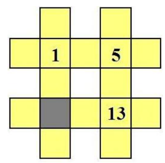
Junior problems

J91. The squares in the figure below are labeled 1 through 16 such that the sum of the numbers in each row and each column is the same. The positions of $1, 5$, and 13 are given.



Prove that there is only one possibility for the number in the darkened square and find this number.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Roberto Bosch Cabrera, Cuba

Let x the number in the darkened square, and c the sum in each row and each column. We obtain that $(1+2+\cdots+16)+(1+5+13)+x = 4c$. That is to say $155 + x = 4c$, but $155 \equiv 3 \mod 4$ so $x \equiv 1 \mod 4$ and x could be $1, 5, 9, 13$. From this we deduce that $x = 9$ because $1, 5, 13$ are given.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Call x the number on the darkened square, and s the sum of each row and column. Clearly, $4s - x - 1 - 5 - 13 = 1 + 2 + \cdots + 16 = 56$ is a multiple of 4, or x must have a remainder of 1 when divided by 4. But all integers not exceeding 16 with remainder 1 modulus 4 have been used except for 9. So $x = 9$ is the only possible value.

Third solution by Oles Dobosevych, Ukraine

Let us label the empty squares with the numbers $a_1, a_2, \ldots, a_{13}$ from left to right and from up to down. The number in the darkened square is a_9.

Notice that

$$a_1 + 1 + a_6 + a_9 + a_{12} = a_2 + 5 + a_7 + 13 + a_{13},$$

$$a_1 + 1 + a_6 + a_9 + a_{12} = a_3 + 1 + a_4 + 5 + a_5,$$

$$a_1 + 1 + a_6 + a_9 + a_{12} = a_8 + a_9 + a_10 + 13 + a_{11},$$

and

$$a_1 + a_2 + \cdots + a_{13} + 1 + 5 + 13 = 1 + 2 + \cdots + 16 = \frac{16(16+1)}{2}.$$

Thus

$$\begin{aligned} 4(a_1 + 1 + a_6 + a_9 + a_{12}) &= (a_1 + 1 + a_6 + a_9 + a_{12}) + (a_2 + 5 + a_7 + 13 + a_{13}) \\ &+ (a_3 + 1 + a_4 + 5 + a_5) + (a_8 + a_9 + a_{10} + 13 + a_{11}) \\ &= (a_1 + \cdots + a_{13}) + 1 + 5 + 13 + a_{13} + 1 + 5 + 13 \\ &= \frac{16(16+1)}{2} + a_{13} + 19 \\ &= 8 \cdot 17 + a_{13} + 19. \end{aligned}$$

This implies $19 + a_{13} \equiv 0 \ (mod\ 4)$ or $a_{13} \equiv 1 \ (mod\ 4)$.

There are only four numbers, $1, 5, 9, 13$, that are less than 16 and congruent to 1 modulo 4. Since $1, 5, 13$ are already in use then 9 is the only possibility.

Fourth solution by Johan Gunardi, Indonesia.

Let n be the sum of the numbers in each row or column. Let h be the number in the darkened square. So $4n = (1 + 2 + \ldots + 16) + (1 + 5 + 13 + h) = 155 + h$.

So 4 divides $155 + h$, which implies $h = 1, 5, 9, 13$. Since $1, 5, 13$ are already used thus there is only one possibility for h, which is 9.

Also solved by Mihai Miculita, Oradea, Romania; John T. Robinson, Yorktown Heights, NY, USA.

J92. Find all primes $q_1, q_2, \ldots, q_6$ such that $q_1^2 = q_2^2 + \cdots + q_6^2$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Andrea Munaro, Italy

Every square is 0 or 1 modulo 3 and clearly $q_1 \neq 3$. Suppose there are $0 \leq a \leq 5$ primes between $q_2, ..., q_6$ not equal to 3. Then $1 \equiv 1 \cdot a + 0(5-a) \pmod 3$, from which $a = 1$ or $a = 4$.

Suppose $a = 1$.
Then $q_1^2 = q_2^2 + 4 \cdot 3^2$ or $(q_1 - q_2)(q_1 + q_2) = 36$. Since $q_1 + q_2 > q_1 - q_2$, then $q_1 + q_2$ can be only $9, 12, 18, 36$ and it's easy to see that there are no solutions.

Suppose $a = 4$.
Then $q_1^2 = q_2^2 + q_3^2 + q_4^2 + q_5^2 + 9$. Since q_i are primes, their quadratic residues modulo 8 are 1 if q_i is odd or 4 if $q_i = 2$. Clearly $q_1 \neq 2$ and suppose that there are $0 \leq b \leq 4$ primes between $q_2, ..., q_5$ not equal to 2. Then $1 \equiv 1 + 1 \cdot b + 4(4-b) \pmod 8$ or $3b \equiv 0 \pmod 8$, which has the only solution $b = 0$.
Hence the solutions to the problem are $(5, 2, 2, 2, 2, 3)$ and its permutations with 5 fixed.

Second solution by Ivanov Andrei, Moldova

It is clear that $q_1 \neq 2$, because $q_2^2 + \cdots + q_6^2 > 4$, thus q_1 is odd. Then:

$$q_2^2 + \ldots + q_6^2 \equiv 1 \pmod 8$$

Suppose, that none of $q_2, \ldots, q_6$ is 2, then $q_2^2 + \ldots + q_6^2 \equiv 5 \pmod 8$, contradiction. So we can have:

- 1 prime is equal to 2 $\Rightarrow q_2^2 + \ldots + q_6^2 \equiv 4 + 1 + 1 + 1 + 1 \equiv 0 \pmod 8$, contradiction.
- 2 primes are equal to 2 $\Rightarrow q_2^2 + \ldots + q_6^2 \equiv 4 + 4 + 1 + 1 + 1 \equiv 3 \pmod 8$, contradiction.
- 3 primes are equal to 2 $\Rightarrow q_2^2 + \ldots + q_6^2 \equiv 4 + 4 + 4 + 1 + 1 \equiv 6 \pmod 8$, contradiction.
- 4 primes are equal to 2 $\Rightarrow q_2^2 + \ldots + q_6^2 \equiv 4 + 4 + 4 + 4 + 1 \equiv 1 \pmod 8$.

So 4 of these primes are equal to 2. Let $q_2 = \ldots = q_5 = 2$. Then we have to solve the following equation: $q_1^2 = 16 + q_6^2 \Leftrightarrow (q_1 - 4)(q_1 + 4) = q_6^2$. Because q_6 is prime and $q_1 - 4 < q_1 + 4$ we can have the single possibility:

$$\begin{cases} q_1 - 4 = 1 \\ q_1 + 4 = q_6^2 \end{cases} \Leftrightarrow (q_1, q_6) = (5, 3)$$

So $(q_1, q_2, q_3, q_4, q_5, q_6) = (5, 2, 2, 2, 2, 3)$

Also solved by Oles Dobosevych, Ukraine; Daniel Lasaosa, Universidad Publica *de Navarra, Spain; John T. Robinson, Yorktown Heights, NY, USA; Roberto Bosch Cabrera, Cuba.*

J93. Let a and b be positive real numbers. Prove that

$$\frac{a^6+b^6}{a^4+b^4} \geq \frac{a^4+b^4}{a^3+b^3} \cdot \frac{a^2+b^2}{a+b}.$$

Proposed by Arkady Alt, San Jose, California, USA

First solution by Nguyen Manh Dung, Hanoi University of Science, Vietnam

The above inequality is equivalent to

$$(a^6+b^6)(a^3+b^3)(a+b) \geq (a^4+b^4)^2(a^2+b^2).$$

Using the Cauchy-Scharz inequality, we have

$$(a^6+b^6)(a^2+b^2) \geq (a^4+b^4)^2$$

$$(a^3+b^3)(a+b) \geq (a^2+b^2)^2.$$

Multiplying these inequalities, the conclusion follows. Equality occurs when $a = b$.

Second solution by Shamil Asgarli, Canada

After multiplying out the right side the inequality becomes

$$\frac{a^6+b^6}{a^4+b^4} \geq \frac{a^6+a^4b^2+b^4a^2+b^6}{a^4+a^3b+b^3a+b^4}$$

or

$$\frac{a^6+b^6}{a^4+b^4} \geq \frac{a^6+b^6+a^2b^2(a^2+b^2)}{a^4+b^4+ab(a^2+b^2)}.$$

To simplify calculations let use make the following substitutions

$$a^6+b^6 = x$$

$$a^4+b^4 = y$$

$$ab = z$$

$$a^2+b^2 = t.$$

Our inequality transforms to

$$\frac{x}{y} \geq \frac{x+z^2t}{y+zt}$$

or after some algebra $x \geq yz$. If we back substitute we obtain

$$a^6 + b^6 \geq (a^4 + b^4)ab$$

which is exactly the same as

$$(a^5 - b^5)(a - b) \geq 0$$

and thus we are done. Equality occurs when $a = b$.

Third solution by An Zhen-ping, China

Let us start with the following observation

$$(a^6 + b^6)(a^3 + b^3) - (a^5 + b^5)(a^4 + b^4) = a^3 b^3 (a + b)(a - b)^2 \geq 0.$$

The last observation implies that

$$\frac{(a^6 + b^6)}{(a^5 + b^5)} \geq \frac{(a^4 + b^4)}{(a^3 + b^3)}.$$

By the same token we observe that

$$\frac{(a^5 + b^5)}{(a^4 + b^4)} \geq \frac{(a^2 + b^2)}{(a + b)}.$$

The final step is to multiply the last two inequalities to obtain the desired result.

Also solved by Andrea Munaro, Italy; Brian Bradie, Newport News, VA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; John T. Robinson, Yorktown Heights, NY, USA; Michel Batailll, France; Oleh Faynshteyn, Leipzig, Germany; Oles Dobosevych, Ukraine; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Mihai Miculita, Oradea, Romania; Roberto Bosch Cabrera, Cuba.

J94. Prove that the equation $x^3 + y^3 + z^3 + w^3 = 2008$ has infinitely many solutions in integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by John T. Robinson, Yorktown Heights, NY, USA

Since $2008 = 8 \cdot 251 = s^3 \cdot 251$, it suffices to show that $x^3 + y^3 + z^3 + w^3 = 251$ has infinitely many solutions over the integers. Let us note that

$$(30n^3 + 5)^3 - (30n^3 - 5)^3 - (30n^2)^3 + 1 = 251$$

and thus we are done.

Second solution by Jose Hernandez Santiago, Mexico

For every $n \in \mathbb{Z}$, the 4-tuple

$$(x = 10 + 60n^3, \quad y = 10 - 60n^3, \quad z = 2, \quad w = -60n^2)$$

provides us with a solution to the given equation. Since $n^3 = m^3$ implies $n = m$, we have that no two of these solutions can be identical: this terminates our proof.

Also solved by Oles Dobosevych, Ukraine; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Oleh Faynshteyn, Leipzig, Germany; Johan Gunardi, Jakarta, Indonesia; Roberto Bosch Cabrera, Cuba.

J95. Let ABC be a triangle and let I_a, I_b, I_c be its excenters. Denote by O_a, O_b, O_c the circumcenters of triangles I_aBC, I_bAC, I_cAB. Prove that the area of triangle $I_aI_bI_c$ is twice the area of hexagon $O_aCO_bAO_cB$.

Proposed by Mehmet Sahin, Ankara, Turkey

First solution by Andrea Munaro, Italy

It is well-known that the angle-bisectors of ABC are the altitudes of $I_aI_bI_c$ with feet A, B, C. Denote by H the orthocenter of $I_aI_bI_c$. Clearly HAI_bC, HAI_cB and HBI_aC are cyclics and so $I_aO_a = O_aH$, $I_bO_b = O_bH$ and $I_cO_c = O_cH$. Then $[I_aO_aC] = [O_aHC]$ and so on. This leads to $[I_aI_bI_c] = 2[O_aCO_bAO_cB]$.

Second solution by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

Let I denote the incenter of triangle ABC. We have $\angle BI_aC = 90^\circ - \frac{A}{2}$ which implies $\angle BO_aC = 180^\circ - A$. Hence O_a, O_b, O_c lie on the circumcircle (O) of ABC. Again, since $BO_a = CO_a$, we conclude that O_a is the midpoint of minor arc BC of (O). Therefore A, I, O_a, I_a all lie on a line.

Now $\angle BIC + \angle BI_aC = 90^\circ + \frac{A}{2} + 90^\circ - \frac{A}{2} = 180^\circ$. Thus I lies on the circumcircle (O_a) of BI_aC. Therefore O_a is the midpoint of diameter II_a and hence $2[IBO_a] = [IBI_a], 2[ICO_a] = [ICI_a]$ so that $2[IBO_aC] = [IBI_aC]$. Similarly $2[ICO_bA] = [ICI_bA]$ and $2[IAO_cB] = [IAI_cB]$. Adding, we conclude that $2[O_aCO_bAO_cB] = [I_aI_bI_c]$.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Ivanov Andrei, Moldova; Michel Bataille, France; Oleh Faynshteyn, Leipzig, Germany; Ricardo Barroso Campos, Spain; Vicente Vicario Garca, Huelva, Spain; Oles Dobosevych, Ukraine; Mihai Miculita, Oradea, Romania; Roberto Bosch Cabrera, Cuba.

J96. Let n be an integer. Find all integers m such that $a^m + b^m \geq a^n + b^n$ for all positive real numbers a and b with $a + b = 2$.

Proposed by Oleg Mushkarov, Bulgarian Academy of Sciences, Sofia, Bulgaria

First solution by Arkady Alt, San Jose, California, USA

First we will prove that for any positive real numbers x, y and any natural $m \geq n$ following inequality holds

(1) $\quad x^m + y^m \geq \dfrac{(x^n + y^n)(x^{m-n} + y^{m-n})}{2}$.

We have $2(x^m + y^m) - (x^n + y^n)(x^{m-n} + y^{m-n}) = x^m + y^m - x^n y^{m-n} - x^{m-n} y^n =$

$(x^n - y^n)(x^{m-n} - y^{m-n}) \geq 0.$

Combination (1) with PM-AM inequality $\dfrac{x^{m-n} + y^{m-n}}{2} \geq \left(\dfrac{x+y}{2}\right)^{m-n}$ gives inequality

(2) $x^m + y^m \geq (x^n + y^n)\left(\dfrac{x+y}{2}\right)^{m-n}$

Applying (2) to positive real numbers a and b with $a + b = 2$ we obtain

$$a^m + b^m \geq (a^n + b^n)\left(\frac{a+b}{2}\right)^{m-n} = a^n + b^n.$$

Suppose that there is natural $m < n$ such that for any positive real numbers a and b

with $a + b = 2$ holds inequality $a^m + b^m \geq a^n + b^n \iff a^m + (2-a)^m \geq a^n + (2-a)^n$.

Then we obtain inequality $2^m = \lim\limits_{a \to 2}(a^m + (2-a)^m) \geq \lim\limits_{a \to 2}(a^n + (2-a)^n) = 2^n$ which

contradict to inequality $2^m < 2^n \Longleftarrow m < n$.

Thus, answer is $\{m \mid m \in \mathbb{N} \text{ and } m \geq n\}$.

Second solution by Oles Dobosevych, Ukraine

First, let us prove that for any nonnegative positive integer k and for any positive integer n $a^{n+k} + b^{n+k} \geq a^n + b^n$, if $a + b = 2$ and $a \geq 0$, $b \geq 0$. To prove this, we use mathematical induction on k.

The case $k = 0$ is readily checked. Assume that the statemnt is true for $k = l$ and let us prove, that it is true too for k=l+1.

We have to prove that

$$a^{n+l+1} + b^{n+l+1} \geq a^{n+l} + b^{n+l}.$$

If we have that $a + b = 2$ the inequalities is equal to

$$2a^{n+l+1} + 2b^{n+l+1} \geq (a^{n+l} + b^{n+l})(a + b),$$

or

$$a^{n+l+1} + b^{n+l+1} \geq a^{n+1}b + b^{n+1}a.$$

The last inequality is equivalent to
$(a-b)^2(a^{n+l-1} + a^{n+l-2}b + \ldots + b^{n+l-2}a + b^{n+l-1}) \geq 0$ and since $(a-b)^2 \geq 0$ and $a^{n+l-1} + a^{n+l-2}b + \ldots + b^{n+l-2}a + b^{n+l-1} > 0$ we have that inequality holds if and only if $a = b$ or $k = 0$.

Thus $a^{n+l+1} + b^{n+l+1} \geq a^{n+l} + b^{n+l} \geq a^n + b^n$ or $a^{n+l+1} + b^{n+l+1} \geq a^n + b^n$. This completes the induction and proves that the statemnt is true for all nonnegative integers. The last result implies that the given relation is only satisfied for all integers $m \geq n$.

Third solution by Roberto Bosch Cabrera, Cuba

Let $n = 0$ or $n = 1$, we need to find all integers m such that $a^m + b^m \geq 2$. Let $f(x) = x^m + (2-x)^m$ with $0 < x < 2$. Since $g(x) = x^m$ is convex in this interval we have that $f(x) = g(x) + g(2-x) \geq 2g(1) = 2$ by Jensen's inequality, so all integers m satisfy. To note that $g''(x) = m(m-1)x^{m-2}$ and $m(m-1) \geq 0$. If $n \geq 2$ then $m \geq n$. Let $f(x) = x^n + (2-x)^n$. Since $f(x) = f(2-x)$ it suffices consider $1 \leq x < 2$ fixed. Now let $g(n) = x^n + (2-x)^n$, we will prove that $g(n)$ is non decreasing. We have $g'(n) = x^n \ln(x) + (2-x)^n \ln(2-x)$. Now suppose $n \geq 2$ fixed, that is to say $h(x) = x^n \ln(x) + (2-x)^n \ln(2-x)$. We need to prove that $h(x) \geq 0$. Since $t(x) = x^n \ln(x)$ is convex if $1 \leq x < 2$ we have that $h(x) = t(x) + t(2-x) \geq 2t(1) = 0$ by Jensen's inequality and we are done. To note that $t''(x) = x^{n-2}\left[n(n-1)\ln(x) + 2n - 1\right] > 0$. If $n \leq -1$ then $m \leq n$. The idea is the same, but in this case we consider $0 < x \leq 1$.

Senior problems

S91. Find all triples (n, k, p), where n and k are positive integers and p is a prime, satisfying the equation

$$n^5 + n^4 + 1 = p^k.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Andrea Munaro, Italy

It's easy to see that $(1, 1, 3)$ and $(2, 2, 7)$ are solutions. We'll prove that there aren't any other.

$$\begin{aligned} n^5 + n^4 + 1 &= (n^2+1)^2 - n^2 + n^5 - n^2 \\ &= (n^2+1-n)(n^2+1+n) + n^2(n-1)(n^2+1+n) \\ &= (n^2+n+1)(n^3-n+1). \end{aligned}$$

Suppose $n > 2$.
Then $n^3-n+1-(n^2+n+1) = n(n+1)(n-2) > 0$. Hence we have $n^2+n+1 = p^r$ and $n^3 - n + 1 = p^s$ where $r + s = k$ and $s > r$. Subtracting the first relation from the second we get

$$n(n+1)(n-2) = p^r(p^{s-r} - 1).$$

Clearly $r > 0$ and p doesn't divide n. If $n + 1 = p^r$ it contradicts the first relation, and so p divides both $n+1$ and $n-2$. But $(n+1, n-2) = (n+1, 3)$ can be 3 or 1, from which $p = 3$.
Then $n^5 + n^4 + 1 \equiv 0 \pmod 9$, but checking this congruence for every residue modulo 9 it's easy to see that there are no solutions.

Second solution by John T. Robinson, Yorktown Heights, NY, USA

Solution - First note that

$$n^5 + n^4 + 1 = (n^2+n+1) * (n^3-n+1) = r(n) * s(n),$$

and that $s(n) > r(n)$ for $n \geq 3$. Next note that for $n \geq 3$, we cannot have $r(n) = p^a$ and $s(n) = p^b$ for any prime p and positive integers $a, b, b > a$, since otherwise we would have $\frac{s(n)}{r(n)} = p^(b-a)$, but $\frac{s(n)}{r(n)} = n - 1 - \frac{(n-2)}{r(n)}$, and $0 < \frac{(n-2)}{r(n)} < 1$ for $n \geq 3$. Therefore any solution must satisfy $n < 3$, that is there are at most two solutions. Finally, it is trivially verified that $n = 1$ and $n = 2$ both give solutions: $(1)n = 1, k = 1$, and $p = 3$; and $(2)n = 2, k = 2, andp = 7$.

Third solution by Mahmoud Ezzaki, Morocco

For $n = 1$ we have $p = 3$ and $k = 1$. Now suppose that $n \geq 2$, since $p^k = n^5 + n^4 + 1 = (n^2 + n + 1)(n^3 - n + 1)$ there exist two positive integers r and s such that $n^2 + n + 1 = p^2$ and $n^3 - n + 1 = p^r$ with $r \geq s$.

We have that $\gcd(n^2+n+1, n^3-n+1) = \gcd(p^s, p^r) = p^s$. Because $n^3-n+1 = (n-1)(n^2+n+1) - (n-2)$ and $n^2+n+1 = (n-2)(n-3)+7$, we have that $\gcd(n^2+n+1, n^3-n+1) = \gcd(n^2+n+1, n-2) = \gcd(n-2, 7)$ therefore $p^s = 1$ or $p^s = 7$. Examining these two cases we obtain two solutions

$$(n, p, k) = \{(1, 3, 1); (2, 7, 2)\}.$$

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Michel Bataille, France; Ozgur Kircak, Macedonia; Oleh Faynshteyn, Leipzig, Germany; Oles Dobosevych, Ukraine; Roberto Bosch Cabrera, Cuba.

S92. Let ABC be a triangle with altitudes BE and CF and let M be a point on its circumcircle. Denote by P the intersection of MB and CF and by Q the intersection of MC and BE. Prove that EF bisects the segment PQ.

Proposed by Son Ta Hong, Ha Noi University, Vietnam

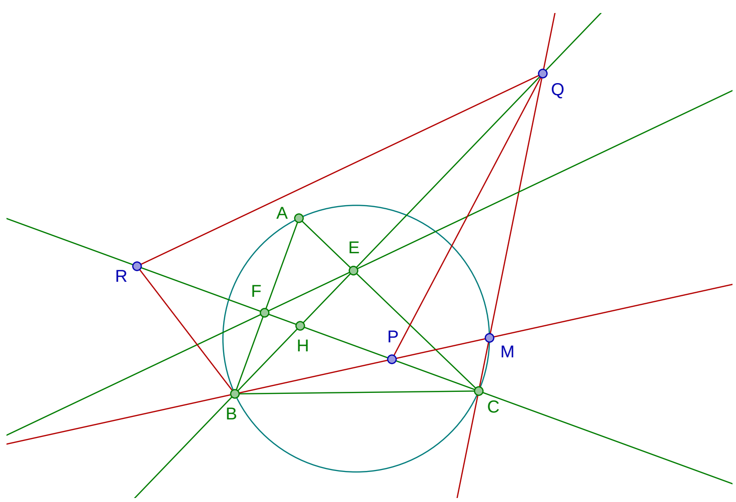
First solution by Pham Huu Duc, Australia

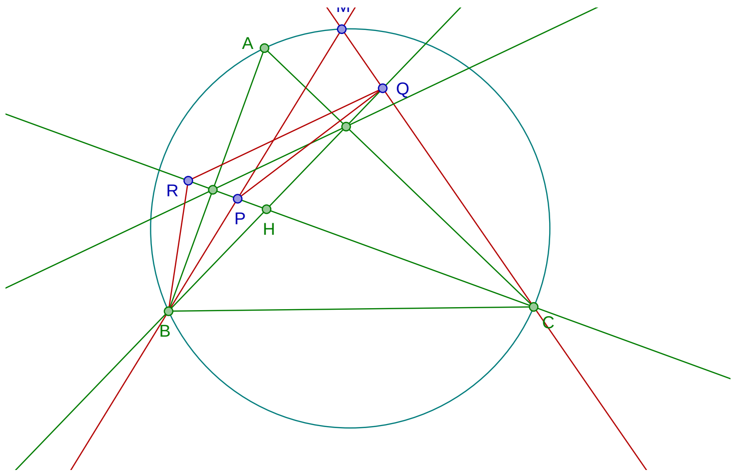
Let H be the orthocenter of $\triangle ABC$. Then $\angle BMC = \angle BAC = \angle EHC$ since $\angle BMC$ and $\angle BAC$ subtend the same arc of the circle ABC and $\angle BAC$ and $\angle EHC$ have their legs respectively perpendicular. It follows that the quadrilateral $HPMQ$ is cyclic and so $\angle HQM = \angle HPB$.

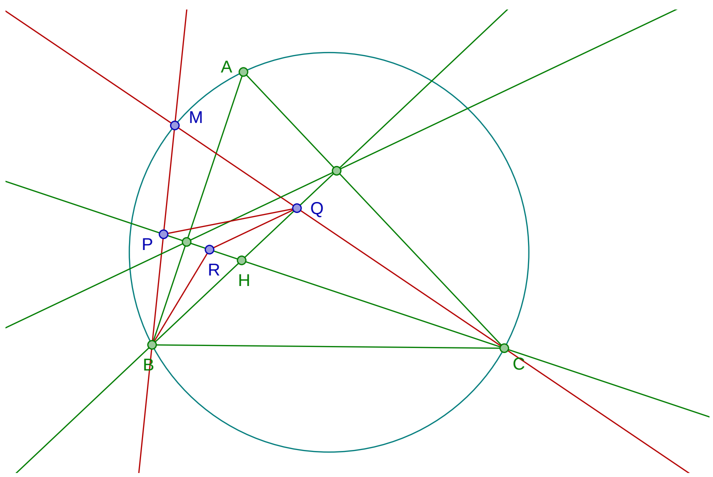
Let R be the reflection of P through F. Then $\angle BRC = \angle BQC$, showing that the quadrilateral $BCQR$ is cyclic. It follows that $\angle QRC = \angle QBC = \angle EFC$, and consequently the two lines QR and EF are parallel. In $\triangle PQR$, since EF is parallel to the base QR and bisects PR, we conclude that EF bisects PQ.

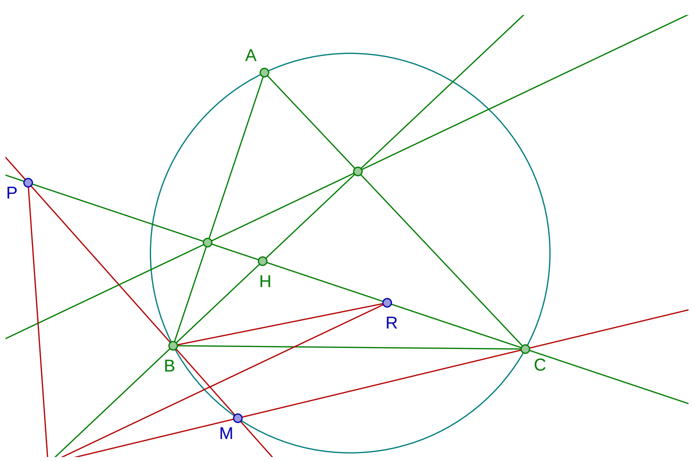
For the case M is on the smaller arc BC, we work with angle congruence mod $180°$ and the proof remains the same.

Note that we implicitly assume that $\angle BAC$ is not a right angle.

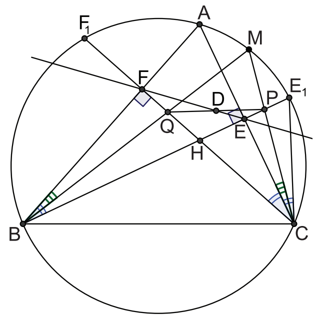








Second solution by Ivanov Andrei, Moldova



Denote by E_1 and F_1 intersections of the altitudes BE and CF with the circumcircle of triangle ABC. Let H be orthocenter of the triangle and D the intersection of the lines PQ and EF.
It is well known that E_1 is reflection of H w.r.t. point E, so $HE = EE_1$ and $\angle E_1HC = \angle HE_1C$. So $\triangle EE_1C \equiv \triangle EHC \sim \triangle FHB$ then $\frac{FH}{EE_1} = \frac{BF}{EC}$. (1)
But $\angle ABM = \angle ACM$, so $\triangle FBQ \sim \triangle ECP \Rightarrow \frac{FQ}{EP} = \frac{BF}{EC}$. (2)
From (1) and (2) we obtain that

$$\frac{FQ}{EP} = \frac{FH}{EE_1} = \frac{FH}{EH} \Leftrightarrow \frac{FQ}{FH} = \frac{EP}{EH}.$$

Then apply Menelaus Theorem to the triangle QHP and transversal $\overline{FDE}$:

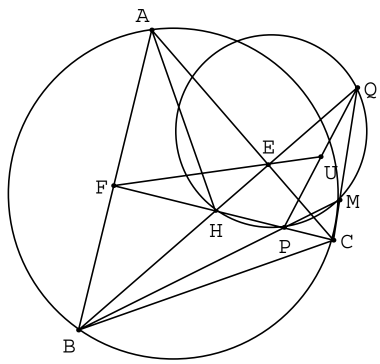
$$\frac{FQ}{FH} \cdot \frac{HE}{EP} \cdot \frac{PD}{DQ} = 1 \Leftrightarrow \frac{PD}{DQ} = 1$$

.

Third solution by Michel Bataille, France;

Let ABC be a triangle with altitudes BE and CF and let M be a point on its circumcircle. Denote by P the intersection of MB and CF and by Q the intersection of MC and BE. Prove that EF bisects the segment PQ.
Let H denote the orthocentre of ΔABC and let EF meet PQ at U. Then, (FEU) is a tranversal of the triangle PHQ and from Menelaus's theorem, $\frac{EQ}{EH} \cdot \frac{FH}{FP} \cdot \frac{UP}{UQ} = 1$. Thus, it suffices to prove $\frac{EH}{FH} = \frac{EQ}{FP}$.
Denote by $\angle(\ell, \ell')$ the directed angle between lines ℓ, ℓ' (measured modulo π).
Since A, F, H, E are concyclic (on the circle with diameter AH), we have

$$\angle(AB, AC) = \angle(AF, AE) = \angle(HF, HE)$$



and so, using the concyclicity of A, B, C, M,

$$\angle(HP, HQ) = \angle(HF, HE) = \angle(AB, AC) = \angle(MB, MC) = \angle(MP, MQ).$$

It follows that H, P, M, Q are concyclic and

$$\angle(QE, QC) = \angle(QH, QM) = \angle(PH, PM) = \angle(PF, PB).$$

As a result, the right-angled triangles QEC and PFB are similar and $\frac{EQ}{FP} = \frac{EC}{FB}$. On the other hand, the right-angled triangles HEC and HFB are also similar, hence $\frac{EH}{FH} = \frac{EC}{FB}$. Finally, $\frac{EH}{FH} = \frac{EQ}{FP}$ $(= \frac{EC}{FB})$ and we are done.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Oles Dobosevych, Ukraine; Mihai Miculita, Oradea, Romania; Roberto Bosch Cabrera, Havana, Cuba.

S93. Let n be an integer greater than 1 and let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers whose sum is $\sqrt{2}$. Determine the maximum, as a function of n, of

$$\frac{x_1^2}{1+x_1^2} + \frac{x_2^2}{1+x_2^2} + \cdots + \frac{x_n^2}{1+x_n^2}.$$

Proposed by Alex Anderson, Washington University in St. Louis, USA

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

For $n = 2$, assume that we may obtain a maximum value not less than $\frac{2}{3}$. Then, for some x_1, x_2, it must hold

$$\frac{x_1^2}{1+x_1^2} + \frac{x_2^2}{1+x_2^2} \geq \frac{2}{3}; \qquad 0 \leq x_1^2 + x_2^2 + 4x_1^2x_2^2 - 2 = 2x_1x_2(2x_1x_2 - 1).$$

But using the AM-GM inequality, $2x_1x_2 \leq \frac{(x_1+x_2)^2}{2} = 1$. The maximum for $n = 2$ is then $\frac{2}{3}$, obtained when either $x_1x_2 = 0$, or when $x_1 = x_2 = \frac{1}{\sqrt{2}}$.

For $n = 3$, assume again that we may obtain a maximum value not less than $\frac{2}{3}$. Since $x_1 + x_2 = \sqrt{2} - x_3$, calling $p = x_1x_2$, we may write that, for some combination of x_3 and p, it must hold

$$\frac{x_1^2}{1+x_1^2} + \frac{x_2^2}{1+x_2^2} + \frac{x_3^2}{1+x_3^2} \geq \frac{2}{3};$$

$$\begin{aligned} 0 &\leq x_1^2 + x_2^2 + x_3^2 + 4(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2) + 7x_1^2x_2^2x_3^2 - 2 = \\ &= 4x_3^4 - 8\sqrt{2}x_3^3 + 10x_3^2 - 2\sqrt{2}x_3 + p(2p-1) + px_3^2(7p-8). \end{aligned}$$

Note that, calling $y = \sqrt{2}x_3$, the first four terms in the RHS may be written as $y^4 - 4y^3 + 5y^2 - 2y = y(y-1)^2(y-2)$. Now, clearly $0 \leq \sqrt{2}x_3 = y \leq 2$, or this sum is always non-positive, reaching a maximum value of 0 when $x_3 = \sqrt{2}$ or when $x_3 = 0$. As in the previous case, we obtain $2p \leq 1$, which results in $p(2p-1)$ being also non-positive, reaching a maximum value of 0 when $x_1x_2 = 0$ or when $x_1 = x_2 = \frac{1}{\sqrt{2}}$. Finally, $7p - 8 < 0$, which means that $px_3^2(7p-8)$ is also always non-positive, reaching a maximum value of 0 when $x_3 = 0$ or when $x_1x_2 = 0$. In conclusion, the maximum may never be larger than $\frac{2}{3}$ for $n = 3$, with equality iff either two of the x_i are zero, or if one of the x_i is zero and the other two are equal.

We complete the proof by stating and demonstrating the following

Claim: The maximum is independent of n for $n \geq 3$.

Proof: Define $f(x) = \frac{x^2}{1+x^2}$. Clearly $f'(x) = \frac{2x}{(1+x^2)^2}$ and $f''(x) = 2\frac{1-3x^2}{(1+x^2)^3}$. Note that $f''(x) \geq 0$ for $x \geq \frac{1}{\sqrt{3}}$, or if $n \geq 3$, and wlog $x_1 \geq x_2 \geq \cdots \geq x_n$, then $x_m \leq \frac{\sqrt{2}}{3} < \frac{1}{\sqrt{3}}$ for all $m \geq 3$, and since $f(x)$ is convex between 0 and $\frac{1}{\sqrt{3}}$, then the maximum of the sum of all but the first two terms is reached when all are zero except for one of them, ie, when $x_m = 0$ for $m \geq 4$. The claim clearly follows.

As a consequence of the claim and the previously demonstrated results, the maximum is always $\frac{2}{3}$ for all $n > 1$. This maximum may only be obtained either when $n-1$ of the x_i are zero, or when $n-2$ of the x_i are zero, the other two being equal to $\frac{1}{\sqrt{2}}$.

Second solution by Oles Dobosevych, Ukraine

Let us prove that if $a \geq 0$, $b \geq 0$, $a + b \leq \sqrt{2}$ then

$$\frac{a^2}{1+a^2} + \frac{b^2}{1+b^2} \leq \frac{(a+b)^2}{1+(a+b)^2}.$$

The last inequality is equivalent to

$$\frac{a^2+b^2+2a^2b^2}{1+a^2+b^2+a^2b^2} \leq \frac{(a^2+b^2+2ab)}{1+a^2+b^2+2ab}$$

which after some algebra transforms into

$$2ab + ab(a+b)^2 \leq 2.$$

The previous inequality is true since $2\sqrt{ab} \leq a + b \leq \sqrt{2}$ and $ab \leq \frac{1}{2}$.

Using this observation $n-1$ times we obtain that

$\frac{x_1^2}{1+x_1^2} + \frac{x_2^2}{1+x_2^2} + \ldots + \frac{x_n^2}{1+x_n^2} \leq \frac{(x_1+x_2+\ldots+x_n)^2}{1+(x_1+x_2+\ldots+x_n)^2} \leq \frac{2}{3}$.

The inequality holds when $x_1 = \sqrt{2}$ and $x_2 = x_3 = \ldots = x_n = 0$ then

$\frac{x_1^2}{1+x_1^2} + \frac{x_2^2}{1+x_2^2} + \ldots + \frac{x_n^2}{1+x_n^2} = \frac{2}{3}$.

So the maximum value is $\frac{2}{3}$.

S94. Consider a quadrilateral that is incribed in a circle and circumscribed about a circle. Prove that the product of its diagonals is a constant.

Proposed by Ivan Borsenco, MIT, USA

First solution by Ivanov Andrei, Moldova

Let $ABCD$ be a bicentric quadrilateral and let O and I be the circumcenter and incenter, respectively. Then we have

$$P \in OI \quad \text{and} \quad OP = \frac{2R^2 \cdot OI}{R^2 + d^2}.$$

These two relations imply that the point P is fixed. We have $AC \cdot BD = \frac{2S}{\sin \alpha}$, where α is the angle between diagonals. On the other side we have $S = \sqrt{abcd}$ and thus

$$\begin{aligned} AC \cdot BD = \frac{2\sqrt{abcd}}{\sin \alpha} &= 2\sqrt{\frac{a}{\sin \alpha} \cdot \frac{b}{\sin \alpha} \cdot cd} \\ &= 2\sqrt{\frac{d}{\sin \angle ABP} \cdot \frac{c}{\sin \angle DBC} \cdot AP \cdot CP} \\ &= 4R\sqrt{AP \cdot CP}. \end{aligned}$$

But the value $4R\sqrt{AP \cdot CP}$ is constant because both $4R$ and $AP \cdot CP$ is constant(power of the point P which is fixed).

Second solution by Michel Bataille, France

We assume that the quadrilateral $ABCD$ is convex and denote by r and R the inradius and circumradius of $ABCD$, respectively. Let $AB = a$, $BC = b$, $CA = c$, $DA = d$ and $s = \frac{a+b+c+d}{2}$. Since $ABCD$ circumscribes a circle, we have $a + c = b + d = s \qquad (1)$.
Since $ABCD$ is inscribed in a circle, the product p of the diagonals is $p = AC \times BD = ac + bd$ (by Ptolemy's theorem) and the area F of $ABCD$ is given by Brahmagupta's formula:

$$F = \sqrt{(s-a)(s-b)(s-c)(s-d)} \qquad (2).$$

Note that (1) and (2) give $F = \sqrt{abcd}$ and that $F = rs = r(a + c) = r(b + d)$.
Now, we prove the following formula: $16R^2F^2 = p(ab + cd)(ad + bc) \qquad (*)$.
Let $A = \angle BAD$. Then $4R^2 = \frac{BD^2}{\sin^2 A}$ and $BD^2 = a^2 + d^2 - 2ad \cos A = b^2 + c^2 +$

$2bc\cos A$, hence $\cos A = \frac{a^2+d^2-b^2-c^2}{2(ad+bc)}$ and $\frac{BD^2-b^2-c^2}{bc} = \frac{a^2+d^2-BD^2}{ad}$. We readily deduce $BD^2 = \frac{(ab+cd)(ac+bd)}{ad+bc}$ and

$$\begin{aligned}
\sin^2 A = 1 - \cos^2 A &= \frac{(2ad+2bc)^2 - (a^2+d^2-b^2-c^2)^2}{4(ad+bc)^2} \\
&= \frac{((a+d)^2-(b-c)^2)((b+c)^2-(a-d)^2)}{4(ad+bc)^2} \\
&= \frac{16(s-a)(s-b)(s-c)(s-d)}{4(ad+bc)^2} = \frac{4F^2}{(ad+bc)^2}.
\end{aligned}$$

Thus, $4R^2 = \frac{(ab+cd)(ac+bd)}{ad+bc} \cdot \frac{(ad+bc)^2}{4F^2}$ and $(*)$ follows.
Now, from $(*)$, we deduce

$$16r^2s^2R^2 = p((b+d)^2ac - 2abcd + (a+c)^2bd - 2abcd) = s^2p^2 - 4pF^2$$

or, after division by s^2,

$$p^2 - 4pr^2 - 16r^2R^2 = 0.$$

Solving for p leads to the constant

$$p = 2r^2 + 2r\sqrt{r^2+4R^2}.$$

Third solution by Oleh Faynshteyn, Leipzig, Germany

Let $ABCD$ be a quadrilateral that is inscribed in a circle and circumscribed about a cirle, then we have

$$AB = CD, BC = DA. \quad (1)$$

According to the Ptolemey theorem, we have

$$AB \cdot CD + BC \cdot DA = AC \cdot BD$$

and using (1) we get that

$$AB^2 + BC^2 = AC \cdot BD$$

which is constant.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Vicente Vicario Garcia, Huelva, Spain; Oles Dobosevych, Ukraine.

S95. There are 16 bad boys in a neghborhood. During one year 69 quarrels have occured among the members of the neighborhood (each quarrel involves exactly two persons). At the end of the year a local wrestling club wants to organize a match between two teams of three people. The boys will fight each other if and only if each member of one team has quarreled with each member of the other team. Prove that the club can always organize such a fight.

Proposed by Iurie Boreico, Harvard University and Ivan Borsenco, MIT, USA

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

For each trio of bad boys, call m_j the number of bad boys that have quarreled with the three of them, where j is some index that numbers the $\binom{16}{3}$ possible trios of bad boys. The problem is then equivalent to showing that there is some m_j that is at least 3. The sum of all the m_j's is obviously equivalent to adding, for each bad boy, the number of trios that may be made out of the bad boys with which he has quarreled. In other words, if each bad boy has been involved in n_i quarrels ($i = 1, 2, \dots, 16$), then the sum of the m_j's equals

$$\sum_{i=1}^{16} \binom{n_i}{3}.$$

Note that

$$\binom{n+1}{3} + \binom{n-1}{3} = \frac{(2n^2 - 4n + 6)(n-1)!}{3!(n-2)!} > \frac{2n(n-2)(n-1)!}{3!(n-2)!} = 2\binom{n}{3},$$

$$\binom{n+2}{3} + \binom{n-1}{3} = \frac{2n^3 - 3n^2 + 13n - 6}{2n^3 - 3n^2 + n}\left(\binom{n+1}{3} + \binom{n}{3}\right) > \binom{n+1}{3} + \binom{n}{3}.$$

Analogously we may show that this result still holds when the difference in the numbers of elements out of which are 3 are chosen is increased. The minimum value in the sum of the m_j's is then achieved when all the n_i differ by at most 1, ie, when 10 of them are equal to 9, and 6 of them are equal to 8, for a total of $10 \cdot 9 + 6 \cdot 8 = 138 = 2 \cdot 69$ quarrels (each one counted twice because two bad boys take part in it). The sum of the m_j's is then at least

$$10\binom{9}{3} + 6\binom{8}{3} = 840 + 336 = 1176 = \sum_j m_j > 1120 = 2\binom{16}{3}.$$

Therefore, some m_j necessarily exceeds 2, and the fight is always possible.

S96. Let n be an integer greater than 2. Prove that $\binom{n-1}{k} \equiv (-1)^k \pmod{n}$ for each $k = 1, 2, \ldots, n-1$, if and only if n is a prime.

Proposed by Dorin Andrica, "Babes-Bolyai" University, and Mihai Piticari, "Dragos Voda" National College, Romania

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Clearly,

$$\binom{n-1}{k} = \frac{(n-1)\cdot(n-2)\cdot\ldots\cdot(n-k)}{k!}.$$

If n is prime, for any $k < n$, then $k!$ is prime with n, and $\binom{n-1}{k} \equiv (-1)^k \pmod{n}$ is equivalent to

$$(n-1)\cdot(n-2)\cdot\ldots\cdot(n-k) \equiv (-1)^k k! \pmod{n}.$$

This is clearly true since $n-j \equiv (-j) \pmod{n}$ for each $j \in \{1, 2, \ldots, n-1\}$.

If n is not prime, let k be the least divider of n which is larger than 1. Then, since $(k-1)!$ is prime with n, and $n-j \equiv (-j) \pmod{n}$ for all $j \in \{1, 2, \ldots, k-1\}$, the result $\binom{n-1}{k} \equiv (-1)^k \pmod{n}$ is equivalent to

$$\frac{n-k}{k} \equiv -1 \pmod{n},$$

or equivalently, $\frac{n}{k} \equiv 0 \pmod{n}$, absurd. This concludes the proof.

Second solution by Michel Bataille, France

Let n be an integer greater than 2. Prove that $\binom{n-1}{k} \equiv (-1)^k \pmod{n}$ for each $k = 1, 2, \ldots, n-1$, if and only if n is a prime.
First, suppose that $\binom{n-1}{k} \equiv (-1)^k \pmod{n}$ for each $k = 1, 2, \ldots, n-1$ and for the purpose of a contradiction, assume that n is not a prime. Let p be the smallest prime dividing n and let m be the integer such that $n = pm$. Note that $n \geq 4$ and $m > 1$. Also $p \in \{2, 3, \ldots, n-1\}$ (since $p \leq \sqrt{n} < n$) and so $\binom{n-1}{p} \equiv (-1)^p \pmod{n}$.
On the other hand,

$$\begin{aligned}
(p-1)!\binom{n-1}{p} &= (n-1)(n-2)\cdots(n-p+1)(m-1) \\
&\equiv (-1)(-2)\cdots(-(p-1))(m-1) \pmod{n} \\
&\equiv (-1)^{p-1}(p-1)!(m-1) \pmod{n}
\end{aligned}$$

which yields

$$\binom{n-1}{p} \equiv (-1)^{p-1}(m-1) \pmod{n},$$

because $(p-1)!$ is coprime to n (any prime factor of $(p-1)!$ is less than p and so cannot divide n). As a result, $m-1 \equiv -1 \pmod{n}$ that is, $m \equiv 0 \pmod{n}$, a contradiction since $1 < m < n$.

Conversely, suppose that n is a prime. If $f(x) = \sum_{k\geq 0} a_k x^k$ and $g(x) = \sum_{k\geq 0} b_k x^k$ are (formal) series with coefficients in $\mathbb{Z}$, let $f(x) \equiv g(x) \pmod{n}$ mean that $a_k \equiv b_k \pmod{n}$ for $k = 0, 1, 2, \ldots$. For example, $(1+x)^n \equiv 1 + x^n \pmod{n}$ (a well-known result). From the equality

$$(1+x)^{n-1} = (1+x)^n \cdot \frac{1}{1+x}$$

we obtain

$$\sum_{k=0}^{n-1} \binom{n-1}{k} x^k \equiv (1+x^n) \sum_{k\geq 0} (-x)^k \equiv \sum_{k\geq 0} (-1)^k x^k + \sum_{k\geq 0} (-1)^k x^{n+k} \pmod{n}.$$

Comparing the coefficients of x^k for $k = 1, 2, \ldots, n-1$, we see that $\binom{n-1}{k} \equiv (-1)^k \pmod{n}$, as desired.

Note that the same method can easily lead to the following general result: if p is a prime and $r \in \{1, 2, \ldots p\}$, then for $k = 0, 1, \ldots, p-r$,

$$\binom{p-r}{k} \equiv (-1)^k \binom{r-1+k}{r-1} \pmod{p}.$$

Third solution by John T. Robinson, Yorktown Heights, NY, USA Note - we can actually start with $n = 2$ since $-1 = 1 \pmod 2$; also we can let k range from 0 to $n-1$.

For $n \geq 2$, since $\binom{n-1}{0} = 1$ and $\binom{n-1}{1} = n-1$, clearly we have $\binom{n-1}{k} = (-1)^k \pmod{n}$ for $k = 0, 1$. Next suppose $\binom{n-1}{k} = (-1)^k \pmod{n}$ and consider $\binom{n-1}{k+1}$. Using

$$(k+1) \cdot \binom{n-1}{k+1} = \binom{n-1}{k} \cdot (n-k-1),$$

we see that

$$(k+1) \cdot \binom{n-1}{k+1} = ((-1)^{k+1}) \cdot (k+1) \pmod{n}.$$

Since the non-zero integers under multiplication modulo n form a group if n is prime, this has a unique solution for $\binom{n-1}{k+1}$ in the case of prime n, namely $\binom{n-1}{k+1} = (-1)^{k+1}$. This establishes the result by induction on k for any prime n.

Next, if $n = 4$ we note that $\binom{3}{2} = 3 = -1 \pmod 4$ (i.e. not $(-1)^2$), establishing the result for this special case. Finally suppose n is composite with $n \geq 6$. Then n can be factored as $(k+1) \cdot m$ for some $k \geq 1$ and $m > 2$. Next suppose $\binom{n-1}{k} = (-1)^k \pmod n$ (note that if the case is otherwise then the result is already established for this value of n and k), and again consider $\binom{n-1}{k+1}$. Using the previous relationship, but writing $n-k-1$ as $(m-1) \cdot (k+1)$ and then dividing both sides by $k+1$, we have

$$\binom{n-1}{k+1} = \binom{n-1}{k} \cdot (m-1).$$

Since by assumption $\binom{n-1}{k} = (-1)^k \pmod n$, we now have

$$\binom{n-1}{k+1} = ((-1)^k) \cdot (m-1) \pmod n.$$

However since $1 < (m-1) < (n-1)$ (because $m > 2$ and $m \leq \frac{n}{2}$), we cannot get $\binom{n-1}{k+1} = (-1)^{k+1} \pmod n$ by multiplying $(-1)^k$ by $(m-1)$, thus establishing the result for all composite n.

Also solved by Vicente Vicario Garcia, Huelva, Spain

Undergraduate problems

U91. Prove that there are no polynomials $P, Q \in \mathbb{R}[x]$ such that

$$\int_0^{\log n} \frac{P(x)}{Q(x)} dx = \frac{n}{\pi(n)},$$

for all $n \geq 1$, where $\pi(n)$ is the prime counting function.

Proposed by Cezar Lupu, University of Bucharest, Romania

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Assume that such polynomials $P(x)$ of degree M and $Q(x)$ of degree N exist, and call

$$f(y) = \int_0^y \frac{P(x)}{Q(x)} dx.$$

Clearly, $f(\log n) = \frac{n}{\pi(n)}$ for any positive integer n. We prove first the following

Claim: The set of positive real numbers is the disjoint union of a finite number of intervals where $f'(y)$ is positive, a finite number of intervals where $f'(y)$ is negative, a finite set of isolated points where $f'(y) = 0$, and a finite set of isolated points where $f'(y)$ cannot be defined.

Proof: The indefinite integral of $\frac{P(x)}{Q(x)}$ exists, and it may be expressed as the sum of a rational function (possibly constant or zero) and a certain number (possibly none) of functions of the form $\arctan(ay)$ and $\log |by + c|$, for appropriate real constants a, b, c. This indefinite integral exists and is continuous and differentiable at least once, at all points except at the zeros of $Q(y)$, where it cannot be defined. For every other positive real y, the derivative $f'(y)$ trivially coincides with $\frac{P(y)}{Q(y)}$. Since $\frac{P(y)}{Q(y)}$ has a finite number of zeros (at most M), and a finite number of discontinuities (at most N), and outside of these zeros and discontinuities $f'(y) = \frac{P(y)}{Q(y)}$ is continuous and does not change signs, the claim follows.

Define now, for each positive integer n, the function $\Delta(n) = \frac{n+1}{\pi(n+1)} - \frac{n}{\pi(n)} = f(\log(n+1)) - f(\log(n))$. Using the claim, we trivially deduce that there must be a finite number of integers m for which $\Delta(m)$ and $\Delta(m+1)$ have opposite signs, since each time that this happens, either there is a discontinuity or a sign change in $f'(y)$, in the interval $(\log(m), \log(m+2))$. But this is not true, since for any prime $p > 2$, it holds $\pi(p+1) = \pi(p) = \pi(p-1) + 1$, and

$$\Delta(p) = \frac{p+1}{\pi(p+1)} - \frac{p}{\pi(p)} = \frac{1}{\pi(p)} > 0,$$

$$\Delta(p-1) = \frac{p}{\pi(p)} - \frac{p-1}{\pi(p-1)} = \frac{1-p}{\pi(p)(\pi(p)-1)} < 0.$$

Contradiction follows, hence no such polynomials $P(x)$ and $Q(x)$ may exist, qed.

U92. Find the maximum value of $F(\mathbf{x},\mathbf{y},\mathbf{z})=\min\left\{\frac{\|\mathbf{y}-\mathbf{z}\|}{\|\mathbf{x}\|},\frac{\|\mathbf{z}-\mathbf{x}\|}{\|\mathbf{y}\|},\frac{\|\mathbf{x}-\mathbf{y}\|}{\|\mathbf{z}\|}\right\}$

Proposed by Arkady Alt, San Jose, California, USA

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Consider non-negative angles α,β,γ such that $\alpha+\beta+\gamma=\pi$, and assume that γ is fixed. Then,

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}+\cos\frac{\alpha-\beta}{2}=2\sin\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2},$$

and if $\gamma\neq 0$, the sine of $\frac{\gamma}{2}$ is positive, and the sum $\cos\alpha+\cos\beta+\cos\gamma$ is maximum when $\alpha=\beta$. Proceeding similarly by assuming β fixed, the sum is maximum when $\gamma=\alpha$. Hence $\cos\alpha+\cos\beta+\cos\gamma\leq 3\cos\frac{\pi}{3}=\frac{3}{2}$. Note that either $\gamma=0$ or $\beta=0$ yield a lower value 1 for the sum, since the two nonzero angles would have cosines with sum zero.

Consider now non-negative angles A,B,C such that $A+B+C=2\pi$, and assume wlog $A\geq B\geq C$, and C fixed. Clearly, $\cos A+\cos B=-2\cos\frac{C}{2}\cos\frac{A-B}{2}$, where $\frac{C}{2}\leq\frac{\pi}{3}$ or the sum $\cos A+\cos B+\cos C$ is minimum when $A=B$. Similarly, $\frac{B}{2}\leq\frac{\pi}{2}$, leading to a minimum sum for given B when $A=C$, or $\cos A+\cos B+\cos C\geq 3\cos\frac{2\pi}{3}=-\frac{3}{2}$. Note that the case $A=B=\pi$ (for $\cos\frac{B}{2}=0$) yields a higher value -1 for the sum.

Call now $x=||\mathbf{x}||$, $y=||\mathbf{x}||$, $z=||\mathbf{x}||$, and call A,B,C the angles non-negative angles respectively between vectors $\mathbf{x},\mathbf{y}$, vectors $\mathbf{y},\mathbf{z}$, and vectors $\mathbf{z},\mathbf{x}$, define

$$F_x=\frac{||\mathbf{y}-\mathbf{z}||}{||\mathbf{x}||}=\frac{\sqrt{y^2+z^2-2yz\cos A}}{x},$$

and define cyclically F_y and F_z. We continue the solution by stating the following

Claim: The maximum occurs when the vectors $\mathbf{x},\mathbf{y},\mathbf{z}$ are coplanar and simultaneously $F_x=F_y=F_z$.

Proof: Clearly $A+B+C\leq 2\pi$, with equality only if vectors $\mathbf{x},\mathbf{y},\mathbf{z}$ are coplanar. Assume wlog $A\geq B\geq C$. If $A\geq\pi$, then we may substitute A by $2\pi-A$, leaving F_x unchanged, and having the possibility of increasing B and C, thus increasing F_y and F_z, hence the maximum happens when A,B,C do not exceed π. We may then increase the value of F by increasing the angle or angles appearing in the expression for the minimum or minima of F_x,F_y,F_z, at the expense of the rest of angles. The maximum cannot hence happen unless $A+B+C=2\pi$ (otherwise the angles could still be increased, and hence the value of F), and $F_x=F_y=F_z$.

Consider now a point O in the plane, and points X,Y,Z such that $\vec{OX}=\mathbf{x}$, $\vec{OY}=\mathbf{y}$, $\vec{OZ}=\mathbf{z}$. Then, $YZ=||\mathbf{y}-\mathbf{z}||$, $ZX=||\mathbf{z}-\mathbf{x}||$, $XY=||\mathbf{x}-\mathbf{y}||$, or $\frac{XY}{OZ}=\frac{YZ}{OX}=\frac{ZX}{OY}$, and a triangle exists with sidelengths x,y,z, similar to triangle XYZ, and whose angles at the respective opposite vertices will be denoted α,β,γ. Then,

$$\frac{x^2(F_x^2-1)}{2yz}=\frac{y^2+z^2-x^2}{2yz}-\cos A=\cos\alpha-\cos A,$$

and similarly for its cyclic permutations. Hence,

$$\frac{3}{2}(F^2-1)=\frac{3}{2}\left((\min\{F_x,F_y,F_z\})^2-1\right)\leq 3\sqrt[3]{\frac{x^2(F_x^2-1)}{2yz}\frac{y^2(F_y^2-1)}{2zx}\frac{z^2(F_z^2-1)}{2xy}}\leq$$

$$\leq\cos\alpha+\cos\beta+\cos\gamma-(\cos A+\cos B+\cos C)\leq 3,$$

with equality iff $\alpha=\beta=\gamma=\frac{\pi}{3}$, $A=B=C=\frac{2\pi}{3}$, $F_x=F_y=F_z$, and $x=y=z$. Clearly $F\leq\sqrt{3}$ follows, with equality iff $\mathbf{x},\mathbf{y},\mathbf{z}$ are vectors of the same length at angles of $\frac{2\pi}{3}$ with each other. where $\mathbf{x},\mathbf{y},\mathbf{z}$ are arbitrary nonzero vectors in $\mathbb{R}^n, n\geq 2$.

U93. Let $x_0 \in (0,1]$ and $x_{n+1} = x_n - \arcsin(\sin^3 x_n)$, $n \geq 0$. Evaluate $\lim_{n\to\infty} \sqrt{n}x_n$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Arin Chaudhuri

We have $x_{n+1} = f(x_n)$ where $f(x) = x - \arcsin(\sin^3 x)$.
Now, if $x \in (0, \pi/2)$ then clearly, $0 < \sin x < 1$ and hence, $0 < \sin^3 x < \sin x < 1$. Noting that arcsin is a strictly increasing function, we have, $0 < \arcsin(\sin^3 x) < \arcsin(\sin x)$. Now since $\arcsin(\sin x) = x$ for $x \in (-\pi/2, \pi/2)$, we have, $0 < \arcsin(\sin^3 x) < x$, or equivalently, $-x < -\arcsin(\sin^3 x) < 0$ or $0 < x - \arcsin(\sin^3 x) < x$, i.e., $0 < f(x) < x$, for $x \in (0, \pi/2)$.

So if, $x_n \in (0, \pi/2)$ for some n, then we have $0 < f(x_n) < x_n$, i.e., $0 < x_{n+1} < x_n$ and hence $x_{n+1} \in (0, \pi/2)$. Since $x_0 \in (0,1] \subset (0, \pi/2)$, from induction we have $x_n \in (0, \pi/2)$ for all n and it also follows that $0 < x_{n+1} = f(x_n) < x_n$. So x_n is a strictly decreasing sequence, bounded below by 0, so its limit exists, call it α. From the continuity of f, we must have $\alpha = f(\alpha)$ and hence $\alpha = \alpha - \arcsin(\sin^3 \alpha)$, i.e., $\sin^3 \alpha = 0$, hence $\sin \alpha = 0$. Also, since $0 \leq x_n \leq x_0 \leq 1$ we have $0 \leq \alpha \leq 1$. Since 0 is the only zero of sin in $[0,1]$ we have $\alpha = \lim_{n\to\infty} x_n = 0$.

Now, we have, from definition of x_n, $\sin(x_n - x_{n+1}) = \sin^3(x_n)$. Now note both $x_n - x_{n+1}$ and x_n are positive (hence non-zero) and tend to 0 and hence,

$$\lim_{n\to\infty} \sin(x_n - x_{n+1})/(x_n - x_{n+1}) = 1$$

and

$$\lim_{n\to\infty} \sin^3(x_n)/x_n^3 = \lim_{n\to\infty} (\sin(x_n)/x_n)^3 = 1.$$

Define d_n as

$$d_n = \frac{x_n - x_{n+1}}{x_n^3} = \frac{\sin^3(x_n)/x_n^3}{\sin(x_n - x_{n+1})/(x_n - x_{n+1})}$$

We have $\lim_{n\to\infty} d_n = 1$. So we can write,

$$x_{n+1} = x_n - d_n x_n^3 \tag{1}$$

where $\lim_{n\to\infty} d_n = 1$.

From equation 1 it also follows that $x_{n+1}/x_n = 1 - d_n x_n^2 \to 1$ as $n \to \infty$.

Squaring equation 1 we get

$$x_{n+1}^2 = x_n^2 + d_n^2 x_n^6 - 2x_n^4 d_n$$

Hence,

$$x_n^2 - x_{n+1}^2 = 2x_n^4 d_n - d_n^2 x_n^6$$

or

$$(x_n^2 - x_{n+1}^2)/(x_{n+1}^2 x_n^2) = (2x_n^4 d_n)/(x_n^2 x_{n+1}^2) - (d_n^2 x_n^6)/(x_n^2 x_{n+1}^2)$$

Now note the first term on the right can be written as

$$2d_n/(x_{n+1}/x_n)^2$$

which converges to 2 as $n \to \infty$, and the second term on the right can be written as

$$d_n x_n^2/(x_{n+1}/x_n)^2$$

which converges to 0 as $n \to \infty$.

Hence,

$$\lim_{n\to\infty}(\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2}) = \lim_{n\to\infty}(x_n^2 - x_{n+1}^2)/(x_{n+1}^2 x_n^2) = 2$$

Define w_n as

$$w_n = \frac{1}{x_{n+1}^2} - \frac{1}{x_n^2}.$$

Since $\lim\limits_{n\to\infty} w_n = 2$, the averages converge to the same limit, and hence,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} w_k = 2.$$

But,

$$\frac{1}{n}\sum_{k=0}^{n-1} w_k = \frac{1}{nx_n^2} - \frac{1}{nx_0^2}.$$

So it follows that

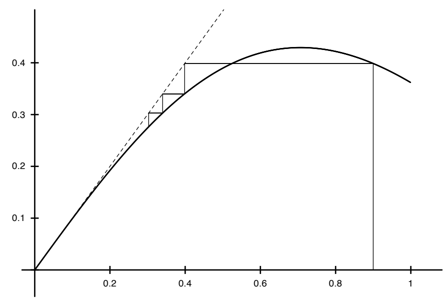
$$\lim_{n\to\infty}\frac{1}{nx_n^2} = \lim_{n\to\infty}(\frac{1}{n}\sum_{k=0}^{n-1} w_k + \frac{1}{nx_0^2}) = 2.$$

Hence,

$$\lim_{n\to\infty}\sqrt{n}x_n = \lim_{n\to\infty}\frac{1}{\sqrt{1/nx_n^2}} = \frac{1}{\sqrt{2}}.$$

Second solution by Brian Bradie, VA, USA

The cobweb diagram shown below illustrates that $x_n \to 0$ for any $x_0 \in (0,1]$.



Now, let $y_n = 2x_n^2$. Then, $y_n > 0$ for all n, $y_n \to 0$ and

$$y_{n+1} = y_n - y_n^2 + O(y_n^3).$$

According to formula (8.5.3) in [1, page 155], it follows that

$$y_n = \frac{1}{n} + O(n^{-2} \ln n)$$

as $n \to \infty$. Therefore,

$$x_n = \frac{1}{\sqrt{2n}} + O(n^{-3/2} \ln n)$$

as $n \to \infty$, and

$$\lim_{n\to\infty} \sqrt{n}\, x_n = \frac{1}{\sqrt{2}}.$$

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Clearly

$$sin(x_n^3) = \int_0^{x_n} 3x^2 \cos(x^3) dx > \int_0^{x_n} 3 \sin^2 x \cos x dx = \sin^3 x_n,$$

or $\arcsin(\sin^3 x_n) < x_n^3$. Since the Taylor series for the sine function when the angle is less than 1 is made of terms with alternating signs and decreasing absolute value, we also know that

$$\sin^3 x_n > \left(x_n - \frac{x_n^3}{3!} \right)^3 > x_n^3 - \frac{x_n^5}{2}.$$

Since furthermore the arc is always greater than the sine for positive angles, we conclude that $\arcsin(\sin^3 x_n) > x_n^3 - \frac{x_n^5}{2}$. It follows that

$$x_n^2 - 2x_n^4 + 2x_n^6 - x_n^8 + \frac{x_n^{10}}{4} > x_{n+1}^2 > x_n^2 - 2x_n^4 + x_n^6.$$

Define therefore sequences $\{y_n\}$ and $\{z_n\}$ by $y_0 = z_0 = x_0^2$, $y_{n+1} = y_n - 2y_n^2 + y_n^3$ and $z_{n+1} = z_n - 2z_n^2 + 2z_n^3 - z_n^4 + \frac{z_n^5}{4}$. Clearly, for all n we have $y_n < x_n^2 < z_n$, while the three sequences are positive, monotonic (strictly decreasing) and bounded, so the three sequences have a non-negative limit that does not exceed 1. In the case of y_n, this limit ℓ clearly satisfies $\ell = \ell - 2\ell^2 + \ell^3$, or $\ell = 0$ since $\ell \le 1$. Similarly, in the case of z_n, the limit ℓ' satisfies $\ell'^2(2-\ell')(4-2\ell'+\ell'^2) = 0$, and $\ell' = 0$. Clearly the limit of x_n is also 0.

Assume now that $y_n < \frac{1}{2n}$. Then,

$$y_{n+1} < \frac{4n^2 - 4n + 1}{8n^3} < \frac{1}{2n+2},$$

where the last inequality may be proved by multiplying both sides by $8n^3(2n+2)$. Assume finally that $z_n < \frac{1}{2n}$. Then,

$$z_{n+1} < \frac{64n^4 - 64n^3 + 32n^2 - 8n + 1}{128n^5} < \frac{1}{2n+2},$$

where the last inequality may be proved by multiplying both sides by $128n^5(2n+2)$. Therefore, the sequences ny_n and nz_n have upper bounds. Furthermore,

$$(n+1)y_{n+1} - ny_n > y_{n+1} - ny_n^2 > y_{n+1} - \frac{y_n}{2} > 0,$$

where the last inequality is true since, if $2y_{n+1} < y_n$, then $y_n > 1 - \frac{1}{\sqrt{2}}$, which is incompatible with $y_n < \frac{1}{2n}$ for $n \ge 2$. Similarly, $(n+1)z_{n+1} - nz_n > 0$. Therefore, sequences $2ny_n$ and $2nz_n$ are positive, monotonic (strictly increasing) and bounded, hence they have limits L and L'. It follows that

$$\lim_{n\to\infty} \frac{y_{n+1} - y_n}{\frac{1}{2n+2} - \frac{1}{2n}} = \lim_{n\to\infty} n(2n+2)\left(2y_n^2 - y_n^3\right) = \left(\lim_{n\to\infty}(2ny_n)\right)^2 = L^2$$

$$\lim_{n\to\infty} \frac{z_{n+1} - z_n}{\frac{1}{2n+2} - \frac{1}{2n}} = \lim_{n\to\infty} n(2n+2)\left(2z_n^2 - 2z_n^3 + z_n^4 - \frac{z_n^5}{4}\right) = \left(\lim_{n\to\infty}(2nz_n)\right)^2 = L'^2.$$

But by the Cesaro-Stolz theorem, these limits are respectively equal to L and L', or since L, L' must be positive, then $L = L' = 1$, and

$$\lim_{n\to\infty} \sqrt{n}x_n = \frac{1}{\sqrt{2}}\sqrt{\lim_{n\to\infty} 2ny_n} = \frac{1}{\sqrt{2}}\sqrt{\lim_{n\to\infty} 2nz_n} = \frac{1}{\sqrt{2}}.$$

Remark: since y_n and z_n have limit 0, we may always choose N large enough, but finite, such that $y_{N+1}, z_{N+1} < \frac{1}{2}$. It suffices then to define $y'_n = y_{n+N}$ and $z'_n = z_{n+N}$ for $n \ge 0$, and obviously the assumptions are then true, leading to

$$1 = \lim_{n\to\infty}(2ny'_n) = \lim_{n\to\infty} \frac{2n}{2n+N}(2ny_n) = \lim_{n\to\infty}(2ny_n),$$

and similarly for z_n and z'_n.

Fourth solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

The answer is the content of the exercise num.174 at page 38 of the book by G. Pólya, G.Szegö, *Problems and Theorems in Analysis, I.*

The exercise is:

Assume that $0 < f(x) < x$ *and* $f(x) = x - ax^k + bx^l + x^l\varepsilon(x)$, $\lim_{x\to 0}\varepsilon(x) = 0$, *for* $0 < x < x_0$ *where* $1 < k < l$ *and* a, b *both positive. The sequence* x_n *defined by* $x_{n+1} = f(x_n)$ *satisfies* $\lim_{n\to\infty} n^{1/(k-1)}x_n = (a(k-1))^{-1/(k-1)}$

We check the hypotheses of the exercise.

$f(x) > 0$ amounts to observe that $\sin x > \sin^3 x > 0$ for $0 < x \leq 1$. $f(x) < x$ is $\arcsin(\sin^3 x) > 0$. Since $\arcsin(\sin^3 x) = x^3 - \frac{x^5}{2} + O(x^7)$, the condition $f(x) = x - ax^k + bx^l + x^l\varepsilon(x)$, $\lim_{x\to 0}\varepsilon(x) = 0$, where $1 < k < l$ is satisfied too with $a = 1$ and $b = 1/2$. The required limit easily follows by the conclusion of the exercise.

U94. Let Δ be the plane domain consisting of all interior and boundary points of a rectangle $ABCD$, whose sides have lengths a and b. Define $f : \Delta \to R$, $f(P) = PA + PB + PC + PD$. Find the range of f.

Proposed by Mircea Becheanu, University of Bucharest, Romania

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Clearly $PA + PC \geq AC$, with equality iff P is in segment AC, and similarly $PB + PD \geq BD$, with equality iff P is in segment BD. Therefore, $f(P) \geq AC + BD = 2\sqrt{a^2 + b^2}$, with equality iff P is the center of rectangle $ABCD$.

Assume that the maximum of f occurs for some point P in the interior of $ABCD$, and consider the ellipses, passing through P, E_1 with foci A and B, and E_2 with foci C and D. Both ellipses intersect at P inside $ABCD$. Consider one of the points Q where E_1 intersects the perimeter of $ABCD$. Clearly, Q is outside E_2, or $QC + QD > PC + PD$, while $QA + QB = PA + PB$ because P and Q belong to E_1. Since $Q \in \Delta$, then $f(Q) > f(P)$, and the maximum of f cannot occur in the interior of $ABCD$.

Wlog, P such that $f(P)$ is maximum, is on AB or on AD. In the first case, $PA + PB = AB$, while $PC + PD$ is maximum for the case of the largest ellipse with foci C and D that may be constructed with some intersection point in segment AB. Clearly, this happens when $P = A$ or $P = B$, for $PC + PD = AD + AC$. The result is the same in the second case by analogous reasoning. Restoring generality, the maximum of f is $a+b+\sqrt{a^2 + b^2}$, attained when P is one of the vertices of $ABCD$.

As P moves continuously from the center of $ABCD$ to one of its vertices, f varies continously, or the range of f is $[2\sqrt{a^2 + b^2}, a + b + \sqrt{a^2 + b^2}]$, where as stated above the maximum occurs at the vertices of $ABCD$, and the minimum occurs at its center.

Also solved by Arin Chaudhuri

U95. Find all monic polynomials P and Q, with real coefficients, such that

$$P(1) + P(2) + \cdots + P(n) = Q(1 + 2 + 3 + \cdots + n),$$

for all $n \geq 1$.

Proposed by Ovidiu Furdui, Cluj-Napoca, Romania

First solution by Arin Chaudhuri

We have the following choices for $P(x)$ and $Q(x)$

(a) $P(x) = Q(x) = x$

(b) $P(x) = x^3 + bx$ and $Q(x) = x^2 + bx$ for some real b.

Proof of the above statement follows:

Lemma 1: If $P(x)$ and $Q(x)$ satisfy the given condition we must have

$$P(x) = Q(\frac{x(x+1)}{2}) - Q(\frac{x(x-1)}{2})$$

.

Proof: Define,

$$R(x) = Q((x+1)(x+2)/2) - Q(x(x+1)/2) - R(x+1).$$

For any positive integer n we have

$$\begin{aligned} R(n) &= Q((n+1)(n+2)/2) - Q(n(n+1)/2) - P(n+1) \\ &= Q(1 + 2 + \cdots + (n+1)) - Q(1 + 2 + \cdots + n) - P(n+1) \\ &= (P(1) + P(2) + \cdots + P(n+1)) - (P(1) + P(2) + \cdots + P(n)) - P(n+1) \\ &= 0. \end{aligned}$$

So, $R(x)$ vanishes at all integers. Now $R(x)$ is clearly a polynomial, and hence can have only finitely many roots if it is not constant, so the the above is possible iff $R(x)$ is identically 0. Hence, $P(x+1) = Q((x+1)(x+2)/2) - Q(x(x+1)/2)$ or equivalently $P(x) = Q(x(x+1)/2) - Q(x(x-1)/2)$.

Lemma 2: If $P(x)$ and $Q(x)$ satisfy the given condition we must have $Q(0) = 0$.

Proof: We have $P(x) = Q(x(x+1)/2) - Q(x(x-1)/2)$. Hence, $P(1) = Q(1) - Q(0)$, but from the given condition $P(1) = Q(1)$, hence we must have $Q(0) = 0$.

Lemma 3: If $P(x)$ and $Q(x)$ satisfy the given condition the degree of Q is

either 1 or 2.
Proof: Let $Q(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Then

$$\begin{aligned} P(x) &= Q(x(x+1)/2) - Q(x(x-1)/2) \\ &= (x(x+1)/2)^n - (x(x-1)/2)^n + a_{n-1}((x(x+1)/2)^{n-1} \\ &- (x(x-1)/2)^{n-1}) + \cdots a_1 + x \end{aligned}$$

The leading term of the $P(x)$ above is easily seen to be $\dfrac{nx^{2n-1}}{2^{n-1}}$, and, since $P(x)$ is monic, we must have $\dfrac{n}{2^{n-1}} = 1$. This condition is clearly true for $n \in \{1, 2\}$ and note for $n \geq 3$ we have $2^{n-1} = (1+1)^{n-1} \geq 1+\binom{n-1}{1}+\binom{n-1}{2} > 1+(n-1) = n$, hence the only possible values for n are 1 and 2.

Since, Q has to be monic, $Q(0) = 0$ and degree of Q is 1 or 2, hence we must either have $Q(x) = x$ or $Q(x) = x^2 + bx$, and the corresponding values of P from Lemma 1 have to be $P(x) = x$ and $P(x) = (x(x+1)/2)^2 + bx(x+1)/2 - (x(x-1)/2)^2 - bx(x-1)/2 = x^3 + bx$ respectively.

We now prove sufficiency, it is trivial that $P(x) = Q(x) = x$ satisfy the conditions in the problem. In case $Q(x) = x^2 + bx$ and $P(x) = x^3 + bx$, using the well known result that the sum of cubes of first n positive integers is the square of their sum, we get

$$\begin{aligned} P(1) + \cdots + P(n) &= 1^3 + \cdots + n^3 + b(1 + \cdots + n) \\ &= (1 + \cdots + n)^2 + b(1 + \cdots + n) \\ &= Q(1 + \cdots + n) \end{aligned}$$

Hence proved.

Second solution by Arkady Alt, San Jose, California, USA

Since 1+2+...+$n = \dfrac{n(n+1)}{2}$ then $P(n) = Q\left(\dfrac{n(n+1)}{2}\right) - Q\left(\dfrac{n(n-1)}{2}\right)$,for any $n \geq 2$ and $P(1) = Q(1)$. Let $R(x) := P(x) - Q\left(\dfrac{x(x+1)}{2}\right) + Q\left(\dfrac{x(x-1)}{2}\right)$.

Supposition that $R(x)$ isn't zero polynomial lead us to contradiction, because $R(x)$ have more roots then it's degree ($R(n) = 0$ for any natural n). Thus,

(1) $\quad P(x) = Q\left(\dfrac{x(x+1)}{2}\right) - Q\left(\dfrac{x(x-1)}{2}\right), x \in \mathbb{R}.$

Let $P(x), Q(x)$ two monic not-zero polynomials which satisfy $\sum_{k=1}^{n} P(k) = Q\left(\dfrac{n(n+1)}{2}\right)$

and $m := \deg Q$.Since $P(1) = Q(1)$ then $Q(0) = Q(1) - P(1) = 0$. Hence, $m \geq 1$,

because from supposition $m = 0$ immediately follows $Q(x)$ is zero polynomial.

Also, due (1) we obtain $\deg P = 2m - 1$.Since $P(x) = x^{2m-1} + P_1(x)$ and $Q(x) =$

$x^m + Q_1(x)$ where $\deg P_1 < 2m - 1$ and $\deg Q_1 < m$,then by (1) we have

$x^{2m-1} + P_1(x) = \dfrac{x^m (x+1)^m}{2^m} - \dfrac{x^m (x-1)^m}{2^m} + Q_1\left(\dfrac{x(x+1)}{2}\right) - Q_1\left(\dfrac{x(x-1)}{2}\right)$.

Since $\dfrac{x^m (x+1)^m}{2^m} - \dfrac{x^m (x+1)^m}{2^m} = \dfrac{x^m}{2^m}\left((x+1)^m - (x-1)^m\right) =$

$\dfrac{x^m}{2^m} \cdot \left(2\binom{m}{1}x^{m-1} + 2\binom{m}{3}x^{m-3} + ...\right) = \dfrac{mx^{2m-1}}{2^{m-1}} + ...$ then coefficient for x^{2m-1} in the right hand side is $\dfrac{m}{2^{m-1}}$ and should be equal 1.Equation $2^{m-1} = m$ have exactly

two solutions $m = 1$ and $m = 2$.That gives us two variant for polynomial Q,namely

$Q(x) = x$ or $Q(x) = x^2 + ax$ where a is arbitrary real number.

$Q(x) = x$ produce $P(x) = \dfrac{x(x+1)}{2} - \dfrac{x(x-1)}{2} = x$ and

$Q(x) = x^2 + ax$ produce $P(x) = \dfrac{x^2 (x+1)^2}{4} - \dfrac{x^2 (x-1)^2}{4} + ax = x^3 + ax$.

Thus solutions of the problem gives the following pairs of polynomials:

$(P, Q) = (x, x)$ and $(P, Q) = \left(x^3 + ax, x^2 + ax\right)$ where a is arbitrary real constant.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Assume that P and Q have respectively degrees u and v. Clearly,

$$P(n) = Q\left(\frac{n(n+1)}{2}\right) - Q\left(\frac{n(n-1)}{2}\right).$$

Since this must be true for an infinitude of values of n, the coefficient multiplying the highest degree of n must be equal at both sides of the equality, ie., $n^u = \frac{n^v}{2^v} 2vn^{v-1}$, yielding $2v - 1 = u$ and $2v = 2^v$. The second equality has solutions $v = 1$ and $v = 2$, and since the derivatives with respect to v of $2v$ and 2^v are 2 and $2^v \ln 2$, the second one being larger for $v \geq 2$, no additional solutions for non-negative integers v may be found. $v = 1$ and $v = 2$ yield respectively $u = 1$ and $u = 3$.

In the first case, we may write $P(n)=n+a$ and $Q(n)=n+a$ since $P(1)=Q(1)$, and

$$n+a=P(n)=Q\left(\frac{n(n+1)}{2}\right)-Q\left(\frac{n(n-1)}{2}\right)=n,$$

resulting in $a=0$. The only solution in this case is then $P(x)=Q(x)=x$.

In the second case, we may write $P(n)=n^3+an^2+bn+c$ and $Q(n)=n^2+dn+e$, resulting in

$$n^3+an^2+bn+c=P(n)=Q\left(\frac{n(n+1)}{2}\right)-Q\left(\frac{n(n-1)}{2}\right)=n^3+dn,$$

yielding $a=c=0$ and $d=b$. Since furthermore $P(1)=Q(1)$, we conclude that $e=0$. Direct substitution shows that $P(x)=x^3+bx$ and $Q(x)=x^2+bx$ are valid solutions for all real b since, as it is well known (or easily provable by induction), that

$$\sum_{m=1}^{n} m^3=\left(\frac{n(n+1)}{2}\right)^2.$$

U96. Let $f:(0,\infty)\to[0,\infty)$ be a bounded function. Prove that if

$$\lim_{x\to 0}\left(f(x)-\frac{1}{2}\sqrt{f(\frac{x}{2})}\right)=0 \quad \text{and} \quad \lim_{x\to 0}\left(f(x)-2f(2x)^2\right)=0,$$

then $\lim_{x\to 0} f(x)=0$.

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania and Mihai Piticari, "Dragos Voda" National College, Romania

First solution by Arkady Alt, San Jose, California, USA

Let $\alpha(x):=f(x)-2f^2(2x)$ and $\beta(x):=f(2x)-\dfrac{1}{2}\sqrt{f(x)}$ then by condition

$\lim\limits_{x\to 0}\alpha(x)=0$ and $\lim\limits_{x\to 0}\beta(x)=\lim\limits_{x\to 0}\left(f(2x)-\dfrac{1}{2}\sqrt{f(x)}\right)=\lim\limits_{t\to 0}\left(f(t)-\dfrac{1}{2}\sqrt{f\left(\dfrac{t}{2}\right)}\right)=0$

where $t:=2x$.Since $f(2x)-\dfrac{1}{2}\sqrt{f(x)}=\beta(x) \implies f(x)=4f^2(2x)-4f(2x)\beta(x)+\beta^2(x)$

and $2f(x)=4f^2(2x)+2\alpha(x)$ then $f(x)=4f^2(2x)+2\alpha(x)-\left(4f^2(2x)-4f(2x)\beta(x)+\beta^2(x)\right)=4f(2x)\beta(x)-\beta^2(x)+2\alpha(x)$.

Since $f(x)$ is bounded and $\lim\limits_{x\to 0}\beta(x)=\lim\limits_{x\to 0}\alpha(x)=0$ then $\lim\limits_{x\to 0}f(2x)\beta(x)=0$, $\lim\limits_{x\to 0}\beta^2(x)=0$, $\lim\limits_{x\to 0}2\alpha(x)=0$,and, therefore, $\lim\limits_{x\to 0}f(x)=0$.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Substituting x by $2x$ in the first condition, we obtain

$$\lim_{x\to 0}\left(f(2x)-\frac{1}{2}\sqrt{f(x)}\right)=0.$$

Since $2f(2x)+\sqrt{f(x)}$ is bounded by hypothesis, multiplying the limit by this function we obtain

$$\lim_{x\to 0}\left(2f(2x)^2-\frac{1}{2}f(x)\right)=0.$$

Adding the second condition yields the proposed result.

Olympiad problems

O91. Let ABC be an acute triangle. Prove that

$$\tan A + \tan B + \tan C \geq \frac{s}{r},$$

where s and r are semiperimeter and inradius of triangle ABC, respectively.

Proposed by Mircea Becheanu, University of Bucharest, Romania

First solution by Michel Bataille, France

In addition to the usual trig formulas, we will make use of the following two known results:

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

and

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

Now,

$$\begin{aligned}
\tan A + \tan B + \tan C &= \frac{\sin A \sin B \sin C}{\cos A \cos B \cos C} \\
&= \frac{1}{4} \cdot \frac{\sin 2A + \sin 2B + \sin 2C}{\cos A \cos B \cos C} \\
&= \frac{1}{2}\left(\frac{\sin A}{\cos B \cos C} + \frac{\sin B}{\cos C \cos A} + \frac{\sin C}{\cos A \cos B}\right) \\
&= \frac{\sin A}{\cos(B-C) - \cos A} + \frac{\sin B}{\cos(C-A) - \cos B} \\
&+ \frac{\sin C}{\cos(A-B) - \cos C} \\
&\geq \frac{\sin A}{1-\cos A} + \frac{\sin B}{1-\cos B} + \frac{\sin C}{1-\cos C} \\
&= \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} \\
&= \frac{s-a}{r} \frac{s-b}{r} + \frac{s-c}{r} = \frac{s}{r}
\end{aligned}$$

and the result follows.

Second solution by Oleh Faynshteyn, Leipzig, Germany

We have

$$\begin{aligned}\tan\alpha+\tan\beta &= \frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}=\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}\\ &= \frac{2\sin\gamma}{\cos(\alpha-\beta)+\cos(\alpha+\beta)} \geq \frac{2\sin\gamma}{1+\cos(\alpha+\beta)}\\ &= \frac{2\sin\gamma}{1-\cos\gamma}=\frac{2\sin\gamma}{\sin^2(\frac{\gamma}{2})}\\ &= \frac{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}{\sin^2(\frac{\gamma}{2})}=2\cot\frac{\gamma}{2}. \quad (1)\end{aligned}$$

Similarly,

$$\tan\beta+\tan\gamma\geq 2\cot\frac{\alpha}{2}, \quad \tan\gamma+\tan\alpha\geq 2\cot\frac{\beta}{2}. \quad (2)$$

Adding (1) and (2) we obtain

$$\tan\alpha+\tan\beta+\tan\gamma\geq\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}.$$

It is known that

$$\cot\frac{\alpha}{2}=\frac{s-a}{r}, \quad \cot\frac{\beta}{2}=\frac{s-b}{r}, \quad \cot\frac{\gamma}{2}=\frac{s-c}{r}$$

therefore

$$\tan\beta+\tan\gamma\geq\frac{s-a}{r}+\frac{s-b}{r}+\frac{s-c}{r}=\frac{s}{r}$$

and we are done.

Also solved by Arkady Alt, San Jose, California, USA; Vicente Vicario Garcia, Huelva, Spain; Samin Riasat, Notre Dame College, Dhaka, Bangladesh; Roberto Bosch Cabrera, Cuba

O92. Let n be a positive integer. Prove that

a) there are infinitely many triples (a, b, c) of distinct integers such that $\min(a, b, c) \geq n$ and $abc + 1$ divides one of the numbers $(a - b)^2, (b - c)^2, (c - a)^2$.

b) there is no triple (a, b, c) of distinct positive integers such that $abc+1$ divides more than one of the numbers $(a - b)^2, (b - c)^2, (c - a)^2$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

For any positive integer n, the following relation holds:

$$(n^3+6n^2+10n+4)(n^2+4n+3)n+1 = n^6+10n^5+37n^4+62n^3+46n^2+12n+1 =$$

$$= (n^3 + 5n^2 + 6n + 1)^2 = \left((n^3 + 6n^2 + 10n + 4) - (n^2 + 4n + 3)\right)^2 .$$

Taking $a = n^3 + 6n^2 + 10n + 4$, $b = n^2 + 4n + 3$ and $c = n$, clearly $\min(a, b, c) = c = n$, and $abc+1$ divides $(a - b)^2 = abc + 1$. The conclusion of part a) follows.

Assume now wlog that $a > b > c$. If $abc+1$ divides $(b-c)^2$, then $a > b > b-c$, and $abc + 1 \leq (b - c)^2 < ab$, clearly impossible. If $abc + 1$ divides $(a - b)^2$ and $(a-c)^2$, then it divides $(a-c)^2 - (a-b)^2 = (2a-b-c)(b-c) < 2ab$, and since $2a > 2a-b-c > 0$ and $b > b-c > 0$, then $2ab > (2a-b-c)(b-c) \geq abc+1$, resulting in $c = 1$ and $(2a - b - 1)(b - 1) = ab + 1$, or $a = \frac{b^2}{b-2} = b + 2 + \frac{4}{b-2}$. Since $b-2$ divides 4, then $b = 3, 4, 6$, yielding respectively $a = 9, 8, 9$; however, in none of these cases does $abc + 1$ divide $(a - b)^2$ or $(a - c)^2$. The conclusion of part b) follows.

O93. Let k be a positive integer. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) + f(y)$ divides $x^k + y^k$ for all $x, y \in \mathbb{N}$.

Proposed by Nguyen Tho Tung, Hanoi University of Education, Vietnam

No solutions has yet been received.

O94. Let ω be a circle with center O and let A be a fixed point outside ω. Choose points B and C on ω, with $AB \neq AC$, such that AO is a symmedian, but not a median, in triangle ABC. Prove that the circumcircle of triangle ABC passes through a second fixed point.

Proposed by Alex Anderson, Washington University in St. Louis, USA

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Let P, Q be the second points where AB, AC respectively intersect ω, let ω' be the circumcircle of ABC with center O', let M be the midpoint of BC, and let N the second point where the internal bisector of angle $\angle BAC$ intersects the circumcircle of ABC. Since BC is a chord both in ω and ω', its midpoint M clearly lies on line OO'. The point N is also clearly the midpoint of arc BC, or it is also on line OO'. Hence, M, N, O, O' are collinear.

Claim: PQ is a diameter of ω.

Proof: Triangles ABC and AQP are clearly similar, hence the internal bisector of angles $\angle BAC$ and $\angle QAP$ is the same. Since AB, AC are the respectively symmetric lines of AQ, AP with respect to this internal bisector, and AO is a symmedian in triangle BAC, then it is a median in triangle APQ. Assume now that PQ is a chord of ω that is not a diameter. Since AO passes through its midpoint and through the center of ω, then AO is the perpendicular bisector of PQ, and since it is also a median, APQ is isosceles in A, and so is ABC. We reach a contradiction, hence PQ is a diameter, qed.

Second solution by David E. Narvaez, Panama

Let ω be a circle with center O and let A be a fixed point outside ω. Choose points B and C on ω with $AB \neq AC$, such that AO is a symmedian, but not a median, in triangle ABC. Prove that the cinrcumcircle of triangle ABC passes through a second fixed point.

Solution. Let S be the circumcircle of triangle ABC and let A' be the second point of intersection of S with line OA. We claim that the point A' is fixed.

Let X and P be the intersections of line BC with the perpendicular bisector of AA' and line OA, respectively. Since AA' is the symedian of triangle ABC, it's also the pole of the point X with respect to S. Then points X, B, P and C are harmonic conjugates, thus X is in the pole of P with respect to ω. Let P' be the midpoint of AA', then P' is the projection of X on the line OP, so it's the inverse of P with respect to ω. Since P is in the radical axis of circles ω and S, we have that

$$\begin{aligned}
PA\cdot PA' &= (r-OP)(r+OP)\\
(OA-OP)\left(OP-OA'\right) &= r^2-OP^2\\
(OA-OP)\left(OP-\left(OA-2P'A\right)\right) &= r^2-OP^2\\
(OA-OP)\left(OP-\left(OA-2\left(OA-OP'\right)\right)\right) &= r^2-OP^2\\
(OA-OP)\left(OP-\left(OA-2\left(OA-\frac{r^2}{OP}\right)\right)\right) &= r^2-OP^2\\
(OA-OP)\left(OP+OA-\frac{2r^2}{OP}\right) &= r^2-OP^2
\end{aligned}$$

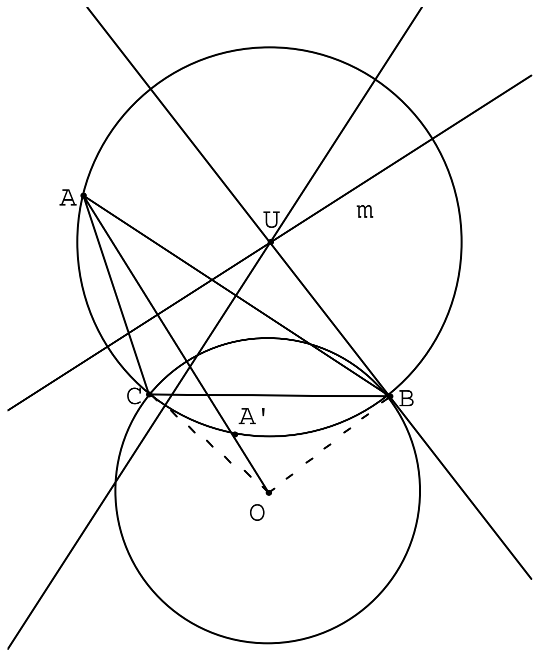
and solving this equation for OP yields

$$OP=\frac{2OAr^2}{OA^2+r^2}$$

so OP is fixed, thus P' is fixed and so is A', which proves our claim.

Third solution by Michel Bataille, France

Let P be the point of intersection of the tangents at B and C to the circumcircle Γ of ΔABC. This point P certainly is on the perpendicular bisector of BC, but it is also on the symmedian through A (a well-known result). Since this symmedian is the line AO, the only possibility is $P=O$. It follows that the centre U of Γ is on the perpendicular to OB at B (and to OC at C). This said, we can now conclude in two ways:



Solution I. Since $UA = UB$ and $UB^2 + r^2 = UO^2$ (where r denotes the radius of ω), we have $UO^2 - UA^2 = r^2$. Thus, U is on the line $m = \{X : XO^2 - XA^2 = r^2\}$, which is orthogonal to OA. This implies that Γ passes by the point A' symmetrical of A in the line m.

Solution II. Under the inversion in the circle ω, the circle Γ, which is orthogonal to ω, is invariant. Since A is on Γ, its inverse A' is on Γ as well.

O95. Prove that there is a sequence $x_1, x_2, \dots$ of integers such that

- For each $n \in \mathbb{Z}$ there exists i such that $x_i = n$.
- $\displaystyle\prod_{d|n} d^{\frac{n}{d}} = \sum_{i=1}^{n} x_i$.

Proposed by Juan Ignacio Restrepo, Universidad de Los Andes, Colombia

No solutions has yet been received.

O96. Let p and q be primes, $q \geq p$. Prove that pq divides $\binom{p+q}{p} - \binom{q}{p} - 1$.

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania

First solution by Vicente Vicario Garcia, Spain

Lemma: If p is prime, then

$$\binom{p}{1}, \binom{p}{2}, \dots, \binom{p}{p-1}$$

are divisible by p.

Proof: We consider the binomial coefficient of $\binom{p}{i}$ where $1 \leq i \leq p-1$. Then

$$\binom{p}{i} = p \cdot \frac{(p-1)(p-2)\cdots(p-i+1)}{i!}$$

and because p has no common factors with $i!$ the conclusion follows.

(a) If $p = q$ then $\binom{p+q}{p} - \binom{q}{p} - 1 = \binom{2p}{p} - 2 \equiv 0 \pmod{p}$ (after we use the lemma). Using the binomial coefficient property $\binom{n}{k} = \binom{n}{n-k}$ and the well known combinatorial identity

$$\binom{m+n}{k} = \sum_{k=0}^{\max(k,n)} \binom{m}{k-l}\binom{n}{l}$$

for $m = n = p$ produces

$$\binom{2p}{p} = \binom{p}{0}^2 + \binom{p}{1}^2 + \cdots + \binom{p}{p-1}^2 + \binom{p}{p}^2 \equiv 2 \pmod{p}.$$

(b) If $q > p$ we obtain

$$\binom{p+q}{p} = \binom{p}{p}\binom{q}{0} + \binom{p}{p-1}\binom{q}{1} + \cdots + \binom{p}{0}\binom{q}{p} \equiv \quad (\text{mod } p)q$$

because $\binom{p}{p} = \binom{p}{0} = \binom{q}{0} = 1$ and we are done.

Second solution by Michel Bataille, France

Let $A = \binom{p+q}{p} - \binom{q}{p} - 1$.
First, we examine the special case when $p = q$. We have to show that $A =$

$\binom{2p}{p} - 2$ is divisible by p^2. Recalling the identity $\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j}^2$ for positive integer n, the result follows from

$$A = \sum_{j=1}^{p-1} \binom{p}{j}^2 \equiv 0 \pmod{p^2}$$

(the latter because $\binom{p}{j} \equiv 0 \pmod p$ for $j = 1, 2, \dots, p-1$, as it is well-known).
Now, suppose that $q > p$. We have to prove (a) A is divisible by q and (b) A is divisible by p.
(a) Since $\binom{q}{p}$ is divisible by q, all amounts to proving that $B = \binom{p+q}{p} - 1$ is divisible by q.
Modulo q, we have

$$p!B = (p+q)(p+q-1)\cdots(q+1) - p! \equiv [p(p-1)\cdots 1] - p! \equiv 0,$$

hence $p!B$ is divisible by q and so is B since q is coprime with $1, 2, \dots, p$.

(b) We have

$$\begin{aligned}(q!)A = {} & (p+q)(p+q-1)\cdots(p+1) \\ & - [q(q-1)\cdots(p+1)] \cdot [(q-p+1)(q-p+2)\cdot\cdots\cdot q] - q!(*).\end{aligned}$$

Let $q = kp + r$ where $k, r \in \mathbb{N}$ with $0 < r < p$ and let

$$K = \frac{q!}{p!} = (p+1)(p+2)\cdot\cdots\cdot(kp+r).$$

Then

$$\begin{aligned}(p+q)(p+q-1)\cdots(p+1) = {} & K(kp+r+1)\cdots(kp+p-1) \\ & \cdot((k+1)p)((k+1)p+1)\cdots((k+1)p+r),\end{aligned}$$

$$\begin{aligned}& [q(q-1)\cdots(p+1)] \cdot [(q-p+1)(q-p+2)\cdot\cdots\cdot q] \\ & = K((k-1)p+r+1)\cdots(kp)(kp+1)\cdots(kp+r)\end{aligned}$$

and dividing both sides of $(*)$ by pK, we obtain

$$\begin{aligned}(p-1)!A = {} & (k+1)(kp+r+1)\cdots(kp+p-1)[((k+1)p+1)\cdots((k+1)p+r)] \\ & - k[((k-1)p+r+1)\cdots((k-1)p+p-1)] \\ & \cdot[(kp+1)\cdots(kp+r)] - (p-1)!.\end{aligned}$$

By Wilson's theorem, we have $(p-1)! \equiv -1 \pmod p$ and so, modulo p,

$$-A \equiv (k+1)(r+1)\cdots(p-1)\cdot(r!) - k[(r+1)\cdots(p-1)]\cdot(r!) + 1 \equiv -(k+1) + k + 1$$

and so $A \equiv 0 \pmod p$, as desired.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Note that the proposed result is not always true when $q = p$, ie, q^2 does not always divide $\binom{2q}{q} - \binom{q}{q} - 1 = \binom{2q}{q} - 2$. Take for example $q = 4$, resulting in $\binom{8}{4} - 2 = 68 = 4 \cdot 17$, divisible by $q = 4$, but not by $q^2 = 16$. We may however guarantee that q will always divide $\binom{2q}{q} - 2$, since

$$\binom{2q}{q} - 2 = 2\frac{(2q-1)(2q-2)\dots(q+1)}{(q-1)(q-2)\dots 1} - 2.$$

Now, obviously $2q - i \equiv q - i \pmod q$ for $i = 1, 2, \dots, q-1$. Therefore, since

$$(2q-1)(2q-2)\dots(q+1) \equiv (q-1)(q-2)\dots 1 \pmod q,$$

all of the terms at both sides being prime with q, the chinese remainder theorem guarantees that only one remainder modulus q is congruent to their ratio, and this remainder is trivially 1. The result follows.

Consider now the case $q > p$, and let a be the largest multiple of p not exceeding q. Clearly, a is the only multiple of p in the set $A = \{q, q-1, q-2, \cdots, q-p+1\}$, while $a + p$ is the only multiple of p in the set $B = \{q+1, q+2, \cdots, q+p\}$. Consider the set $B \backslash \{a+p\}$; the $p-1$ elements in this set are relatively prime with p, and no two of them can be congruent to each other modulus p since the largest difference between any two of them is $p-1$. Hence, each one of the elements of $B \backslash \{a+p\}$ is congruent modulus p to each one of the elements of $\{1, 2, \dots, p-1\}$, all of them being also relatively prime with p. Similarly, each one of the $p-1$ elements of $A \backslash \{a\}$ is congruent to each one of the elements of $\{1, 2, \dots, p-1\}$, all of them being also relatively prime with p. Hence,

$$\binom{p+q}{p} = \frac{(q+p)(q+p-1)\dots(q+1)}{p(p-1)\dots 1} \equiv \frac{a+p}{p} \pmod p;$$

$$\binom{q}{p} = \frac{q(q-1)\dots(q-p+1)}{p(p-1)\dots 1} \equiv \frac{a}{p} \pmod p.$$

It clearly follows that p divides $\binom{p+q}{p} - \binom{q}{p} - 1$. Now, since $0 < p < q$, then q divides $\binom{q}{p} = q\frac{(q-1)(q-2)\dots(q-p+1)}{p(p-1)\dots 1}$, since all the elements in the fraction (numerator and denominator) are less than q, and hence relatively prime with q. Finally, since $1, 2, \dots, p$ are all less than prime q, then $p!$ is prime with q, while $q + i \equiv i \pmod q$ is also prime with q for any $i = 1, 2 \dots, p$. It clearly follows that $\binom{p+q}{p} \equiv 1 \pmod q$, and hence q divides $\binom{p+q}{p} - \binom{q}{p} - 1$. Since p, q are relatively prime, the conclusion follows.

Fourth solution by Roberto Bosch Cabrera, Cuba

The proof is based on the following identity:

$$\binom{p+q}{p} = \sum_{k=0}^{p} \binom{p}{k} \binom{q}{p-k}$$

This identity can be proved as follows.
The left side is the coefficient of x^p in the expansion of $(1+x)^{p+q}$, while the right side is the coefficient of x^p in the expansion of the product $(1+x)^p(1+x)^q$, and of course the two are the same.
We obtain that

$$\binom{p+q}{p} - \binom{q}{p} - 1 = \sum_{k=1}^{p-1} \binom{p}{k} \binom{q}{p-k}$$

since $\binom{p}{k}$ is divisible by p for all $k = 1, 2, ..., p-1$ and $\binom{q}{p-k}$ is divisible by q for all $k = 1, 2, ..., p-1$ each of the terms of the sum is divisible by pq, hence the conclusion.