Discover disc covers!

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A version of the following problem, proposed by the author, appeared at the Macedonian Mathematical Olympiad in 1994 [DMTS94].

Problem 1. Let $m \geq n \geq k \geq 2$ be natural numbers. A set M of m points is given in the plane. Among any n points in M, one can always find k that can be covered by a disc of diameter 1. What is the minimal number of discs or radius 1 that is always sufficient to cover all the points in M?

(Macedonian Mathematical Olympiad, 1994)

Solution. *Weakening of the condition.* First we show that the maximal number of points in M with all mutual distances greater than 1 is $n-k+1$. Indeed, let K be a set of $n-k+2$ points. Extend this set to a set of n points and cover k of them by a disc of diameter 1. Clearly, at least 2 of the k covered points must come from K, but this means that their mutual distance is at most 1.

Next disc, please. We show now that we can always cover all the points in M by $n-k+1$ discs of radius 1. Start with an arbitrary point A_1 in M, and place a disc d_1 of radius 1 centered at A_1. If all the points of M are covered, we are done. Otherwise, there exists a point, say A_2, whose distance to A_1 is greater than 1. Place a disc d_2 of radius 1 centered at A_2. If all the points are covered by d_1 and d_2 we are done. Otherwise, there exists a point, say A_3, whose distance to both A_1 and A_2 is greater than 1. We proceed in the same manner and note that after t discs are placed either all points of M are covered or a set of $t+1$ points in M can be found whose mutual distances are greater than 1. By our previous observation we conclude that after at most $n-k+1$ discs are placed, all points of M are covered.

Demanding example. It remains to show that sometimes $n-k+1$ discs are actually needed. Consider $n-k+1$ points in the plane with mutual distances greater than $2x$, $x \geq 1$. Choose one of these points, say A, and place $m-n+k-1$ more points in the disc d of diameter 1 centered at A. We now have m points and any choice of n of them includes at least k points inside the disc d. Thus, this set satisfies the conditions of the problem. On the other hand, no disc of radius x can cover more than one of the initially chosen $n-k+1$ points, which means that we need at least that many discs to cover all the points. ∎

We note that we could show that no more than $n-k+1$ discs are needed by using a slightly different argument than the one provided by the procedure for adding discs introduced above. Indeed, after the initial observation that no more than $n-k+1$ points can have mutual distances greater than 1, we choose a maximal subset K of points in M whose mutual distances are greater than 1. Maximality here means that no point from M can be added to K that would have distance greater than 1 to all points in K. Thus all points in M are within distance 1 from at least 1 point in K, which means that discs of radius 1 centered at the points of K cover all the points of M.

The difference in the two arguments is subtle and it is a question of taste which one is preferred over the other. We can learn a lot of good mathematics and also learn a lot about ourselves by thinking of and comparing different solutions, as well as from reconsidering our solutions and studying what actually happened.

For example, let us reconsider what we did in the first part of the solution. From the initial condition that k out of any n points can be covered by a disc of diameter 1, we concluded that $2 = k-(k-2)$ out of any $n-(k+2)$ points could be covered by such a disc. The latter condition is a weakening of the original one (indeed, even though we can always find 2 vertices that can be covered by a disc of diameter 1 among any 4 vertices of a regular pentagon of side length 1, we cannot cover 3 of the 5 vertices by such a disc). Therefore, our conclusion was weaker than the given conditions. We may ask ourselves if there are situations in which our conclusions are just as strong as the original conditions. Moreover, we want conclusions that are seemingly stronger than the given conditions.

Definition 1. Let $m \geq n \geq k \geq 2$ be natural numbers. We say that a set M of points in the plane has the property (m, n, k) if M has m points and at least k of any n points in M can be covered by a disc of diameter 1.

We are ready now to state a meaningful and interesting question.

Question 1. Find the triples (m, n, k) such that every set of points in the plane that has the property (m, n, k) also has the property $(m, n+1, k+1)$?

We distinguish an extremely important case.

Problem 2. Four points are placed in the plane. Show that if any three of the four points can be covered by a disc of diameter 1, then all four points can be covered by such a disc.

The connection to Question 1 is obvious. The problem states that if a set of points in the plane has the property $(4, 3, 3)$ then it also has the property $(4, 4, 4)$. We offer the following equivalent reformulation.

Problem 3 (Dual reformulation of Problem 2)**.** Four discs of diameter 1 are placed in the plane. Show that if any three of the four discs have a common point, then all four discs have a common point.

We leave it to the reader to verify that the two problems are stating the same thing. The reason the reformulated version seems easier to handle is that we need to establish the existence of a single point (that lies inside several discs) rather than the existence of a disc (that covers several points). Thus the roles of the points and the discs are somewhat inverted. Before we solve the problem(s), note that they are not as brave as it seems. As an encouragement, note that if any two out of three segments (of any length!) on a line have a common point, so do all three of them.

We now solve the dual version of the problem.

First solution to Problem 3. Let A, B, C and D be the centers of the 4 given discs of diameter 1, denoted d_A, d_B, d_C and d_D, respectively.

Triangle. Assume one of the four points A, B C and D is inside the triangle (including the degenerate cases) determined by the other three points. Without loss of generality assume that D is inside the triangle ABC. The fact that $d_A \cap d_B \cap d_C$ is nonempty means that there exists a point X within distance $1/2$ from each of the points A, B and C. But then the disc of radius $1/2$ (and diameter 1) centered at X covers the triangle ABC and its interior point D. Thus X is also within distance $1/2$ from D an therefore belongs to d_D.

Convex quadrilateral. Otherwise the four points A, B, C and D form a convex quadrilateral q.

Intersection inside. Assume that the intersection $d_B \cap d_D$ lies entirely inside d_A. Then the point in the intersection $d_B \cap d_C \cap d_D$ is also in d_A and we are done.

Reduction to the one-dimensional case. Thus, by symmetry, we may assume that no intersection of two of the discs lies entirely inside any other disc. Consider the disc d_A. Since q is a convex quadrilateral, there exists a line a through A such that the points B C and D are all on the same side of a. Let A' be the point on the boundary circle b_A of d_A such that AA' is orthogonal to a and A' is not on the same side of a as B, C and D (see Figure 1). None of the three arcs

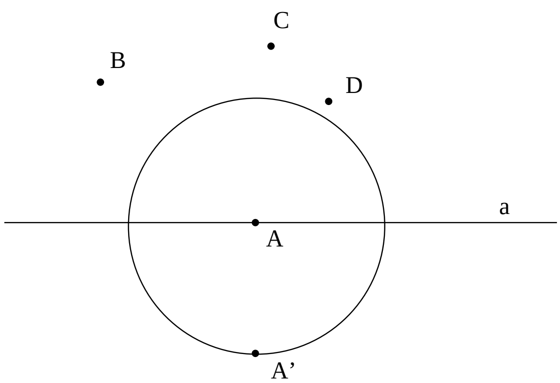

Figure 1: The case of convex quadrilateral

$\ell_X = b_A \cap d_X$, for $X = B, C, D$, can contain A'. This is because the distance from X to A' is larger than $1/2$. Any two of the three arcs ℓ_X, $X = B, C, D$, intersect. This is true since, by our assumption, the intersection $d_X \cap d_Y \cap d_A$ is nonempty and is not contained within d_A. Since none of the three arcs ℓ_X, $X = B, C, D$, contains A' and any two of them intersect, they all must have a common point (we are essentially in the one-dimensional case mentioned just before we started our proof, since a circle with a point removed is essentially a line, as every topologist will readily testify). ■

Note that the diameters of the discs in Problem 3 were not important. i.e., the statements holds regardless of the sizes of the four discs (in the last part of the solution above one needs to choose the disc d_A with the largest diameter in order to make sure that the point A' is not covered by the discs d_X, $X = B, C, D$). In fact, a more general result, known as Helly Theorem, holds.

Theorem 1 (Helly Theorem)**.** *Let $n \geq 3$ and $A_1, A_2, \ldots, A_n$ be convex sets in the plane. If any three of the sets $A_1, A_2, \ldots, A_n$ have a common point, then all of them have a common point.*

A survey of similar results, called Helly type theorems, can be found in [Eck93]. The statements (and the proofs) are often deep, usually involve convex sets and use the language of convex geometry and linear algebra (a set is convex if it contains every segment whose endpoints are in the set). We tried to keep the argument in our solution above as elementary as possible. We also note that the statement can be appropriately generalized to higher dimensions (in dimension d we need any $d+1$ convex sets from a finite family of convex sets to intersect in order to conclude that the whole family has a common point) and to infinite families of sets (by using convex compact sets).

We offer another solution to Problem 3 that uses the idea of convexity and solves the problem in the general setting of a family of four convex sets in the plane.

Second solution to Problem 3. Let c_A, c_B, c_C and c_D be four convex sets in the plane such that any three of them have nonempty intersection. Let A' be a point in $c_B \cap c_C \cap c_D$, B' a point in $c_A \cap c_C \cap c_D$, C' a point in $c_A \cap c_B \cap c_D$ and D' a point in $c_A \cap c_B \cap c_C$.

Triangle. Assume one of the four points A', B' C' and D' is inside the triangle (including the degenerate cases) determined by the other three points. Without loss of generality assume that D' is inside the triangle $A'B'C'$. Since A', B' and C' all belong to c_D and c_D is convex the whole triangle $A'B'C'$, along with the interior point D' is contained in c_D.

Convex quadrilateral. Otherwise the four points A', B', C' and D' form a convex quadrilateral q. Without loss of generality assume that $A'C'$ and $B'D'$ are the diagonals of q. Since both A' and C' belong to c_B and c_D the convexity of c_B and c_D implies that the whole diagonal $A'C'$ belongs to $c_B \cap c_D$. Similarly, the whole diagonal $B'D'$ belongs to $c_A \cap c_C$. But then the intersection of the two diagonals belongs to all four sets c_A, c_B, c_C and c_D. ■

We go back to the demanding example in the solution of Problem 1 and note that even if we were allowed to use discs of radius x larger than 1, we would still sometimes need $n-k+1$ such discs to cover all the points, as long as there is an upper bound on the radius of the discs we are allowed to use. Further, note that the total number of points in M plays no role in the answer. What about smaller discs?

Question 2. Let $m \geq n \geq k \geq 2$. What is the minimal number $\#(m,n,k)$ of discs of diameter 1 that is always sufficient to cover the points of a set with the property (m,n,k)?

By the comments made just before we posed Question 2 we know that

$$\#(m,n,k) \geq n-k+1.$$

Here is a special instance of Question 2.

Problem 4. What is the minimal number of discs of diameter 1 that is always sufficient to cover the points of a set with the property $(6,5,3)$?

In other words, what is $\#(6,5,3)$?

Solution. We claim that $\#(6,5,3) = 3$. Since we already know that $\#(6,5,3) \geq 5-3+1 = 3$ it remains to be shown that 3 discs are always sufficient.

At least 3 of the 6 points can be covered by a disc of diameter 1. Let A, B and C be three such points and denote the other 3 points by X, Y and Z.

If any 2 of the points X, Y and Z are within distance 1 from each other than such 2 points can be covered by a disc of diameter 1 and 3 discs are sufficient to cover all 6 points $(3+2+1=6)$.

Assume that the mutual distances between X, Y and Z are greater than 1. Thus no disc of diameter 1 contains 2 of these three points. Consider the three sets of points $S_A = \{B,C,X,Y,Z\}$, $S_B = \{A,C,X,Y,Z\}$ and $S_C = \{A,B,X,Y,Z\}$. By the property $(6,5,3)$ each contains a subset of three points, denoted T_A, T_B and T_C respectively, that can be covered by a disc of diameter 1. Since no disc of diameter 1 contains two of the points X, Y and Z the subsets T_A, T_B and T_C always involve exactly two of the points A, B and C. If the union of any two of the sets T_A, T_B and T_C contains 5 points we again see that 3 discs of diameter 1 suffice $(5+1=6)$. Otherwise all three sets T_A, T_B and T_C contain the same point among X, Y and Z (why?). Assume, without loss of generality, that X is contained in T_A, T_B and T_C, Then the points A, B, C and X satisfy the property $(4,3,3)$ and, by Problem2, can be covered by a single disc of diameter 1. Thus again 3 discs suffice $(4+1+1=6)$. ■

For the end we offer the following two challenges to the reader.

Problem 5. Determine $\#(6,n,k)$ for $6 \geq n \geq k \geq 2$.

The answers to some of the questions posed in the previous problem that are already answered above (or are trivial) are provided in the following table

$n\backslash k$	2	3	4	5	6
2					
3		1			
4			1		
5		3		1	
6	5	4	3	2	1

As a hint (or warning) we mention that it is not correct that $\#(6, n, k) = n - k + 1$, which may be conjectured from the above partially completed table. For example, try to show that $\#(6, 5, 2) = 5$.

Problem 6. Show that $\#(m, 2, 2) \leq 3$.

We see that almost any triple (m, n, k) presents a meaningful and often challenging problem. An excellent starting point for beginners in combinatorial geometry is [HD64]). The problems in this field are often easy to state and understand, but nevertheless deep and sometimes very difficult. There are still many unsolved problems.

References

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