Junior problems

J121. For an even integer n consider a positive integer N having exactly n^2 divisors greater than 1. Prove that N is the fourth power of an integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J122. Let $ABCD$ be a quadrilateral inscribed in a circle and circumscribed about a circle such that the points of tangency form a quadrilateral $A_1B_1C_1D_1$. Prove that $A_1C_1 \perp B_1D_1$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J123. Solve in prime numbers the equation: $x^y + y^x = z$.

Proposed by Lucian Petrescu, "Henri Coanda" College, Tulcea, Romania

J124. Let a and b be integers such that $|b-a|$ is an odd prime. Prove that $P(x) = (x-a)(x-b)-p$ is irreducible in $\mathbb{Z}[X]$ for any prime p.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J125. Let ABC be an isosceles triangle with $\angle A = 100^\circ$. Denote by BL the angle bisector of angle $\angle ABC$. Prove that $AL + BL = BC$.

Proposed by Andrei Razvan Baleanu, "G. Cosbuc" National College, Romania

J126. Let a, b, c be positive real numbers. Prove that

$$3(a^2b^2 + b^2c^2 + c^2a^2)(a^2 + b^2 + c^2) \geq (a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2).$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Senior problems

S121. Let $f : [1, \infty) \to \{1, 2, \ldots\}$ be a function such that $f(x) = y$, where $y! \leq x < (y+1)!$. Prove that $f(a^2) + f(b^2) \leq 2f(ab)$, for all $a, b \geq 1$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S122. Let P and Q be points on a segment BC such that P lies between B and Q. Suppose that BP, PQ, QC form a geometric progression in some order. Prove that there is a point A in the plane such that AP and AQ are the trisectors of angle BAC if and only if PQ is less than BP and QC.

Proposed by Daniel Campos, Costa Rica

S123. Prove that in any triangle with sidelenghts a, b, c the following inequality holds:

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} + \frac{(b+c-a)(c+a-b)(a+b-c)}{abc} \geq 7.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

S124. Let ABC be a triangle with midpoints M_a, M_b, M_c and let X, Y, Z be the points of tangency of the incircle of triangle $M_aM_bM_c$ with M_bM_c, M_cM_a, M_aM_b, respectively.

a) Prove that the lines AX, BY, CZ are concurrent at some point P.

b) If AA_1, BB_1, CC_1 are cevians through P, then the perimeter of triangle $A_1B_1C_1$ is greater than or equal to the semiperimeter of triangle ABC.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

S125. Find all pairs (p, q) of positive integers that satisfy

$$\left|\frac{p}{q} - \sqrt{2}\right| < \frac{1}{q^2}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S126. Let a, b, c be positive real numbers. Prove that

$$\sqrt{\frac{a^2(b^2+c^2)}{a^2+bc}} + \sqrt{\frac{b^2(c^2+a^2)}{b^2+ca}} + \sqrt{\frac{c^2(a^2+b^2)}{c^2+ab}} \leq a+b+c.$$

Proposed by Pham Huu Duc, Ballajura, Australia

Undergraduate problems

U121. Let p be a prime and let α be a permutation of order p in S_{p+1}. Find the set $C_\alpha = \{\sigma \in S_{p+1} \mid \sigma\alpha = \alpha\sigma\}$.

Proposed by Dorin Andrica, Babes-Bolyai University and Mihai Piticari, National College Dragos-Voda, Romania

U122. Let $f : [0,1] \to \mathbb{R}$ be a twice differentiable function with the second derivative continuous such that

$$\int_0^1 f(x)dx = 3\int_{1/3}^{2/3} f(x)dx.$$

Prove that there exists $x_0 \in (0,1)$ such that $f''(x_0) = 0$.

Proposed by Cezar Lupu, University of Bucharest and Tudorel Lupu, Decebal High School Constanta, Romania

U123. Let C_1, C_2, C_3 be concentric circles with radii $1, 2, 3$, respectively. Consider a triangle ABC with $A \in C_1$, $B \in C_2$, $C \in C_3$. Prove that $\max K_{ABC} < 5$, where $\max K_{ABC}$ denotes the greatest possible area of triangle ABC.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

U124. Let $\{x_n\}_{n\geq 1}$ be a sequence of real numbers such that $\arctan x_n + nx_n = 1$ for all positive integers n. Evaluate $\lim\limits_{n\to\infty} n\ln(2 - nx_n)$.

Proposed by Duong Viet Thong, Nam Dinh University of Technology and Education, Vietnam

U125. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be distinct real numbers. Let A be the matrix with entries $a_{ij} = \frac{u_i+v_j}{u_i-v_j}$ and B be the matrix with entries $b_{ij} = \frac{1}{u_i-v_j}$ for $1 \leq i, j \leq n$. Prove that

$$\det A = 2^{n-1}(u_1u_2\cdots u_n + v_1v_2\cdots v_n)\det B.$$

Proposed by Darij Grinberg, Ludwig Maximilian University of Munich, Germany

U126. Find all continuos and bijective functions $f : [0,1] \to [0,1]$ such that

$$\int_0^1 g(f(x))dx = \int_0^1 g(x)dx,$$

for all continuos functions $g : [0,1] \to \mathbb{R}$.

Proposed by Dorin Andrica, Babes-Bolyai University and Mihai Piticari, National College Dragos-Voda, Romania

Olympiad problems

O121. Let a, b, c be positive real numbers. Prove that

$$\sqrt{ab(a+b)}+\sqrt{bc(b+c)}+\sqrt{ca(c+a)} \geq \frac{5}{4}\sqrt{(a+b)(b+c)(c+a)}+2\sqrt{abc}.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

O122. Let p and q be odd primes such that $q \nmid p-1$ and let $a_1, a_2, \ldots, a_n$ be distinct integers such that $q \mid (a_i - a_j)$ for all pairs (i, j). Prove that

$$P(x) = (x-a_1)(x-a_2)\cdots(x-a_n) - p,$$

is irreducible in $\mathbb{Z}[X]$ for $n \geq 2$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O123. Let ABC be a triangle and let A_1, B_1, C_1 be the points of tangency of its incircle ω with triangle's sides. Medians A_1M, B_1N, C_1P in triangle $A_1B_1C_1$ intersect ω at A_2, B_2, C_2, respectively. Prove that AA_2, BB_2, CC_2 are concurrent at the isogonal conjugate of the Gergonne point Γ.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O124. Let $S(n)$ be the number of pairs of positive integers (x, y) such that $xy = n$ and $\gcd(x, y) = 1$. Prove that

$$\sum_{d \mid n} S(d) = \tau(n^2),$$

where $\tau(s)$ is the number of divisors of s.

Proposed by Dorin Andrica, Babes-Bolyai University and Mihai Piticari, Campulung Moldovenesc, Romania

O125. Let a, b, c be positive real numbers. Prove that

$$4 \leq \frac{a+b+c}{\sqrt[3]{abc}} + \frac{8abc}{(a+b)(b+c)(c+a)}.$$

Proposed by Pham Huu Duc, Ballajura, Australia

O126. Let ABC be a scalene triangle and let $\mathcal{K}_a$ be the A-mixtilinear incircle (the circle tangent to sides AB, AC and internally tangent to the circumcircle Γ of triangle ABC). Denote by A' the tangency point of $\mathcal{K}_a$ with Γ and let A'' be the diametrically opposed point of A' with respect to $\mathcal{K}_a$. Similarly, define B'' and C''. Prove that lines AA'', BB'', CC'' are concurrent.

Proposed by Cosmin Pohoata, National College "Tudor Vianu", Romania