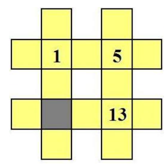
Junior problems

J91. The squares in the figure below are labeled 1 through 16 such that the sum of the numbers in each row and each column is the same. The positions of $1, 5$, and 13 are given.



Prove that there is only one possibility for the number in the darkened square and find this number.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J92. Find all primes $q_1, q_2, \ldots, q_6$ such that $q_1^2 = q_2^2 + \cdots + q_6^2$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J93. Let a and b be positive real numbers. Prove that

$$\frac{a^6+b^6}{a^4+b^4} \geq \frac{a^4+b^4}{a^3+b^3} \cdot \frac{a^2+b^2}{a+b}.$$

Proposed by Arkady Alt, San Jose, California, USA

J94. Prove that the equation $x^3+y^3+z^3+w^3 = 2008$ has infinitely many solutions in integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J95. Let ABC be a triangle and let I_a, I_b, I_c be its excenters. Denote by O_a, O_b, O_c the circumcenters of triangles I_aBC, I_bAC, I_cAB. Prove that the area of triangle $I_aI_bI_c$ is twice the area of hexagon $O_aCO_bAO_cB$.

Proposed by Mehmet Sahin, Ankara, Turkey

J96. Let n be an integer. Find all integers m such that $a^m + b^m \geq a^n + b^n$ for all positive real numbers a and b with $a+b=2$.

Proposed by Oleg Mushkarov, Bulgarian Academy of Sciences, Sofia, Bulgaria

Senior problems

S91. Find all triples (n, k, p), where n and k are positive integers and p is a prime, satisfying the equation
$$n^5 + n^4 + 1 = p^k.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S92. Let ABC be a triangle with altitudes BE and CF and let M be a point on its circumcircle. Denote by P the intersection of MB and CF and by Q the intersection of MC and BE. Prove that EF bisects the segment PQ.

Proposed by Son Ta Hong, Ha Noi University, Vietnam

S93. Let n be an integer greater than 1 and let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers whose sum is $\sqrt{2}$. Determine the maximum, as a function of n, of
$$\frac{x_1^2}{1+x_1^2} + \frac{x_2^2}{1+x_2^2} + \cdots + \frac{x_n^2}{1+x_n^2}.$$

Proposed by Alex Anderson, Washington University in St. Louis, USA

S94. Consider a quadrilateral that is incribed in a circle and circumscribed about a circle. Prove that the product of its diagonals is a constant.

Proposed by Ivan Borsenco, MIT, USA

S95. There are 16 bad boys in a neghborhood. During one year 69 quarrels have occured among the members of the neighborhood (each quarrel involves exactly two persons). At the end of the year a local wrestling club wants to organize a match between two teams of three people. The boys will fight each other if and only if each member of one team has quarreled with each member of the other team. Prove that the club can always organize such a fight.

Proposed by Iurie Boreico, Harvard University and Ivan Borsenco, MIT, USA

S96. Let n be an integer greater than 2. Prove that $\binom{n-1}{k} \equiv (-1)^k \pmod{n}$ for each $k = 1, 2, \ldots, n-1$, if and only if n is a prime.

Proposed by Dorin Andrica, "Babes-Bolyai" University, and
Mihai Piticari, "Dragos Voda" National College, Romania

Undergraduate problems

U91. Prove that there are no polynomials $P, Q \in \mathbb{R}[x]$ such that

$$\int_0^{\log n} \frac{P(x)}{Q(x)} dx = \frac{n}{\pi(n)},$$

for all $n \geq 1$, where $\pi(n)$ is the prime counting function.

Proposed by Cezar Lupu, University of Bucharest, Romania

U92. Find the maximum value of $F(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \min \left\{ \frac{\|\mathbf{y}-\mathbf{z}\|}{\|\mathbf{x}\|}, \frac{\|\mathbf{z}-\mathbf{x}\|}{\|\mathbf{y}\|}, \frac{\|\mathbf{x}-\mathbf{y}\|}{\|\mathbf{z}\|} \right\}$

where $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are arbitrary nonzero vectors in $\mathbb{R}^n, n \geq 2$.

Proposed by Arkady Alt, San Jose, California, USA

U93. Let $x_0 \in (0, 1]$ and $x_{n+1} = x_n - \arcsin(\sin^3 x_n)$, $n \geq 0$. Evaluate $\lim_{n\to\infty} \sqrt{n} x_n$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U94. Let Δ be the plane domain consisting of all interior and boundary points of a rectangle $ABCD$, whose sides have lengths a and b. Define $f : \Delta \to R$, $f(P) = PA + PB + PC + PD$. Find the range of f.

Proposed by Mircea Becheanu, University of Bucharest, Romania

U95. Find all monic polynomials P and Q, with real coefficients, such that

$$P(1) + P(2) + \cdots + P(n) = Q(1 + 2 + 3 + \cdots + n),$$

for all $n \geq 1$.

Proposed by Ovidiu Furdui, Cluj-Napoca, Romania

U96. Let $f : (0, \infty) \to [0, \infty)$ be a bounded function. Prove that if

$$\lim_{x\to 0} \left(f(x) - \frac{1}{2}\sqrt{f(\frac{x}{2})} \right) = 0 \quad \text{and} \quad \lim_{x\to 0} \left(f(x) - 2f(2x)^2 \right) = 0,$$

then $\lim_{x\to 0} f(x) = 0$.

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania and Mihai Piticari, "Dragos Voda" National College, Romania

Olympiad problems

O91. Let ABC be an acute triangle. Prove that

$$\tan A + \tan B + \tan C \geq \frac{s}{r},$$

where s and r are semiperimeter and inradius of triangle ABC, respectively.

Proposed by Mircea Becheanu, University of Bucharest, Romania

O92. Let n be a positive integer. Prove that

a) there are infinitely many triples (a, b, c) of distinct integers such that $\min(a, b, c) \geq n$ and $abc + 1$ divides one of the numbers $(a-b)^2, (b-c)^2, (c-a)^2$.

b) there is no triple (a, b, c) of distinct positive integers such that $abc+1$ divides more than one of the numbers $(a-b)^2, (b-c)^2, (c-a)^2$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O93. Let k be a positive integer. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) + f(y)$ divides $x^k + y^k$ for all $x, y \in \mathbb{N}$.

Proposed by Nguyen Tho Tung, Hanoi University of Education, Vietnam

O94. Let ω be a circle with center O and let A be a fixed point outside ω. Choose points B and C on ω, with $AB \neq AC$, such that AO is a symmedian, but not a median, in triangle ABC. Prove that the circumcircle of triangle ABC passes through a second fixed point.

Proposed by Alex Anderson, Washington University in St. Louis, USA

O95. Prove that there is a sequence $x_1, x_2, \ldots$ of integers such that

- For each $n \in \mathbb{Z}$ there exists i such that $x_i = n$.
- $\prod_{d|n} d^{\frac{n}{d}} = \sum_{i=1}^{n} x_i$.

Proposed by Juan Ignacio Restrepo, Universidad de Los Andes, Colombia

O96. Let p and q be primes, $q \geq p$. Prove that pq divides $\binom{p+q}{p} - \binom{q}{p} - 1$.

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania