Junior problems

J85. Let a and b be positive real numbers. Prove that

$$\sqrt[3]{\frac{(a+b)(a^2+b^2)}{4}} \geq \sqrt{\frac{a^2+ab+b^2}{3}}.$$

Proposed by Arkady Alt, San Jose, California, USA

J86. A triangle is called α-angular if none of its angles exceeds α degrees. Find the least α for which each non α-angular triangle can be dissected into some α-angular triangles.

Proposed by Titu Andreescu, University of Texas at Dallas and Gregory Galperin, Eastern Illinois University, USA

J87. Prove that for any acute triangle ABC, the following inequality holds:

$$\frac{1}{-a^2+b^2+c^2} + \frac{1}{a^2-b^2+c^2} + \frac{1}{a^2+b^2-c^2} \geq \frac{1}{2Rr}.$$

Proposed by Mircea Becheanu, Bucharest, Romania

J88. Find the greatest n for which there are points $P_1, P_2, \ldots, P_n$ in the plane such that each triangle whose vertices are among $P_1, P_2, \ldots, P_n$, has a side less than 1 and a side greater than 1.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

J89. Let A and B lie on circle $\mathcal{C}$ of center O and let C be the point on the small arc AB such that OA is the external angle bisector of $\angle BOC$. Denote by M the midpoint of BC and by N the intersection of AM and OC. Prove that the intersection of the angle bisector of $\angle BOC$ with the circle of center O and radius ON is the center of the circle tangent to lines OB and OC, and also internally tangent to $\mathcal{C}$.

Proposed by Francisco Javier Garcia Capitan, Spain

J90. For a fixed positive integer n let $a_k = 2^{2^{k-n}} + k$, $k = 0, 1, \ldots, n$. Prove that

$$(a_1 - a_0) \cdots (a_n - a_{n-1}) = \frac{7}{a_1 + a_0}.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Senior problems

S85. Find the least real number r such that for each triangle with sidelength a, b, c,

$$\frac{\max(a,b,c)}{\sqrt[3]{a^3+b^3+c^3+3abc}} < r.$$

Proposed by Titu Andreescu, University of Texas at Dallas

S86. An equilateral triangle is dissected into n^2 equilateral triangles of side 1. How many regular hexagons appear?

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

S87. Let ABC be a triangle. The incircle $C(I, r)$ and the excicle $C(I_A, r_a)$ corresponding to the vertex A are tangent to AB at points D and E, respectively. Prove that the lines IE and I_aD intersect on BC if and only if $AB \perp BC$.

Proposed by Ciupan Andrei Laurentiu, Tudor Vianu High School, Romania

S88. Let a, b, c, d be non-negative real numbers. Prove that

$$a^2+b^2+c^2+d^2+1+abcd \geq ab+bc+cd+da+ac+bd.$$

Proposed by Alex Anderson, New Trier Township High School, Winnetka, USA

S89. Let ABC be an acute triangle. Prove that the following conditions are equivalent:

(i) For any point $M \in (AB)$ and any point $N \in (AC)$ one may construct a triangle with sides CM, BN, MN.

(ii) $AB = AC$.

Proposed by Mircea Becheanu, Bucharest, Romania

S90. Prove that

$$\sum_{i=0}^{3n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{n-1+3i-10j}{n-1} = \frac{10^n+2}{3}.$$

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

Undergraduate problems

U85. Evaluate

$$\text{a)} \sum_{k=1}^{\infty} \frac{1}{1^3 + 2^3 + \cdots + k^3} \qquad\qquad \text{b)} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{1^3 + 2^3 + \cdots + k^3}$$

Proposed by Brian Bradie, Christopher Newport University, USA

U86. Determine all non-degenerate triangles with angles α, β, γ in radians and sides $\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma}$.

Proposed by Daniel Campos Salas, Costa Rica

U87. Let $f : (0, \infty) \to (0, \infty)$ be an unbounded function and let β be a positive real number. If for every $\alpha > 0$ we have

$$\lim_{x \to 0^+} (f(x) - \alpha f^{\beta}(\alpha x)) = 0,$$

prove that $\lim_{x \to 0^+} f(x) = 0$.

Proposed by Dorin Andrica, Babes-Bolyai University and Mihai Piticari, Campulung Moldovenesc, Romania

U88. Consider the sequence

$$a_n = \int_1^n \frac{dx}{(1 + x^2)^n}.$$

Evaluate $\lim_{n \to \infty} n \cdot 2^n \cdot a_n$.

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College, Romania

U89. Let $f : [0, \infty) \to [0, a]$ a continuous function on $(0, \infty)$ with a Darboux property on $[0, \infty)$ and $f(0) = 0$. Prove that if

$$x f(x) \geq \int_0^x f(t) dt,$$

for every $x \in [0, \infty)$, then f has an antiderivative.

Proposed by Dorin Andrica, Babes-Bolyai University and Mihai Piticari, Campulung Moldovenesc, Romania

U90. Let α be a real number greater than 2. Evaluate

$$\sum_{n=1}^{\infty} \left(\zeta(\alpha) - \frac{1}{1^\alpha} - \frac{1}{2^\alpha} - \cdots - \frac{1}{n^\alpha} \right),$$

where ζ denotes the Riemann-Zeta function.

Proposed by Ovidiu Furdui, University of Toledo, USA

Olympiad problems

O85. Let a, b, c be non-negative real numbers such that $ab + bc + ca = 1$. Prove that

$$4 \leq \left(\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \right) (a + b + c - abc) .$$

Proposed by Arkady Alt, San Jose, California, USA

O86. The sequence $\{x_n\}$ is defined by $x_1 = 1$, $x_2 = 3$ and $x_{n+1} = 6x_n - x_{n-1}$ for all $n \geq 1$. Prove that $x_n + (-1)^n$ is a perfect square for all $n \geq 1$.

Proposed by Brian Bradie, Christopher Newport University, USA

O87. Let G be a graph with n vertices, $n \geq 5$. The edges of a graph are colored in two colors such that there are no monochromatic cycles of length $3, 4$, and 5. Prove that there are no more than $\left\lfloor \frac{n^2}{3} \right\rfloor$ edges in the graph.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

O88. Determine all pairs (z, n) such that

$$z + z^2 + \cdots + z^n = n|z|,$$

where $z \in C$ and $|z| \in \mathbb{Z}_+$.

Proposed by Dorin Andrica, Babes-Bolyai University and Mihai Piticari, Campulung Moldovenesc, Romania

O89. Let P be an arbitrary point in the interior of a triangle ABC and let P' be its isogonal conjugate. Let I be the incenter of triangle ABC and let X, Y, Z be the midpoints of the small arcs BC, CA, AB. Denote by A_1, B_1, C_1 the intersections of lines AP, BP, CP with sides BC, CA, AB, respectively, and let A_2, B_2, C_2 be the midpoints of the segments IA_1, IB_1, IC_1. Prove that lines XA_2, YB_2, ZC_2 are concurrent on line IP'.

Proposed by Cosmin Pohoata, Tudor Vianu National College, Romania

O90. Find all positive integers n having at most four distinct prime divisors such that

$$n \mid 2^{\phi(n)} + 3^{\phi(n)} + \cdots + n^{\phi(n)}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Gabriel Dospinescu, Ecole Normale Superieure, France