S^1 is a Symmetric Space

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1 Introduction

A symmetric space M, is a manifold* with an involutive† isometry‡ s_p at each point, $p \in M$. The class of Symmetric Spaces includes many of the important and well known spaces—spheres, real and complex projective spaces, the Grassmann Manifolds, and the 'exotic' Lie Groups.

The study of Symmetric Spaces was initiated by Elie Cartan [Ca] and employs many of the sophisticated tools of modern mathematics—differential geometry, group theory, topology, quotient spaces, Lie Groups and Lie Algebras, see [Be], [Ch-Ch], [C -N].

This note presents some of the basic properties of Symmetric Spaces and the tools of differential geometry with the simplest compact example, the unit circle in the plane, S^1. The concepts of symmetries, stereographic projection, isometries in the plane, and Lie Groups are introduced.

The unit circle in the plane, S^1, is used to illustrate these properties of symmetric spaces:

- S^1 is a differentiable manifold.
- At every point p of S^1 there is an involutive isometry, s_p, a reflection of the circle that fixes the point p.
- A reflection at a point p acts as $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -Identity$ on the tangent space to S^1 at p.
- The set of reflections of the points of S^1 generates the Lie Group $SO(2)$, the group of counterclockwise rotations of S^1.
- $S^1 = SO(2)$, i.e., the circle is a Lie Group.

For this note the required knowledge minimal—high school algebra, basic geometry, trigonometry, calculus, and some group theory. The best source of information on Symmetric Spaces is the book by Helgason, [H].

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* An n-dimensional differentiable manifold is a topological space that is 'locally' differentiable to $\mathbb{R}^n$.

† A map σ is involutive if $\sigma^2 = Identity$, e.g., $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an involutive map on the plane, $\mathbb{R}^2$.

‡ An isometry is a distance-preserving map, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is also an isometry on $\mathbb{R}^2$ with the usual distance.

This paper is dedicated to Tadashi Nagano.

2 The Unit Circle is a Differentiable Manifold

S^1 is the circle of radius one, centered at the origin: $S^1 = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\}$.

A space is an n dimensional manifold ([H]) if it is 'locally' isomorphic to a portion of $\mathbb{R}^n$, the n dimensional plane.

To show that S^1 is a manifold, decompose S^1 into two open sets U_1 and U_2 so that $S^1 = U_1 \cup U_2$ and find maps,

i. $\varphi_1 : U_1 \to \mathbb{R}^1$

ii. $\varphi_2 : U_2 \to \mathbb{R}^1$

Then the mapping $\varphi_2 \circ \varphi_1^{-1}$ is a differentiable mapping of $\varphi_1(U_1 \bigcap U_2) \subset \mathbb{R}^1$ onto $\varphi_2(U_1 \bigcap U_2) \subset \mathbb{R}^1$, i.e., $\varphi_2 \circ \varphi_1^{-1}$ is a differentiable mapping of a subset of $\mathbb{R}^1$ onto another subset of $\mathbb{R}^1$. Similarly, the mapping $\varphi_1 \circ \varphi_2^{-1}$ should a differentiable mapping of $\varphi_2(U_1 \bigcap U_2)$ onto $\varphi_1(U_1 \bigcap U_2)$.

2.1 Stereographic Projection

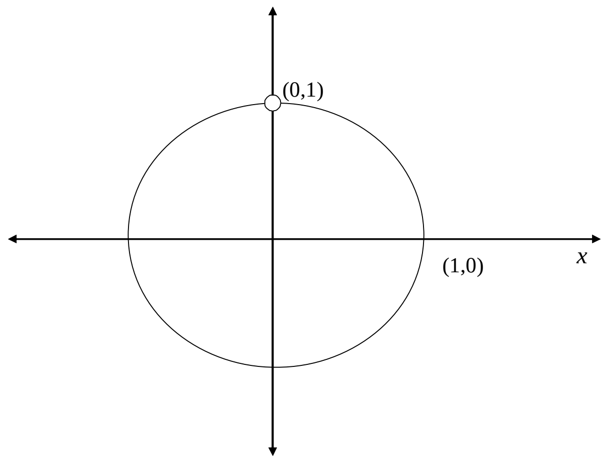
There are several ways to show that S^1 is a manifold, these notes use the method of Stereographic Projection.

In Stereographic Projection, the open sets are:

U1= $\{(x,y): x^2+y^2=1\} - \{(0,1)\}$, the circle minus the North Pole, and

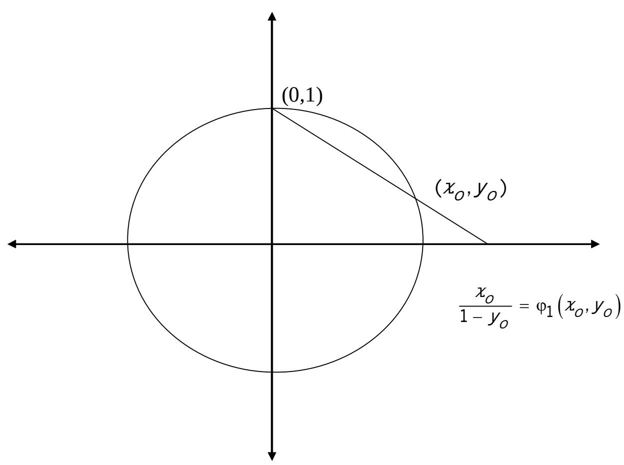
U2= $\{(x,y): x^2+y^2=1\} - \{(0,-1)\}$, the circle minus the South Pole.

Then $S^1 = U_1 \cup U_2$



$U_1 = S^1 - \{(0,1)\}$

The maps φ_1 and φ_2 are the projections from the North and South Poles, respectively, onto the real line. To obtain φ_1, draw the line from the North Pole through (x_o, y_o) and the image of (x_o, y_o) under the map φ_1 is the point where the line crosses the *x*-axis.



Stereographic Projection from the North Pole

<u>Lemma</u> Let $(x_o, y_o) \in$ U$_1$, then $\varphi_1\,(x_o, y_o) = \dfrac{x_o}{1 - y_o} = \dfrac{1 + \sqrt{1 - x_o^2}}{x_o}$.

Proof (**exercise**): find the equation of the line through (0,1) and (x_o, y_o) on the circle. What is the *x*-intercept?

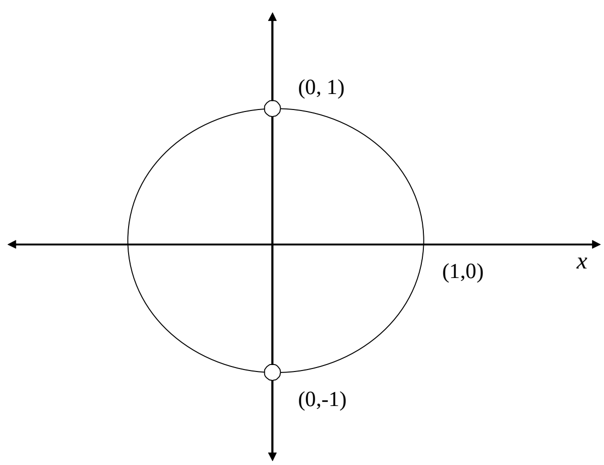
Exercise: Show that if $(x_o, y_o) \in$ U$_2$, then the projection from the South pole gives $\varphi_2\,(x_o, y_o) = \dfrac{1 - \sqrt{1 - x_o^2}}{x_o}$.

Exercises: What are the sets:

i. $\varphi_1(U_1)$?

ii. $\varphi_2(U_1 \bigcap U_2)$?

Note that $U_1 \cap U_2 = S^1 - (0,1) - (0,-1)$, the two halves of the circle, minus the poles.



$U_1 \cap U_2 = S^1 - \{(0,-1)\} - \{(0,1)\}$

To show that S^1 is a manifold we find φ_1^{-1} and show that the composition $\varphi_2 \circ \varphi_1^{-1}$ is a differentiable mapping of $\varphi_1(U_1 \bigcap U_2)$ onto $\varphi_2(U_1 \bigcap U_2)$.

<u>Lemma</u>: If $a \in \mathbb{R}$, $a \neq 0$, then $\varphi_1^{-1}(a) = \left(\dfrac{2a}{a^2 + 1}, \dfrac{a^2 - 1}{a^2 + 1}\right)$

Proof: Let $a \in \mathbb{R}$, we write it as $(a,0)$. Then $\varphi_1^{-1}(a)$ is the intersection of the line through $(a,0)$ and $(0,1)$, $y = \frac{-1}{a}x + 1$, and the circle, S^1.

$$(*) \quad y = \frac{-1}{a}x + 1$$

$$(**) \quad x^2 + y^2 = 1, (x,y) \neq (0,1)$$

$$(*) \Rightarrow \quad y^2 = \left(\frac{-1}{a}x + 1\right)^2 = \frac{1}{a^2}x^2 - \frac{2}{a}x + 1$$

Substitute into (**) and solve for y.

Lemma: $\varphi_2 \circ \varphi_1^{-1}(a) = \frac{1}{a}, a \neq 0,$ is a differentiable map.

$$\varphi_2 \circ \varphi_1^{-1}(a) = \varphi_2\left(\frac{2a}{a^2+1}, \frac{a^2-1}{a^2+1}\right) = \frac{\frac{2a}{a^2+1}}{\frac{a^2-1}{a^2+1}+1} = \frac{2a}{a^2-1+a^2+1} = \frac{2a}{2a^2} = \frac{1}{a}$$, which is differentiable for $a \neq 0$.

[Note that $\varphi_2 \circ \varphi_1^{-1}$ is an inversion through the unit circle, if a is inside the circle, $0 < |a| < 1$, then $\varphi_2 \circ \varphi_1^{-1}(a) = \frac{1}{a}$ is outside the circle, and conversely, see [C], p. 77-85].

Exercise: Find $\varphi_1 \circ \varphi_2^{-1}(a)$ and show that it is differentiable.

3 S^1 has an involution at each point

An *involution* is a map whose square is the identity, $\sigma^2 = Identity$, e.g., $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an involutive map on the plane, $\mathbb{R}^2$. The map $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$ is a reflection of a point in $\mathbb{R}^2$ through the *x*-axis. What points are fixed under this involution?

Common involutions on $\mathbb{R}^2$ are

i. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, reflection through the line $y = x$.

ii. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, reflection through the *y*-axis.

iii. $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, rotation through 180°.

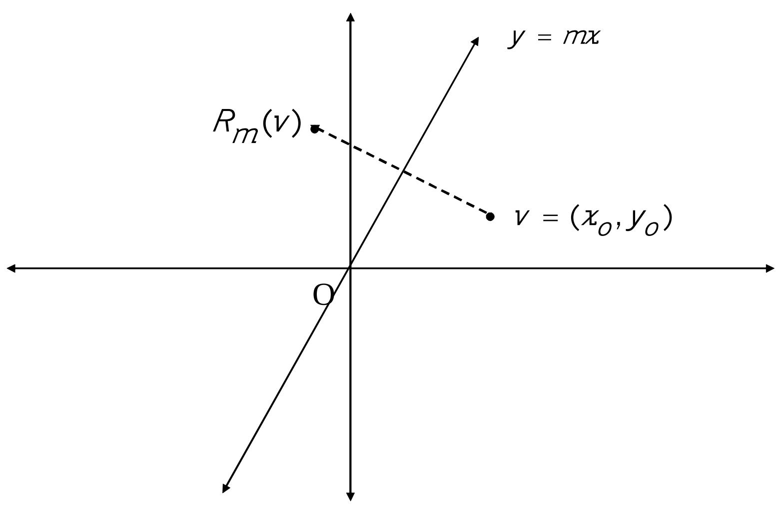
Exercise: For each of i, ii, and iii, show that the map is an involution and determine the set of fixed points.

For each point p on S^1 draw a line through the point and the origin. Then the reflection of the plane through that line is an involution that fixes p. Note that there is a second point on S^1 fixed by the reflection, the antipodal point of p on S^1.

3.1 *Reflections in the plane*

We reflect the point $v = (x_o, y_o)$ through the line $y = mx$, where m is the slope, $m \in \mathbb{R}$. We call this reflection R_m.

The image of v under R_m is the point $v' = R_m(v)$ where the line through v and v' is perpendicular to $y = mx$ and the point of intersection of these two lines is the midpoint of the segment from v to v'.



The reflection of v through the line $y=mx$.

Lemma: The reflection of the point $v = (x_o, y_o)$ through the line $y = mx$ is given by

$$R_m\begin{pmatrix} x_o \\ y_o \end{pmatrix} = \frac{1}{m^2+1}\begin{pmatrix} -m^2+1 & 2m \\ 2m & m^2-1 \end{pmatrix}\begin{pmatrix} x_o \\ y_o \end{pmatrix} = \begin{pmatrix} \left(\dfrac{-m^2+1}{m^2+1}\right)x_o + \left(\dfrac{2m}{m^2+1}\right)y_o \\ \left(\dfrac{2m}{m^2+1}\right)x_o + \left(\dfrac{m^2-1}{m^2+1}\right)y_o \end{pmatrix}.$$

Sketch of proof:

1. Find the equation of the line through $\begin{pmatrix} x_o \\ y_o \end{pmatrix}$ perpendicular to $y = mx$.

2. Find the point of intersection, (a,b), of the line through $\begin{pmatrix} x_o \\ y_o \end{pmatrix}$ perpendicular to $y = mx$ and the line $y = mx$.

3. The midpoint of $R_m\begin{pmatrix} x_o \\ y_o \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x_o \\ y_o \end{pmatrix}$ is (a,b), set up the midpoint equations and solve for x and y.

Exercises:

i. Show $R_m{}^2 = Id$ for all m, i.e., R_m is involutive.

ii. Show $\det R_m = -1$

3.2 Involutive map at each point of the circle

Let $p = (a, \sqrt{1-a^2}), -1 \le a \le 1$ be point on S^1 in the first or second quadrant.

3.2.1 $a \ne 0$

If $a \ne 0$, the equation of the line through the origin and p is $y = \frac{\sqrt{1-a^2}}{a}x$, i.e., the slope is $m = \frac{\sqrt{1-a^2}}{a}$.

So the reflection that fixes this line is

$$R_m = \frac{1}{m^2+1}\begin{pmatrix} -m^2+1 & 2m \\ 2m & m^2-1 \end{pmatrix} = \begin{pmatrix} 2a^2-1 & 2a\sqrt{1-a^2} \\ 2a\sqrt{1-a^2} & -2a^2+1 \end{pmatrix}$$

3.2.2 $a = 0$

If $a = 0$ then $P = (0,1)$, the "North Pole", and the reflection is the reflection through the y-axis, $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -x \\ y \end{pmatrix}$.

The matrix is:

$$\lim_{m\to\infty} R_m\begin{pmatrix} x_o \\ y_o \end{pmatrix} = \lim_{m\to\infty}\begin{pmatrix} \left(\frac{-m^2+1}{m^2+1}\right)x_o + \left(\frac{2m}{m^2+1}\right)y_o \\ \left(\frac{2m}{m^2+1}\right)x_o + \left(\frac{m^2-1}{m^2+1}\right)y_o \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_o \\ y_o \end{pmatrix} = \begin{pmatrix} -x_o \\ y_o \end{pmatrix}$$

Exercises.

i. R_a fixes the points $p = (a, \sqrt{1-a^2})$ and $q = (-a, -\sqrt{1-a^2}), -1 \le a \le 1$.

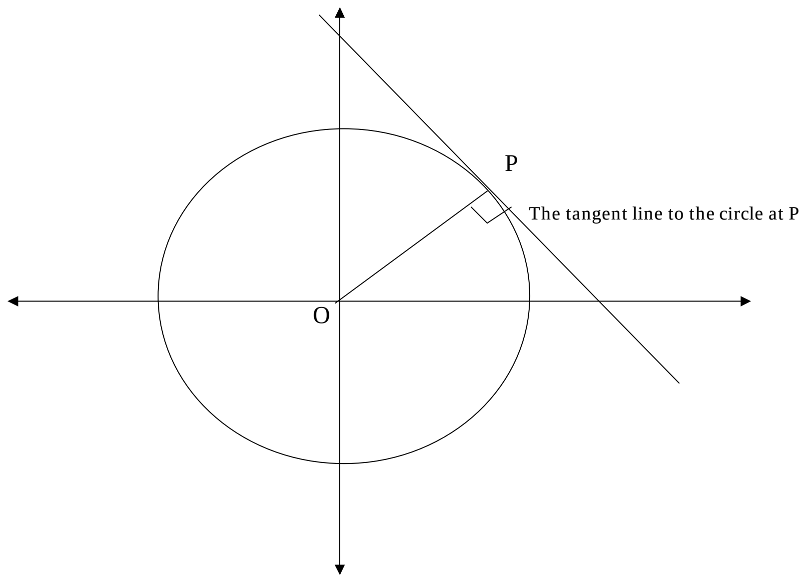
ii. $R_a = \begin{pmatrix} 2a^2 - 1 & 2a\sqrt{1-a^2} \\ 2a\sqrt{1-a^2} & -2a^2+1 \end{pmatrix}$ is involutive, i.e., $R_a R_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4 The involution at p acts as –Id on the tangent space

For a general *n*-dimensional manifold, *M*, the tangent space at a point $p \in M$ is the *n*-dimensional vector space spanned by the tangent vectors to *M* at p.

4.1 The tangent 'space' to S^1

The tangent 'space' to the circle, S^1, at a point is the line tangent to the circle at that point.



The equation of the circle is given by $x^2 + y^2 = 1$ and using implicit differentiation we get $x\,dx + y\,dy = 0$ and $\frac{dy}{dx} = \frac{-x}{y}$, i.e., the slope of the tangent line at a point $p = (a, \sqrt{1-a^2}), -1 < a < 1, a \neq 0,$ on the circle is $\frac{dy}{dx} = \frac{-a}{\sqrt{1-a^2}}$, the tangent line at p is parallel to the line $y = -\frac{a}{\sqrt{1-a^2}}x$ through the origin.

The line $y = -\frac{a}{\sqrt{1-a^2}}x$ through the origin is a 1-dimensional vector space, i.e., the set of vectors $\begin{pmatrix} x \\ -\frac{a}{\sqrt{1-a^2}}x \end{pmatrix}$, i.e., the tangent space to S^1 at p.

Then (**exercise**):

$$R_a\begin{pmatrix} x \\ -\frac{a}{\sqrt{1-a^2}}x \end{pmatrix} = \begin{pmatrix} -x \\ \frac{a}{\sqrt{1-a^2}}x \end{pmatrix}, \text{that is } R_a\begin{pmatrix} x \\ -\frac{a}{\sqrt{1-a^2}}x \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ -\frac{a}{\sqrt{1-a^2}}x \end{pmatrix}.$$

We see that at the point p, the reflection preserves the tangent line, but reverses the direction.

If $a = 1$ (or $a = -1$), the reflection that fixes $(1,0)$ (or $(-1,0)$) is the reflection through the x-axis, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$. The tangent line to $(1,0)$ is the vertical line tangent to the circle at this point, $\begin{pmatrix} 0 \\ y \end{pmatrix}$. The reflection through the *x*-axis reverses this line.

5 Isometries, The Reflections of S^1 Generate $SO(2)$

5.1 The Orthogonal Group

In general, an isometry is a map that preserves distance. In the plane $\mathbb{R}^2$ the usual distance between two points is given by the standard dot product and the Pythagorean identity.

Thinking of $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ as a column vector, then the length of the vector is:

$$|\vec{x}| = \sqrt{\vec{x} \bullet \vec{x}} = \sqrt{x^2 + y^2}, \text{ where } \bullet \text{ is the usual dot product.}$$

The *Orthogonal Group* $O(2,\mathbb{R}) = O(2)$ is the set of 2 x 2 matrices with real entries that preserve the length of a vector, i.e., isometries of the plane that fix the origin.

This means that if A is a 2 x 2 matrix then $A \in O(2) \Leftrightarrow |A\vec{x}| = |\vec{x}|$.

(O(1) is the set of 1 x 1 matrices that preserve distance on the $\mathbb{R}$ line. The operation is multiplication, $A \in O(1) \Rightarrow |A\vec{x}| = |\vec{x}| \Rightarrow A = \{1 \text{ or } -1\}$. Note that multiplication by 1 preserves orientation on $\mathbb{R}$ and multiplication by -1 reverses orientation on $\mathbb{R}$.)

What are the algebraic equations for elements of O(2)?

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2)$,

then $\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right\| = \left\|\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}\right\| = \sqrt{(ax+by)^2 + (cx+dy)^2} = \sqrt{x^2+y^2}$

implies

$$
\begin{aligned}
x^2 + y^2 &= (ax + by)^2 + (cx + dy)^2 \\
&= a^2x^2 + 2abxy + b^2y^2 + c^2x^2 + 2cdxy + d^2y^2 \\
&= (a^2 + c^2)x^2 + 2(ab + cd)xy + (b^2 + d^2)y^2
\end{aligned}
$$

Then

(1) $a^2 + c^2 = 1 \Rightarrow c = \pm\sqrt{1-a^2}$

(2) $b^2 + d^2 = 1 \Rightarrow b = \pm\sqrt{1-d^2}$

(3) $ab + cd = 0$

and

$$
\begin{aligned}
&(3) \Rightarrow b = \frac{-cd}{a}, a \neq 0 \\
&\Rightarrow \sqrt{1-d^2} = \frac{-\sqrt{1-a^2}d}{a} \\
&\Rightarrow 1 - d^2 = \frac{(1-a^2)d^2}{a^2} \\
&\Rightarrow a^2 - a^2d^2 = d^2 - a^2d^2 \\
&\Rightarrow a^2 = d^2 \\
&\Rightarrow a = \pm d
\end{aligned}
$$

Further (1) & (2) $\Rightarrow |a| \leq 1, |b| \leq 1, |c| \leq 1, \textit{and} \ |d| \leq 1$

Therefore $O(2) = \begin{pmatrix} a & \pm\sqrt{1-a^2} \\ \pm\sqrt{1-a^2} & \pm a \end{pmatrix}$.

(3) determines the sign of the various entries, the possible choices are:

(i) $\begin{pmatrix} a & -\sqrt{1-a^2} \\ \sqrt{1-a^2} & a \end{pmatrix}$, rotation counterclockwise

(ii) $\begin{pmatrix} a & \sqrt{1-a^2} \\ -\sqrt{1-a^2} & a \end{pmatrix}$, rotation clockwise

(iii) $\begin{pmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & -a \end{pmatrix}$, reflection through the line with slope, $m = \sqrt{\frac{1-a}{1+a}}$

(iv) $\begin{pmatrix} a & -\sqrt{1-a^2} \\ -\sqrt{1-a^2} & -a \end{pmatrix}$, reflection through the line with slope, $m = -\sqrt{\frac{1-a}{1+a}}$

Exercises:

i. For each of (i) - (iv) the determinant is ± 1.

ii. $A \in O(2) \Leftrightarrow {}^tAA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where ${}^tA$ is the transpose of A.

iii. $R_a = \begin{pmatrix} 2a^2 - 1 & 2a\sqrt{1-a^2} \\ 2a\sqrt{1-a^2} & -2a^2 + 1 \end{pmatrix}$, $-1 \le a \le 1$, is an isometry. Which of i, ii, iii, or iv is R_a?

Definition: SO(2) is the set of elements in O(2) with determinant 1, the "special orthogonal" group.

5.2 SO(2)

From section 5.1 that matrices in O(2) given by:

(i) $\begin{pmatrix} a & -\sqrt{1-a^2} \\ \sqrt{1-a^2} & a \end{pmatrix}$, rotation counterclockwise

(ii) $\begin{pmatrix} a & \sqrt{1-a^2} \\ -\sqrt{1-a^2} & a \end{pmatrix}$, rotation clockwise

have determinant 1. Thus SO(2) consists of the rotations of the plane.

It is more convenient to write these as

(i) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, rotation counterclockwise in an angle θ and

(ii) $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, rotation counterclockwise in an angle θ , where $\theta = \arccos a$.

5.2.1 SO(2) is a group

A <u>group</u> [Cl] is a non-empty set G with a binary operation that satisfies 3 axioms:

1. For all $g, h, k \in G, \quad (gh)k = g(hk)$, associative.
2. There exist an element $e \in G$, the <u>identity</u> element, so that $eg = ge = g$ for all $g \in G$.
3. For every $g \in G$ there exist an element $g^{-1} \in G$, the inverse element, so that $gg^{-1} = g^{-1}g = e$.

Exercise: Show that SO(2) = $\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in \mathbb{R} \right\}$ is a group, with identity element, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If $g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ what is g^{-1}?

Exercises:

i. A group is called <u>Abelian</u> or <u>commutative</u> if $gh = hg \; \forall \, g,h \in G$. Is $SO(2)$ Abelian?

ii. Is $O(2)$ a group?

iii. Is $O(2)$ Abelian?

iv. Is the set of reflections in $O(2)$ a group?

5.3 The reflections on S^1 generate $SO(2)$.

It is a fact of plane geometry that any rotation in the plane can be written as a product of two reflections, see [C]. This fact is illustrated using the representations of the reflections and rotations as matrices.

The points on S^1 can be written as $p = (\cos\theta, \sin\theta)$. Then, as above, there is an involutive isometry at p,

$R_\theta = \begin{pmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & -2\cos^2\theta + 1 \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ that fixes p and its antipodal point. Note that this is a reflection, not a rotation, the determinant is -1.

Let $q = (\cos\alpha, \sin\alpha)$ be another point on S^1, then the involution that fixes q is

$$R_\alpha = \begin{pmatrix} 2\cos^2\alpha - 1 & 2\cos\alpha\sin\alpha \\ 2\cos\alpha\sin\alpha & -2\cos^2\alpha + 1 \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}.$$

Then $R_\alpha R_\theta = \begin{pmatrix} \cos 2\alpha\cos 2\theta + \sin 2\alpha\sin 2\theta & \cos 2\alpha\sin 2\theta - \sin 2\alpha\cos 2\theta \\ \sin 2\alpha\cos 2\theta - \cos 2\alpha\sin 2\theta & \sin 2\alpha\sin 2\theta + \cos 2\alpha\cos 2\theta \end{pmatrix}$

$= \begin{pmatrix} \cos(2\alpha - 2\theta) & -\sin(2\alpha - 2\theta) \\ \sin(2\alpha - 2\theta) & \cos(2\alpha - 2\theta) \end{pmatrix} \in SO(2)$ i.e., a counterclockwise rotation in the angle, $2(\alpha - \theta)$.

That is, products $R_\alpha R_\theta$ generate $SO(2)$.

Exercise: Given $g = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \in SO(2)$ find α and θ so that $R_\alpha R_\theta = g$.

6 $SO(2) = S^1$

6.1 The Action of a Group on a Set

Definition [Cl]: A group G **acts** on a set X as a group of transformations, if for each pair $(g,x) \in G \times X$ there is an associated element $g * x \in X$ so that:

i. $g * (h * x) = (gh) * x,$ for all $g,h \in G$ and $x \in X$.

ii. $e * x = x,$ for all $x \in X$, where e is the identity element in G.

Definition: The **orbit** of $x \in X$ under the action of G is a subset of X, $G * x = \{y \in X \mid y = g * x, \text{ for some } g \in G\}$.

Definition: The action of G on X is **transitive** if for every $x, y \in X$ there is a $g \in G$ so that $g * x = y$. The action is **simply transitive** if for every $x, y \in X$ there is a unique $g \in G$ so that $g * x = y$.

Definition: The **stabilizer** (or isotropy subgroup) of $x \in X$ under the action of G is the set of group actions that fix x, $G_x = \{g \in G \mid g * x = x\}$.

Exercise: G_x is subgroup of G.

6.2 The Action of $SO(2)$ on S^1

Geometrically $SO(2)$ acts on S^1 as a group of rotations, each element of $SO(2)$ rotates the circle. Algebraically the action is matrix multiplication of a vector in $\mathbb{R}^2$. Since $SO(2)$ preserves distance the action takes S^1 onto itself.

The orbit of a point p in S^1 is the locus of the points where p is sent by the action of the members of the group $SO(2)$.

1. The orbit of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under the action of $SO(2)$ is S^1.

   Proof: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$.

2. The action of $SO(2)$ on S^1 is transitive, but not simply transitive.

   Proof: for any two points on S^1 we can find a rotation that take one to another (**exercise**). However, the rotation is not unique since
   $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\theta + 2n\pi) & -\sin(\theta + 2n\pi) \\ \sin(\theta + 2n\pi) & \cos(\theta + 2n\pi) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad n \in \mathbb{Z}.$

3. The stabilizer of $\begin{pmatrix}1\\0\end{pmatrix}$ under the action of $SO(2)$ is

$$SO(2)_1 = \left\{\begin{pmatrix}\cos(0+2n\pi) & -\sin(0+2n\pi)\\ \sin(0+2n\pi) & \cos(0+2n\pi)\end{pmatrix},\quad n \in \mathbb{Z}\right\}.$$

4. $\dfrac{SO(2)}{SO(2)_1}$ is the set of cosets where $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ is in the same coset as $\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$ if $\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} = \begin{pmatrix}\cos(0+2k\pi) & -\sin(0+2k\pi)\\ \sin(0+2k\pi) & \cos(0+2k\pi)\end{pmatrix}$, for some integer k, i.e., $\theta = \alpha - 2k\pi$.
5. $\dfrac{SO(2)}{SO(2)_1}$ is a group. **Exercise**.

Theorem: $\dfrac{SO(2)}{SO(2)_1} \cong S^1$.

Proof: (**exercise**).

7 $S^1 \cong SO(2)$ is a Lie Group

Roughly speaking, a Lie Group is a non-empty set that is both a differentiable manifold and a group. We've seen that S^1 is a manifold and that S^1 can be given a group structure by identifying it with $SO(2)$, the final step is to check that the group operations are differentiable.

Definition ([H]) A *Lie Group* is a group G that is also a differentiable manifold such that the mapping $(g,h) \to gh^{-1}$ of the product manifold $G \times G$ into G is a differentiable map.

Since $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix} = \begin{pmatrix}\cos(\theta+\alpha) & -\sin(\theta+\alpha)\\ \sin(\theta+\alpha) & \cos(\theta+\alpha)\end{pmatrix}$ we can define a multiplication on S^1 by

$$(\cos\theta, \sin\theta) * (\cos\alpha, \sin\alpha) = (\cos(\theta+\alpha), \sin(\theta+\alpha)).$$

Exercise: Check that S^1 with * is a group. What is the identify element? What is $(\cos\theta, \sin\theta)^{-1}$?

Then $\Theta[(\cos\theta,\sin\theta),(\cos\alpha,\sin\alpha)] = (\cos\theta,\sin\theta) * (\cos\alpha,\sin\alpha)^{-1} = (\cos(\theta-\alpha),\sin(\theta-\alpha))$ is a mapping $S^1 \times S^1 \to S^1$.

To check that this map of manifolds is differentiable, [H], use the stereographic projection maps φ_1 and φ_2 and show, e.g., that the mapping $\varphi_1 \circ \Theta \circ \left(\varphi_1^{-1} \times \varphi_1^{-1}\right) : \mathbb{R}^2 \to \mathbb{R}$ is differentiable.

$$\begin{array}{ccccc} S^1 & \times & S^1 & \xrightarrow{\Theta} & S^1 \\ \varphi_1^{-1} \uparrow & & \varphi_1^{-1} \uparrow & & \downarrow \varphi_1 \\ a & & b & & \dfrac{ab + a - b + 1}{ab - a + b + 1} \end{array}$$

Exercises:

i. Show $f(a,b) = \varphi_1 \circ \Theta \circ \left(\varphi_1^{-1} \times \varphi_1^{-1}\right)(a,b) = \dfrac{\cos(\theta - \alpha)}{1 - \sin(\theta - \alpha)} = \dfrac{ab + a - b + 1}{ab - a + b + 1}$, where $\cos\theta = \dfrac{2a}{a^2 + 1}$, and $\cos\alpha = \dfrac{2b}{b^2 + 1}$. Where is this map defined?

ii. Show that $f(a,b)$ is differentiable by calculating $\dfrac{\partial f}{\partial a}$ and $\dfrac{\partial f}{\partial b}$. Where is this map <u>not</u> differentiable?

8 Conclusion

In these notes we introduced many of the major themes of the study of symmetric spaces – involutions, actions of Lie Groups, and the realization of symmetric spaces as quotient spaces.

These notions generalize to n-dimensional spheres, $S^n = \dfrac{SO(n+1)}{SO(n)}$, the Grassmann Manifolds $\dfrac{SO(p+q)}{SO(p) \times SO(q)}$, the projective spaces, a variety of exotic spaces. See [H] for the list of irreducible symmetric spaces and a complete exposition of the theory.

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