The entirely mixing variables method

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In some situations, proofs of conditional inequalities by making two variables equal become difficult, even impossible. It often happens with cyclic problems, when the equality holds for totally different variables. There is a very helpful method which is often used to resolve this matter called the entirely mixing variables method. It is for annulling one variable that's equal to 0, namely subtracting simultaneously from all variables a fixed value. The method can be used when there is a disparity of terms that approximate to 0 (which can be understand is three differences $(a-b), (b-c), (c-a)$ for three-variable problems).

See the following example to know thoroughly about it

Problem 1. *Prove that for all non-negative real numbers a, b, c we have*

$$a^3 + b^3 + c^3 - 3abc \geq 4(a-b)(b-c)(c-a).$$

SOLUTION. The above inequality can be rewritten as follow

$$(a+b+c)\left((a-b)^2 + (b-c)^2 + (c-a)^2\right) \geq 8(a-b)(b-c)(c-a) \quad (1)$$

WLOG, we may assume that $c = \min(a, b, c)$. Fix the differences $a-b, b-c, c-a$ and decrease simultaneously from a, b, c one value c (which means that we supersede a, b, c by $a-c, b-c, 0$), hence $a-b, a-c, b-c$ don't change but $a+b+c$ is decreased. Thus the left hand side of (1) is decreased but the right hand side of (1) is invariable. So we only need to prove the problem in case $a, b \geq c = 0$, the problem becomes

$$a^3 + b^3 \geq 4ab(b-a),$$

But this above one is obviously true because

$$4a(b-a) \leq b^2 \;\Rightarrow 4ab(b-a) \leq b^3 \leq a^3 + b^3.$$

The equality occurs if and only if $a = b = c$. ❏

This is only a simple sample, but represent the idea of method straightforwardly. A remarkable feature of this solution is a small comment, that if we decrease a, b, c at once by a number which is not bigger than $\min(a, b, c)$, one side is decreased but another side is immutable. To prove it is very easy, because the degree (of close-to-0 values) of $(a+b+c)\left((a-b)^2 + (b-c)^2 + (c-a)^2\right)$ is 2, and of $(a-b)(b-c)(c-a)$ is 3. We also have a similar problem of four variables as follow

Problem 2. *Prove that for all non-negative real numbers a, b, c, d then*

$$a^4 + b^4 + c^4 + d^4 - 4abcd \geq 2(a-b)(b-c)(c-d)(d-a),$$

and in case $(a-b)(c-d) \leq 0$, the inequality above is still true if replacing 2 by 17.

SOLUTION. Firstly, we will prove the inequality in case $(a-b)(c-d) \le 0$ and since then, the reader can easily conclude all parts of the problem with the same manner. If $d=0$ hence $a \le b$, the problem becomes

$$a^4+b^4+c^4 \ge 17ac(b-a)(b-c).$$

If $b \le c$, we have done. Otherwise, if $b \ge a, c$, by $AM-GM$ Inequality

$$ac(b-a)(b-c) \le ac(b-t)^2 \le t^2(b-t)^2,$$

for which $t=(a+c)/2 \le b$. Besides $a^4+c^4 \ge 2t^4$, so it suffices to prove that

$$2t^4+b^4 \ge 17t^2(b-t)^2.$$

Let $x=b/t-1 \ge 0$, the above inequality becomes

$$\begin{aligned} &2+(x+1)^4 \ge 17x^2 \\ &\Leftrightarrow\ x^4+(x-1)(4x^2-7x-3) \ge 0 \\ &\Leftrightarrow\ 1 \ge y(1-y)(3y^2+7y-4). \end{aligned}$$

Clearly, if $y \ge 1$ we have done. Otherwise, if $y \le 1$ we have two cases

+, If $3y^2+7y \le 8$ then $RHS \le 4y(1-y) \le 1$ of course.

+, If $3y^2+7y \ge 8$ then $y \ge 0.8$, therefore

$$y(1-y) \le 0.8(1-0.8) \le \frac{1}{6}\ \Rightarrow y(1-y)(3y^2+7y-4) \le \frac{1}{6}(3y^2+7y-4) \le 1.$$

So the problem is proved if $d=0$. Suppose that $a,b,c,d>0, d=\min(a,b,c,d)$, then

$$\begin{aligned} LHS &= a^4+b^4+c^4+d^4-4abcd = (a^2-c^2)^2+(b^2-d^2)^2+2(ac-bd)^2 \\ &= \frac{1}{2}\big((a-c)^2(a+c)^2+(b-d)^2(b+d)^2+(a-b)^2(c+d)^2+ \\ &+(c-d)^2(a+b)^2+(a-d)^2(a+d)^2+(b-c)^2(b+c)^2\big). \end{aligned}$$

Certainly, this step claims that $d=0$ is all work to complete, it ends the proof. ❑

In some instances, when we decrease simultaneously all variables, both hand sides of the inequality are changed simultaneously too, increasing or decreasing. Consider the following examples which are more difficult

Problem 3. *Let a,b,c be non-negative real numbers with sum 3. Find all possible values of k for which the below inequality is always true*

$$a^4+b^4+c^4-3abc \ge k(a-b)(b-c)(c-a).$$

Solution. In case $c = 0$, easy to prove that $-6\sqrt{2} \le k \le 6\sqrt{2}$. Indeed, if $c = 0, a + b = 3$, applying $AM - GM$ Inequality, we obtain

$$LHS = a^4 + b^4 = (a^2 - b^2)^2 + 2a^2b^2 \ge 2\sqrt{2}|ab(b^2 - a^2)| = 6\sqrt{2}|ab(b - a)|.$$

The equality is taken if and only if $|b^2 - a^2| = |ab|$ and $a + b = 3, c = 0$.

Because the equality can be happened if a, b, c are different values, so the positive value of k to contruct a valid inequality must lie between $-6\sqrt{2}$ and $6\sqrt{2}$.

Now we will prove that for all non-negative real numbers a, b, c adding up to 3 then

$$a^4 + b^4 + c^4 - 3abc \ge 6\sqrt{2}(a - b)(b - c)(c - a).$$

To perform the idea of the entirely mixing variables method, the first neccesary thing is normalizing two sides of the inequality

$$a^4 + b^4 + c^4 - abc(a + b + c) \ge 2\sqrt{2}(a - b)(b - c)(c - a)(a + b + c).$$

Decreasing or increasing merely all variables by a number $t \le \min(a, b, c)$, we only realize that both sides of the above inequality are changed. But the feature here is that we can compare these changing values. Indeed, consider the function

$$\begin{aligned} f(t) = {} & (a + t)^4 + (b + t)^4 + (c + t)^4 - \\ & - (a + t)(b + t)(c + t)(a + b + c + 3t) \\ & - k(a - b)(b - c)(c - a)(a + b + c + 3t) \\ \Rightarrow f(t) = {} & A + Bt + Ct^2, \end{aligned}$$

in which the coefficients of $f(t)$ are

$$\begin{aligned} A &= a^4 + b^4 + c^4 - abc(a + b + c) - k(a - b)(b - c)(c - a), \\ B &= 4(a^3 + b^3 + c^3) - (a + b + c)(ab + bc + ca) - 3abc - k(a - b)(b - c)(c - a), \\ C &= 6(a^2 + b^2 + c^2) - (a + b + c)^2 - 3(ab + bc + ca). \end{aligned}$$

Clearly, $C \ge 0$ because

$$6(a^2 + b^2 + c^2) \ge 2(a + b + c)^2 \ge (a + b + c)^2 + 3abc.$$

WLOG, assume that $c = \min(a, b, c)$. We will prove

$$4(a^3 + b^3 + c^3) - (a + b + c)(ab + bc + ca) - 3abc \ge 6\sqrt{2}(a - b)(b - c)(c - a).$$

The left hand side can be analyzied to sum of squares as follow

$$\begin{aligned} LHS &= (a - b)^2(a + b) + (b - c)^2(b + c) + (c - a)^2(c + a) + 2(a^3 + b^3 + c^3 - 3abc) \\ &= (a - b)^2(2a + 2b + c) + (b - c)^2(2b + 2c + a) + (c - a)^2(2c + 2a + b). \end{aligned}$$

From this transformation, we only need to examine one case $c = 0$ (that is based on the first idea of the entirely mixing variables method). If $c = 0$, we need to prove that

$$4(a^3 + b^3) - ab(a + b) \geq 6\sqrt{2}ab(b - a)$$
$$\Leftrightarrow \ a^3 + 4b^3 + (6\sqrt{2} - 1)a^2b \geq (6\sqrt{2} + 1)ab^2.$$

But this last one is obviously true by $AM - GM$ Inequality

$$4b^3 + (6\sqrt{2} - 1)a^2b \geq 4\sqrt{6\sqrt{2} - 1}ab^2 \geq (6\sqrt{2} + 1)ab^2.$$

Thus in the expression of $f(t)$, the coefficients of t, t^2 are positive at once, it implies that $f(t)$ is an increasing function for $t \geq 0$. If $c = \min(a, b, c)$, we have

$$a^4 + b^4 + c^4 - abc(a + b + c) - k(a - b)(b - c)(c - a)(a + b + c)$$
$$\geq a'^4 + b'^4 + c'^4 - a'b'c'(a' + b' + c') - k(a' - b')(b' - c')(c' - a'),$$

in which $a' = a - c, b' = b - c, c' = c - c = 0$. It implies that we only need to check the first problem in case $c = 0$, which was showed as above. The problem is completely solved and the equality holds for $a = b = c$ or the following case with its permutations

$$c = 0, b = \frac{1 + \sqrt{5}}{2}a \ \Leftrightarrow \ a = \frac{3(3 - \sqrt{5})}{2}, b = \frac{3(\sqrt{5} - 1)}{2}, c = 0. \ \square$$

This is clearly a painstaking and complicated solution, but the main content is only involved by the idea of the entirely mixing variables method. Using functions as above is almost a helpful way for problems of this kind. One special example is Jack Grafukel's Inequality, where this method shows the most original, simplest solution

Problem 4. *Prove that for all non-negative real numbers a, b, c then the following inequality holds*

$$\frac{a}{\sqrt{a + b}} + \frac{b}{\sqrt{b + c}} + \frac{c}{\sqrt{c + a}} \leq \frac{5}{4}\sqrt{a + b + c}.$$

(*Jack Grafulkel, Crux*)

SOLUTION. Firstly, we must prove the problem in case one of three numbers a, b, c is 0. Indeed, suppose $c = 0$, the given inequality can be changed into

$$\frac{a}{\sqrt{a + b}} + \sqrt{b} \leq \frac{5}{4}\sqrt{a + b}$$
$$\Leftrightarrow \ \frac{1}{4}a + \frac{5}{4}b \geq \sqrt{b(a + b)}$$
$$\Leftrightarrow \ (a + b) + 4b \geq 4\sqrt{b(a + b)}.$$

By $AM - GM$ Inequality, it's obviously true and has equality when $a = 3b$.

Next, we will solve this problem in the general case. Denote

$$x=\sqrt{\frac{a+b}{2}}, y=\sqrt{\frac{a+c}{2}}, z=\sqrt{\frac{b+c}{2}}, k=\frac{5\sqrt{2}}{4}.$$

WLOG, we may assume that $x=\max(x,y,z)$. The required inequality is equivalent to

$$\frac{y^2+z^2-x^2}{z}+\frac{z^2+x^2-y^2}{x}+\frac{x^2+y^2-z^2}{y}\le\frac{5\sqrt{2}}{4}\sqrt{x+y+z}$$
$$\Leftrightarrow\ x+y+z+(x-y)(x-z)(z-y)\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)\le k\sqrt{x^2+y^2+z^2}\quad(1)$$

Clearly, the problem is proved in case $z\ge y$. Firstly, we will prove that, for every numbers $t\ge 0$ then

$$k\sqrt{(x+t)^2+(b+t)^2+(z+t)^2}\ge k\sqrt{x^2+y^2+z^2}+3t\quad(2)$$
$$\Leftrightarrow\ k\sqrt{x^2+y^2+z^2+2t(x+y+z)+3t^2}\ge k\sqrt{x^2+y^2+z^2}+3t.$$

Note that $k^2\ge 3$ and $x+y+z\ge\sqrt{x^2+y^2+z^2}$ so the above inequality is true. On the other hand, since $x=\max(x,y,z)$ and $x^2\le y^2+z^2$, there exists a positive number $t\le\min(a,b,c)$ for which $(x-t)^2=(y-t)^2+(z-t)^2$. Hence, as the above solved result, (there is one of three numbers a,b,c is 0) inequality (1) is true if we replace x,y,z by $x'=x-t, y'=y-t, z'=z-t$. So we obtain

$$x'+y'+z'+(x'-y')(x'-z')(z'-y')\left(\frac{1}{x'y'}+\frac{1}{y'z'}+\frac{1}{z'x'}\right)\le k\sqrt{x'^2+y'^2+z'^2}\quad(3)$$

The inequality (1) can be rewritten as follow

$$(x'-y')(x'-z')(z'-y')\left(\frac{1}{(x'+t)(y'+t)}+\frac{1}{(y'+t)(z'+t)}+\frac{1}{(z'+t)(x'+t)}\right)+$$
$$+x'+y'+z'+3t\le k\sqrt{(x'+t)^2+(y'+t)^2+(z'+t)^2},$$

but clearly, we obtain this one directly by adding (3) and (2) (in which x,y,z were replaced by x',y',z'). The inequality is completely solved and has equality when $x=3t, y=t, z=0$ or permutations. ❑

Moreover, here is an similar example

Problem 5. *Let a,b,c be three side-lengths of a triangle. Prove that*

$$2\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)\ge a+b+c+\frac{b^2}{a}+\frac{c^2}{b}+\frac{a^2}{c}.$$

Solution. Clearly, this one is equivalent to

$$\sum_{\text{cyc}}\frac{(a-b)^2}{b}\ge\frac{(a-b)(b-c)(c-a)(a+b+c)}{abc}.$$

$$\Leftrightarrow \sum_{\text{cyc}} ac(a-b)^2 \geq (a-b)(a-c)(b-c)(a+b+c).$$

The above form shows that we only need to prove it in case $a \geq b \geq c$ and $a = b+c$ (indeed, we only need to prove $\sum_{cyc}(a+c)(a-b)^2 \geq 3(a-b)(a-c)(b-c)$, applying the mixing variables method again, it remains to prove that $a(a-b)^2+b^2(b+a)+a^2b \geq 3ab(a-b)$, which is obvious). So we only need to prove the initial problem in case (a,b,c) are three lengths of a trivial triangle when $a = b+c$. The inequality becomes

$$2\left((b+c)^3c+c^3b+b^2(a+b)\right) \geq 2bc(b+c)^2+(b+c)^3b+b^3c+c^2(b+c)$$

$$\Leftrightarrow\; b^4-2b^3c-b^2c^2+4bc^3+c^4 \geq 0.$$

Because of the homogeneity, we may assume $c=1$ and prove $f(b) \geq 0$ for

$$f(b) = b^4-2b^3-b^2+4b+1$$

By derivative, it's easy to prove this property. This ends the proof. ❑

For the end, the reader should try proving two hard and interesting inequalities

Problem 6. *Let a,b,c be three side-lengths of a triangle. Prove that*

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right) \geq k\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}-3\right)$$

where $k = 1-\dfrac{2}{\left(2\sqrt{2}-1\right)\sqrt{5+4\sqrt{2}}+1}$.

Problem 7. *Prove that for all non-negative real numbers a,b,c with sum 1 then*

$$1 \leq \frac{a}{\sqrt{a+2b}}+\frac{b}{\sqrt{b+2c}}+\frac{c}{\sqrt{c+2a}} \leq \sqrt{\frac{3}{2}}.$$

The detailed proof will be saved up for the reader. Try it !