Junior problems

J169. If $x, y, z > 0$ and $x + y + z = 1$, find the maximum of

$$E(x,y,z) = \frac{xy}{x+y} + \frac{yz}{y+z} + \frac{zx}{z+x}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

J170. In the interior of a regular pentagon $ABCDE$ consider the point M such that triangle MDE is equilateral. Find the angles of triangle AMB.

Proposed by Catalin Barbu, Vasile Alecsandri College, Bacau, Romania

J171. If different letters represent different digits, could the addition

$$\begin{array}{r} AXXXU \\ BXXV \\ CXXY \\ +DEXXZ \\ --------- \\ XXXXX \end{array}$$

be correct?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J172. Let P be a point situated in the interior of an equilateral triangle ABC and let A', B', C' be the intersectionsof lines AP, BP, CP with sides BC, CA, AB, respectively. Find P such that

$$A'B^2 + B'C^2 + C'A^2 = AB'^2 + BC'^2 + CA'^2.$$

Proposed by Catalin Barbu, Vasile Alecsandri College, Bacau, Romania

J173. Let a and b be rational numbers such that

$$|a| \leq \frac{47}{|a^2 - 3ab^2|} \quad \text{and} \quad |b| \leq \frac{52}{|b^2 - 3a^2|}.$$

Prove that $a^2 + b^2 \leq 17$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J174. The incircle of triangle ABC touches sides BC, CA, AB at D, E, F, respectively. Let K be a point on side BC and let M be the point on the line segment AK such that $AM = AE = AF$. Denote by L and N the incenters of triangles ABK and ACK, respectively. Prove that K is the foot of the altitude from A if and only if $DLMN$ is a square.

Proposed by Bogdan Enescu, B.P.Hasdeu College, Buzau, Romania

Senior problems

S169. Let $k > 1$ be an odd integer such that $a^k + b^k = c^k + d^k$ for some positive integers a, b, c, d. Prove that $\frac{a^k+b^k}{a+b}$ is not a prime.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S170. Consider $n(n \geq 6)$ circles of radius $r < 1$ that are pairwise tangent and all tangent to a circle of radius 1. Find r.

Proposed by Catalin Barbu, Vasile Alecsandri College, Bacau, Romania

S171. Prove that if the polynomial $P \in \mathbf{R}[X]$ has n distinct real zeros, then for any $\alpha \in \mathbf{R}$ the polynomial $Q(X) = \alpha X P(X) + P'(X)$ has at least $n-1$ distinct real zeros.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

S172. Let a, b, c be positive real numbers. Prove that

$$\sum_{cyc} \frac{a^2b^2(b-c)}{a+b} \geq 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S173. Let

$$f_n(x,y,z) = \frac{(x-y)z^{n+2} + (y-z)x^{n+2} + (z-x)y^{n+2}}{(x-y)(y-z)(x-z)}.$$

Prove that $f_n(x,y,z)$ can be written as a sum of monomials of degree n and find $f_n(1,1,1)$ for all positive integers n.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S174. Prove that for each positive integer k the equation

$$x_1^3 + x_2^3 + \cdots + x_k^3 + x_{k+1}^2 = x_{k+2}^4$$

has infinitely many solutions in positive integers with $x_1 < x_2 < \cdots < x_{k+1}$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Undergraduate problems

U169. Sequences $(x_n)_{n\geq1}$ and $(y_n)_{n\geq1}$ are defined by $x_1 = 2, y_1 = 1$, and $x_{n+1} = x_n^2 + 1$, $y_{n+1} = x_n y_n$ for all n. Prove that for all $n \geq 1$

$$\frac{x_n}{y_n} < \frac{651}{250}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

U170. Sequences $(x_n)_{n\geq1}, (y_n)_{n\geq1}$ are defined as follows: $x_1 = \alpha$, $y_1 = \beta$, with $|\alpha| \neq |\beta| \neq 0$, and

$$x_{n+1} = \max\left(x_n - y_n, x_n + y_n\right),$$
$$y_{n+1} = \min\left(x_n - y_n, x_n + y_n\right),$$

for all $n \geq 1$. Prove that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty.$$

Proposed by Bogdan Enescu, B.P.Hasdeu College, Buzau, Romania

U171. Let A be a matrix of order n such that $A^{10} = O_n$. Prove that

$$\frac{1}{4}A^4 + \frac{1}{2}A^3 + \frac{1}{2}A^2 + A + I_n$$

is invertible.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U172. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing invertible function such that for all $x \in \mathbb{R}$, $f(x) + f^{-1}(x) = e^x - 1$ for all $x \in R$. Prove that f has at most one fixed point.

Proposed by Samin Riasat, University of Dhaka, Bangladesh

U173. Let θ be a real number. Prove that

$$\sum_{k=0}^{n-1} \frac{\sin\left(\frac{2k\pi}{n} - \theta\right)}{3 + 2\cos\left(\frac{2k\pi}{n} - \theta\right)} = \frac{(-1)^n n \sin(n\theta)}{5F_n^2 + 4(-1)^n \sin^2\left(\frac{n\theta}{2}\right)},$$

where F_n denotes the n^{th} Fibonacci number.

Proposed by Javier Buitrago, Universidad National de Colombia, Colombia

U174. Let p be a prime. A linear recurrence of degree n in $\mathbb{F}_p$ is a sequence $\{a_k\}_{k\geq0}$ in $\mathbb{F}_p$ satisfying a relation of the form

$$a_{i+n} = c_{n-1}a_{i+n-1} + \cdots + c_1 a_{i+1} + c_0 a_i \text{ for all } i \geq 0,$$

where $c_0, c_1, \ldots, c_{n-1} \in \mathbb{F}_p$ and $c_0 \neq 0$.

(a) What is the maximal possible period of a linear recurrence of degree n in $\mathbb{F}_p$?

(b) How many distinct linear recurrences of degree n have this maximal period?

Proposed by Holden Lee, Massachusetts Institute of Technology

Olympiad problems

U169. Let a, b, c, d be real numbers such that

$$(a^2+1)(b^2+1)(c^2+1)(d^2+1) = 16.$$

Prove that

$$-3 \leq ab + bc + cd + da + ac + bd - abcd \leq 5.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA, and Gabriel Dospinescu, Ecole Normale Superieure, France

U170. Let a and b be positive integers such that a does not divide b and b does not divide a. Prove that there is an integer x such that $1 < x \leq a$ and both a and b divide $x^{\phi(b)+1} - x$, where ϕ is Euler's totient function.

Proposed by Vahgan Aslanyan, Yerevan, Armenia

U171. Prove that in any convex quadrilateral $ABCD$,

$$\sin\left(\frac{A}{3} + 60^\circ\right) + \sin\left(\frac{B}{3} + 60^\circ\right) + \sin\left(\frac{C}{3} + 60^\circ\right) + \sin\left(\frac{D}{3} + 60^\circ\right)$$
$$\geq \frac{1}{3}(8 + \sin A + \sin B + \sin C + \sin D).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U172. Prove that if a 7×7 square board is covered by 38 dominoes such that each domino covers exactly two squares of the board, then it is possible to remove one domino after which the remaining 37 cover the board.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

U173. Find all triples (x, y, z) of integers such that

$$\frac{x^3 + y^3 + z^3}{3} - xyz = 2010 \max\{\sqrt[3]{x-y}, \sqrt[3]{y-z}, \sqrt[3]{z-x}\}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Gabriel Dospinescu, Ecole Normale Superieure, France

U174. The point O is considered inside of the convex quadrilateral $ABCD$ of area S. Suppose that K, L, M, N are interior points of the sides AB, BC, CD, DA, respectively. If $OKBL$ and $OMDN$ are parallelograms of areas S_1 and S_2, respectively, prove that

(a) $\sqrt{S_1} + \sqrt{S_2} < 1.25\sqrt{S}$;

(b) $\sqrt{S_1} + \sqrt{S_2} < C_0\sqrt{S}$, where $C_0 = \max_{0<\alpha<\frac{\pi}{4}} \frac{\sin(2\alpha+\frac{\pi}{4})}{\cos\alpha}$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia