Solutions to Admission Test A

1. Place ten 1's and six 0's in a 4×4 array such that each row has an even number of 1's and each column has an odd number of 1's.

   Solution: There are many correct different arrangements. One of them is shown below

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

2. Let n be a positive integer, and let $S_n = 1 + 2 \cdots + n$. What are all the possible units digits (in decimal representation) of S_n? Prove your result.

   Solution: The possible units digits of S_n are $\boxed{0, 1, 3, 5, 6, 8}$.

   Note that $S_n = \frac{n(n+1)}{2}$. If n and m are two positive integer with their difference a multiple of 20, then S_n and S_m have the same unit digit. Indeed, assume that $n - m = 20k$ for some positive integer k. Then

$$S_n = \frac{(m+20k)(m+20k+1)}{2} = \frac{m(m+1)}{2} + 10k(2m+40k+1) = S_m + 10k(20m+40k+1),$$

   implying that S_n and S_m have the same unit digit. It is easy to check the unit digits of S_n for $n = 1, 2, \ldots, 20$.

   Note: A bit more work with modular arithmetic can simplify the solution.

3. Given a 6×6 square, show it is possible to dissect it into 8 incongruent rectangles with integer side lengths. Prove that if it is dissected into 9 rectangles with integer side lengths, then two of the rectangles must be congruent to each other.

   Solution: We call a rectangle *integral* if it has integer side lengths. The nine smallest incongruent integral rectangles are 1×1, 1×2, 1×3, 1×4, 2×2, 1×5, 1×6, 2×3, and 1×7. Their total area is equal to 38, which is greater than the area of a 6×6 square. Hence if 6×6 square is dissected into 9 integral rectangles, then two of the rectangles must be congruent to each other.

   There are varies ways to tile an 6×6 square with 8 incongruent rectangles. The identity $36 = 1 + 2 + 3 + 4 + 5 + 6 + 6 + 9$ motivates the following tiling:

1	4	4	4	4	6
2	5	7	7	7	6
2	5	7	7	7	6
3	5	8	8	8	6
3	5	8	8	8	6
3	5	8	8	8	6

4. Given distinct positive real numbers x, y and z such that

$$\frac{z}{x+y} < \frac{x}{y+z} < \frac{y}{z+x},$$

write x, y, and z in increasing order from left to right. Justify your result *algebraically*.

Note: This is the corrected version of the problem statement. In the original statement, the term "positive" was missing. Full credit will be given for any solution that either assumed the missing restriction or made significant progress towards surmising that no definite order can be established for x, y, and z from the question as stated. Here is the solution for the intended question.

Solution: The answer is $\boxed{z < x < y}$.

Because x, y, z are positive, the given condition can be rewritten as

$$\frac{x+y}{z} > \frac{y+z}{x} > \frac{z+x}{y}.$$

Adding 1 to each sides of these inequalities gives

$$\frac{x+y}{z} + 1 > \frac{y+z}{x} + 1 > \frac{z+x}{y} + 1,$$

or

$$\frac{x+y+z}{z} > \frac{y+z+x}{x} > \frac{z+x+y}{y},$$

implying that

$$\frac{1}{z} > \frac{1}{x} > \frac{1}{y},$$

that is, $z < x < y$.

5. In the coordinate-plane, given square $ABCD$ with $A = (12, 19)$ and $C = (3, 22)$. Find the coordinates of B and D.

Solution: The coordinates of B and D are $\boxed{(9, 25)}$ and $\boxed{(6, 16)}$.

Let M be the midpoint of segment AC. Then $M = (7.5, 20.5)$ and $\overrightarrow{AM} = [-4.5, 1.5]$. It follows that

$$\{B, D\} = \{M \pm [1.5, 4.5]\} = \{(9, 25), (6, 16)\}.$$

Note: If the reader is not familiar with vector operations, please consider the movement from A to M in the rectangular coordinate system. This movement is perpendicular to the movement from B to M and two movements have the same length. Hence if one movement is $[a, b]$ (moving a units in the along the x-axis and b units the y-axis) then the other movement is either $[-b, a]$ or $[b, -a]$.

6. Let T be a subset of $\{1, 2, \ldots, 2003\}$. An element a of T is called *isolated* if neither $a-1$ nor $a+1$ is in T. Determine the number of five-element subsets of T that contain no isolated elements.

Solution: The answer is $\boxed{3996001}$.

Let $S = \{a, b, c, d, e\}$ be a five-element subsets of T that contain no isolated elements. Assume that $a < b < c < d < e$. Since a and e are not isolated, $b = a + 1$ and $d = e - 1$. Since c is not isolated, either $c = b + 1$ or $c = d - 1$ or both. We consider three cases:

- In this case, we assume that $S = \{c-2, c-1, c, d, d+1\}$ with $c \le d$. Hence (c, d) can be any ordered pair of integers with $3 \le c \le d \le 2002$. There are $\frac{2000 \cdot 1999}{2} = 1999000$ such pairs, and so there are 1999000 subsets of this type.
- In this case, we assume that $S = \{b-1, b, c, c+1, c+2\}$ with $b \le c$. Hence (b, c) can be any ordered pair of integers with $2 \le c \le d \le 2001$. There are $\frac{2000 \cdot 1999}{2} = 1999000$ such pairs, and so there are 1999000 subsets of this type.
- In this case, we assume that $S = \{a, a+1, a+2, a+3, a+4\}$. Clearly, there are 1999 such subsets.

The subsets in the third case appeared in both the first and second cases. Hence the answer is $1999000 \times 2 - 1999 = 1999^2$.

7. Given nonzero real numbers a, b, and c such that the quadratic equations (in x) $ax^2+bx+c = 0$, $bx^2+cx+a = 0$, $cx^2+ax+b = 0$ share a common root, find all possible values of $\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}$.

Solution: The answer is $\boxed{3}$.

Let α be the common root. Then

$$a\alpha^2 + b\alpha + c = 0, \quad b\alpha^2 + c\alpha + a = 0, \quad c\alpha^2 + a\alpha + b = 0.$$

Adding these three equations yields

$$(a + b + c)(\alpha^2 + \alpha + 1) = 0.$$

Note that $\alpha^2 + \alpha + 1 = \left(\alpha + \frac{1}{2}\right)^2 + \frac{3}{4} \ge \frac{3}{4} > 0$. We conclude that if $\alpha^2 + \alpha + 1 \neq 0$, then $a + b + c = 0$. Then by the identity

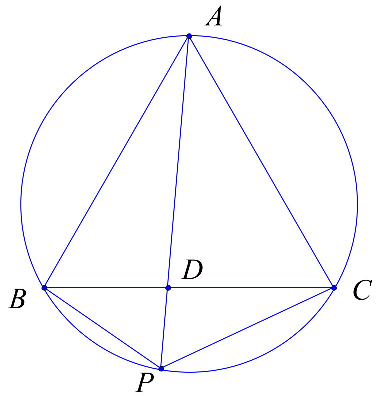
$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

we have $a^3 + b^3 + c^3 = 3abc$, and $\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} = \frac{a^3+b^3+c^3}{abc} = 3$.

If $\alpha^2 + \alpha + 1 = 0$ then $\alpha = \pm\frac{-1+\sqrt{3}i}{2}$. But since a, b, and c are real, $\overline{\alpha}$ is also a root of each original quadratic and so $a = b = c$ and the desired value is again 3.

8. Equilateral triangle ABC is inscribed in circle ω. Point P lies on minor arc $\widehat{BC}$. Segments AP and BC meet at D. Given that $BP = 21$ and $CP = 28$, compute $\frac{BD}{DC}$ and PD.

Solution: The answer is $\boxed{\dfrac{BD}{DC} = \dfrac{3}{4}}$ and $\boxed{PD = 12}$.



Note that both minor arcs $\widehat{AB}$ and $\widehat{AC}$ has $120°$ measure, and so $\angle BPD = \angle BPA = \angle CPA = \angle CPD = 60°$. Hence PD is an interior angle bisector of triangle BPC, and so

$$\frac{BD}{CD} = \frac{BP}{CP} = \frac{3}{4},$$

by the **Angle-bisector Theorem**. It follows that

$$\frac{BD}{BA} = \frac{BD}{BC} = \frac{3}{7}.$$

Note that triangle CDP is similar to triangle ADB,

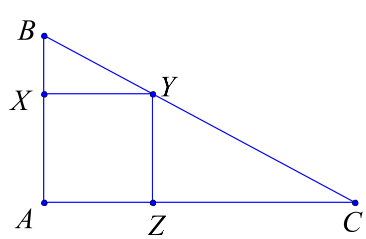
$$\frac{PD}{PC} = \frac{BD}{BA} = \frac{3}{7},$$

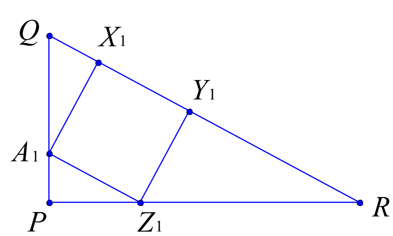
implying that $DP = \frac{3}{7} \cdot CP = \frac{3}{7} \cdot 28 = 12$.

9. A square is *inscribed* in a triangle if the vertices of the square lie on the sides of the triangle. Given a right triangle, there are two obvious ways to inscribe a square. The first way is to place a corner of a square at a vertex of the triangle. The second way is to place a side of a square on the hypotenuse of the right triangle. Which way will result in a bigger square, or will they be equal?

   Note: The first way is always better. We present two approaches.

   First Proof:





We start with two congruent right triangles ABC and PQR. (We label them this way so we can compare these with the figure shown below for the second solution). Let $AB = PQ = c$,

$BC = QR = a$, and $CA = RP = b$, and let $XY = s$ and $X_1Y_1 = t$. It suffice to show that $s^2 > t^2$.

Since triangles BXY and BAC are similar, we have

$$\frac{BY}{BC} = \frac{XY}{AC},$$

implying that $BY = \frac{sa}{b}$. Likewise, we also have $CY = \frac{sc}{a}$. Since $BY + YC = a$, we have $\frac{sa}{b} + \frac{sa}{c} = a$, implying that

$$s = \frac{bc}{b+c} \quad \text{or} \quad s^2 = \frac{b^2c^2}{(b+c)^2}.$$

Note that triangles A_1QX_1, RQP, and RZ_1Y_1 are similar. It follows that

$$QX_1 = \frac{ct}{b} \quad \text{and} \quad RY_1 = \frac{bt}{c}.$$

Because $QR = QX_1 + X_1Y_1 + Y_1R$ or $a = \frac{ct}{b} + t + \frac{bt}{c}$, it follows that

$$t = \frac{abc}{b^2+c^2+bc} \quad \text{or} \quad t^2 = \frac{a^2b^2c^2}{(b^2+c^2+bc)^2} = \frac{(b^2+c^2)b^2c^2}{(b^2+c^2+bc)^2}.$$

It remains to show that

$$\frac{(b^2+c^2)b^2c^2}{(b^2+c^2+bc)^2} = t^2 < s^2 = \frac{b^2c^2}{(b+c)^2},$$

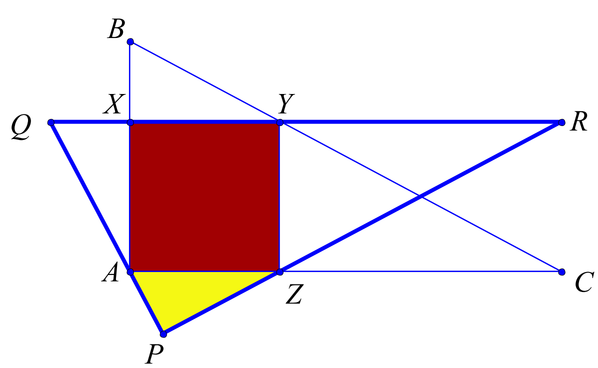
or

$$(b^2+c^2)(b+c)^2 < (b^2+c^2+bc)^2.$$

The inequality can easily be established by expanding its right-hand side as

$$\begin{aligned}(b^2+c^2+bc)^2 &= (b^2+c^2+bc)[(b+c)^2 - bc] \\ &= (b^2+c^2)(b+c)^2 + [(b+c)^2 - (b^2+c^2)]bc \\ &= (b^2+c^2)(b+c)^2 + 2b^2c^2.\end{aligned}$$

Second Proof: We inscribe the same square ($AXYZ$) into two similar right triangles. The second method requires a bigger triangle (triangle PQR), and so the first method (triangle ABC) is better. (Note that triangles YZR and YZC are congruent to each other, and so do triangles BXY and AXQ.)



Note: Given a right triangle, how can you construct these two squares by compass and straight edge?

10. A heap of 2006 balls consists of 1003 10-gram balls and 1003 9.9-gram balls. We wish to pick out two heaps of balls with equal numbers of balls in them but different total weights. What is the minimal number of weighings needed to do this? (The balance scale reports the weight of the objects in the left pan, minus the weight of the objects in the right pan.)

    Proof: The answer is $\boxed{0}$ weighings.

    Consider the total weight of all the balls, which equals $1003 \times 10 + 1003 \times 9.9 = 19959.7$. Then the heap cannot be separated into two piles of equal weight because each new pile would have to weigh $\frac{19959.7}{2} = 9979.85$ grams, which is clearly unattainable with only 10-gram and 9.9 gram balls. So any two piles must have different total weights, and weighing is unnecessary.