Two Applications of RCF, LCF, and EV Theorems

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Abstract

In this paper we present two new and difficult symmetric inequalities with right convex and left concave functions, as applications of RCF-Theorem and LCF-Theorem from [1], [2] and [3]. Moreover, we show that both inequalities can be also proved using the Equal Variable Theorem from [2] and [5].

Proposition 1. If $a_1, a_2, \ldots, a_n$, $n \le 81$ are nonegative real numbers such that

$$a_1^6 + a_2^6 + \cdots + a_n^6 = n,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 \le a_1^5 + a_2^5 + \cdots + a_n^5.$$

Proof. By letting $a_n = 1$, we obtain the initial statement but for $n-1$ numbers. Thus it suffices to prove the inequality for $n = 81$. Let us make the following substitution: $x_i = a_i^{\frac{1}{6}}$ for all i. Now we have to prove that

$$x_1^{\frac{1}{3}} + x_2^{\frac{1}{3}} + \cdots + x_n^{\frac{1}{3}} \le x_1^{\frac{5}{6}} + x_2^{\frac{5}{6}} + \cdots + x_n^{\frac{5}{6}}$$

when $x_1 + x_2 + \cdots + x_{81} = 81$. This inequality is equivalent to

$$f(x_1) + f(x_2) + \cdots + f(x_{81}) \le 81 \cdot f\left(\frac{x_1 + x_2 + \cdots + x_{81}}{81}\right),$$

where $f(u) = u^{\frac{1}{3}} - u^{\frac{5}{6}}, u \ge 0$. The second derivative of $f(u)$ is

$$f''(u) = \frac{1}{36} u^{-\frac{5}{3}} \left(5u^{\frac{1}{2}} - 8\right).$$

It follows that f is concave for $u \le s$, where $s = \dfrac{x_1 + x_2 + \cdots + x_{81}}{81} = 1.$

Thus by the LCF theorem, it suffices to prove the inequality for

$$x_1 = x_2 = \cdots = x_{80} \le 1 \le x_{81}.$$

This requires to prove the original inequality for $a_1 = a_2 = \cdots = a_{80} \le 1 \le a_{81}$. Let us rewrite the original inequality in the homogeneous form

$$81\left(a_1^5 + a_2^5 + \cdots + a_{81}^5\right)^2 \ge \left(a_1^6 + a_2^6 + \cdots + a_{81}^6\right)\left(a_1^2 + a_2^2 + \cdots + a_{81}^2\right)^2.$$

Since the case $a_1 = a_2 = \cdots = a_{80} = 0$ is trivial, we assume that $a_1 = a_2 = \cdots = a_{80} = 1$. Let $a_{81} = x$, then the inequality becomes

$$81(80 + x^5)^2 \geq (80 + x^6)(80 + x^2)^2$$

which is equivalent to

$$(x - 1)^2(x - 2)^2(x^6 + 6x^5 + 21x^4 + 60x^3 + 75x^2 + 60x + 20) \geq 0,$$

and clearly is true. For $a_1 \leq a_2 \leq \cdots \leq a_{81}$, equality occurs in the above homogeneous inequality when $a_1 = a_2 = \cdots = a_{81}$ or $a_1 = a_2 = \cdots = \frac{1}{2}a_{81}$. In the original inequality, equality occurs when $a_1 = a_2 = \cdots = a_n = 1$. Moreover, for $n = 81$, equality occurs when $a_1 = a_2 = \cdots = a_{80} = \sqrt[3]{\frac{3}{4}}$ and $a_{81} = \sqrt[3]{6}$.

Remark 1. The inequality is not valid for $n > 81$. To prove this is is enough to let $a_1 = a_2 = \cdots = a_{n-1} = 1$ and $a_n = 2$ in the homogeneous inequality

$$n\left(a_1^5 + a_2^5 + \cdots + a_n^5\right)^2 \geq \left(a_1^6 + a_2^6 + \cdots + a_n^6\right)\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^2.$$

We would get that $(n - 1)(81 - n) \geq 0$, clearly false for $n > 81$.

Remark 2. We can also prove the inequality in Proposition 1 using the Equal Variable Theorem ([2], [5]). According to the EV theorem, the following statement holds:
If $0 \leq x_1 \leq x_2 \leq \ldots \leq x_{81}$ such that $x_1 + x_2 + \cdots + x_{81} = 81$ and $x_1^{\frac{1}{3}} + x_2^{\frac{1}{3}} + \cdots + x_{81}^{\frac{1}{3}}$ is constant, then the sum $x_1^{\frac{5}{6}} + x_2^{\frac{5}{6}} + \cdots + x_{81}^{\frac{5}{6}}$ is minimal whenever $x_1 = x_2 = \cdots = x_{80} \leq x_{81}$.

Proposition 2. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$a_1 a_2 \cdots a_n = 1.$$

If $m \geq n - 1$, then

$$a_1^m + a_2^m + \cdots + a_n^m + n(2m - n) \geq (2m - n + 1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

Proof. Since the case $n = 2$ and $m = 1$ is trivial, we may assume that $m > 1$. Let $a_i = e^{x_i}$ for all i. We have to prove that

$$e^{mx_1} + e^{mx_2} + \cdots + e^{mx_n} + n(2m - n) \geq (2m - n + 1)(e^{-x_1} + e^{-x_2} + \cdots + e^{-x_n})$$

for $x_1 + x_2 + \cdots + x_n = 0$. This inequality is equivalent to

$$f(x_1) + f(x_2) + \cdots + f(x_n) \leq n \cdot f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right),$$

where $f(u) = e^{mu} + 2m - n - (2m - n + 1)e^{-u}$, $u \in \mathbb{R}$.

We will prove that $f(u)$ is right convex for $u \geq s$, where $s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0$; or $f''(u) \geq 0$ for $u \geq 0$. Taking the second derivative of $f(u)$, we get

$$f''(u) = e^{-u}[m^2 e^{(m+1)u} - 2m + n - 1] > 0,$$

because

$$m^2 e^{(m+1)u} - 2m + n - 1 \geq m^2 - 2m + n - 1 = (m-1)^2 + n - 2 > 0.$$

According to the RCF theorem, it suffices to prove the inequality above for $x_2 = x_3 = \cdots = x_n \geq 0$ or, equivalently, the original inequality for $a_2 = \cdots = a_n \geq 1$:

$$x^m + (n-1)y^m + n(2m - n) \geq (2m - n + 1)\left(\frac{1}{x} + \frac{n-1}{y}\right)$$

for $m \geq n - 1,\ 0 < x \leq 1 \leq y$ and $xy^{n-1} = 1$. Let us rewrite the inequality as $f(y) \geq 0$, where

$$f(y) = \frac{1}{y^{m(n-1)}} + (n-1)y^m + n(2m - n) - (2m - n + 1)\left(y^{n-1} + \frac{n-1}{y}\right).$$

We have $f'(y) = \frac{(n-1)g(y)}{y^{mn-m+1}},\ g(y) = m(y^{mn-1}) - (2m - n + 1)y^{mn-m-1(y^n-1)}$, and

$$g'(y) = y^{mn-m-2}h(y),$$

where

$$h(y) = m^2 n y^{m+1} - (2m - n + 1)[(m+1)(n-1)y^n - mn + m + 1]$$
$$h'(y) = (m+1)ny^{n-1}[m^2 y^{m-n+1} - (2m - n + 1)(n-1)].$$

If $m = n - 1$ and $n \geq 3$, then $h(y) = n(n-1)(n-2) > 0$. Otherwise, if $m > n - 1$ and $n \geq 2$, then

$$m^2 y^{m-n+1} - (2m - n + 1)(n-1) \geq m^2 - (2m - n + 1)(n-1) = (m - n + 1)^2 > 0,$$

and hence $h'(y) > 0$ for $y \geq 1$. Therefore, $h(y)$ is strictly increasing on $[1, \infty)$, and

$$h(y) \geq h(1) = n[(m-1)^2 + n - 2] > 0$$

for $y \geq 1$. Since $h(y) > 0$ implies $g'y > 0$, it follows that $g(y)$ is strictly increasing on $[1, \infty)$. Then $g(y) \geq g(1)$ for $y \geq 1$, and from $y^{mn-m+1}f'(y) = (n-1)g(y) \geq 0$ it follows that $f(y)$ is strictly increasing on $[1, \infty)$.

Consequently, $f(y) \geq f(1) = 0$ for $y \geq 1$. For $n = 2$ and $m = 1$, the original inequality becomes an equality. Otherwise, equlity occurs if and only if $a_1 = a_2 = \cdots = a_n = 1$.

Remark 3. For $m = n - 1$, the following statement is true:
If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \cdots a_n = 1$, then

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \geq (n-1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

This inequality follows from the Generalized Popoviciu's Inequality
If f is a convex function on an interval $\boldsymbol{I}$ and $a_1, a_2, \ldots, a_n \in \boldsymbol{I}$, then

$$f(a_1) + \cdots + f(a_n) + n(n-2) f\left(\frac{a_1 + \cdots + a_n}{n}\right) \geq (n-1)\left(f(b_1) + \cdots + f(b_n)\right),$$

where $b_j = \frac{1}{n-1} \sum_{j \neq i}^{n} a_j$, for all i.

Consider the convex function $f(x) = e^x$, replace $a_1, a_2, \ldots, a_n$ with $(n-1)\ln a_1, (n-1)\ln a_2, \ldots, (n-1)\ln a_n$, and you get the desired result (see [4]).

Remark 4. Replacing $a_1, a_2, \ldots, a_n$ by $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}$ the inequality in Proposition 2 becomes

$$\frac{1}{x_1^m} + \frac{1}{x_2^m} + \cdots + \frac{1}{x_n^m} + (2m-n)n \geq (2m-n+1)(x_1 + x + 2 + \cdots + x_n),$$

where $x_1 x_2 \cdots x_n = 1$. We can also prove the inequality by the Equal Variable theorem:
If $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ such that

$$x_1 + x_2 + \cdots + x_n = \text{constant and } \ x_1 x_2 \cdots x_n = 1,$$

then the sum $\frac{1}{x_1^m} + \frac{1}{x_2^m} + \cdots + \frac{1}{x_n^m}$ is minimal when $0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n$.

References

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