Junior problems

J139. Let $a_0 = a_1 = 1$ and

$$a_{n+1} = \frac{a_n^2}{a_n + a_{n-1}}$$

for $n \geq 1$. Find a_n in closed form.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J140. Let n be a positive integer. Find all real numbers x such that

$$\lfloor x \rfloor + \lfloor 2x \rfloor + \ldots + \lfloor nx \rfloor = \frac{n(n+1)}{2}.$$

Proposed by Mihai Piticari, "Dragos-Voda" National College, Romania

J141. Let a, b, c be the side lengths of a triangle. Prove that

$$0 \leq \frac{a-b}{b+c} + \frac{b-c}{c+a} + \frac{c-a}{a+b} < 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, "Babes-Bolyai" University, Romania

J142. For each positive integer m, define the binomial coefficient $\binom{x}{m} = \frac{x(x-1)\cdots(x-m+1)}{m!}$. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_1 + x_2 + \cdots + x_n \geq n^2$. Prove that

$$\frac{n-1}{2}\left(\sum_{i=1}^{n}\binom{x_i}{3}\right)\left(\sum_{i=1}^{n} x_i\right) \geq \frac{n-2}{3}\left(\sum_{i=1}^{n}\binom{x_i}{2}\right)^2.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J143. Let $x_1 = -2$, $x_2 = -1$ and

$$x_{n+1} = \sqrt[3]{n(x_n^2 + 1) + 2x_{n-1}}$$

for $n \geq 2$. Find x_{2009}.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J144. Let ABC be a triangle with $a > b > c$. Denote by O and H its circumcenter and orthocenter, respectively. Prove that

$$\sin\angle AHO + \sin\angle BHO + \sin\angle CHO \leq \frac{(a-c)(a+c)^3}{4abc \cdot OH}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Senior problems

S139. Let $a_0 = 1$ and $a_{n+1} = a_0 \cdots a_n + 3$ for $n \geq 0$. Prove that

$$a_n + \sqrt[3]{1 - a_n a_{n+1}} = 1,$$

for all $n \geq 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S140. Let a, b, c be integers. Prove that

$$\sum_{cyc} (a-b)(a^2+b^2-c^2)c^2$$

is divisible by $(a+b+c)^2$.

Proposed by Dorin Andrica, "Babeş-Bolyai" University, Cluj-Napoca, Romania

S141. Four squares are laying inside a circle of radius $\sqrt{5}$ such that no two have a common point. Prove that one can place these squares inside a square of side 4, such that no two have a common point.

Proposed by Nairi Sedrakyan, Armenia

S142. Consider two concentric circles $C_1(O, R)$ and $C_2(O, \frac{R}{2})$. Prove that for each point A on the circumference of circle C_1 and for each point Ω inside the circle C_2 there are points B and C on the circumference of C_1 such that Ω is the center of the nine-point circle of triangle ABC.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S143. Let m and n be positive integers, $m < n$. Evaluate

$$\sum_{k=m+1}^{n} k(k^2 - 1^2) \cdots (k^2 - m^2).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S144. Let $ABCD$ be a quadrilateral. We consider the reflection of the lines AB, BC, CD, DA on the respective midpoints of the opposite sides CD, DA, AB, BC. Prove that these four lines bound a quadrilateral $A'B'C'D'$ homothetic with $ABCD$ and find the ratio and center of the homothety.

Proposed by Francisco Javier García Capitán and Juan Bosco Romero Márquez

Undergraduate problems

U139. Find the least interval containing all values of the expression

$$E(x,y,z) = \frac{x}{x+2y} + \frac{y}{y+2z} + \frac{z}{z+2x}.$$

Proposed by Dorin Andrica, "Babeş-Bolyai" University, Cluj-Napoca, Romania

U140. Let $(a_n)_{n\geq 1}$ a decreasing sequence of positive real numbers. Let

$$s_n = a_1 + a_2 + \ldots + a_n,$$

and

$$b_n = \frac{1}{a_{n+1}} - \frac{1}{a_n},$$

for all $n \geq 1$. Prove that if the sequence $(s_n)_{n\geq 1}$ is convergent, then the sequence $(b_n)_{n\geq 1}$ is unbounded.

Proposed by Bogdan Enescu, "B. P. Hasdeu" National College, Buzau, Romania

U141. Find all pairs (x,y) of positive integers such that $13^x + 3 = y^2$.

Proposed by Andrea Munaro, Universit degli Studi di Trento, Italy

U142. Let $f : [0,1] \to \mathbb{R}$ be a continuously differentiable function. Prove that if $\int_0^{\frac{1}{2}} f(x)dx = 0$ then

$$\int_0^1 (f'(x))^2 dx \geq 12 \left(\int_0^1 f(x)dx \right)^2.$$

Proposed by Duong Viet Thong Faculty of Foundation, Nam Dinh University of Technology Education, Phu Nghia Road, Loc Ha Ward, Nam Dinh City, Vietnam

U143. For a positive integer $n > 1$, determine

$$\lim_{x \to 0} \frac{\sin^2(x)\sin^2(nx)}{n^2\sin^2(x) - \sin^2(nx)}.$$

Proposed by N. Javier Buitrago A., Universidad Nacional de Colombia

U144. Let F be the set of all continuous functions $f : [0, \infty) \to [0, \infty)$ satisfying the relation

$$f\left(\int_0^x f(t)dt\right) = \int_0^x f(t)dt$$

for all $x \in [0, \infty)$.

a) Prove that F has infinitely many elements.

b) Find all convex functions f in the set F.

Proposed by Mihai Piticari, "Dragos-Voda" National College, Romania

Olympiad problems

O139. Through point M on the circle circumscribed to the acute triangle ABC, draw parallels to the sides BC, CA, AB which interesect the circle the second time at points A', B', C' ($A' \in \overset{\frown}{BC}, B' \in \overset{\frown}{CA}, C' = \in \overset{\frown}{AB}$). If $\{D\} = A'B' \cap BC, \{E\} = A'B' \cap CA, \{F\} = B'C' \cap CA, \{D'\} = B'C' \cap AB, \{E'\} = A'C' \cap AB, \{F'\} = A'C' \cap BC$, prove that the lines DD', EE', FF' are concurrent.

Proposed by Cătălin Barbu, Colegiul "Vasile Alecsandri", Bacau Romania

O140. Let n be a positive integer and let $x_k \in [-1, 1]$, $1 \leq k \leq 2n$, such that $\sum_{k=1}^{2n} x_k$ is an odd integer. Prove that

$$1 \leq \sum_{k=1}^{2n} |x_k| \leq 2n - 1.$$

Proposed by Bogdan Enescu, "B. P. Hasdeu" National College, Buzau, Romania

O141. Let S_n be the set of all $3n$-digit numbers consisting of n 1^s,n 2^s , and n 5^s. Prove that for each n there are at least 4^{n-1} numbers in S_n that can be written as sum of the cubes of some $n+1$ distinct positive integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O142. If m is a positive integer show that $5^m + 3$ has neither a prime divisor of the form $p = 30k + 11$ nor of the form $p = 30k - 1$.

Proposed by Andrea Munaro, Universita degli Studi di Trento, Italy

O143. Let $ABCDEF$ be a convex hexagon such that the sum of the distances from each of its interior point to the six sides is equal to the sum of the distances between the midpoints of AB and DE, BC and EF, and CD and FA. Prove that $ABCDEF$ is cyclic.

Proposed by Nairi Sedrakyan, Armenia

O144. Find all positive integers a, b, c such that $(2^a - 1)(3^b - 1) = c!$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France