## Junior problems

J199. Prove that there are infinitely many pairs $(p, q)$ of primes such that $p^6 + q^4$ has two positive divisors whose difference is $4pq$.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

J200. Let $x$, $y$, $z$ be positive real numbers with $x \leq 2$, $y \leq 3$, and $x + y + z = 11$. Prove that $xyz \leq 36$.

*Proposed by Mircea Lascu, Zalau, and Marius Stanean, Zalau, Romania*

J201. Let $ABC$ be an isosceles triangle with $AB = AC$. Point $D$ lies on side $AC$ such that $\angle CBD = 3\angle ABD$. If

$$\frac{1}{AB} + \frac{1}{BD} = \frac{1}{BC},$$

find $\angle A$.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

J202. Let $ABC$ be a triangle with incenter $I$ and let $A_1, B_1, C_1$ be the symmetric points of $I$ with respect to the midpoints of sides $BC, CA, AB$. If $I_a, I_b, Ic$ denote the excenters corresponding to sides $BC, CA, AB$, respectively, prove that lines $I_aA_1, I_bB_1, I_cC_1$ are concurrent.

*Proposed by Dorin Andrica, Babes-Bolyai University, Romania*

J203. Let $ABCD$ be a trapezoid $(AB \| CD)$ with acute angles at vertices $A$ and $B$. Denote by $E$ the reflection of $A$ with respect to $B$. Line $BC$ and the tangent lines from $A$ and $E$ to the circle of center $D$ tangent to $AB$ are concurrent at $F$. Prove that $AC$ bisects the segment $EF$ if and only if $AF + EF = 4AB$.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

J204. Give a straightedge and compass construction of a triangle $ABC$ starting with its incenter $I$, the foot of the altitude from $A$, and the midpoint of the side $BC$.

*Proposed by Cosmin Pohoata, Princeton University, Princeton, NJ*

## Senior problems

S199. In triangle $ABC$ let $BB'$ and $CC'$ be the angle bisectors of $\angle B$ and $\angle C$. Prove that

$$B'C' \geq \frac{2bc}{(a+b)(a+c)}\left[(a+b+c)\sin\frac{A}{2} - \frac{a}{2}\right].$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

S200. On each vertex of the regular hexagon $A_1A_2A_3A_4A_5A_6$ we place a rod. On each rod we have $a_i$ rings, where $a_i$ corresponds to the vertex $A_i$. Taking a ring from any three adjacent rods we can create chains of three rings. What is the maximum number of such chains that we can create?

*Proposed by Arkady Alt, San Jose, USA*

S201. Prove that in any triangle,

$$r_a \leq 4R\sin^3\left(\frac{A}{3} + \frac{\pi}{6}\right).$$

*Proposed by Dorin Andrica, Babes-Bolyai University, Romania*

S202. Let $a$ and $b$ be integers such that $a^2m - b^2n = a - b$, for some consecutive integers $m$ and $n$. Prove that $\gcd(a, b) = \sqrt{|a-b|}$.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

S203. Let $ABC$ be a triangle, and $P$ a point not lying on its sides. Call $XYZ$ the cevian triangle of $P$ with respect $ABC$ and consider the points $Y_a$, $Z_a$ of intersection of $BC$ with the parallel lines to $AX$ through $Y$ and $Z$ respectively. Prove that $AX$, $YZ_a$, $Y_aZ$ concur in a point $Q$ that satisfies the cross-ratio

$$(AXPQ) = \frac{AP}{AX}.$$

*Proposed by Francisco Javier Garcia Capitan, Spain*

S204. Find all positive integers $k$ and $n$ such that $k^n - 1$ and $n$ are divisible by precisely the same primes.

*Proposed by Tigran Hakobyan, Yerevan, Armenia*

## Undergraduate problems

U199. Prove that in any triangle $ABC$,

$$3\sqrt{3} \leq \cot\frac{A+B}{4} + \cot\frac{B+C}{4} + \cot\frac{C+A}{4} \leq \frac{3\sqrt{3}}{2} + \frac{s}{2r},$$

where $s$ and $r$ denote the semiperimeter and the inradius of triangle $ABC$, respectively.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

U200. Let $p$ be an odd prime and let $n$ be an integer greater than 1. Find all integers $k$ for which there exists an $n \times n$ matrix $A$ of rank $k$ such that $A + A^2 + ... + A^p = 0$.

*Proposed by Gabriel Dospinescu, Ecole Polytehnique, France*

U201. Evaluate

$$\sum_{n=2}^{\infty} \frac{3n^2 - 1}{(n^3 - n)^2}.$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

U202. The interval $(0, 1]$ is divided into $N$ equal intervals $(\frac{i-1}{N}, \frac{i}{N}]$, where $i \in [1, n]$. An interval is called *special* if it contains at least one number from the set $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N}\}$. Find a good approximation for the number of special intervals.

*Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

U203. Let $P$ be a polynomial of degree 5, with real coefficients, all whose zeros are real. Prove that for each real number $a$ that is not a zero of $P$ or $P'$ there is a real number $b$ such that

$$b^2 P(a) + 4bP'(a) + 5P''(a) = 0.$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

U204. Let $A_1 A_2 \ldots A_n$ be a convex polygon and let $P$ be a point in its interior. Prove that

$$\min_{i \in \{1,2,\ldots,n\}} \angle P A_i A_{i+1} \leq \frac{\pi}{2} - \frac{\pi}{n}.$$

*Proposed by Hojoo Lee, Seoul, South Korea, and Cosmin Pohoata, Princeton University, Princeton, NJ*

## Olympiad problems

O199. Prove that an acute triangle with $A = 20^\circ$ and side-lengths $a, b, c$ satisfying

$$\sqrt[3]{a^3 + b^3 + c^3 - 3abc} = \min(b, c)$$

is isosceles.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

O200. Determine all the primes that do not have a multiple in the sequence $a_n = 2^n n^2 + 1$, $n \geq 1$.

*Proposed by Andrei Ciupan, Harvard University, Cambridge, MA*

O201. Let $ABC$ be a triangle with circumcenter $O$, and let perpendiculars at $B$, $C$ to $BA$, $CA$ intersect the sidelines $CA$, $AB$ at $E$, $F$, respectively. Prove that the perpendiculars to $OB$ and $OC$ at $F$ and $E$, respectively intersect at a point $L$ lying on the altitude $AD$, and satisfying $DL = LA \sin^2 A$.

*Proposed by Francisco Javier Garcia Capitan, Spain*

O202. Find all pairs $(x, y)$ of positive integers for which there is a nonnegative integer $z$ such that

$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) = 1 + \left(\frac{2}{3}\right)^z.$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

O203. Let $M$ be an arbitrary point on the circumcircle of triangle $ABC$ and let the tangents from this point to the incircle meet the sideline $BC$ at $X_1$, and $X_2$. Prove that the second intersection of the circumcircle of triangle $MX_1X_2$ with the circumcircle of $ABC$ coincides with the tangency point of the circumcircle with the $A$-mixtilinear incircle. (As usual, the $A$-mixtilinear incircle names the circle tangent to $AB$, $AC$ and to the circumcircle of $ABC$ internally).

*Proposed by Cosmin Pohoata, Princeton University, Princeton, NJ*

O204. Alice and Bob play the following game: Alice has a pawn in the upper-left unit cell of a $(2m+1) \times (2n+1)$ board that she wants to move to the low-right cell after a finite number of steps. At each step she is allowed to move the pawn to an adjacent square, while Bob chooses a particular cell to "block", but still so that Alice still has a path from her current position to the low-right corner via adjacent cells. What is the maximum number of moves Bob can force Alice to make?

*Proposed by Radu Bumbacea, Bucharest, Romania*