On distances in regular polygons

Abstract

This paper shows a method for solving exercises at the math olympics level involving distances to the vertices in a regular polygon. Using basic expressions, exercises and solutions of differents levels are presented. We also establish a lemma which simplifies the solutions in many cases.

Let the distance between two numbers in the complex plane ($z = a + bi$ and $w = c + di$) be defined by $|z - w|$, equivalent to the ordinary distance

$$|z - w| = \sqrt{(a-c)^2 + (b-d)^2},$$

and let the vertices in a regular $n-$sided polygon be given by

$$A_k = R \cdot e^{i\left(\frac{2k\pi}{n}+\phi\right)} = R\left(\cos\left(\frac{2k\pi}{n}+\phi\right) + i\sin\left(\frac{2k\pi}{n}+\phi\right)\right), \quad k = 0, 1, \ldots, n-1$$

where R is the radius of the polygon's circumcircle, and ϕ is the angle of rotation about the real plane. Furthermore we denote by A_0 the first vertex counting in the counter clockwise direction, by A_1 the second vertex counting in the counter clockwise direction, and so on, until we reach the n^{th} vertex denoted by A_{n-1}.

We can also find that the distance between the arbitrary point M, with coordinates $x = p\cos(\theta)$ and $y = p\sin(\theta)$, and the vertices in a regular polygon is

$$MA_k = \sqrt{\left(R\cos\left(\frac{2k\pi}{n}+\phi\right) - p\cos(\theta)\right)^2 + \left(R\sin\left(\frac{2k\pi}{n}+\phi\right) - p\sin(\theta)\right)^2}.$$

This expression, using trigonometrical identities can be written as

$$MA_k = \sqrt{R^2 + p^2 - 2Rp\cos\left(\frac{2k\pi}{n}+\phi-\theta\right)}, \text{ for } k = 0, 1, \ldots, n-1. \tag{1}$$

In the exercises that we will present, without loss of generality we can let $\phi = 0$ and therefore (1) becomes

$$MA_k = \sqrt{r^2 + p^2 - 2rp\cos\left(\frac{2k\pi}{n}-\theta\right)}, \text{ for } k = 0, 1, \ldots, n-1. \tag{2}$$

If the point M is lies on the circumcircle, it is easy to show that (2) can be written as

$$MA_k = 2R\left|\sin\left(\frac{k\pi}{n} - \frac{\theta}{2}\right)\right|, \quad \text{for} \quad k = 0, 1. \ldots n-1, \tag{3}$$

With this we can solve the following exercices:

1. A regular $n-$gon $A_1A_2A_3\cdots A_n$ inscribed in a circle of radius R is given. If S is a point on the circle, calculate

$$T = \sum_{k=1}^{n} SA_k^2.$$

(IMO longlist 1989)

Solution: From (2) we have

$$\begin{aligned}
\sum_{k=1}^{n} SA_k^2 &= \sum_{k=0}^{n-1}\left(R^2 + l^2 - 2rl\cos\left(\frac{2k\pi}{n} - \theta\right)\right) \\
&= n(R^2 + l^2) - 2Rl\sum_{k=0}^{n-1}\cos\left(\frac{2k\pi}{n} - \theta\right) \\
&= n(R^2 + l^2) - 2Rl\left(\cos\theta\sum_{k=0}^{n-1}\cos\left(\frac{2k\pi}{n}\right) + \sin\theta\sum_{k=0}^{n-1}\sin\left(\frac{2k\pi}{n}\right)\right).
\end{aligned}$$

Since

$$\sum_{k=0}^{n-1}\cos\left(\frac{2k\pi}{n}\right) = \sum_{k=0}^{n-1}\sin\left(\frac{2k\pi}{n}\right) = 0, \tag{4}$$

the sum has value $n(R^2 + l^2)$.

2. Let A, B, C be three consecutive vertices of a regular polygon and let us consider a point M on the major arc AC of the circumcircle. Prove that

$$MA \cdot MC = MB^2 - AB^2.$$

(Andreescu T. and Andrica D. Complex Numbers from A to ... Z)

Solution: Without loss of generality, we let $k = 0$, $k = 1$, and $k = 2$ correspond to the points A, B and C respectively. As M is on the major arc AC we plug $k = 0, k = 1$, and $k = 2$ into (3) to get

$$MA = 2R\sin\left(\frac{\theta}{2}\right), MB = 2R\sin\left(\frac{\theta}{2} - \frac{\pi}{n}\right), \text{ and } MC = 2R\sin\left(\frac{\theta}{2} - \frac{2\pi}{n}\right),$$

because it is clear that $2\pi - \frac{4\pi}{n} \geq \theta \geq \frac{4\pi}{n}$. Now taking $k = 0$ and $\theta = \frac{2\pi}{n}$ in (3) we see that $AB = 2R\sin\left(\frac{\pi}{n}\right)$, i.e., the size of each side of the polygon. Combining the above results (and recalling the identities $\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta, \cos(2\alpha)=1-2\sin^2\alpha$) we have

$$\begin{aligned}
MB^2-AB^2 &= 4R^2\sin^2\left(\frac{\theta}{2}-\frac{\pi}{n}\right)-4R^2\sin^2\left(\frac{\pi}{n}\right)\\
&= 2R^2\left(1-2\sin^2\left(\frac{\pi}{n}\right)-1+2\sin^2\left(\frac{\theta}{2}-\frac{\pi}{n}\right)\right)\\
&= 2R^2\left(\cos\left(\frac{2\pi}{n}\right)-\cos\left(\theta-\frac{2\pi}{n}\right)\right)\\
&= 4R^2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}-\frac{2\pi}{n}\right)\\
&= MA\cdot MC.
\end{aligned}$$

3. Let $A_1, A_2, \ldots, A_n$ be a regular $n-$gon inscribed in a circle with center O and radius R. Prove that for each point M in the plane of the $n-$gon the following inequality holds:

$$\prod_{k=1}^{n} MA_k \leq (OM^2+R^2)^{\frac{n}{2}}.$$

(Mathematical Reflections, problem S128. Proposed by Dorin Andrica)

Solution: Let $d=OM$. Applying in (2) the AM–GM inequality to the numbers MA_k^2 we have

$$\begin{aligned}
\left(\sum_{k=1}^{n}\frac{MA_k^2}{n}\right)^n &\geq \prod_{k=1}^{n} MA_k^2\\
\left(d^2+R^2-\frac{2dR}{n}\left(A\cos\theta+B\sin\theta\right)\right)^n &\geq \prod_{k=1}^{n} MA_k^2,
\end{aligned}$$

where $A=\sum\limits_{k=0}^{n-1}\cos\frac{2k\pi}{n}$ and $B=\sum\limits_{k=0}^{n-1}\sin\frac{2k\pi}{n}$. Finally, we again apply 4) and see

$$\left(d^2+R^2\right)^n=\left(OM^2+R^2\right)^n \geq \prod_{k=1}^{n} MA_k^2,$$

and the conclusion follows.

4. Let $d_1, d_2, \ldots, d_n$ denote the distances of the vertices $A_1, A_2, \ldots, A_n$ of the regular $n-$gon $A_1A_2\ldots A_n$ from an arbitrary point P on the minor arc A_1A_n of the circumcircle. Prove that

$$\frac{1}{d_1d_2}+\frac{1}{d_2d_3}+\cdots+\frac{1}{d_{n-1}d_n}=\frac{1}{d_1d_n}.$$

(The IMO Compendium Group)

Solution: Since P is on the minor arc A_1A_n, it's clear that $-\frac{2\pi}{n} < \theta < 0$. So from (3) we find

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{1}{d_kd_{k+1}} &= \frac{1}{4R^2}\sum_{k=0}^{n-2}\frac{1}{\sin\left(\frac{k\pi}{n}-\frac{\theta}{2}\right)\sin\left(\frac{(k+1)\pi}{n}-\frac{\theta}{2}\right)}\\ &= \frac{1}{4R^2}\sum_{k=0}^{n-2}\csc\left(\frac{k\pi}{n}-\frac{\theta}{2}\right)\csc\left(\frac{(k+1)\pi}{n}-\frac{\theta}{2}\right). \qquad (5)\end{aligned}$$

Using the identity

$$\csc(\alpha)\csc(\beta) = \frac{1}{\sin(\alpha-\beta)}(\cot(\alpha)-\cot(\beta)), \forall\alpha\neq\beta \text{ and } \alpha\neq\frac{n\pi}{2}, \beta\neq\frac{n\pi}{2}, n = 0, \pm1, \pm2, \ldots,$$

where $\alpha = \frac{(k+1)\pi}{n} - \frac{\theta}{2}$ and $\beta = \frac{k\pi}{n} - \frac{\theta}{2}$, (5) can be written

$$\sum_{k=1}^{n-1}\frac{1}{d_kd_{k+1}} = \frac{1}{4R^2}\sum_{k=0}^{n-2}\frac{1}{\sin\left(\frac{\pi}{n}\right)}\left(\cot\left(\frac{(k+1)\pi}{n}-\frac{\theta}{2}\right)-\cot\left(\frac{k\pi}{n}-\frac{\theta}{2}\right)\right).$$

The above sum is telescopic, therefore

$$\sum_{k=1}^{n-1}\frac{1}{d_kd_{k+1}} = \frac{1}{4R^2}\frac{1}{\sin\left(\frac{\pi}{n}\right)}\left(\cot\left(\frac{(n-1)\pi}{n}-\frac{\theta}{2}\right)-\cot\left(-\frac{\theta}{2}\right)\right).$$

Using the identity again,

$$\sum_{k=1}^{n-1}\frac{1}{d_kd_{k+1}} = \frac{1}{4R^2}\csc\left(\frac{(n-1)\pi}{n}-\frac{\theta}{2}\right)\csc\left(-\frac{\theta}{2}\right) = \frac{1}{d_1d_n},$$

since from (3) we see that

$$d_1 = 2r\sin\left(-\frac{\theta}{2}\right) \text{ and } d_n = 2r\sin\left(\frac{(n-1)\pi}{n}-\frac{\theta}{2}\right).$$

Now we shall prove the following lemma:

Lemma: If z_k, for $k = 0, 1, \ldots, n-1$, are the complex roots of unity of order n, where n is an integer, then

$$\prod_{k=0}^{n-1}(A - Bz_k) = A^n - B^n$$

for all complez numbers A and B.

Proof: If $B = 0$, the result is trivial. If $B \neq 0$, taking using the identity

$$\prod_{k=0}^{n-1}(z - z_k) = z^n - 1,$$

with $z = \frac{A}{B}$ we find

$$\prod_{k=0}^{n-1}\left(\frac{A}{B} - z_k\right) = \left(\frac{A}{B}\right)^n - 1 \Rightarrow \prod_{k=0}^{n-1}(A - Bz_k) = A^n - B^n,$$

the desired identity.

Taking the norm on both sides and letting $M = A = pe^{i\theta}$ and $B = R$, we see from (2) that

$$\prod_{k=1}^{n} MA_k = \prod_{k=1}^{n}|M - Bz_k| = \prod_{k=0}^{n-1}\sqrt{R^2 + p^2 - 2Rp\cos\left(\frac{2k\pi}{n} - \theta\right)}.$$

On the other hand,

$$|M^n - B^n| = |p^n e^{in\theta} - R^n| = \sqrt{p^{2n} + R^{2n} - 2R^n p^n \cos(n\theta)}.$$

Equating both expressions we obtain

$$\prod_{k=1}^{n} MA_k = \prod_{k=0}^{n-1}\sqrt{R^2 + p^2 - 2Rp\cos\left(\frac{2k\pi}{n} - \theta\right)} = \sqrt{p^{2n} + R^{2n} - 2R^n p^n \cos(n\theta)}. \qquad (6)$$

If $R = p$, the result is reduced to

$$\prod_{k=1}^{n} MA_k = \prod_{k=0}^{n-1} 2R\left|\sin\left(\frac{k\pi}{n} - \frac{\theta}{2}\right)\right| = 2R^n\left|\sin\left(\frac{n\theta}{2}\right)\right|. \qquad (7)$$

5. $A_1A_2\ldots A_n$ is a regular polygon inscribed in the circle of radius R and center O. P is a point on line OA_1 extended beyond A_1. Show that

$$\prod_{i=1}^{n} PA_i = PO^n - R^n.$$

(Putnam 1955)

Solution: It is enough to take $\theta = 0$ and $p = PO \geq R$ in (6). The conclusion follows.

6. Let $A_1A_2\ldots A_n$ be a regular polygon with circumradius 1. Find the maximum value of $\prod_{k=1}^{n} PA_k$ as P ranges over the circumcircle.

(Romanian Mathematical Regional Contest "Grigore Moisil", 1992)

Solution: Taking $R = 1$ in (7), we see that the maximum value is 2.

7. For a positive integer $n > 1$, determine

$$\lim_{x\to 0} \frac{\sin^2(x)\sin^2(nx)}{n^2\sin^2(x) - \sin^2(nx)}.$$

(Mathematical Reflections, problem U143)

Solution: Taking natural logarithm in (7), differentiating twice with respect to θ and omitting all θ for which $\frac{k\pi}{n} - \frac{\theta}{2} = 0$ we find

$$\sum_{k=0}^{n-1} \csc^2\left(\frac{k\pi}{n} - \frac{\theta}{2}\right) = n^2\csc^2\left(\frac{n\theta}{2}\right).$$

Evaluating at $k = 0$, and taking the limit as $\theta \to 0$ the previous expression is equivalent to

$$\sum_{k=1}^{n-1} \csc^2\left(\frac{k\pi}{n}\right) = \lim_{\theta\to 0}\sum_{k=1}^{n-1} \csc^2\left(\frac{k\pi}{n} - \frac{\theta}{2}\right) = \lim_{\theta\to 0}\left[n^2\csc^2\left(\frac{n\theta}{2}\right) - \csc^2\left(\frac{\theta}{2}\right)\right].$$

By [1],

$$\sum_{k=1}^{n-1} \csc^2\left(\frac{k\pi}{n}\right) = \frac{n^2-1}{3},$$

it follows that

$$\lim_{\theta\to 0}\left[n^2\csc^2\left(\frac{n\theta}{2}\right) - \csc^2\left(\frac{\theta}{2}\right)\right] = \frac{n^2-1}{3}.$$

Therefore, taking $x = \dfrac{\theta}{2}$, the desired limit has a value of $\dfrac{3}{n^2-1}$.

8. A regular $n-$gon inscribed in a circle of radius 1 is given. Let $a_2,\ldots,a_{n-1}$ be the distances from one vertex of the polygon to all other vertices. Show that

$$(5-a_2^2)(5-a_3^2)\cdots(5-a_n^2) = F_n^2,$$

where F_n denotes the n^{th} Fibonacci number.

(Iberoamerican Mathematical Olympiad for University Students, 2006)

Solution: Without loss of generality we can take the vertex A_0 as the reference vertex and multiply both sides by 5 to get

$$\prod_{k=1}^{n}(5-a_k^2)=5F_n^2.$$

Taking in (2) $\theta=0, R=p=1$ we have

$$\prod_{k=1}^{n}(5-a_k^2)=\prod_{k=0}^{n-1}\left(3+2\cos\left(\frac{2k\pi}{n}\right)\right).$$

We need to find the values of A and B satisfying $A^2+B^2=3$ and $AB=-1$. We can take $A>B$, and simultaneously solving the equations above we obtain:

$$A=\frac{1+\sqrt{5}}{2} \quad\text{and}\quad B=\frac{1-\sqrt{5}}{2}.$$

Squaring (6), we obtain for the given values

$$\prod_{k=1}^{n}(5-a_k^2)=\prod_{k=0}^{n-1}\left(3+2\cos\left(\frac{2k\pi}{n}\right)\right)=\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)^2=5F_n^2$$

because

$$F_n=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right),$$

and the conclusion follows.

Exercises:

1. Two regular $n-$gons $A_1A_2\ldots A_n$ and $B_1B_2\ldots B_n$ are in the same plane P and have the same center.

   a) Show that $\prod_{j=1}^{n} B_iA_j = \prod_{i=1}^{n} A_jB_i, \forall i,j \in \{1,2,\ldots,n\}$.

   b) Find $\min_{M\in P}\{MA_1 \cdot MA_2 \cdot \ldots \cdot MA_n + MB_1 \cdot MB_2 \cdot \ldots \cdot MB_n\}$.

(Romanian mathematical competition, shortlist 2008)

2. Let $A_0, A_1, \ldots, A_{2n}$ be a regular polygon with circumradius equal to 1 and consider a point P on the circumcircle. Prove that

$$\sum_{k=0}^{n-1} PA_{k+1}^2 PA_{n+k+1}^2 = 2n.$$

(Andreescu T. and Andrica D. Complex Numbers from A to $\ldots$ Z)

3. Consider an integer $n \geq 3$ and the parabola of equation $y^2 = 4px$, with focus F. A regular $n-$gon $A_1A_2\cdots A_n$ has center at F and no one of its vertices lies on the x axis. The rays $FA_1, FA_2, \ldots, FA_n$ cut the parabola at points $B_1, B_2, \ldots, B_n$.

   Prove that $FB_1 + FB_2 + \cdots + FB_n > np$.

(Romanian mathematical competition 2004)

References

[1] Some remarks on problem U23, Dorin Andrica and Mihai Piticari, Mathematical Reflections 4(2008).

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