Proposed Problems

Secondary Level

Solutions should arrive by July 20, 2006 in order to be considered for publication.

Juniors

J13. Prove that for any positive integer n, the system of equations

$$x + y + 2z = 4n$$
$$x^3 + y^3 - 2z^3 = 6n$$

is solvable in nonnegative integers x, y, and z.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

J14. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$a\left(b^2 - \sqrt{b}\right) + b\left(c^2 - \sqrt{c}\right) + c\left(a^2 - \sqrt{a}\right) \geq 0.$$

Proposed by Zdravko F. Starc, Vršac, Serbia and Montenegro

J15. Find the least positive number α with the following property: in every triangle, one can choose two sides of lengths a, b such that

$$1 \leq \frac{a}{b} < \alpha.$$

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College, Romania

J16. Consider a scalene triangle ABC and let $X \in (AB)$ and $Y \in (AC)$ be two variable points such that $(BX) = (CY)$. Prove that the circumcircle of triangle AXY passes through a fixed point (different from A).

Proposed by Liubomir Chiriac, student, Chişinău, Moldova

J17. Let a, b, c be positive numbers. Prove the following inequality:

$$(ab + bc + ca)^3 \leq 3\left(a^2b + b^2c + c^2a\right)\left(ab^2 + bc^2 + ca^2\right).$$

Proposed by Ivan Borsenco, student, Chişinău, Moldova

J18. Let n be an integer greater than 2. Prove that

$$2^{2^{n+1}} + 2^{2^n} + 1$$

is the product of three integers greater than 1.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

S13. Let k be an integer and let $n = \sqrt[3]{k+\sqrt{k^2+1}} + \sqrt[3]{k-\sqrt{k^2+1}} + 1$. Prove that $n^3 - 3n^2$ is an integer.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

S14. Let a, b, c be the sides of a scalene triangle ABC and let S be its area. Prove that

$$\frac{2a+b+c}{a(a-b)(a-c)} + \frac{a+2b+c}{b(b-a)(b-c)} + \frac{a+b+2c}{c(c-a)(c-b)} < \frac{3\sqrt{3}}{4S}$$

Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Spain

S15. Consider a scalene triangle ABC and let $X \in \overline{AB}$ and $Y \in \overline{AC}$ be two variable points such that $BX = CY$. If $\{Z\} = BY \cap CX$ and the circumcircles of $\triangle AYB$ and $\triangle AXC$ meet each other at A and K, prove that the reflection of K across the midpoint of AZ belongs to a fixed line.

Proposed by Liubomir Chiriac, student, Chişinău, Moldova

S16. Let M_1 be a point inside triangle ABC and let M_2 be its isogonal conjugate. Let R and r denote the circumradius and the inradius of the triangle. Prove that

$$4R^2r^2 \geq \left(R^2 - OM_1^2\right)\left(R^2 - OM_2^2\right).$$

Proposed by Ivan Borsenco, student, Chişinău, Moldova

S17. Let $m > n > 1$ be positive integers. A set of m real numbers is given. We are allowed to pick any n of them, say $a_1, a_2, \ldots, a_n$, and ask: is it true that $a_1 < a_2 < \ldots < a_n$? Determine k such that we can find the order of all m numbers asking at most k questions.

Proposed by Iurie Boreico, student, Chişinău, Moldova

S18. Find the least positive integer n for which the polynomial

$$P(x) = x^{n-4} + 4n$$

can be written as a product of four nonconstant polynomials with integer coefficients.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Undergraduate Level

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U13. Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be two different matrices, at least one of them invertible. Prove that there exist the matrices $X, Y \in \mathcal{M}_n(\mathbb{C})$ such that

$$XAY - YBX = I_n.$$

Proposed by Daniela Petrişan, student, University of Bucharest

U14. Evaluate

$$\int_0^1 \frac{\ln(x)\ln(1-x)}{(1+x)^2}\,dx$$

Proposed by Ovidiu Furdui, Western Michigan University

U15. Let $f : [a, b] \to R$ be a continuous and convex function. Prove that

$$\int_a^b f(x)\,dx \geq 2\int_{\frac{3a+b}{4}}^{\frac{3b+a}{4}} f(x)\,dx \geq (b-a)\,f\left(\frac{a+b}{2}\right)$$

Proposed by Cezar Lupu, University of Bucharest, and Tudorel Lupu, Decebal Highschool, Constanţa

U16. Let $n \geq 1$ be a natural number. Prove that:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)\cdots(k+n)} = -\frac{2^n \ln 2}{n!} + \frac{1}{n!}\sum_{k=1}^{n} \frac{2^{n-k}}{k}.$$

Proposed by Ovidiu Furdui, Western Michigan University

U17. Find all real numbers a such that the sequence $x_n = n\{a \cdot n!\}$ converges.

Proposed by Gabriel Dospinescu, "Louis le Grand" College, Paris, France

U18. Let a and b be two positive real numbers. Evaluate

$$\int_a^b \frac{e^{\frac{x}{a}} - e^{\frac{b}{x}}}{x}\,dx.$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Olympiad Level

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O13. Let ABC be a triangle and P be an arbitrary point inside the triangle. Let A', B', C', respectively, be the intersections of AP, BP, and CP with the triangle's sides. Through P we draw a line perpendicular to PA that intersects BC at A_1. We define B_1 and C_1 analogously. Let P' be the isogonal conjugate of the point P with respect to triangle $A'B'C'$. Show that A_1, B_1, and C_1 lie on a line l that is perpendicular to PP'.

Proposed by Khoa Lu Nguyen, Sam Houston High School, Houston, Texas.

O14. The vertices of a planar graph G have degrees $3, 4$, or 5 and vertices with the same degree are not connected. Suppose that the number of 5-sided faces is greater than the number of 3-sided faces. Denote by v the total number of vertices and by v_3 the number of vertices with degree 3. Prove that

$$v_3 \geq \frac{v+23}{4}.$$

Proposed by Ivan Borsenco, student, Chişinău, Moldova

O15. a) The cells of a $(n^2-n+1)\times(n^2-n+1)$ matrix are colored in n colors. A color is called dominant in a row or column if there are at least n cells of this color on this row or column. A cell if called extremal if its color is dominant both on its row and on its column. Find all $n \geq 2$ for which there is a coloring with no extremal cells.

Proposed by Iurie Boreico, student, Chişinău, Moldova

O16. Let ABC be an acute-angled triangle. Let ω be the center of the nine point circle and let G be its centroid. Let $A', B', C', A", B", C"$ be the projections of ω and G on the corresponding sides. Prove that the perimeter of $A"B"C"$ is not less than the perimeter of $A'B'C'$.

Proposed by Iurie Boreico, student, Chişinău, Moldova

O17. Let α be a root of the polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, where $a_i \in [0,1]$, for $i = 0, 1, \ldots, n-1$. Prove that

$$\operatorname{Re}\alpha < \frac{1+\sqrt{5}}{2}.$$

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College, Romania

O18. Let x, y, z be real numbers such that $0 < y < x < 1$ and $0 < z < 1$. Prove that

$$(x^z - y^z)(1 - x^z y^z) > \frac{x-y}{1-xy}.$$

Proposed by Nikolai Nikolov, Sofia, Bulgaria