Junior problems

J109. Let a, b, c be positive real numbers. Prove that

$$\frac{(a+b)^2}{c} + \frac{c^2}{a} \geq 4b.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Ovidiu Furdui, Cluj, Romania

We will use the following lemma. **Lemma.** *Let a, b, α, β be positive real numbers then*

$$\frac{a^2}{\alpha} + \frac{b^2}{\beta} \geq \frac{(a+b)^2}{\alpha+\beta}.$$

Proof. The lemma can be proved by straight forward calculations. Based on the lemma we have that

$$\frac{(a+b)^2}{c} + \frac{c^2}{a} \geq \frac{(a+b+c)^2}{a+c},$$

so it suffices to prove that

$$\frac{(a+b+c)^2}{a+c} \geq 4b.$$

The preceding inequality is equivalent to $(a-b+c)^2 \geq 0$, and the problem is solved.

Second solution by Filip Stankovski, Skopje, Macedonia

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \text{LHS} \cdot (c+a) &\geq \left(\sqrt{c} \cdot \sqrt{\frac{(a+b)^2}{c}} + \sqrt{a} \cdot \sqrt{\frac{c^2}{a}}\right)^2 \\ &= (a+b+c)^2 = (b+(a+c))^2 \text{ and by AM-GM inequality} \\ &\geq (2\sqrt{b \cdot (a+c)})^2 = 4b \cdot (a+c) \\ &= \text{RHS} \cdot (a+c). \end{aligned}$$

Dividing both sides by $(a+c)$ we obtain the desired result

Third solution by Shamil Asgarli, Howard Ko, Burnaby, Canada

After clearing denominators we need to prove that $f(b) = ab^2 + (2a^2 - 4ac)b + a^3 + c^3 \geq 0$. Considering f as a quadratic function in b we can easily check that the discriminant $(2a^2 - 4ac)^2 - 4a(a^3 + c^3) = -4ac(2a - c)^2 \leq 0$. Since the leading coefficient of this quadratic function is positive, we conclude that $f(b) \geq 0$ for all $b \geq 0$.

Fourth solution by Ercole Suppa, Teramo, Italy

The proposed inequality follows from:

$$\begin{aligned}
\frac{(a+b)^2}{c} + \frac{c^2}{a} - 4b = \frac{a^3 + 2a^2b + ab^2 + c^3 - 4abc}{ac} &= \frac{a\left[b^2 + 2(a-2c)b + a^2\right] + c^3}{ac} \\
&= \frac{a\left[b^2 + 2(a-2c)b + (a-2c)^2 - (a-2c)^2 + a^2\right] + c^3}{ac} \\
&= \frac{a\left[(b+a-2c)^2 + 4ac - 4c^2\right] + c^3}{ac} \\
&= \frac{a(a+b-2c)^2 + 4a^2c - 4ac^2 + c^3}{ac} \\
&= \frac{a(a+b-2c)^2 + c\left(4a^2 - 4ac + c^2\right)}{ac} \\
&= \frac{(a+b-2c)^2}{c} + \frac{(2a-c)^2}{a} \geq 0.
\end{aligned}$$

Also solved by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; John T. Robinson, Yorktown Heights, NY, USA; Baleanu Andrei Razvan, Romania; Brian Bradie, Newport News, USA; Arkady Alt, San Jose, California, USA; Navid Safaei, Tehran, Iran; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Nguyen Manh Dung, Hanoi University of Science, Vietnam; Paolo Leonetti, Milano, Italy.

J110. Let $\tau(n)$ and $\phi(n)$ denote the number of divisors of n and the number of positive integers less than or equal to n that are relatively prime to n, respectively. Find all n such that $\tau(n) = 6$ and $3\phi(n) = 7!$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Brian Bradie, Newport News, USA

If $\tau(n) = 6$, then $n = p^5$ or $n = p_1 p_2^2$ for some primes p, p_1 and p_2. If $n = p^5$, then $\phi(n) = p^4(p-1)$; however, there is no prime p for which

$$p^4(p-1) = \frac{1}{3}7! = 2^4 \cdot 3 \cdot 5 \cdot 7.$$

If $n = p_1 p_2^2$, then $\phi(n) = (p_1 - 1)p_2(p_2 - 1)$. We have four cases to consider:

$p_2 = 2$	$p_2(p_2 - 1) = 2$	$2^3 \cdot 3 \cdot 5 \cdot 7 + 1 = 841$ is not prime
$p_2 = 3$	$p_2(p_2 - 1) = 6$	$2^3 \cdot 5 \cdot 7 + 1 = 281$ is prime
$p_2 = 5$	$p_2(p_2 - 1) = 20$	$2^2 \cdot 3 \cdot 7 + 1 = 85$ is not prime
$p_2 = 7$	$p_2(p_2 - 1) = 42$	$2^3 \cdot 5 + 1 = 41$ is prime

Thus, there are two n such that $\tau(n) = 6$ and $3\phi(n) = 7!$:

$$n = 281 \cdot 3^2 = 2529 \quad \text{and} \quad n = 41 \cdot 7^2 = 2009.$$

Second solution by John T. Robinson, Yorktown Heights, NY, USA

Suppose the prime factorization of n is

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}.$$

Then $\tau(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1) = 6$. Since the only factorizations of 6 into factors greater than 1 are a) $2 \cdot 3$ and b) 6, it follows that n must be of the form a) $p_1 p_2^2$ for distinct primes p_1 and p_2, or b) p^5 for some prime p.

In case b) we have

$$\phi(n) = 2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 1680 = p^4(p-1) = 2^4 \cdot 3 \cdot 5 \cdot 7$$

for some prime p, however since 1680 has only 2 as a fourth power of a prime in its factorization, and $2^4(2-1) \neq 1680$, this case is excluded.

In case a) we have

$$\phi(n) = 2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 1680 = (p_1 - 1)(p_2 - 1)p_2 = 2^4 \cdot 3 \cdot 5 \cdot 7$$

for distinct primes p_1 and p_2. Checking $p_2 = 2$, 3, 5, and 7, the requirement is that $\frac{1680}{(p_2-1)p_2}+1$ be prime; we have (for $p_2 = 2$, 3, 5, and 7 respectively) $841 = 29^2$, 281 which is prime, $85 = 5 \cdot 17$, and 41 which is prime. The result is that there are only two possibilities for n: $n = 281 \cdot 3^2 = 2529$ and $n = 41 \cdot 7^2 = 2009$.

Third solution by Paolo Leonetti, Milano, Italy

If $n = \prod_i p_i^{\alpha_i}$ and the number of divisor is $\prod_i \alpha_i + 1 = 6$ then n is in the form pq^2 with p, q different primes or in the form p^5. Considering the constraint $q(q-1)(p-1) = 2^4 \cdot 3 \cdot 5 \cdot 7$ we have $q \in \{2, 3, 5, 7\}$ (in the other case only $p = 2$ has the fourth power, but it is impossible for the other factors). So the only numbers are $281 \cdot 3^2$ and $41 \cdot 7^2$.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Elmir Sukanovic, Sarajevo, Bosnia and Herzegovina.

J111. Prove that there is no n for which $\prod_{k=1}^{n}(k^4+k^2+1)$ is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by John T. Robinson, Yorktown Heights, NY, USA

Let $g(k) = k^4 + k^2 + 1$. If we evaluate $g(1)$, $g(2)$, $g(3)$, etc., we notice an interesting pattern: $g(1) = 1 \cdot 3$, $g(2) = 3 \cdot 7$, $g(3) = 7 \cdot 13$, $g(4) = 13 \cdot 21$, and so on. This is because $g(k)$ factors as

$$g(k) = (k^2 - k + 1)(k^2 + k + 1), \text{ and}$$

$$\begin{aligned} g(k+1) &= ((k+1)^2 - (k+1) + 1)\,((k+1)^2 + (k+1) + 1) \\ &= (k^2 + k + 1)(k^2 + 3k + 3)\,. \end{aligned}$$

So multiplying $g(1), g(2), \ldots, g(n)$, we see that there are a sequence of squares in the product $(3^2, 7^2, 13^2, \ldots)$, but with a final factor of n^2+n+1 left over at the end. It follows that since n^2+n+1 cannot be a square (since it lies between n^2 and $(n+1)^2$), the product cannot be a square.

Second solution by Brian Bradie, Newport News, USA

First, note that

$$\prod_{k=1}^{1}(k^4+k^2+1) = 3,$$

which is not a perfect square. Now, suppose $n > 1$. Because

$$k^4 + k^2 + 1 = (k^2 + k + 1)(k^2 - k + 1)$$

and

$$(k+1)^2 - (k+1) + 1 = k^2 + k + 1,$$

it follows that

$$\prod_{k=1}^{n}(k^4+k^2+1) = (n^2+n+1)\prod_{k=1}^{n-1}(k^2+k+1)^2.$$

Moreover, because the difference between $(n+1)^2$ and n^2 is $2n+1$, n^2+n+1 cannot be a perfect square for any $n > 1$. Thus, there is no n for which $\prod_{k=1}^{n}(k^4+k^2+1)$ is a perfect square.

Third solution by Ovidiu Furdui, Campia Turzii, Cluj, Romania

Since $k^4 + k^2 + 1 = (k^2 + k + 1)(k^2 - k + 1)$ we obtain that

$$\begin{aligned} P &= \prod_{k=1}^{n} (k^4 + k^2 + 1) \\ &= \prod_{k=1}^{n} \left((k^2 + k + 1)(k^2 - k + 1)\right) \\ &= \left(\prod_{k=1}^{n} (k^2 - k + 1)\right)^2 \cdot (n^2 + n + 1). \end{aligned}$$

Thus it suffices to prove that there is no natural number $n \geq 1$ such that n^2+n+1 is a square. Let $n^2 + n + 1 = u^2$. This implies that $n^2 + n + 1 - u^2 = 0$ and this equation has rational solutions provided that the discriminant of the quadratic equation is a square. Thus, $5 = 4u^2 + \Delta^2$. This implies that $\pm\Delta = 1 = \pm u$. It is easy to see that if $u^2 = 1$ then $n^2 + n = 0$ which does not have solutions in positive integers.

Also solved by Arkady Alt, San Jose, California, USA; Navid Safaei, Tehran, Iran; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Elmir Sukanovic, Sarajevo, Bosnia and Herzegovina; Paolo Leonetti, Milano, Italy.

J112. Let a, b, c be integers such that $\gcd(a,b,c) = 1$ and $ab+bc+ca = 0$. Prove that $|a+b+c|$ can be expressed in the form x^2+xy+y^2, where x and y are integers.

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

First solution by John T. Robinson, Yorktown Heights, NY, USA

Note that a, b, and c can't all be zero ($\gcd(a,b,c)$ is either undefined or zero, depending on which variant of the definition for gcd is used). If two, say b and c, are zero, then $a = \pm 1$ and $|a+b+c| = 1^2 + 1 \cdot 0 + 0^2$. Finally if one, say c, is zero, then from $ab = 0$ we see that a or b or both has to be zero and we are back to one of the earlier cases. So we can assume that a, b, and c are non-zero. Next, for notational convenience, let $g_{ab} = \gcd(a,b)$, $g_{ac} = \gcd(a,c)$, and $g_{bc} = \gcd(b,c)$ (note that by the definition of gcd these are all positive). Since $\gcd(a,b,c) = 1$ we can write

$$a = g_{ab}g_{ac}a',$$

$$b = g_{ab}g_{bc}b',$$

$$c = g_{ac}g_{bc}c',$$

where g_{ab}, g_{ac}, g_{bc}, a', b', and c' are pairwise relatively prime. From $ab+ac+bc = 0$ we have

$$g_{ab}^2 g_{ac} g_{bc} a'b' + g_{ab} g_{ac}^2 g_{bc} a'c' + g_{ab} g_{ac} g_{bc}^2 b'c' = 0\,;$$

$$g_{ab}a'b' + g_{ac}a'c' + g_{bc}b'c' = 0$$

$$-g_{ab}a'b' = c'(g_{ac}a' + g_{bc}b')$$

Not that this implies $a'|c'(g_{ac}a'+g_{bc}b')$ which implies that $a'|(g_{ac}a'+g_{bc}b')$ (since a' and c' are relatively prime) which implies that $a'|g_{bc}b'$ which implies that $a' = \pm 1$ (since a' is relatively prime to g_{bc} and b'). The argument is symmetric, so we also have $b' = \pm 1$ and $c' = \pm 1$.

Since $ab+ac+bc = 0$ we cannot have all of a, b, and c positive or all negative; the choices are one negative and two positive, or two negative and one positive. However if two are negative and one positive, since we are looking at how $|a+b+c|$ can be expressed we can substitute $-a, -b, -c$ for a, b, c to get the one negative and two positive case, so it suffices to consider only this case. Finally, by symmetry it also suffices to consider only the case $a' = -1$, $b', c' = 1$. Substituting these into

$$g_{ab}a'b' + g_{ac}a'c' + g_{bc}b'c' = 0$$

we have

$$-g_{ab} - g_{ac} + g_{bc} = 0\,;$$

$$g_{ac} = g_{bc} - g_{ab}\,.$$

Therefore

$$a+b+c = -g_{ab}g_{ac} + g_{ab}g_{bc} + g_{ac}g_{bc} =$$

$$-g_{ab}(g_{bc} - g_{ab}) + g_{ab}g_{bc} + (g_{bc} - g_{ab})g_{bc} =$$

$$-g_{ab}g_{bc} + g_{ab}^2 + g_{ab}g_{bc} + g_{bc}^2 - g_{ab}g_{bc} =$$

$$g_{ab}^2 - g_{ab}g_{bc} + g_{bc}^2\,.$$

Choosing $x = -g_{ab}$ and $y = g_{bc}$ completes the proof.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Assume without loss of generality $a = 0$. Then, $bc = 0$, and without loss of generality $b = 0$, and since $\gcd(a,b,c) = 1$, then $|c| = |a+b+c| = 1$, for $x = 1$ and $y = 0$ or *vice versa*. Note that if without loss of generality $a+b = 0$, then $ab = 0$, leading again to $a = b = 0$. It suffices therefore to prove the proposed result for nonzero integers (a,b,c) such that $a+b$, $b+c$, and $c+a$ are nonzero.

Since simultaneously inverting the signs of a, b and c does not alter the proposed problem, assume without loss of generality that two of a,b,c are positive, without loss of generality $a,b > 0$. Since $c = -\frac{ab}{a+b}$, then $a+b$ divides ab. Calling d the greatest common divider of a,b, we may write $a = du$ and $b = dv$ for some relatively prime positive integers u,v, and $u+v$ must divide duv. If for some prime p, p^α divides $u+v$ but does not divide d, then p divides uv, or it divides one of them, and hence it divides both, absurd. Therefore, $u+v$ divides d. Let $d = (u+v)q$, where q is a positive integer, which leads to $a = u(u+v)q$, $b = v(u+v)q$ and $c = -uvq$. Clearly, q divides a,b,c, or since $\gcd(a,b,c) = 1$, then $q = 1$, and $|a+b+c| = a+b+c = u^2+uv+v^2$. The conclusion follows.

Note finally that $|a+b+c|$ can be expressed in the form x^2+xy+y^2 in at least three different ways:

$$u^2+uv+v^2 = (-u-v)^2 + (-u-v)v + v^2 = (-u-v)^2 + (-u-v)u + u^2.$$

Also solved by Arkady Alt, San Jose, California, USA; Paolo Leonetti, Milano, Italy.

J113. Call *penta*-sequence a sequence of consecutive positive integers such that each of them can be written as a sum of five nonzero perfect squares. Prove that there are infinitely many penta-sequences of length 7.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by John T. Robinson, Yorktown Heights, NY, USA

It is a result of number theory that every integer greater than or equal to 170 is a sum of five positive squares (for example see Niven and Zuckerman, *An Introduction to the Theory of Numbers*, Theorem 5.6 – in fact 170 can be lowered to 34 in this statement, but the integers in the range 34 to 169 need to be checked separately); the result follows (note that "7" can be replaced by *any* positive integer).

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Consider, for each non-negative integer n, the numbers

$$(2n+6)^2+(n+5)^2+(n+4)^2+(n+2)^2+(n+2)^2=8n^2+50n+85,$$

$$(2n+7)^2+(n+4)^2+(n+4)^2+(n+2)^2+(n+1)^2=8n^2+50n+86,$$
$$(2n+6)^2+(n+5)^2+(n+4)^2+(n+3)^2+(n+1)^2=8n^2+50n+87,$$
$$(2n+7)^2+(n+5)^2+(n+3)^2+(n+2)^2+(n+1)^2=8n^2+50n+88,$$
$$(2n+8)^2+(n+4)^2+(n+2)^2+(n+2)^2+(n+1)^2=8n^2+50n+89,$$
$$(2n+5)^2+(n+6)^2+(n+4)^2+(n+3)^2+(n+2)^2=8n^2+50n+90,$$
$$(2n+8)^2+(n+4)^2+(n+3)^2+(n+1)^2+(n+1)^2=8n^2+50n+91.$$

Clearly, for each non-negative integer n, these numbers form different penta-sequences of length 7. The conclusion follows.

Also solved by Paolo Leonetti, Milano, Italy

J114. Let p be a prime. Find all solutions to the equation $a+b-c-d=p$, where a,b,c,d are positive integers such that $ab=cd$.

Proposed by Iurie Boreico, Harvard University, USA

First solution by John T. Robinson, Yorktown Heights, NY, USA

Let $x=ab=cd$, and let $y=\gcd(a,c)$, so that $a=ya'$ and $c=yc'$ (where a' and c' are relatively prime). Since $a|x$ and $c|x$ we can write

$$x=za'c'y\,.$$

Next, $b=x/a=zc'$ and $d=x/c=za'$. Substituting into $a+b-c-d=p$ we have:

$$ya'+zc'-yc'-za'=p\,;$$
$$(y-z)(a'-c')=p$$

or equivalently

$$(z-y)(c'-a')=p\,.$$

Since p is prime, an exhaustive list of possible solutions in positve integers y,z,a',c' is as follows:

$$y=z+1 \text{ and } a'=c'+p\,;$$
$$y=z+p \text{ and } a'=c'+1\,;$$
$$z=y+1 \text{ and } c'=a'+p\,;$$
$$z=y+p \text{ and } c'=a'+1\,.$$

These solutions all work out to be equivalent if, considering z (or y) and c' (or a') to be free parameters, we allow the resulting expressions for a and b, and c and d, to be switched. For example, considering z and c' to be parameters in the first solution above, we have $a=(z+1)(c'+p)$, $b=zc'$, $c=(z+1)c'$, $d=z(c'+p)$, and similarly for the other cases. All of the above solutions can be expressed in a more concise parametric form as follows, where u and v are any two positive integers:

$$\{a,\,b\}=\{uv,\;(u+1)(v+p)\}\,;$$
$$\{c,\,d\}=\{(u+1)v,\;u(v+p)\}\,.$$

Second solution by Arkady Alt, San Jose, California, USA

We will use the notation $x\perp y$ when integers x and y are relatively prime, i.e. $\gcd(x,y)=1$ and $x\mid y$ if x divides y. Let $s=\gcd(a,c)$. Since $a=ms$,

$c = sc_1$ then $\gcd(sm, sc_1) = s \iff \gcd(m, c_1) = 1$. Let $t = \gcd(b, c_1)$. Since $b = nt$, $c_1 = tc_2$ then $\gcd(tn, tc_2) = t \iff \gcd(n, c_2) = 1$. Thus we obtain $a = ms, b = tn, c = stc_2$, where $c_2 \perp n$ and $c_2 \perp m$ (because $c_2 \mid c_1$ and $c_1 \perp m$. Hence, $c_2 \perp mn$ and

$$\gcd(ab, c) = \gcd(smtn, stxc_2) = st \gcd(mn, c_2) = st$$

and, since $c \mid ab$ yields $\gcd(ab, c) = c$,then $st = c$. Let $a = ms, b = tn, c = st$. Then $ab = cd$ yields $d = mn$ and, therefore, $a + b - c - d = p \iff ms + nt - st - mn = p \iff (m - t)(s - n) = p$. Thus we have four types of solutions:

1. $\begin{cases} m - t = -1 \\ s - n = -p \end{cases}$ then $\begin{cases} m = t - 1 \\ n = s + p \end{cases}$ and

   $a = s(t-1), b = t(s+p), c = st, d = (t-1)(s+p)$;

2. $\begin{cases} m - t = -p \\ s - n = -1 \end{cases}$ then $\begin{cases} m = t - p \\ n = s + 1 \end{cases}$ and

   $a = s(t-p), b = t(s+1), c = st, d = (t-p)(s+1)$;

3. $\begin{cases} m - t = 1 \\ s - n = p \end{cases}$ then $\begin{cases} m = t + 1 \\ n = s - p \end{cases}$ and

   $a = s(t+1), b = t(s-p), c = st, d = (s-p)(t+1)$;

4. $\begin{cases} m - t = p \\ s - n = 1 \end{cases}$ then $\begin{cases} m = t + p \\ n = s - 1 \end{cases}$ and

   $a = s(t+p), b = t(s-1), c = st, d = (s-1)(t+p)$,

where s and t are any non-zero integers.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Assume without loss of generality that $a \geq b$ and $c \geq d$, since a, b are interchangeable, and so are c, d. Clearly $a > c \geq d > b$, since if $a = c$, then $b = d$ and $p = 0$ absurd, and similarly if $d = b$. Moreover, since $ab = cd$ but $a + b > c + d$, then $a - b > c - d$, and if $a \leq c$, then $b < d$, yielding $ab < cd$ absurd. Clearly, $p(a + b + c + d) = (a + b)^2 - (b + d)^2 = (a + c)(a - c) + (b + d)(b - d)$, or

$$a + c = \frac{(b + d)(p + d - b)}{a - c - p} = b + d + \frac{p(b + d)}{d - b},$$

yielding $a - d = \frac{pd}{d-b}$ and $c - b = \frac{pb}{d-b}$.

Assume first that $d - b$ divides d and b. Hence, we may write $d - b = k$, and $d = (u + 1)k$, $b = uk$ for some positive integer u. Then, $a = (u + 1)(k + p)$ and $c = u(k + p)$. Note that these necessary forms for a, b, c, d satisfy both conditions, since $ab = cd = u(u + 1)k(k + p)$. All possible solutions in this case may then be written as $(a, b, c, d) = ((u+1)(k+p), uk, u(k+p), (u+1)k)$ for any positive integers u, k. Generality is restored by allowing permutations of a, b, of c, d, and of both.

Also solved by Paolo Leonetti, Milano, Italy

Senior problems

S109. Solve the system of equations

$$\sqrt{x}-\frac{1}{y}=\sqrt{y}-\frac{1}{z}=\sqrt{z}-\frac{1}{x}=\frac{7}{4}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Ercole Suppa, Teramo, Italy

Adding the three equations we obtain:

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{21}{4} \qquad \Rightarrow$$

$$\left(\sqrt{x}-2\right)f(x)+\left(\sqrt{y}-2\right)f(y)+\left(\sqrt{z}-2\right)f(z)=0 \tag{1}$$

where

$$f(t)=\frac{2+\sqrt{t}+4t}{4t}>0, \qquad \forall t>0$$

Now $\sqrt{x}\geq 2$. In fact if $\sqrt{x}<2$ then

$$x<4 \quad \Rightarrow$$
$$\sqrt{z}=\frac{1}{x}+\frac{7}{4}>\frac{1}{4}+\frac{7}{4}=2 \quad \Rightarrow$$
$$\sqrt{y}=\frac{1}{z}+\frac{7}{4}<\frac{1}{4}+\frac{7}{4}=2 \quad \Rightarrow$$
$$\sqrt{x}=\frac{1}{y}+\frac{7}{4}>\frac{1}{4}+\frac{7}{4}=2$$

which is impossible. Similarly $\sqrt{y}<2$ or $\sqrt{z}<2$ leads to contradiction. Thus

$$\sqrt{x}-2\geq 0 \quad , \quad \sqrt{y}-2\geq 0 \quad , \quad \sqrt{z}-2\geq 0$$

Therefore, since $f(x)\geq 0$, $f(y)\geq 0$, $f(z)\geq 0$ from (1) follows that:

$$\sqrt{x}=\sqrt{y}=\sqrt{z}=2 \quad \Rightarrow \quad x=y=z=4$$

and we are done.

Second solution by Francisco Javier García Capitán, Spain

We have $x=f(y), y=f(z), z=f(x)$ where $f(x)=(\frac{1}{x}+\frac{7}{4})^2$ is a strictly decreasing function over the positive real numbers. Hence we have $f(f(f(x)))=x$ and the same for y and z.

Suppose that $f(x) > x$. Then we have $f(f(x)) < f(x)$ and $x = f(f(f(x))) > f(f(x))$. And, applying f again, we have $f(x) < x$, contradiction. In a similar way we discard the case $f(x) < x$ and we have $f(x) = x$, and the same for y and z.

Solving $f(x) = x$ we get only a real solution $x = 4$, so the solution of our system is $x = y = z = 4$.

Third solution by Magkos Athanasios, Kozani, Greece

Set $\sqrt{x} = a, \sqrt{y} = b, \sqrt{z} = c$. We have $a, b, c > 0$ and the system becomes

$$a - \frac{1}{b^2} = \frac{7}{4}, b - \frac{1}{c^2} = \frac{7}{4}, c - \frac{1}{a^2} = \frac{7}{4}.$$

Pairwise subtractions of these equations yield

$$a - b = \frac{(c-b)(c+b)}{(bc)^2}, b - c = \frac{(a-c)(a+c)}{(ac)^2}, c - a = \frac{(b-a)(b+a)}{(ab)^2}.$$

Multiplying these equations we obtain

$$(a-b)(b-c)(c-a) = \frac{(b-a)(a-c)(c-b)(a+b)(b+c)(c+a)}{(abc)^4}.$$

Now, if the system had a solution (a, b, c) with $a \neq b \neq c \neq a$, the last equation would imply $(a+b)(b+c)(c+a) = -(abc)^4$, which is impossible since $a, b, c > 0$. Hence two of a, b, c are equal, say $a = b$. Then from the first equation we easily find $a = b = 2$ and from the third equation we find $c = 2$. Hence, the original system has the unique solution $(4, 4, 4)$.

Also solved by Baleanu Andrei Razvan, Romania; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; John T. Robinson, Yorktown Heights, NY, USA; Oleh Faynshteyn, Leipzig, Germany; Navid Safaei, Tehran, Iran; Daniel Lasaosa, Universidad Publica de Navarra, Spain.

S110. Let X be a point on the side BC of a triangle ABC. The parallel through X to AB meets CA at V and the parallel through X to AC meets AB at W. Let $D = BV \cap XW$ and $E = CW \cap XV$. Prove that DE is parallel to BC and

$$\frac{1}{DE} = \frac{1}{BX} + \frac{1}{CX}.$$

Proposed by Francisco Javier García Capitán, Spain

First solution by Miguel Amengual Covas, Mallorca, Spain

By Thales' theorem,

$$\frac{AV}{VC} = \frac{WE}{EC} \tag{1}$$

Since $WX \parallel CA$, we get $\frac{AV}{WD} = \frac{BV}{BD} = \frac{VC}{DX}$, so

$$\frac{AV}{VC} = \frac{WD}{DX} \tag{2}$$

so that we have from (1) and (2),

$$\frac{WE}{EC} = \frac{WD}{DX}$$

Thus $DE \parallel XC$, that is $DE \parallel BC$.

We extend DE to meet AB at F and obtain

$$\frac{FD}{BX} = \frac{WD}{WX} = \frac{DE}{XC}$$

Therefore

$$\begin{aligned}\frac{DE}{BX} + \frac{DE}{XC} &= \frac{DE}{BX} + \frac{FD}{BX} \\ &= \frac{DE + FD}{BX} \\ &= \frac{FE}{BX} \\ &= 1\end{aligned}$$

since $FE = BX$ follows because the quadrilateral $FBXE$ is a parallelogram.

Hence

$$\frac{1}{DE} = \frac{1}{BX} + \frac{1}{CX}$$

as desired.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Call $\vec{u} = \vec{AB}$ and $\vec{v} = \vec{AC}$. Clearly, $\vec{BC} = \vec{v} - \vec{u}$, and for any $X \in BC$, a real ρ exists such that $\vec{BX} = \rho\vec{BC}$, $\vec{AX} = (1-\rho)\vec{u} + \rho\vec{v}$. Since $V \in AC$, then $\vec{AV} = \alpha\vec{v}$ for some real κ. Moreover, since $\vec{XV} \parallel \vec{AB}$, the coefficient of $\vec{v}$ is the same for $\vec{AV}$ and $\vec{AX}$, or $\vec{AV} = \rho\vec{v}$. Similarly, $\vec{AW} = (1-\rho)\vec{u}$. Since $D \in BV$, then $\vec{AD} = \kappa\vec{AB} + (1-\kappa)\vec{AV}$ for some real κ, or $\vec{AD} = \kappa\vec{u} + (1-\kappa)\rho\vec{v}$, and since $D \in XW$, then the coefficient of $\vec{u}$ in $\vec{AD}$ is $1-\rho$. Or, $\kappa = 1-\rho$ and $\vec{AD} = (1-\rho)\vec{u} + \rho^2\vec{v}$. Similarly, $\vec{AE} = (1-\rho)^2\vec{u} + \rho\vec{v}$, or $\vec{DE} = \vec{AE} - \vec{AD} = \rho(1-\rho)\left(\vec{v} - \vec{u}\right) = \rho(1-\rho)\vec{BC}$. Hence, $\vec{DE} \parallel \vec{BC}$, and

$$\frac{1}{DE} = \frac{1}{\rho(1-\rho)BC} = \frac{1}{\rho BC} + \frac{1}{(1-\rho)BC} = \frac{1}{BX} + \frac{1}{CX}.$$

Note that the result is true for any real ρ except for $\rho = 0, 1$, ie, we may take X on any point of line BC except for B or C, and the result will always hold as long as we take signed distances for BX and CX, in other words, the last relation will be valid as long as $BX \leq 0$ if and only if X is on line BC but not on ray BC, and similarly for CX.

Also solved by Baleanu Andrei Razvan, Romania; Salem Malikic, Bosnia and Herzegovina.

S111. Prove that there are infinitely many positive integers n that can be expressed as $a^4 + b^4 + c^4 + d^4 - 4abcd$, where a, b, c, d are positive integers, such that n is divisible by the sum of its digits.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by John T. Robinson, Yorktown Heights, NY, USA

For $j \geq 1$, let

$$n = 196 \cdot 10^{4j-2} + 2 \cdot 10^{4j-4}, \text{ then}$$

$$n = (10^j)^4 + (10^j)^4 + (10^{j-1})^4 + (10^{j-1})^4 - 4(10^j)(10^j)(10^{j-1})(10^{j-1}),$$

and n is divisible by the sum of its digits (18) since n is even and also divisible by 9 by the decimal integer divisibility test for 9 (i.e. the sum of the digits is divisible by 9).

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Note that it suffices to find one, since if $n = a^4 + b^4 + c^4 + d^4 - 4abcd$, then $n' = (10^m a)^4 + (10^m b)^4 + (10^m c)^4 + (10^b d)^4 - 4(10^m a)(10^m b)(10^m b)(10^m c) = 10^{4m} n$ has the same sum of digits as n, and if n is divisible by the sum of its digits, so is n'. Note now that $(10^5)^4 + (10^4)^4 + (2 \cdot 10^3)^4 + 1^4 - 4 \cdot 2 \cdot 10^{5+4+3} = 10^{20} + 10^{16} + 8 \cdot 10^{12} + 1$ has sum of digits 11, and is clearly divisible by 11 since the sum of its digits in odd positions is 11, and the sum of its digits in even positions is 0. The conclusion follows.

S112. Let a, b, c be the side lengths and let s be the semiperimeter of a triangle ABC. Prove that

$$(s-c)^a(s-a)^b(s-b)^c \leq \left(\frac{a}{2}\right)^a \left(\frac{b}{2}\right)^b \left(\frac{c}{2}\right)^c.$$

Proposed by Johan Gunardi, Jakarta, Indonesia

First solution by Arkady Alt, San Jose, California, USA

By the weighted AM-GM inequality we have

$$\left(\frac{s-c}{a}\right)^a \left(\frac{s-a}{b}\right)^b \left(\frac{s-b}{c}\right)^c \leq \left(\frac{\frac{s-c}{a}\cdot a + \frac{s-a}{b}\cdot b + \frac{s-b}{c}\cdot c}{a+b+c}\right)^{a+b+c} =$$

$$\frac{1}{2^{a+b+c}} \iff (s-c)^a\,(s-a)^b\,(s-b)^c \leq \left(\frac{a}{2}\right)^a \left(\frac{b}{2}\right)^b \left(\frac{c}{2}\right)^c.$$

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

If the triangle is degenerate, then the LHS is identically zero, while the RHS is non-negative, the inequality being trivially true. We need to consider thus only non-degenerate triangles. Dividing by the RHS and taking logarithm, the inequality may be rewritten in the following equivalent form:

$$a\log\left(1+\frac{b-c}{a}\right) + b\log\left(1+\frac{c-a}{b}\right) + c\log\left(1+\frac{a-b}{c}\right) \leq 0.$$

Clearly, $g(x) = x - \log(1+x)$ is zero for $x = 0$, while $\frac{dg(x)}{dx} = 1 - \frac{1}{1+x}$. Note that if $x > 0$, then $g(x)$ strictly increases, while if $x < 0$, then $g(x)$ strictly decreases, or the maximum of $g(x)$ is 0, occurring iff $x = 0$. As a consequence, $x \geq \log(1+x)$ for all $x > -1$, with equality iff $x = 0$. The triangular inequality guarantees that $-1 < \frac{b-c}{a} < 1$ for non-degenerate triangles, or $a\log\left(1+\frac{b-c}{a}\right) \leq b-c$ with equality iff $b = c$. Adding this inequality to its cyclic permutations we obtain the proposed inequality. Equality holds if and only if, either $a = b = c$, or the triangle is degenerate with two equal sides and one side of length 0.

Third solution by Nguyen Manh Dung, Hanoi University of Science, Vietnam

The inequality is equivalent to

$$(a+b-c)^a(b+c-a)^b(c+a-b)^c \leq a^a b^b c^c$$

It suffices to shows that

$$a\ln\frac{a+b-c}{a} + b\ln\frac{b+c-a}{b} + c\ln\frac{c+a-b}{c} \leq 0$$

Since the function $f(x) = \ln x$ is concave, by Jensen's Inequality we get

$$\begin{aligned} a \ln \frac{a+b-c}{a} &+ b \ln \frac{b+c-a}{b} + c \ln \frac{c+a-b}{c} \\ &\leq (a+b+c) \ln \left(\frac{a+b-c+b+c-a+c+a-b}{a+b+c} \right) \\ &= (a+b+c) \ln 1 = 0 \end{aligned}$$

Hence we are done.
The equality holds when $a = b = c$.

Fourth solution by Magkos Athanasios, Kozani, Greece

Observe that the inequality we have to prove is equivalent to

$$\left(1 + \frac{b-c}{a}\right)^a \left(1 + \frac{c-a}{b}\right)^b \left(1 + \frac{a-b}{c}\right)^c \leq 1.$$

Employing the well known inequality $e^x \geq 1 + x$ and setting $x = \frac{b-c}{a}$, we get

$$e^{\frac{b-c}{a}} \geq 1 + \frac{b-c}{a} \Rightarrow \left(1 + \frac{b-c}{a}\right)^a \leq e^{b-c}.$$

Analogously, we have

$$\left(1 + \frac{c-a}{b}\right)^b \leq e^{c-a}, \left(1 + \frac{a-b}{c}\right)^c \leq e^{a-b}.$$

Multiplying these three inequalities we get the desired result. Since $e^x = 1+x \Leftrightarrow x = 0$, it is clear that the equality holds iff $a = b = c$.

S113. Prove that for different choices of signs + and − the expression

$$\pm 1 \pm 2 \pm 3 \pm \cdots \pm (4n+1),$$

yields all odd positive integers less than or equal to $(2n+1)(4n+1)$.

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania

First solution by John T. Robinson, Yorktown Heights, NY, USA

If we take all signs positive we have

$$1+2+3+\cdots+4n+(4n+1)=(2n+1)(4n+1)=S\,.$$

Consider now what happens if on the left hand side we change the sign of exactly one summand: if we replace 1 by -1 this subtracts 2 from S; if we replace 2 by -2 this subtracts 4 from S; and so on, up to replacing $(4n+1)$ by $-(4n+1)$ which subtracts $(8n+2)$ from S. Clearly this process can be repeated, leaving the rightmost terms with the signs changed alone on the next iteration (that is for the first iteration all terms are initially positive, for the second iteration the $(4n+1)$ term starts out negative, for the third iteration the $4n$ and $(4n+1)$ terms start out negative, and so on). This gives a decreasing sequence of consecutive odd positive integers starting with S and ending with $-S$, and this sequence includes $1, 3, 5, \ldots, S$ as a sub-sequence.

Second solution by Brian Bradie, Newport News, USA

If we take all of the signs to be +, we generate $(2n+1)(4n+1)$ because

$$\sum_{j=1}^{4n+1} j = \frac{(4n+1)(4n+2)}{2} = (2n+1)(4n+1).$$

Now, let x be an odd integer less than $(2n+1)(4n+1)$. Suppose

$$x \geq (2n+1)(4n+1) - 2(4n+1) = (2n-1)(4n+1).$$

Because $(2n+1)(4n+1)$ and x are both odd, it follows that there exists a natural number k with $1 \leq k \leq 4n+1$ such that $(2n+1)(4n+1) - x = 2k$. Thus,

$$x = (2n+1)(4n+1) - 2k = \sum_{j=1}^{4n+1} j - 2k = \sum_{j=1, j\neq k}^{4n+1} j - k;$$

that is, take the + sign for all integers except k for which we take the − sign. Next, suppose $x < (2n-1)(4n+1)$ and let ℓ be the smallest natural number for which

$$\sum_{j=1}^{\ell} j - \sum_{j=\ell+1}^{4n+1} j > x.$$

Because

$$\sum_{j=1}^{\ell} j - \sum_{j=\ell+1}^{4n+1} j = \sum_{j=1}^{4n+1} j - 2\sum_{j=\ell+1}^{4n+1} j$$

is odd and x is also odd, there exists a natural number k with $1 \le k \le \ell$ such that

$$\sum_{j=1}^{\ell} j - \sum_{j=\ell+1}^{4n+1} j - x = 2k.$$

Thus,

$$x = \sum_{j=1}^{\ell} j - \sum_{j=\ell+1}^{4n+1} j - 2k = \sum_{j=1, j\neq k}^{\ell} j - k - \sum_{j=\ell+1}^{4n+1} j;$$

that is, take the $+$ sign on 1 through ℓ except k and the $-$ sign on k and $\ell+1$ through $4n+1$.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Clearly, the result is true for $n = 1$ since $15 = 5+4+3+2+1$, $13 = 5+4+3+2-1$, $11 = 5+4+3-2+1$, $9 = 5+4-3+2-1$, $7 = 5-4+3+2+1$, $5 = 5-4+3+2-1$, $3 = 5-4+3-2+1$ and $1 = 5-4-3+2+1$. Assume now that the result is true for $n-1$. Then, since $(4n+1) + 4n + (4n-1) + (4n-2) = 16n-2$, adding the sum of the remaining $4n-3$ elements with all combinations of signs that produce all positive integers between 1 and $(2n-1)(4n-3)$, we obtain all odd numbers between $16n-1$ and $8n^2+6n+1 = (4n+1)(2n+1)$. Moreover, subtracting from $16n-2$ the same sums of the remaining $4n-3$ elements with the same combinations of signs, we obtain all odd integers between $16n-3$ and $-8n^2+10n-3 = -(2n-1)(4n-3)$. Clearly, this last integer is negative for any positive integer n, so all odd numbers between 1 and $(2n+1)(4n+1)$ have thus been generated. The conclusion follows.

S114. Consider triangle ABC with angle bisectors AA_1, BB_1, CC_1. Denote by U the intersection of AA_1 and B_1C_1. Let V be the projection from U to BC. Let W be the intersection of the angle bisectors of $\angle BC_1V$ and $\angle CB_1V$. Prove that A, V, W are collinear.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

If ABC is isosceles at A, then $AA_1 = AU = AV$ is an axis of symmetry, and the internal bisectors of $\angle BC_1V$ and $\angle CB_1V$ are parallel to each other and symmetric around AV. In this sense, W is a point "at infinity" on AV. Otherwise, assume withoit loss of generality that $b > c$ and thus $\angle B > \angle C$. We begin by calculating BV and CV. The theorem of the transversals guarantees that

$$\frac{CA_1 \cdot BC_1}{AC_1} + \frac{BA_1 \cdot CB_1}{AB_1} = \frac{BC \cdot UA_1}{AU},$$

or since, as it is well known, $\frac{BA_1}{CA_1} = \frac{BA}{CA}$, and similarly for its cyclic permutations, then $\frac{2a}{b+c} = \frac{UA_1}{AU} = \frac{UA_1}{AA_1 - UA_1}$, and $UA_1 = \frac{2aAA_1}{2a+b+c}$. Now, applying Stewart's theorem to the internal bisector of $\angle BAC$, we find the well-known result $AA_1 = \frac{2bc\cos\frac{A}{2}}{b+c}$, or

$$VA_1 = UA_1 \sin\frac{B-C}{2} = \frac{4abc\sin\frac{B+C}{2}\sin\frac{B-C}{2}}{(2a+b+c)(b+c)} = \frac{2abc(\cos C - \cos B)}{(2a+b+c)(b+c)},$$

where we have used that UV is perpendicular to BC and VA_1 is the internal bisector of $\angle BAC$ to deduce that $\angle VUA_1 = \frac{B-C}{2}$. Finally,

$$BV = BA_1 - VA_1 = \frac{a^2+ac+c^2-b^2}{2a+b+c}, \qquad CV = a - BV = \frac{a^2+ab+b^2-c^2}{2a+b+c}.$$

Let now the internal bisector of $\angle BC_1V$ intersect BC at X and AV at W_1, and let the internal bisector of $\angle CB_1V$ intersect BC at Y and AV at W_2. Clearly, $\frac{XB}{XV} = \frac{C_1B}{C_1V}$, and $\frac{YC}{YV} = \frac{B_1C}{B_1V}$, while Menelaus' theorem guarantees that

$$1 = \frac{VW_1}{W_1A}\frac{AC_1}{C_1B}\frac{BX}{XV}, \qquad 1 = \frac{VW_2}{W_2A}\frac{AB_1}{B_1C}\frac{CY}{YV}.$$

The proposed result is clearly equivalent to $W_1 = W_2$, and it suffices to show that $\frac{VW_1}{W_1A} = \frac{VW_2}{W_2A}$, or equivalently,

$$\frac{B_1V}{C_1V} = \frac{AB_1}{AC_1} = \frac{a+b}{a+c}, \qquad (a+c)^2 B_1V^2 = (a+b)^2 C_1V^2.$$

Now, the Cosine Law allows us to write $\cos C = \frac{a^2+b^2-c^2}{2ab}$ and

$$C_1V^2 = BV^2 + BC_1^2 - 2BV \cdot BC_1 \cos C = BV^2 + BC_1^2 - \frac{BV \cdot BC_1}{a}(a^2+b^2-c^2).$$

After some algebra,

$$(a+b)^2 C_1V^2 = \frac{a^2(a+b)^2(a+c)^2}{(2a+b+c)^2} - \frac{(a+b)(a+c)(b^2-c^2)^2}{(2a+b+c)^2} -$$

$$-\frac{a(a+b)(a+c)(b+c)(b-c)^2}{(2a+b+c)^2} + a^2\frac{(b^2+c^2) - a(a+b)(a+c) + (b^3+c^3)}{2a+b+c}.$$

Note that the RHS is invariant under exchange of b and c, or $(a+c)^2 B_1V^2 = (a+b)^2 C_1V^2$. The result follows.

Second solution by Ercole Suppa, Teramo, Italy

Denote as usual by a, b, c, R the sides BC, CA, AB and the circumradius of $\triangle ABC$, respectively. Let X, W_1 be the points where the the angle bisector of $\angle BC_1V$ intersects BC, AV and let Y, W_2 be the points where the the angle bisector of $\angle VB_1C$ intersects BC, AV (refere to FIGURE 1). In order to complete the proof is enough to show that $W_1 = W_2$.

Applying Menelaus's theorem to triangle $\triangle ABV$ and the line XC_1W_1, and taking into account that $BX : XV = C_1B : C_1V$ (internal bisector theorem), we have:

$$\frac{AC_1}{C_1B} \cdot \frac{C_1B}{C_1V} \cdot \frac{VW_1}{W_1A} = 1 \tag{1}$$

In similar way applying Menelaus's theorem to triangle $\triangle AVC$ and the line YB_1W_2, and taking into account that $VY : YC = B_1V : B_1C$, we have:

$$\frac{AB_1}{B_1C} \cdot \frac{B_1C}{B_1V} \cdot \frac{VW_2}{W_2A} = 1 \tag{2}$$

From (1), (2) follows that:

$$\frac{AC_1}{C_1V} \cdot \frac{VW_1}{W_1A} = \frac{AB_1}{B_1V} \cdot \frac{VW_2}{W_2A} \tag{3}$$

In order to compute the ratio $\frac{C_1V}{B_1V}$, we draw the perpendiculars from C_1, B_1 to BC and denote with C_2, B_2 the projections of C_1, B_1, respectively.

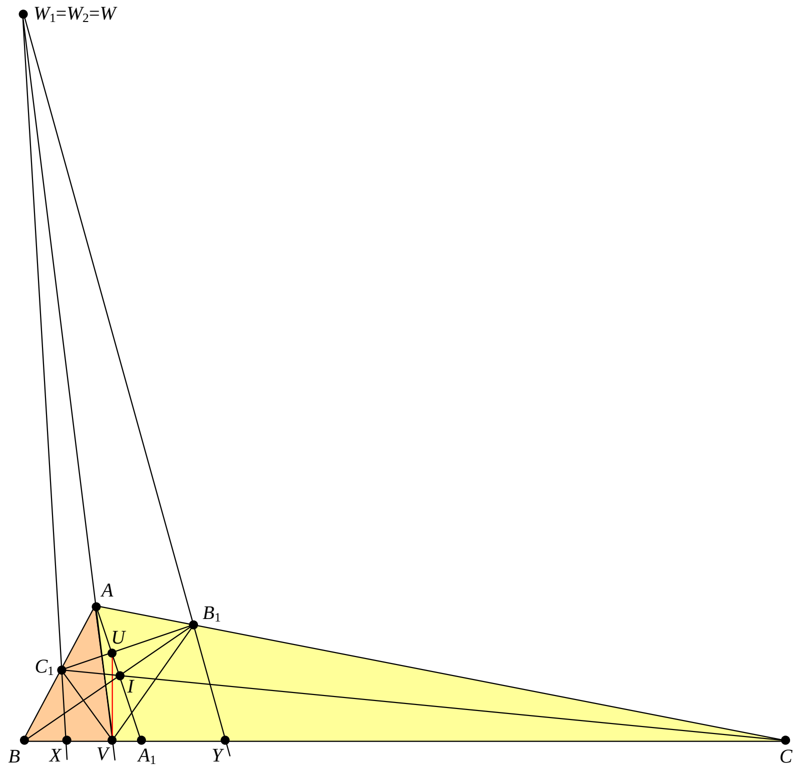

Figure 1

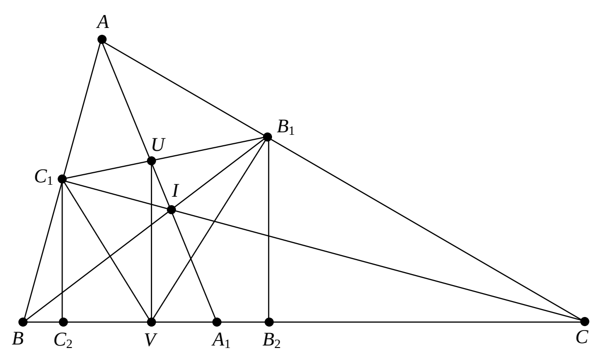

Figure 2

Using the Thales theorem and the well known relations $AC_1 = \frac{bc}{a+b}$, $BC_1 = \frac{ac}{a+b}$, $AB_1 = \frac{bc}{a+c}$, $CB_1 = \frac{ab}{a+c}$ we have:

$$\frac{C_1C_2}{B_1B_2} = \frac{BC_1 \sin B}{CB_1 \sin C} = \frac{\frac{ac}{a+b}\frac{2R}{a}}{\frac{ab}{a+c}\frac{2R}{b}} = \frac{a+c}{a+b} \tag{4}$$

$$\frac{C_2V}{VB_2} = \frac{C_1U}{UB_1} = \frac{AC_1}{AB_1} = \frac{\frac{bc}{a+b}}{\frac{bc}{a+c}} = \frac{a+c}{a+b} \tag{5}$$

From (4), (5) follows that:

$$\frac{C_1C_2}{B_1B_2} = \frac{C_2V}{VB_2} \tag{6}$$

Thus the right triangles $\triangle C_1C_2V$, $\triangle B_1B_2V$ are similar, so:

$$\frac{C_1V}{B_1V} = \frac{C_2V}{VB_2} = \frac{C_1U}{UB_1} = \frac{AC_1}{AB_1} \tag{7}$$

From (3), (7) follows that

$$\frac{VW_1}{W_1A} = \frac{VW_2}{W_2A} \qquad \Longrightarrow \qquad W_1 = W_2$$

and the proof is completed.

Undergraduate problems

U109. Find all pairs (m, n) of integers such that $m^2 + 2mn - n^2 = 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania

Solution by John T. Robinson, Yorktown Heights, NY, USA

The equation can be written as

$$(m+n)^2 - 2n^2 = 1 .$$

Let $x = m + n$, then we have

$$x^2 - 1 = (x-1)(x+1) = 2n^2 .$$

We see from this that x must be odd, so substituting $x = 2y + 1$:

$$2y(2y+2) = 2n^2 ;$$

$$2y(y+1) = n^2 .$$

We now see that n must be even, so substituting $n = 2z$:

$$2y(y+1) = 4z^2 ;$$

$$\frac{y(y+1)}{2} = z^2 .$$

From this we see that z^2, if non-zero, must be a square triangular number. There are various formulas for these numbers; one is the following: let p_k/q_k be the kth convergent in the continued fraction expansion for $\sqrt{2}$, then the kth square triangular number is $p_k^2 q_k^2$. Therefore, letting $z = \pm p_k q_k$ and working backwards, we have:

$$n = \pm 2 p_k q_k ;$$

$$(m+n)^2 = 1 + 2n^2 = 1 + 8p_k^2 q_k^2 ;$$

$$m + n = \pm\sqrt{1 + 8p_k^2 q_k^2} ;$$

$$m = \mp 2p_k q_k \pm \sqrt{1 + 8p_k^2 q_k^2} .$$

So in addition to $(m, n) = (1, 0)$, a general solution is

$$(m, n) = (\mp 2p_k q_k \pm \sqrt{1 + 8p_k^2 q_k^2}\, , \pm 2p_k q_k)$$

where p_k/q_k is the kth convergent in the continued fraction expansion for $\sqrt{2}$, and where the signs in this expression are chosen as $++-$, $-++$, $+--$, or $--+$ (that is, the first and last signs are opposite, and the middle sign can be chosen independently).

Also solved by Arkady Alt, San Jose, California, USA; Brian Bradie, Newport News, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Minghua Lin, China, Paolo Leonetti, Milano, Italy.

U110. Let $a_1, a_2, \ldots, a_n$ be real numbers with $a_n,\ a_0 \neq 0$ such that the polynomial $P(X) = (-1)^n a_n X^n + (-1)^{n-1} a_{n-1} X^{n-1} + \cdots + a_2 X^2 - a_1 X + a_0$ has all of its roots in the interval $(0, \infty)$, and let $f : \mathbb{R} \to \mathbb{R}$ be an n-time differentiable function. Prove that if

$$\lim_{x\to\infty} \left(a_n f^{(n)}(x) + a_{n-1} f^{(n-1)}(x) + \cdots + a_2 f^{''}(x) + a_1 f^{'}(x) + a_0 f(x) \right) = L \in \overline{\mathbb{R}},$$

then $\lim_{x\to\infty} f(x)$ exists and $\lim_{x\to\infty} f(x) = \frac{L}{a_0}$.

Proposed by Radu Ţiţiu, Targu-Mures, Romania

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

The proposed result will be first proved by induction over n in the particular case $L = 0$. For $n = 0$, the result is trivially true, since the condition is actually

$$\lim_{x\to\infty} a_0 f(x) = 0 \text{ with } a_0 \neq 0.$$

For $n = 1$, $P(X) = -a_1 X + a_0$ has a positive real root, ie, a_0 and a_1 have the same sign, and we may assume without loss of generality that $a_0, a_1 > 0$. The condition $\lim_{x\to\infty} (a_1 f'(x) + a_0 f(x)) = 0$ is equivalent to stating that, for any positive $\epsilon > 0$, there exists one real value M such that, for all $x > M$, then $|a_1 f'(x) + a_0 f(x)| < \epsilon$. Define now the (possibly empty) set X_0 such that any $x_0 \in X_0$ satisfies $x_0 > M$ and $f'(x_0) = 0$. Clearly, for all $x_0 \in X_0$, $|f(x_0)| < \frac{\epsilon}{a_0}$. If X_0 does not have an upper bound, then for any x which is larger than some x_0 we may find $x_- < x_+ \in X_0$ such that $x \in (x_-, x_+)$. If $|f(x)| > \frac{\epsilon}{a_0}$ for some x larger than some x_0, then clearly there must be a local maximum or minimum in (x_-, x_+) where $|f(x)| > \frac{\epsilon}{a_0}$. In this local maximum or minimum, $f'(x) = 0$, reaching a contradiction. Hence if $f(x)$ does not converge to 0, the set X_0 has an upper bound, and some real N exists such that $f'(x)$ does not change signs for $x > N$, ie, $f(x)$ is monotonous for $x > N$. Since we may exchange $f(x)$ by $-f(x)$ without altering the problem, assume without loss of generality that $f(x)$ is increasing, hence $f'(x) > 0$, for all $x > N$. Therefore, for all $x > N$, either $f(x) > 0$ and $f(x) < \frac{\epsilon - a_1 f'(x)}{a_0} < \frac{\epsilon}{a_0}$, or $f(x)$ is negative. In either case, $f(x)$ is an increasing function with an upper bound ϵ, hence it has a limit. Calling ℓ this limit, clearly $\lim_{x\to\infty} f'(x) = -\frac{\ell a_0}{a_1} = 0$, since otherwise $f(x)$ would not have a limit. Therefore, $\ell = 0$, and the result is proved for $n = 1$.

Assume now that the result is proved for $1, 2, \ldots, n-1$. Choose any root $r \in \mathbb{R}^+$ of $P(X)$. Clearly, $P(X) = (X - r)Q(X)$, where $Q(x)$ is an $n-1$-degree polynomial with all roots real and positive. Defining the operator $\Delta = \frac{d}{dx}$, clearly the condition is written as

$$\lim_{x\to\infty} (P(-\Delta) f(x)) = 0, \qquad \text{or equivalently} \qquad \lim_{x\to\infty} (Q(-\Delta)(\Delta + r) f(x)) = 0.$$

Note therefore that $(\Delta + r)f(x) = f'(x) + rf(x)$ is a real function, $n-1$-times differentiable, and such that $\lim_{x\to\infty}(Q(-\Delta)f(x)) = 0$ for some polynomial $Q(X)$ with all its roots real and positive. Therefore, $(\Delta + r)f(x)$ satisfies the condition of the problem for $n-1$, and by hypothesis of induction $\lim_{x\to\infty}((\Delta + r)f(x)) = 0$, where clearly $-X + r$ is a linear polynomial with real and positive root r, and $f(x)$ is differentiable at least once. Hence $f(x)$ satisfies the condition of the problem for $n = 1$. The result for $L = 0$ follows.

If $L \neq 0$, define $g(x) = f(x) - \frac{L}{a_0}$. Clearly, $g(x)$ satisfies the conditions of the problem with $L = 0$, and $\lim_{x\to\infty} g(x) = 0$. The conclusion follows.

U111. Let n be a given positive integer and let $a_k = 2\cos\frac{\pi}{2^{n-k}}$, $k = 0, 1, \ldots, n-1$. Prove that

$$\prod_{k=0}^{n-1}(1-a_k) = \frac{(-1)^{n-1}}{1+a_0}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Brian Bradie, Newport News, USA

This problem is very similar to Problem 1097 which appeared in the Pi Mu Epsilon Journal (volume 12, number 2, Spring 2005, pp. 113-114). Note that

$$\begin{aligned}
1-a_k &= \left(1-2\cos\frac{\pi}{2^{n-k}}\right)\frac{1+2\cos\frac{\pi}{2^{n-k}}}{1+2\cos\frac{\pi}{2^{n-k}}} = \frac{1-4\cos^2\frac{\pi}{2^{n-k}}}{1+2\cos\frac{\pi}{2^{n-k}}} \\
&= \frac{1-2\left(1+\cos\frac{\pi}{2^{n-k-1}}\right)}{1+2\cos\frac{\pi}{2^{n-k}}} = -\frac{1+2\cos\frac{\pi}{2^{n-k-1}}}{1+2\cos\frac{\pi}{2^{n-k}}} \\
&= -\frac{1+a_{k+1}}{1+a_k}.
\end{aligned}$$

Thus, the indicated product telescopes, and

$$\prod_{k=0}^{n-1}(1-a_k) = (-1)^n\frac{1+2\cos\pi}{1+a_0} = \frac{(-1)^{n+1}}{1+a_0} = \frac{(-1)^{n-1}}{1+a_0}.$$

Second solution by Darij Grinberg, Germany

Lemma 1. For every $t\in\mathbb{R}$, we have

$$(2\cos t-1)(2\cos t+1) = 2\cos(2t)+1.$$

Proof. We have

$$(2\cos t-1)(2\cos t+1) = 4\cos^2 t - 1 = 2\underbrace{\left(2\cos^2 t-1\right)}_{=\cos(2t)}+1 = 2\cos(2t)+1,$$

and Lemma 1 is proved.

Lemma 2. For every $k\in\{0,1,...,n-1\}$, we have

$$a_k - 1 = \frac{a_{k+1}+1}{a_k+1},$$

where we set $a_n = -2$ (so that $a_k = 2\cos\dfrac{\pi}{2^{n-k}}$ holds for all $k\in\{0,1,...,n\}$).

Proof. We have $a_k + 1 \neq 0$ (since $a_k = 2\cos \underbrace{\frac{\pi}{2^{n-k}}}_{\in[0,\pi/2]} > 0$) and

$$(a_k - 1)(a_k + 1) = \left(2\cos\frac{\pi}{2^{n-k}} - 1\right)\left(2\cos\frac{\pi}{2^{n-k}} + 1\right) = 2\cos\left(2\frac{\pi}{2^{n-k}}\right) + 1 \qquad \text{(by Lemma 1)}$$
$$= 2\cos\frac{\pi}{2^{n-(k+1)}} + 1 = a_{k+1} + 1,$$

so that $a_k - 1 = \dfrac{a_{k+1}+1}{a_k+1}$. Lemma 2 is proved.

Now,

$$\prod_{k=0}^{n-1}(1-a_k) = \prod_{k=0}^{n-1}(-(a_k-1)) = (-1)^n\prod_{k=0}^{n-1}(a_k-1) = (-1)^n\prod_{k=0}^{n-1}\frac{a_{k+1}+1}{a_k+1} \qquad \text{(by Lemma 2)}$$
$$= (-1)^n\frac{\prod_{k=0}^{n-1}(a_{k+1}+1)}{\prod_{k=0}^{n-1}(a_k+1)} = (-1)^n\frac{\prod_{k=1}^{n}(a_k+1)}{\prod_{k=0}^{n-1}(a_k+1)} = (-1)^n\frac{a_n+1}{a_0+1}$$
$$= (-1)^n\frac{-2+1}{a_0+1} = (-1)^n\frac{-1}{a_0+1} = \frac{-(-1)^n}{1+a_0} = \frac{(-1)^{n-1}}{1+a_0}.$$

Also solved by Arkady Alt, San Jose, California, USA; Navid Safaei, Tehran, Iran; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Paolo Leonetti, Milano, Italy.

U112. Let x, y, z be real numbers greater than 1. Prove that

$$x^{x^3+2xyz} \cdot y^{y^3+2xyz} \cdot z^{z^3+2xyz} \geq (x^x y^y z^z)^{xy+yz+zx}.$$

Proposed by Cezar Lupu, University of Bucharest, Romania and Valentin Vornicu, San Diego, USA

First solution by Darij Grinberg, Germany

Lemma 1. Let x, y, z, a, b, c be nonnegative reals such that $x \geq y \geq z$ and $ax \geq by$. Then,

$$\left(x^3 + 2xyz\right) a + \left(y^3 + 2xyz\right) b + \left(z^3 + 2xyz\right) c \geq (yz + zx + xy)(xa + yb + zc).$$

Proof. Since $x \geq y \geq z$ and $ax \geq by$, the Vornicu-Schur inequality[1], applied to $A = x$, $B = y$, $C = z$, $X = ax$, $Y = by$, $Z = cz$, yields

$$ax(x-y)(x-z) + by(y-z)(y-x) + cz(z-x)(z-y) \geq 0.$$

This rewrites as

$$\left(x^3 + 2xyz\right) a + \left(y^3 + 2xyz\right) b + \left(z^3 + 2xyz\right) c - (yz + zx + xy)(xa + yb + zc) \geq 0.$$

Thus,

$$\left(x^3 + 2xyz\right) a + \left(y^3 + 2xyz\right) b + \left(z^3 + 2xyz\right) c \geq (yz + zx + xy)(xa + yb + zc),$$

proving Lemma 1.

Let $a = \ln x$, $b = \ln y$, $c = \ln z$. Then, a, b, c are nonnegative (since x, y, z are greater or equal to 1). Without loss of generality assume that $x \geq y \geq z$ (we can assume this since the inequality is symmetric). Then, $ax \geq by$ (since a, b, x, y are nonnegative and $a \geq b$ and $x \geq y$, where $a \geq b$ is because $x \geq y$ yields $\underbrace{\ln x}_{=a} \geq \underbrace{\ln y}_{=b}$). Hence, Lemma 1 yields

$$\left(x^3 + 2xyz\right) a + \left(y^3 + 2xyz\right) b + \left(z^3 + 2xyz\right) c \geq (yz + zx + xy)(xa + yb + zc).$$

[1] The "Vornicu-Schur inequality" that we use here is the following fact:

Let A, B, C be three reals, and let X, Y, Z be three nonnegative reals. If $A \geq B \geq C$ and $X \geq Y$, then

$$X(A-B)(A-C) + Y(B-C)(B-A) + Z(C-A)(C-B) \geq 0.$$

This is Theorem 1 **a)** in [1]. The proof is fairly easy (just show that $X(A-B)(A-C) + Y(B-C)(B-A) \geq 0$ and $Z(C-A)(C-B) \geq 0$).

Since

$$\begin{aligned}\left(x^3+2xyz\right)a+\left(y^3+2xyz\right)b+\left(z^3+2xyz\right)c\\ &= \left(x^3+2xyz\right)\ln x+\left(y^3+2xyz\right)\ln y+\left(z^3+2xyz\right)\ln z\\ &= \ln\left(x^{x^3+2xyz}y^{y^3+2xyz}z^{z^3+2xyz}\right)\end{aligned}$$

and

$$\begin{aligned}(yz+zx+xy)(xa+yb+zc) &= (yz+zx+xy)(x\ln x+y\ln y+z\ln z)\\ &= (yz+zx+xy)\ln\left(x^xy^yz^z\right)=\ln\left(\left(x^xy^yz^z\right)^{yz+zx+xy}\right),\end{aligned}$$

this becomes $\ln\left(x^{x^3+2xyz}y^{y^3+2xyz}z^{z^3+2xyz}\right)\geq\ln\left(\left(x^xy^yz^z\right)^{yz+zx+xy}\right)$. Since the ln function is strictly increasing, this yields

$$x^{x^3+2xyz}y^{y^3+2xyz}z^{z^3+2xyz}\geq\left(x^xy^yz^z\right)^{yz+zx+xy}.$$

References

[1] Darij Grinberg, *The Vornicu-Schur inequality and its variations*, version 13 August 2007.
`http://de.geocities.com/darij_grinberg/VornicuS.pdf`

Second solution by Minghua Lin, China

It suffices to show

$$(x^3+2xyz)\ln x+(y^3+2xyz)\ln y+(z^3+2xyz)\ln z$$

$$\geq x(xy+yz+xz)\ln x+y(xy+yz+xz)\ln y+z(xy+yz+xz)\ln z,$$

i.e.

$$x(x-y)(x-z)\ln x+y(y-x)(y-z)\ln y+z(z-x)(z-y)\ln z. \quad (1)$$

without loss of generality, let $x\geq y\geq z$, then $(x-y)(x-z)\ln x+(y-x)(y-z\ln y$. Therefore, (1) is valid.
The equality holds if and only if $x=y=z$.

Also solved by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Arkady Alt, San Jose, California, USA; Navid Safaei, Tehran, Iran; Daniel Lasaosa, Universidad Publica de Navarra, Spain.

U113. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that f is a periodic function but it does not have a least period.

Proposed by Radu Ţiţiu, Targu-Mures, Romania

First solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

Not having a least period means that for all $r > 0$ there exists $T_r < r$ such that $f(x + T_r) = f(x)$ for all $x \in \mathbf{R}$. Let's suppose that f is not constant. This means that there exists two points, $x_1 > x_0$ such that $f(x_1) > f(x_0)$. Without loss of generality we may suppose $f(x_0) = 0$ and $f(x_1) > 0$. Let d be $x_1 - x_0$. The continuity of f implies that in an open neighborhood of x_1, say (a, b), $f(x) > f(x_1)/2$ holds. Starting by x_0 we move toward right doing jumps of lengths $T_{d/2} < d/2$. The value assumed by f after every jump is 0 by periodicity but after a while, inevitably we enter (a, b) and we fall in contradiction because f should assume the value 0 and a value greater than $f(x_1)/2$.

Second solution by Arkady Alt, San Jose, California, USA

Let Ω_+ be set of all positive periods of function f and let $\omega_* = \inf \Omega_+$.

Since by definition "least period" mean least positive period then by problem's condition $\omega_* \notin \Omega_+$.Consider two cases:

1. $\omega_* = 0$. Then for any natural n there is $\omega_n \in \Omega_+$ such that $0 < \omega_n < \dfrac{1}{n}$.

Let x by any real number and let $x_n := x - \omega_n \left\lfloor \dfrac{x}{\omega_n} \right\rfloor$ then $0 \le x < \omega_n$ and,

therefore, $\lim\limits_{n\to\infty} x_n = 0$. Since $f(x) = f\left(x_n + \omega_n \left\lfloor \dfrac{x}{\omega_n} \right\rfloor\right) = f(x_n)$ and f is continuous

function then $f(x) = \lim\limits_{n\to\infty} f(x) = \lim\limits_{n\to\infty} f(x_n) = f\left(\lim\limits_{n\to\infty} x_n\right) = f(0)$.

2. $\omega_* > 0$. Then for any natural n there is $\omega_n \in \Omega_+$ such that $\omega_* < \omega_n < \omega_* + \dfrac{1}{n}$.

Let x be any real number. Since $\lim\limits_{n\to\infty} \omega_n = \omega_*$ and f is continuous function then

$f\ (x) = \lim\limits_{n\to\infty} f(x) = \lim\limits_{n\to\infty} f(x + \omega_n) = f\left(\lim\limits_{n\to\infty} (x + \omega_n)\right) = f(\omega_*)$.

Thus we obtain that ω_* is positive period of f , that is contradiction with $\omega_* \notin \Omega_+$

and this prove that only constant functions is solution of problem.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

It is well known, that, given any two periods p_1, p_2 of a periodic function f, then for any pair of integers m, n, not both zero, $|mp_1 + np_2|$ is also a period. The fact that $|mp_1|$ is a period for any m may be trivially shown by induction, while since $|np_2|$ and $|mp_1|$ are periods, then $f(x+mp_1+np_2) = f(x+mp_1) = f(x)$.

Claim: For any positive integer N, there is a period p of f such that $p \leq \frac{1}{N}$.

Proof: If any period of f is a rational multiple of a given period p_0, then every period may be written as $\frac{p_0 u}{v}$, where u, v are relatively prime positive integers. If the v's have a least common multiple V, then every period is a multiple of $\frac{p_0}{V}$, and a least period (which is one of the multiples of $\frac{p_0}{V}$) may be found. Contradiction, hence the v's do not have a least common multiple, and one may always find an integer $W \geq Np_0$ such that $\frac{p_0}{W} \leq \frac{1}{N}$ is a period. If periods p_1 and p_2 of f are not rational multiples of one another, then $|mp_1+np_2|$ does not have a lower bound; indeed, for any m, take n as the closest integer to $-\frac{mp_1}{p_2}$. This results in $0 < |mp_1+np_2| < p_2$. When at least Np_2 such pairs (m, n) have been chosen, by Dirichlet's principle there are two, (m_1, n_1) and (m_2, n_2), such that $|m_1p_1 + n_1p_2| - |m_2p_1 + n_2p_2| < \frac{1}{N}$, and in one of the cases where (m, n) equals $(m_1 - m_2, n_1 - n_2)$ or $(m_1 + m_2, n_1 + n_2)$, then $p = |mp_1 + np_2| < \frac{1}{N}$ will be a period of f.

Consider now any real $x_0 \neq 0$. Since f is continuous, then given x_0, and for any $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. For given x_0 and ϵ, and its corresponding δ, take N as the smallest integer which is at least $\frac{1}{\delta}$, and take any period $p < \frac{1}{N}$. Clearly, there is an integer value m such that $|x_0 - mp| < \delta$, since otherwise it would be $2\delta \leq p < \frac{1}{N}$. Since p is a period, then $|f(x_0) - f(0)| = |f(x_0) - f(mp)| < \epsilon$. Hence $f(x_0) = f(0)$ for all x_0, and the only such functions f are constant functions.

U114. Let a, b, c be nonnegative real numbers. Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i,j=1}^{n} \frac{1}{\sqrt{i^2 + j^2 + ai + bj + c}}.$$

Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania

First solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

We show that the limit is independent on a, b, c allowing us to set $a = b = c = 0$ for evaluating it. If $Q = [0,1] \times [0,1]$, the limit becomes

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i,j=1}^{n} \frac{1}{\sqrt{i^2 + j^2}} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^{n} \frac{1}{\sqrt{\frac{i^2}{n^2} + \frac{j^2}{n^2}}} = \int\int_Q \frac{1}{\sqrt{x^2+y^2}} dxdy$$

By writing the integral as $2\int_0^1 \left(\int_0^x \frac{1}{\sqrt{x^2+y^2}} dy \right) dx$ and passing to polar coordinates we have

$$2\int_{\pi/4}^{\pi/2} \left(\int_0^{1/\sin\theta} \frac{\rho}{\rho} d\rho \right) d\theta = 2\int_{\pi/4}^{\pi/2} \frac{1}{\sin\theta} d\theta = 2\ln\tan\frac{\theta}{2}\Big|_{\pi/4}^{\pi/2} = 2\ln(\sqrt{2}+1)$$

To show that the limit is independent by a, b, c, we prove

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i,j=1}^{n} \frac{1}{\sqrt{i^2 + j^2 + ai + bj + c}} = \lim_{n\to\infty} \frac{1}{n} \sum_{i,j=1}^{n} \frac{1}{\sqrt{i^2 + j^2 + a'i + b'j + c'}}$$

for any a, b, c, a', b', c'. We introduce a number of positive constants C_k, $k = 0, 1, \dots$. Since $i|a'-a| + j|b'-b| + |c'-c| \le C_0(i+j)$ and $i^2 + j^2 + ai + bj + c \le C_1(i^2 + j^2)$ we have the bound

$$\left| \frac{1}{\sqrt{i^2+j^2+ai+bj+c}} - \frac{1}{\sqrt{i^2+j^2+a'i+b'j+c'}} \right| =$$
$$\left| \frac{i(a'-a)+j(b'-b)+c'-c}{(i^2+j^2+ai+bj+c)(i^2+j^2+a'i+b'j+c')} \right| \left(\frac{1}{\sqrt{i^2+j^2+ai+bj+c}} + \frac{1}{\sqrt{i^2+j^2+a'i+b'j+c'}} \right)^{-1} \le$$
$$\frac{C_0(i+j)}{(i^2+j^2)^2} \frac{\sqrt{i^2+j^2}}{C_1} = C_2 \frac{i+j}{(i^2+j^2)^{3/2}}$$

Thus

$$\frac{1}{n} \sum_{i,j=1}^{n} \frac{i+j}{(i^2+j^2)^{3/2}} \le \frac{1}{n} \sum_{j=1}^{\infty} \sum_{i=1}^{n} \frac{i}{(2ij)^{3/2}} + \frac{1}{n} \sum_{i=1}^{\infty} \sum_{j=1}^{n} \frac{j}{(2ij)^{3/2}} \le C_3/\sqrt{n}$$

and it follows that for any a, b, c, a', b', c'

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i,j=1}^{n} \left(\frac{1}{\sqrt{i^2+j^2+ai+bj+c}} - \frac{1}{\sqrt{i^2+j^2+a'i+b'j+c'}} \right) = 0$$

In particular we can take $a' = b' = c' = 0$ and write

$$\frac{1}{\sqrt{i^2+j^2+ai+bj+c}} = \left(\frac{1}{\sqrt{i^2+j^2+ai+bj+c}} - \frac{1}{\sqrt{i^2+j^2}} \right) + \frac{1}{\sqrt{i^2+j^2}}$$

The conclusion is that for any a, b, c the limit assumes the same value $2\ln(\sqrt{2}+1)$

Second solution by John T. Robinson, Yorktown Heights, NY, USA

Let

$$F(n) = \sum_{1\le i,j\le n} \frac{1}{\sqrt{i^2+j^2+ai+bj+c}},$$

$$f_1(n) = \sum_{1\le i\le n-1} \frac{1}{\sqrt{n^2+i^2+ai+bn+c}},$$

$$f_2(n) = \sum_{1\le j\le n-1} \frac{1}{\sqrt{n^2+j^2+an+bj+c}}, \text{ and}$$

$$g(n) = \frac{1}{\sqrt{2n^2+an+bn+c}}.$$

Then

$$F(n) = F(n-1) + f_1(n) + f_2(n) + g(n).$$

Claim: $\lim\limits_{n\to\infty}(f_1(n)+f_2(n)+g(n)) = 2\ln(1+\sqrt{2})$. Since $g(n)$ goes to zero, we need only consider $f_1(n)$ and $f_2(n)$. We can rewrite $f_1(n)$ as follows, where $x = i/n$ and $\Delta x = 1/n$:

$$f_1(n) = \sum_{1\le i\le n-1} \frac{1}{\sqrt{1+x^2+ax\Delta x+b\Delta x+c\Delta x^2}} \Delta x.$$

Therefore

$$\lim_{n\to\infty} f_1(n) = \int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx$$

$$= \left[\ln(x+\sqrt{1+x^2})\right]_0^1 = \ln(1+\sqrt{2}).$$

The argument for f_2 is the same, establishing the claim. Finally, expressing $F(n)$ as

$$F(n) = \sum_{1\le k\le n} (f_1(k)+f_2(k)+g(k)),$$

we see that $\lim_{n\to\infty}(1/n)F(n) = 2\ln(1+\sqrt{2})$.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain.

Olympiad problems

O109. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a+b+1}{a+b^2+c^3}+\frac{b+c+1}{b+c^2+a^3}+\frac{c+a+1}{c+a^2+b^3}\le\frac{(a+1)(b+1)(c+1)+1}{a+b+c}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Gheorghe Pupazan, Chisinau, Moldova

From the Cauchy-Schwartz inequality and because $abc = 1$ we obtain that

$$(a+b^2+c^3)(a+1+ab)\ge(a+b+c)^2 \iff \frac{1}{a+b^2+c^3}\le\frac{1+a+ab}{(a+b+c)^2}.$$

So we get that

$$\frac{a+b+1}{a+b^2+c^3}\le\frac{(a+b+1)(1+a+ab)}{(a+b+c)^2}.$$

In a similar way we obtain the following inequalities

$$\frac{b+c+1}{b+c^2+a^3}\le\frac{(b+c+1)(1+b+bc)}{(a+b+c)^2}$$

and

$$\frac{c+a+1}{c+a^2+b^3}\le\frac{(c+a+1)(1+c+ca)}{(a+b+c)^2}.$$

So it is enough only to prove that

$$\frac{(a+b+1)(1+a+ab)+(b+c+1)(1+b+bc)+(c+a+1)(1+c+ca)}{(a+b+c)^2}$$
$$\le\frac{(a+1)(b+1)(c+1)+1}{a+b+c}.$$

The last one is equivalent to

$$\sum_{cyc}(a+b+1)(1+a+ab)\le(a+b+c)(a+1)(b+1)(c+1)+a+b+c$$

$$\iff 3\sum a+3+\sum a^2+2\sum ab+\sum ab(a+b)$$
$$\le abc\cdot\sum a+3abc+\sum ab(a+b)+\sum a^2+2\sum ab+2\sum a$$

, which is true.

Second solution by Manh Dung Nguyen, Vietnam

By Cauchy-Schwarz Inequality, we have:

$$(a + b^2 + c^3)(a + 1 + ab) \geq (a + b + c)^2$$

Similarly, we obtain

$$\begin{aligned} LHS &\leq \frac{(a+b+1)(a+1+ab) + (b+c+1)(b+1+bc) + (c+a+1)(c+1+ca)}{(a+b+c)^2} \\ &= \frac{(a+b+c)(ab+bc+ca+a+b+c+3)}{(a+b+c)^2} \\ &= \frac{(a+1)(b+1)(c+1)+1}{a+b+c} = RHS \end{aligned}$$

Equality holds when $a = b = c = 1$.

Also solved by Navid Safaei, Tehran, Iran; Daniel Lasaosa, Universidad Publica de Navarra, Spain.

O110. Hexagon $A_1A_2A_3A_4A_5A_6$ is inscribed in a circle $C(O,R)$ and at the same time circumscribed about a circle $\omega(I,r)$. Prove that if

$$\frac{1}{A_1A_2}+\frac{1}{A_3A_4}+\frac{1}{A_5A_6}=\frac{1}{A_2A_3}+\frac{1}{A_4A_5}+\frac{1}{A_6A_1}$$

then one of its diagonals coincides with OI.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

We will prove that, either $I=O$ and the hexagon is regular, its three diagonals meeting at $I=O$, or $I\neq O$ and IO is one of the diagonals of the hexagon which is also an axis of symmetry. We will use cyclic notation throughout the problem, ie, $A_{i+6}=A_i$ for all i, and begin by proving the following

Lemma: Let $A_1A_2A_3A_4A_5A_6$ be a circuminscribed hexagon, such that the incircle touches side A_iA_{i+1} at point $T_{i,i+1}$. Denote $d_i=A_iT_{i,i+1}=A_iT_{i-1,i}$, where clearly $A_iA_{i+1}=d_i+d_{i+2}$. Then, it is equivalent that (1) the hexagon is regular and consequently its three diagonals meet at $I=O$, (2) $I=O$, (3) at least three of the d_i are equal, and (4) each two opposite sides have the same length.

Proof: If the hexagon is regular, by symmetry its three diagonals clearly meet at $I=O$. If (1) is true, clearly so are (2), (3) and (4). Since $IA_i^2=d_i^2+r^2$, if (3) is true, then three of the distances IA_i are equal, and $I=O$, or (2) is true. If (2) is true, then $d_i^2=IA_i^2-r^2=R^2-r^2$ for all i, and the hexagon is regular with sidelengths $2\sqrt{R^2-r^2}$. Since the hexagon admits an inscribed circle, Brianchon's theorem guarantees that its diagonals A_iA_{i+3} meet at a point P, where clearly $\angle A_iA_{i+1}P=\angle PA_{i+3}A_{i+4}$ and $\angle A_{i+1}A_iP=\angle PA_{i+4}A_{i+3}$ because the hexagon is cyclic, and triangles A_iPA_{i+1} and $A_{i+3}PA_{i+4}$ are similar. Therefore, if (4) is true, triangles A_iPA_{i+1} and $A_{i+3}PA_{i+4}$ are equal, and $A_iA_{i+1}A_{i+4}A_{i+3}$ is a rectangle with center $P=O$ and whose diagonals have thus length $2R$, and such that the incircle touches its sides A_iA_{i+1} and $A_{i+3}A_{i+4}$. Thus $A_{i+1}A_{i+4}=A_iA_{i+3}=2r$, and $A_iA_{i+1}=A_{i+3}A_{i+4}=2\sqrt{R^2-r^2}$ for all i. The lemma follows.

Since triangles A_iPA_{i+1} and $A_{i+3}PA_{i+4}$ are similar, then $\frac{d_i+d_{i+1}}{d_{i+3}+d_{i+4}}=\frac{PA_i}{PA_{i+1}}=\frac{PA_{i+1}}{PA_{i+3}}$, or

$$1=\frac{PA_1}{PA_5}\frac{PA_5}{PA_3}\frac{PA_3}{PA_1}=\frac{(d_1+d_2)(d_3+d_4)(d_5+d_6)}{(d_2+d_3)(d_4+d_5)(d_6+d_1)},$$

and we may call $p = (d_1 + d_2)(d_3 + d_4)(d_5 + d_6) = (d_2 + d_3)(d_4 + d_5)(d_6 + d_1)$. Now,

$$\frac{d_6 - d_2}{(d_1 + d_2)(d_1 + d_6)} = \frac{1}{A_1A_2} - \frac{1}{A_6A_1} = \frac{1}{A_2A_3} + \frac{1}{A_4A_5} - \frac{1}{A_3A_4} - \frac{1}{A_5A_6} =$$

$$= \frac{d_2 + d_3 + d_4 + d_5}{(d_2 + d_3)(d_4 + d_5)} - \frac{d_3 + d_4 + d_5 + d_6}{(d_3 + d_4)(d_5 + d_6)} = \frac{(d_3 + d_4 + d_5 - d_1)(d_6 - d_2)}{p}.$$

If $d_2 \neq d_6$, then $0 = (d_6+d_1)(d_1+d_2)(d_3+d_4+d_5-d_1)-p = (d_3-d_1)(A_1A_2 - A_4A_5)$, and similarly for its cyclic permutations. Hence, if the hexagon is not regular, wlog $A_1A_2 \neq A_4A_5$, resulting in $d_2 = d_6 \neq d_4$; similarly d_1, d_3, d_5 cannot be all distinct. If $d_1 = d_3$, then $(d_2 - d_4)(d_5 - d_1) = 0$ and the hexagon is regular, or $d_1 \neq d_3$, and similarly $d_1 \neq d_5$. Hence $d_3 = d_5 \neq d_1$, leading to $A_2A_3 = A_5A_6$, $A_1A_2 = A_1A_6$, and $A_3A_4 = A_4A_5$, or triangles $A_6A_1A_2$ and $A_3A_4A_5$ are isosceles at A_1 and A_4 respectively, while $A_3A_5A_6A_2$ is an isosceles trapezoid with $A_3A_5 \parallel A_6A_2$. The circumcenters of these three polygons lie then on A_1A_4, the common perpendicular bisector of A_2A_6 and A_3A_5. The hexagon is clearly symmetric with respect to A_1A_4, and I and O lie on it. The result follows.

O111. Prove that for each positive integer n the number

$$\left(\binom{n}{0}+2\binom{n}{2}+2^2\binom{n}{4}+\cdots\right)^2\left(\binom{n}{1}+2\binom{n}{3}+2^2\binom{n}{5}+\cdots\right)^2$$

is triangular.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Ercole Suppa, Teramo, Italy

We denote

$$f(n)=\left(\binom{n}{0}+2\binom{n}{2}+2^2\binom{n}{4}+\cdots\right)^2\cdot\left(\binom{n}{1}+2\binom{n}{3}+2^2\binom{n}{5}+\cdots\right)^2$$

For each real number x the the Binomial Theorem yields:

$$(x+1)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\cdots \tag{1}$$

$$(x-1)^n=\binom{n}{0}-\binom{n}{1}x+\binom{n}{2}x^2-\cdots \tag{2}$$

By summing (1) and (2) we have

$$\frac{1}{2}\left[(x+1)^n+(x-1)^n\right]=\binom{n}{0}+\binom{n}{2}x^2+\binom{n}{4}x^4+\cdots \tag{3}$$

$$\frac{1}{2x}\left[(x+1)^n-(x-1)^n\right]=\binom{n}{1}+\binom{n}{3}x^2+\binom{n}{5}x^4\cdots \tag{4}$$

From (3) and (4), taking $x=\sqrt{2}$, we obtain:

$$\frac{1}{2}\left[(\sqrt{2}+1)^n+(\sqrt{2}-1)^n\right]=\binom{n}{0}+\binom{n}{2}2+\binom{n}{4}2^2+\cdots \tag{5}$$

$$\frac{1}{2\sqrt{2}}\left[(\sqrt{2}+1)^n-(\sqrt{2}-1)^n\right]=\binom{n}{1}+\binom{n}{3}2+\binom{n}{5}2^2+\cdots \tag{6}$$

Thus

$$f(n)=\left(\frac{a^n+b^n}{2}\right)^2\cdot\left(\frac{a^n-b^n}{2\sqrt{2}}\right)^2=\frac{1}{2}\left(\frac{a^n+b^n}{2}\right)^2\cdot\left(\frac{a^n-b^n}{2}\right)^2 \tag{7}$$

where $a=\sqrt{2}+1$ and $b=\sqrt{2}-1$. Let $k=\left(\frac{a^n-b^n}{2}\right)^2$. Since $ab=1$ we have:

$$k+1=\left(\frac{a^n-b^n}{2}\right)^2+1=\frac{a^{2n}+b^{2n}-2a^nb^n}{4}+1=$$
$$=\frac{a^{2n}+b^{2n}-2+4}{4}=\frac{a^{2n}+b^{2n}+2a^nb^n}{4}=\left(\frac{a^n+b^n}{2}\right)^2$$

Therefore
$$f(n)=\frac{k(k+1)}{2}=\binom{k+1}{2}$$
and we are done.

Second solution by Darij Grinberg, Germany

First, we consider a more general setting:

> **Theorem 1.** Let q be a real. Define a sequence $(f_0, f_1, f_2, ...)$ by $f_n=\sum\limits_{k\in\mathbb{N}}\binom{n}{2k}q^k$ for every $n\in\mathbb{N}$, and define a sequence $(g_0, g_1, g_2, ...)$ by $g_n=\sum\limits_{k\in\mathbb{N}}\binom{n}{2k+1}q^k$ for every $n\in\mathbb{N}$. [2] Then, for every $n\in\mathbb{N}$, we have the matrix equality
> $$\begin{pmatrix} f_n & qg_n \\ g_n & f_n \end{pmatrix}=\begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^n \tag{3}$$
> and the equalities
> $$f_n^2-qg_n^2=(1-q)^n; \tag{4}$$
> $$2f_ng_n=g_{2n}. \tag{5}$$
> For any $a\in\mathbb{N}$ and $b\in\mathbb{N}$, we have
> $$f_{a+b}=f_af_b+qg_ag_b; \tag{6}$$
> $$g_{a+b}=f_ag_b+g_af_b. \tag{7}$$

[2] A sum of the form $\sum\limits_{k\in\mathbb{N}}a(k)$ (where $a:\mathbb{N}\to\mathbb{R}$ is some map) only makes sense if all but finitely many $k\in\mathbb{N}$ satisfy $a(k)=0$. But this condition is easily verified for our sum $\sum\limits_{k\in\mathbb{N}}\binom{n}{2k}q^k$ (in fact, all $k\in\mathbb{N}\setminus\{0,1,...,n\}$ satisfy $k>n$, thus $2k\geq k>n$, thus $\binom{n}{2k}=0$, thus $\binom{n}{2k}q^k=0$; thus, all but finitely many $k\in\mathbb{N}$ satisfy $\binom{n}{2k}q^k=0$) and (similarly) for our sum $\sum\limits_{k\in\mathbb{N}}\binom{n}{2k+1}q^k$. Similar arguments can show that all other sums of the form $\sum\limits_{k\in\mathbb{N}}a(k)$ that we will encounter in our solution will be well-defined.

Remark. Here, $\mathbb{N}$ means the set $\{0, 1, 2, ...\}$.

Proof of Theorem 1. We will prove (1) by induction:

Induction base. We have

$$\begin{aligned} f_0 &= \sum_{k\in\mathbb{N}} \binom{0}{2k} q^k = \sum_{k\in\mathbb{N}} \left\{ \begin{array}{l} 1, \text{ if } k=0; \\ 0, \text{ if } k\neq 0 \end{array} \right. \cdot q^k \\ &= 1 \cdot q^0 = 1 \cdot 1 = 1 \end{aligned}$$

$$\left(\text{since } \binom{0}{2k} = \left\{ \begin{array}{l} 1, \text{ if } 2k=0; \\ 0, \text{ if } 2k\neq 0 \end{array} \right. = \left\{ \begin{array}{l} 1, \text{ if } k=0; \\ 0, \text{ if } k\neq 0 \end{array} \right. \right)$$

and

$$\begin{aligned} g_0 &= \sum_{k\in\mathbb{N}} \binom{0}{2k+1} q^k = \sum_{k\in\mathbb{N}} 0 \cdot q^k \\ &= 0, \end{aligned} \qquad \left(\begin{array}{c} \text{since } \binom{0}{2k+1} = \left\{ \begin{array}{l} 1, \text{ if } 2k+1=0; \\ 0, \text{ if } 2k+1\neq 0 \end{array} \right. = 0, \\ \text{because } k\in\mathbb{N} \text{ yields } 2k+1\neq 0 \end{array} \right)$$

so that

$$\left(\begin{array}{cc} f_0 & qg_0 \\ g_0 & f_0 \end{array} \right) = \left(\begin{array}{cc} 1 & q\cdot 0 \\ 0 & 1 \end{array} \right) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = \left(\begin{array}{cc} 1 & q \\ 1 & 1 \end{array} \right)^0.$$

In other words, (1) holds for $n = 0$. This completes the induction base.

Induction step. Let $N \in \mathbb{N}$. Assume that (1) holds for $n = N$. We have to show that (1) holds for $n = N + 1$ as well.

Since (1) holds for $n = N$, we have

$$\left(\begin{array}{cc} f_N & qg_N \\ g_N & f_N \end{array} \right) = \left(\begin{array}{cc} 1 & q \\ 1 & 1 \end{array} \right)^N.$$

But

$$
\begin{aligned}
f_{N+1} &= \sum_{k\in\mathbb{N}} \binom{N+1}{2k} q^k = \sum_{k\in\mathbb{N}} \left(\binom{N}{2k} + \binom{N}{2k-1} \right) q^k \\
&\qquad \left(\text{as } \binom{N+1}{2k} = \binom{N}{2k} + \binom{N}{2k-1} \text{ by the recurrence of the binomial coefficients} \right) \\
&= \sum_{k\in\mathbb{N}} \binom{N}{2k} q^k + \sum_{k\in\mathbb{N}} \binom{N}{2k-1} q^k = \sum_{k\in\mathbb{N}} \binom{N}{2k} q^k + \sum_{\substack{k\in\mathbb{N};\\ k\geq 1}} \binom{N}{2k-1} q^k \\
&= \sum_{k\in\mathbb{N}} \binom{N}{2k} q^k + \sum_{k\in\mathbb{N}} \underbrace{\binom{N}{2(k+1)-1}}_{=\binom{N}{2k+1}} \underbrace{q^{k+1}}_{=q\cdot q^k} \\
&\qquad \text{(here we substituted } k+1 \text{ for } k \text{ in the second sum)} \\
&= \underbrace{\sum_{k\in\mathbb{N}} \binom{N}{2k} q^k}_{=f_N} + q \underbrace{\sum_{k\in\mathbb{N}} \binom{N}{2k+1} q^k}_{=g_N} = 1 \cdot f_N + q \cdot g_N
\end{aligned}
$$

$$
\left(\begin{array}{c} \text{here we replaced the } \sum\limits_{k\in\mathbb{N}} \text{ sign by an } \sum\limits_{\substack{k\in\mathbb{N};\\ k\geq 1}} \text{ sign, since the addend for } k=0 \text{ is zero} \\ \left(\text{as } \binom{N}{2k-1} = \binom{N}{2\cdot 0-1} = \binom{N}{-1} = 0 \text{ for } k=0\right) \end{array} \right)
$$

and

$$
\begin{aligned}
g_{N+1} &= \sum_{k\in\mathbb{N}} \binom{N+1}{2k+1} q^k = \sum_{k\in\mathbb{N}} \left(\binom{N}{2k} + \binom{N}{2k+1} \right) q^k \\
&\qquad \left(\text{as } \binom{N+1}{2k+1} = \binom{N}{2k} + \binom{N}{2k+1} \text{ by the recurrence of the binomial coefficients} \right) \\
&= \underbrace{\sum_{k\in\mathbb{N}} \binom{N}{2k} q^k}_{=f_N} + \underbrace{\sum_{k\in\mathbb{N}} \binom{N}{2k+1} q^k}_{=g_N} = 1 \cdot f_N + 1 \cdot g_N,
\end{aligned}
$$

so that

$$\begin{aligned}\begin{pmatrix} f_{N+1} & qg_{N+1} \\ g_{N+1} & f_{N+1} \end{pmatrix} &= \begin{pmatrix} 1\cdot f_N + q\cdot g_N & q\left(1\cdot f_N + 1\cdot g_N\right) \\ 1\cdot f_N + 1\cdot g_N & 1\cdot f_N + q\cdot g_N \end{pmatrix} \\ &= \begin{pmatrix} 1\cdot f_N + q\cdot g_N & 1\cdot qg_N + q\cdot f_N \\ 1\cdot f_N + 1\cdot g_N & 1\cdot qg_N + 1\cdot f_N \end{pmatrix} \\ &= \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix} \begin{pmatrix} f_N & qg_N \\ g_N & f_N \end{pmatrix} = \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^N \\ &= \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^{N+1}.\end{aligned}$$

In other words, (1) holds for $n = N+1$. This completes the induction step. Thus, the induction proof is complete, so that (1) is proven for all $n \in \mathbb{N}$.

Now, (2) follows from

$$\begin{aligned}f_n^2 - qg_n^2 = f_n\cdot f_n - qg_n\cdot g_n = \det\begin{pmatrix} f_n & qg_n \\ g_n & f_n \end{pmatrix} &= \det\left(\begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^n\right) \qquad \text{(by (1))} \\ = \left(\det\begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}\right)^n &= \left(1\cdot 1 - q\cdot 1\right)^n = \left(1-q\right)^n.\end{aligned}$$

For any $a \in \mathbb{N}$ and $b \in \mathbb{N}$, we have

$$\begin{aligned}\begin{pmatrix} f_{a+b} & qg_{a+b} \\ g_{a+b} & f_{a+b} \end{pmatrix} &= \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^{a+b} \qquad \text{(by (1), applied to } n = a+b\text{)} \\ &= \underbrace{\begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^{a}}_{\substack{=\begin{pmatrix} f_a & qg_a \\ g_a & f_a \end{pmatrix} \\ \text{(by (1), applied to } n=a\text{)}}} \underbrace{\begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^{b}}_{\substack{=\begin{pmatrix} f_b & qg_b \\ g_b & f_b \end{pmatrix} \\ \text{(by (1), applied to } n=b\text{)}}} \\ &= \begin{pmatrix} f_a & qg_a \\ g_a & f_a \end{pmatrix}\begin{pmatrix} f_b & qg_b \\ g_b & f_b \end{pmatrix} = \begin{pmatrix} f_a\cdot f_b + qg_a\cdot g_b & f_a\cdot qg_b + qg_a\cdot f_b \\ g_a\cdot f_b + f_a\cdot g_b & g_a\cdot qg_b + f_a\cdot f_b \end{pmatrix} \\ &= \begin{pmatrix} f_af_b + qg_ag_b & q\left(f_ag_b + g_af_b\right) \\ f_ag_b + g_af_b & f_af_b + qg_ag_b \end{pmatrix}.\end{aligned}$$

Thus, $f_{a+b} = f_af_b + qg_ag_b$ and $g_{a+b} = f_ag_b + g_af_b$, so that (4) and (5) are proven. For every $n \in \mathbb{N}$, we have

$$\begin{aligned}g_{2n} = g_{n+n} &= f_ng_n + g_nf_n \qquad \text{(by (5))} \\ &= 2f_ng_n,\end{aligned}$$

and (3) follows.

Altogether, we have now proven Theorem 1.

From now on, we set $q = 2$. Then,

$$\begin{aligned}
f_n^2 g_n^2 &= (f_n g_n)^2 = \frac{1}{4}(2 f_n g_n)^2 = \frac{1}{4} g_{2n}^2 \qquad \text{(by (3))} \\
&= \frac{1}{4}\cdot\frac{1}{q}\cdot q g_{2n}^2 = \frac{1}{4}\cdot\frac{1}{q}\cdot\left(f_{2n}^2 - \left(f_{2n}^2 - q g_{2n}^2\right)\right) \\
&= \frac{1}{4}\cdot\frac{1}{q}\cdot\left(f_{2n}^2 - (1-q)^{2n}\right) \\
&\qquad \left(\text{since } f_{2n}^2 - q g_{2n}^2 = (1-q)^{2n}\text{, what results if we substitute } 2n \text{ for } n \text{ in (2)}\right) \\
&= \frac{1}{4}\cdot\frac{1}{2}\cdot\left(f_{2n}^2 - \underbrace{(1-2)^{2n}}_{\substack{=(-1)^{2n}=1,\\ \text{since } 2n \text{ is even}}}\right) = \frac{1}{8}\left(f_{2n}^2 - 1\right) = \frac{1}{8}\left(f_{2n}-1\right)\left(f_{2n}+1\right) \\
&= \frac{1}{2}\cdot\frac{f_{2n}-1}{2}\cdot\frac{f_{2n}+1}{2} = \frac{1}{2}\cdot\frac{f_{2n}-1}{2}\cdot\left(\frac{f_{2n}-1}{2}+1\right)
\end{aligned}$$

for every $n \in \mathbb{N}$. Since $\dfrac{f_{2n}-1}{2} \in \mathbb{Z}$ for every $n \in \mathbb{N}$ (since

$$\begin{aligned}
\frac{f_{2n}-1}{2} &= \frac{\sum\limits_{k\in\mathbb{N}} \binom{2n}{2k} q^k - 1}{2} = \frac{\sum\limits_{k\in\mathbb{N}} \binom{2n}{2k} 2^k - 1}{2} = \frac{\left(\binom{2n}{2\cdot 0} 2^0 + \sum\limits_{\substack{k\in\mathbb{N};\\ k\geq 1}} \binom{2n}{2k} 2^k\right) - 1}{2} \\
&= \frac{\left(1 + \sum\limits_{\substack{k\in\mathbb{N};\\ k\geq 1}} \binom{2n}{2k} 2^k\right) - 1}{2} \qquad \left(\text{since } \binom{2n}{2\cdot 0} 2^0 = \binom{2n}{0} 2^0 = 1\cdot 1 = 1\right) \\
&= \frac{\sum\limits_{\substack{k\in\mathbb{N};\\ k\geq 1}} \binom{2n}{2k} 2^k}{2} = \sum_{\substack{k\in\mathbb{N};\\ k\geq 1}} \binom{2n}{2k} 2^{k-1} \in \mathbb{Z}
\end{aligned}$$

), this yields that $f_n^2 g_n^2$ is a triangular number for every $n \in \mathbb{N}$. This is exactly what the problem asked us to prove.

Remark. Theorem 1 could be proved more quickly using the binomial formula applied to $\left(1+\sqrt{q}\right)^n$ and $\left(1-\sqrt{q}\right)^n$. However, such a proof would fail if we replace $\mathbb{R}$ by a field of characteristic 2 and q is a square in that field. The proof given above works over any field and for any q. (Then again, from a deeper viewpoint, it is just a straightforward elementarization of the proof using the binomial formula.)

Also solved by John T. Robinson, Yorktown Heights, NY, USA; Brian Bradie, Newport News, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain.

O112. Let a, b, c be real positive numbers. Prove that

$$\frac{a^3+abc}{(b+c)^2}+\frac{b^3+abc}{(c+a)^2}+\frac{c^3+abc}{(a+b)^2}\geq\frac{3}{2}\cdot\frac{a^3+b^3+c^3}{a^2+b^2+c^2}.$$

Proposed by Cezar Lupu, University of Bucharest, Romania and Pham Huu Duc, Ballajura, Australia

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Call $u=(a-b)(a-c)$, $v=(b-c)(b-a)$, $w=(c-a)(c-b)$. After some algebra,

$$(2a+b+c)u+(2b+c+a)v+(2c+a+b)w=$$
$$=2(a^3+b^3+c^3)-(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b)=$$
$$=3(a^3+b^3+c^3)-(a+b+c)(a^2+b^2+c^2),$$

while at the same time

$$\frac{a^3+abc}{(b+c)^2}-\frac{a}{2}=\frac{a(2a-b-c)}{2(b+c)}+\frac{au}{(b+c)^2}.$$

Now,

$$\frac{a(2a-b-c)}{2(b+c)}+\frac{b(2b-c-a)}{2(c+a)}+\frac{c(2c-a-b)}{2(a+b)}=$$
$$=(a+b+c)\frac{3(a^3+b^3+c^3)-(a+b+c)(a^2+b^2+c^2)}{2(a+b)(b+c)(c+a)}=$$
$$=(a+b+c)\frac{(2a+b+c)u+(2b+c+a)v+(2c+a+b)w}{2(a+b)(b+c)(c+a)}.$$

Since

$$\frac{a+b+c}{(a+b)(b+c)(c+a)}-\frac{1}{a^2+b^2+c^2}=\frac{a^3+b^3+c^3-2abc}{(a+b)(b+c)(c+a)(a^2+b^2+c^2)}$$

is clearly positive because $a^3+b^3+c^3\geq 3abc$, and

$$(2a+b+c)u+(2b+c+a)v+(2c+a+b)w=(a+b+c)(u+v+w)+au+bv+cw$$

is clearly non-negative because $a^mu+b^mv+c^mw\geq 0$ for any $m\geq 0$ as a result of Schur's inequality, then it suffices to show that

$$\frac{au}{(b+c)^2}+\frac{bv}{(c+a)^2}+\frac{cw}{(a+b)^2}\geq 0.$$

Finally,

$$a(a+b)^2(a+c)^2u = a^5 + 2a^3(ab+bc+ca) + a(ab+bc+ca)^2,$$

and it suffices to show that

$$(a^5u+b^5v+c^5w)+2(ab+bc+ca)(a^3u+b^3v+c^3w)+(ab+bc+ca)^2(au+bv+cw) \geq 0,$$

clearly true again by Schur's inequality. The conclusion follows, equality being reached if and only if $a = b = c$.

Also solved by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Gheorghe Pupazan, Chisinau, Moldova; Oleh Faynshteyn, Leipzig, Germany; Ercole Suppa, Teramo, Italy; Darij Grinberg, Germany.

O113. Let P be a point on the circumcircle Γ of a triangle ABC. The tangents from P to the incircle of ABC meet again the circumcircle at X and Y, respectively. Prove that the line XY is parallel to a side of triangle ABC if and only if P is the tangency point of Γ with some mixtilinear incircle of triangle ABC.

Proposed by Cosmin Pohoata, Bucharest, Romania

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Claim 1: If XY is a chord tangent to γ, and P is a point on Γ such that $IP' \perp X'Y'$, where P', X', Y' are the respective inverses of P, X, Y with respect to γ, then γ is the inscribed circle to PXY.

Proof: Let M be the midpoint of the chord XY that does not contain P. It is well known that M is the circumcenter of IXY, hence the inverse of the circumcircle of IXY with respect to γ is line $X'Y'$. Moreover, line IP is its own inverse IP' with respect to γ. If $IP' \perp X'Y'$, then IP is perpendicular to the circumcircle of IXY, or IP passes through M, ie, IP is the internal bisector of $\angle XPY$. By Poncelet's porism, there is a triangle PUV that admits I as its incircle, hence M is the midpoint of chord UV that does not contain P. We conclude that UV and XY are tangent to γ, such that γ is inside triangles PUV and PXY, and with $XY \parallel UV$, hence $XY = UV$, and PXY has inscribed circle γ. The claim follows.

Claim 2: Let A' be the inverse of A with respect to γ. Then, circle $\gamma'_A(A', r)$ is the inverse with respect to γ of the mixtilinear incircle $\gamma_A(I_A, r_A)$ which is tangent to Γ, AB and AC. Moreover, calling Q the point where γ_A and Γ touch, and Q' its inverse with respect to γ, then circle $\Gamma'\left(O', \frac{r}{2}\right)$ with diameter $A'Q'$ passes through A', is internally tangent to γ'_A, and is the inverse of Γ with respect to γ.

Proof: It is well known that A, A', I, I_A are collinear and they lie on the internal bisector of $\angle BAC$. It is also well known (or easily provable) that $r = r_A \cos^2 \frac{A}{2}$. Denote by S, T the two points where the mixtilinear incircle γ_A intersects AI. Then, $IS = II_A - r_A$, $IT = II_A + r_A$, where

$$II_A = AI_A - AI = \frac{r}{\sin \frac{A}{2} \cos^2 \frac{A}{2}} - \frac{r}{\sin \frac{A}{2}} = \frac{r \sin \frac{A}{2}}{\cos^2 \frac{A}{2}}.$$

Moreover, denoting S', T' the respective inverses of S, T with respect to γ, $IS' \cdot IS = r^2$ and $IT' \cdot IT = r^2$, and since $IA' \cdot IA = r^2$, we have

$$A'S' = IS' - IA' = \frac{r \cos^2 \frac{A}{2}}{1 - \sin \frac{A}{2}} - r \sin \frac{A}{2} = r,$$

and similarly $A'T' = IT' + IA' = r$. We conclude that circle γ_A with diameter ST transforms upon inversion into circle γ'_A with diameter $S'T'$. Now, since γ_A is internally tangent to Γ at Q, its inverse Γ' is internally tangent to γ'_A at Q', or line $A'Q'$ contains a diameter of Γ'. But Γ' passes through A', or $A'Q'$ is a diameter of Γ', and the length of the radius of Γ' is $\frac{r}{2}$. The claim follows.

Call now D the point where BC touches γ, and D' the diametrally opposite point to D in γ. Let the parallel to BC through D' intersect Γ at X and Y. The inverse of line XY with respect to γ is clearly circle $\Gamma_{XY}\left(N, \frac{r}{2}\right)$ with diameter ID', where N is the midpoint of ID'.

Claim 3: $A'O'IN$ is a parallelogram.

Proof: By the second claim, $A'O' = \frac{r}{2}$ is a radius of Γ', while $NI = \frac{r}{2}$ is a radius of Γ_{XY}. Now, since IA' is the internal bisector of $\angle BAC$ and $IN = ID \perp BC$, then $\angle A'ID' = \frac{B-C}{2}$, or the Cosine Law yields

$$A'N^2 = IA'^2 + IN'^2 - 2IA' \cdot IN' \cos\frac{B-C}{2} = r^2\sin^2\frac{A}{2} + \frac{r^2}{4} - r^2\sin\frac{A}{2}\cos^2\frac{B-C}{2} =$$

$$\frac{r^2}{4} - r^2\sin\frac{A}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right) = \frac{r^2(R-2r)}{4R},$$

or $A'N = \frac{rOI}{2R}$, where we have used that $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ and $OI^2 = R^2 - 2Rr$. Finally, points J, K where the diameter OI intersects Γ are at distances $R - OI, R + OI$ from I, or their inverses J', K' are at distances $\frac{r^2}{R-OI}, \frac{r^2}{R+OI}$ from I. Their midpoint is clearly O', or

$$IO' = \frac{IJ' - IK'}{2} = \frac{r^2 OI}{R^2 - OI^2} = \frac{rOI}{2R} = A'N.$$

The claim follows.

Consider now an arbitrary point P on Γ, and construct points X, Y as described in the proposed problem. By Poncelet's porism, XY is tangent to γ. Hence, if $XY \parallel BC$, XY must be the parallel to BC through D', which defines P univocally. Moreover, the hypothesis of the third claim holds, and $A'O'IN$ is a parallelogram, or A', O', Q' are collinear, and so are I, N, D', with $A'O' = O'Q' = IN = ND' = \frac{r}{2}$. Hence $IQ' \parallel O'N \parallel A'D'$. Now, $X'Y'$ is a common chord of Γ' and Γ_{XY}, or $X'Y' \perp O'N$, and $IQ' \perp X'Y'$. We may then apply the first claim, and QXY is a triangle inscribed into Γ and circumscribed around γ, and such that $XY \parallel BC$. By Poncelet's porism, there cannot be another such point. By cyclic permutations, $XY \parallel CA$ and $XY \parallel AB$ iff P is the point where Γ touches respectively the mixtilinear incircles touching AB, BC and BC, CA. The result follows.

O114. Prove that for all real numbers x, y, z the following inequality holds

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) \geq 3(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)$$

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

First solution by Darij Grinberg, Germany

The inequality in question immediately follows from the identity

$$\begin{aligned}&\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right)\left(x^2+xy+y^2\right)\\&=3\left(x^2y+y^2z+z^2x\right)\left(xy^2+yz^2+zx^2\right)+\left(\left(x-y\right)\left(y-z\right)\left(z-x\right)\right)^2.\end{aligned}$$

What remains is to prove this identity. Of course, we can prove it by expanding, but here is a more conceptual proof:

Denote $a = x^2y + y^2z + z^2x$ and $b = xy^2 + yz^2 + zx^2$.

We work in $\mathbb{C}$. Let $\zeta = \dfrac{1+\sqrt{3}i}{2}$. Then, $\zeta^3 = -1$ and thus

$$\begin{aligned}&\left(x+\zeta y\right)\left(y+\zeta z\right)\left(z+\zeta x\right)\\&=\left(\underbrace{\zeta^3+1}_{=-1+1=0}\right)xyz+\zeta\left(\underbrace{x^2y+y^2z+z^2x}_{=a}\right)+\zeta^2\underbrace{\left(xy^2+yz^2+zx^2\right)}_{=b}=\zeta a+\zeta^2 b\\&=\zeta\left(a+\zeta b\right).\end{aligned}$$

The same computation with ζ replaced by $\dfrac{1}{\zeta}$ everywhere (and using $\left(\dfrac{1}{\zeta}\right)^3 = -1$ instead of $\zeta^3 = -1$) proves

$$\left(x+\frac{1}{\zeta}y\right)\left(y+\frac{1}{\zeta}z\right)\left(z+\frac{1}{\zeta}x\right)=\frac{1}{\zeta}\left(a+\frac{1}{\zeta}b\right).$$

But any two complex numbers u and v satisfy

$$\left(u+\zeta v\right)\left(u+\frac{1}{\zeta}v\right)=u^2+uv+v^2 \tag{8}$$

(since $\left(u+\zeta v\right)\left(u+\dfrac{1}{\zeta}v\right) = u^2 + \left(\zeta+\dfrac{1}{\zeta}\right)uv + v^2$ and $\zeta + \dfrac{1}{\zeta} = 1$ as we can easily see).

Hence,

$$\begin{aligned}
&\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right)\left(x^2+xy+y^2\right)\\
&=(y+\zeta z)\left(y+\frac{1}{\zeta}z\right)(z+\zeta x)\left(z+\frac{1}{\zeta}x\right)(x+\zeta y)\left(x+\frac{1}{\zeta}y\right)\\
&\qquad\left(\begin{array}{c}\text{since (2) yields }(y+\zeta z)\left(y+\frac{1}{\zeta}z\right)=y^2+yz+z^2,\\ (z+\zeta x)\left(z+\frac{1}{\zeta}x\right)=z^2+zx+x^2\text{ and }(x+\zeta y)\left(x+\frac{1}{\zeta}y\right)=x^2+xy+y^2\end{array}\right)\\
&=(x+\zeta y)(y+\zeta z)(z+\zeta x)\cdot\left(x+\frac{1}{\zeta}y\right)\left(y+\frac{1}{\zeta}z\right)\left(z+\frac{1}{\zeta}x\right)\\
&=\zeta(a+\zeta b)\cdot\frac{1}{\zeta}\left(a+\frac{1}{\zeta}b\right)=(a+\zeta b)\left(a+\frac{1}{\zeta}b\right)=a^2+ab+b^2 \qquad\text{(by (2))}\\
&=3ab+(b-a)^2\\
&=3\left(x^2y+y^2z+z^2x\right)\left(xy^2+yz^2+zx^2\right)+\left((x-y)(y-z)(z-x)\right)^2
\end{aligned}$$

(since $a=x^2y+y^2z+z^2x$, $b=xy^2+yz^2+zx^2$, and a quick computation shows that

$$b-a=\left(xy^2+yz^2+zx^2\right)-\left(x^2y+y^2z+z^2x\right)=(x-y)(y-z)(z-x)$$

), qed.

Second solution by Magkos Athanasios, Kozani, Greece

If $xyz=0$, say $z=0$ the inequality is equivalent to $x^2y^2(x-y)^2\geq 0$, which clearly holds. Assume now that $xyz\neq 0$. Dividing both sides of the inequality by $(xyz)^2$ we have to prove that

$$\left(\left(\frac{x}{y}\right)^2+\frac{x}{y}+1\right)\left(\left(\frac{y}{z}\right)^2+\frac{y}{z}+1\right)\left(\left(\frac{z}{x}\right)^2+\frac{z}{x}+1\right)\geq 3\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right).$$

Set

$$\frac{x}{y}=a,\frac{y}{z}=b,\frac{z}{x}=c.$$

Then $abc=1$ and we have to prove that

$$(a^2+a+1)(b^2+b+1)(c^2+c+1)\geq 3(a+b+c)(ab+bc+ca).$$

multiplying out we arrive at

$$[ab+bc+ca-(a+b+c)]^2\geq 0.$$

Third solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

It is trivial to observe that the l.h.s. and the r.h.s. are equal for $x = y$, for $y = z$ and for $z = x$. The ratio

$$\frac{(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) - 3(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)}{(x-y)(y-z)(z-x)}$$

yields

$$-xy^2 + zx^2 - z^2x + xy^2 - y^2z + yz^2$$

which annihilates for $x = y$, for $y = z$ and for $z = x$ as well. We have

$$\frac{-xy^2 + zx^2 - z^2x + xy^2 - y^2z + yz^2}{(x-y)(y-z)(z-x)} = 1$$

and then

$$\begin{aligned}&(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)+\\ &\qquad\qquad -3(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2) =\\ &= (x-y)^2(y-z)^2(z-x)^2\end{aligned}$$

and we are done.

Also solved by Hoang Quoc Viet; Gheorghe Pupazan, Chisinau, Moldova; Arkady Alt, San Jose, California, USA; Navid Safaei, Tehran, Iran; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Nguyen Manh Dung, Hanoi University of Science, Vietnam.