Junior problems

J127. Let $a_1, \ldots, a_n$ be positive real numbers such that $\sum_{i=1}^{n} \frac{1}{a_i^2+1} = n-1$. Prove that

$$\sum_{1\le i<j\le n} a_i a_j \le \frac{n}{2}.$$

Proposed by Tuan Le, Fairmont High-School, Anaheim, USA

J128. Consider the sequences are given

(a) $(a_n)_{n\in\mathbb{N}^*} : 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots, 1, 2, 3, \ldots, p-1, p, \ldots$
(b) $(b_n)_{n\in\mathbb{N}^*} : 1, 2, 1, 3, 2, 1, 4, 3, 2, 1, \ldots, p, p-1, p-2, \ldots, 2, 1, \ldots$

How many of the first 2009 terms of these sequences are equal?

Proposed by Marian Teler, Costesti, Romania and Marin Ionescu, Pitesti, Romania

J129. Given a nondegenerate triangle ABC, consider circles $\Gamma_a, \Gamma_b, \Gamma_c$ with diameters BC, CA, and AB, respectively. For which triangles ABC are $\Gamma_a, \Gamma_b, \Gamma_c$ concurrent?

Proposed by Daniel Lasaosa, Universidad Publica de Navarra, Spain

J130. Consider a triangle ABC. Let D the orthogonal projection of A onto BC and let E and F be points on lines AB and AC respectively such that $\angle ADE = \angle ADF$. Prove that the lines AD, BF, and CE are concurrent.

Proposed by Francisco Javier García Capitán, Spain

J131. Let P be a point inside a triangle ABC and let d_a, d_b, d_c be the distances from point P to the triangle's sides. Prove that

$$d_a h_a^2 + d_b h_b^2 + d_c h_c^2 \ge (d_a + d_b + d_c)^3,$$

where h_a, h_b, h_c are the altitudes of the triangle.

Proposed by Magkos Athanasios, Kozani, Greece

J132. Consider a regular hexagon $A_1A_2A_3A_4A_5A_6$ with center O. In how many different ways up to rotation can one color regions A_iOA_{i+1} (take i mod 6) in n colors?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Senior problems

S127. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 \geq 3$. Prove that

$$\frac{x^3}{\sqrt{y^2+z^2+7}} + \frac{y^3}{\sqrt{z^2+x^2+7}} + \frac{z^3}{\sqrt{x^2+y^2+7}} \geq 1.$$

Proposed by Orif Ibrogimov, Samarqand State University, Uzbekistan

S128. Let $A_1, A_2, \ldots, A_n$ be a regular $n-$gon inscribed in a circle of center O and radius R. Prove that for each point M in the plane of the $n-$gon the following inequality holds:

$$\prod_{k=1}^{n} MA_k \leq (OM^2 + R^2)^{\frac{n}{2}}.$$

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania

S129. Let $a_1, a_2, \ldots, a_n \in [0, 1]$ and λ be real numbers such that $a_1 + a_2 + \cdots + a_n = n + 1 - \lambda$. For any permutation $(b_i)_{i=1}^n$ of $(a_i)_{i=1}^n$ prove that

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq n + 1 - \lambda^2.$$

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

S130. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} \cos[(n-k)x + ky] = \left(2\cos\frac{x-y}{2}\right)^n \cos n\frac{x+y}{2}$$

for all positive integers n and all real numbers x and y.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, "Babes-Bolyai" University, Romania

S131. Let P be a point in the interior of a triangle ABC and let P_a, P_b, and P_c be the symmetrical points of A, B, and C with respect to P. Parallels to PB and PC drawn through P_a intersect lines AB and AC at A_b and A_c, respectively. In the same way we define points B_a, B_c, C_a, and C_b. Prove that points A_b, A_c, B_a, B_c, C_a, and C_b are on an ellipse.

Proposed by Catalin Barbu, Bacau, Romania

S132. Let G be a 4-partite graph on n vertices. Prove that the number of k-cliques, $k \geq 3$, in G is less than or equal to $\frac{n^4+16n^3}{256}$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Undergraduate problems

U127. Let $(a_n)_{n\geq 1}$ be a convergent sequence. Evaluate

$$\lim_{n\to\infty}\left(\frac{a_1}{n+1}+\frac{a_2}{n+2}+\cdots+\frac{a_n}{2n}\right).$$

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania and Mihai Piticari, "Dragos-Voda" National College, Romania

U128. Let f be a twice differentiable continuous real-valued function defined on $[0,1]$ such that $f(0)=f(1)=f'(1)=0$ and $f'(0)=1$. Prove that

$$\int_0^1 \left(f''(x)\right)^2 dx \geq 4.$$

Proposed by Duong Viet Thong, Nam Dinh University of Technology Education, Vietnam

U129. Let $a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_n>0$ such that

$$a_1^x+a_2^x+\cdots+a_n^x\geq b_1^x+b_2^x+\cdots+b_n^x,$$

for all x in $\mathbb{R}$. Prove that the function $f:\mathbb{R}\to(0,\infty)$,

$$f(x)=\left(\frac{a_1}{b_1}\right)^x+\left(\frac{a_2}{b_2}\right)^x+\cdots+\left(\frac{a_n}{b_n}\right)^x$$

is increasing.

Proposed by Cezar Lupu, University of Bucharest, Romania

U130. Let f be a three times differentiable real-valued function defined on $(0,1)$ such that $|f'''(x)|\geq 1$ for all $0<x<1$. Consider the set

$$M=\{x\in(0,1):|f'(x)|\leq 2\}.$$

Prove that for the measure of the set M the following inequality holds:

$$\mu(M)\leq 4\sqrt{2}.$$

Proposed by Orif Ibrogimov, Samarqand State University, Uzbekistan

U131. Prove that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\arctan\frac{k}{n}}{n+k}\cdot\frac{\varphi(k)}{k}=\frac{3\log 2}{4\pi},$$

where φ denotes the Euler totient function.

Proposed by Cezar Lupu, University of Bucharest, Romania

U132. Let $P \in \mathbb{R}[X]$ be a nonconstant polynomial and let $f : \mathbb{R} \to \mathbb{R}$ be a function with the intermediate value property such that $P \circ f$ is continuous. Prove that f is continuous.

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania and Gabriel Dospinescu, Ecole Normale Superieure, Paris, France

Olympiad problems

O127. Let n be an integer greater than 1. A set A is called stable if there is at least one positive real number in A and whenever $x_1, x_2, \ldots, x_n$ are elements of A, not necessarily distinct and such that $x_1^2 + x_2^2 + \cdots + x_n^2 \in A$, so does $x_1 + x_2 + \cdots + x_n$. Find all subsets A of $\mathbb{R}$ such that A is stable and for any stable subset B of $\mathbb{R}$ we have $A \subseteq B$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris, France

O128. Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be real numbers that sum up to 1. Let $b_k = \sqrt{1 - \frac{1}{4^k}}\sqrt{a_1^2 + a_2^2 + \cdots + a_k^2}$. Find the minimum value of

$$b_1 + b_2 + \cdots + b_{n-1} + 2b_n.$$

as a function of n.

Proposed by Alex Anderson, Washington University in St. Louis, USA

O129. Let ABC be a triangle and let points P and Q lie on sides AB and AC, respectively. Let M and N be the midpoints of BP and CQ, respectively. Prove that the centers of the nine-point circles of triangles ABC, APQ, and AMN are collinear.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O130. Let $a_1, a_2, \ldots, a_{2009}$ be distinct positive integers not exceeding 10^6. Prove that there are indices i, j such that $|\sqrt{ia_i} - \sqrt{ja_j}| \geq 1$.

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

O131. Let G be a graph on n vertices such that there are no K_4 subgraphs in it. Prove that G contains at most $\left(\frac{n}{3}\right)^3$ triangles.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O132. Let m and n be integers greater than 1. Prove that

$$\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_1,k_2,\ldots,k_n\geq 0}} \frac{1}{k_1!k_2!\cdots k_n!}\cos(k_1+2k_2+\cdots+nk_n)\frac{2\pi}{n} = 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, "Babes-Bolyai" University, Romania