Solutions for Admission Test B

Problem 1. The top of a ladder resting against a vertical wall is 24 feet above the ground. By moving the base of the ladder 8 feet farther out, the ladder slides down resting against the wall at a point 20 feet above the ground. How long is the ladder?

Solution: Let l be the length of the ladder and let x be the initial distance from the base to the wall. Then

$$l^2 = x^2 + 24^2 \quad \text{and} \quad l^2 = (x+8)^2 + 20^2.$$

It follows that

$$x^2 + 576 = x^2 + 16x + 64 + 400,$$

yielding $x = \frac{112}{16} = 7$. Then $l = \sqrt{7^2 + 24^2} = \underline{\mathbf{25}}$.

Problem 2. What is the greatest positive integer n for which $n!$ ends in exactly 33 zeros?

Solution: Let us explain why 135! ends in exactly 33 zeros. Clearly, the number of zeros at the end of $n!$ is equal to the greatest power of 5 that divides $n!$. Counting the divisors of the form 5^k we get

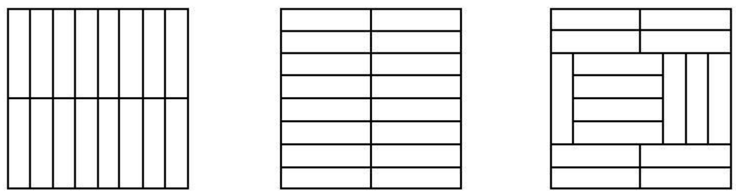
$$\lfloor \frac{135}{5} \rfloor + \lfloor \frac{135}{25} \rfloor + \lfloor \frac{135}{125} \rfloor = 27 + 5 + 1 = 33.$$

We also know that 140! has more than 33 zeros at the end. Therefore the greatest positive integer to end with 33 zeros in its factorial representation is $\underline{\mathbf{139}}$.

Problem 3. Find the number of different scalene triangles all whose sides are of integral length and the longest of which is 13.

Solution: Let the sides be $a < b < c = 13$. Then, by the triangle inequality, $8 \leq b \leq 12$ and $c - b < a < b$. As b decreases by 1, the range of a decreases by 2. For $b = 12$ we have $2 \leq a \leq 11$, hence the number of triangles is $10 + 8 + 6 + 4 + 2 = \underline{\mathbf{30}}$.

Problem 4. We want to tile an 8×8 chess board with 16 4×1 rectangular pieces. In how many ways can this be done? Here are three ways:



Solution: Answer: 100.

The chess board has eight rows and eight columns. Any row can be covered by two horizontal tiles, one horizontal tile, and four vertical tiles, or by eight vertical tiles. It follows that the number of vertical tiles in a tiling must be either $0, 4, 8, 12, 16$. There is exactly one tiling which uses 0 vertical tiles.

If there are four vertical tiles, then all four of the vertical tiles must be in the span the same rows of the chess board. In fact, the four horizontal tiles in those rows must appear together as a square block. That square block can appear in any one of five sets of columns. There are five sets of rows which can be contain these vertical tiles, so there is a total of $5 \times 5 = 25$ ways to tile using four vertical lines.

If there are eight tiles, one of two situations exist. Either four of the vertical tiles appear as a square block in the left four columns of the chessboard, while the other four vertical tiles appear as a square block in the right four columns of the chess board, or four of the vertical tiles appear as a square block in the upper four rows of the chess board, while the other four vertical tiles appear as a square block in the bottom four rows of the chess board. There are five ways to place a block of four vertical tiles on the left side of the chess board and five ways to place a block of four vertical tiles on the right side of the chess board giving $5 \times 5 = 25$ ways to locate the eight vertical tiles. Similarly, there are 25 ways of placing four vertical tiles on the upper half of the chess board. This would give a total of $25 + 25 = 50$ ways of tiling except that two of the tiling patterns have been counted twice each. They are the pattern where there are four vertical tiles in the upper left and four in the lower right and the pattern where there are four vertical tiles in the upper right and four in the lower left. Hence there are only 48 ways of tiling the chess board with eight vertical tiles.

The number of ways of tiling the chess board with twelve vertical tiles is the same as the number of ways of tiling the board with four vertical tiles. This can be seen by taking each of the tilings with four vertical tiles and rotating the chess board 90 degrees. Similarly, there are the same number of tilings using sixteen vertical tiles as there are using no vertical tiles.

Thus the total number of tilings is $1 + 25 + 48 + 25 + 1 = \underline{\mathbf{100}}$.

Problem 5. Find the greatest perimeter of a Pythagorean triangle with one of its sides of length 2007.

Solution: Because $2007 \equiv 3 (\bmod 4)$, 2007 cannot be the sum of two perfect squares, so it must be a leg of the right triangle. Recall the fact that all Pythagoreans triples are of the form

$$(m^2 - n^2,\ 2mn,\ m^2 + n^2),$$

where m, n are positive integers. It follows that

$$2007 = m^2 - n^2 = (m - n)(m + n).$$

The perimeter of the triangle is equal to

$$(m^2 - n^2) + 2mn + (m^2 + n^2) = (m^2 - n^2) + (m + n)^2 = 2007 + (m + n)^2,$$

so we want to maximize $m + n$. Observe that the greatest possible value for $m + n$ is 2007. That gives us $m - n = 1$, therefore $m = 2006$, $n = 1$. The greatest perimeter is equal to $2007^2 + 2007 = 2007 \cdot 2008 = \underline{\mathbf{4030056}}$.

Problem 6. Let a, b, and c be distinct prime numbers such that $a+b+c$, $-a + b + c$, $a - b + c$, and $a + b - c$ are also primes. If $b + c = 200$, find a.

Solution: Note that $b + c \equiv 2 (\bmod 3)$. If $b \equiv 0 (\bmod 3)$, then $b = 3$ and the distinct primes $-a + b + c$ and $a + b - c$ would add up to 6, which is impossible.

If $b \equiv 2 (\bmod 3)$, then $c \equiv 0 (\bmod 3)$, implying $c = 3$. From the fact that the distinct primes $-a+b+c$ and $a-b+c$ add up to 6 we obtain a contradiction. Hence $b \equiv 1 (\bmod 3)$ and $c \equiv 1 (\bmod 3)$.

It is not difficult to see that the primes $a+b+c, -a+b+c, a-b+c, a+b-c$ are also distinct from a, b, and c. As we have seen above, $a \not\equiv 0 (\bmod 3)$. If $a \equiv 1 (\bmod 3)$, then $a+b+c \equiv 1+1+1 \equiv 0 (\bmod 3)$, implying that the sum of the primes a, b, and c is 3, impossible. Hence $a \equiv 2 (\bmod 3)$ and $-a+b+c \equiv 0$ $(\bmod 3)$. Thus $-a + b + c = 3$, yielding $a = 200 - 3 = \underline{\mathbf{197}}$. For the choice $b = 7$, $c = 193$, the seven prime numbers are $3, 7, 11, 193, 197, 383, 397$, while for $b = 43$, $c = 157$, the seven primes are $3, 43, 83, 157, 197, 311, 397$.

Problem 7. Each face of a cube is to be painted with one of six colors such that every two adjacent faces have different colors. Two colorings are distinguishable if one cannot be obtained from the other by rotations of the cube. In how many distinguishable ways can the faces be painted?

Solution: Answer: 230.

If only three colors are used, each must be used to paint a pair of opposite faces. There are $\binom{6}{3} = 20$ ways of choosing the colors, and it does not matter how they are applied. If only four colors are used, two must be used to paint two pairs of opposite faces. They can be chosen, in $\binom{6}{2} = 15$ ways. Two more colors are then chosen, in $\binom{4}{2} = 6$ ways.

Because it does not matter how the colors are applied, the number of ways in this case is $15 \times 6 = 90$. If five colors are used, one must be used to paint a pair of opposite faces. It can be chosen in six ways. The other four colors can be chosen in $\binom{5}{4} = 5$ ways, and divided into two pairs in three ways, each pair being used to paint a pair of opposite. The number of ways in this case is $6 \times 5 \times 3 = 90$ also.

Finally, suppose all six colors are used. They can be divided into three pairs in $5 \times 3 = 15$ ways, each pair being used a pair of opposite faces. It is only when we come to the last pair that we have to make a distinction which color is used on which face. Hence the number of ways in this case is $15 \times 2 = 30$ ways. The total number of ways is

$$20 + 90 + 90 + 30 = \underline{\mathbf{230}}.$$

Problem 8. For each positive integer n let

$$a_n = \frac{n^3}{n^2 - 15n + 75}.$$

Prove that $a_1 + a_2 + ... + a_{15}$ is a whole number and find its value without direct calculation.

Solution: Note that the denominator $n(15 - n) + 75$ does not change if we replace n by $15 - n$. Define $a_0 = 0$. Then for $n = 0, 1, ..., 7$ we have

$$a_n + a_{15-n} = \frac{n^3 + (15 - n)^3}{n^2 - 15n + 75} = \frac{15^3 - 3 \cdot 15^2 n + 3 \cdot 15 n^2}{n^2 - 15n + 75} =$$

$$= \frac{45(n^2 - 15n + 75)}{n^2 - 15n + 75} = 45.$$

Summing up these eight inequalities yields $a_1 + a_2 + ... + a_{15} = 8 \cdot 45 = \underline{\mathbf{360}}$.

Problem 9. Find all pairs (x, y) of integers such that

$$xy + \frac{x^3 + y^3}{3} = 2007.$$

Solution: We write the equation as

$$xy + \frac{(x+y)^3}{3} - xy(x+y) = 2007.$$

Note that $3|x + y$, and since $3|\frac{(x+y)^3}{3}$, $3|xy(x + y)$ and $3|2007$, it follows that $3|xy$. Because 3 also divides $x + y$, we deduce that $3|x$ and $3|y$.

Let $x = 3u$ and $y = 3v$ for some integers u and v and let $s = u + v$ and $p = uv$. Then $p(3s - 1) = s^3 - 223$, hence $3s - 1|s^3 - 223$, and so $3s - 1|27(s^3 - 223)$. But $3s - 1$ divides $(3s - 1)(9s^2 + 3s + 1) = 27s^3 - 1$, thus $3s - 1$ divides $27 \cdot 223 - 1 = 6020 = 2^2 \cdot 5 \cdot 7 \cdot 43$.

On the other hand, $s^2 \geq 4p$ (which follows from $(u - v)^2 \geq 0$), so

$$s^2 \geq \frac{4(s^3 - 223)}{3s - 1}$$

It follows that $\dfrac{s^3 + s^2 - 892}{3s - 1} \leq 0$.

If $s = 0$, then $p = 223$, yielding nonintegral s and p.

If $s \leq -1$, then the denominator $3s - 1$ is negative and $s^2(s + 1) - 892$ is also negative, a contradiction.

Thus $s \geq 1$, implying $3s - 1 > 0$ and $s^2(s + 1) \leq 892$. It follows that $1 \leq s \leq 9$, so $2 \leq 3s - 1 \leq 26$. The only divisors congruent to $-1 \pmod 3$ of $2^2 \cdot 5 \cdot 7 \cdot 43$ in the interval $[2, 26]$ are $2, 5, 14, 20$, yielding $3s - 1 \in \{2, 5, 14, 20\}$. Hence

1) $s = 1$, $p = -111$

2) $s = 2$, $p = -43$

3) $s = 5$, $p = -7$

4) $s = 7$, $p = 6$

The first three systems do not have solutions in integers. The fourth system yields $\{u, v\} = \{1, 6\}$ so $(x, y) =$ **(3,18)** and $(x, y) =$ **(18,3)** are the only solutions. They satisfy the given equation.

Problem 10. The www.awesomemath.org homepage displays the equilateral triangles "Summer Program", "Year-round", and "Math Reflections". Prove that the centers of these triangles are the vertices of another equilateral triangle.

Solution: Consider triangle ABC with equilateral triangle BCA_1, CAB_1, and ABC_1 constructed in its exterior. Denote by X, Y, Z the centers of triangles BCA_1, CAB_1, ABC_1, respectively. Observe that

$$\angle YAZ = \angle YAC + \angle BAC + \angle ZAB = 60 + A.$$

Using the Law of Cosines in triangle YAZ we get

$$YZ^2 = AY^2 + AZ^2 - 2AZ \cdot AY \cos(60 + A).$$

Clearly $AY = \frac{b}{\sqrt{3}}$, $AZ = \frac{c}{\sqrt{3}}$. Therefore

$$3YZ^2 = b^2 + c^2 - 2bc \cos(60 + A).$$

From the fact $\cos(60 + A) = \frac{1}{2} \cos A - \frac{\sqrt{3}}{2} \sin A$ we obtain

$$3YZ^2 = b^2 + c^2 - bc \cos A + \sqrt{3} \cdot bc \sin A.$$

Then using the Law of Cosines in triangle ABC, $2bc \cos A = b^2 + c^2 - a^2$, we have

$$3YZ^2 = \frac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3} \cdot \text{Area}_{ABC}.$$

It follows that XY, YZ, XY are symmetric representation in a, b, c and triangle XYZ is equilateral.