Junior problems

J121. For an even integer n consider a positive integer N having exactly n^2 divisors greater than 1. Prove that N is the fourth power of an integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by John T. Robinson, Yorktown Heights, NY, USA

The following results will be of use.

Lemma 1: If an odd prime p divides x^2+1 for some integer x, then $p = 1 \bmod 4$.

Proof: Since p is odd, p is of the form $4y+1$ or $4y+3$; showing that $p = 3 \bmod 4$ is impossible will establish the claim. Assuming to the contrary that $p = 4y+3$, we have

$$x^2 = -1 \bmod p\,;$$

$$(x^2)^{2y+1} = x^{4y+2} = x^{p-1} = (-1)^{2y+1} = -1 \bmod p$$

which is impossible since (noting that p cannot divide x) $x^{p-1} = 1$ (by Fermat's "little" theorem).

Lemma 2. If $4x^2+1 = f_1 f_2 ... f_k$, then for each factor f_i, $f_i = 1 \bmod 4$.

Proof. Since $4x^2+1$ is odd, it can only have odd divisors. Each factor is either 1 or some product of (odd) prime factors of $4x^2+1$, each of which must be congruent to 1 mod 4 by lemma 1. It follows that every factor is congruent to 1 mod 4.

Considering now the original problem, let $n = 2m$, then the condition that N has n^2 divisors greater than 1 is equivalent to

$$\tau(N) = (e_1+1)(e_2+1)...(e_k+1) = 4m^2+1$$

where τ is the number of divisors function and the prime factorization of N is

$$N = p_1^{e_1} p_2^{e_2} ... p_k^{e_k}\ .$$

By lemma 2 each of the factors e_i+1 is of the form $4y_i+1$; therefore each e_i is a multiple of 4. It follows that N is a fourth power.

Second solution by Neal Wu

Let N be factored (into primes) as $N = p_1^{e_1} \cdots p_k^{e_k}$. It is well known that N has

$$(e_1+1) \times \cdots \times (e_k+1) = n^2+1$$

divisors (including 1). Since n is even, $n^2+1 \equiv 1 \pmod 4$, so e_i+1 must be either 1 mod 4 or 3 mod 4 for $1 \le i \le k$. We will show that n^2+1 has no factors of the form 3 mod 4; thus $e_i+1 \equiv 1 \pmod 4$ so e_i is divisible by 4 for all i, implying that N is a fourth power.

Proof: Assume for a contradiction that there exists some number p such that $p = 4k+3$ and n^2+1 is divisible by p. WLOG let p be prime; otherwise it must have some prime factor also of the form 3 mod 4 (since p is 3 mod 4, its prime factors cannot all be 1 mod 4). Thus $n^2 \equiv -1 \pmod p$. Note that n must then be relatively prime to p; thus by Fermat's Little Theorem, $n^{p-1} \equiv 1 \pmod p$. However,

$$n^{p-1} \equiv n^{4k+2} \equiv \left(n^2\right)^{2k+1} \equiv (-1)^{2k+1} \equiv -1 \pmod p.$$

Thus $1 \equiv -1 \pmod p$, but since $p \ge 3$, we have the desired contradiction.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; José H. Nieto S., Universidad del Zulia, Venezuela.

J122. Let $ABCD$ be a quadrilateral inscribed in a circle and circumscribed about a circle such that the points of tangency form a quadrilateral $A_1B_1C_1D_1$. Prove that $A_1C_1 \perp B_1D_1$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Neal Wu

Let the tangency points of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ intersect the inner circle at points A_1, B_1, C_1, and D_1, respectively, and let P be the intersection point of $\overline{A_1C_1}$ and $\overline{B_1D_1}$. Then, since $ABCD$ is cyclic,

$$\begin{aligned}180^\circ &= \angle DAB + \angle BCD \\ &= \frac{\widehat{A_1B_1} + \widehat{B_1C_1} + \widehat{C_1D_1} - \widehat{D_1A_1}}{2} + \frac{\widehat{C_1D_1} + \widehat{D_1A_1} + \widehat{A_1B_1} - \widehat{B_1C_1}}{2} \\ &= \widehat{A_1B_1} + \widehat{C_1D_1} \\ &= 2\angle A_1PB_1\end{aligned}$$

where $\widehat{XY}$ denotes the (minor) arc between points X and Y. Thus $\angle A_1PB_1 = 90^\circ$, and $A_1C_1 \perp B_1D_1$ as desired.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We will use in this solution the following

Lemma: The diagonals of a cyclic quadrilateral $TUVW$ are perpendicular iff $TU^2 + VW^2 = UV^2 + WT^2$.

Proof: Call P the point where diagonals TV and UW meet, and $\alpha = \angle TPU = \angle VPW = \pi - \angle UPV = \pi - \angle WPT$. Using the Cosine Law, $TU^2 = TP^2 + PU^2 - 2TP \cdot PU \cos\alpha$, and similarly for its cyclic permutations, yielding

$$TU^2 + VW^2 - UV^2 - WT^2 = -2(TP\cdot UP + UP\cdot VP + VP\cdot WP + WP\cdot TP)\cos\alpha.$$

Clearly, $TU^2 + VW^2 = UV^2 + WT^2$ iff $\cos\alpha = 0$. The lemma follows.

Call $\gamma(I, r)$ the incircle of $ABCD$ with center I and radius r. Assume wlog that A_1, B_1, C_1, D_1 are the points where γ touches respectively sides AB, BC, CD, DA. Clearly A_1ID_1 is isosceles at I with $IA_1 = ID_1 = r$ and $\angle A_1ID_1 = \pi - A$, where A denotes $\angle D_1AA_1 = \angle DAB$. Thus $D_1A_1 = r\cos\frac{A}{2}$, and similarly for its cyclic permutations, or $D_1A_1^2 + B_1C_1^2 = r^2$ because $A + C = \pi$, and similarly $A_1B_1^2 + C_1D_1^2 = r^2$. Using the lemma, the conclusion immediately follows.

Also solved by Magkos Athanasios, Kazani, Greece; Ercole Suppa, Teramo, Italy; Vicente Vicario García, Huelva, Spain.

J123. Solve in prime numbers the equation: $x^y + y^x = z$.

Proposed by Lucian Petrescu, "Henri Coanda" College, Tulcea, Romania

First solution by Dmitri Skjorshammer, Harvey Mudd College, Claremont, CA

We first note that neither x and y equal nor are they both odd, for otherwise z wouldn't be prime. Without loss of generality, suppose that $x = 2$. The equation then reduces to $2^y + y^2 = z$, or,

$$\begin{aligned} z &= 2^y - 2 + y^2 + 2, \\ &= 2(2^{y-1} - 1) + (y+1)(y+2) - 3y. \end{aligned}$$

First, $(y+1)(y+2)$ is divisible by three since all primes greater than 3 are either one or two less than a multiple of three (if some prime was three less, then it would contradict the prime property). Second, since y is an odd prime of the form $2k+1$ (for some specific integer k), we rewrite the first term as

$$2^{(2k+1)-1} - 1 = 4^k - 1.$$

Since $4 \equiv 1 \mod 3$, it is easy to show that $4^k \equiv 1 \mod 3$. Thus, $4^k - 1$ is divisible by three.

We've shown that z is divisible by three for $y > 3$. It follows that the only solution in prime numbers is $x = 2, y = 3, z = 17$ or $x = 3, y = 2, z = 17$.

Second solution by John T. Robinson, Yorktown Heights, NY, USA

We must have one of x, y even and the other odd (since otherwise z would be even and greater than 2). Therefore a solution must be of the form

$$2^p + p^2 = z$$

where p is an odd prime and z is prime. We see that

$$2^3 + 3^2 = 17$$

is a solution; are there any others? No: all primes greater than 3 are of the form $6n \pm 1$, and since

$$2^p = (-1)^p = -1 \bmod 3$$

for p odd, and

$$(6n \pm 1)^2 = 36n^2 \pm 12n + 1 = 1 \bmod 3\,,$$

it follows that $2^{6n\pm1} + (6n \pm 1)^2$ is divisible by 3.

Third solution by Andrea Cameli, Universit degli Studi di L'Aquila, Italy

First of all, we can observe that x, y can't be both odd primes, or z would be even; so, one of them must be equal to 2, say x. We now have the equation $2^y + y^2 = z$. The following lemmas hold:

LEMMA 1. If a is an odd number then $2^a \equiv 2 \pmod 3$.

Proof.

$$2 \not\equiv 0 \pmod 3 \;\Rightarrow\; 2^k \not\equiv 0 \pmod 3 \;\Rightarrow\; \left(2^k - 1\right)\left(2^k + 1\right) \equiv 0 \pmod 3$$

Thus

$$2^{2k} \equiv 1 \pmod 3 \quad \Rightarrow \quad 2^{2k+1} \equiv 2 \pmod 3 \qquad \square$$

LEMMA 2. If $p \neq 3$ is a prime then $p^2 \equiv 1 \pmod 3$.

Proof. If p is such a prime then $(p, 3) = 1$: we can apply Fermat's Theorem, so

$$p^{3-1} = p^2 \equiv 1 \pmod 3 \qquad \square$$

LEMMA 3. If $p > 3$ is a prime then $2^p + p^2 \equiv 0 \pmod 3$.

Proof. It follows from LEMMA 1 and LEMMA 2. $\square$

It follows from LEMMA 3 that y must be equal to 3: the only one solution is $(2, 3, 17)$, or $(3, 2, 17)$ by symmetry between x and y.

Also solved by EL ALAMI Anass, Meknes, Maroc; Bedri Hajrizi, Albania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; José H. Nieto S., Universidad del Zulia, Venezuela; Magkos Athanasios, Kozani, Greece; Mario Ynocente Castro, National University of Ingeneering, Peru; Neal Wu; Irfan Besic, Sarajevo, Bosnia and Herzegovina.

J124. Let a and b be integers such that $|b-a|$ is an odd prime. Prove that $P(x) = (x-a)(x-b)-p$ is irreducible in $\mathbb{Z}[X]$ for any prime p.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by John T. Robinson, Yorktown Heights, NY, USA

We want to show that

$$(x-a)(x-b)-p = x^2-(a+b)x+ab-p = 0$$

has no solutions in integers when $|b-a|$ is an odd prime and p is prime. The discriminant of this quadratic is

$$(a+b)^2-4ab+4p = (a-b)^2+4p = q^2+4p$$

where $q = |b-a|$, so the problem is equivalent to showing that q^2+4p cannot be a square. Suppose to the contrary that for some integer x

$$q^2+4p = x^2$$

$$4p = x^2-q^2 = (x-q)(x+q)$$

In order for $(x-q)(x+q)$ to be divisible by 4, x must have the same parity as q, so letting $x = q+2y$ for some positive integer y we now have

$$4p = 2y(2q+2y)$$

$$p = y(q+y)$$

Since q is an odd prime we cannot have $y = 1$ (since then $p = q+1$ which is imposible for p, q prime with q odd), therefore $y > 1$, which is also impossible since then the prime p would factor. It follows that the discriminant cannot be a square, therefore the polynomial is irreducible over the integers.

Second solution by Neal Wu

Assume for a contradiction that $P(x) = (x-a)(x-b)-p$ is indeed reducible over $\mathbb{Z}[X]$ for some prime p. Then, since $P(x)$ is a quadratic, there must be some integers r and s such that $P(x) = (x-r)(x-s)$. Thus, $P(r) = 0$, or $(r-a)(r-b) = p$. However, since $|(r-a)-(r-b)| = |a-b|$ is odd, their product, p, must be even, so $p = 2$.

Let $x = r-a$ and $y = r-b$, and without loss of generality let $x < y$. Since $xy = p > 0$, either x, y are both positive or x, y are both negative. Without loss of generality let them both be positive. Then since $|x-y|=|a-b|$, which is an odd prime,

$$xy \geq 1 \cdot (1+3) > 2 = p,$$

the desired contradiction.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Andrea Cameli, Universit degli Studi di L'Aquila, Italy; Magkos Athanasios, Kozani, Greece; Vicario Garca, Huelva, Spain.

J125. Let ABC be an isosceles triangle with $\angle A = 100°$. Denote by BL the angle bisector of angle $\angle ABC$. Prove that $AL + BL = BC$.

Proposed by Andrei Razvan Baleanu, "G. Cosbuc" National College, Romania

First solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

Let D be a point on BL produced beyond L such that $LD = LA$ and let E be a point on BC such that LE bisects $\angle BLC$.

Since $\angle ABC = \angle BCA = 40°$, we have $\angle ABL = \angle LBE = 20°$, $\angle BLA = 60°$, $\angle BLC = 120°$ and $\angle BLE = \angle ELC = \frac{1}{2}\angle BLC = 60° = \angle DLC$.

Thus, triangles ABL and EBL are congruent (angle-side-angle), giving $LA = LE$. Therefore, $LD = LE$.

We also have that LC is the bisector of the vertical angle in isosceles triangle DLE. Hence LC is actually the perpendicular bisector of the base DE, and accordingly it crosses it at right angles at its midpoint.

Consequently, $\angle LCD = \angle ECL = 40°$ and $\angle EDC = 90° - \angle LCD = 50°$.

So we have

$$\angle BDC = \angle BDE + \angle EDC = 30° + 50° = 40° + 40° = \angle BCL + \angle LCD = \angle BCD,$$

making $\triangle BCD$ isosceles with

$$\begin{aligned} BC &= BD \\ &= BL + LD \\ &= BL + LA \end{aligned}$$

and we are done.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Clearly, $\angle B = \angle C = 40°$, or $\angle ABL = \angle CBL = 20°$, leading to $\angle ALB = 60°$ and $\angle CLB = 120°$. Using the Sine Law,

$$\frac{AL + BL}{AB} = \frac{\sin\angle ABL + \sin\angle BAL}{\sin\angle ALB} = \frac{\sin 20° + \sin 100°}{\sin 60°} =$$

$$= \frac{2\sin\frac{100°+20°}{2}\cos\frac{100°-20°}{2}}{\sin 60°} = 2\cos 40° = \frac{\sin 80°}{\sin 40°} = \frac{\sin 100°}{\sin 40°} = \frac{\sin A}{\sin C} = \frac{BC}{AB}.$$

The conclusion follows.

Third solution by José H. Nieto S., Universidad del Zulia, Venezuela

First we note that $\angle CBA = \angle BCA = 40°$, hence $\angle LBA = 20°$ and $\angle BLA = 60°$. Let M be the point (on BC) symmetric of A with respect to BL. Then $\angle BLM = \angle BLA = 60°$, hence also $\angle MLC = 60°$. On the prolongation of BL take P such that $AL = LP$. Since $\angle CLP = \angle ALB = 60°$ and $LP = LA = LM$ we see that M and P are symmetrical with respect to AC, hence $\angle PCB = 2\angle ACB = 80°$. Since $\angle CBP = 20°$, we have $\angle CPB = 80°$ and BCP is isosceles, therefore $BC = BP = BL + LP = BL + AL$.

Also solved by Magkos Athanasios, Kozani, Greece; Mario Ynocente Castro, National University of Ingeneering, Peru; Irfan Besic, Sarajevo, Bosnia and Gerzegovina; Ercole Suppa, Teramo, Italy; Vicente Vicario Garca, Huelva, Spain.

J126. Let a, b, c be positive real numbers. Prove that

$$3(a^2b^2+b^2c^2+c^2a^2)(a^2+b^2+c^2) \geq (a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2).$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Arkady Alt, San Jose, California, USA

Dividing original inequality by $a^2b^2c^2$ we obtain

$$3\left(a^2+b^2+c^2\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geq \left(\frac{a}{b}+\frac{b}{a}+1\right)\left(\frac{b}{c}+\frac{c}{b}+1\right)\left(\frac{c}{a}+\frac{a}{c}+1\right) \iff$$

(1) $9+3\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right) \geq \prod_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)+\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{b}{c}+\frac{c}{b}\right)+\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)+1.$

Since

$$\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{b}{c}+\frac{c}{b}\right)=\sum_{cyc}\left(\frac{a}{c}+\frac{c}{a}+\frac{b^2}{ca}+\frac{ca}{b^2}\right)=\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)+\sum_{cyc}\left(\frac{a^2}{bc}+\frac{bc}{a^2}\right),$$

$$\prod_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)=2+\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)$$

then (1) $\iff$ $6+2\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right) \geq 2\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)+\sum_{cyc}\left(\frac{a^2}{bc}+\frac{bc}{a^2}\right),$

where latter inequality holds, because

$$6+2\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)-2\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)-\sum_{cyc}\left(\frac{a^2}{bc}+\frac{bc}{a^2}\right)=\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)^2+$$

$$\frac{1}{2}\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{a^2}{c^2}\right)+\frac{1}{2}\sum_{cyc}\left(\frac{b^2}{a^2}+\frac{b^2}{c^2}\right)-2\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)-\sum_{cyc}\left(\frac{a^2}{bc}+\frac{bc}{a^2}\right)=$$

$$\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{a}{b}+\frac{b}{a}-2\right)+\frac{1}{2}\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{a^2}{c^2}-\frac{2a^2}{bc}\right)+\frac{1}{2}\sum_{cyc}\left(\frac{b^2}{a^2}+\frac{b^2}{c^2}-\frac{2bc}{a^2}\right)=$$

$$\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2+\frac{1}{2}\sum_{cyc}\left(\frac{a}{b}-\frac{a}{c}\right)^2+\frac{1}{2}\sum_{cyc}\left(\frac{b}{a}-\frac{b}{c}\right)^2 \geq 0.$$

Second solution by Gheorghe Pupazan, Chisinau, Republic of Moldova

After opening the brackets, the inequality becomes equivalent to:

$$2\sum a^2b^2(a^2+b^2)+6a^2b^2c^2 \geq abc(a^3+b^3+c^3)+\sum a^3b^3+2abc\cdot\sum ab(a+b).$$

Let $T = \sum c^2(a-b)^4 + (a-b)^2(b-c)^2(c-a)^2$. It's obvious that $T \geq 0$ and since

$$\frac{1}{2}\cdot T = \sum a^2b^2(a^2+b^2)+6a^2b^2c^2-abc(a^3+b^3+c^3)-\sum a^3b^3-abc\cdot\sum ab(a+b) \geq 0$$

it suffices only to show that:

$$\sum a^2b^2(a^2+b^2) \geq abc \cdot \sum ab(a+b).$$

But the proof for this one is simple, as from Muirhead's inequality we know that $(4,2,0) \succ (3,2,1)$, so the conclusion follows.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Magkos Athanasios, Kozani, Greece; Marius Mainea, Romania; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma; Italy; Irfan Besic, Sarajevo, Bosnia and Gerzegovina.

Senior problems

S121. Let $f : [1, \infty) \to \{1, 2, \dots\}$ be a function such that $f(x) = y$, where $y! \leq x < (y+1)!$. Prove that $f(a^2) + f(b^2) \leq 2f(ab)$, for all $a, b \geq 1$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

The proposed result is not true; take for example $a = 4$, $b = 5$. Clearly, since $3! = 6$, $4! = 24$ and $5! = 120$, $f(a^2) = f(16) = 3$, $f(b^2) = f(25) = 4$, and $2f(ab) = 2f(20) = 6 < f(a^2) + f(b^2)$.

Note that the opposite to the proposed result is note true either, since taking $a = 10$, $b = 12$, and since $6! = 720$, results in $f(a^2)+f(b^2) = 4+5 < 10 = 2f(ab)$.

S122. Let P and Q be points on a segment BC such that P lies between B and Q. Suppose that BP, PQ, QC form a geometric progression in some order. Prove that there is a point A in the plane such that AP and AQ are the trisectors of angle BAC if and only if PQ is less than BP and QC.

Proposed by Daniel Campos, Costa Rica

No solution has yet been received.

S123. Prove that in any triangle with sidelenghts a, b, c the following inequality holds:

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}+\frac{(b+c-a)(c+a-b)(a+b-c)}{abc}\geq 7.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

First solution by Arkady Alt, San Jose, California, USA

Let $s=\dfrac{a+b+c}{2}$ is semiperimeter of a triangle with sidelengths a, b, c.Then,due to triangle inequalites $a, b, c<s$ and setting $x:=s-a, y:=s-b, z:=s-c$ we obtain

$a=y+z, b=z+x, c=x+y, s=x+y+z$, where $x, y, z>0$.

Thus, original inequality becomes

$\displaystyle\sum_{cyc}\frac{2x+y+z}{y+z}+\frac{8xyz}{(y+z)(z+x)(x+y)}\geq 7 \iff \sum_{cyc}\frac{s+x}{s-x}+\frac{8xyz}{(s-x)(s-y)(s-z)}\geq 7 \iff$

$\displaystyle\sum_{cyc}(s+x)(s-y)(s-z)+8xyz\geq 7(s-x)(s-y)(s-z) \iff$

$\displaystyle\sum_{cyc}(s+x)(sx+yz)+8xyz\geq 7(xy+yz+zx-xyz) \iff 2s^3-8s(xy+yz+zx)+18xyz\geq 0$

$(x+y+z)^3-4(x+y+z)(xy+yz+zx)+9xyz\geq 0 \iff$

$\displaystyle\sum_{cyc}x(x-y)(x-z)\geq 0$ **(Schure Inequality)**.

Second solution by Gheorghe Pupazan, Chisinau, Republic of Moldova

We make the well-known substitution $a=x+y, b=y+z$ and $c=z+x$, where $x, y, z>0$. The inequality becomes equivalent to:

$$\frac{2x+y+z}{y+z}+\frac{x+2y+z}{z+x}+\frac{x+y+2z}{x+y}+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq 7.$$

After multiplying both sides by $(x+y)(y+z)(z+x)$, the inequality becomes equivalent with:

$$2(x^3+y^3+z^3)+6xyz\geq 2xyz\cdot\sum xy(x+y)$$

$$\iff 2\sum x(x-y)(x-z)\geq 0$$

which is just Schur's inequality.

Third solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

The inequality to be proved may be written in the form

$$\frac{b+c-a}{a}+\frac{c+a-b}{b}+\frac{a+b-c}{c}+\frac{(b+c-a)(c+a-b)(a+b-c)}{abc}\geq 4 \quad (1)$$

Let $x=\frac{b+c-a}{2}$, $y=\frac{c+a-b}{2}$, $z=\frac{a+b-c}{2}$; note that x, y, z are positive, since a, b, c are sides of a triangle. Then $a=y+z$, $b=z+x$, $c=x+y$. We substitute these into (1) and obtain, after multiplying both sides by $(x+y)(y+z)(z+x)$,

$$x(z+x)(x+y)+y(y+z)(x+y)+z(y+z)(z+x)+4xyz\geq 2(x+y)(y+z)(z+x)$$

which is equivalent to

$$x^3+y^3+z^3-xy^2-yz^2-zx^2-x^2y-y^2z-z^2x+3xyz\geq 0$$

This, in turn, is equivalent to

$$x(x-y)(x-z)+y(y-z)(y-x)+z(z-x)(z-y)\geq 0$$

which follows by Schur's inequality.

Equality occurs only if $x=y=z$, only if $a=b=c$, i.e., only if $\triangle ABC$ is equilateral.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Magkos Athanasios, Kozani, Greece; Marius Mainea, Romania; Irfan Besic, Sarajevo, Bosnia and Gerzegovina; Vicente Vicario García, Huelva, Spain.

S124. Let ABC be a triangle with midpoints M_a, M_b, M_c and let X, Y, Z be the points of tangency of the incircle of triangle $M_aM_bM_c$ with M_bM_c, M_cM_a, M_aM_b, respectively.

a) Prove that the lines AX, BY, CZ are concurrent at some point P.

b) If AA_1, BB_1, CC_1 are cevians through P, then the perimeter of triangle $A_1B_1C_1$ is greater than or equal to the semiperimeter of triangle ABC.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

a) By Thales' theorem, and since $M_cM_b \parallel BC$, we have $\frac{BX}{XC} = \frac{M_cD}{DM_b}$, where D is the point where the incircle of $M_aM_bM_c$ touches M_bM_c. Now, it is well known or easily provable that $M_cD = \frac{M_bM_c+M_cM_a-M_aM_b}{2} = \frac{a+b-c}{4}$, and similarly $DM_b = \frac{M_aM_b+M_bM_c-M_cM_a}{2} = \frac{c+a-b}{4}$, or $\frac{BX}{XC} = \frac{a+b-c}{c+a-b}$, and similarly for its cyclic permutations. By the reciprocal of the Menelaus' theorem, AX, BY, CZ meet at a point P. Incidentally, since X may be identified as the point where side BC touches the excircle that touches side BC and the extensions of sides AB and AC, and similarly for Y and Z, then the point P where AX, BY, CZ meet is the Nagel point of triangle ABC.

b) Applying twice the Cosine Law and operating results in

$$B_1C_1^2 = AB_1^2 + AB_1^2 - 2AC_1 \cdot AC_1 \cos A =$$

$$= \frac{(a+b-c)^2}{4} + \frac{(c+a-b)^2}{4} - \frac{(a+b-c)(c+a-b)(b^2+c^2-a^2)}{4bc} = a^2(1-\sin B \sin C),$$

and similarly for its cyclic permutations, where Hero's formula for the area of ABC has been used. Note now that

$$2\sin B \sin C = \cos(B-C) - \cos(B+C) \leq 1 + \cos A = 2 - 2\sin^2 \frac{A}{2}, \qquad B_1C_1 \geq a \sin \frac{A}{2},$$

and similarly for its cyclic permutations. It therefore suffices to show that

$$\frac{a}{a+b+c} \sin \frac{A}{2} + \frac{b}{a+b+c} \sin \frac{B}{2} + \frac{c}{a+b+c} \sin \frac{C}{2} \geq \frac{1}{2}.$$

Since the second derivative of $\sin x$ is $-\sin x$, negative for half-angles in a triangle, applying Jensen's inequality it suffices to show that

$$\sin \frac{aA+bB+cC}{2(a+b+c)} \geq \frac{1}{2}, \qquad \frac{aA+bB+cC}{a+b+c} \geq \frac{A+B+C}{3},$$

$$2aA + 2bB + 2cC \geq (aB + bC + cA) + (aC + bA + cB),$$

this last relation being true because of the reordering inequality, since wlog $a \geq b \geq c$ iff $A \geq B \geq C$.

S125. Find all pairs (p, q) of positive integers that satisfy

$$\left|\frac{p}{q} - \sqrt{2}\right| < \frac{1}{q^2}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Ercole Suppa, Teramo, Italy

We need the following lemmas

Lemma 1. A pair (p, q) of positive integers, with $q \geq 2$, satisfies

$$\left|\frac{p}{q} - \sqrt{2}\right| < \frac{1}{q^2} \tag{1}$$

if and only if $\left|p^2 - 2q^2\right| \leq 2$ and $q \geq 2$.

Proof. ($\Rightarrow$) If (p, q) satisfies inequality (1) we have

$$\left|p - q\sqrt{2}\right| < \frac{1}{q} \qquad , \qquad \frac{p}{q} < \sqrt{2} + \frac{1}{q^2}$$

so

$$\left|p^2 - 2q^2\right| < \frac{1}{q}\left(p + q\sqrt{2}\right) = \frac{p}{q} + \sqrt{2} < 2\sqrt{2} + \frac{1}{q^2} \tag{2}$$

We now consider the following cases

- If $q = 2$ then $\left|p - 2\sqrt{2}\right| < 1/2$, hence

$$2\sqrt{2} - \frac{1}{2} < p < 2\sqrt{2} + \frac{1}{2} \quad \Rightarrow \quad p = 3 \quad \Rightarrow \quad \left|p^2 - 2q^2\right| \leq 2$$

- If $q \geq 3$, by (2) we have

$$\left|p^2 - 2q^2\right| \leq 2\sqrt{2} + \frac{1}{q^2} \leq 2\sqrt{2} + \frac{1}{9} \approx 2.9 \quad \Rightarrow$$

  and, since $p^2 - 2q^2$ is integer, $\left|p^2 - 2q^2\right| \leq 2$.

($\Leftarrow$) If (p, q) is a pair of positive integers, with $q \geq 2$, such that $\left|p^2 - 2q^2\right| \leq 2$, then we have

$$2q^2 - p^2 \leq \left|p^2 - 2q^2\right| \leq 2 \quad \Rightarrow \quad \frac{p^2}{q^2} \geq 2 - \frac{2}{q^2} \geq 1 \quad \Rightarrow \quad \frac{p}{q} \geq 1$$

Therefore

$$\left|\frac{p}{q}-\sqrt{2}\right| = \frac{\left|p^2-2q^2\right|}{q^2\left(\frac{p}{q}+\sqrt{2}\right)} \leq \frac{2}{q^2\left(1+\sqrt{2}\right)} < \frac{1}{q^2}$$

and the lemma is proven. ■

Lemma 2. Every positive solution of the equation $\left|x^2-2y^2\right| = 1$ is given by (x_n, y_n), where $x_n$and y_n are the integers determined from

$$x_n + y_n\sqrt{2} = \left(1+\sqrt{2}\right)^n \quad , \quad n \in \mathbb{N}$$

Furthermore if n in odd $x_n^2 - 2y_n^2 = -1$, whereas if n in even $x_n^2 - 2y_n^2 = 1$.

Proof. It can be found in a number theory book which deals with Pell's equation theory. ■

Now we will solve the problem.

If $q = 1$ we find the pairs $(1,1)$ and $(2,1)$.

If $q \geq 2$, due to the Lemma 1, it is enough to find all pairs (p,q) of positive integers that satisfy $\left|p^2-2q^2\right| \leq 2$. Since $p^2-2q^2 \neq 0$ we must solve the equations $\left|p^2-2q^2\right| = 1$ and $\left|p^2-2q^2\right| = 2$.

By using the Lemma 2, the solutions of $\left|p^2-2q^2\right| = 1$ are the pairs (p_n, q_n) such that

$$p_n + q_n\sqrt{2} = \left(1+\sqrt{2}\right)^n \quad , \quad n \in \mathbb{N}$$

Since

$$\begin{aligned} p_{n+1} + q_{n+1}\sqrt{2} &= \left(1+\sqrt{2}\right)^{n+1} = \left(1+\sqrt{2}\right)^n \cdot \left(1+\sqrt{2}\right) = \\ &= \left(p_n + q_n\sqrt{2}\right) \cdot \left(1+\sqrt{2}\right) = p_n + 2q_n + (p_n+q_n)\sqrt{2} \end{aligned}$$

the solutions (p_n, q_n) are given by the following recurrence

$$A \ : \ \begin{cases} p_{n+1} = p_n + 2q_n \\ q_{n+1} = p_n + q_n \end{cases} \quad , \quad n \in \mathbb{N} \tag{3}$$

with initial conditions $p_1 = 1$ and $q_1 = 1$.

In order to solve the equation $\left|p^2 - 2q^2\right| = 2$, let us observe that p must be even. Thus, by setting $q = u$ and $p = 2v$ the equation turns into

$$\left|u^2 - 2v^2\right| = 1$$

whose solutions (u_n, v_n) are given by the recurrences $u_{n+1} = u_n + 2v_n$, $v_{n+1} = u_n + v_n$ $(n \in \mathbb{N})$, with $u_1 = 1$ and $v_1 = 1$.

Then, after an easy calculation, we find that the solutions of $\left|p^2 - 2q^2\right| = 2$ are

$$B \ : \quad \begin{cases} p_{n+1} = p_n + 2q_n \\ q_{n+1} = p_n + q_n \end{cases} \quad , \qquad n \in \mathbb{N} \tag{4}$$

with initial conditions $p_1 = 2$ and $q_1 = 1$.

Finally we have proved that the solutions of the proposed inequality can be obtained by means of the recurrences (3) and (4) which, obviously, include also the pairs $(1, 1)$ and $(2, 1)$.

By means of Mathematica we have listed the first thirteen solutions given by A and B:

$$A = \{\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \frac{1393}{985}, \frac{3363}{2378}, \frac{8119}{5741}, \frac{19601}{13860}, \frac{47321}{33461}, \dots\}$$

$$B = \{\frac{2}{1}, \frac{4}{3}, \frac{10}{7}, \frac{24}{17}, \frac{58}{41}, \frac{140}{99}, \frac{338}{239}, \frac{816}{577}, \frac{1970}{1393}, \frac{4756}{3363}, \frac{11482}{8119}, \frac{27720}{19601}, \frac{66922}{47321}, \dots\}$$

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We will first prove the following:

Claim: The given condition is equivalent to $|p^2 - 2q^2| \leq 2$.

Proof: Assume first that $\left|\frac{p}{q} - \sqrt{2}\right| < \frac{1}{q^2}$; clearly $p < q\sqrt{2} + \frac{1}{q}$, or

$$|p^2 - 2q^2| = q(p + q\sqrt{2}) \left|\frac{p}{q} - \sqrt{2}\right| < \frac{p + q\sqrt{2}}{q} < 2\sqrt{2} + \frac{1}{q^2}.$$

If $q \geq 3$, and since $3 - \frac{1}{9} = \sqrt{\frac{676}{81}} > \sqrt{\frac{648}{81}} = 2\sqrt{2}$, clearly $|p^2 - 2q^2| < 3$. If $q = 2$, then $2 < 2\sqrt{2} - \frac{1}{2} < p < 2\sqrt{2} + \frac{1}{2} < 4$ because $2 + \frac{1}{2} = \sqrt{\frac{25}{4}} < \sqrt{8} = 2\sqrt{2}$ and $4 - \frac{1}{2} = \sqrt{\frac{49}{4}} > \sqrt{8} = 2\sqrt{2}$, or $p = 3$, and $|p^2 - 2q^2| = 1 \leq 2$. Finally, if $q = 1$, then $0 < \sqrt{2} - 1 < p < \sqrt{2} + 1 < 3$ for $p = 1$ or $p = 2$, and $|p^2 - 2q^2| = 1$ or $|p^2 - 2q^2| = 2$.

Reciprocally, if $|p^2 - 2q^2| \leq 2$, assume that $\left|\frac{p}{q} - \sqrt{2}\right| > \frac{1}{q^2}$. Then,

$$p + q\sqrt{2} = \frac{|p^2 - 2q^2|}{|p - q\sqrt{2}|} < 2q, \qquad |p - q\sqrt{2}| > 2q(\sqrt{2} - 1) = \frac{2q}{\sqrt{2} + 1},$$

and since $p, q \geq 1$, then $p + q\sqrt{2} \geq \sqrt{2} + 1$, and $|p^2 - 2q^2| > 2q \geq 2$, absurd. The claim follows.

Since $p^2 = 2q^2$ is impossible for positive integers p, q because $\sqrt{2}$ is irrational, then by the claim any and all solutions satisfy either $p^2 - 2q^2 = \pm 1$ or $p^2 - 2q^2 = \pm 2$. These are Pell or Pell-like equations, with easily computable solutions by standard means (see for example the entry on Pell equation in http://mathworld.wolfram.com), resulting in infinite sequences of solutions $\{p_n, q_n\}_{n \geq 1}$, defined as follows:

$$p^2 - 2q^2 = -2 \leftrightarrow p_n = \frac{(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}}{\sqrt{2}}, q_n = \frac{(1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n}}{2};$$

$$p^2 - 2q^2 = -1 \leftrightarrow p_n = \frac{(1+\sqrt{2})^{2n-1} + (1-\sqrt{2})^{2n-1}}{2}, q_n = \frac{(1+\sqrt{2})^{2n-1} - (1-\sqrt{2})^{2n-1}}{2\sqrt{2}};$$

$$p^2 - 2q^2 = 1 \leftrightarrow p_n = \frac{(1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n}}{2}, q_n = \frac{(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}}{2\sqrt{2}};$$

$$p^2 - 2q^2 = 2 \leftrightarrow p_n = \frac{(1+\sqrt{2})^{2n-1} - (1-\sqrt{2})^{2n-1}}{\sqrt{2}}, q_n = \frac{(1+\sqrt{2})^{2n-1} + (1-\sqrt{2})^{2n-1}}{2}.$$

Also solved by EL ALAMI Anass, Meknes, Maroc;Bedri Hajrizi, Albania.

S126. Let a, b, c be positive real numbers. Prove that

$$\sqrt{\frac{a^2(b^2+c^2)}{a^2+bc}}+\sqrt{\frac{b^2(c^2+a^2)}{b^2+ca}}+\sqrt{\frac{c^2(a^2+b^2)}{c^2+ab}} \leq a+b+c.$$

Proposed by Pham Huu Duc, Ballajura, Australia

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Call $x=\frac{a^2(b^2+c^2)}{a^2+bc}$, $y=\frac{b^2(c^2+a^2)}{b^2+ac}$, $z=\frac{c^2(a^2+b^2)}{c^2+ab}$. The proposed inequality is then equivalent to $x+y+z+2(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}) \leq a^2+b^2+c^2+2(ab+bc+ca)$. Moreover, since clearly

$$2(\sqrt{xy}-ab) \leq \frac{xy-a^2b^2}{ab} \quad \text{iff} \quad xy-2ab\sqrt{xy}+a^2b^2 \geq 0,$$

the second inequality being true with equality iff $\sqrt{xy}=ab$, then it suffices to show that

$$\frac{xy-a^2b^2}{ab}+\frac{yz-b^2c^2}{bc}+\frac{zx-c^2a^2}{ca} \leq a^2+b^2+c^2-(x+y+z).$$

This inequality will be equivalent to the proposed one when $xy=a^2b^2$, $yz=b^2c^2$ and $zx=c^2a^2$ simultaneously, which is equivalent to $x=a^2$, $y=b^2$ and $z=c^2$, or to $a^2+b^2+c^2=2a^2+bc=2b^2+ca=2c^2+ab$, and finally to $a=b=c$. Now, the common denominator D in $x+y+z$ equals

$$(a^2+bc)(b^2+ca)(c^2+ab)=a^3b^3+b^3c^3+c^3a^3+abc(a^3+b^3+c^3)+2a^2b^2c^2,$$

while the product of the numerator of x and the denominators of y and z equals

$$a^2(b^2+c^2)(b^2+ca)(c^2+ab)=a^2b^2c^2(b^2+c^2)+(a^3b^3+a^3c^3)(b^2+c^2)+a^4bc(b^2+c^2+a^2)-a^6bc.$$

After some algebra, the cyclic sum of this product equals

$$D(x+y+z)=D(a^2+b^2+c^2)-abc\left(a^5+b^5+c^5-abc(ab+bc+ca)\right).$$

Furthermore, also after some algebra,

$$xy-a^2b^2=a^2b^2c\frac{c^3+a^2c+b^2c-a^3-b^3-abc}{c^2ab+c(a^3+b^3)+a^2b^2}=$$

$$=a^2b^2c\frac{c^5+a^2c^3+b^2c^3-a^3c^2-b^3c^2+a^3bc+ab^3c-a^4b-ab^4-a^2b^2c}{D}.$$

Adding over all cyclic permutations finally yields

$$\frac{xy-a^2b^2}{ab}+\frac{yz-b^2c^2}{bc}+\frac{zx-c^2a^2}{ca}=$$

$$= abc\frac{a^5+b^5+c^5-(ab+bc+ca)(a^3+b^3+c^3)+3abc(a^2+b^2+c^2)-abc(ab+bc+ca)}{D} =$$

$$= abc\frac{-(ab+bc+ca)(a^3+b^3+c^3)+3abc(a^2+b^2+c^2)}{D}+a^2+b^2+c^2-(x+y+z).$$

Therefore, it suffices to show that

$$(ab+bc+ca)(a^3+b^3+c^3) \geq 3abc(a^2+b^2+c^2),$$

which is clearly true since in virtue of the inequality between quadratic and cubic means, the Cauchy-Schwarz inequality, and the AM-GM inequality,

$$\left(\frac{a^3+b^3+c^3}{3}\right)^{\frac{2}{3}} \geq \frac{a^2+b^2+c^2}{3} \geq \frac{ab+bc+ca}{3} \geq (abc)^{\frac{2}{3}}.$$

The result clearly follows, with equality iff $a=b=c$.

Second solution by Gheorghe Pupazan, Chisinau, Republic of Moldova

After bringing all the expressions to a common denominator and after squaring the both sides, the inequality becomes equivalent to:

$$\left(\sum\sqrt{c^2(a^2+b^2)(a^2+bc)(b^2+ca)}\right)^2 \leq (a+b+c)^2(a^2+bc)(b^2+ca)(c^2+ab).$$

From the Cauchy-Schwartz inequality we get that:

$$\left(\sum\sqrt{c^2(a^2+b^2)(a^2+bc)(b^2+ca)}\right)^2 \leq \left(\sum c(a^2+b^2)\right)\left(\sum c(a^2+bc)(b^2+ca)\right).$$

So it suffices to prove that:

$$\left(\sum c(a^2+b^2)\right)\left(\sum c(a^2+bc)(b^2+ca)\right) \leq (a+b+c)^2(a^2+bc)(b^2+ca)(c^2+ab)$$

$$\iff \sum a^3b^3(a^2+b^2)+2\sum a^5bc(b+c)+2\sum a^4b^4+3\sum a^4bc(b^2+c^2)+4a^2b^2c^2(a^2+b^2+c^2)+$$

$$+6a^2b^2c^2(ab+bc+ca) \leq abc(a^5+b^5+c^5)+\sum a^3b^3(a^2+b^2)+2\sum a^5bc(b+c)+2\sum a^4b^4+$$

$$+3\sum a^4bc(b^2+c^2)+4a^2b^2c^2(a^2+b^2+c^2)+5a^2b^2c^2(ab+bc+ca)$$

$$\iff \sum a^2b^2c^2(ab+bc+ca) \leq abc(a^5+b^5+c^5)$$

$$\iff \sum abc(ab+bc+ca) \leq a^5+b^5+c^5.$$

But the proof for this one is simple, as from Muirhead's inequality we know that $(5,0,0) \succ (2,2,1)$, so the conclusion follows.

Undergraduate problems

U121. Let p be a prime and let α be a permutation of order p in S_{p+1}. Find the set $C_\alpha = \{\sigma \in S_{p+1} | \sigma\alpha = \alpha\sigma\}$.

Proposed by Dorin Andrica, Babes-Bolyai University and Mihai Piticari, National College Dragos-Voda, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We begin the proof with the following

Claim: If two permutations $\alpha, \sigma \in S_n$ commute, and $\alpha(i) = i$ is a fixed element in α, then either $\sigma(i) = i$ is a fixed element in σ, or if $\sigma(i) = j$, then $\alpha(j) = j$ and $\alpha(\sigma(j)) = \sigma(j)$ are also fixed points in α.

Proof: Assume that $\sigma(i) = j$. Therefore, $\alpha(\sigma(i)) = \alpha(j)$, while $\sigma(\alpha(i)) = \sigma(i) = j$. Therefore, either $i = j$, or $\alpha(j) = j$. Additionally, if $i \neq j$, $\sigma(\alpha(j)) = \sigma(j) = \alpha(\sigma(j))$.

It is well known that any permutation of a finite number of elements can be written as the product of a finite number of cycles, the order of the permutation being the least common multiple of the lengths of the cycles. Hence, any permutation of order p in S_{p+1} is clearly the product of one 1-cycle and one p-cycle, ie, α has exactly one fixed point, the other p elements forming one cycle of length p. We will assume wlog that $\alpha(p+1) = p+1$ is the only fixed point in α, the other p elements forming a p-cycle. Since α has no additional fixed points, by the claim σ commuting with α has the same fixed point, ie, $\sigma(p+1) = p+1$.

Consider now the cyclic subgroup $A \subset S_{p+1}$ generated by α, ie, the subgroup $A = \{\alpha, \alpha^2, \dots, \alpha^{p-1}, \alpha^p = I\}$ where I is the identity permutation. All elements of this subgroup clearly commute with α, since $\alpha^k\alpha = \alpha^{k+1} = \alpha\alpha^k$ for all $k = 1, 2, \dots, p$. Assume now that $\sigma \notin A$ commutes with α. By the associative property, α^k commutes with σ^ℓ for any positive integer k and ℓ. Let ℓ be the order of the minimum-length cycle of σ other than $\sigma(p+1) = p+1$. Clearly, σ^ℓ has at least ℓ fixed points, of which we may choose one $\sigma(i) = i$ with $i \neq p+1$. Since $\alpha(i)$ is part of a p-cycle, then $\alpha(i), \alpha^2(i), \dots, \alpha^{p-1}(i)$ takes all values from 1 to p, or by direct application of the claim $\sigma^\ell = I$ is the identity permutation. Therefore, all cycles in σ have order which is a divisor of ℓ. Clearly $\ell > 1$, since otherwise $\sigma = I \in A$, and since ℓ is minimum, all other cycles in σ have length ℓ, or ℓ divides p. Therefore, any other permutation σ that commutes with α is also the product of a p-cycle, and the same 1-cycle present in α. Now, for any $i \neq p+1$, we know that $\alpha(i) = j \neq i$. Since $\sigma(j), \sigma^2(j), \dots, \sigma^{p-1}(j), \sigma^p(j) = j$ takes all possible values in $\{1, 2, \dots, p\}$, clearly $m < p$ exists such that $\sigma^m(j) = i$. Note therefore that $\sigma^m(\alpha(i)) = i$, and since $\sigma^m(\alpha)$ commutes with α, then by previous results $\sigma^m\alpha = I$, absurd if $\sigma^m \notin A$. Therefore, $C_\alpha = \{\alpha, \alpha^2, \dots, \alpha^{p-1}, \alpha^p = I\}$.

U122. Let $f : [0,1] \to \mathbb{R}$ be a twice differentiable function with the second derivative continuous such that

$$\int_0^1 f(x)dx = 3\int_{\frac{1}{3}}^{\frac{2}{3}} f(x)dx.$$

Prove that there exists $x_0 \in (0,1)$ such that $f''(x_0) = 0$.

Proposed by Cezar Lupu, Romania

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let $F : [0,1] \to \mathbb{R}$ be a function continuous in $[0,1]$, three times differentiable in $(0,1)$ with continuous third derivative, and define $g : [0,1] \to \mathbb{R}$ as $g(x) = x$ and $u : \left[\frac{1}{3}, \frac{2}{3}\right] \to \mathbb{R}$ as $u(x) = F\left(x + \frac{1}{3}\right) - 2F(x) + F\left(x - \frac{1}{3}\right)$. By the Cauchy generalization of the intermediate value theorem, $x_2 \in \left(\frac{1}{3}, \frac{2}{3}\right)$ exists such that

$$F'\left(x_2 + \frac{1}{3}\right) - 2F'(x_2) + F'\left(x_2 - \frac{1}{3}\right) = \frac{u'(x_2)}{g'(x_2)} = \frac{u\left(\frac{2}{3}\right) - u\left(\frac{1}{3}\right)}{\frac{2}{3} - \frac{1}{3}} =$$

$$= 3F(1) - 9F\left(\frac{2}{3}\right) + 9F\left(\frac{1}{3}\right) - 3F(0).$$

Define now $v : \left[\frac{1}{6}, \frac{5}{6}\right] \to \mathbb{R}$ as $v(x) = F'\left(x + \frac{1}{6}\right) - F'\left(x - \frac{1}{6}\right)$. Again by the Cauchy generalization of the intermediate value theorem, $x_1 \in \left(x_2 - \frac{1}{6}, x_2 + \frac{1}{6}\right) \subset \left(\frac{1}{6}, \frac{5}{6}\right)$ exists such that

$$F''\left(x_1 + \frac{1}{6}\right) - F''\left(x_1 - \frac{1}{6}\right) = \frac{v'(x_1)}{g'(x_1)} = \frac{v\left(x_2 + \frac{1}{6}\right) - v\left(x_2 - \frac{1}{6}\right)}{x_2 + \frac{1}{6} - x_2 + \frac{1}{6}} =$$

$$= 3F'\left(x_2 + \frac{1}{3}\right) - 6F'(x_2) + 3F'\left(x - \frac{1}{3}\right).$$

Define finally $w : [0,1] \to \mathbb{R}$ as $F''(x)$. Once more by the Cauchy generalization of the intermediate value theorem, $x_0 \in \left(x_1 - \frac{1}{6}, x_1 + \frac{1}{6}\right) \subset \left(x_2 - \frac{1}{3}, x_2 + \frac{1}{3}\right) \subset (0,1)$ exists such that

$$F'''(x_0) = \frac{w'(x_0)}{g'(x_0)} = \frac{w\left(x_1 + \frac{1}{6}\right) - w\left(x_1 - \frac{1}{6}\right)}{x_1 + \frac{1}{6} - x_1 + \frac{1}{6}} = 3F''\left(x_1 + \frac{1}{6}\right) - 3F''\left(x_1 - \frac{1}{6}\right).$$

Therefore, $x_0 \in (0,1)$ exists such that

$$F'''(x_0) = 27\left(F(1) - 3F\left(\frac{2}{3}\right) + 3F\left(\frac{1}{3}\right) - F(0)\right).$$

Call now $F(x) = \int_0^x f(x)dx$, which is clearly continuous in $[0,1]$ and three times differentiable with continuous third derivative in $(0,1)$. Since

$$F(1) - F(0) = \int_0^1 f(x)dx = 3\int_{\frac{1}{3}}^{\frac{2}{3}} f(x)dx = 3F\left(\frac{2}{3}\right) - 3F\left(\frac{1}{3}\right),$$

then $F'''(x_0) = f''(x_0) = 0$. The conclusion follows.

Second solution by José H. Nieto S., Universidad del Zulia, Venezuela

If $f''(x) \neq 0$ for all $x \in (0,1)$ then, since f'' is continuous, it must have constant sign in $(0,1)$. Let us assume that $f''(x) > 0$ for all $x \in (0,1)$ (the case $f'' < 0$ is similar). Let $g(x) = f(x) - f(1/3) - f'(1/3)(x - 1/3)$. It is easily verified that $g(1/3) = 0$, $g'(1/3) = 0$, $g''(x) = f''(x)$ for all $x \in (0,1)$ and $\int_0^1 g(x)\,dx = 3\int_{1/3}^{2/3} g(x)\,dx$. Since $g''(x) = f''(x) > 0$ for all $x \in (0,1)$, g' is strictly increasing. Therefore $g'(x) < 0$ for $x \in (0,1/3)$ and $g'(x) > 0$ for $x \in (1/3,1)$. Hence g is strictly decreasing in $[0,1/3]$ and strictly increasing in $[1/3,1]$. Observe that, since g is continuous, $\int_0^{1/3} g(x)\,dx > 0$ and $\int_{1/3}^{2/3} g(x)\,dx > 0$. Also, since g is convex, $\int_{1/3}^{2/3} g(x)\,dx \leq (1/2)(g(1/3) + g(2/3))(1/3) = g(2/3)/6$.

Now, since $g'(t) < g'(t + 1/3)$ for $t \in [1/3, 2/3)$, integrating from $1/3$ to x we obtain that $g(x) < g(x + 1/3) - g(2/3)$ for $t \in [1/3, 2/3]$. It follows that

$$\begin{aligned}
\int_0^1 g(x)\,dx &> \int_{1/3}^1 g(x)\,dx = \int_{1/3}^{2/3} g(x)\,dx + \int_{2/3}^1 g(x)\,dx \\
&= \int_{1/3}^{2/3} g(x)\,dx + \int_{1/3}^{2/3} g(x + \frac{1}{3})\,dx \\
&\geq \int_{1/3}^{2/3} g(x)\,dx + \int_{1/3}^{2/3} (g(\frac{2}{3}) + g(x))\,dx \\
&\geq 2\int_{1/3}^{2/3} g(x)\,dx + \frac{1}{3}g(\frac{2}{3}) \geq 4\int_{1/3}^{2/3} g(x)\,dx,
\end{aligned}$$

arriving at a contradiction, hence f'' must be 0 at some point in $(0,1)$.

Third solution by Vinh Quynh Anh, Belarusian State University, Belarus

Letting

$$g(t) = \int_0^{t/3} f(x)dx + \int_{2t/3}^{t} f(x)dx - 2\int_{t/3}^{2t/3} f(x)dx$$

then $g(0) = 0$ and the required condition is equivalent to $g(1) = 0$.

Hence, by the Mean Value Theorem there is $t_0 \in (0,1)$ such that $g'(t_0) = 0$ or

$$\begin{aligned} g'(t_0) &= \tfrac{1}{3} f\left(t_0/3\right) + f\left(t_0\right) - \tfrac{2}{3} f\left(2t_0/3\right) - \tfrac{4}{3} f\left(2t_0/3\right) + \tfrac{2}{3} f\left(t_0/3\right) \\ &= \left(f\left(t_0\right) - f\left(2t_0/3\right)\right) - \left(f\left(2t_0/3\right) - f\left(t_0/3\right)\right) = 0 \end{aligned}$$

Note that, by the Mean Value Theorem there are $\theta_+ \in (2t_0/3, t_0) \subset (0,1)$ and $\theta_- \in (t_0/3, 2t_0/3) \subset (0,1)$ such that

$$0 = g'(t_0) = f'(\theta_+)\frac{t_0}{3} - f'(\theta_-)\frac{t_0}{3}.$$

By applying one more time the Mean Value Theorem there is $x_0 \in (\theta_-, \theta_+) \subset (0,1)$ such that

$$0 = f''(x_0)\left(\theta_+ - \theta_-\right)\frac{t_0}{2}.$$

Thus, $f''(x_0) = 0$.

U123. Let C_1, C_2, C_3 be concentric circles with radii $1, 2, 3$ respectively. Consider a triangle ABC with $A \in C_1$, $B \in C_2$, $C \in C_3$. Prove that $\max K_{ABC} < 5$, where $\max K_{ABC}$ denotes the greatest possible area of triangle ABC.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Assume that A, B such that the area of ABC is maximum are known, and denote by h_C the length of the altitude from C and by P_C the foot of the altitude from P onto AB. Clearly, $h_C \leq PC + PP_C$, with equality iff C, P, P_C are collinear with P inside segment CP_C. Now, $PC = 3$ and PP_C is fixed for known A, B, or the area is maximum when P is inside segment CP_C, which is perpendicular to AB. By cyclic symmetry, P is the orthocenter of ABC, and it is inside ABC, or ABC is acute. The claim follows.

It is well known, or easily provable, that P being the orthocenter of acute triangle ABC, then $PA = 2R\cos A$, $PB = 2R\cos B$ and $PC = 2R\cos C$, where R is the circumradius of ABC. Now,

$$3R = 2R^2 \cos C = 2R^2 \sin A \sin B - 2R^2 \cos A \cos B = \sqrt{4R^2 - 1}\sqrt{R^2 - 1} - 1,$$

or $4R^3 - 14R - 6 = 0$. At the same time, using this relation yields

$$a^2b^2c^2 = (4R^2 - 1)(4R^2 - 4)(4R^2 - 9) = 4(24R^3 + 49R^2 - 9),$$

$$S^2 = \frac{a^2b^2c^2}{16R^2} = 6R + \frac{49}{4} - \frac{9}{4R^2}.$$

If $S \geq 5$, then $32R + 12 = 8R^3 \leq 17R^2 - 3$, or $17R^2 \geq 32R + 15 = \frac{16}{7}(14R + 6 + \frac{9}{16}) \geq \frac{64R^3}{7}$, and $R \leq \frac{119}{64} < 2$. But if $R \leq 2$, $4R^3 - 14R - 6 \leq 16R - 14R - 6 \leq -2$, contradiction, hence the area of triangle ABC must be smaller than 5.

Note: Numerically solving the equation $4R^3 - 14R - 6 = 0$ and plugging the result into the expression for the maximum area of ABC as a function only of R, a better estimate $4.90482 < \max K_{ABC} < 4.90483$ may be obtained.

Second solution by John T. Robinson, Yorktown Heights, NY, USA

he conditions define a triangle such that the three distances from a point P (the center of the concentric circles) to the vertices are 1, 2, and 3. Drawing line segments from P to the vertices, let θ be the angle between the line segments of lengths 1 and 3, and ϕ be the angle between the line segments of lengths 2 and 3. The area of the triangle is then

$$A(\theta, \phi) = 3\sin(\phi) + \frac{3}{2}\sin(\theta) + \sin(2\pi - \theta - \phi)\,.$$

If we let $x = \sin(\theta)$ and $y = \sin(\phi)$ then the area K as a function of x and y is

$$K(x, y) = 3y + \frac{3}{2}x - \left(\pm y\sqrt{1-x^2} \pm x\sqrt{1-y^2}\right) ,$$

where the signs for the square roots are chosen as + or - depending on whether θ and ϕ are less than or greater than $90°$; we see that we get a larger area by choosing both greater than $90°$ so assuming that this is the case we have

$$K(x, y) = 3y + \frac{3}{2}x + y\sqrt{1-x^2} + x\sqrt{1-y^2}$$

for $0 \le x, y \le 1$. Let

$$f(x, y) = \frac{3}{2}y + \frac{3}{2}x + y\sqrt{1-x^2} + x\sqrt{1-y^2} .$$

Since this is symmetric in x, y its maximum is more easily analyzed (since it must occur where $x = y$), and we have

$$K(x, y) = \frac{3}{2}y + f(x, y) \le \frac{3}{2} + \max(f(x, y)) .$$

Finding the maximum of $f(x, y)$ for $0 \le x, y \le 1$, we have

$$g(x) = f(x, x) = 3x + 2x\sqrt{1-x^2}$$

$$g'(x) = 3 + 2\sqrt{1-x^2} - \frac{2x^2}{\sqrt{1-x^2}} = 0$$

$$3\sqrt{1-x^2} + 2 - 2x^2 = 2x^2$$

$$3\sqrt{1-x^2} = 4x^2 - 2$$

$$9 - 9x^2 = 16x^4 - 16x^2 + 4$$

$$16x^4 - 7x^2 - 5 = 0$$

$$x_{\max} = \sqrt{\frac{7+\sqrt{369}}{32}} = 0.90500988793...$$

$$g(x_{\max}) = f(x_{\max}, y_{\max}) = 3.48499493499... .$$

It follows that $\max K_{ABC} \le 1.5 + 3.48499493499... = 4.98499493499... < 5$.

U124. Let $\{x_n\}_{n\geq 1}$ be a sequence of real numbers such that $\arctan x_n + nx_n = 1$ for all positive integers n. Evaluate $\lim_{n\to\infty} n\ln(2 - nx_n)$.

Proposed by Duong Viet Thong, Nam Dinh University of Technology and Education, Vietnam

First solution by Arkady Alt, San Jose, California, USA

First note that for any positive real x holds inequality $\arctan x < x$.(This immmediatelly

follows from inequality $x < \tan x, x \in \left(0, \frac{\pi}{2}\right)$).

Since function $f(x) := \arctan x + nx$ is odd in $\mathbb{R}$, then from $f(x) = 1$ follows $x > 0$.

Thus, all terms of sequence $\{x_n\}_{n\geq 1}$ determined by equation $\arctan x_n + nx_n = 1$ should

be positive and for any natural n holds inequality

$$\arctan x_n < x_n \iff 1 - nx_n < x_n \iff \frac{1}{n+1} < x_n.$$

From the other hand since $x_n > 0$ then $\arctan x_n > 0 \implies 1 - nx_n > 0 \iff x_n < \frac{1}{n}$.

Thus, $\frac{1}{n+1} < x_n < \frac{1}{n}, n \in \mathbb{N}$ and, therefore, $\lim\limits_{n\to\infty} nx_n = 1$, $\lim\limits_{n\to\infty} \arctan x_n = 0$.

Moreover, $\lim\limits_{n\to\infty} n \arctan x_n = \lim\limits_{n\to\infty} \left(\frac{\arctan x_n}{x_n} \cdot nx_n\right) = \lim\limits_{n\to\infty} \frac{\arctan x_n}{x_n} \cdot \lim\limits_{n\to\infty} nx_n = 1 \cdot 1 = 1.$

Using this we obtain $\lim\limits_{n\to\infty} n \ln(2 - nx_n) = \lim\limits_{n\to\infty} n \ln(1 + \arctan x_n) =$
$\lim\limits_{n\to\infty} \left(\frac{\ln(1 + \arctan x_n)}{\arctan x_n} \cdot n \arctan x_n\right) = \lim\limits_{n\to\infty} \frac{\ln(1 + \arctan x_n)}{\arctan x_n} \cdot \lim\limits_{n\to\infty} n \arctan x_n = 1 \cdot 1 = 1.$

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote $f(x) = \arctan x$ and $g(x) = \ln(1 + x)$. Since $1 - x^2 < f'(x) = \frac{1}{1+x^2} < 1$ and $1 - x < g'(x) = \frac{1}{1+x} < 1$ for $x > 0$, then $x - \frac{x^3}{3} < \arctan x < x$ and $x - \frac{x^2}{2} < \ln(1 + x) < x$ for $x > 0$.

If $x_n < 0$, then $-\frac{\pi}{2} < \arctan x_n < 0$, and $\arctan x_n + nx_n < 0$, absurd, hence $x_n > 0$ and $\arctan x_n > 0$, or $nx_n = 1 - \arctan x_n < 1$. Therefore, $x_n - \frac{x_n^3}{3} <$

$\arctan x_n = 1 - nx_n < x_n$, and $x_n > \frac{1}{n+1}$. But if $x_n \geq \frac{1}{n+1} + \frac{1}{3(n+1)^3}$, then

$$1 = \arctan x_n + nx_n \geq (n+1)x_n - \frac{x_n^3}{3} > 1 + \frac{1}{3(n+1)^2} - \frac{x_n^3}{3},$$

and $\frac{1}{n^3} > x_n^3 \geq \frac{1}{(n+1)^2}$, clearly false for $n \geq 3$. Therefore,

$$1 + \frac{1}{n+1} - \frac{n}{3(n+1)^3} < 2 - nx_n < 1 + \frac{1}{n+1},$$

$$\frac{n}{n+1} - \frac{n}{2(n+1)^2} - \frac{n^2}{3(n+1)^3} + \frac{n^2}{3(n+1)^4} - \frac{n^3}{18(n+1)^6} <$$

$$< n\ln\left(1 + \frac{1}{n+1} - \frac{n}{3(n+1)^3}\right) < n\ln(2 - nx_n) < n\ln\left(1 + \frac{1}{n+1}\right) < \frac{n}{n+1},$$

and since the limit when $n \to \infty$ of both sides of the inequality is 1, then

$$\lim_{n\to\infty} n\ln(2 - nx_n) = 1.$$

Third solution by Vinh Quynh Anh, Departmaent of Mathematics ,Belarusian State University

Consider the function $f_n(x) = \arctan x + nx - 1$, $x \in R, n = 1, 2, ...$

We obtain $f_n'(x) = \frac{1}{1+x^2} + n > 0$, $\forall x \in R$ or $f_n(x)$ is strictly increasing on R. Note that $f_n(0) = -1 < 0$ and $f_n(1) = \frac{\pi}{4} + n - 1 > 0$, $\forall n \geq 1$. Hence, by the Intermediate value theorem there exist unique $x_n \in (0, 1)$ such that $f_n(x_n) = 0$.

On the other hand,

$$f_{n+1}(x_n) = \arctan x_n + nx_n - 1 + x_n = f_n(x_n) + x_n = x_n > 0$$

Since $f_{n+1}(x_{n+1}) = 0 < f_{n+1}(x_n)$ and $f_n(x)$ is strictly increasing on R, we obtain $x_n > x_{n+1}, \forall n = 1, 2, ...$ Thus, there is $\lim\limits_{n\to\infty} x_n = x \in [0, 1)$. If $x > 0$ then $\lim\limits_{n\to\infty} (\arctan x_n + nx_n - 1) = +\infty$.Hence, $x = 0$ and $\lim\limits_{x\to\infty} nx_n = 1$,we get

$$\lim_{n\to\infty} n\ln(2 - nx_n) = \lim_{n\to\infty} n\ln(1 + \arctan x_n) = \lim_{x\to\infty} nx_n = 1.$$

U125. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be distinct real numbers. Let A be the matrix with entries $a_{ij} = \frac{u_i+v_j}{u_i-v_j}$ and B be the matrix with entries $b_{ij} = \frac{1}{u_i-v_j}$ for $1 \le i, j \le n$. Prove that

$$\det A = 2^{n-1}\left(u_1u_2\ldots u_n + v_1v_2\ldots v_n\right)\det B.$$

Proposed by Darij Grinberg, Ludwig Maximilian University of Munich, Germany

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote by $A^{(n)}, B^{(n)}$ matrices A, B with size $n \times n$, and denote by $C^{(n-1,k)}$ and $D^{(n-1,k)}$ the result of eliminating, in matrices $A^{(n)}$ and $B^{(n)}$ respectively, the first row and the k-th column, for $k = 1, 2, \ldots, n$. We will prove the proposed result by induction on n, where the base case $n = 1$ is trivially true, since $\det A^{(1)} = \frac{u_1+v_1}{u_1-v_1} = (u_1+v_1)\det B^{(1)}$. Now for the step, note that

$$\det A^{(n)} = \sum_{k=1}^{n}(-1)^{k+1}\left(\frac{u_1+v_k}{u_1-v_k}\det C^{(n-1,k)}\right),$$

$$\det B^{(n)} = \sum_{k=1}^{n}(-1)^{k+1}\left(\frac{1}{u_1-v_k}\det D^{(n-1,k)}\right).$$

Denoting $U_n = u_1u_2\ldots u_n$ and $V_n = v_1v_2\ldots v_n$, if the proposed result is true for matrices $A^{(n-1)}$ and $B^{(n-1)}$ of size $n-1 \times n-1$, then clearly $\det C^{(n,k)} = 2^{n-2}\left(\frac{U_n}{u_1} + \frac{V_n}{v_k}\right)\det D^{(n-1,k)}$, or

$$\det A^{(n)} =$$

$$= 2^{n-2}\sum_{k=1}^{n}(-1)^{k+1}\left(\left(2U_n + 2V_n + \frac{(u_1-v_k)V_n}{v_k} - \frac{(u_1-v_k)U_n}{u_1}\right)\frac{\det D^{(n-1,k)}}{u_1-v_k}\right) =$$

$$= 2^{n-1}(U_n+V_n)\sum_{k=1}^{n}(-1)^{k+1}\left(\frac{1}{u_1-v_k}\det D^{(n-1,k)}\right) +$$

$$+2^{n-1}\sum_{k=1}^{n}(-1)^{k+1}\left(\left(\frac{V_n}{v_k} - \frac{U_n}{u_1}\right)\det D^{(n-1,k)}\right).$$

It remains therefore only to be proved that

$$0 = \sum_{k=1}^{n}(-1)^{k+1}\left(\left(\frac{V_n}{v_k} - \frac{U_n}{u_1}\right)\det D^{(n-1,k)}\right) =$$

$$= V_n \sum_{k=1}^{n} (-1)^{k+1} \frac{\det D^{(n-1,k)}}{v_k} - \frac{U_n}{u_1} \sum_{k=1}^{n} (-1)^{k+1} \det D^{(n-1,k)}.$$

Note that this last expression is the weighted sum of two determinants, with respective weights V_n and $-\frac{U_n}{u_1}$. The determinants are the result of substituting, in the determinant of $B^{(n)}$, the first row by vector $\left(\frac{1}{v_1}, \frac{1}{v_2}, \ldots, \frac{1}{v_n}\right)$, and by vector $(1, 1, \ldots, 1)$, respectively. The value of the first determinant does not change when we add the first row to each one of the remaining rows. Since $\frac{1}{v_k} + \frac{1}{u_i - v_k} = \frac{u_i}{v_k(u_i - v_k)}$, we may take out common factor u_i from each row for $i = 2, 3, \ldots, n$, and common factor $\frac{1}{v_k}$ from each column for $k = 1, 2, \ldots, n$, for a total factor of $\frac{U_n}{u_1 V_n}$, while the resulting determinant is equal to the second determinant. The conclusion follows.

U126. Find all continuous and bijective functions $f : [0,1] \to [0,1]$ such that

$$\int_0^1 g(f(x))dx = \int_0^1 g(x)dx,$$

for all continuous functions $g : [0,1] \to \mathbb{R}$.

Proposed by Dorin Andrica, Babes-Bolyai University and Mihai Piticari, National College Dragos-Voda, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Assume that $0 \le a < b < c \le 1$ exist such that $f(a), f(c) \le f(b)$. Clearly $f(a), f(c) \ne f(b)$ since f is bijective. Since f is continuous, by the intermediate value theorem $x_1 \in (a,b)$ and $x_2 \in (b,c)$ exist such that $f(x_1) = f(x_2) = \frac{f(b)+\max\{f(a),f(c)\}}{2}$, contradiction. Therefore, f is either strictly increasing or strictly increasing, where in the first case clearly $f(0) = 0$ and $f(1) = 1$, and in the second case $f(0) = 1$ and $f(1) = 0$. Note also that, taking $g(x) = h(1-x)$ for any continuous function $h : [0,1] \to \mathbb{R}$, we have

$$\int_0^1 h(x)dx = \int_0^1 g(1-x)dx = \int_0^1 g(x)dx = \int_0^1 g(f(x))dx = \int_0^1 h(1-f(x))dx,$$

where the second equality is obtained after changing variable x into $1-x$. Therefore, $f(x)$ is a solution iff $1-f(x)$ is a solution, or we may assume wlog that f is increasing, and generality is restored by considering, for each strictly increasing solution $f(x)$, that an additional solution $1-f(x)$ exists.

For any $0 < a < 1$, let $f(a) = b$, and let $g(x) = \frac{b-x}{b}$ if $0 \le x \le b$, and $g(x) = 0$ if $b \le x \le 1$. Clearly, $g : [0,1] \to [0,1]$ is continuous, or

$$\frac{b}{2} = \int_0^b \frac{b-x}{b}dx = \int_0^1 g(x)dx = \int_0^1 g(f(x))dx = \int_0^a \frac{b-f(x)}{b}dx = a - \frac{1}{b}\int_0^a f(x)dx,$$

thus for all $0 < a < 1$,

$$\int_0^a f(x)dx = af(a) - \frac{(f(a))^2}{2}.$$

Define $F : [0,1] \to \mathbb{R}$ as $F(x) = \int_0^x f(y)dy$. Clearly $F(x)$ is differentiable with first derivative $f(x)$ continuous, and $F(x) = xF'(x) - \frac{(F'(x))^2}{2}$. Denote now $F(x) = \frac{x^2}{2} + \Delta(x)$, where $\Delta(x)$ is a function to be determined, which is however clearly differentiable with continuous first derivative. Inserting this form of $F(x)$ into its differential equation, we obtain $\Delta(x) = -\frac{(\Delta'(x))^2}{2}$, or $\Delta(x) \le 0$ for all $x \in [0,1]$, and whenever $\Delta(x) \neq 0$,

$$1 = \frac{\Delta'(x)}{\sqrt{-2\Delta(x)}} = -\frac{d\sqrt{-2\Delta(x)}}{dx}.$$

Assume that $\Delta(x_0) \neq 0$ for some $x_0 \in (0,1)$. Since $\Delta(x_0)$ has continuous first derivative, an interval (u,v) such that $x_0 \in (u,v)$ and $\Delta(x) \neq 0$ for all $x \in (u,v)$ exists. Therefore, a constant K exists such that, for all $x \in (u,v)$, we have $\sqrt{-2\Delta(x)} = K - x$. Hence, $F(x) = Kx - \frac{K^2}{2}$, or $F'(x) = f(x) = K$ for all $x \in (u,v)$, and f would not be bijective, reaching a contradiction. Therefore, $\Delta(x)$ cannot be nonzero in any interval, and $F(x) = \frac{x^2}{2}$, or $f(x) = x$ is the only strictly increasing solution. We conclude that all possible solutions are $f(x) = x$ and $f(x) = 1 - x$, for which the condition of the problem clearly holds.

Olympiad problems

O121. Let a, b, c be positive real numbers. Prove that

$$\sqrt{ab(a+b)}+\sqrt{bc(b+c)}+\sqrt{ca(c+a)} \geq \frac{5}{4}\sqrt{(a+b)(b+c)(c+a)}+2\sqrt{abc}.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

The proposed inequality is not true; it suffices to set $a = b = c$, and the inequality becomes $3\sqrt{2} \geq \frac{5\sqrt{2}}{2}+2$, equivalent to $4 \leq \sqrt{2}$, clearly false. It is true however that, for any positive integers a, b, c, the following inequality holds:

$$\sqrt{ab(a+b)}+\sqrt{bc(b+c)}+\sqrt{ca(c+a)} \geq \sqrt{(a+b)(b+c)(c+a)}+\sqrt{2abc}.$$

Consider the limit of both sides of the inequality when $a \to 0$, which is clearly in this case $\sqrt{bc(b+c)} \geq \sqrt{bc(b+c)}$; if the coefficient of $\sqrt{(a+b)(b+c)(c+a)}$ is raised beyond 1 without altering the coefficients of the terms in the LHS, the inequality will become false for sufficiently small a. We will now prove the inequality proposed in this solution.

Define $x = \sqrt{\frac{b+c}{2a}}$, $y = \sqrt{\frac{c+a}{2b}}$ and $z = \sqrt{\frac{a+b}{2c}}$, so that the inequality transforms into $x+y+z \geq 2xyz+1$, or equivalently $2(xy+yz+zx) = 1+4xyz+4x^2y^2z^2-(x^2+y^2+z^2)$. Now,

$$x^2y^2z^2 = \frac{(a+b)(b+c)(c+a)}{8abc} = \frac{(a+b+c)(ab+bc+ca)-abc}{8abc},$$

$$x^2+y^2+z^2 = \frac{(a+b+c)(ab+bc+ca)-3abc}{2abc} = 4x^2y^2z^2-1.$$

The inequality is then equivalent to $xy+yz+zx \geq 1+2xyz$. But

$$xy = \frac{\sqrt{c(a+b+c)+ab}}{2\sqrt{ab}} \geq \frac{\sqrt{3c(a+b+c)}}{4\sqrt{ab}}+\frac{1}{4},$$

where the inequality between weighted arithmetic and quadratic means has been applied to $\sqrt{\frac{c(a+b+c)}{3}}$ and $\sqrt{ab}$, with respective weights 3 and 1, equality holding iff $abc(a+b+c) = 3a^2b^2$. Since the cyclic permutations of this inequality also hold, we obtain

$$xy + yz + zx - 1 \geq \frac{(a+b+c)\sqrt{3(a+b+c)} - \sqrt{abc}}{4\sqrt{abc}},$$

with equality iff $a = b = c$, and it suffices to show that

$$\frac{3(a+b+c)^3 + abc - 2(a+b+c)\sqrt{3abc(a+b+c)}}{16abc} \geq 4x^2y^2z^2 =$$

$$= \frac{(a+b+c)(ab+bc+ca) - abc}{2abc},$$

or equivalently,

$$3(a+b+c)^3 + 9abc - 8(a+b+c)(ab+bc+ca) \geq 2(a+b+c)\sqrt{3abc(a+b+c)}.$$

Now,

$$(a+b+c)^3+9abc-4(a+b+c)(ab+bc+ca) = a^3+b^3+c^3+6abc-(a+b+c)(ab+bc+ca) =$$

$$= a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \geq 0,$$

the last step holding because it is Schur's inequality, with equality iff $a = b = c$. It therefore suffices to show that $(a+b+c)^3 \geq 9abc + 2\sqrt{3abc(a+b+c)^3}$, clearly true since $(a+b+c)^3 \geq 3\sqrt{3abc(a+b+c)^3} \geq 27abc$ because of the AM-GM inequality applied to a, b, c. The conclusion follows, equality holding in the result proposed in this solution iff $a = b = c$.

O122. Let p and q be odd primes such that $q \nmid p-1$ and let $a_1, a_2, \dots, a_n$ be distinct integers such that $q \mid (a_i - a_j)$ for all pairs (i, j). Prove that

$$P(x) = (x - a_1)(x - a_2)\dots(x - a_n) - p,$$

is irreducible in $\mathbb{Z}[X]$ for $n \geq 2$.

Proposed by Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

If $n = 2$, and $P(x)$ is not irreducible in $\mathbb{Z}[X]$, and since $P(x)$ is monic, then $P(x) = (x-r)(x-s)$ for integers r, s. Then, $P(a_1) = (a_1 - r)(a_1 - s) = -p$ and $P(a_2) = (a_2 - r)(a_2 - s) = -p$, where wlog $a_1 - r = 1$ and $a_1 - s = -p$ since we may exchange r and s, and we may exchange x by $-x$ without altering the problem. Clearly $a_2 - r \neq 1$ because a_1 and a_2 are distinct. If $a_2 - r = -1$, then $a_1 - a_2 = 2$, not divisible by any odd prime p. Hence $a_2 - r = p$ or $a_2 - r = -p$. In the first case, $a_1 - a_2 = 1 - p$, hence $q|(a_2 - a_1) = p - 1$, absurd, or $a_2 - r = -p$ and $a_2 - s = 1$. Substitution yields $r = a_1 - 1 = a_2 - p$ and $s = a_1 - p = a_2 - 1$. Note that $r - s = p - 1 = 1 - p$, yielding $p = 1$, absurd, hence $P(x)$ is irreducible for $n = 2$. We will assume in the rest of the problem that $n \geq 2$.

Assume that $P(x)$ is not irreducible in $\mathbb{Z}[X]$. Then, polynomials $Q(x), R(x) \in \mathbb{Z}[X]$ exist such that $Q(x)R(x) = P(x)$, where wlog $1 \leq \deg(Q(x)) \leq \deg(R(x))$, ie, $1 \leq \deg(Q(x)) \leq \frac{n}{2}$. Clearly, $Q(a_i)R(a_i) = P(a_i) = -p$ for $i = 1, 2, \dots, n$, where $Q(a_i)$ and $R(a_i)$ are integers, or $Q(a_i) \in \{-1, 1, -p, p\}$ for all $i = 1, 2 \dots, n$. Assume now that $Q(a_i)$ takes only two values $k_1, k_2 \in \{-1, 1, -p, p\}$ when i takes all possible values between 1 and n. There are thus at least $\frac{n}{2}$ values of i for which wlog $Q(a_i) = k_1$. Therefore, $Q(x) - k_1$, which has degree at most $\frac{n}{2}$, has at least $\frac{n}{2}$ roots, or it is identically zero, and $Q(x) = k_1$, absurd. Therefore, $Q(a_i)$ takes at least three different values when i takes all possible values between 1 and n. We conclude that distinct $i, j \in \{1, 2, \dots, n\}$ exist such that, either $Q(a_i) = p$ and $Q(a_j) = 1$, or $Q(a_i) = -1$ and $Q(a_j) = -p$. Therefore,

$$q|(a_i - a_j)|\,(Q(a_i) - Q(a_j)) = p - 1,$$

contradiction. The result follows.

O123. Let ABC be a triangle and let A_1, A_2, A_3 be the points of tangency of its incircle ω with the triangle's sides. Medians A_1M, B_1N, C_1P in triangle $A_1B_1C_1$ intersect ω at A_2, B_2, C_2, respectively. Prove that AA_2, BB_2, CC_2 are concurrent at the isogonal conjugate of the Gergonne point.

Proposed by Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Clearly, M is the inverse of A with respect to ω, because IM is the internal bisector of A and the perpendicular bisector of B_1C_1, hence the inverse of line A_1A_2 with respect to ω is the circumcircle of A_1IA_2, which passes also through A. Since A_1IA_2A is cyclic, and $A_1I = A_2I$ are radii of ω, then $\angle IAA_2 = \angle IAA_1$, and AA_2 is the symmetric of AA_1 with respect to the angle bisector IA. Thus, since the Gergonne point lines on AA_1, its isogonal conjugate lies on AA_2, and by cyclic permutation, also on BB_2 and CC_2. The conclusion follows.

Second solution by Ercole Suppa, Teramo, Italy

We begin by proving the following

Lemma. The tangents at the extremities of a chord PQ of a circle ω meet at A; any line through A cuts the circle at B, C, where A and C lie on opposite sides of line PQ; the line through C and the mid-point M of PQ intersects ω again in D. Then the lines AB, AD are isogonal conjugates.

Proof.
Let O be the center of ω and $R = AB \cap PQ$.

Claim 1. PQ bisects angle $\angle BMC$. (Fig. 1)
Proof. Since $AO \perp PQ$ it is enough to prove that $\angle BMA = \angle CMO$.

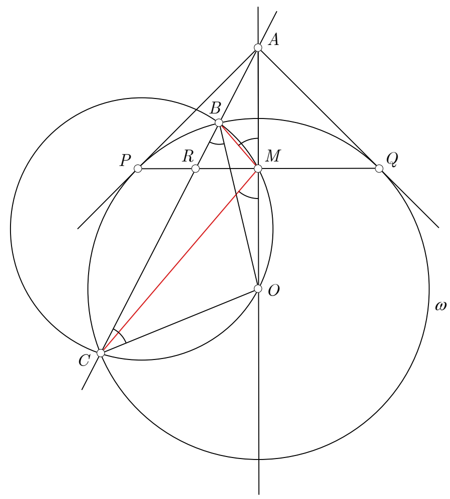


Fig. 1

By Euclid and the Power of a Point theorems we have

$$AO \cdot AM = AP^2 = AB \cdot AC$$

Thus the quadrilateral $BMOC$ is cyclic and, since $OB = OC$, we get

$$\angle BMA = 180^\circ - \angle BMO = \angle BCO = \angle CBO = \angle CMO$$

establishing the claim. ■

Claim 2. The points B, M, E are collinear. (Fig. 2)
Proof.

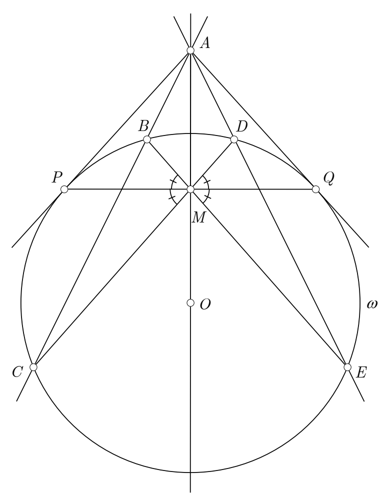


Fig. 2

By *Claim 1* we have

$$\angle BMP = \angle PMC \qquad , \qquad \angle DMQ = \angle QME$$

Since $\angle PMC = \angle DMQ$ we get $\angle BMP = \angle QME$, which proves our claim. ∎

Claim 3. $\angle CAM = \angle EAM$. (Fig. 3)
Proof. Let us consider triangles $\triangle ABM$ and $\triangle ADM$.

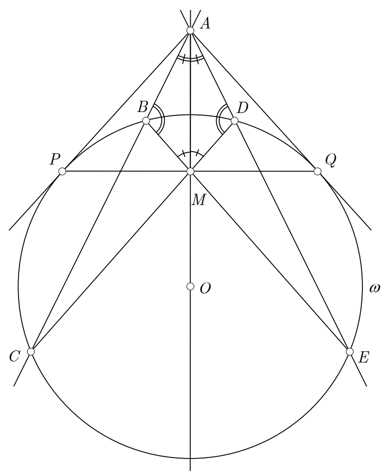


Fig. 3

The *Claim 1* yields

$$\angle PMB = \angle PMC = \angle DMQ \qquad \Rightarrow \qquad \angle AMB = \angle AMD$$

and, since B, C, E, D are concyclic, we have

$$\angle CBE = \angle CDE \qquad \Rightarrow \qquad \angle ABM = \angle ADM$$

Then we get $\angle CAM = \angle EAM$ and the the Lemma is proven. ∎

Now by using the Lemma we can easily solve the given problem. (Fig. 4)

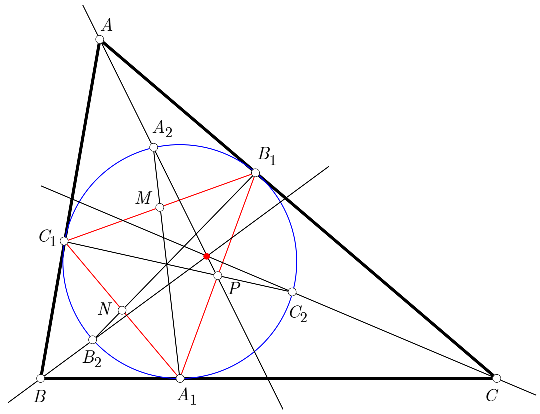


Fig. 4

In fact the pairs of lines AA_1 and AA_2, BB_1 and BB_2, CC_1 and CC_2, are isogonal conjugate and since AA_1, BB_1, CC_1 are concurrent at the Gergonne point Γ, the lines AA_2, BB_2, CC_2 are concurrent too at the point Γ' isogonal conjugate of Γ (due to a well known result). This ends the proof.

Third solution by Ivanov Andrei, Moldova

We need to prove that AA_2 and AA_1 are isogonal cevians in ABC, i.e. $\angle C_1AA_1 = \angle A_2AB_1$.

Let ω meets AA_1 at D. By the well known property, A_1D is symmedian in $\triangle A_1B_1C_1$, so $\angle C_1A_1D = \angle B_1A_1A_2$. Therefore $C_1D = B_1A_2$, and because of symmetry, $\triangle AC_1D \equiv \triangle AB_1A_2 \Longrightarrow \angle C_1AA_1 = \angle A_2AB_1$.

O124. Let $S(n)$ be the number of pairs of positive integers (x, y) such that $xy = n$ and $\gcd(x, y) = 1$. Prove that

$$\sum_{d|n} S(d) = \tau(n^2),$$

where $\tau(s)$ is the number of divisors of s.

Proposed by Dorin Andrica, Babes-Bolyai University and Mihai Piticari, Campulung Moldovenesc, Romania

Solution by José H. Nieto S., Universidad del Zulia, Venezuela

Let $n = p_1^{a_1} \dots p_k^{a_k}$ be the prime decomposition of n. It is well known that $\tau(n) = (a_1 + 1) \dots (a_k + 1)$, so $\tau(n^2) = (2a_1 + 1) \dots (2a_k + 1)$. On the other hand, if for each subset M of $\{p_1^{a_1}, \dots, p_k^{a_k}\}$ we put $x = \prod_{m \in M} m$ and $y = n/x$, then $xy = n$ and $\gcd(x, y) = 1$. Clearly all such pairs (x, y) may be obtained in this way, hence $S(n) = 2^k$.

The divisors of n are the summands in the expansion of the product

$$(1 + p_1 + p_1^2 + \dots + p_1^{a_1})(1 + p_2 + \dots + p_2^{a_2}) \cdots (1 + p_k + \dots + p_k^{a_k}).$$

Now, if we substitute each $p_i^{a_i}$ (with $a_i > 0$) in the above expresion by 2, the expansion will contain a summand 2^r corresponding to each divisor of n with exactly r distinct prime factors, i.e., the result will be $\sum_{d|n} S(d)$. Therefore,

$$\begin{aligned}\sum_{d|n} S(d) &= (1 + \underbrace{2 + 2 + \dots + 2}_{a_1\ 2's}) \cdots (1 + \underbrace{2 + 2 + \dots + 2}_{a_k\ 2's}) \\ &= (1 + 2a_1)(1 + 2a_2) \cdots (1 + 2a_k) = \tau(n^2).\end{aligned}$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Irfan Besic, Sarajevo, Bosnia and Herzegovina.

O125. Let a, b, c be positive real numbers. Prove that

$$4 \leq \frac{a+b+c}{\sqrt[3]{abc}} + \frac{8abc}{(a+b)(b+c)(c+a)}.$$

Proposed by Pham Huu Duc, Ballajura, Australia

Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

Defining $a = x^3$, $b = y^3$, $c = z^3$ we come to

$$4 \leq \frac{x^3+y^3+z^3}{xyz} + \frac{8(xyz)^3}{(x^3+y^3)(y^3+z^3)(z^3+x^3)}$$

Now

$$(x^3+y^3)(y^3+z^3)(z^3+x^3) \leq (2(x^3+y^3+z^3))^3/27$$

so we prove

$$\frac{x^3+y^3+z^3}{xyz} + \frac{27(xyz)^3}{(x^3+y^3+z^3)^3} \geq 4$$

Defining $S \doteq (x^3+y^3+z^3)/(xyz)$ we have

$$S + 27/S^3 \geq 4 \qquad \text{or} \qquad S^4 - 4S^3 + 27 \geq 0 \qquad \text{or} \qquad (S-3)^2(S^2+2S+3) \geq 0$$

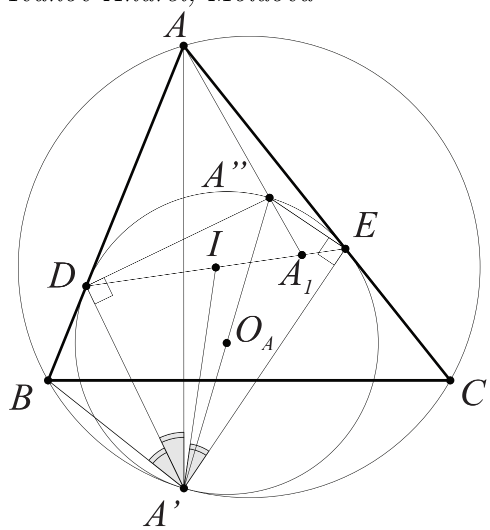
and we are done.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Gheorghe Pupazan, Chisinau, Republic of Moldova; Lyou Ikseung, South Korea; Irfan Besic, Sarajevo, Bosnia and Herzegovina; Tigran Hakobyan, Armenia.

O126. Let ABC be a scalene triangle and let κ_a be the A-mixtilinear incircle (the circle tangent to sides AB, AC and internally tangent to the circumcircle Γ of triangle ABC). Denote by A' the tangency point of κ_a with Γ and let A'' be the diametrically opposed point of A' with respect to κ_a. Similarly, define B'' and C''. Prove that lines AA'', BB'', CC'' are concurrent.

Proposed by Cosmin Pohoata, National College "Tudor Vianu", Romania

First solution by Ivanov Andrei, Moldova



Denote by I the incenter of ABC, O_A the center of $\mathcal{K}_a$, D and E the tangency points of $\mathcal{K}_a$ with AB and AC respectively, A_1 the inersection point of AA'' with DE.

By a well known property, AA_1 is symmedian in $\triangle DA''E$. So

$$\frac{DA_1}{EA_1} = \frac{A''D^2}{A''E^2} = \frac{\sin^2 \angle A''ED}{\sin^2 \angle A''DE} = \frac{\cos^2 \angle A'ED}{\cos^2 \angle A'DE}$$

We know that I is midpoint of DE, $A'D$ bisects $\angle BA'A$ and $A'A$ is symmedian in $\triangle A'DE$. Therefore,

$$\angle BA'D = \angle AA'D = \angle EA'I$$

Also, $\angle BDA' = \angle IEA'$, so $\triangle BDA' \sim \triangle IEA'$ and

$$\frac{BD}{IE} = \frac{A'D}{A'E}$$

But

$$BD = c - AD = c - \frac{AI}{\cos \frac{A}{2}} = c - \frac{AI}{\cos \frac{A}{2}} = c - \frac{2r}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = c - \frac{2S}{p \sin A} = c - \frac{bc}{p} = \frac{c(p-b)}{p} \quad (1)$$

and

$$IE = AI\tan\tfrac{A}{2} = \frac{r}{\cos\frac{A}{2}} = \frac{S}{p\sqrt{\frac{p(p-a)}{bc}}} = \frac{\sqrt{bc(p-b)(p-c)}}{p} \quad (2)$$

Dividing (1) and (2) we get

$$\frac{A'D}{A'E} = \frac{BD}{IE} = \frac{\frac{c(p-b)}{p}}{\frac{\sqrt{bc(p-b)(p-c)}}{p}} = \sqrt{\frac{c(p-b)}{b(p-c)}} = k$$

From

$$\angle IA'B = \angle IA'D + \angle DA'B = \angle CA'E + \angle IA'E = \angle CA'I$$

we get that DI bisects $BA'C$ and, therefore, $\angle DA'E = 90^\circ - \frac{\angle A}{2}$. By cosine theorem in $\triangle DA'E$ we get

$$4IE^2 = DE^2 = A'D^2 + A'E^2 - 2A'D\cdot A'E\sin\tfrac{A}{2} = A'E^2(k^2 + 1 - 2k\sin\tfrac{A}{2})$$

So

$$AE^2 = \frac{4IE^2}{k^2+1-2k\sin\frac{A}{2}} = \frac{4bc(p-c)(p-b)}{p^2\left(\frac{c(p-b)}{b(p-c)} + 1 - 2\sqrt{\frac{c(p-b)}{b(p-c)}\cdot\frac{(p-b)(p-c)}{bc}}\right)} = \frac{4bc(p-c)(p-b)}{p^2\left(\frac{c(p-b)}{b(p-c)} + 1 - \frac{2(p-b)}{b}\right)} =$$

$$= \frac{4b^2c(p-c)^2(p-b)}{p^2\Big((b-c)^2 + a(p-a)\Big)}$$

Similiarly, $A'D^2 = \dfrac{4bc^2(p-c)(p-b)^2}{p^2\Big((b-c)^2+a(p-a)\Big)}$.

Also,

$$R(\mathcal{K}_a) = O_AD = AD\tan\tfrac{A}{2} = \frac{AI\tan\frac{A}{2}}{\cos\frac{A}{2}} = \frac{r}{\cos^2\frac{A}{2}} = \frac{S}{p\cdot\frac{p(p-a)}{bc}} = \frac{bc\sqrt{(p-b)(p-c)}}{p\sqrt{p(p-a)}}$$

So

$$\frac{\sin\angle BAA''}{\sin\angle CAA''} = \frac{DA_1}{EA_1} = \frac{1-\sin^2\angle A'ED}{1-\sin^2\angle A'DE} = \frac{4R^2(\mathcal{K}_a) - A'D^2}{4R^2(\mathcal{K}_a) - A'E^2} = \frac{\frac{4b^2c^2(p-b)(p-c)}{p^3(p-a)} - \frac{4bc^2(p-c)(p-b)^2}{p^2((b-c)^2+a(p-a))}}{\frac{4b^2c^2(p-b)(p-c)}{p^3(p-a)} - \frac{4b^2c(p-c)^2(p-b)}{p^2((b-c)^2+a(p-a))}} =$$

$$= \frac{\frac{bc}{p(p-a)} - \frac{c(p-b)}{(b-c)^2+a(p-a)}}{\frac{bc}{p(p-a)} - \frac{b(p-c)}{(b-c)^2+a(p-a)}} = \frac{bc(b-c)^2 + abc(p-a) - pc(p-b)(p-a)}{bc(b-c)^2 + abc(p-a) - pb(p-c)(p-a)} = \frac{c(p-c)(p-2b)}{b(p-b)(p-2c)}$$

Similiarly, $\dfrac{\sin\angle ACC''}{\sin\angle BCC''} = \dfrac{b(p-b)(p-2a)}{a(p-a)(p-2b)}$ and $\dfrac{\sin\angle CBB''}{\sin\angle ABB''} = \dfrac{a(p-a)(p-2c)}{c(p-c)(p-2a)}$.

So

$$\frac{\sin\angle BAA''}{\sin\angle CAA''}\cdot\frac{\sin\angle ACC''}{\sin\angle BCC''}\cdot\frac{\sin\angle CBB''}{\sin\angle ABB''} = \frac{c(p-c)(p-2b)}{b(p-b)(p-2c)}\cdot\frac{b(p-b)(p-2a)}{a(p-a)(p-2b)}\cdot\frac{a(p-a)(p-2c)}{c(p-c)(p-2a)} = 1,$$

and by Ceva's theorem (trigonometric form) lines AA'', BB'' and CC'' are concurrent.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

It is well known that AA', BB' and CC' intersect at the external similitude center, whose trilinear coordinates are $\sin^2\frac{A}{2} : \sin^2\frac{B}{2} : \sin^2\frac{c}{2}$. Therefore, A' is the second point of intersection of line AA' (in trilinear coordinates $\beta\sin^2\frac{C}{2} = \gamma\sin^2\frac{B}{2}$), and the circumcircle of ABC (in trilinear coordinates $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$). Substitution of the relation between β and γ into the equation for the circumcircle yields

$$\beta\left(a\beta\sin^2\frac{C}{2} + b\alpha\sin^2\frac{C}{2} + c\alpha\sin^2\frac{B}{2}\right) = 0,$$

$$\alpha = -\frac{a\sin^2\frac{C}{2}}{b\sin^2\frac{C}{2} + c\alpha\sin^2\frac{B}{2}}\beta = -\frac{2a\sin^2\frac{C}{2}}{b+c-a}\beta = -\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\sin\frac{B}{2}}\beta,$$

the solution $\beta = 0$ being discarded because it corresponds to A, and where we have used that $a = b\cos C + c\cos B$, that $b + c - a = 8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, the Sine Law, and well known trigonometrical identities, R being the circumradius of ABC. Using the additional condition $a\alpha + b\beta + c\gamma = 2S$, where S is the area of ABC, in exact trilinear coordinates we obtain

$$4S = 2\frac{b\sin^2\frac{B}{2} + c\sin^2\frac{C}{2} - a\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin^2\frac{B}{2}}\beta =$$

$$= (a+b+c)\frac{\sin^2\frac{B}{2} + \sin^2\frac{C}{2} - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin^2\frac{B}{2}}\beta,$$

where we have used that

$$b\sin^2\frac{B}{2} = b\sin\frac{B}{2}\cos\frac{C+A}{2} = \frac{a+b+c}{2}\sin^2\frac{B}{2} - b\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

and similarly for $c\sin^2\frac{C}{2}$, since $a+b+c = 8R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$. Hence, calling $\rho_A = \frac{r}{\cos^2\frac{A}{2}}$ the radius of κ_A, r being the inradius of ABC, in exact trilinear coordinates we obtain

$$\frac{\beta}{2\rho_A} = \frac{(a+b+c)\cos^2\frac{A}{2}}{4S}\beta = \frac{\sin^2\frac{B}{2}\cos^2\frac{A}{2}}{\sin^2\frac{B}{2} + \sin^2\frac{C}{2} - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}.$$

Now, the exact trilinear coordinates of the center of κ_A are $\frac{2S-(b+c)\rho_A}{a} : \rho_A : \rho_A$, or calling $\alpha' : \beta' : \gamma'$ the exact trilinear coordinates of A'', we find $\beta' + \beta = 2\rho_A$ because the center of κ_A is the midpoint of $A'A''$, and

$$\frac{\beta'}{2\rho_A} = 1 - \frac{\beta}{2\rho_A} = \frac{\sin^2\frac{A}{2}\sin^2\frac{B}{2} + \sin^2\frac{C}{2} - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin^2\frac{B}{2} + \sin^2\frac{C}{2} - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}.$$

Denote by $f(A,B,C)$ the numerator of the previous expression, and by $g(A,B,C)$ the denominator, and note that $f(A,B,C) = f(B,A,C)$ and $g(A,B,C) = g(A,C,B)$. Similarly, by permutation of B and C, $\frac{\gamma'}{2\rho_A} = \frac{f(A,C,B)}{g(A,C,B)}$, or $\frac{\beta'}{\gamma'} = \frac{f(A,B,C)}{f(A,C,B)}$. Now, calling D the point where AA'' intersects BC, clearly the ratio between the distances from D to AC and to AB is $\frac{\beta'}{\gamma'}$, while $BD \sin B$ equals the distance from D to AB, and $CD \sin C$ equals the distance from D to AC. Therefore,

$$\frac{BD}{DC} = \frac{\sin C}{\sin B}\frac{\gamma'}{\beta'} = \frac{c}{b}\frac{f(A,C,B)}{f(A,B,C)},$$

and by cyclic permutation, calling E, F the points where BB'', CC'' respectively meet CA, AB, we find

$$\frac{CE}{EA} = \frac{a}{c}\frac{f(B,A,C)}{f(B,C,A)}, \qquad \frac{AF}{FB} = \frac{b}{a}\frac{f(C,B,A)}{f(C,A,B)},$$

or clearly, since $f(B,A,C) = f(A,B,C)$, $f(B,C,A) = f(C,B,A)$ and $f(A,C,B) = f(C,A,B)$, then $\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = 1$, and by the reciprocal of Menelaus' theorem, the conclusion follows.