Junior problems

J157. Evaluate
$$1^2+2^2+3^2-4^2-5^2+6^2+7^2+8^2-9^2-10^2+\cdots-2010^2,$$
where each three consecutive signs $+$ are followed by two signs $-$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Bedri Hajrizi, Albania

Notice that
$$(5k-4)^2+(5k-3)^2+(5k-2)^2-(5k-1)^2-(5k)^2=25k^2-80k+28.$$
Hence
$$1^2+2^2+3^2-4^2-5^2+6^2+7^2+8^2-9^2-10^2+\cdots-2010^2$$
is equal to
$$25\sum_{k=1}^{402}k^2-80\sum_{k=1}^{402}k+28\sum_{k=1}^{402}1=25\frac{402\cdot 403\cdot 805}{6}-40\cdot 402\cdot 403+28\cdot 402=536926141.$$

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; G. C. Greubel, Newport News, VA; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Raul A. Simon, Chile; Vicente Vicario Garcia, Huelva, Spain.

J158. Let n be a positive integer relatively prime with 10. Prove that the hundreds digit of n^{20} is even.

Proposed by Badar Al-Ghamdi, Saudi Arabia

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

If n is prime with 10 it is prime with 5 and 2. Since $\varphi(5^2) = 5 \cdot 4 = 20$ and $\varphi(2^3) = 2^2 \cdot 1 = 4$, where $\varphi(n)$ is Euler's totient function, then $n^{20} \equiv 1 \pmod{25}$ and $n^{20} = (n^4)^5 \equiv 1^5 \equiv 1 \pmod{8}$, or $n^{20} \equiv 1 \pmod{200}$, and uniqueness of this residue modulus 200 is guaranteed by the Chinese Remainder Theorem, hence the last three digits of n^{20} are $a01$, where a is even. The conclusion follows.

Also solved by Arkady Alt, San Jose, California, USA.

J159. Find all integers n for which $9n + 16$ and $16n + 9$ are both perfect squares.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by José Hernández Santiago, Oaxaca, México

If $9n+16$ and $16n+9$ are both perfect squares then $n \geq 0$ and the number $p_n = (9n+16)(16n+9) = (12n)^2 + (9^2 + 16^2)n + 12^2$ is also a perfect square. Since

$$(12n + 12)^2 \leq (12n)^2 + (9^2 + 16^2)n + 12^2 < (12n + 15)^2$$

it follows that if $n > 0$ then we must have $p_n = (12n + 13)^2$ or $p_n = (12n + 14)^2$. The former condition gives $n = 1$ and the latter, $n = 52$. Therefore, $n = 0$, $n = 1$, and $n = 52$ are the only integers n for which the expressions $9n + 16$ and $16n + 9$ simultaneously return perfect squares.

Also solved by Bedri Hajrizi, Albania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Raul A. Simon, Chile; Stefania Garasto, Università di Roma "Tor Vergata", Roma, Italy; Sayan Mukherjee, Kolkata, India.

J160. Let ABC be a triangle with $\widehat{A} = 90^\circ$ and let d be a line passing trough the incenter of the triangle and intersecting sides AB and AC in P and Q, respectively. Find the minimum of $AP \cdot AQ$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Magkos Athanasios, Kozani, Greece

Let M, N be the projections of I on AC, AB respectively. We have $IM = IN = r$ and from the similarity of the triangles PMI, INQ we find

$$PM \cdot NQ = r^2,$$

where r is the inradius of triangle ABC.

Then, we have

$$AP \cdot AQ = (AM + MP)(AN + NQ) = AM \cdot AN + AM \cdot NQ + MP \cdot AN + MP \cdot NQ =$$

$$= 2r^2 + r(NQ + MP) \geq 2r^2 + 2r\sqrt{MP \cdot NQ} = 2r^2 + 2r^2 = 4r^2.$$

The sign of equality holds iff $MP = NQ \Leftrightarrow AP = AQ \Leftrightarrow \measuredangle APQ = \measuredangle AQP = 45^\circ$.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Vicente Vicario Garcia, Huelva, Spain; Raul A. Simon, Chile.

J161. Let a, b, c be positive real numbers such that $a+b+c+2=abc$. Find the minimum of

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Proposed by Abdulmajeed Al-Gasem, Saudi Arabia

First solution by Arkady Alt, San Jose, California, USA

Note that $a+b+c+2=abc$ implies

$$3+2\left(a+b+c\right)+\left(ab+bc+ca\right)=abc+ab+bc+ca+a+b+c+1.$$

Then

$$\left(a+1\right)\left(b+1\right)\left(c+1\right)=\sum_{cyc}\left(a+1\right)\left(b+1\right)$$

and dividing by the left-hand side, we obtain

$$\sum_{cyc}\frac{1}{a+1}=1 \iff \sum_{cyc}\frac{1}{1+\dfrac{1}{a}}=2.$$

By the Cauchy-Schwarz inequality

$$\sum_{cyc}\left(1+\frac{1}{a}\right)\sum_{cyc}\frac{1}{1+\dfrac{1}{a}}\geq 9 \iff \sum_{cyc}\left(1+\frac{1}{a}\right)\geq\frac{9}{2} \iff \sum_{cyc}\frac{1}{a}\geq\frac{3}{2}$$

hence the minimum is $\frac{3}{2}$.

Second solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

Let $x=\frac{1}{1+a}$, $y=\frac{1}{1+b}$, $z=\frac{1}{1+a}$ (see T.Andreescu, G.Dospinescu "Problems from the Book" XYZ Press, 2008). By trivial algebra we observe that $a+b+c+2=abc$ is equivalent to $x+y+z=1$ and then $a=\frac{1-x}{x}=\frac{y+z}{x}$, $b=\frac{1-y}{y}=\frac{x+z}{y}$, $c=\frac{1-z}{z}=\frac{y+x}{z}$. In terms of the variable x, y, z the inequality is

$$\frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y}, \qquad x+y+z=1$$

That the minimum of the above expression is $3/2$ is the content of Nesbitt's inequality which is well known. One of the many proof available is

$$\frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y},\geq\frac{(x+y+z)^2}{2(xy+yz+zx)}=\frac{1}{2(xy+yz+zx)}\geq\frac{3}{2}$$

having employed Cauchy–Schwarz. Thus we have $xy+yz+zx\leq\frac{1}{3}$ which is obvious.

Also solved by Anthony Erb Lugo, San Juan, Puerto Rico; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Magkos Athanasios, Kozani, Greece; Jérôme Nicolas, Collège Versailles and Université Paul Cézanne (Faculté d'Économie Appliquée), France; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Sayan Mukherjee, Kolkata, India.

J162. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\frac{a_1}{(1+a_1)^2} + \frac{a_2}{(1+a_1+a_2)^2} + \cdots + \frac{a_n}{(1+a_1+\cdots+a_n)^2} \le \frac{a_1+\cdots+a_n}{1+a_1+\cdots+a_n}.$$

Proposed by Neculai Stanciu, Buzau, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We will prove the result by induction. For $n = 1$, the result is equivalent to

$$\frac{a_1}{(1+a_1)^2} \le \frac{a_1}{1+a_1},$$

clearly true with a strict inequality since $1 + a_1 > 1$. If the result is true with a strict inequality for $n - 1$, then for the result to be true with a strict inequality for n, it suffices to show that

$$\frac{s_n - a_n}{1+s_n-a_n} + \frac{a_n}{(1+s_n)^2} \le \frac{s_n}{1+s_n}, \qquad (s_n-a_n)(1+s_n)^2 \le (s_n+s_n^2-a_n)(1+s_n-a_n),$$

where we have denoted $s_n = a_1 + a_2 + \cdots + a_n$, and which after performing the products in both sides and simplifying, transforms into $a_n^2 \ge 0$, trivially true. The conclusion follows, the inequality being strict for all n. One may get arbitrarily close to equality when $a_1, a_2, \ldots, a_n$ all tend to 0, both sides being arbitrarily close to $a_1 + a_2 + \cdots + a_n$, which is in turn arbitrarily close to zero.

Also solved by Arkady Alt, San Jose, California, USA; G. C. Greubel, Newport News, USA; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

Senior problems

S157. Let ABC be at triangle. Find the locus of points X on line BC such that

$$AB^2 + AC^2 = 2(AX^2 + BX^2).$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

By the median theorem,

$$AM^2 + BM^2 = AM^2 + \frac{BC^2}{4} = \frac{AB^2 + AC^2}{2},$$

where M is the midpoint of side BC. Clearly $X = M$ is one solution. Any other solution is on the intersection of side BC and the circle with center the midpoint of AB passing through M, since it is well known that $AX^2 + BX^2 = k$, where $k > \frac{AB^2}{2}$, is the equation of a circle with center the midpoint of AB, and M is on that circle. There is therefore at most one more solution to the equation, which is the symmetric of M with respect to the perpendicular to BC through the center of the circle, ie the symmetric of M with respect to the foot of the perpendicular from the midpoint of AB onto BC. There is exactly one solution, ie exactly one point X that satisfies the given condition, when the midpoint of BC is also the foot of the perpendicular from the midpoint of AB onto BC, ie by Thales' theorem when $\angle C = 90°$.

Also solved by Arkady Alt, San Jose, California, USA.

S158. Is there an integer n such that exactly two of the numbers $n+8, 8n-27, 27n-1$ are perfect cubes?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

The first positive perfect cubes are 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728, 2197, 2744, 3375, 4096, the difference of any two perfect cubes one of which is in absolute value larger than 4096 being clearly larger than $4096 - 3375 = 721$.

Assume that $8n-27 = u^3$ and $27n-1 = v^3$ are both perfect cubes for some integers u, v. In this case, $(2v)^3 - (3u)^3 = 27^2 - 8 = 721$, and 721 must be the difference between two perfect cubes. By inspection, the pairs of perfect cubes for which this happens are $(4096, 3375)$, $(729, 8)$, $(-8, -729)$ and $(-3375, -4096)$, yielding respectively $2v = 16, 9, -2, -15$ and $3u = 15, 2, -9, -16$, with solutions $(u, v) = (5, 8)$ and $(u, v) = (-3, -1)$ (the other two cases yield non-integral values for u, v). It follows respectively that $n = 19$ and $n = 0$, for $n+8 = 3^3$ and $n+8 = 2^3$, perfect cubes in both cases, or whenever $8n-27$ and $27n-1$ are simultaneously perfect cubes, so is $n+8$.

We proceed similarly in the cases where $n+8 = w^3$ and $8n-27 = u^3$ are both perfect cubes (hence $(2w)^3 - u^3 = 91$ is the difference between two perfect cubes) and $n+8 = w^3$ and $27n-1 = v^3$ are both perfect cubes (hence $(3w)^3 - v^3 = 217$ is the difference between two perfect cubes). In both cases, we find by inspection that the only integral values of w, u and w, v that make this possible, also yield, respectively, $27n-1 = v^3$ a perfect cube, and $8n-27 = u^3$ a perfect cube. In fact, the values obtained for u, v, w, n in these two cases are exactly the values found in the case where we assume that $8n-27 = u^3$ and $27n-1 = v^3$ are both perfect cubes.

It follows that either at most one of the given numbers is a perfect cube, or all three are perfect cubes, hence no n exists such that exactly two of the given numbers are perfect cubes.

S159. In triangle ABC, lines AA', BB', CC' are concurrent at P, where points A', B', C' are situated on sides BC, CA, AB, respectively. Consider points A'', B'', C'' on segments $B'C', C'A', A'B'$, respectively. Prove that AA'', BB'', CC'' are concurrent if and only if $A'A'', B'B'', C'C''$ are concurrent.

Proposed by Dorin Andrica, Babes-Bolyai University Cluj-Napoca, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let D, E, F the second points where AA'', BB'', CC'' intersect lines BC, CA, AB, respectively. Applying the Sine Law to triangles ADB, ADC, $C'AA''$ and $B'AA''$, we find:

$$BD = \frac{AB \sin \angle BAD}{\sin \angle ADB}, \qquad CD = \frac{AC \sin \angle CAD}{\sin \angle ADC},$$

$$C'A'' = \frac{AC' \sin \angle C'AA''}{\sin \angle AA''C'}, \qquad B'A'' = \frac{AB' \sin \angle B'AA''}{\sin \angle AA''B'}.$$

Now, since $180^\circ = \angle ADB + \angle ADC = \angle AA''C' + \angle AA''B'$, $\angle BAD = \angle C'AA''$ and $\angle CAD = \angle B'AA''$, we find

$$\frac{BD}{CD} = \frac{AB}{AC} \cdot \frac{AB'}{AC'} \cdot \frac{C'A''}{B'A''},$$

and similarly for its cyclic permutations. We then conclude that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \left(\frac{AB'}{B'C} \cdot \frac{CA'}{A'B} \cdot \frac{BC'}{C'A} \right) \left(\frac{A'B''}{B''C'} \cdot \frac{C'A''}{A''B'} \cdot \frac{B'C''}{C''A'} \right).$$

By Ceva's theorem, the first term in the RHS equals 1, and again by Ceva's theorem, AA'', BB'', CC'' are concurrent iff the LHS equals 1, iff the second term in the RHS equals 1, if and only if $A'A'', B'B'', C'C''$ are concurrent. The conclusion follows.

Also solved by Raul A. Simon, Chile

S160. Let ABC be a triangle with $\widehat{B} \geq 2\widehat{C}$. Denote by D the foot of the altitude from A and by M be the midpoint of BC. Prove that $DM \geq \frac{AB}{2}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Clearly $CD = b\cos C$ and $2CM = a$, or since by the Sine Law $a = 2R\sin A = b\cos C + c\cos B$, we have

$$2DM = 2b\cos C - a = b\cos C - c\cos B = 2R\sin(B-C) \geq 2R\sin C = AB,$$

where we have used that $\pi - \hat{C} > \hat{B} - \hat{C} \geq \hat{C}$. The conclusion follows, and equality is reached iff $\hat{B} = 2\hat{C}$.

Also solved by Arkady Alt, San Jose, California, USA; Raul A. Simon, Chile; Vicente Vicario Garcia, Huelva, Spain.

S161. Let ABC be a triangle inscribed in a circle of center O and radius R. If d_A, d_B, d_C are the distances from O to the sides of the triangle, prove that

$$R^3 - (d_A^2 + d_B^2 + d_C^2)R - 2d_Ad_Bd_C = 0.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

First solution by Arkady Alt, San Jose, California, USA

Since $\triangle COB$ is isosceles and$\angle COB = 2\widehat{A}$ then $\dfrac{d_a}{R} = \cos A$, and cyclicly $\dfrac{d_b}{R} = \cos B, \dfrac{d_c}{R} = \cos C$. By substitution in the well known identity $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$ we obtain

$$\frac{d_a^2}{R^2} + \frac{d_b^2}{R^2} + \frac{d_c^2}{R^2} + \frac{2d_ad_bd_c}{R^3} = 1 \iff R^3 - \left(d_a^2 + d_b^2 + d_c^2\right) R - 2d_ad_bd_c = 0.$$

Since the function $\varphi(x) \ := 1 - \dfrac{2d_ad_bd_c}{x^3} - \dfrac{d_a^2 + d_b^2 + d_c^2}{x^2}$ is increasing on $(0, \infty)$ then R is a single positive root of the cubic equation $x^3 - x\left(k^2 + l^2 + m^2\right) - 2klm = 0.$

Remark. This problem is part of the problem #11443 published in **The American Mathematical Monthly, Vol.116, N.6, June-July 2009.**

Problem 11443. Proposed by Eugen Ionascu, Columbus state University, Columbus, GA.

Consider a triangle ABC with circumcenter O and circumradius R.Denote the distances from O to the sides AB, BC, CA,respectively, by x, y, z. Prove that if ABC is acute then $R^3 - \left(x^2 + y^2 + z^2\right) R = 2xyz$,and $\left(x^2 + y^2 + z^2\right) R - R^3 = 2xyz$ otherwise.

A stronger statement, namely

Let x, y, z be arbitrary real positive numbers, then there exist a unique acute triangle with sidelenghts $a = 2\sqrt{R^2 - x^2}, b = 2\sqrt{R^2 - y^2}, c = 2\sqrt{R^2 - z^2}$, where R (circumradius) is the only positive root of a cubic equation $t^3 - t\left(x^2 + y^2 + z^2\right) - 2xyz = 0$, and for which numbers x, y, z be the distances from a circumcenter to the sides of this triangle

is **part i**. of the **Theorem 1** in **[1].**

[1] Arkady Alt, An independent parametrization of an acute triangle and its applications,

Mathematical Reflections 2009, Issue 4.

Second solution by Sayan Mukherjee, Kolkata, India

Note that we have, $\frac{d_A}{OC} = \sin \angle BOC = \sin\left(90^\circ - \frac{1}{2}\angle BOC\right) = \cos\alpha$. Then

$$d_A = R\cos\alpha, d_B = R\cos\beta, d_C = R\cos\gamma,$$

which, in turn, gives us

$$\begin{aligned} R^3 - (d_A^2 + d_B^2 + d_C^2)R - 2dAdBdC &= R^3\left[1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma\right] \\ &= 0; \end{aligned}$$

Whereas the last line follows from the well-known identity in any triangle ABC; i.e. $\cos^2\alpha + \cos^{\beta} + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma = 1$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Vicente Vicario Garcia, Huelva, Spain.

S162. Alice has a pair of scales that display the weight in grams. At step n she cuts a square of side n from a very large laminated sheet and places it on one of the two scales. A square of side 1 weighs 1 gram.

(a) Prove that for each integer g Alice can place the laminated squares on the scales such that after a certain number of steps the difference between the aggregate weights on the two scales is g grams.

(b) Find the least number of steps necessary to reach a difference of 2010 grams.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

(a) Clearly the weight added to one scale at step n is n^2. Note first that for any integer n, we have $n^2 - (n+1)^2 - (n+2)^2 + (n+3)^2 = 4$, ie, if by step $n-1$ we have managed to obtain a difference of g, by step $n+3$ we can obtain differences $g+4$ and $g-4$, by placing on one scale the squares with sides $n, n+3$, and on the other scale the squares with sides $n+1, n+2$, choosing which squares go on which scale according to whether we want the difference to increase or decrease. It therefore suffices to show that we may obtain differences $1, 2, 3$ (difference 0 is present at the initial condition and may be clearly obtained again after 8 steps, 4 steps to add 4, and 4 steps to substract 4). Difference 1 is easily obtained after 1 step, and difference 3 after 2 steps, by placing the squares of sides 1 and 2 in opposite scales (with respective weights 1 and 4). Finally, difference 2 is obtained by placing in opposite scales the squares with sides $1, 2, 3$ (total weight 14 grams), and the square with side 4 (weight 16 grams). The conclusion to part (a) follows.

(b) It is well known (or easily checked by induction) that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. Since

$$1^2 + 2^2 + \cdots + 17^2 = 1785 < 2010 < 2109 = 1^2 + 2^2 + \cdots + 18^2,$$

we need a minimum of 18 steps. In 18 steps the task is impossible, since there are 9 odd perfect squares among the first 18 perfect squares, or the difference of weight between both scales will be odd, because one will contain an even number of odd weights and the other an odd number of odd weights. A difference of 2010 may be however obtained after 19 steps, since

$$1^2 + 12^2 + \cdots + 19^2 - 2(15^2 + 2^2 + 1^2) = \frac{19 \cdot 20 \cdot 39}{6} - 2(225 + 4 + 1) = 2470 - 460 = 2010,$$

ie a difference of 2010 grams may be obtained by placing on one of the scales all squares with sides 3 to 19 inclusive, except for the square with side 15, which is placed on the other scale together with the squares with sides $1, 2$. The minimum is thus 19 steps.

Undergraduate problems

U157. Let $(A, +, \cdot)$ be a finite ring such that $1 + 1 = 0$. Prove that the number of solutions to the equation $x^2 = 0$ is equal to the number of solutions to the equation $x^2 = 1$.

Proposed by Mihai Piticari, Dragos Voda National College, Campulung Moldovenesc, Romania

Solution by Ajat Adriansyah, Universitas Indonesia, Indonesia

For every $x \in A$, $2x = x + x = (1+1)x = 0$. Let $U = \{x \in A | x^2 = 0\}$ and $V = \{x \in A | x^2 = 1\}$. We define the map $\tau : A \to A$ as follow: $\tau(x) = x + 1$. Since

$$x \in U \Rightarrow x^2 = 0 \Rightarrow x^2 + 1 = 1 \Rightarrow x^2 + 2x + 1 = 1 \Rightarrow$$
$$(x+1)^2 = 1 \Rightarrow [\tau(x)]^2 = 1 \Rightarrow \tau(x) \in V$$

we see that $\tau(U) \subset V$. On the other hand, if $y \in V$ then set $x = y - 1$, we have $x^2 = y^2 - 2y + 1 = 1$, that is $x \in U$ and $\tau(x) = y$. We conclude that $\tau(U) = V$. Since $a + 1 = b + 1$ implies $a = b$, the map τ is one-to-one. Hence, $\tau(U) = V$ implies $|U| = |V|$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Daniel Lopez Aguayo, Puebla, Mexico; José Hernández Santiago, Oaxaca, México.

U158. Let $(a_n)_{n\geq 0}$ be a sequence with $a_0 > 0$ and $a_{n+1} = a_n + \frac{1}{a_n}$ for $n = 0, 1, \dots$.

(a) Prove that $\lim\limits_{n\to\infty} a_n = +\infty$.

(b) Find $\lim\limits_{n\to\infty} \dfrac{a_n}{\sqrt{n}}$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

The sequence (a_n) is not decreasing and then converges to $\sup_n a_n \doteq S$. If S is finite we would have $S = S + \frac{1}{S}$ yielding a contradiction unless $S = +\infty$

As for (b) we employ the Cesaro-Stolz theorem:

Theorem *Let (a_n) and (b_n) be two real sequences, such that b_n is positive, increasing and unbounded. Then*

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = L \qquad \Longrightarrow \qquad \lim_{n\to\infty} \frac{a_n}{b_n} = L$$

We apply the theorem to the two sequences a_n^2 and n obtaining

$$\frac{a_{n+1}^2 - a^2 n}{(n+1) - n} = \frac{a_{n+1} + a_n}{a_n} = \frac{a_{n+1}}{a_n} + 1 = 1 + \frac{1}{a_n^2} + 1 \ \to\ 2$$

and the limit is $\sqrt{2}$ by the positivity of a_n.

Also solved by Arkady Alt, San Jose, California, USA; Ajat Adriansyah, Universitas Indonesia, Indonesia; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Moubinool Omarjee, Paris, France; Jérôme Nicolas, Collège Versailles and Université Paul Cézanne (Faculté d'Économie Appliquée), France; Younghan Bae, Sejong Science High School, South Korea.

U159. Let x and y be positive real numbers. Prove that

$$x^y y^x \le \left(\frac{x+y}{2}\right)^{x+y}.$$

Proposed by Samuel G. Moreno, Universidad de Jaén, Spain

First solution by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

From weighted AM–GM inequality we conclude that

$$x^y y^x \le \left(\frac{xy+yx}{x+y}\right)^{x+y} = \left(\frac{2xy}{x+y}\right)^{x+y} \le \left(\frac{x+y}{2}\right)^{x+y},$$

where the last inequality again follows from AM–GM

$$\frac{2xy}{x+y} \le \frac{x+y}{2} \Leftrightarrow \sqrt{xy} \le \frac{x+y}{2}.$$

Second solution by Arkady Alt, San Jose, California, USA

Let us take a more indepth look at this inequality. Firstly, we will prove the stronger inequality

$$x^y y^x \le (xy)^{\frac{x+y}{2}}.$$

Indeed,

$$x^y y^x \le (xy)^{\frac{x+y}{2}} \iff x^{2y} y^{2x} \le x^{x+y} y^{x+y} \iff x^y y^x \le x^x y^y \iff 1 \le \left(\frac{x}{y}\right)^{x-y}.$$

Since

$$(xy)^{\frac{1}{2}} \le \frac{x+y}{2} \iff (xy)^{\frac{x+y}{2}} \le \left(\frac{x+y}{2}\right)^{x+y}$$

we obtain $x^y y^x \le \left(\frac{x+y}{2}\right)^{x+y}$.

Secondly, by weighted AM–GM,

$$x^{\frac{y}{x+y}} y^{\frac{x}{x+y}} \le x \cdot \frac{y}{x+y} + y \cdot \frac{x}{x+y} = \frac{2xy}{x+y}$$

then original inequality immediately follows from the obtained stronger inequality

$$x^y y^x \le \left(\frac{2xy}{x+y}\right)^{x+y}$$

and the inequality $\frac{2xy}{x+y} \le \frac{x+y}{2}$. Thus we obtained the chain of inequalities

$$x^y y^x \le \left(\frac{2xy}{x+y}\right)^{x+y} \le (xy)^{\frac{x+y}{2}} \le \left(\frac{x+y}{2}\right)^{x+y}.$$

Remark. Normalization of the inequality $x^y y^x \le \left(\dfrac{2xy}{x+y}\right)^{x+y}$ by the condition $x+y=1$ yields

$$x^y y^x \le 2xy \iff \frac{1}{2} \le x^x y^y \iff \frac{1}{2} \le x^x (1-x)^{1-x}, x \in (0,1)$$

which can proved by employing calculus. Let

$$h(x) = x \ln x + (1-x) \ln (1-x)$$

then

$$h'(x) = \ln \frac{x}{1-x}$$

and

$$h'(x) < 0 \iff x \in \left(0, \frac{1}{2}\right), h'(x) > 0 \iff x \in \left(\frac{1}{2}, 1\right), h'(x) = 0 \iff x = \frac{1}{2}.$$

Thus

$$\min_{x\in(0,1)} h(x) = h\left(\frac{1}{2}\right),$$

i.e.,

$$h(x) \ge h\left(\frac{1}{2}\right) = \ln \frac{1}{2} \iff \frac{1}{2} \le x^x (1-x)^{1-x}.$$

Also solved by Ajat Adriansyah, Universitas Indonesia, Indonesia; Daniel Lasaosa, Universidad Pública de Navarra, Spain; G. C. Greubel, Newport News, USA; Magkos Athanasios, Kozani, Greece; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Samuel G. Moreno, Universidad de Jaen, Spain; Samin Riasat, Notre Dame College, Dhaka, Bangladesh; Sayan Mukherjee, Kolkata, India.

U160. Let p be a prime and let s and n be positive integers. Prove that

$$\sum_{k=0 \pmod p} (-1)^k \cdot \binom{n}{k} \cdot k^s$$

is a multiple of p^d, where $d = \left\lfloor \frac{n-s-1}{p-1} \right\rfloor$ and $\lfloor x \rfloor$ is the integer part of x.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

We will denote by v_p the p-adic valuation. Recall that for $x \in \mathbb{Z}$, $v_p(x)$ is simply the exponent of p in the prime factorization of x, while $v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b)$ for integers a, b. It is well-known (but nontrivial) that v_p extends to the ring of algebraic numbers $\bar{\mathbb{Q}}$ by $v_p(x) = \frac{1}{d} v_p(f(0))$, where f is the minimal polynomial of x and $d = \deg(f)$. In particular, if z is a primitive root of order p of 1, the minimal polynomial of $1 - z$ is $\frac{(X-1)^p+1}{X}$. Thus $v_p(1-z) = \frac{1}{p-1}$.

Now, let us start the proof. Fix in what follows a primitive root of order p of the unity. Note that $\frac{1}{p} \sum_{j=0}^{p-1} z^{kj} = 0$ if k is not a multiple of p and 1 otherwise. We deduce that

$$\sum_{0 \le k \le n, p|n} (-1)^k \cdot k^s \binom{n}{k} = \frac{1}{p} \sum_{j=0}^{p-1} \sum_{k=0}^{n} (-z^j)^k k^s \binom{n}{k}.$$

Now, let $n - s - 1 = q(p-1) + r$ for some $0 \le r < p - 1$. We will prove that

$$v_p\left(\sum_{k=0}^{n} (-z^j)^k k^s \binom{n}{k}\right) > q$$

for all $0 \le i \le p - 1$. This will imply that

$$v_p\left(\sum_{0 \le k \le n, p|n} (-1)^k \cdot k^s \binom{n}{k}\right) > q - 1$$

and since this p-adic valuation is an integer, the conclusion will follow.

Now, to prove the crucial claim, we will use the following:

Lemma. The polynomial $\sum_{k=0}^{n} k^s X^k \binom{n}{k}$ is a multiple of $(1 + X)^{n-s}$ for all $s < n$.

Proof. This is very easy: for $s = 0$ this follows from

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = (1-1)^n = 0$$

and if

$$\sum_{k=0}^{n} k^s X^k \binom{n}{k} = (1 + X)^{n-s} f(X),$$

it is enough to differentiate the previous relation and to multiply it by X to get the inductive step.

Coming back to the proof, write

$$\sum_{k=0}^{n} k^s X^k \binom{n}{k} = (1+X)^{n-s} f(X)$$

for some $f \in \mathbb{Z}[X]$. Then for $1 \le j < p$ (note that by the lemma the term for $j = 0$ vanishes) we have

$$\sum_{k=0}^{n} (-z^j) k^s \binom{n}{k} = (1 - z^j)^{n-s} f(-z^j)$$

and so

$$v_p\left(\sum_{k=0}^{n} (-z^j) k^s \binom{n}{k}\right) \ge \frac{n-s}{p-1} = q + \frac{r+1}{p-1} > q.$$

Thus, the claim is proved and the conclusion follows.

U161. Let $f : (0, \infty) \to (0, \infty)$ be a function satisfying $f(f(x)) = x^2$ for all $x \in (0, \infty)$.

(a) Find $f(1)$.

(b) Determine the function f if it is differentiable at $x = 1$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Define $g : \mathbb{R} \to \mathbb{R}$ as $g(y) = \ln(f(e^y))$. Note that for all positive real x, $y = \ln(x)$ exists and is real, while conversely for all real y, $x = e^y$ is a positive real. Now,

$$g(g(\ln(x))) = g(\ln(f(x))) = \ln(f(f(x))) = \ln(x^2) = 2\ln(x),$$

or for all real y, $g(g(y)) = 2y$. Moreover, if f is differentiable at x_0, then g is differentiable at $y_0 = \ln(x_0)$, since

$$\frac{dg(y)}{dy} = \frac{d(\ln(f(x)))}{dx}\frac{dx}{dy} = \frac{x}{f(x)} \cdot \frac{df(x)}{dx},$$

and if $\frac{df(x)}{dx}$ exists and is real for some x_0, then $\frac{dg(y)}{dy}$ exists and is real for $y_0 = \ln(x_0)$. Since f is differentiable at $x = 1$, then g is differentiable at $y = 0$.

Note first that $g(g(g(y))) = g(2y) = 2g(y)$, or making $y = 0$ (ie $x = 1$) we find $g(0) = 2g(0)$, or $g(0) = 0$ and consequently $f(1) = e^0 = 1$. Define now for any nonzero real y_0, the sequence such that for all $n \geq 1$, $y_n = \frac{y_{n-1}}{2} = \frac{y_0}{2^n}$. Clearly,

$$g\left(\frac{y_0}{2^n}\right) = \frac{1}{2}g\left(\frac{y_0}{2^{n-1}}\right), \qquad g(y_n) = g\left(\frac{y_0}{2^n}\right) = \frac{g(y_0)}{2^n},$$

where the second equality is obtained from the first after trivial induction. Equivalently, for all $n \geq 1$ we have $\frac{g(y_n)}{y_n} = \frac{g(y_0)}{y_0}$. But $\lim_{n\to\infty} y_n = 0$, and by the definition of the derivative,

$$g'(0) = \lim_{y\to 0}\frac{g(y) - g(0)}{y - 0} = \lim_{n\to\infty}\frac{g(y_n)}{y_n} = \lim_{n\to\infty}\frac{g(y_0)}{y_0} = \frac{g(y_0)}{y_0},$$

or for all real y, $g(y) = g'(0)y$. Inserting into $g(g(y)) = 2y$ yields $(g'(0))^2 = 2$, resulting in $g(y) = \sqrt{2}y$ for all real y or $g(y) = -\sqrt{2}y$ for all real y, and equivalently $f(x) = x^{\sqrt{2}}$ for all positive real x, or $f(x) = x^{-\sqrt{2}}$ for all positive real x. The conclusion follows.

Also solved by Sayan Mukherjee, Kolkata, India.

U162. Let $f:\mathbf{R}\to\mathbf{R}$ be a monotonic function and let $F:\mathbf{R}\to\mathbf{R}$,

$$F(x)=\int_0^x f(t)dt.$$

Prove that if F is differentiable, then f is continuous.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, and Mihai Piticari, Dragos Voda National College, Campulung Moldovenesc, Romania

Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

We make use of the following two Lemmas

Lemma 1. *A monotonic function $f:\mathbf{R}\to\mathbf{R}$ can have at most countable discontinuities. Also at any point it admits right band left limit.*

Lemma 2. *Let $f:\mathbf{R}\to\mathbf{R}$ be a differentiable function. If $\lim_{x\to x_0^\pm} f'(x)=l^\pm$ ($\pm\infty$ allowed), then $\lim_{h\to 0^\pm}\frac{1}{h}(f(x_0+h)-f(x_0))=l^\pm$*

Let A be the set of point where f is continuous and let x_0 be a point of discontinuity for $f(x)$. Of course Fundamental–theorem of calculus (FTC) we have $F'(x)=f(x)$ for any $x\in A$.

Since by the first statement of Lemma 1 x_0 is an accumulation point of A, there are two sequences in A, $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ such that $x_n\searrow x_0$ and $y_n\nearrow x_0$. We have $F'(x_n)=f(x_n)\to l^+$ and $F'(y_n)=f(y_n)\to l^-$ where the first equal is due to the FTC while the second to the second statement of Lemma 1. By Lemma 2 the limits l^+ and l^- are the right and left derivatives of $F(x)$ but both they are equal to $F'(x_0)$ since $F(x)$ is differentiable and then $l^+=l^-$ that is f is continuous at x_0.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

Olympiad problems

O157. A frog jumps on the real axis, from the origin towards point $(1,0)$ such that the length of the nth jump is $1/p_n$ times its distance to the point $(1,0)$, where p_n is the nth prime ($p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots$). Can the frog reach point $(1,0)$?

Proposed by Moreno Miguel Marano, Universidad de Jaén, Spain

First solution by G.R.A.20 Problem Solving Group, Roma, Italy

Let x_n be the distance to the point $(1,0)$ after the nth jump. Hence $x_0 = 1$ and for $n > 0$, x_n is a strictly decreasing sequence such that

$$x_n = x_{n-1} - \frac{x_{n-1}}{p_n} = \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right).$$

It is well known that

$$x_n = \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right) \to \frac{1}{\zeta(1)} = 0,$$

so, after a finite number of jups, the frog can arrive arbitrarily close to $(1,0)$ but it will never reach it.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

After the n-th jump, the distance from $(1,0)$ will be at least $\frac{p_n-1}{p_n}$ times the distance before the n-th jump, hence always positive. The frog never reaches point $(1,0)$.

O158. For each positive integer n define

$$a_n = \frac{(n+1)(n+2)\cdots(n+2010)}{2010!}.$$

Prove that there are infinitely many n such that a_n is an integer with no prime factors less than 2010.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let N be the least common multiple of $\{1, 2, 3, \ldots, 2010\}$, P the product of all primes less than 2010, and take $n = kNP$, where k is any positive integer. Note that, for any $m \in \{1, 2, \ldots, 2010\}$, m divides N, and $\frac{n+m}{m} = \frac{kNP+m}{m} = kP\frac{N}{m} + 1$ is an integer relatively prime with P, hence relatively prime with all primes less than 2010. We may therefore express, for all positive integers k, a_{kNP} as a product of integers, all relatively prime with all primes less than 2010. The conclusion follows.

Second solution by G.R.A.20 Problem Solving Group, Roma, Italy

Let $\mathcal{P}$ be the set of primes less than 2010. If $p \in \mathcal{P}$ then the base p representation of 2010 has at most 11 digits:

$$(2010)_p = c_{10}c_9 \ldots c_1 c_0.$$

For any $a > 10$, let

$$n = \prod_{p \in \mathcal{P}} p^a.$$

Hence the first 11 digits of $n + 2010$ and 2010 in base p coincide, and by Lucas' Theorem

$$a_n = \binom{n+2010}{2010} \equiv \binom{c_s}{0} \cdots \binom{c_{11}}{0}\binom{c_{10}}{c_{10}} \cdots \binom{c_1}{c_1}\binom{c_0}{c_0} = 1 \pmod{p}$$

which means that p does not divide a_n for any $p \in \mathcal{P}$.

O159. Let G be a graph with $n \geq 5$ vertices. The edges of G are colored in two colors such that there are no monochromatic cycles C_3 and C_5. Prove that there are no more than $\frac{3}{8}n^2$ edges in the graph.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy

Assume that the number of edges $|E|$ is greater than $\frac{3}{8}n^2$. By Turan's Theorem,

$$|E| > \frac{3}{8}n^2 = \frac{n^2}{2}\left(1 - \frac{1}{5-1}\right)$$

implies that there is a K_5 in G. The 10 edges of K_5 are colored in two colors, and the degree of each vertex is 4. If a vertex has at least three monochromatic edges then there is a monochromatic C_3 and we have a contradiction. Hence any vertex has exactly two edges of one color and two edges of the other one. After a few diagrams it is easy to see that also in this case K_5 contains a monochromatic C_3 or a monochromatic C_5.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

O160. Let $a_1, a_2, \dots a_n, \cdots$ be a sequence of positive integers, such that for each prime p there are infinitely many terms in the sequence that are divisible by p. Prove that every positive rational number less than 1 can be represented as

$$\frac{b_1}{a_1} + \frac{b_2}{a_1 a_2} + \cdots + \frac{b_n}{a_1 a_2 \cdots a_n},$$

where $b_1, b_2, \dots b_n$ are integers such that $0 \le b_i \le a_i - 1$, $i = 1, \dots, n$.

Proposed by Nairi Sedrakyan, Armenia

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let q_1 be any positive rational less than 1, and define recursively, b_n as the largest non-negative integer such that $\frac{b_n}{a_n} \le q_n$, and $q_{n+1} = a_n q_n - b_n$. Note that q_n is clearly a positive rational, and by trivial double induction, $b_n \le a_n - 1$ since otherwise $b_n \ge a_n$ and $q_n \ge \frac{b_n}{a_n} \ge 1$ absurd, and simultaneously q_n is less than 1, because if $q_{n+1} \ge 1$, then $\frac{b_n+1}{a_n} \le q_n$ in contradiction with the definition of b_n. Note further that

$$q_1 = \frac{b_1}{a_1} + \frac{q_2}{a_1} = \frac{b_1}{a_1} + \frac{b_2}{a_1 a_2} + \frac{q_3}{a_1 a_2} = \cdots = \frac{b_1}{a_1} + \frac{b_2}{a_1 a_2} + \cdots + \frac{b_n}{a_1 a_2 \dots a_n} + \frac{q_{n+1}}{a_1 a_2 \dots a_n}.$$

Note that sequences $b_1, b_2, \dots$ and $q_1, q_2, \dots$ are therefore well defined and unique for each positive rational q_1 less than 1, and that it suffices to show that eventually a value $q_{n+1} = 0$ will appear, since then $q_{n+1} = b_{n+1} = q_{n+2} = b_{n+2} = \cdots = 0$, and q_1 will be expressed in the desired form. In order to prove that eventually a $q_{n+1} = 0$ will appear, write $q_n = \frac{u_n}{v_n}$, where $0 \le u_n < v_n$ are integers (clearly possible since q_n is a non-negative rational less than 1), where furthermore u_n, v_n are relatively prime (possible since we may divide both by their greatest common divider if they are not relatively prime). We show next that v_{n+1} divides v_n, both being equal iff a_n is prime with v_n. Clearly,

$$\frac{u_{n+1} v_n}{v_{n+1}} = a_n u_n - b_n v_n,$$

and v_{n+1} must divide $u_{n+1} v_n$ because the RHS is an integer. But u_{n+1} and v_{n+1} are relatively prime by definition, or v_{n+1} divides v_n. Assume now that $v_{n+1} = v_n$; since $u_{n+1} = a_n u_n - b_n v_n$ is relatively prime with $v_{n+1} = v_n$, then $a_n u_n$ is relatively prime with v_n, in particular a_n is relatively prime with v_n. Therefore, if a_n and v_n are not relatively prime, v_{n+1} is a proper divisor of v_n. Note therefore that the process cannot continue indefinitely; call k the number of (not necessarily different) prime factors of v_1, then after at most the k-th time that v_n and a_n are not relatively prime, we would have $v_{n+1} = 1$, impossible unless $u_{n+1} = 0$, ie $q_{n+1} = 0$. Moreover, if after a certain N we would encounter only values of a_n that are prime with v_n, then for all $n \ge N$, a_n is prime with v_N, absurd since there are infinitely many a_n's that are divisible by any prime that divides v_N. The conclusion follows.

O161. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^5(b+2c)^2} + \frac{1}{b^5(c+2a)^2} + \frac{1}{c^5(a+2b)^2} \geq \frac{1}{3}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Magkos Athanasios, Kozani, Greece

We will make use of the following lemma which was proven in the solution of problem J131: If $x, y, z, a, b, c > 0$ we have

$$\frac{x^3}{a^2} + \frac{y^3}{b^2} + \frac{z^3}{c^2} \geq \frac{(x+y+z)^3}{(a+b+c)^2}.$$

We will also relax the condition $abc = 1$ to $abc \leq 1$. Set

$$a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z}.$$

Then, we have $xyz \geq 1$ and the left hand side of the inequality is equal to

$$K = (xyz)^2 \sum_{cyc} \frac{x^3}{(2y+z)^2}.$$

From the above lemma (and since $xyz \geq 1$)we have

$$K \geq \frac{(x+y+z)^3}{9(x+y+z)^2} = \frac{x+y+z}{9} \geq \frac{3\sqrt[3]{xyz}}{9} \geq \frac{1}{3}.$$

Second solution by Arkady Alt, San Jose, California, USA

Since

$$\frac{1}{a^5\left(b+2c\right)^2} = \frac{1}{a^5b^2c^2\left(\dfrac{1}{c} + \dfrac{2}{b}\right)^2} = \frac{\left(\dfrac{1}{a}\right)^3}{\left(\dfrac{1}{c} + \dfrac{2}{b}\right)^2}$$

then by replacing $\left(\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}\right)$ in original inequality with (a, b, c) we obtain the equivalent inequality

$$\sum_{cyc} \frac{a^3}{\left(2b+c\right)^2} \geq \frac{1}{3},$$

with $abc = 1$. Note that $\dfrac{a^2}{2b+c} \geq \dfrac{2}{3}a - \dfrac{2b+c}{9}$; hence

$$\begin{aligned}
\sum_{cyc} \frac{a^3}{(2b+c)^2} &\geq \sum_{cyc} \frac{a}{2b+c}\left(\frac{2}{3}a - \frac{2b+c}{9}\right) = \frac{2}{3}\sum_{cyc}\frac{a^2}{2b+c} - \sum_{cyc}\frac{a}{9} \\
&\geq \frac{2}{3}\sum_{cyc}\left(\frac{2}{3}a - \frac{2b+c}{9}\right) - \sum_{cyc}\frac{a}{9} = \frac{a+b+c}{3} - \frac{2}{9}\sum_{cyc}(2b+c) \\
&= \frac{a+b+c}{9} \geq \frac{3\sqrt[3]{abc}}{9} = \frac{1}{3}.
\end{aligned}$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; G. C. Greubel, USA;

Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

O162. In a convex hexagon $ABCDEF$, $AB||DE$, $BC||EF$, $CD||FA$ and $AB+DE=BC+EF=CD+FA$. Denote the midpoints of sides AB,BC,DE,EF by A_1,B_1,D_1,E_1, respectively. Prove that $\widehat{D_1OE_1}=\frac{1}{2}\widehat{DEF}$, where O is the point of intersection of segments A_1D_1 and B_1E_1.

Proposed by Nairi Sedrakyan, Armenia

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Define $\alpha=\pi-\angle FAB=\pi-\angle CDE$, $\beta=\pi-\angle ABC=\pi-\angle DEF$ and $\gamma=\pi-\angle BCD=\angle EFA$. Clearly $\alpha+\beta+\gamma=\pi$, or at least two of them are acute. Define also O',O'' as the respective intersections of B_1E_1 and C_1F_1, and of C_1F_1 and A_1D_1, where C_1,F_1 are the respective midpoints of CD,FA, and call $\alpha'=\angle F_1O''A_1=\angle C_1O''D_1$, $\beta'=\angle A_1OB_1=\angle D_1OE_1$ and $\gamma'=\angle B_1O'C_1=\angle E_1O'F_1$. Clearly, $\alpha'+\beta'+\gamma'=\pi$, since we may displace C_1F_1 parallel to itself until it passes through O, and the angles between the lines do not change. Moreover, by the cyclic symmetry in the problem, it is equivalent that $\beta'=\angle D_1OE_1=\frac{1}{2}\angle DEF=\frac{\pi-\beta}{2}$, that $\alpha'=\frac{\pi-\alpha}{2}$, and that $\gamma'=\frac{\pi-\gamma}{2}$; therefore, we may cyclically rotate the vertices without altering the problem, or we may choose wlog α and β acute, denoting $m=\tan\beta$ and $M=\tan\alpha$, clearly positive.

Let us now choose a system of coordinates such that, wlog $A\equiv(-d,-h)$, $B\equiv(d,-h)$, $D\equiv(d'+\Delta,h)$, $E\equiv(-d'+\Delta,h)$, where d,d',h are positive and $d'\geq d$. After some algebra (finding the equation of lines BC,CD,EF,FA and solving for their relevant intersections, and using that BC,CD have respective slopes $m,-M$), we find

$$C\equiv\left(\frac{Md'+M\Delta+md+2h}{M+m},\frac{Mmd'+Mm\Delta-Mmd-Mh+mh}{M+m}\right),$$

$$F\equiv\left(-\frac{md'-m\Delta+Md+2h}{M+m},\frac{Mmd'-Mm\Delta-Mmd+Mh-mh}{M+m}\right).$$

Note that the vertical coordinate for C,F must be contained strictly in $(-h,h)$ for the hexagon to be convex, which yields

$$\frac{2h}{m}>d'-d+\Delta>-\frac{2h}{M},\qquad\frac{2h}{M}>d'-d-\Delta>-\frac{2h}{m}.$$

After some more algebra, and taking into account the previous results, we have

$$BC=\frac{M(d'-d+\Delta)+2h}{M+m}\sqrt{m^2+1},\qquad EF=\frac{2h-M(d'-d-\Delta)}{M+m}\sqrt{m^2+1},$$

$$CD=\frac{2h-m(d'-d+\Delta)}{M+m}\sqrt{M^2+1},\qquad FA=\frac{m(d'-d-\Delta)+2h}{M+m}\sqrt{M^2+1}.$$

Therefore,

$$d+d'=\frac{AB+DE}{2}=\frac{BC+EF}{2}=\frac{CD+FA}{2}=$$
$$=\frac{2h+M\Delta}{M+m}\sqrt{m^2+1}=\frac{2h-m\Delta}{M+m}\sqrt{M^2+1}.$$

Now, from the equality of the last two terms, we find

$$\left(\frac{\Delta}{2h}\right)^2 + 2\frac{Mm+1}{M-m}\left(\frac{\Delta}{2h}\right) - 1 = 0,$$

$$\frac{\Delta}{2h} = \frac{-Mm-1+\sqrt{(M^2+1)(m^2+1)}}{M-m} = \tan\frac{\alpha-\beta}{2},$$

where we have assumed that $M \neq m$; if $M = m$, then $\Delta = 0$, yielding $d + d' = \frac{h\sqrt{m^2+1}}{m} = \frac{h}{\sin\beta} = \frac{h}{\sin\alpha}$. Note that we have neglected the negative root, not only by continuity with the solution when $M = m$, but also since the positive root, after some calculations, yields $2h = (d+d')(\sin\alpha + \sin\beta)$, clearly true since $2h = BC\sin\beta + CD\sin\alpha = EF\sin\beta + FA\sin\alpha$.

Now, we may easily find

$$A_1 \equiv (0, -h), \qquad D_1 \equiv (\Delta, h),$$

$$B_1 \equiv \left(\frac{Md' + Md + M\Delta + 2md + 2h}{2(M+m)}, \frac{Mmd' + Mm\Delta - Mmd - 2Mh}{2(M+m)}\right),$$

$$E_1 \equiv \left(-\frac{2md' - 2m\Delta - M\Delta + Md + Md' + 2h}{2(M+m)}, \frac{Mmd' - Mm\Delta - Mmd + 2Mh}{2(M+m)}\right).$$

Now, β' is also the angle formed by vectors $\overrightarrow{A_1D_1}$ and $\overrightarrow{B_1E_1}$, where

$$\overrightarrow{A_1D_1} \equiv (\Delta, 2h), \qquad \overrightarrow{B_1E_1} \equiv \left(-\frac{(M+m)(d+d') + 2h - m\Delta}{M+m}, -M\frac{m\Delta - 2h}{M+m}\right)$$

Now, since $\frac{2h}{\Delta} = \cot\frac{\alpha-\beta}{2}$, $\overrightarrow{A_1D_1}$ forms an angle of $\frac{\pi-\alpha+\beta}{2}$ with the horizontal axis. Moreover,

$$\frac{M(m\Delta - 2h)}{(M+m)(d+d') + 2h - m\Delta} = \frac{\tan\alpha(\tan\beta\tan\frac{\alpha-\beta}{2} - 1)}{\frac{\tan\alpha+\tan\beta}{\sin\alpha+\sin\beta} + 1 - \tan\beta\tan\frac{\alpha-\beta}{2}} =$$

$$= \frac{-\sin\alpha\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2}(1+\cos\alpha)} = -\tan\frac{\alpha}{2},$$

or the angle formed by $\overrightarrow{B_1E_1}$ is $\pi - \frac{\alpha}{2}$. Therefore, the angle between $\overrightarrow{B_1E_1}$ and $\overrightarrow{A_1D_1}$ is $\beta' = \pi - \frac{\alpha}{2} - \frac{\pi-\alpha+\beta}{2} = \frac{\pi-\beta}{2}$. The conclusion follows.

Note that if $M = m$, then $\overrightarrow{A_1D_1} \equiv (0, 2h)$ forms an angle of $\frac{\pi}{2}$ with the horizontal, while $\overrightarrow{B_1E_1} \equiv \left(-\frac{m(d+d')+h}{m}, h\right)$, and since

$$-\frac{mh}{m(d+d')+h} = -\frac{\sin\beta}{1+\cos\beta} = -\tan\frac{\beta}{2},$$

then $\overrightarrow{B_1E_1}$ forms an angle of $\pi - \frac{\beta}{2}$ with the horizontal, or again $\beta' = \pi - \frac{\beta}{2} - \frac{\pi}{2} = \frac{\pi-\beta}{2}$.