Junior problems

J151. Let $a \geq b \geq c > 0$. Prove that

$$(a-b+c)\left(\frac{1}{a}-\frac{1}{b}+\frac{1}{c}\right) \geq 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Hoang Quoc Viet, Dang Huyen Trang, University of Auckland, New Zealand

We write the inequality as follows

$$\frac{1}{a}-\frac{1}{b}+\frac{1}{c} \geq \frac{1}{a-b+c}.$$

It is equivalent to

$$\frac{a+c}{ac} \geq \frac{a+c}{b(a-b+c)}.$$

Therefore it is enough to check that

$$ac \leq b(a-b+c)$$

or

$$(b-a)(b-c) \leq 0.$$

The last inequality is true due to the given condition. Hence, the proof is completed.

Also solved by Arkady Alt, San Jose, California, USA; Neacsu Adrian, Pitesti, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Jamie D. Haley, Auburn University Montgomery, USA; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; G.R.A.20 Problem Solving Group, Roma, Italy; Sayan Mukherjee, Kolkata, India.

J152. Let $a, b, c > 0$. Prove that the following inequality holds

$$\frac{a+b}{a+b+2c} + \frac{b+c}{b+c+2a} + \frac{c+a}{c+a+2b} + \frac{2(ab+bc+ca)}{3\left(a^2+b^2+c^2\right)} \leq \frac{13}{6}.$$

Proposed by Andrei Răzvan Băleanu, "George Coşbuc" College, Motru, Romania

Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

Since

$$\sum_{\text{cyc}} \frac{a+b}{a+b+2c} = \sum_{\text{cyc}} \Big(1 - \frac{2c}{a+b+2c}\Big)$$

the inequality is

$$\sum_{\text{cyc}} \frac{2a}{2a+b+c} \geq \frac{5}{6} + \frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)}$$

by Cauchy–Schwarz

$$\frac{(a+b+c)^2}{2(a^2+b^2+c^2)+2(ab+bc+ca)} - \frac{1}{3}\frac{ab+bc+ca}{a^2+b^2+c^2} \geq \frac{5}{12}$$

Now define $ab+bc+ca = x$, and take $a+b+c = 1$ by homogeneity. We get

$$\frac{1}{2}\frac{1}{1-x} - \frac{1}{3}\frac{x}{1-2x} \geq \frac{5}{12}, \qquad 0 \leq x \leq 1/3$$

Simplifying we come to

$$\frac{1}{12}\frac{(2x+1)(3x-1)}{(1-x)(2x-1)} \geq 0, \qquad 0 \leq x \leq 1/3$$

which clearly holds.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Jamie D. Haley, Auburn University Montgomery, USA; Sayan Mukherjee, Kolkata, India; Hoang Quoc Viet, Dang Huyen Trang, University of Auckland, New Zealand.

J153. Find all integers n such that $n^2 + 2010n$ is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

The trivial solutions are $n = 0$ and $n = -2010$ for which $n^2 + 2010n = 0^2$. If $n^2 + 2010n$ is a nonzero perfect square, let d be the greatest common divider of n and 2010, and let $m = \frac{n}{d}$ and $k = \frac{2010}{d}$ (clearly relatively prime). Then $n^2 + 2010n = d^2m(m + k)$, where m and $m + k$ are relatively prime since otherwise their greatest common divider would divide both m and k. Since m and $m + k$ are relatively prime and their product is a perfect square, each one of them must be a perfect square. Since $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, and 67 is clearly prime since it is not divisible by $2, 3, 5, 7$ and it is less than $11^2 = 121$, we conclude that we may then find all possible roots by taking all divisors k of 2010, and for each one of them, finding all integers m such that $m, m + k$ are perfect squares $v^2 < u^2$, or $-m, -m - k$ are perfect squares $u^2 > v^2$. Note that $k = u^2 - v^2 = (u + v)(u - v)$ is the difference of two perfect squares, hence either a multiple of 4 (absurd, since 4 does not divide 2010) or odd. Now,

- if $k = 3$ and $d = 670$, $u + v = 3$ and $u - v = 1$, for $u = 2$ and $u = 1$, yielding $m = 1$ or $m = -4$, for $n = 670$ or $n = -2680$, and in both cases $n^2 + 2010n = 1340^2$.
- if $k = 5$ and $d = 402$, $u + v = 5$ and $u - v = 1$, for $u = 3$ and $v = 2$, yielding $m = 4$ or $m = -9$, for $n = 1608$ or $n = -3618$, and in both cases $n^2 + 2010n = 2412^2$.
- if $k = 15$ and $d = 134$, $u + v = 5$ and $u - v = 3$ for $u = 4$ and $v = 1$, or $u + v = 15$ and $u - v = 1$ for $u = 8$ and $v = 7$. This results respectively in $n = 134$ or $n = -2144$ for $n^2 + 2010n = 536^2$, and in $n = 6566$ or $n = -8576$ for $n^2 + 2010n = 7504^2$.
- if $k = 67$ and $d = 30$, $u + v = 67$ and $u - v = 1$ for $u = 34$ and $v = 33$, yielding $n = 32670$ and $n = -34680$ for $n^2 + 2010n = 33660^2$.
- if $k = 201$ and $d = 10$, $u + v = 67$ and $u - v = 3$ for $u = 35$ and $v = 32$, or $u + v = 201$ and $u - v = 1$ for $u = 101$ and $v = 100$, yielding respectively $n = 10240$ and $n = -12250$ for $n^2 + 2010n = 11200^2$, and $n = 100000$ and $n = -102010$ for $n^2 + 2010n = 101000^2$.
- if $k = 335$ and $d = 6$, $u + v = 67$ and $u - v = 5$ for $u = 36$ and $v = 31$, or $u + v = 335$ and $u - v = 1$ for $u = 168$ and $v = 167$, yielding respectively $n = 5766$ and $n = -7776$ for $n^2 + 2010n = 6696^2$, and $n = 167334$ and $n = -169344$ for $n^2 + 2010n = 168336^2$.
- if $k = 1005$ and $d = 2$, $u + v = 67$ and $u - v = 15$ for $u = 41$ and $v = 26$, or $u + v = 201$ and $u - v = 5$ for $u = 103$ and $v = 98$, or $u + v = 335$ and $u - v = 3$ for $u = 169$ and $v = 166$, or $u + v = 1005$ and $u - v = 1$ for $u = 503$ and $v = 502$. This results respectively in $n = 1352$ and $n = -3362$ for $n^2 + 2010n = 2132^2$, $n = 19208$ and $n = -21218$ for $n^2+2010n = 20188^2$, $n = 55112$ and $n = -57122$ for $n^2+2010n = 56108^2$, and $n = 504008$ and $n = -506018$ for $n^2 + 2010n = 505012^2$.

The values of n are then -506018, -169344, -102010, -57122, -34680, -21218, -12250, -8576, -7776, -3618, -3362, -2680, -2144, -2010, 0, 134, 670, 1352, 1608, 5766, 6566, 10240, 19208, 32670, 55112, 100000, 167334, 504008.

Also solved by Raul A. Simon, Chile.

J154. Let ABC be an acute triangle and let $MNPQ$ be a rectangle inscribed in the triangle such that $M, N \in BC, P \in AC, Q \in AB$. Prove that

$$\text{area}MNPQ \leq \frac{1}{2}\text{area}ABC.$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

First solution by Arkady Alt, San Jose, California, USA

Let $x = NP$ and $a = BC$. Since $PQ \parallel BC$ then $\triangle QAP \simeq \triangle BAC$ and, therefore,

$$\frac{PQ}{CB} = \frac{h_a - x}{h_a} \iff PQ = \frac{a\left(h_a - x\right)}{h_a}.$$

Hence,

$$\text{area}MNPQ = \frac{a}{h_a} \cdot x\left(h_a - x\right) \leq \frac{a}{h_a} \cdot \left(\frac{x + \left(h_a - x\right)}{2}\right)^2 = \frac{ah_a}{4} = \frac{1}{2}\text{area}ABC.$$

Second solution by G.R.A.20 Problem Solving Group, Roma, Italy

Since ABC is an acute triangle then the rectangle $MNPQ$ is contained into the triangle ABC. Let AH be the height from A to BC, then

$$|QM| = |PN| = t|AH|,\ |BM| = t|BH|,\ |CN| = t|CH| \quad \text{for some } t \in [0,1].$$

Therefore

$$\text{area}(MNPQ) = |QM|(|BC| - |BM| - |CN|) = t(1-t)|AH||BC| \leq 2t(1-t)\text{area}(ABC).$$

The desired inequality follows by noting that

$$\min_{t\in[0,1]} 2t(1-t) = \frac{1}{2}.$$

Also solved by Ercole Suppa, Teramo, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Raul A. Simon, Chile; Hoang Quoc Viet, Dang Huyen Trang, University of Auckland, New Zealand; Sayan Mukherjee, Kolkata, India.

J155. Find all n for which there are n consecutive integers whose sum of squares is a prime.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Neacsu Adrian, Pitesti, Romania

If $n = 1$ any integer square cannot be a prime, so n is at least 2. In cases $n = 2$ and $n = 3$ we can find consecutive numbers 1,2 and 2,3,4 for which sum of squares is 5 and 29, so prime numbers. Suppose now $n > 3$. If $a \in Z$ such that $A = a^2 + (a+1)^2 + \cdots + (a+n-1)^2$ is prime, then $A = na^2+2(1+2+\cdots+(n-1))a+1^2+2^2+\cdots+(n-1)^2 == n(a^2+(n-1)a+\frac{(n-1)(2n-1)}{6}) = p \cdot B$.

<u>Case 1</u>: n is prime.

Because $n > 3$, we have $2 \nmid n$, $3 \nmid n$, so $6 \nmid n$, and because A is prime it follows $6 \mid \frac{(n-1)(2n-1)}{6}$, $B = 1$ and $A = n$. But A is a sum of $n > 3$ integer consecutive squares, so $A > n$, contradiction.

<u>Case 2</u>: n is not prime.

If p prime, $p \mid n$, $p > 3$, we get $n = pm$, $A = p(ma^2 + m(pm-1)a + \frac{m(pm-1)(2pm-1)}{6}) = pB = p$, because A is prime. But A is a sum of $n > 3$ integer consecutive squares, so $A > n > p$, contradiction. We get $n = 2^a 3^b$, $a \geq 1$, $b \geq 1$ If $a \geq 2$, $n = 4m$, $A = 2(2ma^2 + 2m(4m-1)a + \frac{m(4m-1)(8m-1)}{3}) = 2B = 2$, because A is prime. But for the same above reason, $A > n > 2$, contradiction. If $b \geq 2$ we have the same approach. If $n = 2 \cdot 3 = 6$, we can find integers -1, 0, 1, 2, 3, 4 for which the sum of squares is 31, prime number. Finally the answer is $n \in \{2, 3, 6\}$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

J156. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) + f(x+y)$ is a rational number for all real numbers x and all $y > 0$. Prove that $f(x)$ is a rational number for all real numbers x.

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College, Buzau, Romania

Solution by Ercole Suppa, Teramo, Italy

For each real number x, consider a real number $y > 0$ and define u, v, w in the following way

$$u = f(x) + f(x+y), \quad v = f(x-y) + f(x), \quad w = f(x-y) + f(x+y)$$

Since $x = (x-y) + y$ and $x + y = (x-y) + 2y$, the numbers u, v, w are rational by hypothesis. Therefore

$$f(x) = \frac{1}{2}(u + v - w)$$

is a rational number and we are done.

Also solved by Neacsu Adrian, Pitesti, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

Senior problems

S151. Find all triples (x, y, z) of real numbers such that

$$x^2 + y^2 + z^2 + 1 = xy + yz + zx + |x - 2y + z|.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

We can write

$$x^2 + y^2 + z^2 + 1 = xy + yz + zx + |x - y + z - y|,$$

hence

$$(x - y)^2 + (y - z)^2 + (z - x)^2 + 2 = 2|x - y + z - y|.$$

It follows

$$(x - y)^2 + (y - z)^2 + (z - x)^2 + 2 \le 2|x - y| + 2|y - z|.$$

The last relation is equivalent to

$$(|x - y| - 1)^2 + (|y - z| - 1)^2 + (z - x)^2 \le 0.$$

We get $|x - y| = 1$, $|y - z| = 1$ and $x = z$. The desired triples (x, y, z) are $(a, a - 1, a)$, $(a, a + 1, a)$, where $a \in \mathbb{R}$.s

Second solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

By means of the transformation

$$x = \frac{a}{\sqrt{3}} + \frac{b}{\sqrt{6}} - \frac{c}{\sqrt{2}}, \quad y = \frac{a}{\sqrt{3}} + \frac{b}{\sqrt{6}} + \frac{c}{\sqrt{2}}, \quad z = \frac{a}{\sqrt{3}} - \frac{2b}{\sqrt{6}} \tag{1}$$

The quadratic form $xy + yz + zx$ becomes $\frac{a^2}{2} + \frac{b^2}{2} - a^2$ and then $x^2 + y^2 + z^2 - (xy + yz + zx) + 1 - |x - 2y + z| = \frac{3b^2+3c^2}{2} + 1 - \left|\frac{3c+\sqrt{3}b}{\sqrt{2}}\right| = 0$ is the equation we will solve.

The inverse of (1) is

$$a = \frac{x + y + z}{\sqrt{6}}, \quad b = \frac{x + y - 2z}{\sqrt{6}}, \quad c = \frac{y - x}{\sqrt{2}} \tag{2}$$

If $b \ge -\sqrt{3}c$ we have

$$\frac{3b^2 + 3c^2}{2} + 1 - \left|\frac{3c + \sqrt{3}b}{\sqrt{2}}\right| = \left(\frac{\sqrt{3}b}{\sqrt{2}} - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}c}{\sqrt{2}} - \frac{\sqrt{3}}{2}\right)^2 = 0$$

that is $b = 1/\sqrt{6}$, $c = 1/\sqrt{2}$. These conditions yield by (2) $x = z$ and $y = 1 + x$ while $b \geq -\sqrt{3}c$ yields $2y \geq x + z$ and then

$$\{x = z, y = 1 + x\} \tag{3}$$

If $b \leq -\sqrt{3}c$ we have

$$\frac{3b^2 + 3c^2}{2} + 1 - \left|\frac{3c + \sqrt{3}b}{\sqrt{2}}\right| = \left(\frac{\sqrt{3}b}{\sqrt{2}} + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}c}{\sqrt{2}} + \frac{\sqrt{3}}{2}\right)^2 = 0$$

that is $b = -1/\sqrt{6}$, $c = -1/\sqrt{2}$ yielding $x = z$ and $y = x - 1$ by (2). The condition $b \leq -\sqrt{3}c$ yields $2y \leq x + z$ and then

$$\{x = z, y = x - 1\} \tag{4}$$

and we are done.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Neacsu Adrian, Pitesti, Romania.

S152. Let $k \geq 2$ be an integer and let $m, n \geq 2$ be relatively prime integers. Prove that the equation

$$x_1^m + x_2^m + \cdots + x_k^m = x_{k+1}^n$$

has infinitely many solutions in distinct positive integers.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

First solution by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

Because m and n are relatively prime one can find positive integers α and β such that $\alpha n - \beta m = 1$. Consider $a_1, .., a_k$ distinct positive integers and take $x_i = a_i S^\beta$, for $i = 1, ..., k$, and $x_{k+1} = S^\alpha$, where $S = a_1^m + a_2^m + ... + a_k^m$. It is clear that the positive integers $x_1, x_2, ..., x_{k+1}$ are pairwise distinct and we have

$$x_1^m + x_2^m ... + x_k^m = a_1^m S^{\beta m} + a_2^m S^{\beta m} + ... + a_k^m S^{\beta m} = S^{\beta m+1} = S^{\alpha n} = x_{k+1}^n.$$

Second solution by G.R.A.20 Problem Solving Group, Roma, Italy

Let $a_1, \ldots, a_k$ be distinct positive integers. Let

$$S = \sum_{i=1}^{k} a_i^m = \prod_{h=1}^{j} p_h^{e_h}$$

where the last equality gives the prime factorization of the positive integer S.

Since m, n are relatively prime then for any $1 \leq h \leq j$ there is a positive integer g_h such that $g_h m \equiv -e_h \pmod{n}$. Now let's define

$$T = \prod_{h=1}^{j} p_h^{g_h},$$

hence $T^m S$ is an n-power by construction. Let $x_i = Ta_i$ for any $1 \leq i \leq k$, and let $x_{k+1} = (T^m S)^{1/n}$ then

$$x_1^m + \cdots + x_k^m = x_{k+1}^n.$$

It's clear that by using this method we are able to find infinitely many such solutions.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S153. Let X be a point interior to a convex quadrilateral $ABCD$. Denote by P, Q, R, S the orthogonal projections of X onto AB, BC, CD, DA, respectively. Prove that

$$PA \cdot AB + RC \cdot CD = \frac{1}{2}(AD^2 + BC^2)$$

if and only if

$$QB \cdot BC + SD \cdot DA = \frac{1}{2}(AB^2 + CD^2).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note first that the two given relations are obtained from one another after a cyclic permutation of the vertices of the quadrilateral $A \to B, B \to C, C \to D, D \to A$, and a corresponding permutation of points P, Q, R, S. Now, we may write the first relation as

$$0 = 2PA \cdot AB + 2RC \cdot CD - AD^2 - BC^2$$

$$= 2(PA \cdot PB - QB \cdot QC + RC \cdot RD - SD \cdot SA) + 2PA^2 + 2RC^2 - SA^2 - SD^2 - QB^2 - QC^2.$$

Note that an even number of cyclic permutations of the vertices (A, B, C, D) (for example $A \to C, B \to D, P \to R, \dots$) leaves the term in brackets unchanged, whereas an odd number of cyclic permutations (for example $A \to D, B \to A, P \to S, \dots$) changes its sign. It suffices therefore to prove that an odd number of cyclic permutations of the vertices inverts the sign of the remaining terms, ie, that

$$2PA^2 + 2RC^2 - SA^2 - SD^2 - QB^2 - QC^2 = -2QB^2 - 2SD^2 + PB^2 + PA^2 + RC^2 + RD^2,$$

$$PA^2 + RC^2 + QB^2 + SD^2 = SA^2 + QC^2 + PB^2 + RD^2.$$

But $PA^2 = OA^2 - OP^2$ since $OP \perp AP$, and similarly for the rest of the terms in both sides, hence both sides are equal to $OA^2 + OB^2 + OC^2 + OD^2 - OP^2 - OQ^2 - OR^2 - OS^2$. The conclusion follows. Note that if some of the points P, Q, R, S are not on the corresponding segments AB, BC, CD, DA, signed distances may be used (eg, if P is on ray AB but not inside segment AB, then $PB < 0$) and the proof is still valid.

S154. Let $k \geq 2$ be an integer and let $n_1, \ldots, n_k$ be positive integers. Prove that there are no rational numbers $x_1, \ldots, x_k, y_1, \ldots, y_k$ such that

$$(x_1 + y_1\sqrt{2})^{2n_1} + \cdots + (x_k + y_k\sqrt{2})^{2n_k} = 5 + 4\sqrt{2}.$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

First solution by Neacsu Adrian, Pitesti, Romania

We note that if $x, y \in Q$ such that

$$[1] : x + y\sqrt{2} = 5 + 4\sqrt{2}$$

then $x = 5$ and $y = 4$. Indeed, if $y \neq 4$, $\sqrt{2} = \frac{5-x}{y-4} \in Q$, contradiction. So $y = 4$ and from [1], $x = 5$. The given relation can be written as

$$\sum_{i=1}^{k} \left\{ \binom{2n_i}{0} {x_i}^{2n_i} + \binom{2n_i}{2} {x_i}^{2n_i-2} 2{y_i}^2 + \cdots \right\}$$
$$+ \sum_{i=1}^{k} \left\{ \binom{2n_i}{1} {x_i}^{2n_i-1} y_i + \binom{2n_i}{3} {x_i}^{2n_i-3} 2{y_i}^3 + \cdots \right\} \sqrt{2}$$
$$= 5 + 4\sqrt{2}, X + Y\sqrt{2} = 5 + 4\sqrt{2}.$$

From the above observation it follows $X = 5$, $Y = 4$ and: $X - Y\sqrt{2} = 5 - 4\sqrt{2}$, that can be written back as

$$\sum_{i=1}^{k} (x_i - y_i\sqrt{2})^{2n_i} = 5 - 4\sqrt{2}.$$

The left-hand side is a positive number and right-hand side is a negative one, contradiction.

Second solution by Arkady Alt, San Jose, California, USA

Consider the set $\mathbb{Q}\left(\sqrt{2}\right) = \left\{ x + y\sqrt{2} \vdots\, x, y \in \mathbb{Q} \right\}$ (quadratic extension of $\mathbb{Q}$) which is closed with respect to addition and multiplication. Note that 1 and $\sqrt{2}$ are linearly independent since $x + y\sqrt{2} = 0$ implies $y = 0$ (otherwise $\sqrt{2} = -\frac{x}{y} \in \mathbb{Q}$) and then $x = 0$. Therefore if $x_1, y_1, x_2, y_2 \in \mathbb{Q}$ then $x_1 + y_1\sqrt{2} = x_2 + y_2\sqrt{2} \iff \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases}$.

For any number $z = x + y\sqrt{2}$ from $\mathbb{Q}\left(\sqrt{2}\right)$ we consider the number $\overline{z} = x - y\sqrt{2}$ which we call the conjugate of z. For conjugation, due to linear independency, 1 and $\sqrt{2}$ satisfy the properties

1. $\overline{u + v} = \overline{u} + \overline{v}$, $u, v \in \mathbb{Q}\left(\sqrt{2}\right)$;
2. $\overline{u \cdot v} = \overline{u} \cdot \overline{v}$, $u, v \in \mathbb{Q}\left(\sqrt{2}\right)$.

Since $\left(x + y\sqrt{2}\right)^n = a_n(x, y) + b_n(x, y)\sqrt{2}$ for any positive integer n, where $a_n(x, y)$ and $b_n(x, y)$ are polynomials with integer coefficients then for any rational x, y numbers $a_n = a_n(x, y)$ and

$b_n = b_n(x,y)$ are rational as well. Then

$$\left(x - y\sqrt{2}\right)^n = \overline{\left(x + y\sqrt{2}\right)^n} = \overline{a_n(x,y) + b_n(x,y)\sqrt{2}} = a_n(x,y) - b_n(x,y)\sqrt{2}.$$

Hence, $\left(x_1 - y_1\sqrt{2}\right)^{2n_1} + ... + \left(x_k - y_k\sqrt{2}\right)^{2n_k} = 5 - 4\sqrt{2}$ and, therefore,

$$\sum_{i=1}^{k}\left(x_i + y_i\sqrt{2}\right)^{2n_i}\sum_{i=1}^{k}\left(x_i - y_i\sqrt{2}\right)^{2n_i} = \left(5 + 4\sqrt{2}\right)\left(5 - 4\sqrt{2}\right) = 25 - 32 = -7,$$

which is a contradiction because the left hand side of equality is obviously positive.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S155. Let a, b, c, d be the complex numbers corresponding to the vertices A, B, C, D of a convex quadrilateral $ABCD$. Given that $a\overline{c} = \overline{a}c$, $b\overline{d} = \overline{b}d$ and $a + b + c + d = 0$, prove that $ABCD$ is a parallelogram.

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College, Buzau, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote $a = \rho_a e^{i\alpha} = \rho_a(\cos\alpha + i\sin\alpha)$, $\rho_a = |a|$ being a nonnegative real, and express similarly b, c, d with respective radii ρ_b, ρ_c, ρ_d and angles β, γ, δ. Since $\overline{a}c = \overline{a\overline{c}}$, we conclude that $a\overline{c}$ is real, yielding either $\gamma = \alpha$ or $\gamma = \alpha + \pi$. Similarly, $\delta = \beta$ or $\delta = \beta + \pi$. Note that, if $\alpha = \gamma$,

$$a + c = (\rho_a + \rho_c)\cos\alpha + i(\rho_a + \rho_c)\sin\alpha,$$

whereas if $\gamma = \alpha + \pi$,

$$a + c = (\rho_a - \rho_c)\cos\alpha + i(\rho_a - \rho_c)\sin\alpha,$$

and similarly for $b + d$. In either case, $a + c$ is either 0 or collinear with 0, a and c, while $b + d$ is also either 0 or collinear with 0, b and d. If $a + c$ and $b + d$ are both nonzero, they are both collinear with 0 since they add up to 0, hence a, b, c, d are collinear, absurd. It follows that at least one of $a + c$, $b + d$, is zero, and clearly $a + c = b + d = 0$, or $\rho_a = \rho_c$, $\rho_b = \rho_d$, $\gamma = \alpha + \pi$, $\delta = \beta + \pi$. Clearly 0 is the midpoint of diagonals AC and BD, it follows that $ABCD$ is a parallelogram.

Also solved by Arkady Alt, San Jose, California, USA.

S156. Let $f : \mathbf{N} \to [0, \infty)$ be a function satisfying the following conditions:

(a) $f(100) = 10$;

(b) $\frac{1}{f(0)+f(1)} + \frac{1}{f(1)+f(2)} + \cdots + \frac{1}{f(n)+f(n+1)} = f(n+1)$, for all nonnegative integers n.

Find $f(n)$ in closed form.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

Solution by Emanuele Natale e Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy

By the condition (b) we have that

$$f(n+1) - f(n) = \frac{1}{f(n) + f(n+1)}, \forall n \in \mathbb{N},$$

that is

$$f(n+1)^2 = 1 + f(n)^2 = 2 + f(n-1)^2 = \cdots = n + 1 + f(0)^2.$$

Letting $n = 99$, by (a)

$$100 = f(100)^2 = 100 + f(0)^2$$

that is $f(0) = 0$ and finally (note that $f(n) \geq 0$ by hypothesis)

$$f(n) = \sqrt{n}.$$

Also solved by Arkady Alt, San Jose, California, USA; Neacsu Adrian, Pitesti, Romania; Bedri Hajrizi; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

Undergraduate problems

U151. Let n be a positive integer and let

$$f(x) = x^{n+8} - 10x^{n+6} + 2x^{n+4} - 10x^{n+2} + x^n + x^3 - 10x + 1.$$

Evaluate $f(\sqrt{2}+\sqrt{3})$.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

First solution by John Mangual, UC Santa Barbara, USA

Let $x = \sqrt{2}+\sqrt{3}$. Then $x^2 = 5 + 2\sqrt{6}$ and more manipulation gives $x^4 - 10x^2 + 1 = 0$. The polynomial to be evaluated can be written

$$(x^n + x^{n+4})(x^4 - 10x^2 + 1) + x^3 - 10x + 1.$$

The first term vanishes and we evaluate the remaining trinomial. Using the binomial theorem $(\sqrt{2}+\sqrt{3})^3 = 11\sqrt{2} + 9\sqrt{3}$. Finally

$$(\sqrt{2}+\sqrt{3})^3 - 10(\sqrt{2}+\sqrt{3}) + 1 = (11\sqrt{2}+9\sqrt{3}) - 10(\sqrt{2}+\sqrt{3}) + 1 = \sqrt{2} - \sqrt{3} + 1.$$

Second solution by Arkady Alt, San Jose, California, USA

Note that $\sqrt{2}+\sqrt{3}$ is a root of the polynomial

$$x^4 - 10x^2 + 1 = \left(x - \sqrt{2} - \sqrt{3}\right)\left(x + \sqrt{2} - \sqrt{3}\right)\left(x + \sqrt{2} + \sqrt{3}\right)\left(x - \sqrt{2} + \sqrt{3}\right),$$

Because

$$x^{n+8} - 10x^{n+6} + 2x^{n+4} - 10x^{n+2} + x^n + x^3 - 10x + 1 = \left(x^4 - 10x^2 + 1\right)\left(x^{n+4} + x^n\right) + x^3 - 10x + 1$$
$$= \left(x^4 - 10x^2 + 1\right)\left(x^{n+4} + x^n + \frac{1}{x}\right) + 1 - \frac{1}{x}$$

then

$$f\left(\sqrt{2}+\sqrt{3}\right) = 1 - \frac{1}{\sqrt{3}+\sqrt{2}} = 1 + \sqrt{2} - \sqrt{3}.$$

Also solved by Américo Tavares, Queluz, Portugal; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Daniel Lopez Aguayo, Puebla, Mexico; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Emanuele Natale, Università di Roma "Tor Vergata", Roma, Italy.

U152. Prove that for $n \geq 3$,

$$\varphi(2) + \varphi(3) + \cdots + \varphi(n) \geq \frac{n(n-1)}{4} + 1,$$

where φ is the Euler's totient function.

Proposed by Yufei Zhao, Massachusetts Institute of Technology, USA

First solution by Yufei Zhao, Massachusetts Institute of Technology, USA

The cases $n = 3, 4$ can be checked manually. So we assume that $n \geq 5$. We see that

$$\#\{(i,j) : 1 \leq i < j \leq n, \gcd(i,j) = \phi(2) + \cdots + \phi(n)\}.$$

So that

$$\#\{(i,j) : 1 \leq i < j \leq n, \gcd(i,j) = 1\} = 1 + 2(\phi(2) + \cdots + \phi(n)).$$

So it is equivalent to show that

$$\#\{(i,j) : 1 \leq i < j \leq n, \gcd(i,j) = 1\} \geq \frac{n(n-1)}{2} + 3.$$

Let $f(n)$ denote the LHS quantity. Observe that $f(n)$ counts the number of pairs $(x, y) \in \{1, 2, \ldots, n\}^2$ such that there does not exist any prime p such that p divides both x and y. Usint the principle of inclusion-exclusion, we find that

$$f(n) = n^2 - \sum_p \left\lfloor \frac{n}{p} \right\rfloor + \sum_{p<q} \left\lfloor \frac{n}{pq} \right\rfloor^2 - \sum_{p<q<r} \left\lfloor \frac{n}{pqr} \right\rfloor^2 + \cdots$$

where $p, q, r, \ldots$ are prime numbers. It follows that

$$\begin{aligned} f(n) &= n^2 - \sum_p \left(\frac{n}{p}\right)^2 - \sum_{p<q<r} \left(\frac{n}{pqr}\right)^2 - \cdots \\ &= n^2 \left(1 - \sum_p \frac{1}{p^2} - \sum_{p<q<r} \frac{1}{p^2q^2r^2}\right) \end{aligned}$$

(where only the sums with odd number of primes appear). The THS can be computed exactly, as we shall explain. We know that

$$\prod_p \left(1 - \frac{1}{p^2}\right) = 1 - \sum_p \frac{1}{p^2} + \sum_{p<q} \frac{1}{p^2q^2} - \sum_{p<q<r} \frac{1}{p^2q^2r^2} + \cdots$$

but also

$$\prod_p \left(1 - \frac{1}{p^2}\right)^{-1} = \sum_{n \geq 1} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}.$$

So

$$1 - \sum_p \frac{1}{p^2} + \sum_{p<q} \frac{1}{p^2q^2} - \sum_{p<q<r} \frac{1}{p^2q^2r^2} + \cdots = \frac{6}{\pi^2}.$$

On the other hand

$$1+\sum_p \frac{1}{p^2}+\sum_{p<q}\frac{1}{p^2q^2}+\sum_{p<q<r}\frac{1}{p^2q^2r^2}+\cdots=\prod_p\left(1+\frac{1}{p^2}\right)$$
$$=\prod_p \frac{1-\frac{1}{p^4}}{1-\frac{1}{p^2}}=\frac{\zeta(2)}{\zeta(4)}=\frac{\frac{\pi^2}{6}}{\frac{\pi^4}{90}}=\frac{15}{\pi^2}.$$

Subtracting the above results from each other, we find that

$$1-\sum_p \frac{1}{p^2}-\sum_{p<q<r}\frac{1}{p^2q^2r^2}-\cdots=1-\frac{1}{2}\left(\frac{15}{\pi^2}-\frac{6}{\pi^2}\right)=1-\frac{9}{2\pi^2}.$$

Therefore

$$f(n)\ge n^2\left(1-\frac{9}{2\pi^2}\right)\ge 0.544n^2$$

which is greater than $\frac{n(n-1)}{2}+3$ for all $n\ge 5$. This completes the proof.

Second solution by Sunil Ghosh, Royal Grammar School, Guildford, UK

Theorem.

$$\sum_{k=2}^{n}\varphi(k)\ge \frac{n(n-1)}{4}+1.$$

Proof. The inequality is equivalent to

$$-1+2\sum_{k=1}^{n}\varphi(k)\ge\frac{n(n-1)}{2}+3$$

and we notice that the LHS is in fact equivalent to the number of lattice points $(a,b)\in\mathbb{N}^2$ such that $\gcd(a,b)=1$ in the square grid $[1,n]\times[1,m]$.

Lemma 1. *The number of lattice points $(a,b)\in\mathbb{N}^2$ such that $\gcd(a,b)=1$ in the square grid $[1,n]\times[1,m]$ is equal to*

$$\sum_{k=1}^{n}\varphi(k)\left\lfloor\frac{n}{k}\right\rfloor^2.$$

Proof. Let A_p, where p is a prime less than or equal to n, be the set of pairs (a,b) such that p divides both a and b. By the inclusion-exclusion principle, the following counts non-coprime pairs (a,b)

$$\left|\bigcup_p A_p\right|=\sum_p|A_p|-\sum_{p<q}|A_p\cap A_q|+\sum_{p<q<r}|A_p\cap A_q\cap A_r|-\cdots$$

where the signs alternate; and we have that

$$|A_p|=\left\lfloor\frac{n}{p}\right\rfloor^2, |A_p\cap A_q|=\left\lfloor\frac{n}{pq}\right\rfloor^2, |A_p\cap A_q\cap A_r|=\left\lfloor\frac{n}{pqr}\right\rfloor^2,\ldots.$$

Hence, the number of points with coprime co-ordinates is

$$\begin{aligned}\varphi(1)n^2 + \sum_{p} \varphi(p) \left\lfloor \frac{n}{p} \right\rfloor^2 + \sum_{p<q} \varphi pq \left\lfloor \frac{n}{pq} \right\rfloor^2 + \cdots \\ = \sum_{k=1}^{n} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor^2 .\end{aligned}$$

So, by Lemma 1, the original inequality is equivalent to

$$\sum_{k=1}^{n} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor^2 \geq \frac{n(n-1)}{2} + 3.$$

Let us consider the LHS as follows

$$\begin{aligned}\sum_{k=1}^{n} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor^2 &= \sum_{k=1, \varphi(k)=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor^2 - \sum_{k=1, \varphi(k)=-1}^{n} \left\lfloor \frac{n}{k} \right\rfloor^2 \\ &\geq \sum_{k=1, \varphi(k)=1}^{n} \left(\frac{n^2}{k^2} - \frac{2n}{k} + 1 \right) - \sum_{k=1, \varphi(k)=-11}^{n} \frac{n^2}{k^2} \\ &= n^2 \sum_{k=1}^{n} \frac{\varphi(k)}{k^2} - 2n \sum_{k=1, \varphi(k)=1}^{n} \frac{1}{k} + \sum_{k=1, \varphi(k)=1}^{n} 1. \quad (*)\end{aligned}$$

Now, concentrating on the sum in the first term of (*), we invoke the following lemma

Lemma 2. *For all $n > 125$,*

$$\sum_{k=1}^{n} \frac{\varphi(k)}{k^2} > 0.6.$$

Proof.

$$\begin{aligned}\sum_{k=1}^{n} \frac{\varphi(k)}{k^2} &= \sum_{k=1}^{\infty} \frac{\varphi(k)}{k^2} - \sum_{k=n+1}^{\infty} \frac{\varphi(k)}{k^2} > \sum_{k=1}^{\infty} \frac{\varphi(k)}{k^2} - \sum_{k=n+1}^{\infty} \frac{1}{k^2} \\ &= \sum_{k=1}^{\infty} \frac{\varphi(k)}{k^2} - \left(\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} \right) = \frac{1}{\varsigma(2)} - \varsigma(2) + \sum_{k=1}^{n} \frac{1}{k^2}.\end{aligned}$$

And so,

$$0.6 - \frac{1}{\varsigma(2)} + \varsigma(2) < \sum_{k=1}^{n} \frac{1}{k^2} \Rightarrow \sum_{k=1}^{n} \frac{\varphi(k)}{k^2} > 0.6.$$

Note that the RHS of the right hand inequality is less than 1.637010, so let us find the least n such that the right hand inequality holds. By direct computation, the least such n is 126, with the sum of the reciprocals being explicitly greater than 1.637020, and as the sum is strictly increasing, the inequality holds for all $n > 125$ and this proves Lemma 2. Also, in $(*)$ it is clear that the second summation is less than $\log n$, for $n > 5$, and the third sum for $n > 10$ is greater than or equal to 3. Hence, due to these two facts and Lemma 2,

$$n^2 \sum_{k=1}^{n} \frac{\varphi(k)}{k^2} - 2n \sum_{k=1, \varphi(k)=1}^{n} \frac{1}{k} + \sum_{k=1, \varphi(k)=1}^{n} 1 > \frac{3n^2}{5} - 2n \log n + 3$$

for all $n > 125$. Let us prove that for all n in the interval $[126, \infty)$, the following inequality holds

$$\frac{3n^2}{5} - 2n \log n + 3 \geq \frac{n(n-1)}{2} + 3 \Leftrightarrow n - 20 \log n + 5 \geq 0.$$

Noting that the derivative of the function on the LHS of the right hand inequality is $1 - \frac{20}{n}$, and so is increasing for $n \geq 20$ and by direct calculation, $22 - 20 \log 22 + 5 > 0$, then this inequality holds for all $n \geq 22$, and so the original inequality holds for all n greater than 125.

U153. Let a, b, c, d be non-zero complex numbers such that $ad - bc \neq 0$ and let n be a positive integer. Consider the equation

$$(ax+b)^n + (cx+d)^n = 0.$$

(a) Prove that for $|a| = |c|$ the roots of the equation are situated on a line.

(b) Prove that for $|a| \neq |c|$ the roots of the equation are situated on a circle.

(c) Find the radius of the circle when $|a| \neq |c|$.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

First solution by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

If there is a root x such that $cx + d = 0$, then we have $ax + b = 0$. It follows $ad - bc = 0$, relation which is contrary to the hypothesis. Therefore we can assume that $cx + d \neq 0$. We can write the equation in the equivalent form

$$(\frac{ax+b}{cx+d})^n = -1. \tag{1}$$

This is in fact the binomial equation $z^n = -1$, where $z = \frac{ax+b}{cx+d}$. The roots of this equation are $z_k = cos\frac{(2k+1)\pi}{n} + isin\frac{(2k+1)\pi}{n}$, where $k = 0, 1, ..., n-1$.

It is clear that the relation between the roots of our equation and the roots of the binomial equation $z^n = -1$ is $z_k = \frac{ax_k+b}{cx_k+d}$, $k = 0, 1, ..., n-1$. Because $|z_k| = 1$, it follows that $|\frac{ax_k+b}{cx_k+d}| = 1$ for $k = 0, 1, ..., n-1$. The last relation is equivalent to

$$|\frac{x_k + \frac{b}{a}}{x_k + \frac{d}{c}}| = \frac{|c|}{|a|}. \tag{2}$$

If $|a| = |c|$, then $|x_k + \frac{b}{a}| = |x_k + \frac{d}{c}|$, i.e. the roots x_k are situated on the perpendicular bisector of the segment determined by the points of complex coordinates $-\frac{b}{a}$ and $-\frac{d}{c}$.

If $|a| \neq |c|$, then from (2) it follows that x_k belongs to the Apollonius circle corresponding to the constant $\frac{|c|}{|a|}$.

In order to find the radius of this circle we will use the following known result which can be obtained from Steward's theorem: Let α, β, and $K \geq 0$ be fixed real numbers, let A and B be fixed points in the plane. If $K > \frac{\alpha\beta}{\alpha+\beta} \cdot AB^2$, then the locus of points M in the plane with the property

$$\alpha \cdot MA^2 + \beta \cdot MB^2 = K, \tag{3}$$

ia a circle of radius $R = \sqrt{\frac{K}{\alpha+\beta} - \frac{\alpha\beta}{(\alpha+\beta)^2} \cdot AB^2}$. In our case we have just to take $K = 0$, $\alpha = |a|$, and $\beta = -|c|$ and the fixed points $A(-\frac{b}{a})$, and $B(-\frac{d}{c})$. We get

$$R = \frac{|b| \cdot |ad - bc|}{|c| \cdot ||a| - |b||}. \tag{4}$$

Second solution by G.R.A.20 Problem Solving Group, Roma, Italy

The equation is equivalent to $T(x)^n = -1$ where

$$T(x) = \frac{ax+b}{cx+d}$$

is a Möbius transform which it is known that it preserve circles in the Riemann sphere. Since the n-th complex roots of -1 are on the unit circle $\{|z| = 1\}$ we have to consider the set $T^{-1}(\{|z| = 1\})$. Now

$$T^{-1}(z) = \frac{dz-b}{-cz+a}$$

and the denominator becomes zero for some z such that $|z| = 1$ ($ad - bc \neq 0$ implies that in this case the numerator is not zero) iff $|a| = |c|$. Hence $T^{-1}(z)$ goes to ∞ and the circle $T^{-1}(C)$ is actually a line. If $|a| \neq |c|$ and $c \neq 0$ then by decomposing $T^{-1}(z) = T_3(T_2(T_1(z)))$ where

$$T_1(z) = -cz + a, \quad T_2(z) = \frac{1}{z}, \quad T_3(z) = \frac{(ad-bc)z - d}{c}$$

we find easily that the radius of the desired circle is

$$\frac{|ad-bc|}{||a|^2 - |c|^2|}$$

The case $c = 0$ is easier and gives the same result.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

U154. Find sufficient and necessary conditions on $a, b \in \mathbb{R}$ so that the set

$$S_{a,b} = \{(\{na\}, \{nb\})|n \in \mathbb{N}\}$$

is dense in the unit square $[0,1]^2$.

Proposed by Holden Lee, Massachusetts Institute of Technology, USA

First solution by Holden Lee, Massachusetts Institute of Technology, USA

We will prove that the desired condition is $a, b, 1$ are linearly independent over $\mathbb{Q}$.
Suppose that $a, b, 1$ are linearly independent over $\mathbb{Q}$.

It suffices to show that for any point $P = (x_0, y_0) \in (0,1)^2$ and any $\epsilon_1 > 0$ there exists $X \in S_{a,b}$ so that $PX < \epsilon_1$. We may assume that ϵ_1 is less than the distance between P and the boundary of the square. We use the following:

<u>Dirichlet's Theorem:</u> Let $x_1, ...x_k$ be real numbers and let $\epsilon > 0$. There exists $n \in \mathbb{N}$ and integers $p_1, ..., p_k$ such that $|nx_i - p_i| < \epsilon$ for all i.

For $k = 2$, this says that for any $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$\{na\}, \{nb\} \in [0, \epsilon) \cup (1 - \epsilon, 1)$$

Since a, b are irrational, $\{na\}, \{nb\} \neq 0$. Take $\epsilon = \dfrac{\epsilon_1}{\sqrt{2}}$ and choose n as above. Let $A = \{na\}, B = \{nb\}$. Consider 4 cases:

(a) $A, B \in (0, \epsilon)$. Note that $\{m\{na\}\} = \{mna\}$ for any $m, n \in \mathbb{N}$ and $a \in \mathbb{R}$, implying $S_{A,B} \subseteq S_{a,b}$. Since $a, b, 1$ are linearly independent over $\mathbb{Q}$,

$$K := \frac{A}{A+B} = \frac{na - \lfloor na \rfloor}{na + nb - \lfloor na \rfloor - \lfloor nb \rfloor}$$

is irrational. Let the line passing through P with slope K intersect the line $x + y = 1$ at $Q = (x_1, y_1)$. The set $\{\{Km\}|m \in \mathbb{N}\}$ is dense in $[0,1]$ (in fact, $\{Km\}$ is equidistributed); thus for any $\epsilon_2 > 0$ there exists $m \in \mathbb{N}$ so that $|\{Km\} - x_1| < \epsilon_2$. Let $x_- = \lfloor Km \rfloor$, $x_+ = \lceil Km \rceil$, $y_- = \left\lfloor \dfrac{B}{A} Km \right\rfloor$, $y_+ = \left\lceil \dfrac{B}{A} Km \right\rceil$. Now, the points $\{(NA, NB)|N \in \mathbb{N}\}$ are spaced at equal distances $d := \sqrt{A^2 + B^2} < \epsilon\sqrt{2} = \epsilon_1$ along the ray $y = \dfrac{B}{A}x$, $x > 0$, in particular, along the segment of the ray with $x_- < x < x_+$ and $y_- < y < y_+$. Let

$$\begin{aligned} l &= \left\{(x,y)|y + y_- = \frac{B}{A}(x + x_-)\right\} \\ l' &= l \cap (0,1)^2 \end{aligned}$$

Let $Z = \{(\{NA\}, \{NB\})|N \in \mathbb{N}, x_- < NA < x_+, y_- < NB < y_+\}$. Then $Z \subseteq S_{A,B} \cap l' \subseteq S_{a,b}$ and the points in Z divide l' into segments of length d, except for the two segments at the ends. (We are basically translating the square $(x_-, x_+) \times (y_-, y_+)$ to $(0,1)^2$.) Let l meet $x + y = 1$ at R, and let S be the intersection point of l and the line parallel to $x + y = 1$ through P. R is the unique point $(x, y) \in l, x + y = 1$; it is easy to check that $\left(\{Km\}, \left\{\frac{B}{A}Km\right\}\right)$ satisfies both of these conditions. Since $PQRS$ is a parallelogram, $PS = QR = \sqrt{2}\,|\{Km\} - x_1| < \sqrt{2}\epsilon_2$. Choosing $\epsilon_2 = \frac{\epsilon_1}{2\sqrt{2}}$, we get $PS < \frac{\epsilon_1}{2} < \epsilon_1$ and S must be in $(0,1)^2$ and hence in l'. Now let T, U be on l so that $T \neq U$, $ST = SU = \frac{d}{2}$. Since

$$
\begin{aligned}
PT &\leq PS + ST < \frac{\epsilon_1}{2} + \frac{d}{2} < \epsilon_1 \\
PU &\leq PS + SU < \frac{\epsilon_1}{2} + \frac{d}{2} < \epsilon_1
\end{aligned}
$$

$T, U \in (0,1)^2$ and $\overline{TU}$ is contained in l'. Since l' is divided into segments of length d by the points of Z, and $TU = d$, there exists $X \in Z \cap \overline{TU} \subseteq S_{a,b}$; then

$$PX \leq PS + SX < \frac{\epsilon_1}{2} + \frac{d}{2} < \epsilon_1$$

as desired.

(b) $A, B \in (1 - \epsilon, 1)$. Note that $1 - A, 1 - B, 1$ are linearly independent over $\mathbb{Q}$; repeat the argument in 1 on $1 - A, 1 - B$ instead of A, B to find that there exists $X = (x', y') \in S_{1-A,1-B}$ less than distance ϵ away from $P' = (1 - x_0, 1 - y_0)$. Using

$$
\begin{aligned}
1 - \{N(1 - A)\} &= \{NA\} \\
1 - \{N(1 - B)\} &= \{NB\}
\end{aligned}
$$

for any $N \in \mathbb{N}$, we get that $(1 - x', 1 - y') \in S_{A,B}$ is less than distance ϵ away from $P = (x_0, y_0)$.

(c) $A \in (0, \epsilon), B \in (1 - \epsilon, 1)$. Repeat the argument in 1 on $A, 1 - B$ to find that there exists $X = (x', y') \in S_{A,1-B}$ less than ϵ distance away from $(x_0, 1 - y_0)$. Then $(x', 1 - y') \in S_{A,B}$ is less than ϵ distance away from $P = (x_0, y_0)$.

(d) $A \in (1 - \epsilon, 1), B \in (0, \epsilon)$. Just switch the roles of a and b, x and y, and use 3.

Now suppose that $pa + qb + r = 0$ for $p, q, r \in \mathbb{Q}$ not all zero. One of p, q must be nonzero. Then

$$
\begin{aligned}
p(na - \lfloor na \rfloor) + q(nb - \lfloor nb \rfloor) + nr &= -p\lfloor na \rfloor - q\lfloor nb \rfloor \\
p\{na\} + q\{nb\} &= -nr - p\lfloor na \rfloor - q\lfloor nb \rfloor \qquad (5)
\end{aligned}
$$

The LHS of (1) is bounded so the RHS can only attain a finite number of values: If $r = \frac{r_1}{r_2}$ in lowest terms, then the RHS must be in the form $\frac{s}{r_2}, s \in \mathbb{Z}$. This says that the points of $S_{a,b}$ can only lie on a finite number of lines. Any point of $[0,1]^2$ not on one of these lines is not a limit point, so $S_{a,b}$ is not dense in $[0,1]^2$.

Second solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

Consider the two–dimensional torus $\mathbf{T}^2$ where $\mathbf{T} = \mathbf{R}\backslash\mathbf{N}$. We observe that it is the same problem of proving the density in $\mathbf{T}^2$ of the set

$$\tilde{S}_{a,b} = \{(na, nb) \mid n \in \mathbf{N}\}$$

On $\mathbf{T}$ we define the distance $\rho(\varphi, \varphi') = \min_{k\in\mathbf{Z}} |\varphi - \varphi' - k|$. We need the

Lemma 1 *The set $U_a = \{(na) \mid n \in \mathbf{N}\}$ is dense on $\mathbf{T}$ if and only if a is irrational.*

Proof Let's suppose $a = p/q$ where p and q integers. Since $a \cdot q = p = 0(\text{mod}.p)$ the set U_a is made of a finite set of points and then cannot be dense. So let's suppose a irrational and observe that we have $na \neq ma$ (on $\mathbf{T}$) for any $n \neq m$ integers. Indeed if $na = ma + N$, n, m, N integers, we conclude that a would be rational. Let a irrational. For any $\varepsilon > 0$, let k be the smallest integer such that $1/k < \varepsilon$ and define k subintervals of $\mathbf{T}$ of equal lengths. By the pigeonhole–principle, the points of the set $\{a, 2a, \ldots, (k+1)a\}$ are such that $\rho(ia, ja) < 1/k$ occurs at least for two of them, say ia and ja, $1 \leq i, j \leq k+1$. Defining $\Delta \doteq i - j$ and $\tilde{U}_a \doteq \{(n\Delta a) \mid n \in \mathbf{N}\} \subset U_a$ we conclude that $\rho(n\Delta a, (n+1)\Delta a) < 1/k < \varepsilon$ that is the density of $\tilde{U}_a$ on $\mathbf{T}$ and a fortiori the density of U_a.

The next step is

Lemma 2 *The set $V_{a,b} = \{(ta, tb) \mid t \in \mathbf{R}\}$ is dense on $\mathbf{T}^2$ if and only if a/b is irrational.*

Proof Let $\mathbf{T}^2 = [0,1]^2$ with the usual identification of the points $(x, y) = (x+1, y) = (x, y+1)$. If a/b is rational, a and b integers, it would be $ta = t'a + N$, $tb = t'b + M$ for any $t - t' = N/a$ and $t - t' = M/b$. This implies that $V_{a,b}$ has only a finite number of points. Thus consider a/b irrational. The equation $tb = 1$ is solved by $t_n = n/b$ since we are on a torus and then $ta\Big|_{t=t_n} = na/b$. This means that the track of $V_{a,b}$ on the vertical segment $y \equiv 1$ is made of the discrete set of points $W_{a,b} \doteq \{(na/b, 1) | n \in \mathbf{N}\}$. Lemma 1 implies $\overline{W}_{a,b} = [0,1]$. For any point of $(w_1, 1) \in W_{a,b}$, consider the set $\{w_1 + ta, 1 + tb\}$. These are segments of slope b/a which form a dense set of segments in $\mathbf{T}^2$ since $W_{a,b}$ is dense on $\mathbf{T}$. The conclusion is $\overline{V}_{a,b} = \mathbf{T}^2$.

Finally we have the

Corollary *The set $\tilde{S}_{a,b}$ is dense in $V_{a,b}$*

Proof $\tilde{S}_{a,b} \subset V_{a,b}$ and it is of course an infinite set. Bolzano–Weierstrass theorem assures that there exists in $\mathbf{T}^2$ an accumulation point of $\tilde{S}_{a,b}$ and then for any $\varepsilon > 0$, we have $\rho((ra, rb), (r'a, r'b)) < \varepsilon$ for any $r, r' > n_\varepsilon$. Then consider the subset $\hat{S}_{a,b}$ of $\tilde{S}_{a,b}$ defined by $\hat{S}_{a,b} = \{(\{n\Delta a\}, \{n\Delta b\}) \mid n \in \mathbf{N}\}$ $\Delta = r - r'$. By construction $\hat{S}_{a,b}$ is dense on $V_{a,b}$.

Also solved by Daniel Lopez Aguayo, Puebla, Mexico; Emanuele Natale, Università di Roma "Tor Vergata", Roma, Italy.

U155. Evaluate

$$\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\arctan 2x - \operatorname{arccot} 3x}{x} dx.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Titu Andreescu, University of Texas at Dallas, USA

Let $I = \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\arctan 2x - \operatorname{arccot} 3x}{x} dx$. Let $x = \frac{1}{y}$, then

$$\begin{aligned} I &= \int_3^2 \frac{\arctan \frac{2}{y} - \operatorname{arccot} \frac{3}{y}}{\frac{1}{y}} \frac{1}{-y^2} dy = \int_2^3 \frac{\arctan \frac{2}{y} - \operatorname{arccot} \frac{3}{y}}{y} dy \\ &= \int_2^3 \frac{\operatorname{arccot} \frac{y}{2} - \arctan \frac{y}{3}}{y} dy. \end{aligned}$$

Letting $y = 6x$ we get that

$$\int_2^3 \frac{\operatorname{arccot} \frac{y}{2} - \arctan \frac{y}{3}}{y} dy = 6 \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\arctan 2x - \operatorname{arccot} 3x}{x} dx$$

which implies that $i = -6I$ and hence $I = 0$.

U156. Let $f : [a, b] \to \mathbf{R}$ be a continuous function such that $\int\limits_0^1 xf(x)dx = 0$. Prove that

$$\left|\int\limits_0^1 x^2 f(x)dx\right| \le \frac{1}{6}\max_{x\in[0,1]}|f(x)|.$$

Proposed by Duong Viet Thong, National Economics University, Hanoi, Vietnam

Solution by Federico Buonerba, Università di Roma "Tor Vergata", Roma, Italy

Since $\int_0^1 xf(x)\,dx = 0$ then for any $a \in \mathbb{R}$

$$\left|\int_0^1 x^2 f(x)\,dx\right| = \left|\int_0^1 (x^2 - ax)f(x)\,dx\right| \le \int_0^1 \left|(x^2 - ax)\right|\,dx \max_{x\in[0,1]}|f(x)|.$$

Now we select the best constant by minimizing the integral $C_a = \int_0^1 \left|(x^2 - ax)\right|\,dx$. It is easy to verify that $C_1 = 1/6$. Note that we obtain a better constant for $a = 1/\sqrt{2}$:

$$C_{1/\sqrt{2}} = \frac{2 - \sqrt{2}}{6} < \frac{1}{6}.$$

Also solved by Neacsu Adrian, Pitesti, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

Olympiad problems

O151. Consider a triangle ABC and a point P in its interior. Lines PA, PB, PC intersect BC, CA, AB at A', B', C', respectively. Prove that

$$\frac{BA'}{BC} + \frac{CB'}{CA} + \frac{AC'}{AB} = \frac{3}{2}$$

if and only if at least two of the triangles PAB, PBC, PCA have the same area.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Sayan Mukherjee, Kolkata, India

Note that we have

$$\begin{aligned}\frac{BA'}{BC} = \frac{\triangle ABA'}{\triangle ABC} &= \frac{\triangle PBA'}{\triangle PBC} \\ &= \frac{\triangle ABA' - \triangle PBA'}{\triangle ABC - \triangle PBC} \\ &= \frac{\triangle PAB}{\triangle PAB + \triangle PAC}\end{aligned}$$

Denoting $\triangle PAB = x, \triangle PBC = y, \triangle PCA = z$ we have, from the given condition that

$$\frac{x}{z+x} + \frac{y}{x+y} + \frac{z}{y+z} = \frac{3}{2}.$$

This also implies $\dfrac{x}{x+y} + \dfrac{y}{y+z} + \dfrac{z}{z+x} = \dfrac{3}{2}$; and so we obtain

$$\sum_{cyc} \left(\frac{x}{x+y} - \frac{x}{z+x} \right) = 0;$$

Which, on simple calculations, is equivalent to with

$$\frac{(x-y)(z-y)(z-x)}{(x+y)(y+z)(z+x)} = 0;$$

Which is true if and only if $x = y$ or $y = z$ or $z = x$; i.e when two of $\triangle PAB, \triangle PAC, \triangle PBC$ have the same area.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

O152. Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be sequences defined by $a_{n+3} = a_{n+2} + 2a_{n+1} + a_n$, $n = 0, 1, ...,$ $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ and $b_{n+3} = b_{n+2} + 2b_{n+1} + b_n$, $n = 0, 1, ...,$ $b_0 = 3$, $b_1 = 2$, $b_2 = 1$. How many integers do the sequences have in common?

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Clearly $a_3 = b_3 = 8$, while $a_4 = 16$, $a_5 = 35$, $a_6 = 75$, and $b_4 = 12$, $b_5 = 29$, $b_6 = 61$. Note that for $n = 4, 5, 6$, $a_n > b_n > a_{n-1}$, or by trivial induction, for any $n \geq 3$,

$$a_{n+3} = a_{n+2} + 2a_{n+1} + a_n > b_{n+3} = b_{n+2} + 2b_{n+1} + b_n > a_{n+1} + 2a_n + a_{n-1} = a_{n+2}.$$

Therefore, since both sequences are clearly strictly increasing, no b_n for $n \geq 4$ may appear in (a_n), and the only values that appear in both sequences are $\{1, 2, 3, 8\}$, while the only n's for which $a_n = b_n$ are $n = 1$ and $n = 3$ with $a_1 = b_1 = 2$ and $a_3 = b_3 = 8$.

O153. Find all triples (x, y, z) of integers such that $x^2y + y^2z + z^2x = 2010^2$ and $xy^2 + yz^2 + zx^2 = -2010$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Neacsu Adrian, Pitesti, Romania

Substract both relations and get $(x-y)(y-z)(x-z) = 2010 \cdot 2011$. If $a = x - y$ and $b = y - z$, we have $a \cdot b \cdot (a + b) = 2 \cdot 3 \cdot 5 \cdot 67 \cdot 2011$, primes decomposition. If $P = a \cdot b$, $S = a + b$, we get $P \cdot S > 0$, so both P, S are positive or both are negative. In first case $a = 2010$, $b = 1$ and in second $a = -2011$, $b = 1$ and there permutations.

Case 1: $a = x - y = 2010 = p$, $b = y - z = 1$

We can write: $y = z + 1$, $x = z + p + 1$ From first relation we have $[1] : 3z^3 + 3z^2(p + 2) + z(p+2)^2 + 2p + 1 = 0$ From here, $z \mid 2p + 1 = 4021$, prime number, so $z \in \{-1, 1, -4021, 4021\}$ None of these verifies [1]

Case 2: $a = x - y = -2011 = -p - 1$, $b = y - z = 1$

We can write: $y = z + 1$, $x = z - p$ From first relation we have $3z^3 + 3z^2(1 - p) + z(p - 1)^2 = 0$ We get $z = 0$ or $3z^2 + 3z(1 - p) + (p - 1)^2 = 0$, which has negative discriminant. From $z = 0$, we have $x = -p = -2010$, $y = 1$ Finally all triples (x, y, z) are $(-2010, 1, 0)$, $(1, 0, -2010)$, $(0, -2010, 1)$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

O154. A non-isosceles acute triangle ABC is given. Let O, I, H be the circumcenter, the incenter, and the orthocenter of the triangle ABC, respectively. Prove that $\angle OIH > 135°$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

It is relatively well known that

$$OI^2 = R(R-2r);\ OH^2 = 9R^2 - (a^2+b^2+c^2);\ IH^2 = 4R^2 + 2r^2 - \frac{a^2+b^2+c^2}{2},$$

where as usual R, r denote the circumradius and inradius of ABC with sidelengths a, b, c. The first two relations may be found through the respective powers of I and H with respect to the circumcircle, the second one requiring some algebra using the Cosine Law and expressions of the area of the triangle. The third one may be deduced therefrom, applying the median theorem to triange OIH, since the nine-point center N is the midpoint of OH and $IN = R - \frac{r}{2}$ because the incircle and nine-point circle are tangent at the Feuerbach point. Note that $OH^2 - IH^2 - OI^2 = IH^2 + 2r(R-2r) > 0$ because $R > 2r$ and $IH^2 > 0$, approaching equality when ABC approaches an equilateral triangle, or OIH will be obtuse at I.

We may also write after some algebra $a^2+b^2+c^2 = 8R^2 + 8R^2\cos A\cos B\cos C$, or since ABC is acute, $a^2+b^2+c^2 = 8R^2+4\delta$ for some $\delta > 0$. Therefore, $OH^2 = R^2 - 4\delta$ and $IH^2 = 2r^2 - 2\delta$, or

$$\frac{OH^2 - IH^2 - OI^2}{2OI \cdot IH} = \frac{Rr - r^2 - \delta}{\sqrt{2}\sqrt{r^2-\delta}\sqrt{R(R-2r)}}.$$

Assume that this quantity is smaller than or equal to $\frac{1}{\sqrt{2}}$. Then,

$$\delta^2 + (R^2+2r^2)\delta + r^4 < 0,$$

clearly impossible. Hence $\cos\angle OIH < -\frac{1}{\sqrt{2}} = \cos 135°$, and $\angle OIH > 135°$.

Also solved by Daniel Campos Salas, Costa Rica.

O155. Prove that the equation

$$x^2 + y^3 = 4z^6$$

is not solvable in integers.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Ivan Borsenco, Massachusetts Institute of Technology, USA

Let $p \mid \gcd(x, y, z)$ and assume that $\text{ord}_p(x) = \alpha$, $\text{ord}_p(y) = \beta$, $\text{ord}_p(z) = \gamma$. From the given equation, two numbers from $(2\alpha, 3\beta, 6\gamma)$ are equal. Considering three possible cases:

- $6u = 2\alpha = 3\beta \le 6\gamma$,
- $6u = 2\alpha = 6\gamma \le 3\beta$,
- $6u = 3\beta \le 6\gamma \le 2\alpha$,

we obtain that it is enough to prove the statement in the case when $\gcd(x, y, z) = 1$.

Now, $y^3 = (2z^3 - x)(2z^3 + x)$, so $2z^2 - x = u^3$, $2z^3 + x = v^3$, where $\gcd(u, v) = 1$. Thus $u^3 + v^3 = 4z^3$, which does not have solutions in integers modulo 2.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We will understand that the problem statement should read "*is not solvable in nonzero integers* x, y, z", since clearly $y = 0$, $x = \pm 2z^3$ produces infinitely many integer solutions, two for each integer z. We prove first the following

Claim: solutions to $x^2 + y^3 = 4z^6$ in nonzero integers exist iff solutions to $u^3 + v^3 = 4t^3$ exist in nonzero pairwise relatively prime integers.

Proof: Assume solutions to $x^2 + y^3 = 4z^6$ in nonzero integers exist where $2z^3 - x, 2z^3 + x$ are not pairwise relatively prime. If a prime p divides x, z, it clearly divides y, or p^3 divides x^2, hence p^4 divides y^3, or p^6 divides y^3 and x^2. Clearly, $\left(\frac{x}{p^3}, \frac{y}{p^2}, \frac{z}{p}\right)$ is a new solution, hence proceeding this way, after a finite number of steps we may find solutions such that x, z are pairwise relatively prime. For any such solution, write $y^3 = (2z^3 - x)(2z^3 + x)$. If $2z^3 - x, 2z^3 + x$ have a common prime divisor, it must divide both $4z^3$ and $2x$, or it must be 2. Now, if 4 divides x, clearly y is even and 8 divides $x^2 + y^3$, and z is even, absurd, hence x may be even but not a multiple of 4. Now, if x is odd, clearly $2z^3 - x, 2z^3 + x$ are both odd, hence relatively prime integers whose product is a cube y^3, or relatively prime integers u, v, t exist such that $2z^3 - x = u^3$, $2z^3 + x = v^3$, $z = t$, and clearly $u^3 + v^3 = 4z^3$. If x is even, then y is even and $\frac{x}{2}$ is odd, and $2\left(\frac{y}{2}\right)^3 = \left(z^3 - \frac{x}{2}\right)\left(z^3 + \frac{x}{2}\right)$. Now, $z^3 - \frac{x}{2}, z^3 + \frac{x}{2}$ are mutually prime, and their product is twice a cube, hence wlog since we may change the sign of x, z without altering the problem, relatively prime integers u, v, w exist such that $z^3 - \frac{x}{2} = w^3$, $z^3 + \frac{x}{2} = -2u^3$, $z = v$, and $2u^3 + 2v^3 = w^3$. Clearly w is even, or an integer t exist such that $w = 2t$, and $u^3 + v^3 = 4t^3$, where u, v, t are relatively prime. Conversely, if $u^3 + v^3 = 4t^3$ for some nonzero relatively prime integers u, v, t, then u, v must be both odd, or we may define the nonzero integers $x = \frac{u^3-v^3}{2}$, $y = uv$, $z = t$, and $x^2 + y^3 = \frac{u^6+v^6+2u^3v^3}{4} = \frac{(u^3+v^3)^2}{4} = 4t^6 = 4z^6$.

By the previous claim, it suffices to show that $u^3 + v^3 = 4t^3$ has no solutions in nonzero relatively prime integers. This is relatively well known and has been proved, among others, by E. S. Selmer in its work about solutions to the diophantine equation $ax^3 + by^3 + cz^3 = 0$.

O156. In a cyclic quadrilateral $ABCD$ with $AB = AD$ points M, N lie on the sides BC and CD, respectively so that $MN = BM + DN$. Lines AM and AN meet the circumcircle of $ABCD$ again at points P and Q, respectively. Prove that the orthocenter of the triangle APQ lies on the segment MN.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

First solution by Nairi Sedrakyan, Yerevan, Armenia

Choose a point K on the extension of MD beyond D such that $DK = NB$. It follows from $\angle KDA = 180 - \angle ADC = \angle ABN, DA = AB$, and $DK = NB$ that triangles KDA and NBA are congruent. Therefore $KA = AN$ and $MK = MD + DK = MD + NB = MN$. Hence triangles KMA and NMA are also congruent. Then $\angle DMA = \angle NMA$. Similarly, we can prove that $\angle MNA = \angle BNA$. Take a point H on the segment MN such that $MH = MD$. Then $NH = BN$. Since $\angle DMA = \angle HMA$ and $MD = MH$, the points D and H are summetric with respect to AP. Similarly B and H are summetric with respect to AN. Hence $\angle DAB = 2\angle MAN$. Therefore $\angle HPA = \angle DPA = \angle ABD = 90 - \frac{1}{2}\angle DAB = 90 - \angle MAN$. It means that $PH \perp AQ$. In the same way we can prove that $QH \perp AP$. Thus the altitudes of APQ meet at H.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Since $\angle ACB = \angle ACD$, AC is the internal bisector of $\angle BCD$. Let E be the point in segment MN such that $BM = EM$ and $EN = DN$, and let C' be the diametrally opposite point to C on the circumcircle of $ABCD$. Clearly, the respective perpendiculars to BC, DC through B and D pass through C' because CC' is a diameter of the circumcircle of BCD. The internal bisector of $\angle BME$ is clearly the center of all circles tangent simultaneously to BC and MN, and since $BM = EM$, the circle with center O_B on this bisector, which is tangent to BC at B, is also tangent to MN at E. Similarly, let O_D be the point on the internal bisector of $\angle DNE$ which is the center of the circle simultaneously tangent to CD at D and to MN at E. Clearly, O_B, O_D, E are collinear and on the perpendicular to MN through E.

Now, call X the point where the bisectors of $\angle BME$ and $\angle DNE$ meet. Clearly, this point is at the same distance from BC and MN, and from MN and DC, or it is on the internal bisector of angle $\angle BCD$, hence on AC. Since triangles XBM and XEM are equal ($BM = EM$ and $\angle XMB = \angle XME$ because X is on the bisector or $\angle BME$), $XB = XE$, and similarly $XD = XE$. Now, if $ABCD$ is not a kite with $BC = DC$, X is on the perpendicular bisector of BD, which does not coincide with the bisector of $\angle BCD$ and clearly passes through A, or $A = X$, whereas if $ABCD$ is a kite with $BC = DC$, $\angle ABC = \angle ADC = \frac{\pi}{2}$, and the circle with center A passing through B and D is tangent to BC at B and BD at D, hence also to MN at E. In either case, A is the circumcenter of BDE, hence on the internal bisectors of $\angle BME$ and $\angle DNE$, and AP, AQ are these respective bisectors, while triangles ABM and AEM are equal, and triangles ADN and AEN are also equal.

Let now F be the second point where AE meets the circumcircle of $ABCD$. Now, $\angle DAQ = \angle DAN = \angle NAE = \angle QAF$, and similarly $\angle BAP = \angle PAF$, or $ABPFQD$ is a hexagon with

$AB = AD$, $BP = PF$ and $FQ = QD$. Now,

$$\angle PAQ = \angle PAF + \angle FAQ = \frac{\angle BAF + \angle FAD}{2} = \frac{\angle BAD}{2} =$$

$$= \frac{\pi - \angle ADB - \angle ABD}{2} = \frac{\pi}{2} - \angle AQB,$$

since $\angle ADB = \angle ABD = \angle AQB$. Moreover,

$$\angle ABQ = \angle ABD + \angle DBQ = \frac{\pi - \angle BAD + \angle DAF}{2} = \frac{\pi - \angle BAE}{2} = \angle ABE,$$

since $\angle ABE = \angle AEB$ because $AE = AB$. We conclude that QE is the altitude from Q onto AP. Similarly, PE is the altitude from P onto AQ, or $E \in MN$ is the orthocenter of APQ. The conclusion follows.