Proposed Problems

Secondary Level

Solutions should arrive by May 1, 2006 in order to be considered for publication.

Juniors

J7. Prove that

$$\sum_{n=1}^{9999} \frac{1}{\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt[4]{n}+\sqrt[4]{n+1}\right)} = 9.$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

J8. Let a, b be distinct real numbers such that

$$|a-1| + |b+1| = |a| + |b| = |a-1| + |b+1|.$$

Find the minimal possible value of $|a+b|$.

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College, Romania

J9. Show that the following equation does not have integral solutions

$$(x-y)^2 - 5(x+y) + 25 = 0.$$

Proposed by Ovidiu Pop, Satu Mare, Romania

J10. Let $A = 1! \cdot 2! \cdot \ldots \cdot 1002!$ and $B = 1004! \cdot 1005! \cdot \ldots \cdot 2006!$. Prove that $2AB$ is a square and that $A+B$ is not a square.

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College, Romania

J11. Consider an arbitrary parallelogram $ABCD$ with center O and let P be a point different from O, that satisfies $PA \cdot PC = OA \cdot OC$ and $PB \cdot PD = OB \cdot OD$. Show that the sum of lengths of two of the segments PA, PB, PC, PD equals the sum of lengths of the other two.

Proposed by Iurie Boreico, student, Chisinau, Moldova

J12. Let $ABCD$ be a convex quadrilateral. A square is called inscribed in it if its vertices lie on different sides of $ABCD$. If there are two different squares inscribed in $ABCD$, prove that there are infinitely many squares inscribed in $ABCD$.

Proposed by Iurie Boreico, student, Chisinau, Moldova

S7. Let $x_1, x_2, \ldots, x_n$ be real numbers greater than or equal to $\frac{1}{2}$. Prove that

$$\prod_{i=1}^{n}\left(1+\frac{2x_i}{3}\right)^{x_i} \geq \left(\frac{4}{3}\right)^{n} \sqrt[4]{(x_1+x_2)(x_2+x_3)\ldots(x_{n-1}+x_n)\,(x_n+x_1)}.$$

Proposed by Iurie Boreico and Marcel Teleucă, Chisinau, Moldova

S8. Let O, I, and r be the circumcenter, incenter, and inradius of a triangle ABC. Let M be a point inside the triangle, and let d_1, d_2, d_3, be the distances from M to the sides BC, AC, AB. Prove that if $d_1 \cdot d_2 \cdot d_3 \geq r^3$, then M lies inside the circle with center O and radius OI.

Proposed by Ivan Borsenco, student, Chisinau, Moldova

S9. Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Prove that

$$\prod_{k=1}^{n}\left(\sum_{k=1}^{n} a_k^{T_k}\right) \geq \left(\sum_{k=1}^{n} a_k^{\frac{T_{n+1}}{3}}\right)^{n},$$

where $T_k = \frac{k(k+1)}{2}$ is the k^{th} triangular number.

Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Spain

S10. Let $(a_n)_{n\geq 1}$ be a sequence of positive numbers such as $a_{n+1} = a_n^2 - 2$ for all $n \geq 1$. Show that for all $n \geq 1$ we have $a_n \geq 2$.

Proposed by Dr. Laurenţiu Panaitopol, University of Bucharest, Romania

S11. Consider the sequences given by

$$\begin{aligned} &a_0 = 1, \quad a_{n+1} = \frac{3a_n + \sqrt{5a_n^2 - 4}}{2}, \quad n \geq 1, \\ &b_0 = 0, \quad b_{n+1} = a_n - b_n, \quad n \geq 1. \end{aligned}$$

Prove that $(a_n)^2 = b_{2n+1}$.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

S12. Let a be a real number. Prove that

$$5\left(\sin^3 a + \cos^3 a + 3\sin a\cos a\right) = .04$$

if and only if

$$5\,(\sin a + \cos a + 2\sin a\cos a) = .04.$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Undergraduate Level

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U7. Evaluate

$$\int_1^e \frac{1+x^2\ln x}{x+x^2\ln x}dx.$$

Proposed by Zdravko Starc, Vrsac, Serbia and Montenegro

U8. Let ω_n be an n-th primitive root of the unity and let

$$H_n = \prod_{i=0}^{n-1}(1+\omega_n^i - \omega_n^{2i}).$$

Show that H_n is an integer and give a formula for H_n which uses only integers.

Proposed by Mietek Dabkowski, University of Texas at Dallas, and Josef Przytycki, George Washington University

U9. Let $\|\cdot\|$ be a norm on $\mathcal{M}_n(\mathbb{C})$ and let $A_1, A_2, \ldots, A_p$ be complex matrices of order n. Prove that for every $x > 0$ there exists $z \in \mathbb{C}$, with $|z| < x$, such that

$$\left\|(I_n - zA_1)^{-1} + (I_n - zA_2)^{-1} + \ldots + (I_n - zA_p)^{-1}\right\| \geq p.$$

Proposed by Gabriel Dospinescu, "Louis le Grand" College, Paris

U10. Find all functions $f : [0, +\infty) \longrightarrow [0, +\infty)$, differentiable at $x = 1$ and satisfying

$$f\left(x^3\right) + f\left(x^2\right) + f\left(x\right) = x^3 + x^2 + x,$$

for all $x \geq 0$.

Proposed by Mihai Piticari, Campulung, Romania

U11. Two players, A and B, play the following game: player A divides an $n \times n$ square into stripes of unit width (and various lengths). After that, player B picks an integer k, $1 \leq k \leq n$, and removes all stripes of length k. Let $l(n)$ be the largest area B can remove, regardless the way A divides the square into strips. Evaluate $\lim\limits_{n\to+\infty} \frac{l(n)}{n}$ and find $l(10)$.

Proposed by Iurie Boreico, student, Chisinau, Moldova

U12. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that $e^{a_n} + na_n = 2$, for all positive integers n. Evaluate

$$\lim_{n\to\infty} n\left(1 - na_n\right).$$

Proposed by Teodora Liliana Rădulescu, "Fraţii Buzeşti" College, Craiova, Romania

Olympiad Level

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O7. In the convex hexagon $ABCDEF$ the following equalities hold:

$$AD = BC + EF,\ BE = AF + CD,\ CF = AB + DE.$$

Prove that

$$\frac{AB}{DE} = \frac{CD}{AF} = \frac{EF}{BC}.$$

Proposed by Nairi Sedrakyan, Armenia

O8. Let a, b, c, x, y, z be real numbers and let $A = ax + by + cz$, $B = ay + bz + cx$ and $C = az + bx + cy$. Suppose that $|A - B| \geq 1$, $|B - C| \geq 1$ and $|C - A| \geq 1$. Find the smallest possible value of the product

$$\left(a^2 + b^2 + c^2\right)\left(x^2 + y^2 + z^2\right).$$

Proposed by Adrian Zahariuc, student, Bacău, Romania

O9. Let a be a positive integer such that for any positive integer n, the number $a + n^2$ can be written as a sum of two squares. Prove that a is a square.

Proposed by Gabriel Dospinescu, "Louis le Grand" College, Paris

O10. Let P be an integer polynomial such that $P(2^m)$ is a n-th power of a integer number for any positive integer m. Prove that P itself is a n-th power of an integer polynomial.

Proposed by Iurie Boreico, student, Chisinau, Moldova

O11. Let a, b, c be distinct positive integers. Prove the following inequality:

$$\frac{a^2b + a^2c + b^2a + b^2c + c^2a + c^2b - 6abc}{a^2 + b^2 + c^2 - ab - bc - ac} \geq \frac{16abc}{(a+b+c)^2}.$$

Proposed by Iurie Boreico and Ivan Borsenco , Chisinau, Moldova

O12. Consider the system

$$\begin{cases} x_1x_2x_3 - x_4 = a_1 \\ x_2x_3x_4 - x_1 = a_2 \\ x_3x_4x_1 - x_2 = a_3 \\ x_4x_1x_2 - x_3 = a_4, \end{cases}$$

where $x_i \in \left[\sqrt{2} - 1, 1\right]$ are not all equal. Prove that

$$a_1 + a_2 + a_3 + a_4 \neq 0.$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas