Junior problems

J97. Let a, b, c, d be integers such that $a+b+c+d=0$. Prove that $a^5+b^5+c^5+d^5$ is divisible by 30.

Proposed by Johan Gunardi, Jakarta, Indonesia

J98. Find all primes p and q such that 24 does not divide $q+1$ and p^2q+1 is a perfect square.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J99. In a triangle ABC, let ϕ_a, ϕ_b, ϕ_c be the angles between medians and altitudes emerging from the same vertex. Prove that one the numbers $\tan\phi_a, \tan\phi_b, \tan\phi_c$ is the sum of the other two.

Proposed by Oleh Faynshteyn, Leipzig, Germany

J100. Consider the set of points from the plane such that the distance between any two points is a real number from the interval $[a, b]$. Prove that the number of these points is finite.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J101. Consider triangle ABC with circumcenter O and orthocenter H. Let A_1 be the projection of A onto BC and let D be the intersection of AO with BC. Denote by A_2 the midpoint of AD. Similarly, we define B_1, B_2 and C_1, C_2. Prove that A_1A_2, B_1B_2, C_1C_2 are concurrent.

Proposed by Andrea Munaro, Italy and Ivan Borsenco, MIT, USA

J102. Evaluate

$$\binom{2008}{3} - 2\binom{2008}{4} + 3\binom{2008}{5} - 4\binom{2008}{6} + \cdots - 2004\binom{2008}{2006} + 2005\binom{2008}{2007}.$$

Proposed by Zuming Feng, Phillips Exeter Academy, USA

Senior problems

S97. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n \geq \left(\sqrt[n]{x_1x_2\cdots x_n}\right)^{n-1}\sqrt{\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}}.$$

Proposed by Arkady Alt, San Jose, California, USA

S98. Let n be a positve integer. Prove that $\prod_{d|n}\frac{\phi(d)}{d} \geq \left(\frac{\phi(n)}{n}\right)^{\frac{\tau(n)}{2}}$, where $\tau(n)$ is the number of divisors of n and $\phi(n)$ is Euler's totient function.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S99. Let ABC be an acute triangle. Prove that

$$\frac{1-\cos A}{1+\cos A}+\frac{1-\cos B}{1+\cos B}+\frac{1-\cos C}{1+\cos C} \leq \left(1-\frac{1}{\cos A}\right)\left(1-\frac{1}{\cos B}\right)\left(1-\frac{1}{\cos C}\right).$$

Proposed by Daniel Campos Salas, Costa Rica

S100. Let ABC be an acute triangle with altitudes BE and CF. Points Q and R lie on segments CE and BF, respectively, such that $\frac{CQ}{QE}=\frac{FR}{RB}$. Determine the locus of the circumcenter of triangle AQR when Q and R vary.

Proposed by Alex Anderson, Washington University in St. Louis, USA

S101. Let a, b, c be distinct real numbers. Prove that

$$\left(\frac{a}{a-b}+1\right)^2+\left(\frac{b}{b-c}+1\right)^2+\left(\frac{c}{c-a}+1\right)^2 \geq 5.$$

Proposed by Roberto Bosch Cabrera, University of Havana, Cuba

S102. Consider triangle ABC with circumcenter O and incenter I. Let E and F be the points of tangency of the incircle with AC and AB, respectively. Prove that EF, BC, OI are concurrent if and only if $r_a^2 = r_b r_c$, where r_a, r_b, r_c are the radii of the excircles.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Undergraduate problems

U97. Prove that

$$f(x)=\begin{cases} 1, & x \geq 0 \\ \operatorname{arccot}\dfrac{1}{x}, & x<0, \end{cases}$$

does not have antiderivatives.

Proposed by Dinu Ovidiu Gabriel, Valcea, Romania

U98. Let $f:[0,1]\to\mathbb{R}$ be a differentiable function with continuous derivative such that

$$\int_0^1 f(x)dx=\int_0^1 xf(x)dx.$$

Prove that there exists $\xi\in(0,1)$ such that $f(\xi)=f'(\xi)\int_0^\xi f(x)dx$.

Proposed by Cezar Lupu, Univeristy of Bucharest, Romania

U99. Let a and b be positive real numbers such that $a+b=a^4+b^4$. Prove that

$$a^a b^b \leq 1 \leq a^{a^3} b^{b^3}.$$

Proposed by Vasile Cartoaje, University of Ploiesti, Romania

U100. Let $f\colon[0,1]\to\mathbf{R}$ be an integrable function such that

- $|f(x)|\leq 1$ and $\int_0^1 xf(x)dx=0$,
- $F(x)\doteq\int_0^x f(y)dy\geq 0$.

Prove that $\displaystyle\int_0^1 f^2(x)dx+5\int_0^1 F^2(x)dx\geq 6\int_0^1 f(x)F(x)dx.$

Proposed by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

U101. Consider a sequence of positive real numbers $a_1,a_2,\ldots$ such that for each term in the sequence we have $Aa_n^k\leq a_{n+1}\leq Ba_n^k$, where $A,B,k\in\mathbb{R}^+$. Prove that for all terms $e^{\alpha+\gamma k^n}\leq a_n\leq e^{\beta+\gamma k^n}$, for some $\alpha,\beta,\gamma\in\mathbb{R}^+$.

Proposed by Zoran Sunic, Texas A&M University, USA

U102. Points on the real axis are colored red and blue. We know there exists a function $f\colon\mathbb{R}\to\mathbb{R}^+$ such that if x,y have distinct color then $\min\{f(x),f(y)\}\leq|x-y|$. Prove that every open interval contains a monochromatic open interval.

Proposed by Iurie Boreico, Harvard University, USA

Olympiad problems

O97. Find all odd primes p such that both of the numbers

$$1 + p + p^2 + \cdots + p^{p-2} + p^{p-1} \text{ and } 1 - p + p^2 + \cdots - p^{p-2} + p^{p-1}$$

are primes.

Proposed by Xiaoshen Mou, Shanghai, China

O98. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \le \sqrt[3]{3(3 + a + b + c + ab + bc + ca)}.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

O99. Let AB be a chord that is not a diameter of circle ω. Let T be a mobile point on AB. Construct circles ω_1 and ω_2 that are externally tangent to each other at T and internally tangent to ω at T_1 and T_2, respectively. Let $X_1 \in AT_1 \cap TT_2$ and $X_2 \in AT_2 \cap TT_1$. Prove that X_1X_2 passes through a fixed point.

Proposed by Alex Anderson, Washington University in St. Louis, USA

O100. Let p be a prime. Prove that $p(x) = x^p + (p-1)!$ is irreducible in $\mathbb{Z}[X]$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O101. Let $a_0, a_1, \ldots, a_6$ be real numbers greater than -1. Prove that

$$\frac{a_0^2 + 1}{\sqrt{a_1^5 + a_1^4 + 1}} + \frac{a_1^2 + 1}{\sqrt{a_2^5 + a_2^4 + 1}} + \cdots + \frac{a_6^2 + 1}{\sqrt{a_0^5 + a_0^4 + 1}} \ge 5$$

whenever

$$\frac{a_0^3 + 1}{\sqrt{a_1^5 + a_1^4 + 1}} + \frac{a_1^3 + 1}{\sqrt{a_2^5 + a_2^4 + 1}} + \cdots + \frac{a_6^3 + 1}{\sqrt{a_0^5 + a_0^4 + 1}} \le 9.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O102. A hive is placed in the Cartesian plane and its cells are regular hexagons with two unit sides parallel to y axis. A bee lives in a cell centered at the origin. It wants to visit another bee whose cell contains the point of coordinates $(2008, 2008)$. The bee can move from a cell to any of the six neighboring cells in one second. What is the minimum number of seconds needed for the bee to reach the other bee? Find how many different routes of optimal time exist.

Proposed by Iurie Boreico, Harvard University and Ivan Borsenco, MIT, USA