On the AM-GM Inequality

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It is a well known fact that, if a, b, c are positive real numbers, then by the $AM-GM$ Inequality

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}} = 3.$$

We denote $G(a,b,c) = \dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}-3$, thus $G(a,b,c) \geq 0$ for all $a,b,c>0$.

Observe that $G(a,b,c) \geq 0$ is a cyclic inequality, but not symmetric. Therefore we cannot assume any pairwise order between a, b, c, but we can assume that one of them is the minimum or maximum for all of them. Also it is not difficult to prove that, if $a \geq b \geq c$, then $G(a,b,c) \leq G(a,c,b)$. The purpose of this article is to present some nice properties of the function G through the olympiad inequalities.

Problem 1. *For all positive real numbers a, b, c, k, the following inequality holds*

$$G(a,b,c) \geq G(a+k, b+k, c+k).$$

Solution. Assume without loss of generality $c = \min(a,b,c)$. We rewrite $G(a,b,c)$ in the following way

$$G(a,b,c) = \left(\frac{a}{b}+\frac{b}{a}-2\right)+\left(\frac{b}{c}+\frac{c}{a}-\frac{b}{a}-1\right) = \frac{(a-b)^2}{ab}+\frac{(a-c)(b-c)}{ac}.$$

It is enough to prove that

$$\frac{(a-b)^2}{ab}+\frac{(a-c)(b-c)}{ac} \geq \frac{(a-b)^2}{(a+k)(b+k)}+\frac{(a-c)(b-c)}{(a+k)(c+k)}.$$

But this is true, because $k>0$ and $(a-c)(b-c) \geq 0$. The proof is completed and the equality holds when $a=b=c$. $\square$

The next problem was given on the Mathlinks Contest 2003.

Problem 2. *Let a, b, c be positive integer numbers. Prove that*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{a+b}{a+c}+\frac{a+c}{b+c}+\frac{b+c}{a+b}.$$

Solution. The statement of the problem is equivalent to

$$G(a,b,c) \geq G(a+c, b+c, a+b).$$

Assume without loss of generality $c = \min(a,b,c)$. Using the identity

$$G(a,b,c) = \frac{(a-b)^2}{ab} + \frac{(a-c)(b-c)}{ac},$$

we have to prove that

$$\frac{(a-b)^2}{ab} + \frac{(a-c)(b-c)}{ac} \geq \frac{(a-b)^2}{(a+c)(b+c)} + \frac{(a-c)(b-c)}{(a+b)(a+c)}.$$

Clearly the inequality is true, using the above assumption of the minimality of c. □

Problem 3. *Let a, b, c be real numbers. Prove that*

$$G\left((a-b)^2, (b-c)^2, (c-a)^2\right) \geq 2.$$

(Darij Grinberg)

Solution. For the experience solver the statement of the problem may appear quite strange. The inequality is equivalent to

$$\frac{(a-b)^2}{(b-c)^2} + \frac{(b-c)^2}{(c-a)^2} + \frac{(c-a)^2}{(a-b)^2} \geq 5,$$

and the appearance of a constant 5 in the cyclic inequality with three variables is unclear. The solution is to observe the following indentity

$$\frac{(a-b)^2}{(b-c)^2} + \frac{(b-c)^2}{(c-a)^2} + \frac{(c-a)^2}{(a-b)^2} = 5 + \left(1 + \frac{a-b}{b-c} + \frac{b-c}{c-a} + \frac{c-a}{a-b}\right)^2,$$

and the conclusion follows immediately. □

Problem 4. *Let a, b, c be positive real numbers and $k \geq \max(a^2, b^2, c^2)$, then*

$$G(a,b,c) \geq G(a^2+k, b^2+k, c^2+k).$$

(Pham Kim Hung)

Solution. Assume without loss of generality $c = \min(a, b, c)$. Applying the same procedure as in the first problem, we get equivalent to

$$\frac{(a-b)^2}{ab} + \frac{(a-c)(b-c)}{ac} \geq \frac{(a-b)^2(a+b)^2}{(a^2+k)(b^2+k)} + \frac{(a-c)(b-c)(a+c)(b+c)}{(a^2+k)(c^2+k)}.$$

Therefore it suffices to prove that

$$(a^2+k)(b^2+k) \geq ab(a+b)^2$$
$$(a^2+k)(c^2+k) \geq ac(a+c)(b+c)$$

The first inequality is true, because

$$(a^2+k)(b^2+k) \geq (a^2+b^2)^2 \geq ab(a+b)^2.$$

The second one is equivalent to

$$c^2(k-ac) + a^2(k-bc) + k^2 - abc^2 \geq 0,$$

which is clear, as $k \geq \max(a^2, b^2, c^2)$. The equality holds when $a = b = c$. □

The statement of the problem can be sharpened with $k \geq \max(ab, bc, ca)$. In this case, equality holds for $a \geq b = c,\ k = ab$.

As we mentioned in the beginning of the article, if $a \geq b \geq c$, then $G(a, b, c) \leq G(a, c, b)$. A logic question may appear: is there a constant k such that $G(a, b, c) \geq kG(c, b, a)$, for all positive real numbers a, b, c? The answer for this question is negative. However, if a, b, c are side-lengths of a triangle, we can find such value for k. Consider the following problem

Problem 5. *Find the best positive real constant k such that*

$$G(a, b, c) \geq kG(c, b, a)$$

holds for all a, b, c, which are side-lengths of a triangle.

(Pham Kim Hung, after a problem of Vasile Cirtoaje)

Solution. We use the entirely mixing variables method. Clearly we need to consider only the case when $a \geq b \geq c$. Thus $k \leq 1$ and the inequality transforms into

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 3 \geq k\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c} - 3\right)$$

$$\Leftrightarrow a^2c + c^2b + b^2a - 3abc \geq k(a^2b + b^2c + c^2a - 3abc)$$

$$\Leftrightarrow \frac{1+k}{2}\sum_{cyc} a^2(c-b) + \frac{1-k}{2}\sum_{sym} a^2(b+c) \geq 3(1-k)abc$$

$$\Leftrightarrow (1-k)\sum_{sym} c(a-b)^2 - (1+k)(a-b)(a-c)(b-c) \geq 0.$$

The mixing variables method affirms that it is enough to prove the inequality in case a, b, c are side-lengths of a degenerate triangle, or $a = b + c$. Therefore the inequality reduces to

$$(1-k)\left(b^3 + c^3 + (b+c)(b-c)^2\right) \geq (1+k)bc(b-c).$$

Let $x = b + c,\ y = b - c$, then we have

$$(1-k)(x^3 + 7xy^2) \geq (1+k)y(x^2 - y^2).$$

Consider the following function for $x > 1$,

$$f(x) = \frac{x^3 + 7x}{x^2 - 1}, \quad \Rightarrow f'(x) = \frac{(3x^2+7)(x^2-1) - 2x(x^3+7x)}{(x^2-1)^2}.$$

Hence $f'(x) = 0 \Leftrightarrow (3x^2 + 7)(x^2 - 1) = 2(x^4 + 7x^2) \Leftrightarrow x^4 - 10x^2 - 7 = 0$, or $x = x_0 = \sqrt{5 + 4\sqrt{2}}$, as $x > 1$. It is not difficult to prove that $f(x) \geq f(x_0)$ for all $x \geq 1$, thus the best constant k must satisfy

$$\frac{1+k}{1-k} = f(x_0) \Leftrightarrow k = \frac{f(x_0) - 1}{f(x_0) + 1}.$$

After making direct computations we get

$$k = 1 - \frac{2}{\left(2\sqrt{2} - 1\right)\sqrt{5 + 4\sqrt{2}} + 1} \approx 0.713...$$

Equality holds for $a = b = c$ and $a = b + c,\ \dfrac{b}{c} = \sqrt{5 + 4\sqrt{2}}$ up to their permutations.

$\square$

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