Junior problems

J157. Evaluate

$$1^2+2^2+3^2-4^2-5^2+6^2+7^2+8^2-9^2-10^2+\cdots-2010^2,$$

where each three consecutive signs $+$ are followed by two signs $-$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J158. Let n be a positive integer relatively prime with 10. Prove that the hundreds digit of n^{20} is even.

Proposed by Badar Al-Ghamdi, Saudi Arabia

J159. Find all integers n for which $9n+16$ and $16n+9$ are both perfect squares.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J160. Let ABC be a triangle with $\widehat{A}=90^\circ$ and let d be a line passing trough the incenter of the triangle and intersecting sides AB and AC in P and Q, respectively. Find the minimum of $AP\cdot AQ$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

J161. Let a,b,c be positive real numbers such that $a+b+c+2=abc$. Find the minimum of

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Proposed by Abdulmajeed Al-Gasem, Saudi Arabia

J162. Let $a_1,a_2,\ldots,a_n$ be positive real numbers. Prove that

$$\frac{a_1}{(1+a_1)^2}+\frac{a_2}{(1+a_1+a_2)^2}+\cdots+\frac{a_n}{(1+a_1+\cdots+a_n)^2}\le\frac{a_1+\cdots+a_n}{1+a_1+\cdots+a_n}.$$

Proposed by Neculai Stanciu, Buzau, Romania

Senior problems

S157. Let ABC be at triangle. Find the locus of points X on line BC such that

$$AB^2 + AC^2 = 2(AX^2 + BX^2).$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S158. Is there an integer n such that exactly two of the numbers $n+8, 8n-27, 27n-1$ are perfect cubes?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S159. In triangle ABC, lines AA', BB', CC' are concurrent at P, where points A', B', C' are situated on sides BC, CA, AB, respectively. Consider points A'', B'', C'' on segments $B'C', C'A', A'B'$, respectively. Prove that AA'', BB'', CC'' are concurrent if and only if $A'A'', B'B'', C'C''$ are concurrent.

Proposed by Dorin Andrica, Babes-Bolyai University Cluj-Napoca, Romania

S160. Let ABC be a triangle with $\widehat{B} \geq 2\widehat{C}$. Denote by D the foot of the altitude from A and by M be the midpoint of BC. Prove that $DM \geq \frac{AB}{2}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S161. Let ABC be a triangle inscribed in a circle of center O and radius R. If d_A, d_B, d_C are the distances from O to the sides of the triangle, prove that

$$R^3 - (d_A^2 + d_B^2 + d_C^2)R - 2d_A d_B d_C = 0.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

S162. Alice has a pair of scales that display the weight in grams. At step n she cuts a square of side n from a very large laminated sheet and places it on one of the two scales. A square of side 1 weighs 1 gram.

(a) Prove that for each integer g Alice can place the laminated squares on the scales such that after a certain number of steps the difference between the aggregate weights on the two scales is g grams.

(b) Find the least number of steps necessary to reach a difference of 2010 grams.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Undergraduate problems

U157. Let $(A, +, \cdot)$ be a finite ring such that $1 + 1 = 0$. Prove that the number of solutions to the equation $x^2 = 0$ is equal to the number of solutions to the equation $x^2 = 1$.

Proposed by Mihai Piticari, Dragos Voda National College, Campulung Moldovenesc, Romania

U158. Let $(a_n)_{n\geq 0}$ be a sequence with $a_0 > 0$ and $a_{n+1} = a_n + \frac{1}{a_n}$ for $n = 0, 1, \ldots$.

(a) Prove that $\lim\limits_{n\to\infty} a_n = +\infty$.

(b) Find $\lim\limits_{n\to\infty} \dfrac{a_n}{\sqrt{n}}$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

U159. Let x and y be positive real numbers. Prove that

$$x^y y^x \leq \left(\frac{x+y}{2}\right)^{x+y}.$$

Proposed by Samuel G. Moreno, Universidad de Jaén, Spain

U160. Let p be a prime and let s and n be positive integers. Prove that

$$\sum_{k=0 \pmod p} (-1)^k \cdot \binom{n}{k} \cdot k^s$$

is a multiple of p^d, where $d = \left\lfloor \frac{n-s-1}{p-1} \right\rfloor$ and $\lfloor x \rfloor$ is the integer part of x.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

U161. Let $f : (0, \infty) \to (0, \infty)$ be a function satisfying $f(f(x)) = x^2$ for all $x \in (0, \infty)$.

(a) Find $f(1)$.

(b) Determine the function f if it is differentiable at $x = 1$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

U162. Let $f : \mathbf{R} \to \mathbf{R}$ be a monotonic function and let $F : \mathbf{R} \to \mathbf{R}$,

$$F(x) = \int_0^x f(t)dt.$$

Prove that if F is differentiable, then f is continuous.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, and Mihai Piticari, Dragos Voda National College, Campulung Moldovenesc, Romania

Olympiad problems

O157. A frog jumps on the real axis, from the origin towards point $(1,0)$ such that the length of the nth jump is $1/p_n$ times its distance to the point $(1,0)$, where p_n is the nth prime ($p_1 = 2$, $p_2 = 3$, $p_3 = 5,\ldots$). Can the frog reach point $(1,0)$?

Proposed by Moreno Miguel Marano, Universidad de Jaén, Spain

O158. For each positive integer n define

$$a_n = \frac{(n+1)(n+2)\cdots(n+2010)}{2010!}.$$

Prove that there are infinitely many n such that a_n is an integer with no prime factors less than 2010.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

O159. Let G be a graph with $n \geq 5$ vertices. The edges of G are colored in two colors such that there are no monochromatic cycles C_3 and C_5. Prove that there are no more than $\frac{3}{8}n^2$ edges in the graph.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O160. Let $a_1, a_2, \ldots a_n, \cdots$ be a sequence of positive integers, such that for each prime p there are infinitely many terms in the sequence that are divisible by p. Prove that every positive rational number less than 1 can be represented as

$$\frac{b_1}{a_1} + \frac{b_2}{a_1 a_2} + \cdots + \frac{b_n}{a_1 a_2 \cdots a_n},$$

where $b_1, b_2, \ldots b_n$ are integers such that $0 \leq b_i \leq a_i - 1$, $i = 1, \ldots, n$.

Proposed by Nairi Sedrakyan, Armenia

O161. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^5(b+2c)^2} + \frac{1}{b^5(c+2a)^2} + \frac{1}{c^5(a+2b)^2} \geq \frac{1}{3}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O162. In a convex hexagon $ABCDEF$, $AB||DE$, $BC||EF$, $CD||FA$ and $AB+DE = BC+EF = CD+FA$. Denote the midpoints of sides AB, BC, DE, EF by A_1, B_1, D_1, E_1, respectively. Prove that $\widehat{D_1 O E_1} = \frac{1}{2}\widehat{DEF}$, where O is the point of intersection of segments $A_1 D_1$ and $B_1 E_1$.

Proposed by Nairi Sedrakyan, Armenia