Junior problems

J193. Let $ABCD$ be a square of center O. The parallel through O to AD intersects AB and CD at M and N and a parallel to AB intersects diagonal AC at P. Prove that

$$OP^4 + \left(\frac{MN}{2}\right)^4 = MP^2 \cdot NP^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J194. Let a, b, c be the side-lenghts of a triangle with the largest side c. Prove that

$$\frac{ab\,(2c+a+b)}{(a+c)\,(b+c)} \leq \frac{a+b+c}{3}.$$

Proposed by Arkady Alt, San Jose, California, USA

J195. Find all primes p and q such that both $pq - 555p$ and $pq + 555q$ are perfect squares.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J196. Let I be the incenter of triangle ABC and let A', B', C' be the feet of altitudes from vertices A, B, C. If $IA' = IB' = IC'$, then prove that triangle ABC is equilateral.

Proposed by Dorin Andrica and Liana Topan, Babes-Bolyai University, Romania

J197. Let x, y, z be positive real numbers. Prove that

$$\sqrt{2\left(x^2y^2 + y^2z^2 + z^2x^2\right)\left(\frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3}\right)} \geq x\sqrt{\frac{1}{y} + \frac{1}{z}} + y\sqrt{\frac{1}{z} + \frac{1}{x}} + z\sqrt{\frac{1}{x} + \frac{1}{y}}.$$

Proposed by Vazgen Mikayelyan, Yerevan, Armenia

J198. Find all pairs (x, y) for which $x! + y! + 3$ is a perfect cube.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Senior problems

S193. Find all pairs (x, y) of positive integers such that $x^2 + y^2 = p^6 + q^6 + 1$, for some primes p and q.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S194. Let p be a prime of the form $4k + 3$ and let n be a positive integer. Prove that for each integer m there are integers a and b such that $a^{2^n} + b^{2^n} \equiv m \pmod{p}$.

Proposed by Tigran Hakobyan, Yerevan, Armenia

S195. Let ABC be a triangle with incenter I and circumcenter O and let M be the midpoint of BC. The bisector of angle A intersects lines BC and OM at L and Q, respectively. Prove that

$$AI \cdot LQ = IL \cdot IQ.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S196. Find the least prime that can be written as $\frac{a^3+b^3}{2011}$ for some positive integers a and b.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S197. Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence. Prove that for any prime $p \geq 3$, p divides $F_{2p} - F_p$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

S198. Let x, y, z be positive real numbers such that $(x-2)(y-2)(z-2) \geq xyz - 2$. Prove that

$$\frac{x}{\sqrt{x^5 + y^3 + z}} + \frac{y}{\sqrt{y^5 + z^3 + x}} + \frac{z}{\sqrt{z^5 + x^3 + y}} \leq \frac{3}{\sqrt{x + y + z}}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Undergraduate problems

U193. Let n be a positive integer. Find the largest constant $c_n > 0$ such that, for all positive real numbers $x_1, \dots, x_n$,

$$\frac{1}{x_1^2} + \cdots + \frac{1}{x_n^2} + \frac{1}{(x_1 + \cdots + x_n)^2} \geq c_n \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 + \ldots + x_n} \right)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania

U194. Prove that the set of positive integers n for which n divides $2^{n^2+1} + 3^n$ has density 0.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

U195. Given a positive integer n, let $f(n)$ be the square of the number of its digits. For example $f(2) = 1$ and $f(123) = 9$. Show that $\sum_{n=1}^{\infty} \frac{1}{nf(n)}$ is convergent.

Proposed by Roberto Bosch Cabrera, Florida, USA

U196. Let $A, B \in M_2(\mathbb{Z})$ be commuting matrices such that for any positive integer n there exists $C \in M_2(\mathbb{Z})$ such that $A^n + B^n = C^n$. Prove that $A^2 = 0$ or $B^2 = 0$ or $AB = 0$.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

U197. Let $n \geq 2$ be an integer. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x_1, x_2, ..., x_n \in \mathbb{R}$,

$$\sum_{i=1}^{n} f(x_i) - \sum_{1 \leq i < j \leq n} f(x_i + x_j) + \cdots + (-1)^{n-1} f(x_1 + ... + x_n) = 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U198. Define a sequence $(x_n)_n$ by $x_0 = 1$ and $x_{n+1} = 1 + x_n + \frac{1}{x_n}$ for $n \geq 0$. Prove that there is a real number a such that

$$\lim_{n \to \infty} \frac{n}{\log n} \cdot (a + n + \log n - x_n) = 1.$$

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Olympiad problems

O193. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a+b+\frac{1}{abc}+1}+\frac{1}{b+c+\frac{1}{abc}+1}+\frac{1}{c+a+\frac{1}{abc}+1} \leq \frac{a+b+c}{a+b+c+1}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O194. Let A be a set of nonnegative integers, containing 0 and let a_n be the number of solutions of the equation $x_1 + x_2 + \cdots + x_n = n$, with $x_1, \ldots, x_n \in A$, $a_0 = 1$. Find A, if for all $n \geq 0$,

$$\sum_{k=0}^{n} a_k a_{n-k} = \frac{3^{n+1} + (-1)^n}{4}.$$

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

O195. Let O, I, H be the circumcenter, incenter, and orthocenter of a triangle ABC, and let D be an interior point to triangle ABC such that $BC \cdot DA = CA \cdot DB = AB \cdot DC$. Prove that A, B, D, O, I, H are concyclic if and only if $\angle C = 60°$.

Proposed by Titu Andreescu, Dorin Andrica, and Catalin Barbu

O196. Let ABC be a triangle such that $\angle ABC > \angle ACB$ and let P be an exterior point in its plane such that

$$\frac{PB}{PC} = \frac{AB}{AC}$$

Prove that

$$\angle ACB + \angle APB + \angle APC = \angle ABC.$$

Proposed by Mircea Becheanu, Bucharest, Romania

O197. Let x, y, z be integers such that $3xyz$ is a perfect cube. Prove that $(x+y+z)^3$ is a sum of four cubes of nonzero integers.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

O198. Let a, b, c be positive real numbers such that

$$(a^2+1)(b^2+1)(c^2+1)\left(\frac{1}{a^2b^2c^2}+1\right) = 2011.$$

Find the greatest possible value of $\max(a(b+c), b(c+a), c(a+b))$.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA and
Gabriel Dospinescu, Ecole Polytechnique, France*