Junior problems

J49. Find the least k such that any k-element subset of $\{1, 2, ..., 10\}$ contains numbers whose sum is divisible by 11.

Proposed by Ivan Borsenco, University of Texas at Dallas

J50. Let $\overline{abc}$ be a prime. Prove that $b^2 - 4ac$ cannot be a perfect square.

Proposed by Ivan Borsenco, University of Texas at Dallas

J51. Let a, b, c the sides of a triangle. Prove that

$$(a+b)(b+c)(c+a) + (-a+b+c)(a-b+c)(a+b-c) \geq 9abc.$$

Proposed by Virgil Nicula and Cosmin Pohoata, Romania

J52. In the Cartesian plane, mark the point with coordinates (x, y) if $x, y > 0$ and $x^2 + y^2$ is a prime number. Let l_n be the lines given by $x + y = n$. Find all positive integers n such that line l_n is fully marked in the first quadrant.

Proposed by Ivan Borsenco, University of Texas at Dallas

J53. Consider a triangle ABC. Let I be its incenter and let M, N, P be the midpoints of triangle's sides. Prove that

$$IM^2 + IN^2 + IP^2 \geq r(R + r),$$

where R and r are the circumradius and the inradius, respectively.

Proposed by Cosmin Pohoata, Bucharest, Romania

J54. For each positive integer n, find the exponent of 2 in the prime factorization of the numerator of

$$1 + \frac{1}{3} + \frac{1}{5} + ... + \frac{1}{2n-1}.$$

Proposed by John Selfridge, USA

Senior problems

S49. Find all pairs (x, y) of integers such that

$$xy + \frac{x^3 + y^3}{3} = 2007.$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

S50. Let $p \geq 5$ be a prime and let $q_1^{\beta_1} q_2^{\beta_2} \cdot \ldots \cdot q_n^{\beta_n}$ be the prime factorization of $(p-1)^p + 1$.

Prove that $\sum_{i=1}^{n} q_i \beta_i > p^2$.

Proposed by Ivan Borsenco, University of Texas at Dallas

S51. Consider a quadrilateral $ABCD$ with no two sides parallel. Let O be the intersection of its diagonals and let $E \in AB \cap CD$ and $F \in AD \cap BC$. Parallels through O to the sides CD, DA, AB, BC intersect lines AB, BC, CD, DA at M, N, P, Q, respectively. Prove that M, N, P, Q are collinear and that the line that contain them is parallel to EF.

Proposed by Mihai Miculita, Oradea, Romania

S52. Let a, b, c, d be prime numbers such that $a \neq b$ and $1 < a \leq c$. Suppose that for all sufficiently large n the numbers $an + b$ and $cn + d$ have the same sum of digits in all bases $2, 3, ..., a-1$. Prove that $a = c$ and $b = d$.

(***)

S53. Let ABC be a triangle and let E, F be the feet of the angle bisectors of B and C, respectively. Denote by O the circumcenter of triangle ABC and by I_a the center of the excircle corresponding to vertex A. Prove that $OI_a \perp EF$.

Proposed by Cosmin Pohoata, Bucharest, Romania

S54. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2 - bc}{4a^2 + 4b^2 + c^2} + \frac{b^2 - ca}{4b^2 + 4c^2 + a^2} + \frac{c^2 - ab}{4c^2 + 4a^2 + b^2} \geq 0$$

and find all equality cases.

Proposed by Vasile Cartoaje, University of Ploiesti, Romania

Undergraduate problems

U49. Let $f : [0,1] \to [0,\infty)$ be an integrable function. Prove that

$$\int_0^1 f(x)dx \cdot \int_0^1 x^3 f(x)dx \geq \int_0^1 xf(x)dx \cdot \int_0^1 x^2 f(x)dx.$$

Proposed by Cezar Lupu, Bucharest and Mihai Piticari, Campulung, Romania

U50. Let A, B, C be $n \times n$ matrices such that

$$A^2 = B^2 = (AB)^2,\ A^2C = C^2A,$$

and A is invertible. Prove that $A^4 = B^4 = I_n$ and $AC = CA$.

Proposed by Magkos Athanasios, Kozani, Greece

U51. Let $P(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_0 \in \mathbb{R}[X]$. Suppose $P(x)$ has only real zeros. Prove that $Q(X) = \frac{a_n}{n!}X^n + \frac{a_{n-1}}{(n-1)!}X^{n-1} + \cdots + a_0$ has only real zeros.

Proposed by Jean-Charles Mathieux, Dakar University, Sénégal

U52. Let m be a positive integer. Prove that

$$\sum_{k=0}^{\infty}(-1)^k\binom{2m-2k}{m-k}\binom{m-k}{k} = 2^m.$$

Proposed by Gabriel Alexander Reyes, San Salvador, El Salvador

U53. Let $f, g \in C[X]$ be two nonconstant polynomials and suppose for each $z \in C$, $f(z)$ is a root of a unity and $g(z)$ is also root of a unity, but not necessarily of the same order. What can we say about f and g?

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

U54. Find the best constant c such that for all n, if $f(x) \in \mathbb{R}[X]$ of degree n satisfies

$$\int_0^1\int_0^1 (f(x) - f(y))^2 dxdy = 1,$$

then the function $g : [0,1] \to \mathbb{R}$, $g(x) = x(1-x)f'(x)$ has a Lipschitz constant at most cn^3.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Olympiad problems

O49. Let A_1, B_1, C_1 be points on the sides BC, CA, AB of a triangle ABC. Lines AA_1, BB_1, CC_1 intersect again the circumcircle of triangle ABC at A_2, B_2, C_2, respectively. Prove that

$$\frac{AA_1}{A_1A_2} + \frac{BB_1}{B_1B_2} + \frac{CC_1}{C_1C_2} \geq \frac{3s^2}{r\,(4R+r)},$$

where s, r, R are the semiperimeter, inradius, and circumradius of triangle ABC, respectively.

Proposed by Cezar Lupu, Romania and Darij Grinberg, Germany

O50. Find the least k for which there exist integers $a_1, a_2, ..., a_k$, different from -1, such that numbers $x^2 + a_i y^2, x, y \in \mathbb{Z}$, $i = 1, 2, ..., k$, cover the set of prime numbers.

Proposed by Iurie Boreico, Moldova and Ivan Borsenco, University of Texas at Dallas

O51. Find a closed form for $p(x) = \prod_{a=1}^{M}\prod_{b=1}^{N}(x - e^{\frac{2\pi i\ a}{M}} \cdot e^{\frac{2\pi i\ b}{N}})$,

where M and N are positive integers.

Proposed by Alex Anderson, New Trier High School, Winnetka, IL

O52. Suppose n is not a multiple of 3. Find all integer solutions of

$$(a^2 - bc)^n + (b^2 - ca)^n + (c^2 - ab)^n = 1.$$

Proposed by H. van der Berg

O53. Let ABC be a triangle and let w be its incircle. Denote by D, E, F the intersections of w with BC, CA, AB, respectively. Let $T \in AD \cap w$, $M \in BT \cap w$, $N \in CT \cap w$. Let p_1 be a circle tangent to w at T, and p_2 a circle tangent to w at D, so that p_1 and p_2 intersect on chord (XY). Prove that X, Y, M, N lie on the same circle.

Proposed by Cosmin Pohoata, Bucharest, Romania

O54. Let $p = 2q + 1$ be a prime number greater than 3. Prove that p divides the numerator of

$$\sum_{1 \leq i,j \leq q,\ i+j>q} \frac{1}{ij},$$

where the sum is taken over all ordered pairs (i, j).

Proposed by Iurie Boreico, Moldova