Junior problems

J163. Let a, b, c be nonzero real numbers such that $ab + bc + ca \geq 0$. Prove that

$$\frac{ab}{a^2+b^2} + \frac{bc}{b^2+c^2} + \frac{ca}{c^2+a^2} \geq -\frac{1}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ercole Suppa, Teramo, Italy
We have

$$\begin{aligned}
\sum_{cyc} \frac{ab}{a^2+b^2} &= \sum_{cyc} \left(\frac{ab}{a^2+b^2} + \frac{1}{2}\right) - \frac{3}{2} = \sum_{cyc} \frac{(a+b)^2}{2\left(a^2+b^2\right)} - \frac{3}{2} \\
&\geq \sum_{cyc} \frac{(a+b)^2}{2\left(a^2+b^2+c^2\right)} - \frac{3}{2} = \frac{2\left(a^2+b^2+c^2\right) + 2(ab+bc+ca)}{2\left(a^2+b^2+c^2\right)} - \frac{3}{2} \\
&= 1 + \frac{ab+bc+ca}{a^2+b^2+c^2} - \frac{3}{2} = \frac{ab+bc+ca}{a^2+b^2+c^2} - \frac{1}{2} \geq -\frac{1}{2}
\end{aligned}$$

where in the last step we have used the fact that $ab + bc + ca \geq 0$.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Prithwijit De, HBCSE, India; Andrea Ligori, Università di Roma "Tor Vergata", Italy; Piriyathumwong P., Bangkok, Thailand.

J164. If x and y are positive real numbers such that $\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)=2011$, find the minimum possible value of $x+y$.

Proposed by Neculai Stanciu, "George Emil Palade", Buzau, Romania

First solution by Michel Bataille, France The required minimum value is $\frac{2010}{\sqrt{2011}}$.

Write $x=\sinh(a)$ and $y=\sinh(b)$ where $a=\ln(x+\sqrt{x^2+1})>0$ and $b=\ln(y+\sqrt{y^2+1})>0$. From the hypothesis, we have $a+b=\ln(2011)$ and using a known formula,

$$x+y=\sinh(a)+\sinh(b)=2\sinh\left(\frac{a+b}{2}\right)\cosh\left(\frac{a-b}{2}\right)\geq 2\sinh\left(\frac{a+b}{2}\right)=2\sinh(\ln(\sqrt{2011})$$

where the inequality follows from $\cosh(t)\geq 1$ for all t and $\sinh(u)>0$ for $u>0$.
Since $2\sinh(\ln(\sqrt{2011})=\sqrt{2011}-\frac{1}{\sqrt{2011}}=\frac{2010}{\sqrt{2011}}$, we obtain

$$x+y\geq\frac{2010}{\sqrt{2011}}.$$

Clearly equality holds when $a=b$ (since $\cosh(0)=1$), that is, when $x=y$. The result follows.

Second solution by the authors

Let $z=x+\sqrt{x^2+1}$. We have $z>0$ and (1) $x=\frac{z^2-1}{2z}$. From hypothesis $y+\sqrt{y^2+1}=\frac{2011}{z}$, we get (2) $y=\frac{2011^2-z^2}{2\cdot 2011\cdot z}$. From (1) and (2),

$$x+y=\frac{z^2-1}{2z}+\frac{2011^2-z^2}{2\cdot 2011\cdot z}=\frac{2010}{2\cdot 2011}\left(z+\frac{2011}{z}\right)\geq\frac{2010}{2011}\sqrt{z\cdot\frac{2011}{z}}.$$

The equality occurs for $z\frac{2011}{z}$ or equivalently $z^2=2011$. Then from (1) and

(2)

we obtain

$$x=y=\frac{2010}{2\sqrt{2011}}=\frac{1005}{\sqrt{2011}}.$$

So $\min(x+y)=\frac{2010}{\sqrt{2011}}$.

Also solved by Arkady Alt, San Jose, California, USA; Francisco Javier Garcia Capitan, Spain; Ercole Suppa, Teramo, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

J165. Find all triples (x, y, z) of integers satisfying the system of equations

$$\begin{cases} \left(x^2+1\right)\left(y^2+1\right)+\frac{z^2}{10}=2010 \\ (x+y)(xy-1)+14z=1985. \end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, California, USA

Note that $z = 10k$ for some integer k because $\dfrac{z^2}{10} = 2010 - \left(x^2+1\right)\left(y^2+1\right)$ is an integer. Let $p = x + y$ and $q = xy - 1$. Then

$$\left(x^2+1\right)\left(y^2+1\right) = x^2y^2 + x^2 + y^2 + 1 = (xy-1)^2 + (x+y)^2 = p^2 + q^2$$

and the system becomes

$$\begin{cases} p^2+q^2+10k^2 = 2010 \\ pq + 140k = 1985 \end{cases} \iff \begin{cases} p^2+q^2 = 2010-10k^2 \\ pq = 1985-140k \end{cases} \tag{1}$$

Since $(p-q)^2 = 2010 - 10k^2 - 2(1985-140k) = -10(k-14)^2$ then only $k = 14$ can provide solvability to (1). And for $k = 14$, (1) becomes $\begin{cases} p^2+q^2 = 50 \\ pq = 25 \end{cases} \iff p = q = 5.$

Hence, $\begin{cases} x+y=5 \\ xy=4 \end{cases} \iff \begin{cases} x=4 \\ y=1 \end{cases}$ or $\begin{cases} x=1 \\ y=4 \end{cases}$ and triples $(5, 1, 140), (1, 5, 140)$ are all integer solutions of the original system in integers.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Piriyathumwong P., Bangkok, Thailand.

J166. Let P be a point inside triangle ABC and let d_a, d_b, d_c be the distances from point P to the sides of the triangle. Prove that

$$\frac{K}{d_a d_b d_c} \geq \frac{s}{Rr}$$

where K is the area of the pedal triangle of P and s, R, r are the semiperimeter, circumradius, and inradius of triangle ABC.

Proposed by Andrei Razvan Baleanu, "George Cosbuc", Motru, Romania

Remark: The problem contains a typ and the inequality that needs to be proven is

$$\frac{K}{d_a d_b d_c} \geq \frac{s}{2Rr}.$$

Many readers have solved the correct inequality.

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain The proposed inequality is not true, since if $P = I$ is the incenter of equilateral triangle ABC, then $K = \frac{rs}{4}$ is one quarter the area of ABC, while $d_a = d_b = d_c = r$, and the proposed inequality would be equivalent to $R \geq 4r = 2R$, absurd. We show that the correct inequality that always holds is

$$\frac{2K}{d_a d_b d_c} = \frac{s}{Rr}.$$

Now, denoting by P_A, P_B, P_C the respective projections of P on sides BC, CA, AB, we have $\angle CP_AP = \angle CP_BP = 90°$, or $\angle P_APP_B = 180° - C$, and the area of PP_AP_B is

$$\frac{d_a d_b \sin C}{2} = \frac{d_a d_b d_c}{4R} \frac{c}{d_c}.$$

Adding the analogous expressions for the areas of PP_BP_C and PP_CP_A, we find

$$K = \frac{d_a d_b d_c}{4R} \left(\frac{a}{d_a} + \frac{b}{d_b} + \frac{c}{d_c} \right),$$

or the proposed inequality is equivalent to

$$\frac{a}{d_a} + \frac{b}{d_b} + \frac{c}{d_c} \geq \frac{2s}{r}.$$

Now, $\frac{1}{x}$ is convex for positive x because $\frac{d^2}{dx^2}\left(\frac{1}{x}\right) = \frac{2}{x^3} > 0$, or by Jensen's inequality,

$$\frac{a}{d_a} + \frac{b}{d_b} + \frac{c}{d_c} \geq (a+b+c)\frac{a+b+c}{ad_a + bd_b + cd_c} = \frac{4s^2}{S} = \frac{2s}{r},$$

where $2S = ad_a + bd_b + cd_c = 2rs$ is twice the area of ABC because ad_a is twice the area of BPC, and similarly for its cyclic permutations, and equality is reached iff $d_a = d_b = d_c$, ie iff P is the incenter of ABC, in which case we easily find $K = \frac{r^2(\sin A + \sin B + \sin C)}{2} = \frac{r^2 s}{2R}$.

Second solution by G.R.A.20 Math Problems Group, Roma, Italy Since

$$4KR = cd_a d_b + ad_b d_c + bd_c d_a \quad \text{and} \quad 2sr = ad_a + bd_b + cd_c,$$

it follows that the inequality becomes

$$\left(\frac{a}{d_a} + \frac{b}{d_b} + \frac{c}{d_c}\right) \cdot (ad_a + bd_b + cd_c) \geq (a + b + c)^2$$

which holds by Cauchy-Schwarz.

Also solved by Arkady Alt, San Jose, California, USA; Michel Bataille, France; Ercole Suppa, Teramo, Italy.

J167. Let a, b, c be real numbers greater than 1 such that

$$\frac{b+c}{a^2-1}+\frac{c+a}{b^2-1}+\frac{a+b}{c^2-1}\geq 1.$$

Prove that

$$\left(\frac{bc+1}{a^2-1}\right)^2+\left(\frac{ca+1}{b^2-1}\right)^2+\left(\frac{ab+1}{c^2-1}\right)^2\geq\frac{10}{3}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Prithwijit De, HBCSE, India

Observe that

$$\left(\frac{bc+1}{a^2-1}\right)^2-\left(\frac{b+c}{a^2-1}\right)^2=\frac{(b^2-1)(c^2-1)}{(a^2-1)^2};$$
$$\left(\frac{ca+1}{b^2-1}\right)^2-\left(\frac{c+a}{b^2-1}\right)^2=\frac{(c^2-1)(a^2-1)}{(b^2-1)^2};$$
$$\left(\frac{ab+1}{c^2-1}\right)^2-\left(\frac{a+b}{c^2-1}\right)^2=\frac{(a^2-1)(b^2-1)}{(c^2-1)^2}.$$

Therefore $\sum\left(\frac{bc+1}{a^2-1}\right)^2=\sum\left(\frac{b+c}{a^2-1}\right)^2+\sum\frac{(b^2-1)(c^2-1)}{(a^2-1)^2}\ldots(1)$

Now observe that $\sum\left(\frac{b+c}{a^2-1}\right)^2\geq\frac{\left(\sum\frac{b+c}{a^2-1}\right)^2}{3}\geq\frac{1}{3}\ldots(2)$

and by A.M-G.M inequality we get

$$\sum\frac{(b^2-1)(c^2-1)}{(a^2-1)^2}\geq 3\sqrt[3]{\frac{(a^2-1)^2(b^2-1)^2(c^2-1)^2}{(a^2-1)^2(b^2-1)^2(c^2-1)^2}}=3.\quad\ldots(3)$$

By virtue of (1),(2) and (3) we obtain $\sum\left(\frac{bc+1}{a^2-1}\right)^2\geq 3+\frac{1}{3}=\frac{10}{3}.$

Second solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

We observe that $bc+1=b+c+(b-1)(c-1)$ and then $(bc+1)^2=(b+c)^2+2(b+c)(b-1)(c-1)+(b-1)^2(c-1)^2$. By power–means–inequality and the constraint on a, b, c we have

$$\sum_{\text{cyc}}\frac{(b+c)^2}{(a^2-1)^2}\geq\frac{1}{3}\left(\sum_{\text{cyc}}\frac{b+c}{a^2-1}\right)^2\geq\frac{1}{3}$$

thus the inequality becomes

$$\sum_{\text{cyc}}\left(2\frac{(b+c)(b-1)(c-1)}{(a^2-1)^2}+\frac{(b-1)^2(c-1)^2}{(a^2-1)^2}\right)\geq 3$$

or

$$\sum_{\text{cyc}} \frac{(b-1)(c-1)(bc+b+c+1)}{(a^2-1)^2} \geq 3$$

Since the inequality in the statement is symmetric, we can set $a \geq b \geq c$. Then we observe that

$$\left(\frac{(b-1)(c-1)}{(a-1)^2}, \frac{(c-1)(a-1)}{(b-1)^2}, \frac{(a-1)(b-1)}{(c-1)^2}\right)$$

and

$$\left(\frac{bc+b+c+1}{(a+1)^2}, \frac{ca+c+a+1}{(b+1)^2}, \frac{ab+a+b+1}{(c+1)^2}\right)$$

are equally sorted. This allows us to employ Chebyshev–inequality

$$\sum_{\text{cyc}} \frac{(b-1)(c-1)(bc+b+c+1)}{(a^2-1)^2} \geq \frac{1}{3}\sum_{\text{cyc}} \frac{(b-1)(c-1)}{(a-1)^2} \cdot \sum_{\text{cyc}} \frac{bc+b+c+1}{(a+1)^2}$$

Moreover by AGM we have

$$\sum_{\text{cyc}} \frac{(b-1)(c-1)}{(a-1)^2} \geq 3$$

and

$$\sum_{\text{cyc}} \frac{bc+b+c+1}{(a+1)^2} = \sum_{\text{cyc}} \frac{(b+1)(c+1)}{(a+1)^2} \geq 3$$

Bringing together the last three inequalities we obtain

$$\sum_{\text{cyc}} \frac{(b-1)(c-1)(bc+b+c+1)}{(a^2-1)^2} \geq \frac{1}{3}\ddot{3} \cdot 3 = 3$$

and we are done.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Prithwijit De, HBCSE, India.

J168. Let n be a positive integer. Find the least positive integer a such that the system

$$\begin{cases} x_1 + x_2 + \cdots + x_n = a \\ x_1^2 + x_2^2 + \cdots + x_n^2 = a \end{cases}$$

has no integer solutions.

Proposed by Dorin Andrica, "Babeş-Bolyai University", Cluj-Napoca, Romania

Solution by Lorenzo Pascali, Universita di Roma La Sapienza, Italy

First, we notice that if $x_i \neq 0, 1$ for an integer component x_i then $x_i^2 > x_i$ and we have a contradiction

$$a = x_1^2 + x_2^2 + \cdots + x_n^2 > x_1 + x_2 + \cdots + x_n = a.$$

Hence any component x_i is 0 or 1 and the system has integer solutions for $a = 1, \ldots, n$: take $x_1 = \cdots = x_a = 1$ and $x_{a+1} = \cdots = x_n = 0$. Therefore the least positive integer a such that the system has no integer solutions is $n + 1$:

$$x_1 + x_2 + \cdots + x_n \leq x_1^2 + x_2^2 + \cdots + x_n^2 \leq n < a = n + 1.$$

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy.

Senior problems

S163. (a) Prove that for each positive integer n there is a unique positive integer a_n such that

$$(1+\sqrt{5})^n = \sqrt{a_n} + \sqrt{a_n + 4^n}.$$

(b) When n is even, prove that a_n is divisible by $5 \cdot 4^{n-1}$ and find the quotient.

Proposed by Dorin Andrica, "Babeş-Bolyai University", Cluj-Napoca, Romania

First solution by G. C. Greubel, Newport News, VA

Let $2\alpha = 1+\sqrt{5}$. With this we have

$$2^n \alpha^n = \sqrt{a_n} + \sqrt{a_n + 4^n}. \tag{1}$$

Squaring both sides leads to

$$4^n(\alpha^{2n} - 1) = 2a_n + 2\sqrt{a_n(a_n + 4^n)}. \tag{2}$$

Subtracting $2a_n$ from both sides and squaring the resulting value leads to

$$[4^n(\alpha^{2n} - 1) - 2a_n]^2 = 4a_n(a_n + 4^n). \tag{3}$$

This is reduced to

$$\begin{aligned} a_n &= 4^{n-1}\left(\frac{\alpha^{2n} - 1}{\alpha^n}\right)^2 \\ &= 4^{n-1}\left(\alpha^n - (-1)^n \beta^n\right)^2 \\ &= 4^{n-1}\left(\alpha^{2n} + \beta^{2n} - 2\right) \\ &= 4^{n-1}(L_{2n} - 2) \end{aligned} \tag{4}$$

where L_m is the m^{th} Lucas number. Hence it has been shown that a_n is a positive integer and is given by

$$a_n = 4^{n-1}(L_{2n} - 2).$$

B) If n is an even value, say $n = 2m$, then

$$\begin{aligned} a_{2m} &= 4^{2m-1}(L_{4m} - 2) \\ &= 4^{2m-1} \cdot 5F_{2n}^2 \\ &= 5 \cdot 4^{m-1} \cdot (2^m F_{2m})^2. \end{aligned} \tag{5}$$

From this relation it is shown that a_{2m} is divisible by $5 \cdot 4^{m-1}$ and has the quotient value $(2^m F_{2m})^2$.

Second solution by the authors

(a) Let $(1+\sqrt{5})^n = x_n + y_n\sqrt{5}$, where x_n, y_n are positive integers, $n = 1, 2, \dots$ Then

$$(1-\sqrt{5})^n = x_n - y_n\sqrt{5}, n = 1, 2, \dots,$$

hence

$$x_n^2 - 5y_n^2 = (-4)^n, n = 1, 2, \dots \tag{1}$$

If n is even, consider $a_n = x_n^2 - 4^n$ and we have

$$\begin{aligned}\sqrt{a_n} + \sqrt{a_n + 4^n} &= \sqrt{x_n^2 - 4^n} + \sqrt{x_n^2} = \sqrt{5y_n^2} + \sqrt{x_n^2}\\ &= y_n\sqrt{5} + x_n = (1+\sqrt{5})^n.\end{aligned}$$

If n is odd, consider $a_n = 5y_n^2 - 4^n$ and we have

$$\begin{aligned}\sqrt{a_n} + \sqrt{a_n + 4^n} &= \sqrt{5y_n^2 - 4^n} + \sqrt{5y_n^2} = \sqrt{x_n^2} + \sqrt{5y_n^2}\\ &= x_n + y_n\sqrt{5} = (1+\sqrt{5})^n.\end{aligned}$$

(b) If n is even, then we have $a_n = x_n^2 - 4^n = 5y_n^2$, where

$$y_n = \frac{1}{2\sqrt{5}}[(1+\sqrt{5})^n - (1-\sqrt{5})^n]$$

$$= \frac{2^n}{2\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] = 2^{n-1}F_n,$$

where F_n is the n^{th} Fibonacci number. In this case we get $a_n = 5 \cdot 4^{n-1}F_n^2$, hence $5 \cdot 4^{n-1}|a_n$ and the quotient is F_n^2.

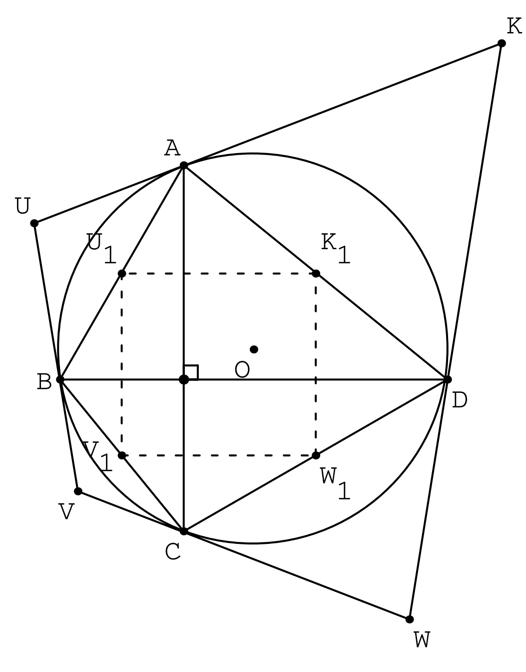
Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S164. Let $ABCD$ be a cyclic quadrilateral whose diagonals are perpendicular to each other. For a point P on its circumscribed circle denote by ℓ_P the line tangent to the circle at P. Let $U = \ell_A \cap \ell_B, V = \ell_B \cap \ell_C, W = \ell_C \cap \ell_D, K = \ell_D \cap \ell_A$. Prove that $UVWK$ is a cyclic quadrilateral.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Michel Bataille, France

Let U_1, V_1, W_1, K_1 be the midpoints of AB, BC, CD, DA, respectively. The Varignon parallelogram $U_1V_1W_1K_1$ of the quadrilateral $ABCD$ is a rectangle (because $AC \perp BD$), hence U_1, V_1, W_1, K_1 lie on a circle γ centered at the centre of the rectangle. Note that O, U_1, U are collinear (on the perpendicular bisector of AB) and that AB is the polar of U with respect to Γ. Similar results hold for V_1, W_1, K_1 and it follows that the inverses of U_1, V_1, W_1, K_1 in the circle Γ are U, V, W, K, respectively, so that U, V, W, K all lie on the inverse of the circle γ. Since U, V, W, K clearly cannot be collinear, the inverse of γ is a circle and so $UVWK$ is a cyclic quadrilateral.



Also solved by Ercole Suppa, Teramo, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Prithwijit De, HBCSE, India.

S165. Let I be the incenter of triangle ABC. Prove that

$$AI \cdot BI \cdot CI \geq 8r^3,$$

where r is the inradius of triangle ABC.

Proposed by Dorin Andrica, "Babeş-Bolyai University", Cluj-Napoca, Romania

Solution by Piriyathumwong P.,Bangkok, Thailand
Since

$$AI = \frac{r}{\sin\frac{A}{2}},\ BI = \frac{r}{\sin\frac{B}{2}},\ CI = \frac{r}{\sin\frac{C}{2}},$$

the inequality above is equivalent to $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{1}{8}$,which is immediately true because of the two well-known facts below:

$$\frac{r}{R} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\ \ , R \geq 2r$$

,where R is the circumradius of triangle ABC.

Also solved by Arkady Alt, San Jose, California, USA; Michel Bataille, France; Scott H. Brown, Auburn University Montgomery, USA; Ercole Suppa, Teramo, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; G.R.A.20 Math Problems Group, Roma, Italy.

S166. If $a_1, a_2, \dots, a_k \in (0,1)$, and k, n are integers such that $k > n \geq 1$, prove that the following inequality holds

$$\min\{a_1(1-a_2)^n, a_2(1-a_3)^n, \dots, a_k(1-a_1)^n\} \leq \frac{n^n}{(n+1)^{n+1}}.$$

Proposed by Marin Bancos, North University of Baia Mare, Romania

First solution by Arkady Alt, San Jose, California, USA

Let $M = \min\left\{a_1 (1-a_2)^n, a_2 (1-a_3)^n, \dots, a_k (1-a_1)^n\right\}$ and for any function $f(x,y)$ let

$$\sum_{cyc}^{k} f(a_1, a_2) = f(a_1, a_2) + f(a_2, a_3) + \cdots + f(a_k, a_1).$$

Since for any $x, y \in (0,1)$ by the AM–GM inequality

$$\sqrt[n+1]{nx(1-y)^n} \leq \frac{nx + n - ny}{n+1} = \frac{n(x-y)+n}{n+1}.$$

Then

$$\begin{aligned}
\sqrt[n+1]{M} &= \min\left\{\sqrt[n+1]{a_1(1-a_2)^n}, \sqrt[n+1]{a_2(1-a_3)^n}, \dots, \sqrt[n+1]{a_k(1-a_1)^n}\right\} \\
&\leq \frac{1}{k}\sum_{cyc}^{k} \sqrt[n+1]{a_1(1-a_2)^n} \\
&= \frac{1}{k\sqrt[n+1]{n}}\sum_{cyc}^{k} \sqrt[n+1]{na_1(1-a_2)^n} \\
&\leq \frac{1}{k(n+1)\sqrt[n+1]{n}}\sum_{cyc}^{k} (n(a_1-a_2)+n) \\
&= \frac{nk}{k(n+1)\sqrt[n+1]{n}} = \frac{n}{(n+1)\sqrt[n+1]{n}} \\
&= \frac{\sqrt[n+1]{n^n}}{(n+1)}.
\end{aligned}$$

Then $M \leq \dfrac{n^n}{(n+1)^{n+1}}$.

Second solution by the author

"Reductio ad absurdam"

Let's suppose that the inequality doesn't hold.

Therefore

$$a_1(1-a_2)^n > \frac{n^n}{(n+1)^{n+1}}$$

$$a_2(1-a_3)^n > \frac{n^n}{(n+1)^{n+1}}$$

.....

$$a_k(1-a_1)^n > \frac{n^n}{(n+1)^{n+1}}$$

Multiplying these relations up, we get

$$a_1 \cdot a_2 \cdot \ldots \cdot a_k \cdot (1-a_1)^n \cdot (1-a_2)^n \cdot \ldots \cdot (1-a_k)^n > \left[\frac{n^n}{(n+1)^{n+1}}\right]^k \quad (*)$$

But, for $a \in (0,1)$, we have

$$a(1-a)^n \le \frac{n^n}{(n+1)^{n+1}}$$

Let's prove this inequality.

$$a\cdot(1-a)^n = \frac{1}{n}\cdot n\cdot a\cdot(1-a)^n \overset{AM-GM}{\le} \frac{1}{n}\cdot\left[\frac{na+\overbrace{(1-a)+\cdots+(1-a)}^{n\ times}}{n+1}\right]^{n+1} = \frac{1}{n}\cdot\left(\frac{n}{n+1}\right)^{n+1} = \frac{n^n}{(n+1)^{n+1}}$$

The equality holds for: $na = 1-a \;\Leftrightarrow\; a = \frac{1}{n+1} \in (0,1)$

Using the proved inequality for $a_1, a_2, \ldots, a_k$, we get:

$$a_1(1-a_1)^n \le \frac{n^n}{(n+1)^{n+1}}$$

$$a_2(1-a_2)^n \le \frac{n^n}{(n+1)^{n+1}}$$

.....

$$a_k(1-a_k)^n \le \frac{n^n}{(n+1)^{n+1}}$$

Multiplying these relations up, we get

$$a_1 \cdot a_2 \cdot \cdots \cdot a_k \cdot (1-a_1)^n \cdot (1-a_2)^n \cdot \cdots \cdot (1-a_k)^n \le \left[\frac{n^n}{(n+1)^{n+1}}\right]^k$$

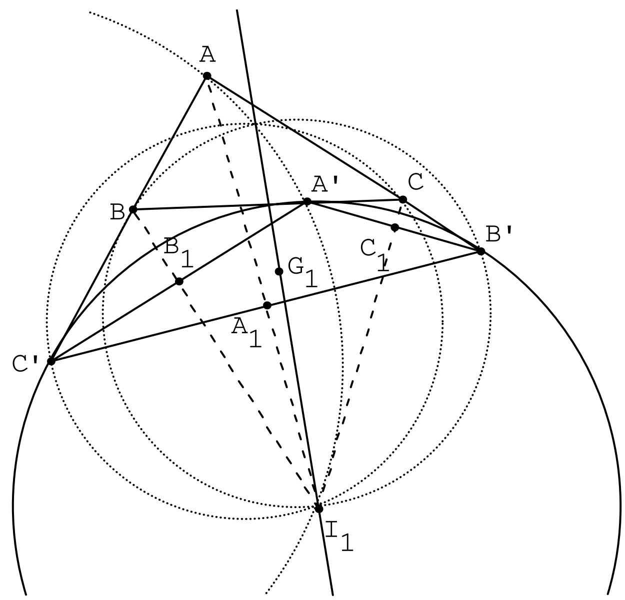
This inequality contradicts $(*)$, which follows from the initial assumption. Therefore, that assumption is false.

Also solved by Michel Bataille, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

S167. Let I_a be the excenter corresnponding to the side BC of triangle ABC. Denote by A', B', C' the tangency points of the excircle of center I_a with the sides BC, CA, AB, respectively. Prove that the circumcircles of triangles AI_aA', BI_aB', CI_aC' have a common point, different from I_a, situated on the line G_aI_a, where G_a is the centroid of triangle $A'B'C'$.

Proposed by Dorin Andrica, "Babeş-Bolyai University", Cluj-Napoca, Romania

First solution by Michel Bataille, France



For typographical reasons, I_a and G_a are denoted by I_1 and G_1 on the figure above.

Let γ be the excircle. Since $I_aA' = I_aC'$ an $BA' = BC'$, the line I_aB is the perpendicular bisector of $A'C'$ and intersects $A'C'$ in its midpoint B_1 . Since $A'C'$ is the polar of B with respect to γ, the inversion in the circle γ exchanges B_1 and B. Since B' is invariant under this inversion, the circumcircle of $\Delta I_aBB'$ inverts into the median $B'B_1$ of triangle $A'B'C'$. Similarly, the circumcircles of $\Delta I_aAA', \Delta I_aCC'$ invert into the medians $A'A_1, C'C_1$. As a result, the three circumcircles all pass through I_a and through the inverse of G_a (because G_a lies on the three medians $A'A_1, B'B_1, C'C_1$). The second result follows from the fact that the inverse of G_a is on the line through I_a and G_a.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain Let D' be the midpoint of $B'C'$. Now $AB' \perp I_aB'$, while $AI_a \perp B'C'$ where $B'D' = C'D'$ by symmetry around the internal bisector of angle A. Thus, triangles $AB'D'$ and $B'I_aD'$ are similar, hence $B'D' \cdot C'D' = B'D'^2 = I_aD' \cdot AD'$, and the power of D' with respect to the circumcircles of $A'B'C'$ and AI_aA' is the same, or D' lies on the radical axis of both circles, which is median $A'D'$.

Let E' be the midpoint of $C'A'$. $BA' \perp I_aA'$, while $BI_a \perp A'C'$ where $A'E' = C'E'$ by symmetry around the external bisector of angle B. Thus, triangles $BE'A'$ and $A'E'I_a$ are similar, hence $A'E' \cdot C'E' = A'E'^2 = BE' \cdot I_aE'$, and median $B'E'$ is the radical axis of the circumcircles of $A'B'C'$ and BI_aB'. Similarly, median $C'F'$ (F' is the midpoint of $A'B'$) is the radical axis of the circumcircles of $A'B'C'$ and CI_aC'.

Clearly, the point G_a where the medians $A'D'$, $B'E'$ and $C'F'$ meet, has the same power with respect to the four circumcircles; consider now the second point P where I_aG_a meets the circumcircle of AI_aA'. Since I_aG_a is the radical axis of the circumcircles of AI_aA' and BI_aB' because I_a, G_a have the same power with respect to both, then P also has the same power with respect to both, but since it is on the circumcircle of AI_aA', it is also on the circumcircle of BI_aB'. Similarly, it is also on the circumcircle of CI_AC'. The conclusion follows.

S168. Let $a_0 \geq 2$ and $a_{n+1} = a_n^2 - a_n + 1, n \geq 0$. Prove that

$$\log_{a_0}(a_n - 1)\log_{a_1}(a_n - 1)\cdots\log_{a_{n-1}}(a_n - 1) \geq n^n,$$

for all $n \geq 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

Proof Induction

$$\log_{a_0}(a_{n+1} - 1)\log_{a_1}(a_{n+1} - 1)\cdots\log_{a_{n-1}}(a_{n+1} - 1)\log_{a_n}(a_{n+1} - 1) \geq (n+1)^{n+1}$$

$$a_{n+1} - 1 = a_n(a_n - 1) \geq (a_n - 1)^2 \quad \Longrightarrow \quad \log_x(a_{n+1} - 1) \geq 2\log_x(a_n - 1)$$

This implies

$$\begin{aligned}
&\log_{a_0}(a_{n+1} - 1)\log_{a_1}(a_{n+1} - 1)\cdots\log_{a_{n-1}}(a_{n+1} - 1)\log_{a_n}(a_{n+1} - 1) \geq \\
&2^n\big(\log_{a_0}(a_n - 1)\log_{a_1}(a_n - 1)\cdots\log_{a_{n-1}}(a_n - 1)\big)\log_{a_n}(a_{n+1} - 1) \geq \\
&2^n n^n \log_{a_n}(a_{n+1} - 1) = 2^n n^n \log_{a_n}(a_n(a_n - 1)) = \\
&(2n)^n(1 + \log_{a_n}(a_n - 1)) \geq (n+1)^{n+1}
\end{aligned} \tag{1}$$

For $n \geq 4$ we have $(2n)^n \geq (n+1)^{n+1}$. Indeed

$$2^n \geq (n+1)e \geq (n+1)\left(1 + \frac{1}{n}\right)^n, \qquad \forall\, n \geq 4$$

We need to show yet the validity of our inequality for $n = 1, 2, 3$.

For $n = 1$ the inequality is

$$\log_{a_0}(a_0(a_0 - 1)) = 1 + \log_{a_0}(a_0 - 1) \geq 1$$

being $a_0 - 1 \geq 1$.

For $n = 2$ we have

$$\log_{a_0}(a_2 - 1)\log_{a_1}(a_2 - 1) \geq 4$$

or

$$\log_{a_0}\big(a_1(a_1 - 1)\big)\log_{a_1}\big(a_1(a_1 - 1)\big) \geq 4$$

namely

$$\log_{a_0}\big(a_1 a_0(a_0 - 1)\big)\log_{a_1}\big(a_1 a_0(a_0 - 1)\big) \geq 4 \tag{2}$$

We rewrite (2) as

$$\big(1 + \log_{a_0} a_1 + \log_{a_0}(a_0 - 1)\big)\big(1 + \log_{a_1} a_0 + \log_{a_1}(a_0 - 1)\big) \geq 4$$

which is implied by, use again $a_0 - 1 \geq 1$,

$$\left(1 + \log_{a_0} a_1\right)\left(1 + \log_{a_1} a_0\right) \geq 4$$

and this holds true since it may be written as $(1+x)(1+1/x) \geq 4$, and $x + 1/x \geq 2$, $x > 0$.

The last integer still remaining is $n = 3$ and by (1) we need to show

$$6^3(1 + \ln_{a_3}(a_3 - 1)) \geq 4^4$$

The first step is: $\ln_{a_3}(a_3 - 1)$ increases with $a_3 \geq 7$. To prove this let's write $\log_{a_3}(a_3 - 1) = \frac{\ln(a_3-1)}{\ln a_3}$ so that

$$\frac{d}{da_3}\frac{\ln(a_3-1)}{\ln a_3} = \frac{1}{\ln a_3}\left(\frac{1}{a_3-1} - \frac{\ln(a_3-1)}{a_3 \ln a_3}\right) > 0 \qquad \text{for} \quad a_3 \geq 7 \tag{3}$$

The monotonicity of $\ln_{a_3}(a_3 - 1)$ for $a_3 \geq 7$ implies that it is greater than or equal to $\ln_7 6$ and this in turn implies that it suffices to show

$$6^3(1 + \ln_7 6) \geq 4^4$$

which evidently holds true and we are done.

Also solved by Michel Bataille, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Prithwijit De, HBCSE, India; Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy; Piriyathumwong P., Bangkok, Thailand.

Undergraduate problems

U163. Find the minimum of $f(x,y,z) = x^2+y^2+z^2-xy-yz-zx$ over all triples (x,y,z) of positive integers for which 2010 divides $f(x,y,z)$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

If wlog z is odd and x,y are even, then z^2 is the only odd term, $f(x,y,z)$ is odd, hence not a multiple of 2010, while if wlog x,y are odd and z is even, then x^2, y^2, xy are the only odd terms, $f(x,y,z)$ is again odd. Therefore, x,y,z have the same parity, and we may define $u = \frac{x-y}{2}$, $v = \frac{y-z}{2}$, or

$$3s^2 + d^2 = 4(u^2+v^2+uv) = (x-y)^2 + (y-z)^2 + (x-y)(y-z) = f(x,y,z),$$

where $s = u+v$ and $d = u-v$, and if 2010 divides $f(x,y,z)$, then $4020 = 2^2 \cdot 3 \cdot 5 \cdot 67$ divides $f(x,y,z)$. Now, any perfect square leaves a remainder equal to $-1,0,1$ modulus 5, hence if d,s are not both multiples of 5, then $3s^2+d^2$ cannot be a multiple of 5, hence 5^2 divides $3s^2+d^2 = f(x,y,z)$, and 20100 divides $f(x,y,z)$. Defining $s' = \frac{s}{5}$ and $d' = \frac{d}{5}$, we find that

$$3s'^2 + d'^2 = \frac{f(x,y,z)}{25} = \frac{20100k}{25} = 804k.$$

But taking $s' = 16$, $d' = 6$, we find $3s'^2 + d'^2 = 768 + 36 = 804$, or $f(x,y,z) \geq 25 \cdot 804 = 20100$, with equality for example for $s = 80$ and $d = 30$, ie $u = 55$ and $v = 25$, or $f(z+160, z+50, z) = 20100$ for all positive integer z as it is easily checked by direct calculation.

Note: We have restricted ourselves to *positive* values of $f(x,y,z)$, since clearly $f(x,x,x) = 0$ is a multiple of 2010 for all positive integer x, making the problem trivial.

U164. Prove that $\varphi\left(2^{2010!}-1\right)$ ends in at least 499 zeros.

Proposed by Dorin Andrica, "Babeş-Bolyai University", Cluj-Napoca, Romania

Solution by G.R.A.20 Math Problems Group, Roma, Italy

We will prove that $\varphi(2^{2010!}-1)$ ends with 501 zeros by showing that it is divisible by 5^{501} and 2^{501}.

Since

$$\sum_{k=1}^{\infty}\left\lfloor\frac{2010}{5^k}\right\rfloor = 501 \qquad \text{and} \qquad \sum_{k=1}^{\infty}\left\lfloor\frac{2010}{2^k}\right\rfloor = 2002,$$

it follows that $2010! = 4\cdot 5^{501}\cdot a = 2^{2002}\cdot b$ for some positive integers a and b.

By Euler's Theorem

$$2^{2010!}-1 = 2^{4\cdot 5^{501}\cdot a}-1 = (2^a)^{\varphi(5^{502})}-1 \equiv 0 \pmod{5^{502}}$$

which implies that 5^{501} divides $\varphi(2^{2010!}-1)$.

Now we show that 2^k divides $\varphi(2^{2^k\cdot b}-1)$ for all $k\geq 1$. For $k=1$ we have that $2^{2\cdot b}-1$ is odd and 2 divides $\varphi(2^{2\cdot b}-1)$. Moreover, since $\gcd(2^{2^{k-1}\cdot b}-1, 2^{2^{k-1}\cdot b}+1)=1$,

$$\varphi(2^{2^k\cdot b}-1) = \varphi(2^{2^{k-1}\cdot b}-1)\cdot\varphi(2^{2^{k-1}\cdot b}+1)$$

and 2^{k-1} divides $\varphi(2^{2^{k-1}\cdot b}-1)$ by inductive hypothesis and 2 divides $\varphi(2^{2^{k-1}\cdot b}+1)$ because $2^{2^{k-1}\cdot b}+1$ is odd. Hence 2^{2002} divides $\varphi(2^{2010!}-1)$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Neacsu Adrian, Pitesti, Romania.

U165. Let $G = \{A_1, A_2, \dots, A_m\} \subset M_n(\mathbb{R})$ such that $(G, \cdot)$ is a group. Prove that $Tr(A_1 + A_2 + \cdots + A_m)$ is an integers divisible by m.

Proposed by Mihai Piticari, "Dragos Voda" National College, Campulung Moldovenesc, Romania

Solution by Michel Bataille, France

Let $B = A_1 + A_2 + \cdots + A_m$. Since $(G, \cdot)$ is a group, for any fixed $j \in \{1, 2, \dots, m\}$ the mapping $A \mapsto AA_j$ is a bijection from G onto G. It follows that $BA_j = B$, and since this is true for $j = 1, 2, \dots, m$, we have

$$B^2 = B(A_1 + +A_2 + \cdots + A_m) = BA_1 + BA_2 + \cdots + BA_m = mB.$$

Now, the matrix $C = \frac{1}{m} B$ is idempotent since

$$C^2 = \frac{1}{m^2} B^2 = \frac{1}{m^2} (mB) = C,$$

hence $Tr(C) = \text{rank}(C)$. Thus, $Tr(B) = m \cdot (\text{rank}(C))$, a multiple of m.

Also solved by Moubinool Omarjee, Paris, France; Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy

U166. Find all functions $f : [0, \infty) \to [0, \infty)$ such that

(a) f is multiplicative

(b) $\lim_{x\to\infty} f(x)$ exists, is finite, and different from 0.

Proposed by Mihai Piticari, "Dragos Voda" National College, Campulung Moldovenesc, Romania

First solution by Emanuele Natale, Università di Roma "Tor Vergata", Roma, Italy Let $\lim_{t\to\infty} f(t) = c \neq 0$ and let $a > 0$, then

$$c = \lim_{t\to\infty} f(at) = \lim_{t\to\infty} f(a)f(t) = f(a) \lim_{t\to\infty} f(t) = f(a)c$$

which implies that $f(a) = 1$. Hence f is identically equal to 1 in $(0, +\infty)$, whereas $f(0)$ can assume any nonnegative real number. It's trivial to check that such functions verify the assumptions.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain Since f is multiplicative, for any non-negative real x we have

$$f(x)f(1) = f(x \cdot 1) = f(x), \qquad f(x)\left(f(1) - 1\right) = 0.$$

If $f(1) \neq 1$, then $f(x) = 0$ for all non-negative real x, in contradiction with condition (b), hence $f(1) = 1$.

Since $f(xy) = f(x)f(y)$ for all non-negative reals x, y, after trivial induction we find that, for any positive integer n and any non-negative integer x, we have $f(x^n) = (f(x))^n$. Take any $y > 1$, and assume that $f(y) = a > 1$. Then,

$$\lim_{x\to\infty} f(x) = \lim_{n\to\infty} f(y^n) = \lim_{n\to\infty} a^n = \infty,$$

in contradiction with condition (b). Similarly, if $f(y) = a < 1$, we find $\lim_{x\to\infty} f(x) = 0$, again in contradiction with condition (b), or $f(y) = 1$ for all $y > 1$, and $\lim_{x\to\infty} f(x) = 1$, finite and nonzero.

For any $0 < z < 1$, $y = \frac{1}{z} > 1$, or $f(z) = f(y)f(z) = f(yz) = f(1) = 1$. We conclude that $f(x) = 1$ for all positive real x. Now, $f(0) = f(0^2) = (f(0))^2$, and either $f(0) = 0$ or $f(0) = 1$. In the second case, $f(x)$ is trivially multiplicative since $f(xy) = 1 = 1^2 = f(x)f(y)$ for all non-negative reals x, y, while in the first case, if wlog $y = 0$, we find for any non-negative real x that $f(x \cdot 0) = f(0) = 0 = f(x) \cdot 0 = f(x)f(0)$. There are therefore two functions that satisfy simultaneously both conditions, $f(x) = 1$ for any non-negative real x, and $f(x) = 1$ for all positive x and $f(0) = 0$.

U167. Let $f:[0,1]\to\mathbb{R}$ be a continuously differentiable function such that $f(1)=0$. Prove that

$$\left|\int_0^1 xf(x)dx\right|\le\frac{1}{6}\max_{x\in[0,1]}|f'(x)|.$$

Proposed by Duong Viet Thong, National Economics University, Ha Noi, Vietnam

Solution by Arkady Alt, San Jose, California, USA

Using integration by parts we obtain $\int_0^1 xf(x)\,dx=\left(\frac{x^2}{2}\cdot f(x)\right)_0^1-\int_0^1\frac{x^2f'(x)}{2}dx=$ $-\frac{1}{2}\int_0^1 x^2f'(x)\,dx$. Since by condition $f(x)$ is continuously differentiable then $M:=\max\limits_{x\in[0,1]}|f'(x)|$ and, therefore, $\left|\int_0^1 xf(x)\,dx\right|=\left|-\frac{1}{2}\int_0^1 x^2f'(x)\,dx\right|=$ $\frac{1}{2}\left|\int_0^1 x^2f'(x)\,dx\right|\le\frac{1}{2}\int_0^1 x^2|f'(x)|\,dx\le\frac{M}{2}\int_0^1 x^2dx=\frac{1}{6}\max\limits_{x\in[0,1]}|f'(x)|..$

Also solved by Michel Bataille, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

U168. Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function on (a, b) and let $\max_{x\in[a,b]} |f''(x)| = M$. Prove that

$$\left|\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a)\right| \leq \frac{(b-a)^3}{24}M.$$

Proposed by Duong Viet Thong, National Economics University, Ha Noi, Vietnam

Solution by Michel Bataille, France

Let $\Delta = \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a)$. From the easily verified result $\int_a^b \left(x - \frac{a+b}{2}\right) dx = 0$, we deduce

$$\Delta = \int_a^b \left(f(x) - f\left(\frac{a+b}{2}\right) - \left(x - \frac{a+b}{2}\right) f'\left(\frac{a+b}{2}\right)\right) dx.$$

Now, for any $x \in [a, b]$, we have

$$f(x) = f\left(\frac{a+b}{2}\right) + \left(x - \frac{a+b}{2}\right) f'\left(\frac{a+b}{2}\right) + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 f''(\theta)$$

for some real θ between x and $\frac{a+b}{2}$, hence

$$\left|f(x) - f\left(\frac{a+b}{2}\right) - \left(x - \frac{a+b}{2}\right) f'\left(\frac{a+b}{2}\right)\right| \leq \frac{M}{2}\left(x - \frac{a+b}{2}\right)^2.$$

It follows that

$$|\Delta| \leq \frac{M}{2}\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx = \frac{M}{6}\left(\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3\right) = \frac{(b-a)^3}{24} M.$$

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; G.R.A.20 Math Problems Group, Roma, Italy; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

Olympiad problems

O163. Prove that the equation

$$\frac{x^3+y^3}{x-y}=2010$$

is not solvable in positive integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, "Babeş-Bolyai University", Cluj-Napoca, Romania

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain Assume that x, y have opposite parity, then x^3+y^3 is odd, and 2010 is odd, contradiction, hence x, y have the same parity and $x-y$ is even. If x, y are both odd, $x^2-xy+y^2=\frac{x^3+y^3}{x+y}$ is odd, hence 2 divides $x+y$ with multiplicity $a+1$, where a is the multiplicity with which 2 divides $x-y$. Now, $(x+y)+(x-y)=2x$ is divided by 2 with multiplicity 1, hence $a=1$, or if x, y are both odd, $x-y$ is divisible by 2 but not by 4, and $x+y$ is divisible by 4 but not by 8. If x, y are both even, and both are divided by 2 with different multiplicity, then $x+y$ and $x-y$ are both divided by the lowest of both multiplicities, hence $\frac{xy(x+y)}{x-y}=2010-x^2+y^2$ is a multiple of 4, or since x^2, y^2 are also multiples of 4, 2010 is a multiple of 4, absurd. Therefore, x, y are divisible by 2 with the same multiplicity a, and $x+y$, $x-y$ are divisible by 2 with multiplicity at least $a+1$, hence $x-y$ is divisible by 2 with multiplicity at least $3a \geq 3$.

It is well known that any perfect cube leaves remainder $-1, 0, 1$ modulus 9, or since x^3+y^3 is a multiple of 3 because 3 divides 2010, then x^3+y^3 is a multiple of 9, and $x-y$ is a multiple of 3 because 2010 is not a multiple of 9. If x, y are not multiples of 3, since $x-y$ divides $xy(x+y)$, then $x+y$ must be a multiple of 3, or $2x=(x+y)+(x-y)$ is a multiple of 3, contradiction, hence x, y are multiples of 3, hence x^3+y^3 is a multiple of 27, and $x-y$ must be a multiple of 9.

Since $45^2=2025>2010=\frac{x^3+y^3}{x-y}>x^2$, we have $x \leq 44$, or since $y \geq 1$, then $x-y \leq 43$ must be an even multiple of 9, divisible by 2 if x, y are both odd, or divisible by at least $2^3=8$ if x, y are both even. We conclude that $x-y=18$, and x, y are both odd. Writing x as a function of y, the proposed equation becomes

$$\frac{y^3}{27}+y^2+18y=562, \qquad , \qquad z^3+9z^2+54z=562,$$

where we have defined $z=\frac{y}{3}$ because y is clearly divisible by 3. Note now that the LHS increases strictly with z, and if $z=5$ then $z^3+9z^2+54z=125+225+270=620>562$, while if $z=4$ then $z^3+9z^2+54z=64+144+216=424<562$. It follows that no positive integral solutions exist for the proposed equation.

Second solution by the authors

Assume that the equation is solvable in positive integers. It is clear that $x>y$. We can write

$$2010=\frac{x^3+y^3}{x-y}>\frac{x^3-y^3}{x-y}=x^2+xy+y^2=(x-y)^2+3xy>(x-y)^2,$$

and get $x - y < \sqrt{2010}$. It follows $x - y \le 44$.

On the other hand, we have

$$2010 = \frac{x^3 + y^3}{x - y} \ge \frac{x^3 + y^3}{44} = \frac{3}{44}(\frac{x^3 + y^3}{3}) \ge \frac{3}{44}(\frac{x + y}{3})^3,$$

hence we obtain $x + y \le 96$.

The equation is equivalent to

$$(x + y)(x^2 - xy + y^2) = 2 \cdot 3 \cdot 5 \cdot 67 \cdot (x - y).$$

If $x+y$ is divisible by 67, then necessarily $x+y = 67$, since $x+y \le 96$ and 67 is a prime number. In this case we get $x^2 - xy + y^2 = 30(x - y)$ and $x + y = 67$, hence $(x + y)^2 - 3xy = 30(x - y)$. That is $67^2 = 30(x-y)+3x(67-x)$, equation with no integer solutions, since 67 is not divisible by 3.

If $x^2 - xy + y^2$ is divisible by 67, then $x^2 - xy + y^2 = 67k$ for some positive integer k. The equation is equivalent to

$$k(x + y) = 30(x - y),$$

that is $(30 - k)x = (30 + k)y$. It follows $y = \frac{30-k}{30+k}x$, hence

$$x^2[(30 + k)^2 - (30 - k)(30 + k) + (30 - k)^2] = 67k(30 + k)^2.$$

and we get

$$x^2(3k^2 + 30^2) = 67k(30 + k)^2.$$

It is clear that k is divisible by 3, hence we have $k = 3a$ for some positive integer $1 \le a \le 9$. Then

$$x^2(3a^2 + 100) = 67 \cdot a \cdot (a + 10)^2.$$

Because x^2 can't be divisible by 67 it follows that $3a^2 + 100$ is divisible by 67. Replacing $a = 1, 2, ..., 9$ is easy to see that $3a^2 + 100$ has no this property, hence the equation is not solvable.

Also solved by Raul A. Simon, Chile.

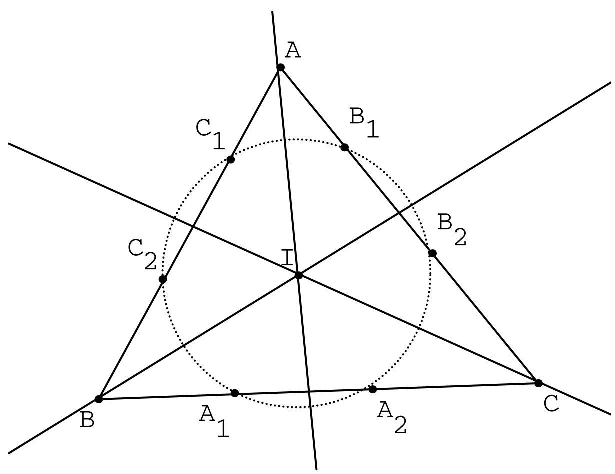
O164. Let ABC be a triangle and let A_1 be a point on the side BC. Starting with A_1 construct reflections in one of the angle bisectors of triangle such that the next point lies on the other side of the triangle. The process is done in one direction: either clockwise or counterclockwise. Thus at the first step we construct an isosceles triangle A_1CB_1 with point B_1 lying on AC. At the second step we construct an isosceles triangle B_1AC_1 with point C_1 on AB. In fact we get a sequence of points $A_1, B_1, C_1, A_2, \ldots$.

(a) Prove that the process terminates in six steps, that is $A_1 \equiv A_3$

(b) Prove that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on the same circle.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Michel Bataille, France



Let R_{MN} denote the reflection in the line MN and let I be the incentre of ΔABC.
(a) As the product of three opposite isometries (reversing the orientation), the isometry $R = R_{BI} \circ R_{AI} \circ R_{CI}$ is opposite as well and since $R(I) = I$, R must be a reflection in a line ℓ. Since $R(A_1) = A_2$, ℓ must be the perpendicular bisector of the line segment A_1A_2 in the general case when $A_1 \neq A_2$ (and IA_1 if $A_1 = A_2$). Thus, $R = R_\ell$ and $R_{BI} \circ R_{AI} \circ R_{CI} \circ R_{BI} \circ R_{AI} \circ R_{CI} = R_\ell \circ R_\ell = Id$ where Id denotes the identity of the plane. As a result, $A_3 = Id(A_1) = A_1$.
(b) Since ℓ is the the perpendicular bisector of A_1A_2, we have $IB_1 = IA_1 = IA_2 = IC_1$. Similarly, if ℓ' denote the perpendicular bisector of B_1B_2, we have $R_{\ell'} = R_{CI} \circ R_{BI} \circ R_{AI}$ and so $IC_2 = IB_2 = IB_1$.
In conclusion, $IA_1 = IA_2 = IB_1 = IB_2 = IC_1 = IC_2$ and the six points $A_1, A_2, B_1, B_2, C_1, C_2$ all lie on a circle with centre I.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Raul A. Simon, Chile.

O165. Let R and r be the circumradius and the inradius of a triangle ABC with the lengths of sides a, b, c. Prove that

$$2 - 2\sum_{cyc}\left(\frac{a}{b+c}\right)^2 \leq \frac{r}{R}.$$

Proposed by Dorin Andrica, "Babeş-Bolyai University", Cluj-Napoca, Romania

Solution by Arkady Alt, San Jose, California, USA

Note that $2 - 2\sum_{cyc}\left(\frac{a}{b+c}\right)^2 \leq \frac{r}{R} \iff 6 - 2\sum_{cyc}\left(\frac{a}{b+c}\right)^2 \leq 4 + \frac{r}{R} \iff$

$2\left(3 - \sum_{cyc}\left(\frac{a}{b+c}\right)^2\right) \leq 4 + \frac{r}{R} \iff 2\sum_{cyc}\frac{(b+c)^2 - a^2}{(b+c)^2} \leq 4 + \frac{r}{R}.$

Since $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$ and $\frac{1}{(b+c)^2} \leq \frac{1}{4bc}$ then $\frac{(b+c)^2 - a^2}{2bc} = 1 + \cos A$

$2\sum_{cyc}\frac{(b+c)^2 - a^2}{(b+c)^2} \leq \sum_{cyc}\frac{(b+c)^2 - a^2}{2bc} = \sum_{cyc}(1 + \cos A) = 4 + \frac{r}{R}.$

Remark.

Let l_a, l_b, l_c be angle bisectors of a triangle ABC.Noting that $\frac{(b+c)^2 - a^2}{(b+c)^2} = \frac{al_a^2}{abc}$ we can rewrite original inequality in such form $2\sum_{cyc}\frac{al_a^2}{abc} \leq 4 + \frac{r}{R} \iff 2\sum_{cyc}\frac{al_a^2}{4Rrs} \leq 4 + \frac{r}{R} \iff$

$$\frac{al_a^2 + bl_b^2 + cl_c^2}{a+b+c} \leq r\left(4R + r\right).$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Michel Bataille, France.

O166. The incircle σ of triangle ABC with incenter I is tangent to sides BC and AC at points A_1 and B_1, respectively. Points A_2 and B_2 are diametrically opposite to A_1 and B_1 in σ. Let A_3 and B_3 be the intersection points of AA_2 with BC and BB_2 with AC, respectively. Let M be the midpoint of side AC and let N be the midpoint of A_1A_3. Line MI meets BB_1 in T and line AT meets BC in P. Let $Q \in (BC), R$ be the intersection of lines AB and QB_1 and $NR \cap AC = \{S\}$. Prove that $[AS] = 2[SM]$ if and only if $[BP] = [PQ]$.

Proposed by Andrei Razvan Baleanu, "George Cosbuc", Motru, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Consider the parallel to BC through A_2, clearly the incircle of ABC is the excircle of the triangle formed by this line, AB and AC, and this triangle is homothetic to ABC, with center of homothety in A, or $A_3 = AA_2 \cap BC$ is the contact point of the excircle of ABC tangent to segment BC and to lines AB and AC. It is well known that the contact points of the incircle and the excircle on a given triangle side are symmetric with respect to its midpoint, so N is also the midpoint of BC. Note that $AS = 2SM$ iff $3AS = 2AM = AC$, or $\frac{AS}{SC} = \frac{1}{2}$. Therefore, since $\frac{CN}{NB} = 1$, and R, S, N are collinear, by Menelaus' theorem, $AS = 2SM$ iff $BR = 2AR$. Now, it is well known that $AB_1 = \frac{b+c-a}{2}$ and $CB_1 = \frac{a+b-c}{2}$, or $\frac{AB_1}{B_1C} = \frac{b+c-a}{a+b-c}$, and again by Menelaus' theorem, and using that Q, B_1, R are collinear, $AS = 2SM$ iff $\frac{BQ}{CQ} = \frac{2(b+c-a)}{a+b-c}$, or equivalently, iff $BQ = \frac{2a(b+c-a)}{3b+c-a}$.

Using exact trillinear coordinates, $M \equiv \left(\frac{h_a}{2}, 0, \frac{h_c}{2}\right)$, where h_a, h_b, h_c are the lengths of the respective altitudes from A, B, C, while $I \equiv (r, r, r)$, r being the inradius. It follows that line MI has equation $a\alpha + (c-a)\beta - c\gamma = 0$. Using again exact trillinear coordinates, $B \equiv (0, h_b, 0)$, while $B_1 \equiv \left(\frac{a+b-c}{2}\sin C, 0, \frac{b+c-a}{2}\sin A\right)$, or line BB_1 has equation $a(b+c-a)\alpha = c(a+b-c)\gamma$, or point T has (non-exact) trillinear coordinates $T \equiv (c(a+b-c), 2ac, a(b+c-a))$. Since $A \equiv (1, 0, 0)$ in non-exact trillinear coordinates, line AT has equation $2c\gamma = (b+c-a)\beta$, or point P satisfies

$$\frac{b \cdot BP}{c \cdot CP} = \frac{BP \sin B}{CP \sin C} = \frac{\gamma}{\beta} = \frac{b+c-a}{2c},$$

or $\frac{CP}{BP} = \frac{2b}{b+c-a}$, yielding $\frac{BC}{BP} = \frac{3b+c-a}{b+c-a}$, and $BP = \frac{a(b+c-a)}{3b+c-a}$. It clearly follows that $BQ = 2BP$, and equivalently $BP = PQ$, iff $AS = 2SM$.

O167. Prove that in any convex quadrilateral $ABCD$,

$$\cos\frac{A-B}{4}+\cos\frac{B-C}{4}+\cos\frac{C-D}{4}+\cos\frac{D-A}{4}\geq 2+\frac{1}{2}(\sin A+\sin B+\sin C+\sin D).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We can write

$$2+\sin A+\sin B=2+2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\leq 2+2\cos\frac{A-B}{2}=4\cos^2\frac{A-B}{4}\leq$$

$$\leq 4\cos\frac{A-B}{4},$$

with equality iff $A+B=180^\circ$ and simultaneously $A=B$, ie, equality holds iff $A=B=90^\circ$. Adding the cyclic permutations of both sides of this resulting inequality, we obtain the proposed inequality with both sides multiplied by 4. The conclusion follows, equality holds in the proposed inequality iff $ABCD$ is a rectangle.

O168. Given a convex polygon $A_1A_2\ldots A_n, n \geq 4$, denote by R_i the radius of the circumcircle of triangle $A_{i-1}A_iA_{i+1}$, where $i = 2, 3, \ldots, n$ and A_{n+1} is the vertex A_1. Given that $R_2 = R_3 = \cdots = R_n$, prove that the polygon $A_1A_2\ldots A_n$ is cyclic.

Proposed by Nairi Sedrakyan, Armenia

Solution by Raul A. Simon, Chile

That the polygon $A_1A_2\ldots A_n$ is convex means that $\angle A_{i-1}A_iA_{i+1}$ is obtuse - if you take the smallest of the two angles formed at the vertex. Therefore, the circumcenter O_i of $\angle A_{i-1}A_iA_{i+1}$ is exterior to it. In fact, all circumcenters O_i lie in a zone Z that is exterior to all triangles $A_{i-1}A_iA_{i+1}$ (where $i = 2, 3, \ldots, n$ and A_{n+1} is the vertex A_1) and interior to the polygon $A_1A_2\ldots A_n$. We have then

$$O_2A_1 = O_2A_2 = O_2A_3 = R,$$

and

$$O_3A_2 = O_3A_3 = O_3A_4 = R,$$

etc. We see that O_2 is the intersection of two arcs of circle of radius R, centered at A_2 and A_3; and that O_3 is determined in exactly the same way. Therefore, since O_2 and O_3 lie on the same side of A_2A_3, O_2 and O_3 must coincide. Repeating this reasoning, we find that all circumcenters must coincide in a unique circumcenter O common to all vertices. The circle $O(O, R)$ is the circumcircle of the polygon; therefore, the latter is cyclic.