Junior problems

J187. Let $m \geq 1$ and $f : [m, \infty) \to [1, \infty),\ f(x) = x^2 - 2mx + m^2 + 1$.

(a) Prove that f is bijective;

(b) Solve the equation $f(x) = f^{-1}(x)$;

(c) Solve the equation $x^2 - 2mx + m^2 + 1 = m + \sqrt{x-1}$.

Proposed by Bogdan Enescu, B.P. Hasdeu National College, Romania

J188. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{10a+11b+11c} + \frac{1}{11a+10b+11c} + \frac{1}{11a+11b+10c} \leq \frac{1}{32a} + \frac{1}{32b} + \frac{1}{32c}.$$

Proposed by Tigran Hakobyan, Armenia

J189. Find all primes q_1, q_2, q_3, q_4, q_5 such that $q_1^4 + q_2^4 + q_3^4 + q_4^4 + q_5^4$ is the product of two consecutive even integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J190. Points A', B', C' are chosen on sides BC, CA, AB of triangle ABC such that lines AA', BB', CC' are concurrent at M and

$$\frac{AM}{MA'} \cdot \frac{BM}{MB'} \cdot \frac{CM}{MC'} = 2011.$$

Evaluate

$$\frac{AM}{MA'} + \frac{BM}{MB'} + \frac{CM}{MC'}$$

Proposed by Bogdan Enescu, B.P. Hasdeu National College, Romania

J191. Find all positive integers n for which $(n-2)! + (n+2)!$ is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J192. Consider an acute triangle ABC. Let $X \in AB$ and $Y \in AC$ such that quadrilateral $BXYC$ is cyclic and let R_1, R_2, R_3 be the circumradii of triangles AXY, BXY, ABC, respectively. Prove that if $R_1^2 + R_2^2 = R_3^2$, then BC is the diameter of the circle $(BXYC)$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Senior problems

S187. Find all positive integers n for which the interval

$$\left(\frac{1+\sqrt{5+4\sqrt{24n-23}}}{2}, \frac{1+\sqrt{5+4\sqrt{24n+25}}}{2}\right)$$

contains at least one integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S188. Let $a \geq b \geq c$ be the side-lengths of a triangle ABC in which $b+c \geq 2a\cos\frac{\pi}{5}$. Denote by O and I the circumcenter and the incenter of this triangle, respectively. Prove that circle centered at O and having radius OI lies entirely inside triangle ABC.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S189. Let a, b, c be real numbers such that $a < 3$ and all zeros of the polynomial $p(x) = x^3+ax^2+bx+c$ are negative real numbers. Prove that $b + c \neq 4$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S190. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{2a^2+bc} + \frac{1}{2b^2+ca} + \frac{1}{2c^2+ab} \leq \frac{1}{9}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2.$$

Proposed by Arkady Alt, San Jose, California, USA

S191. Prove that for any positive integer k the sequence $(\tau(k+n^2))_{n\geq 1}$ is unbounded, where $\tau(m)$ denotes the number of divisors of m.

Proposed by Al-Yazeed Ibrahim Basyoni, Saudi Arabia

S192. Let s, R, r and r_a, r_b, r_c be the semiperimeter, circumradius, inradius, and exradii of a triangle ABC. Prove that

$$s\sqrt{\frac{2}{R}} \leq \sqrt{r_a} + \sqrt{r_b} + \sqrt{r_c} \leq \frac{s}{\sqrt{r}}.$$

Proposed by Arkady Alt, San Jose, California, USA

Undergraduate problems

U187. Let p be a a prime such that $p \equiv 3 (\bmod 8)$ or $p \equiv 5 (\bmod 8)$, and $p = 2q + 1$ with qa prime. Evaluate $\omega^2 + \omega^4 + \cdots + \omega^{2^{p-1}}$, where $\omega \neq 1$ is a root of order p of unity.

Proposed by Dorin Andrica and Mihai Piticari, Romania

U188. Let G be a finite group in which for every positive integer m the number of solutions in G of the equation $x^m = e$ is at most m. Prove that G is cyclic.

Proposed by Roberto Bosch Cabrera, Florida, USA

U189. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be distinct complex numbers such that $a_k + b_l \neq 0$ for all $k, l = 1, 2, \ldots, n$. Solve the system of equations

$$\frac{x_1}{a_k + b_1} + \frac{x_2}{a_k + b_2} + \cdots + \frac{x_n}{a_k + b_n} = \frac{1}{a_k},\ k = 1, 2, \ldots, n.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

U190. Evaluate

$$\lim_{n \to \infty} \left(n \frac{\sqrt[n+1]{(n+1)!)}}{\sqrt[n]{n!}} - (n-1) \frac{\sqrt[n]{n!)}}{\sqrt[n-1]{(n-1)!}} \right).$$

Proposed by Arkady Alt, San Jose, California, USA

U191. For a positive integer n define $a_n = \displaystyle\prod_{k=1}^{n} \left(1 + \frac{1}{2^k}\right)$. Prove that

$$2 - \frac{1}{2^n} \leq a_n < 3 - \frac{1}{2^{n-1}}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

U192. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with finite lateral limits at any point in $\mathbb{R}$. Prove that

(a) f is integrable on any interval $[a, b]$;

(b) If $F(x) = \int_0^x f(t)dt$ is differentiable at any point in $\mathbb{R}$, then f has finite limit at any point in $\mathbb{R}$.

Proposed by Sorin Radulescu and Mihai Piticari, Romania

Olympiad problems

O187. Points A, B, C, D are situated in this order on a line. Through A, B and C, D construct parallel lines a, b and c, d such that their points of intersection are vertices of a square and find the side-length of this square in terms of u, v, w, where $u = AB$, $v = BC$, $w = CD$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

O188. Let $a_1, a_2, \ldots, a_n$ be nonzero real numbers, not necessarily distinct. Find the maximum number of subsets A of $\{1, 2, \ldots, n\}$ such that $\sum_{i \in A} a_i = 0$ if (a) $n = 2010$; (b) $n = 2011$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

O189. Find the locus of the orthocenter of triangle ABC, where A, B, C are distinct points on a given sphere.

Proposed by Jesik Min, Korean Minjok Leadership Academy

O190. Let ABC be a triangle with side-lengths a, b, c and medians m_a, m_b, m_c. Prove that

$$m_a + m_b + m_c \leq \frac{1}{2}\sqrt{7(a^2 + b^2 + c^2) + 2(ab + bc + ca)}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

O191. Let I be the incenter of a triangle ABC and let IA_1, IB_1, IC_1 be symmedians of triangles BIC, CIA, AIB, respectively. Prove that AA_1, BB_1, CC_1 are concurrent at a point that lies on line $G\Gamma$, where G is centroid and Γ is Gergonne point of triangle ABC.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O192. Let p be a prime such that $p \equiv 2 (\text{mod } 3)$. Prove that there are no integers a, b, c of the same parity such that

$$\left(\frac{a}{2} + \frac{b}{2} i\sqrt{3}\right)^p = c + i\sqrt{3}.$$

Proposed by Dorin Andrica and Mihai Piticari, Romania