Equiangular polygons

An algebraic approach

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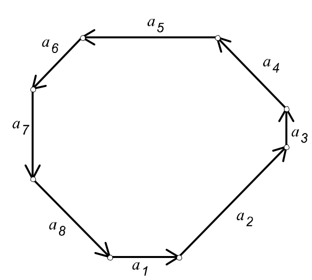
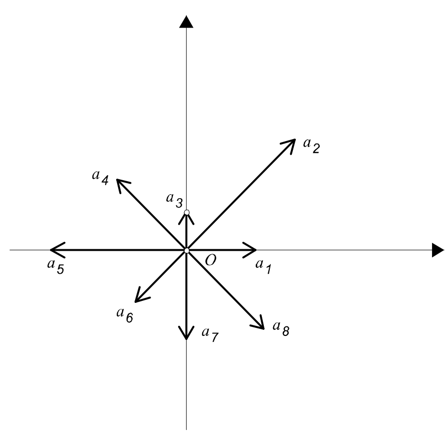
We call a convex polygon equiangular if its angles are congruent. Thus, an equiangular triangle is an equilateral one, and an equiangular quadrilateral is a rectangle (or a square). One interesting algebraic characterisation of the equiangular polygons is stated in the following lemma.

Lemma 1 *Let $a_1, a_2, \ldots, a_n$ be positive real numbers and let ε be a primitive nth root of the unity, e.g. $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. If the sides of an equiangular polygon have lengths $a_1, a_2, \ldots, a_n$ (in counterclockwise order) then*

$$a_1 + a_2\varepsilon + a_3\varepsilon^2 + \ldots + a_n\varepsilon^{n-1} = 0.$$

Proof. Consider the polygon's sides as vectors, oriented counterclockwise (see figure for the case $n = 8$). Then the sum of the vectors equals zero. Now, translate all vectors such that they have the same origin O. If we look at the complex numbers corresponding to their extremities, choosing a_1 on the positive real axis, we see that these are $a_1, a_2\varepsilon, a_3\varepsilon^2, \ldots, a_n\varepsilon^{n-1}$, respectively. We deduce that $a_1 + a_2\varepsilon + a_3\varepsilon^2 + \ldots + a_n\varepsilon^{n-1} = 0$. ■





The converse of the statement is not true. For instance, if a, b, c, d are the side lengths of a quadrilateral and $a+bi+ci^2+di^3 = 0$, then $(a-c)+i(b-d) = 0$; this equality is fulfilled if the quadrilateral is a parallelogram (and not necessarily an equiangular quadrilateral, that is, a rectangle). However, from the proof we see that if $a_1, a_2, \ldots, a_n$ are positive numbers and $a_1 + a_2\varepsilon + a_3\varepsilon^2 + \ldots + a_n\varepsilon^{n-1} = 0$, then there exists an equiangular polygon with sides of lengths $a_1, a_2, \ldots, a_n$.

We examine now several contest problems involving equiangular polygons, presenting in some cases two solutions: a geometric solution and an algebraic solution. Some of them are discussed in [1].

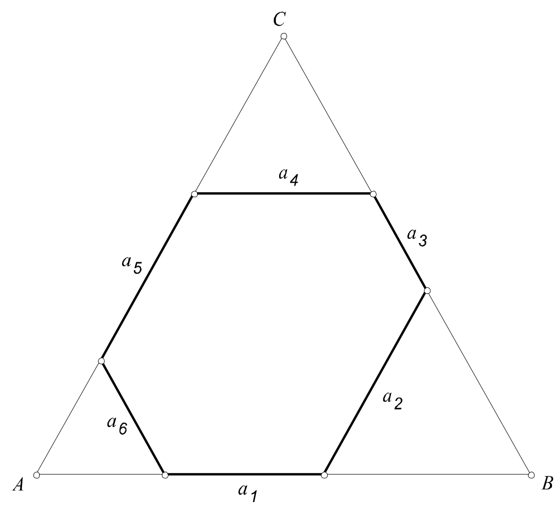
Problem 1. Prove that if an equiangular hexagon has side lengths $a_1, a_2, \ldots, a_6$ (in this order) then

$$a_1 - a_4 = a_5 - a_2 = a_3 - a_6.$$

(Romanian Selection Test)

First solution. Expand every second side of the hexagon to obtain a triangle ABC (see figure below). Since the hexagon is equiangular, triangle ABC is equilateral. Moreover, triangle ABC is the union of the hexagon and three other smaller triangles, which are equilateral as well. We see that

$$AB = a_1 + a_2 + a_6,$$
$$BC = a_2 + a_3 + a_4,$$
$$CA = a_4 + a_5 + a_6.$$



Since $AB = BC = CA$, we obtain the desired result.

Second solution. Let $\varepsilon = \cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6}$ be a primitive sixth root of unity. Then

$$a_1 + a_2\varepsilon + a_3\varepsilon^2 + a_4\varepsilon^3 + a_5\varepsilon^4 + a_6\varepsilon^5 = 0.$$

But $\varepsilon^3 = \cos\pi + i\sin\pi = -1$, so $\varepsilon^4 = -\varepsilon$ and $\varepsilon^5 = -\varepsilon^2$. We deduce that

$$(a_1 - a_4) + (a_2 - a_5)\varepsilon + (a_3 - a_6)\varepsilon^2 = 0.$$

On the other hand, since $\varepsilon^3 = -1$ (and $\varepsilon \neq -1$) we see that $\varepsilon^2 - \varepsilon + 1 = 0$. Thus, ε is a common root of the equations $(a_1 - a_4) + (a_2 - a_5)z + (a_3 - a_6)z^2 = 0$ and

$z^2 - z + 1 = 0$, both with real coefficients. Since $\varepsilon \notin \mathbb{R}$, it follows that the two equations share another common root $\overline{\varepsilon}$, so the coefficients of the two equations must be proportional; that is,

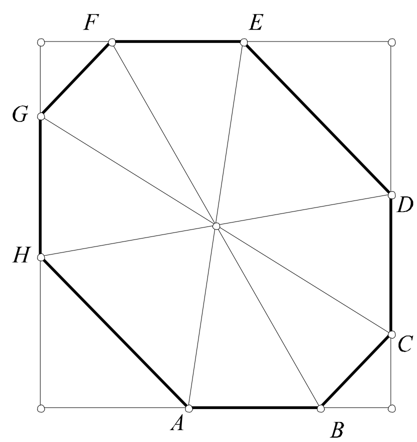
$$(a_1 - a_4) = -(a_2 - a_5) = (a_3 - a_6),$$

as desired.

Problem 2. The side lengths of an equiangular octagon are rational numbers. Prove that the octagon has a symmetry center.

(Russian Olympiad)

First solution. Let $ABCDEFGH$ be the octagon. The angles of an equiangular octagon are equal to $135°$, thus, the lines containing the segments AB, CD, EF and GH determine a rectangle.



Because the opposite sides of this rectangle are equal, we obtain

$$AB + \frac{\sqrt{2}}{2}(AH + BC) = EF + \frac{\sqrt{2}}{2}(DE + FG),$$

or, equivalently,

$$AB - EF = \frac{\sqrt{2}}{2}(DE + FG - AH - BC).$$

Because the side lengths of the octagon are rational numbers, the above equality can hold if and only if

$$AB - EF = DE + FG - AH - BC = 0.$$

In a similar way, we obtain

$$CD - GH = FG + AH - DE - BC = 0.$$

From these equalities it follows that $AB = EF$, $CD = GH$, $BC = FG$ and $DE = AH$, so the opposite sides of the octagon are equal and parallel. It follows that the quadrilaterals $ABEF$, $BCFG$, $CDGH$ and $DEHA$ are parallelograms, hence the midpoints of the segments AE, BF, CG, DH and DH coincide. Obviously, this common point is a symmetry center of the octagon.

Second solution. Denote by $a_1, \ldots, a_8$ the octagon's side lengths. Then

$$a_1 + a_2\varepsilon + \ldots + a_8\varepsilon^7 = 0,$$

where $\varepsilon = \cos\frac{2\pi}{8} + i\sin\frac{2\pi}{8}$. Observing that $\varepsilon^4 = -1$, the above equality becomes

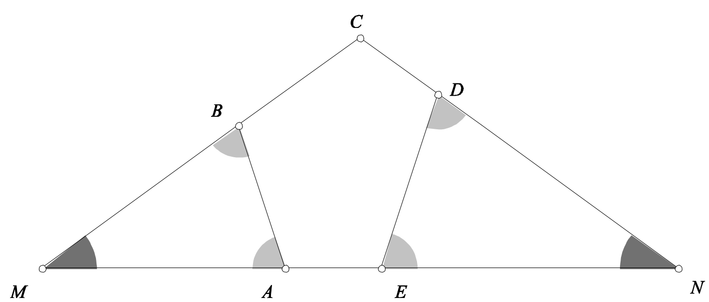
$$a_1 - a_5 + (a_2 - a_6)\,\varepsilon + (a_3 - a_7)\,\varepsilon^2 + (a_4 - a_8)\,\varepsilon^3 = 0.$$

Thus, ε is a common root of the polynomials with rational coefficients $f(X) = X^4 + 1$ and $g(X) = a_1 - a_5 + (a_2 - a_6)\,X + (a_3 - a_7)\,X^2 + (a_4 - a_8)\,X^3$. If g is a non-constant polynomial, we derive that $\gcd(f, g)$ is a non-constant rational polynomial, which is a contradiction since f is irreducible in $\mathbb{Q}[X]$ (to see this, observe that its roots are nonreal, so the only way it can be written as a product of polynomials with *real* coefficients is $f = \left(x^2 + x\sqrt{2} + 1\right)\left(x^2 - x\sqrt{2} + 1\right)$). Therefore, g must be a constant and this implies that the opposite sides of the octagon are equal and parallel. The solution ends like the previous one.

Problem 3. Let $ABCDE$ be an equiangular pentagon whose side lengths are rational numbers. Prove that the pentagon is regular.

(Balkan Mathematical Olympiad)

First solution. Let M and N be the intersection points of the line AE with BC and CD, respectively.



Because $ABCDE$ is equiangular, triangles AMB and DNE are isosceles, with $\angle M = \angle N = 36^\circ$, therefore triangle CMN is also isosceles and $CM = CN$. It follows that $BC + BM = CD + DN$. But $AB = 2\cos 72^\circ BM$ and $DE = 2\cos 72^\circ DN$, whence

$$AB - DE = 2\cos 72^\circ\,(BM - DN) = 2\cos 72^\circ\,(CD - BC)\,.$$

Since $\cos 72^\circ$ is not a rational number, the above equalities imply both $AB = DE$ and $CD = BC$. The conclusion follows easily.

Second solution. We can prove more:

Lemma 2 *The positive integer p is a prime number if and only if every equiangular polygon with p sides of rational lengths is regular.*

Proof. Suppose p is a prime number and let the rational numbers $a_1, a_2, \ldots, a_p$ be the side lengths of an equiangular polygon. We have seen that

$$\varepsilon = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}$$

is a root of the polynomial

$$P(X) = a_1 + a_2X + \ldots a_pX^{p-1}.$$

On the other hand, ε is also a root of the polynomial

$$Q(X) = 1 + X + X^2 + \ldots + X^{p-1}.$$

Because the two polynomials share a common root, their greatest common divisor must be a non-constant polynomial with rational coefficients.This implies that Q can be factorized as a product of two non-constant polynomials with rational coefficients, which is impossible (to prove that, one applies the Eisenstein's criterion to the polynomial $Q(X+1)$). Conversely, suppose p is not a prime number and let $p = mn$, for some positive integers $m, n > 1$. It results that ζ^n is a root of order m of the unity, hence $1+\zeta^n+\zeta^{2n}+\ldots+\zeta^{(m-1)n} = 0$. If we add this equality to $1+\zeta+\zeta^2+\zeta^3+\ldots+\zeta^{p-1} = 0$, we deduce that ζ is the root of a polynomial of degree $p-1$, with some coefficients equal to 1 and the others equal to 2. This means that there exists an equiangular polygon with p sides, some of length 1 and the rest of length 2. Because such a polygon is not regular, our claim is proved. ■

Problem 4. Let $a_1, a_2, \ldots, a_n$ be the side lengths (in order) of an equiangular polygon. Prove that if $a_1 \geq a_2 \geq \ldots \geq a_n$, then the polygon is regular.

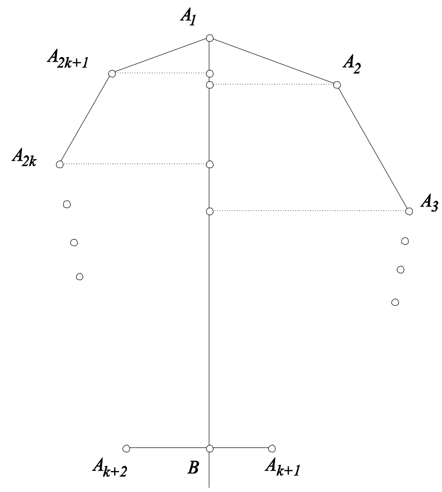
(International Mathematical Olympiad)

First solution. We examine two cases: n odd and n even. If n is odd, say $n = 2k+1$, consider the angle bisector of $\angle A_{2k+1}A_1A_2$. It is not difficult to see that it is perpendicular to the side $A_{k+1}A_{k+2}$. Project all the sides of the polygon on this line. If we denote by x_i the length of the projection of the side A_iA_{i+1} (with the usual convention $A_{2k+2} = A_1$), then

$$x_1 + x_2 + \ldots + x_k = x_{k+2} + x_{k+3} + \ldots + x_{2k+1} = A_1B$$

(see figure below). On the other hand, the angle between A_iA_{i+1} and A_1B is equal to the angle between $A_{2k+2-i}A_{2k+3-i}$ and A_1B, thus $x_i \geq x_{2k+2-i}$, for all $1 \leq i \leq k$. It follows that the above equality can be reached only if the sides of the polygon are equal.

If n is even, there is a similar argument, but instead of the angle bisector of $\angle A_{2k+1}A_1A_2$, one considers the perpendicular on the sides A_1A_2 and $A_{k+1}A_{k+2}$ (it is easy to see that these two sides are in this case parallel).



Second solution. Let

$$\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

be a primitive root of the unity. Then ε is a root of the polynomial

$$P(X) = a_1 + a_2X + \ldots + a_nX^{n-1}.$$

The conclusion is obtained from the following:

Lemma 3 *Let $P(X) = a_1 + a_2X + \ldots + a_nX^{n-1}$, where $a_1 \geq a_2 \geq \ldots \geq a_n > 0$. If α is a root of P, then $|\alpha| \geq 1$, and $|\alpha| = 1$ only if $a_1 = a_2 = \ldots = a_n$.*

Proof. We have

$$a_1 + a_2\alpha + \ldots + a_n\alpha^{n-1} = 0.$$

If we multiply this equality with $\alpha - 1$, we obtain

$$-a_1 + \alpha(a_1 - a_2) + \alpha^2(a_2 - a_3) + \ldots + \alpha^{n-1}(a_{n-1} - a_n) + a_n\alpha^n = 0,$$

or, equivalently,

$$a_1 = \alpha(a_1 - a_2) + \alpha^2(a_2 - a_3) + \ldots + \alpha^{n-1}(a_{n-1} - a_n) + a_n\alpha^n.$$

Now, suppose that $|\alpha| = 1$. It results

$$a_1 = |\alpha(a_1 - a_2) + \alpha^2(a_2 - a_3) + \ldots + \alpha^{n-1}(a_{n-1} - a_n) + a_n\alpha^n| \leq$$
$$\leq |\alpha|(a_1 - a_2) + |\alpha|^2(a_2 - a_3) + \ldots + |\alpha|^{n-1}(a_{n-1} - a_n) + a_n|\alpha|^n \leq$$
$$\leq (a_1 - a_2) + (a_2 - a_3) + \ldots + (a_{n-1} - a_n) + a_n = a_1.$$

Consequently, all inequalities must be equalities. Because $\alpha \notin \mathbb{R}$, this is possible only if $a_1 = a_2 = \ldots = a_n$. ■

We conclude that the polygon is regular.

Problem 5. Let n be a positive integer which is not a power of a prime number. Prove that there exists an equiangular polygon whose side lengths are $1, 2, \ldots, n$ in some order.

Solution. All we have to do is to prove that there exists a polynomial

$$f(x) = a_1 + a_2 x + \ldots + a_n x^{n-1},$$

such that $a_1, a_2, \ldots, a_n$ is a permutation of the numbers $1, 2, \ldots, n$ and $f(\varepsilon) = 0$, where $\varepsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$. Equivalently, we show that there is a permutation $\sigma(0), \sigma(1), \ldots, \sigma(n-1)$ of the numbers $0, 1, \ldots, n-1$ such that

$$\varepsilon^{\sigma(0)} + 2\varepsilon^{\sigma(1)} + 3\varepsilon^{\sigma(2)} + \ldots + n\varepsilon^{\sigma(n-1)} = 0.$$

First, observe that since n is not a power of a prime, n can be written as $n = pq$, where p and q are relatively prime positive integers.

Let k be an integer, $0 \leq k \leq n-1$ and denote $a = \left\lfloor \frac{k}{q} \right\rfloor$, $b = k - q \left\lfloor \frac{k}{q} \right\rfloor$. We define $\sigma(k) = aq + bp \pmod{n}$. Since p and q are relatively prime numbers it follows that σ is well defined.

We then have

$$\begin{aligned} f(\varepsilon) = {} & 1 + 2\varepsilon^{p} + 3\varepsilon^{2p} + \ldots + q\varepsilon^{(q-1)p} \\ & + (q+1)\,\varepsilon^{q} + (q+2)\,\varepsilon^{q+p} + \ldots + 2q\varepsilon^{q+(q-1)p} \\ & + (2q+1)\,\varepsilon^{2q} + (2q+2)\,\varepsilon^{2q+p} + \ldots + 3q\varepsilon^{2q+(q-1)p} \\ & + \ldots \\ & + ((p-1)\,q+1)\,\varepsilon^{(p-1)q} + \ldots + pq\varepsilon^{(p-1)q+(q-1)p}. \end{aligned}$$

If we denote

$$\begin{aligned} \zeta &= 1 + 2\varepsilon^{p} + 3\varepsilon^{2p} + \ldots + q\varepsilon^{(q-1)p}, \\ \xi &= 1 + \varepsilon^{p} + \varepsilon^{2p} + \ldots + \varepsilon^{(q-1)p}, \end{aligned}$$

we have

$$f(\varepsilon) = \zeta + q\varepsilon^{q}\xi + \varepsilon^{q}\zeta + 2q\varepsilon^{2q}\xi + \varepsilon^{2q}\zeta + \ldots + (p-1)\,q^{(p-1)q}\xi + \varepsilon^{(p-1)q}\zeta.$$

Clearly, $\xi = 0$, therefore

$$f(\varepsilon) = \zeta\left(1 + \varepsilon^{q} + \varepsilon^{2q} + \ldots + \varepsilon^{(p-1)q}\right) = 0,$$

as desired.

References

[1] *Mathematical Olympiad Treasures*, Titu Andreescu, Bogdan Enescu, Birkhäuser, Boston, Basel Berlin, 2004

[2] *Poligoane echiangulare*, Titu Andreescu, Bogdan Enescu, Gazeta Matematică, 107, 2002, 11, pp.422-427 (Romanian)