Problems for Mathematical Reflections 6

Juniors

J31. Find the least perimeter of a right-angled triangle whose sides and altitude are integers.

Proposed by Ivan Borsenco, University of Texas at Dallas

J32. Let a and b be real numbers such that

$$9a^2 + 8ab + 7b^2 \leq 6.$$

Prove that $7a + 5b + 12ab \leq 9$.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

J33. Consider the sequence: 31, 331, 3331,... whose nth term has n $3s$ followed by a 1. Prove that this sequence contains infinitely many composite numbers.

Proposed by Wing Sit, University of Texas at Dallas

J34. Let ABC be a triangle and let I be its incenter. Prove that at least one of IA, IB, IC is greater than or equal to the diameter of the incircle of ABC.

Proposed by Magkos Athanasios, Kozani, Greece

J35. Prove that among any four positive integers greater than or equal to 1 there are two, say a and b, such that

$$\frac{\sqrt{(a^2-1)(b^2-1)}+1}{ab} \geq \frac{\sqrt{3}}{2}$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

J36. Let a, b, c, d be integers such that $gcd(a, b, c, d) = 1$ and $ad - bc \neq 0$. Prove that the greatest possible value of $gcd(ax + by, cx + dy)$ over all pairs (x, y) of relatively prime is $|ad - bc|$.

Proposed by Iurie Boreico, Moldova

Seniors

S30. Prove that for all positive real numbers a, b, and c,

$$\frac{1}{a+b+c}\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \geq \frac{1}{ab+bc+ca}+\frac{1}{2(a^2+b^2+c^2)}$$

Proposed by Pham Huu Duc, Australia

S31. Let ABC be a triangle and let P, Q, R be three points lying inside ABC. Suppose quadrilaterals $ABPQ$, $ACPR$, $BCQR$ are concyclic. Prove that if the radical center of these circles is the incenter I of triangle ABC, then the Euler line of the triangle PQR coincides with OI, where O is the circumcenter of triangle ABC.

Proposed by Ivan Borsenco, University of Texas at Dallas

S33. Let a, b, c be nonnegative real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(a+b)}+\frac{4(ab+bc+ca)}{(a+b)(b+c)(a+c)} \geq ab+bc+ca.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

S34. Let ABC be an equilateral triangle and let P be a point on its circumcircle. Find all positive integers n such that

$$PA^n + PB^n + PC^n$$

does not depend upon P.

Proposed by Oleg Mushkarov, Bulgarian Academy of Sciences, Sofia

S35. Let ABC be a triangle with the largest angle at A. On line AB consider the point D such that A lies between B and D and $AD = \dfrac{AB^3}{AC^2}$. Prove that $CD \leq \sqrt{3} \cdot \dfrac{BC^3}{AC^2}$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

S36. Let P be a point in the plane of a triangle ABC, not lying on the lines AB, BC, or CA. Denote by A_b, A_c the intersections of the parallels through A to the lines PB, PC with the line BC. Define analogously B_a, B_c, C_a, C_b. Prove that $A_b, A_c, B_a, B_c, C_a, C_b$ lie on the same conic.

Proposed by Mihai Miculita, Oradea, Romania

Undergraduate

U31. Find the minimum of the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{(x^2 - x + 1)^2}{x^6 - x^3 + 1}.$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

U32. Let $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \cdots, b_n$ be sequences of complex numbers. Prove that

$$\operatorname{Re}\left(\sum_{k=0}^{n} a_k b_k\right) \le \frac{1}{3n+2}\left(\sum_{k=0}^{n} |a_k|^2 + \frac{9n^2+6n+2}{2}\sum_{k=0}^{n} |b_k|^2\right)$$

Proposed by José Luis Díaz-Barrero, Barcelona, Spain

U33. Let n be a positive integer. Evaluate

$$\sum_{r=1}^{\infty} \frac{((n-1)!+1)^r (2\pi i)^r}{r! \cdot n^r} \cdot \prod_{u=0}^{n-1}\prod_{v=0}^{n-1}(n - uv)$$

Proposed by Paul Stanford, University of Texas at Dallas

U34. Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $f(1) = 0$. Prove that there is a $c \in (0,1)$ such that

$$f(c) = \int_0^c f(x)dx$$

Proposed by Cezar Lupu, University of Bucharest, Romania

U35. Find all linear maps $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that $f(XY) = f(X)f(Y)$ for all nilpotent matrices X and Y.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

U36. Let n be an even number greater than 2. Prove that if the symmetric group $\mathfrak{S}_n$ contains an element of order m, then $\mathrm{GL}_{n-2}(\mathbb{Z})$ contains an element of order m.

Proposed by Jean-Charles Mathieux, Dakar University, Sénégal

Olympiad

O31. Let n is a positive integer. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} = \sum_{k=0}^{n} 2^k \binom{n}{k}^2$$

Proposed by Jean-Charles Mathieux, Dakar University, Sénégal

O32. 18. Let $a, b, c > 0$. Prove that

$$\sqrt{\frac{a^2}{4a^2+ab+4b^2}} + \sqrt{\frac{b^2}{4b^2+bc+4c^2}} + \sqrt{\frac{c^2}{4c^2+ca+4a^2}} \leq 1$$

Proposed by Bin Zhao, University of Technology and Science, China

O33. 23. Let ABC be a triangle with cicrumcenter O and incenter I. Consider a point M lying on the small arc BC. Prove that

$$AM + 2OI \geq MB + MC \geq MA - 2OI$$

Proposed by Hung Quang Tran, Ha Noi University, Vietnam

O34. Suppose that $f \in \mathbb{Z}[X]$ is a nonconstant monic polynomial such that for infinitely many integers a, the polynomial $f(X^2+aX)$ is reducible in $\mathbb{Q}[X]$. Does it follow that f is also reducible in $\mathbb{Q}[X]$?

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

O35. Let $0 < a < 1$. Find, with proof, the greatest real number b_0 such that if $b < b_0$ and $(A_n \subset [0;1])_{n\in\mathbb{N}}$ are finite unions of disjoint segments with total length a, then there are two different $i, j \in \mathbb{N}$ such that $A_i \bigcap A_j$ is a union of segments with total length at least b. Generalize this result to numbers greater than 2: if $k \in \mathbb{N}$ find the least b_0 such that whenever $b < b_0$ and $(A_n \subset [0;1])_{n\in\mathbb{N}}$ are finite unions of disjoint segments with total length a, then there are k different $i_1, i_2, \ldots, i_k \in \mathbb{N}$ such that $A_{i_1} \bigcap A_{i_2} \bigcap \ldots \bigcap A_{i_k}$ is a union of segments with total length at least b.

Proposed by Iurie Boreico, Moldova

O36. Let $a_1, a_2, .., a_n$ and $b_1, b_2, ..., b_n$ be real numbers and let x_{ij} be the number of indices k such that $b_k \geq \max(a_i, a_j)$. Suppose that $x_{ij} > 0$ for any i and j. Prove that we can find an even permutation f and an odd permutation g such that $\sum\limits_{i=1}^{n} \frac{x_{if(i)}}{x_{ig(i)}} \geq n$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris