligned}$$

that is, $(b^2+c^2)^2 k_a^2 = 4m_a^2 b^2 c^2$. Thus,

$$\frac{m_a}{k_a} = \frac{b^2+c^2}{2bc} \quad \text{and similarly,} \quad \frac{m_b}{k_b} = \frac{c^2+a^2}{2ca} \quad \text{and} \quad \frac{m_c}{k_c} = \frac{a^2+b^2}{2ab}.$$

As a result, the required inequalities become

$$\frac{3R}{r} \geq \frac{b^2+c^2}{bc} + \frac{c^2+a^2}{ca} + \frac{a^2+b^2}{ab} \geq 6.$$

The right inequality rewrites as $a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2) \geq 6abc$, which holds since $b^2+c^2 \geq 2bc,\ c^2+a^2 \geq 2ca$ and $a^2+b^2 \geq 2ab$.
As for the left inequality, it rewrites as

$$a^2(b+c) + b^2(c+a) + c^2(a+b) \leq 6R^2(a+b+c) \qquad (1)$$

(since $abc = 2Rr(a+b+c)$).
Assuming that $a \leq b \leq c$, we have $a^2 \leq b^2 \leq c^2$ and $b+c \geq c+a \geq a+b$ so that

$$a^2(b+c) + b^2(c+a) + c^2(a+b) \leq \frac{(a^2+b^2+c^2)(2a+2b+2c)}{3} \qquad (2)$$

by Chebyshev's inequality.
Now, let H and O be the orthocentre and circumcentre of ΔABC. We know that $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, from which we deduce $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$. Thus, $a^2 + b^2 + c^2 \leq 9R^2$ and taking (2) into account, we readily obtain (1).

Also solved by Arkady Alt, San Jose, California, USA; Ercole Suppa, Teramo, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Roberto Bosch Cabrera, Havana, Cuba; Vicente Vicario Garcia, Huelva, Spain.

S147. Let $x_1, \ldots, x_n, a, b > 0$. Prove that the following inequality holds

$$\frac{x_1^3}{(ax_1+bx_2)(ax_2+bx_1)} + \cdots + \frac{x_n^3}{(ax_n+bx_1)(ax_1+bx_n)} \geq \frac{x_1+\cdots+x_n}{(a+b)^2}.$$

Proposed by Marin Bancos, North University of Baia Mare, Romania

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We have

$$4x_1^3 - (x_1+x_2)^2(2x_1-x_2) = 2x_1^3 - 3x_1^2x_2 + x_2^3 = (x_1-x_2)^2(2x_1+x_2) \geq 0, \quad (1)$$

with equality if and only if $x_1 = x_2$. It follows that

$$x_1 + \cdots + x_n = (2x_1 - x_2) + \cdots + (2x_n - x_1) \leq \frac{4x_1^3}{(x_1+x_2)^2} + \cdots + \frac{4x_n^3}{(x_n+x_1)^2}.$$

It then suffices to show that

$$\frac{4}{(x_1+x_2)^2(a+b)^2} \leq \frac{1}{(ax_1+bx_2)(ax_2+bx_1)},$$

which is equivalent to $(x_1-x_2)^2(a-b)^2 \geq 0$, clearly true with equality if and only if $x_1 = x_2$ or $a = b$. The conclusion follows. Note that, since $x_1 = x_2 = \cdots = x_n$ is necessary and sufficient for equality in (1), then equality holds in the proposed inequality if and only if $x_1 = x_2 = \cdots = x_n$.

Second solution by Roberto Bosch Cabrera, Havana, Cuba

Note that

$$(ax_1+bx_2)(ax_2+bx_1) = (a^2+b^2)x_1x_2 + ab(x_1^2+x_2^2) \leq \frac{(x_1^2+x_2^2)(a+b)^2}{2}$$

Hence

$$\frac{x_1^3}{(ax_1+bx_2)(ax_2+bx_1)} \geq \frac{2x_1^3}{(a+b)^2(x_1^2+x_2^2)}$$

so we need to prove that

$$\frac{2x_1^3}{x_1^2+x_2^2} + \frac{2x_2^3}{x_2^2+x_3^2} + \cdots + \frac{2x_n^3}{x_n^2+x_1^2} \geq x_1 + \cdots + x_n$$

We suppose the contrary, that is to say, there exist $x_1, ..., x_n$ positive real numbers such that

$$\frac{2x_1^3}{x_1^2+x_2^2} + \frac{2x_2^3}{x_2^2+x_3^2} + \cdots + \frac{2x_n^3}{x_n^2+x_1^2} < x_1 + \cdots + x_n$$

by symmetry we have the following inequality

$$\frac{2x_2^3}{x_1^2+x_2^2}+\frac{2x_3^3}{x_2^2+x_3^2}+\cdots+\frac{2x_1^3}{x_n^2+x_1^2}<x_1+\cdots+x_n$$

summing up we obtain

$$\frac{x_1^3+x_2^3}{x_1^2+x_2^2}+\frac{x_2^3+x_3^3}{x_2^2+x_3^2}+\cdots+\frac{x_n^3+x_1^3}{x_n^2+x_1^2}<x_1+\cdots+x_n$$

contradiction since

$$\frac{x^3+y^3}{x^2+y^2}\geq\frac{x+y}{2}$$

So the original inequality is true, with equality if and only if $x_1=x_2=...=x_n$.

Also solved by Arkady Alt, San Jose, California, USA; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; M. Ramchandran, Tamil Nadu, India.

S148. Let n be a positive integer and let a, b, c be real numbers such that $a^2b \geq c^2$. Find all real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ for which

$$x_1^2 + \cdots + x_n^2 + b\Big(y_1^2 + \cdots + y_n^2\Big) = c.$$

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let $S_x = x_1^2 + \cdots + x_n^2$ and $S_y = y_1^2 + \cdots + y_n^2$. By the Cauchy–Schwarz inequality, $\frac{a}{2} \leq \sqrt{S_x S_y}$, or $4S_xS_y \geq a^2$, with equality iff $y_i = kx_i$ for some real constant k and all $i = 1, 2, \ldots, n$. Moreover, by the AM–GM inequality, $S_x^2 + b^2S_y^2 \geq 2bS_xS_y$, with equality if and only $S_x = bS_y$, and finally

$$4bS_xS_y \geq a^2b \geq c^2 = S_x^2 + b^2S_y^2 + 2bS_xS_y \geq 4bS_xS_y.$$

Therefore, all inequalities need to be equalities, and we conclude that $c^2 = a^2b$ and

$$x_1^2 + \cdots + x_n^2 = b(y_1^2 + \cdots + y_n^2) = bk^2(x_1^2 + \cdots + x_n^2),$$

or $k = \frac{1}{\sqrt{b}} = \frac{a}{c}$. Note that

$$\frac{a}{2} = x_1y_1 + \cdots + x_ny_n = \frac{a}{c}(x_1^2 + \cdots + x_n^2),$$

or $x_1^2+\cdots+x_n^2 = \frac{c}{2}$, and $y_1^2+\cdots+y_n^2 = \frac{a^2}{2c} = \frac{c}{2b}$. Therefore, $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ will be a solution as long as $a^2b = c^2$ and whenever $x_1^2 + \cdots + x_n^2 = \frac{c}{2}$, and $y_i = \frac{ax_i}{c}$. If $a^2b > c^2$, there will be no solution.

Also solved by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Roberto Bosch Cabrera, Havana, Cuba.

S149. Prove that in any acute triangle ABC,

$$\frac{1}{2}\left(1+\frac{r}{R}\right)^2 - 1 \le \cos A\cos B\cos C \le \frac{r}{2R}\left(1-\frac{r}{R}\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Michel Bataille, France

(a) $\frac{1}{2}\left(1+\frac{r}{R}\right)^2 - 1 \le \cos A\cos B\cos C$

From $\cos A+\cos B+\cos C = 1+\frac{r}{R}$, this inequality rewrites as $(\cos A+\cos B+\cos C)^2 \le 2+2\cos A\cos B\cos C$. Since the angles of a nonobtuse triangle may be written as $A=\frac{\pi-A'}{2}$, $B=\frac{\pi-B'}{2}$, $C=\frac{\pi-C'}{2}$ where A', B', C' are the angles of some triangle, it suffices to prove that

$$\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2 \le 2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \qquad (1)$$

holds for any triangle. Using the familiar notations, we transform (1) into

$$\left(\sqrt{\frac{(s-b)(s-c)}{bc}}+\sqrt{\frac{(s-c)(s-a)}{ca}}+\sqrt{\frac{(s-a)(s-b)}{ab}}\right)^2 \le 2+\frac{r}{2R}$$

or, observing that $\frac{(s-a)(s-b)(s-c)}{abc}=\frac{r}{4R}$,

$$\left(\sqrt{\frac{a}{s-a}}+\sqrt{\frac{b}{s-b}}+\sqrt{\frac{c}{s-c}}\right)^2 \le 2+\frac{8R}{r} \qquad (2).$$

Now,

$$\begin{aligned}
\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c} &= \frac{s((s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b))}{s(s-a)(s-b)(s-c)}\\
&= \frac{abc+(s-a)(s-b)(s-c)}{(rs)^2}\\
&= \frac{4srR+sr^2}{r^2s^2} = \frac{4R+r}{rs}
\end{aligned}$$

so that (2) is readily obtained from the Cauchy-Schwarz Inequality as follows

$$\begin{aligned}
\left(\sqrt{\frac{a}{s-a}}+\sqrt{\frac{b}{s-b}}+\sqrt{\frac{c}{s-c}}\right)^2 &\le (a+b+c)\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)\\
&= 2s\cdot\frac{4R+r}{rs} = 2+\frac{8R}{r}.
\end{aligned}$$

(b) $\cos A \cos B \cos C \leq \frac{r}{2R}\left(1-\frac{r}{R}\right)$

We prove that this inequality actually holds in any triangle. Since $2\cos A\cos B\cos C = \sin^2 A + \sin^2 B + \sin^2 C - 2$, we have

$$\begin{aligned} 2\cos A\cos B\cos C &= \frac{a^2+b^2+c^2-8R^2}{4R^2} = \frac{2s^2-2r^2-8rR-8R^2}{4R^2} \\ &= \frac{s^2-r^2-4rR-4R^2}{2R^2} \end{aligned}$$

and the inequality is equivalent to

$$s^2 \leq 4R^2+6rR-r^2.$$

Now, if I and H are the incenter and orthocentre of the triangle, we have $IH^2 = 4R^2+4rR+3r^2-s^2$ so that

$$s^2 \leq 4R^2+4rR+3r^2$$

(this is known as an inequality of Gerratsen). Thus, it suffices to prove that $4R^2+4rR+3r^2 \leq 4R^2+6rR-r^2$ or $0 \leq 2r(R-2r)$. This completes the proof since $R \geq 2r$.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Roberto Bosch Cabrera, Havana, Cuba; Scott H. Brown, Auburn University Montgomery, USA.

S150. Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in a circle $C(O, R)$ and circumscribed about a circle $\omega(I, r)$. Denote by R_i the radius of the circle tangent to A_iA_{i+1} and tangent to the extension of the sides $A_{i-1}A_i$ and $A_{i+1}A_{i+2}$. Prove that the sum $R_1 + R_2 + R_3 + R_4$ does not depend on the position of points A_1, A_2, A_3, A_4.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Lemma: Let $ABCD$ be a bicentric quadrilateral, a, b, c, d the respective lengths of its sides AB, BC, CD, DA and p, q the respective lengths of its diagonals AC, BD. The product pq depends only on the circumradius and inradius, respectively R and r, of $ABCD$.

Proof: Since $ABCD$ is bicentric, its area is $S = \sqrt{abcd} = rs = r(a+c) = r(b+d)$, where s is its semiperimeter. The area of ABC is $\frac{abp}{4R}$, and the area of CDA is $\frac{cdp}{4R}$, or $4R\sqrt{abcd} = p(ab + cd)$. Similarly, $4R\sqrt{abcd} = q(bc + da)$, and

$$\frac{16R^2}{pq} + 4 = \frac{(ab + cd)(bc + da)}{abcd} + 4 = \frac{da^2b + ab^2c + bc^2d + cd^2a + 4abcd}{r^2s^2} =$$

$$= \frac{dab + abc + bcd + cda}{r^2s} = \frac{ac + bd}{r^2} = \frac{pq}{r^2},$$

where we have used Ptolemy's equality for cyclic quadrilaterals. Therefore, the product pq satisfies the relation

$$(pq)^2 - 4r^2(pq) - 16R^2r^2 = 0,$$

which clearly has exactly one positive root which may be written as a function only of R and r.

Denote $a = A_1A_2$, $b = A_2A_3$, $c = A_3A_4$, $d = A_4A_1$, $p = A_1A_3$ and $q = A_2A_4$, and call C_i the circle with radius R_i defined in the problem statement. Assume first that $A_2A_3 \parallel A_4A_1$. Then, circles ω, C_1 and C_3 are tangent to two parallel lines, hence they all have the same radius equal to half the distance between these lines. Moreover, $A_1A_2A_3A_4$ is a cyclic quadrilateral with two parallel sides A_2A_3 and A_4A_1, hence a trapezium with $a = A_1A_2 = A_3A_4 = c$, or $R_1 = \frac{ar}{c}$ and $R_3 = \frac{cr}{a}$. Assume now that A_2A_3 and A_4A_1 intersect at a point P such that wlog $PA_3 < PA_2$ and $PA_4 < PA_1$. Triangles PA_1A_2 and PA_4A_3 are similar since $A_1A_2A_3A_4$ is cyclic, hence their respective inradii r and R_3 are in the same proportion as their sides A_1A_2 and A_4A_3, or $R_3 = \frac{cr}{a}$. Now C_1 and

ω are the respective excircles of PA_1A_2 and PA_4A_3, tangent respectively to sides A_1A_2 and A_4A_3, and to the extensions of sides PA_1, PA_2 and PA_4, PA_3, hence their radii are again in the same proportion as sides A_1A_2 and A_4A_3, or $R_1 = \frac{ar}{c}$. Similarly, $R_2 = \frac{br}{d}$ and $R_4 = \frac{dr}{b}$. Finally,

$$\frac{R_1 + R_2 + R_3 + R_4}{r} = \frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b} = \frac{(bc + da)(ab + cd)}{abcd} = \frac{16R^2}{pq}.$$

Hence $R_1 + R_2 + R_3 + R_4$ depends only on R and r. The conclusion follows.

Also solved by Ercole Suppa, Teramo, Italy; Raul A. Simon, Chile; Roberto Bosch Cabrera, Havana, Cuba.

Undergraduate problems

U145. Consider the following determinant:

$$D_n = \begin{vmatrix} 1 & 2 & \cdots & n \\ 1 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^n & \cdots & n^n \end{vmatrix}.$$

Find $\lim\limits_{n\to\infty} (D_n)^{\frac{1}{n^2 \ln n}}$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Ercole Suppa, Teramo, Italy

Taking into account the Vandermonde formula, we have

$$D_n = \begin{vmatrix} 1 & 2 & \cdots & n \\ 1 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^n & \cdots & n^n \end{vmatrix} = 2 \cdot 3 \cdots n \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2^1 & \cdots & n^1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{n-1} & \cdots & n^{n-1} \end{vmatrix} =$$
$$= n! \prod_{i>j} (i-j) = n!(n-1)! \cdots 2!1!$$

Hence

$$\lim_{n\to\infty} (D_n)^{\frac{1}{n^2 \log n}} = \lim_{n\to\infty} e^{\frac{\log[n!(n-1)!\cdots 2!1!]}{n^2 \log n}}$$

In order to calculate the limit of the exponent $\frac{\log[n!(n-1)!\cdots 2!1!]}{n^2 \log n}$, we can use the theorem of Cesàro-Stolz:

Let (x_n) and (y_n) be two sequences of real numbers with (y_n) strictly positive, increasing, and unbounded. If

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = L$$

then the limit $\lim\limits_{n\to\infty} \frac{x_n}{y_n}$ exists and is equal to L.

Now, calling $x_n = \log[n!(n-1)!\cdots 2!1!]$ and $y_n = n^2 \log n$, we have that the sequence y_n is strictly positive, increasing and unbounded; furthermore

$$\begin{aligned}
\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} &= \lim_{n\to\infty} \frac{\log[(n+1)!n!\cdots 2!1!] - \log[n!(n-1)!\cdots 2!1!]}{(n+1)^2\log(n+1) - n^2\log n} \\
&= \lim_{n\to\infty} \frac{\log(n+1)!}{(n+1)^2\log(n+1) - n^2\log n} \\
&= \lim_{n\to\infty} \frac{\log(n+1) + \log n!}{n^2\left(\log(n+1) - \log n\right) + (2n+1)\log(n+1)} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n} + \frac{\log n!}{n\log(n+1)}}{\frac{1}{\log(n+1)}\log\left(1+\frac{1}{n}\right)^n + \frac{2n+1}{n}} = \frac{1}{2}
\end{aligned}$$

where in the last step we have used the limits $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ and

$$\lim_{n\to\infty} \frac{\log n!}{n\log(n+1)} = 1$$

The above limit also can be proved by means of Cesàro-Stolz theorem; in fact, setting $u_n = \log n!$ and $v_n = n\log(n+1)$ we have:

$$\begin{aligned}
\lim_{n\to\infty} \frac{u_{n+1}-u_n}{v_{n+1}-v_n} &= \lim_{n\to\infty} \frac{\log(n+1)! - \log n!}{(n+1)\log(n+2) - n\log(n+1)} \\
&= \lim_{n\to\infty} \frac{\log(n+1)}{n\log\frac{n+2}{n+1} + \log(n+2)} \\
&= \lim_{n\to\infty} \frac{\log(n+1)}{\log\left(1+\frac{1}{n+1}\right)^n + \log(n+2)} = 1
\end{aligned}$$

Finally, the required limit is

$$\lim_{n\to\infty} (D_n)^{\frac{1}{n^2\log n}} = \lim_{n\to\infty} e^{\frac{\log[n!(n-1)!\cdots 2!1!]}{n^2\log n}} = e^{\frac{1}{2}} = \sqrt{e}$$

and we are done.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Michel Bataille, France; Roberto Bosch Cabrera, Havana, Cuba; Moubinool Omarjee.

U146. Let n be a positive integer. For all $i, j = 1, \ldots, n$ define $S_n(i,j) = \sum_{k=1}^{n} k^{i+j}$, Evaluate the determinant $\Delta = |S_n(i,j)|$.

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania

Solution by Roberto Bosch Cabrera, Havana, Cuba

Note that

$$\Delta = \left| M * M^t \right|$$

where

$$M = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1^2 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1^n & 2^n & \cdots & n^n \end{pmatrix}$$

and hence

$$\Delta = (D_n)^2 = (1!\ 2!\ \cdots\ n!)^2.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Moubinool Omarjee; Pedro Henrique O. Pantoja, Natal, Brazil.

U147. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, and let $c \in \mathbb{R}$ such that

$$\int_a^b f(x)\,dx \neq (b-a)\,f(c),$$

for all $a, b \in \mathbb{R}$. Prove that

$$f'(c) = 0.$$

Proposed by Bogdan Enescu, "B. P. Hasdeu" National College, Romania

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Define

$$F(x) = \int_c^x f(y)dy - (x-c)f(c), \qquad F'(x) = f(x) - f(c), \qquad F''(x) = f'(x).$$

Note that $F(c) = F'(c) = 0$. Now, for any $b > c$, take $a = c$, or

$$F(c) = 0 \neq \int_a^b f(x)dx - (b-a)f(c) = \int_c^b f(x)dx - (b-c)f(c) = F(b)$$

Since $F(x)$ is clearly continuous, $F(x)$ cannot change signs for $x > c$, otherwise by the intermediate value theorem, $F(b) = 0$ for some $b > c$, contradiction. Similarly, taking $b = c$, $F(x)$ cannot change signs for $x < c$, otherwise $F(a) = 0$ for some $a < c$, contradiction. Moreover, for any $a < b < c$,

$$0 \neq \int_a^b f(x)dx - (b-a)f(c) = F(b) - F(a),$$

or $F(x)$ must have opposite signs for $x < c$ and $x > c$, otherwise if wlog $F(b), F(a) > 0$ for $a < c < b$ (since we may always exchange $f(x)$ by $-f(x)$ without altering the problem), then by the intermediate value theorem, x_- and x_+ exist such that $a < x_- < c < x_+ < b$ and $F(x_-) = F(x_+) = \frac{1}{2}\min\{F(a), F(b)\}$, contradiction. Thus, wlog $F(x) < F(c) = 0$ for all $x < c$ and $F(x) > F(c) = 0$ for all $x > c$. Therefore, an open interval I exists such that $c \in I$ and $F'(x) > 0$ for all $x \in I$ such that $x \neq c$. Hence c is a local minimum of $F'(x) = f(x) - f(c)$, and $F''(c) = f'(c) = 0$. The conclusion follows.

Second solution by Roberto Bosch Cabrera, Havana, Cuba

We consider the function $F(y) = \int_c^y f(x)dx - f(c)(y-c)$ for $y \geq c$. We suppose that exist y_1, y_2 with $c < y_1 < y_2$ such that $F(y_1) > 0$ and $F(y_2) < 0$. $F(y)$

is continuos on $[y_1, y_2]$ since $f(y)$ is differentiable. So there is $\alpha \in (y_1, y_2)$ with $F(\alpha) = 0$ by Bolzano's theorem, contradiction. Hence $F(y) \geq 0$ for $y \geq c$, or $F(y) \leq 0$ for $y \geq c$. We consider only the first case, the second one is analogous.

Now let

$$G(y) = \begin{cases} f(c) & \text{if } y = c \\ \frac{\int_c^y f(x)dx}{y-c} & \text{if } y > c \end{cases}$$

We have that $G(y) \geq f(c) = G(c)$ for $y \geq c$, hence

$$\begin{aligned} 0 = G'(c) &= \lim_{y \to c} \frac{G(y) - G(c)}{y - c} = \lim_{y \to c} \frac{\frac{\int_c^y f(x)dx}{y-c} - f(c)}{y - c} \\ &= \lim_{y \to c} \frac{\int_c^y f(x)dx - f(c)(y-c)}{(y-c)^2} = \lim_{y \to c} \frac{f(y) - f(c)}{2(y-c)} \end{aligned}$$

by L'Hopital. Finally we have

$$0 = \lim_{y \to c} \frac{f(y) - f(c)}{2(y-c)} = \frac{1}{2} \lim_{y \to c} \frac{f(y) - f(c)}{y - c} = \frac{1}{2} f'(c).$$

We are done.

U148. Let $f : [0,1] \Rightarrow \mathbb{R}$ be a continuous non-decreasing function. Prove that

$$\frac{1}{2}\int_0^1 f(x)dx \le \int_0^1 xf(x)dx \le \int_{\frac{1}{2}}^1 f(x)dx.$$

Proposed by Duong Viet Thong, Hanoi University of Science, Vietnam

First solution by Arkady Alt, San Jose, California, USA

To prove the left hand side of the inequality we will use the following lemma.

Lemma. Let $f\ ,h\ : [a,b] \longrightarrow \mathbb{R}$ be two continuous non-decreasing functions. Then

$$\int_a^b f(x)\,h(x)\,dx \ge \frac{1}{b-a}\left(\int_a^b f(x)\,dx \cdot \int_a^b h(x)\,dx\right).$$

Proof. Let $x_k = a + \dfrac{(b-a)\,k}{n},\ k = 1, 2, \ldots, n$. Then for any integrable function g on the interval $[a,b]$ (in particular for any continuous function on $[a,b]$) we have

$$\lim_{n\to\infty} \frac{b-a}{n}\sum_{k=1}^{n} g(x_k) = \int_a^b g(x)\,dx.$$

Since n-tuples $(f(x_1), f(x_2), ..., f(x_n))$ and $(h(x_1), h(x_2), ..., h(x_n))$ are sorted similarly then by the Chebishev inequality

$$\sum_{k=1}^{n} h(x_k)\,f(x_k) \ge \frac{1}{n}\sum_{k=1}^{n} h(x_k) \cdot \sum_{k=1}^{n} f(x_k)$$

$$\Longleftrightarrow$$

$$\frac{b-a}{n}\sum_{k=1}^{n} h(x_k)\,f(x_k) \ge \frac{1}{b-a}\left(\frac{b-a}{n}\sum_{k=1}^{n} h(x_k) \cdot \frac{b-a}{n}\sum_{k=1}^{n} f(x_k)\right)$$

$$\Longrightarrow$$

$$\lim_{n\to\infty}\frac{b-a}{n}\sum_{k=1}^{n} h(x_k)\,f(x_k) \ge \frac{1}{b-a}\lim_{n\to\infty}\frac{b-a}{n}\sum_{k=1}^{n} h(x_k) \cdot \lim_{n\to\infty}\frac{b-a}{n}\sum_{k=1}^{n} f(x_k)$$

$$\Longleftrightarrow$$

$$\int_a^b f(x)\,h(x)\,dx \ge \frac{1}{b-a}\left(\int_a^b f(x)\,dx \cdot \int_a^b h(x)\,dx\right).$$

Since $\int\limits_0^1 x dx = \frac{1}{2}$ then applying the lemma to $h(x) = x$ and $a = 0, b = 1$ we obtain

$$\frac{1}{2}\int\limits_0^1 f(x)\,dx \le \int\limits_0^1 xf(x)\,dx.$$

Let us now prove the right hand side of the inequality.

Since $\frac{1}{2} \le x \le 1 \implies 1 - x \le x \implies f(1-x) \le f(x)$ then

$$(1-x) f(1-x) + xf(x) \le (1-x) f(x) + xf(x) = f(x)$$

and, therefore,

$$\int\limits_{\frac{1}{2}}^1 ((1-x) f(1-x) + xf(x))\,dx \le \int\limits_{\frac{1}{2}}^1 f(x)\,dx.$$

On the other hand, since $\int\limits_{\frac{1}{2}}^1 (1-x) f(1-x)\,dx = \int\limits_0^{\frac{1}{2}} xf(x)\,dx$ then

$$\int\limits_{\frac{1}{2}}^1 ((1-x) f(1-x) + xf(x))\,dx = \int\limits_0^1 xf(x)\,dx$$

and, therefore,

$$\int\limits_0^1 xf(x)\,dx \le \int\limits_{\frac{1}{2}}^1 f(x)\,dx.$$

Second solution by Michel Bataille, France

Let $F(x) = \int\limits_0^x f(t)\,dt$. The function F satisfies $F(0) = 0$ and $F'(x) = f(x)$ for all x in $[0,1]$. Also, from the hypothesis on f, F is a convex C^1-function. Integrating by parts, we have

$$\int_0^1 xf(x)\,dx = [xF(x)]_0^1 - \int_0^1 F(x)\,dx = F(1) - \int_0^1 F(x)\,dx = \int_0^1 f(x)\,dx - \int_0^1 F(x)\,dx \qquad (1).$$

Now, the left-hand inequality is equivalent to

$$\int_0^1 F(x)\,dx \le \frac{1}{2}F(1) \qquad (2).$$

From the convexity of F, the curve $y = F(x)$ is below the line though $(0, F(0))$ and $(1, F(1))$, hence $F(x) \le xF(1)$ for all x in $[0, 1]$. Thus,

$$\int_0^1 F(x)\,dx \le F(1)\int_0^1 x\,dx = \frac{1}{2}F(1)$$

and (2) holds.
Similarly, from (1) the right-hand inequality rewrites as

$$\int_0^1 F(x)\,dx \ge F\left(\frac{1}{2}\right) \qquad (3).$$

Again from the convexity of F, the curve $y = F(x)$ is above its tangent at $(\frac{1}{2}, F(\frac{1}{2}))$ and we have $F(x) \ge f(\frac{1}{2})(x - \frac{1}{2}) + F(\frac{1}{2})$ for all x in $[0, 1]$. Thus,

$$\int_0^1 F(x)\,dx \ge f\left(\frac{1}{2}\right)\int_0^1 \left(x - \frac{1}{2}\right)\,dx + F\left(\frac{1}{2}\right) = F\left(\frac{1}{2}\right)$$

and (3) follows.

Third solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

L.H.S. It is consequence of Chebysev's inequality for integrals. If $f\colon [a, b] \to \mathbf{R}$, $g\colon [a, b] \to \mathbf{R}$, are equally monotone, the following inequality holds

$$\frac{1}{b-a}\int_a^b f(x)dx \int_a^b g(x)dx \le \int_a^b f(x)g(x)dx$$

Since $\int_0^1 xdx = \frac{1}{2}$ the l.h.s. is clear by taking $g(x) \equiv x$ and observing that it is non decreasing as well as $f(x)$.

R.H.S.

$$\int_{\frac{1}{2}}^1 f(x)dx - \int_0^1 xf(x)dx = \int_{\frac{1}{2}}^1 f(x)dx - \int_0^{\frac{1}{2}} xf(x)dx - \int_{\frac{1}{2}}^1 xf(x)dx =$$
$$\int_{\frac{1}{2}}^1 (1-x)f(x)dx - \int_0^{\frac{1}{2}} xf(x)dx \ge f(1)\int_{\frac{1}{2}}^1 (1-x)dx - \int_0^{\frac{1}{2}} xf(x)dx =$$
$$f(1)\frac{1}{8} - \int_0^{\frac{1}{2}} xf(x)dx = f(1)\int_0^{\frac{1}{2}} xdx - \int_0^{\frac{1}{2}} xf(x)dx = \int_0^{\frac{1}{2}} x(f(1) - f(x))dx \ge 0$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Roberto Bosch Cabrera, Havana, Cuba; Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy; Shai Covo, Kiryat-Ono, Israel.

U149. Find all real numbers a for which there are functions $f, g : [0,1] \to \mathbb{R}$ such that

$$(f(x) - f(y))(g(x) - g(y)) \geq |x - y|^a$$

for all $x, y \in [0, 1]$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

No solution has yet been received.

U150. Let (a_n) and (b_n) be sequences of positive transcendental numbers such that for all positive integers p the series $\sum_n \left(a_n^p + b_n^p\right)$ converges. Suppose that for all positive integers p there is a positive integer q such that $\sum_n a_n^p = \sum_n b_n^q$. Prove that there is an integer r and a permutation σ of the set of positive integers such that

$$a_n = b_{\sigma(n)}^r.$$

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We may assume wlog that $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are non-increasing sequences, through the choice of an appropriate permutation $\sigma(n)$; we need then to prove that $a_n = b_n^r$ for some positive integer r and all $n \geq 1$. Under these conditions, we begin the solution by proving two lemmas:

Lemma 1: Let $(a_n)_{n\geq 1}$ be an infinite non-increasing sequence of positive reals, and u a positive integer, such that $a_n < a_1$ for all $n > u$. If the sum of all the elements in the sequence is finite, then for every $\epsilon > 0$ and every positive real constant K, a positive integer P exists such that

$$\frac{1}{a_1^p} \sum_{n>u} a_n^p < K\epsilon.$$

for all $p > P$.

Proof: By hypothesis, $\frac{1}{a_1} \sum_{n>u} a_n = S$ is clearly finite, and

$$\frac{1}{a_1^p} \sum_{n>u} a_n^p \leq \left(\frac{a_{u+1}}{a_1}\right)^{p-1} S,$$

with equality iff $a_n = a_{u+1}$ for all $n > u$. If $K\epsilon > S$, the result is trivially true for all $p \geq 1$, otherwise denote $L = \log\left(\frac{S}{K\epsilon}\right)$, hence it suffices to find P such that $(P-1)\log\left(\frac{a_1}{a_{u+1}}\right) > L$, clearly possible since $\frac{a_1}{a_{u+1}} > 1$. The lemma 1 follows.

Lemma 2: Let $y = mx + h$ be the equation of a line, such that, for all $\epsilon > 0$, a positive integer x_0 exists such that, for each integer $x > x_0$, the interval $[mx + h - \epsilon, mx + h + \epsilon]$ contains an integer. Then, m, h are integers.

Proof: For any integer x, denote by $z(x)$ the closest integer to $mx + h$, define $\delta(x) = mx + h - z(x)$, and assume wlog that $\epsilon < \frac{1}{8}$. Denoting by $\{x\}$ the fractional part of x, for all positive integer a and all $x > x_0$ we find that $\frac{1}{4} > 2\epsilon > |\delta(x+a) - \delta(x)| = |am - z(x+1) + z(x)|$, and for each positive integer a, either $\{am\} < \frac{1}{4}$ or $\{am\} > \frac{3}{4}$, in particular either $\{m\} < \frac{1}{4}$ or $\{-m\} < \frac{1}{4}$; exchanging m by $-m$ does not change the problem, so wlog $\{m\} < \frac{1}{4}$. Now, if $\{m\} \neq 0$, there is at least one integer A in the interval $(\frac{1}{4\{m\}}, \frac{3}{4\{m\}})$, since $\frac{3}{4\{m\}} - \frac{1}{4\{m\}} = \frac{1}{2\{m\}} > 2$, or $\frac{3}{4} > \{Am\} = \{A\{m\}\} > \frac{1}{4}$, contradiction, hence $\{m\} = 0$ and m is an integer. Moreover, if $\{h\} \neq 0$, taking $\epsilon < \min\{\{h\}, \{-h\}\}$, for all x we have $|\delta(x)| = \min\{\{h\}, \{-h\}\} > \epsilon$, contradiction, hence $\{h\} = 0$ and h is also an integer. The lemma 2 follows.

Denote now $q(p)$ the value of q such that $\sum_n a_n^p = \sum_n b_n^{q(p)}$ for a particular value of p, and denote by u, v the number of elements of (a_n) and (b_n) respectively equal to a_1 and b_1. We may then write

$$ua_1^p \left(1 + \frac{1}{a_1^p} \sum_{n>u} a_n^p\right) = vb_1^{q(p)} \left(1 + \frac{1}{b_1^{q(p)}} \sum_{n>v} b_1^{q(p)}\right).$$

Now, given any $\epsilon > 0$, and taking $K = |\log(b_1)|$, which is finite because $b_1 \neq 1$ and $b_1 > 0$, lemma 1 guarantees that a P exists such that, for any $p > P$,

$$\log(u) + p\log(a_1) < \log\left(ua_1^p \left(1 + \sum_{n>u} a_n^p\right)\right) <$$

$$< \log(u) + p\log(a_1) + \log\left(1 + |\log(b_1)|\epsilon\right) < \log(u) + p\log(a_1) + |\log(b_1)|\epsilon,$$

where we have used that for all positive x, $0 < \log(1+x) < x$. Similarly, again by lemma 1 applied to (b_n), a Q exists such that, for all $q > Q$,

$$\log(v) + q\log(b_1) < \log\left(vb_1^q \left(1 + \frac{1}{b_1^q} \sum_{n>v} b_n^q\right)\right) < \log(v) + q\log(b_1) + |\log(b_1)|\epsilon.$$

Clearly, whenever $p > P$ and simultaneously $q(p) > Q$,

$$-|\log(b_1)|\epsilon < \log(v) + q(p)\log(b_1) - \log(u) - p\log(a_1) < |\log(b_1)|\epsilon,$$

or equivalently,

$$p\frac{|\log(a_1)|}{|\log(b_1)|} \pm \frac{|\log(u) - \log(v)|}{|\log(b_1)|} - \epsilon < q(p) < p\frac{|\log(a_1)|}{|\log(b_1)|} \pm \frac{|\log(u) - \log(v)|}{|\log(b_1)|} + \epsilon.$$

Take now $m = \frac{|\log(a_1)|}{|\log(b_1)|}$ and $h = \pm\frac{|\log(u) - \log(v)|}{|\log(b_1)|}$, and we may apply the lemma 2. Now, if $u \neq v$, either $\frac{u}{v} = b_1^h$ or $\frac{u}{v} = \sqrt[h]{b_1}$, where h is an integer. Now, b_1 would not be trascendental, contradiction, or $u = v$, which in turn yields $h = 0$ and $q(p) = mp$. Now, we have assumed that for sufficiently large p, $q(p)$ is also sufficiently large. If $a_1 < 1$, note that $\sum_n a_n^p$ decreases with p without a positive lower bound, hence $b_1 < 1$, and $\sum_n b_n^q$ also decreases with q, or $q(p)$ increases with p. Conversely, if $a_1 > 1$, for sufficiently large p, the sum $\sum_n a_n^p$ is dominated by the terms with $a_n > 1$, and increases without upper positive bound with p, hence $b_1 > 1$, and the number of values of $q < Q$ such that $\sum_n b_n^q$ is at least equal to $\sum_n b_n^Q$ is finite, or for sufficiently large p, $q(p)$ must also be sufficiently large and may not take anymore any of those "low" values, and it must also increase with p. This shows first that our assumption that $q(p)$ increases with p for sufficiently large p is true, and moreover that $\log(a_1)$ and $\log(b_1)$ have the same sign, hence $a_1 = b_1^m$.

Now, we have found that, for sufficiently large p, $q(p) = mp$. For all these values of p, the first $u = v$ terms in $\sum_n a_n^p$ and $\sum_n b_n^{q(p)}$ cancel out, and we may repeat this process again with the following largest value of a_n and b_n. However, if $a_N \neq b_N$ for some N, repeating the previous process N times we find that $a_N = b_N$, contradiction. Hence $a_n = b_n$ for all n. The conclusion follows.

Olympiad problems

O145. Find all positive integers n for which

$$\left(1^4+\frac{1}{4}\right)\left(2^4+\frac{1}{4}\right)\cdots\left(n^4+\frac{1}{4}\right)$$

is the square of a rational number.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Arkady Alt, San Jose, California, USA

Let $P=\prod_{k=1}^{n}\left(k^4+\frac{1}{4}\right)$ and let $a_k=k^2-k+\frac{1}{2}, k=1,2,\ldots,n$. Since $a_{k+1}=k^2+k+\frac{1}{2}$ and $k^4+\frac{1}{4}=\left(k^2+\frac{1}{4}\right)-k^2=a_ka_{k+1}, k=1,2,\ldots,n$ then

$$P=a_1a_{n+1}Q^2=\frac{1}{4}\cdot\left(2n^2+2n+1\right)Q^2,$$

where $Q=\prod_{k=2}^{n}a_k$. Therefore, P is the square of a rational number if and only if $2n^2+2n+1$ is the square of a positive integer, i.e. if and only if $2n^2+2n+1=m^2$ for some positive integer m. Therefore $2m^2-(2n+1)^2=1$ and then our problem is finding the solutions to the equation

$$x^2-2y^2=-1$$

in positive integers. Let

$$\mathbb{Z}\left(\sqrt{2}\right)=\left\{x+y\sqrt{2}\,\vdots\,x,y\in\mathbb{Z}\right\},\ \mathbb{N}\left(\sqrt{2}\right):=\left\{x+y\sqrt{2}\,\vdots\,x,y\in\mathbb{N}\right\}$$

and let $s=3+2\sqrt{2}$ and for any $z=x+y\sqrt{2}\in\mathbb{Z}\left(\sqrt{2}\right)$ denote $\overline{z}=x-y\sqrt{2}$. In this notation equation $x^2-2y^2=-1,\ \ x,y\in\mathbb{N}$ becomes $z\overline{z}=-1, z\in\mathbb{N}\left(\sqrt{2}\right)$. Denote the set of all such solutions by Sol, i.e.

$$Sol=\left\{z\,\vdots\,z\in\mathbb{N}\left(\sqrt{2}\right)\text{ and }z\overline{z}--1\right\}.$$

Note that for $z_0=1+\sqrt{2}\in\mathbb{N}\left(\sqrt{2}\right)$ we have $z_0\overline{z}_0=-1$ and since $s\overline{s}=1$ then for $z_k=s^kz_0$ we also have $z_k\overline{z}_k=-1,\ k\in\mathbb{N}$. Also it is clear that z_0 is smallest element in Sol. Note that if $z \in Sol$ (that is $z\overline{z} = -1, z \in \mathbb{N}\left(\sqrt{2}\right))$ and $z \neq z_0$ then $\overline{s}z \in Sol$. Indeed, $\overline{s}z \cdot s\overline{z} = \overline{s}s \cdot z\overline{z} = 1 \cdot (-1) = -1$. It remains to prove $\overline{s}z \in \mathbb{N}\left(\sqrt{2}\right)$. Let $z = x + y\sqrt{2}$ then $x^2 = 2y^2 - 1$ and

$$\overline{s}z = \left(3 - 2\sqrt{2}\right)\left(x + y\sqrt{2}\right) = 3x - 4y + (-2x + 3y)\sqrt{2}.$$

Since $z \neq z_0 \implies z > z_0 \implies x, y \geq 2$, and moreover, $x, y \geq 3$ because x is odd and $2y^2 - 1$ isn't square of integer for $y = 2$, we have

$$3x \geq 4y \iff 9x^2 \geq 16y^2 \iff 9\left(2y^2 - 1\right) \geq 16y^2 \iff 2y^2 \geq 9 \impliedby y \geq 3$$

and

$$3y \geq 2x \iff 9y^2 \geq 4x^2 \iff 9y^2 \geq 4\left(2y^2 - 1\right) \impliedby y^2 + 4 \geq 0.$$

We will prove that $Sol = \left\{z_k \vdots\ k \in \mathbb{N} \cup \{0\}\right\}$. Suppose that exist $z \in Sol$ which not belong to the sequence $z_0 < z_1 < z_2 < \cdots < z_k <, \ldots$. Since $(z_k)_{k\geq 0}$ is unbounded from above then there is k such that $z_k < z < z_{k+1}$. Since $z_0 < \overline{s}^k z < z_1 = sz_0 = 7 + 5\sqrt{2}$ then $\overline{s}^{k+1}z < z_0$ and $\overline{s}^{k+1}z \in Sol$. That is the contradiction because z_0 is smallest element in Sol. Thus,

$$Sol = \left\{z_k \vdots\ k \in \mathbb{N} \cup \{0\}\right\}.$$

Let $z_k = x_k + y_k\sqrt{2}, k \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned} z_{k+1} = sz_k \iff & \begin{cases} x_{k+1} = 3x_k + 4y_k \\ y_{k+1} = 2x_k + 3y_k \end{cases} \\ \implies & x_{k+2} - 3x_{k+1} = 2x_k + 3\left(x_{k+1} - 3x_k\right) \\ \iff & x_{k+2} - 6x_{k+1} + 7x_k = 0, x_0 = 1, x_1 = 7. \end{aligned}$$

Since $x = 2n + 1$ then all natural n for which $2n^2 + 2n + 1$ is a square of integer should be elements of set $\left\{n_k \vdots n_k = \dfrac{x_k - 1}{2},\ k = 1, 2, \ldots\right\}$. By substitution $x_k = 2n_k + 1$ in $x_{k+2} - 6x_{k+1} + 7x_k = 0$ we obtain

$$2n_{k+2} + 1 - 12n_{k+1} - 6 + 14n_k + 7 = 0 \iff n_{k+2} = 6n_{k+1} - 7n_k - 1$$

Thus, all solutions of problem are the terms of the sequence $(n_k)_{k\geq 1}$ defined recursively by

$$n_{k+2} = 6n_{k+1} - 7n_k - 1, n_0 = 0, n_1 = 3.$$

In particular, $n_1 = 3, n_2 = 17, n_3 = 80, n_4 = 360, \ldots$

Second solution by Maxim Ignatiuc,liceul teoretic "Orizont", Moldova

Multiplying by 4 does not affect the property of our number M, we can consider

$$M = \prod \left(4k^4 + 1\right)$$

Observe,that our number can be written in this form:

$$\prod \left(4k^4 + 1\right) = \prod (2k^2 + 2k + 1)(2k^2 - 2k + 1)$$

As the factors will be repeated, we will obtain squares, except the last factor.

$$\prod \left(4k^4 + 1\right) = \prod \left((2k^2 + 2k + 1)(2k^2 - 2k + 1)\right) = \prod (2k^2-2k+1)^2 \cdot (2n^2+2n+1)$$

Let $2n^2 + 2n + 1$ be equal to c^2

Then,

$$(2n + 1)^2 + 1 = 2(2n^2 + 2n + 1) = 2c^2$$

Consequently,we have the equation:

$$(2n + 1)^2 - 2c^2 = -1$$

Using Pell's equation,we can consider two general cases:

$$x^2 - 2y^2 = -1 \qquad (1)$$

and

$$x^2 - 2y^2 = 1$$

Obviously,that minimal roots of first equation are $x_0 = 1$ $y_0 = 1$ and the minimal roots of the second ecuation are $x'_0 = 3$ $y'_0 = 2$

$$x_k\sqrt{1} + y_k\sqrt{2} = (3\sqrt{1} + 2\sqrt{2})^n \cdot (1\sqrt{1} + \sqrt{2}) = (1 + \sqrt{2})^{2k+1} = A \qquad (2)$$

Consequently,

$$x_k\sqrt{1} - y_k\sqrt{2} = (1 - \sqrt{2})^{2k+1} = B$$

is the result of division of (2) by (1)

Then,

$$2n + 1 = x_k = (A + B)/2$$

$$n = (A + B - 2)/4$$

Substitute the values of A and B and derive

$$n = \frac{(1+\sqrt{2})^{2k+1} - (1-\sqrt{2})^{2k+1} - 2}{4}$$

Also solved by Aravind Srinivas L, Chennai, India; Ercole Suppa, Teramo, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Michel Bataille, France; Mohammed Kharbach, Dolphin Energy, UAE; Roberto Bosch Cabrera, Havana, Cuba; Simon Morris.

O146. Find all pairs (m, n) of positive integers such that $\phi(\phi(n^m)) = n$, where ϕ Euler's totient function.

Proposed by Marco Antonio Avila Ponce de Leon, Mexico

Solution by Carlo Pagano, Universit'a di Roma Tor Vergata, Roma, Italy

We will show that the pairs are $(m, 1)$ for all positive integers m, $(3, 2)$ and $(2, 4)$.

We first note that:

1) since $\phi(x)$ is odd iff $x = 1$ or $x = 2$ then $\phi(\phi(n^m)) = n$ for n odd iff $n = 1$ and for all positive integers m;
2) since $\phi(n) < n$ when $n > 1$, then if $m = 1$ then $n = 1$.

So we can assume that $n = 2^k j$, with $k \geq 1$, $j \geq 1$ odd and $m \geq 2$.

We first consider the case when $j \neq 1$. Then, by 1), $\phi(j^m) = 2^s h$ with $s \geq 1$, $h \geq 1$ odd and

$$\phi(\phi(n^m)) = \phi(\phi(2^{km} j^m)) = \phi(2^{km-1} \cdot 2^s h) = 2^{km+s-2}\phi(h) = n = 2^k j.$$

Hence $km + s - 2 \leq k$, that is $k(m-1) \leq 2 - s \leq 1$ which implies that $k = 1$, $m = 2$, and $s = 1$. Then $\phi(h) = j$ and by 1) we have $h = j = 1$ which is a contradiction.

Finally assume that $j = 1$. Then $n = 2^k$ and

$$\phi(\phi(2^{km})) = 2^{km-2} = 2^k$$

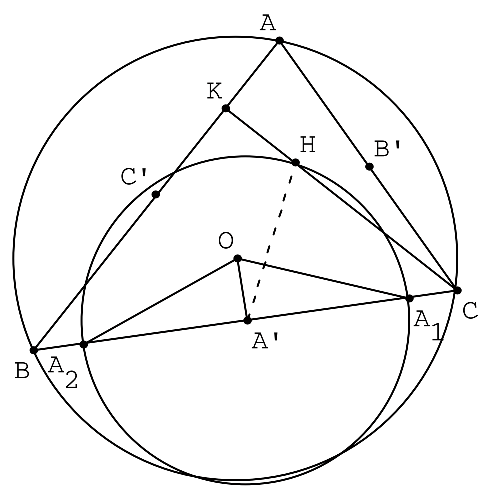
which implies that $k(m-1) = 2$, that is $(n, m) = (2, 3)$ or $(n, m) = (4, 2)$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Mohammed Kharbach, Dolphin Energy, UAE; Roberto Bosch Cabrera, Havana, Cuba; Vahagn Aslanyan, Yerevan, Armenia.

O147. Let H be the orthocenter of an acute triangle ABC, and let A', B', C' be the midpoints of sides BC, CA, AB. Denote by A_1 and A_2 the intersections of circle $C(A', A'H)$ with side BC. In the same way we define points B_1, B_2 and C_1, C_2, respectively. Prove that points $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclic.

Proposed by Catalin Barbu, Bacau, Romania

First solution by Michel Bataille, France



Considering the power of A_1 with respect to the circumcircle $C(O,R)$ of ΔABC, we obtain

$$OA_1^2 - R^2 = \overrightarrow{A_1C} \cdot \overrightarrow{A_1B} = A_1A'^2 - \frac{BC^2}{4} \qquad (1).$$

Let A, B, C denote the angles of the triangle. If K is the orthogonal projection of C onto AB, we clearly have $\angle BCK = 90^\circ - B$, hence the law of cosines in triangle CHA' yields

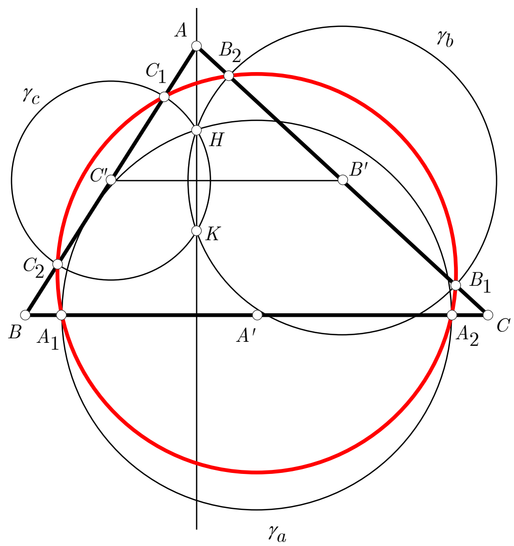
$$A'H^2 = A'C^2 + CH^2 - 2A'C \cdot CH \cdot \cos(90^\circ - B) = \frac{BC^2}{4} + 4OC'^2 - 2BC \cdot OC' \sin B \qquad (2)$$

where we have used the well-known $CH = 2OC'$.
Observing that $OC' = R\cos C$ (since C is acute), $BC = 2R\sin A$ and $A'H = A_1A'$, (1) and (2) readily give $OA_1^2 = R^2 + 4R^2\cos C(\cos C - \sin A \sin B)$.
Now, $\cos C - \sin A \sin B = -\cos(A+B) - \sin A \sin B = -\cos A \cos B$ so that

$$OA_1^2 = R^2(1 - 4\cos A\cos B\cos C).$$

Owing to the symmetry of the result, we see that $OA_1 = OA_2 = OB_1 = OB_2 = OC_1 = OC_2$ and $A_1, A_2, B_1, B_2, C_1, C_2$ are all on the same circle (with centre O).

Second solution by Ercole Suppa, Teramo, Italy



Let γ_a, γ_b, γ_c be the circles with centers A', B', C' and radii $A'H$, $B'H$, $C'H$ respectively. Denote by K the second intersection point of γ_b and γ_c, besides H.

Since $B'C' \parallel BC$ and $AH \perp BC$ we have $AH \perp B'C'$. Thus AH is the radical axis of γ_b and γ_c, so $K \in AH$. The power of a point theorem implies

$$AB_1 \cdot AB_2 = AH \cdot AK = AC_1 \cdot AC_2$$

so the points B_1, B_2, C_1, C_2 are concyclic. Since the axes of B_1B_2 and C_1C_2 intersect at the circumcenter O of $\triangle ABC$, it follows that B_1, B_2, C_1, C_2 lies on the circle with center O and radius OB_1.

Similarly, we can prove that B_1, B_2, A_1, A_2 lies on the circle with center O and radius OB_1. Therefore A_1, A_2, B_1, B_2, C_1, C_2 are concyclic and the proof is complete.

Also solved by Aravind Srinivas L, Chennai, India; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Roberto Bosch Cabrera, Havana, Cuba; Raul A. Simon, Chile.

O148. Let ABC be a triangle and let A_1, A_2 be the intersections of the trisectors of angle A with the circumcircle of ABC. Define analogously points B_1, B_2, C_1, C_2. Let A_3 be the intersection of lines B_1B_2 and C_1C_2. Define analogously B_3 and C_3. Prove that the incenters and circumcenters of triangles ABC and $A_3B_3C_3$ are collinear.

Proposed by Daniel Campos Salas, Costa Rica

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We will show that the proposed result is true iff ABC is isosceles. If ABC is isosceles, wlog at A, then ABC is symmetric with respect to the perpendicular bisector t of BC, hence line $A_3B_3 = C_1C_2$ is the symmetric of $A_3C_3 = B_1B_2$ with respect to t, hence $A_3 \in t$. Moreover, line $B_3C_3 = A_1A_2 \perp t$, or B_3 is the symmetric of C_3 with respect to t, and $A_3B_3C_3$ is symmetric with respect to t. Clearly the circumcenter and incenter of an isosceles triangle lie on its axis of symmetry, hence the incenters and circumcenters of ABC and $A_3B_3C_3$ lie on t.

Assume now that ABC is not isosceles. Denote $X \equiv (\alpha_X, \beta_X, \gamma_X)$ the exact trilinear coordinates of point X. Assume wlog that A_1, A_2 are chosen so that cyclic quadrilateral BCA_2A_1 is convex. Since $BA_1 = CA_2$, then BCA_2A_1 is an isosceles trapezium, or line $B_3C_3 = A_1A_2 \parallel BC$. Moreover, since $\angle A_1AB = \frac{A}{3}$ and $\angle A_1BC = \angle A_1AC = \frac{2A}{3}$, then the distance from BC to B_3C_3 is $BA_1 \sin\frac{2A}{3} = 2R\sin\frac{A}{3}\sin\frac{2A}{3} = R\cos\frac{A}{3} - R\cos A$, and similarly for its cyclic permutations. Calling d_a, d_b, d_c the respective distances between B_3C_3 and BC, between C_3A_3 and CA, and between A_3B_3 and AB, we conclude that the exact trilinear coordinates of A_3 are

$$A_3 \equiv \left(\frac{2S + bd_b + cd_c}{a}, -d_b, -d_c\right),$$

where S is the area of ABC, hence line AA_3 is defined in trilinear coordinates by $\frac{\beta}{d_b} = \frac{\gamma}{d_c}$. By cyclic permutation, the point P where AA_3, BB_3, CC_3 meet satisfies $\frac{\alpha_P}{d_a} = \frac{\beta_P}{d_b} = \frac{\gamma_P}{d_c} = \lambda$ for some positive real λ. Since $a\alpha + b\beta + c\gamma = 2S$ for exact trilinear coordinates, it follows that $\lambda = \frac{2S}{ad_a+bd_b+cd_c}$, or

$$P \equiv \left(\frac{2d_aS}{ad_a + bd_b + cd_c}, \frac{2d_bS}{ad_a + bd_b + cd_c}, \frac{2d_cS}{ad_a + bd_b + cd_c}\right).$$

Now, since $O \equiv (R\cos A, R\cos B, \cos C)$ and $I \equiv (r, r, r)$, where R, r are the respective circumcenter and incenter of ABC, P will be on line OI iff

$$\begin{vmatrix} r & r & r \\ Kd_a & Kd_b & Kd_c \\ R\cos A & R\cos B & R\cos C \end{vmatrix} = 0,$$

$$0 = d_a(\cos B - \cos C) + d_b(\cos C - \cos A) + d_c(\cos A - \cos B) =$$

$$\cos\frac{A}{3}(\cos B - \cos C) + \cos\frac{B}{3}(\cos C - \cos A) + \cos\frac{C}{3}(\cos A - \cos B) = 0,$$

where we have defined $K = \frac{2S}{ad_a+bd_b+cd_c}$. Now, denote $x = \cos\frac{A}{3}$, $y = \cos\frac{B}{3}$ and $z = \cos\frac{C}{3}$. Clearly $\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3} = 4x^3 - 3x$, and similarly for its cyclic permutations, or $P \in OI$ iff

$$0 = x(4y^3 - 3y - 4z^3 + 3z) + y(4z^3 - 3z - 4x^3 + 3x) + z(4x^3 - 3x - 4y^3 + 3y),$$

$$0 = xy^3 - xz^3 + yz^3 - yx^3 + zx^3 - y^3z = (x-y)(y-z)(z-x)(x+y+z).$$

Now, x, y, z are clearly positive, and since ABC is not isosceles, $x \neq y$, $y \neq z$ and $z \neq x$, or P does not lie on line OI. Now, if the incenter I_3 and circumcenter O_3 of $A_3B_3C_3$ are collinear with O, I, they would lie on OI. But since ABC and $A_3B_3C_3$ are homologous, P would lie on lines $OO_3 = OI$ and in $II_3 = IO$, contradiction. Hence O, I, O_3, I_3 are not collinear if ABC is not isosceles.

For a numerical example, consider a triangle with $A = 90°$, $B = 60°$ and $C = 30°$. The condition

$$0 = d_a(\cos B - \cos C) + d_b(\cos C - \cos A) + d_c(\cos A - \cos B)$$

then rewrites as $-\frac{3}{4} + \frac{\sqrt{3}\cos 20°}{2} - \frac{\cos 10°}{2} + \frac{\sqrt{3}}{4}$, which is not true, since the LHS can be calculated to be positive but less than $5 \cdot 10^{-3}$, ie, very small in comparison with the terms whose sum results in this value. No wonder that any approximate drawing, made either by hand or with any software such as Cabri or Geogebra (approximate since an angle cannot be trisected using ruler and compass), places P so close to IO that it seems to be on it...

Also solved by Raul A. Simon, Chile.

O149. A circle is divided into n equal sectors. We color the sectors in $n-1$ colors using each of the colors at least once. How many such colorings are there?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Roberto Bosch Cabrera, Havana, Cuba

There exists at least an color repeated by pigeonhole principle, and this color appear exactly two times, because in other case we have $n-k$ sectors remaining and $n-2$ colors, impossible, since all colors are used. Let $c_1, ..., c_n$ the n colors, we can enumerate all colorings counting these that involve each pair, $(c_1 - c_1), ..., (c_n - c_n)$ because if we have a coloring with two pairs $(c_i - c_i), (c_j - c_j)$ then the $n-4$ sectors remaining can not be colored using $n-3$ colors. By each pair we obtain $\binom{n}{2}(n-2)!$ colorings, so in total we have

$$\frac{n!(n-1)}{2}$$

colorings.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

O150. Let n be a positive integer, $\epsilon_0, \ldots, \epsilon_{n-1}$ be the nth roots of unity, and a, b complex numbers. Evaluate the product

$$\prod_{k=0}^{n-1}\left(a+b\epsilon_k^2\right).$$

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania

First solution by Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy

1) If n is odd then multiplication by 2 acts as a permutation on $\mathbb{Z}_n$ and therefore

$$P=\prod_{i=0}^{n-1}(a+b\epsilon_i)=\sum_{i=0}^{n-1}a^{n-i}b^i\sigma_i.$$

where $\sigma_i=\sum_{0\le j_1<\cdots<j_i\le n-1}\epsilon_{j_1}\cdots\epsilon_{j_i}$. Since

$$\sigma_0=1, \quad \sigma_{n-1}=(-1)^{n-1}, \quad \sigma_j=0 \text{ for } j=1,\ldots,n-2$$

we have that $P=a^n+b^n$ for n odd.

2) If n is even, since $\epsilon_{n/2+i}=-\epsilon_i$ then

$$P=\left(\prod_{i=0}^{n/2-1}(a+b(\epsilon_i)^2)\right)^2=\left(\prod_{i=0}^{n/2-1}(a+b\omega_i)\right)^2$$

where $\omega_0,\ldots,\omega_{n/2-1}$ are the $n/2$-th roots of the unity. Hence, as before we find that $P=(a^{n/2}-(-b)^{n/2})^2$ for n even.

Second solution by Arkady Alt, San Jose, California, USA

From the given we know that $\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{n-1}$ are roots of equation $u^n-1=0, u\in\mathbb{C}$ then

$$\prod_{k=1}^{n-1}(u-\varepsilon_k)=z^n-1.$$

Let z be a solution of the equation $z^2=-\frac{a}{b}$ then

$$\prod_{k=1}^{n-1}\left(a+b\varepsilon_k^2\right)=\prod_{k=1}^{n-1}(-b)\left(\frac{a}{-b}-\varepsilon_k^2\right)=(-b)^n\prod_{k=1}^{n-1}\left(z^2-\varepsilon_k^2\right)$$

$$=(-b)^n\prod_{k=1}^{n-1}(z-\varepsilon_k)\prod_{k=1}^{n-1}(z+\varepsilon_k)=b^n\prod_{k=1}^{n-1}(z-\varepsilon_k)\prod_{k=1}^{n-1}((-z)-\varepsilon_k)$$

$$=b^n(z^n-1)((-z)^n-1)=b^n\left((-1)^n z^{2n}-z^n((-1)^n+1)+1\right).$$

If n is even, that is $n = 2m$, then

$$\prod_{k=1}^{n-1}\left(a+b\varepsilon_k^2\right) = b^{2m}\left(z^{4m}-2z^{2m}+1\right) = b^{2m}\left(\left(-\frac{a}{b}\right)^{2m}-2\left(-\frac{a}{b}\right)^{m}+1\right)$$
$$= a^{2m}-2a^m(-b)^m+b^{2m} = \left(a^m-(-b)^m\right)^2.$$

If n is odd then

$$\prod_{k=1}^{n-1}\left(a+b\varepsilon_k^2\right) = b^n\left(-z^{2n}+1\right) = b^n\left(-\left(-\frac{a}{b}\right)^n+1\right) = a^n+b^n.$$

Thus,

$$\prod_{k=1}^{n-1}\left(a+b\varepsilon_k^2\right) = \begin{cases} \left(a^{\frac{n}{2}}-(-b)^{\frac{n}{2}}\right)^2 \text{ if } n \text{ is even} \\ a^n+b^n \text{ if } n \text{ is odd} \end{cases}.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Michel Bataille, France; Roberto Bosch Cabrera, Havana, Cuba.