Junior problems

J127. Let $a_1,\ldots,a_n$ be positive real numbers such that $\displaystyle\sum_{i=1}^{n}\frac{1}{a_i^2+1}=n-1$. Prove that

$$\sum_{1\le i<j\le n} a_ia_j \le \frac{n}{2}.$$

Proposed by Tuan Le, Fairmont High-School, Anaheim, USA

Solution by Manh Dung Nguyen, Hanoi, Vietnam

From the hypothesis and the Cauchy-Schwarz Inequality, we have

$$1=\sum_{i=1}^{n}\frac{a_i^2}{a_i^2+1}\ge\frac{\left(\sum_{i=1}^{n}a_i\right)^2}{n+\sum_{i=1}^{n}a_i^2}$$

Thus

$$\left(\sum_{i=1}^{n}a_i\right)^2-\sum_{i=1}^{n}a_i^2\le n$$

On the other hand, we have

$$2\sum_{1\le i<j\le n}a_ia_j=\left(\sum_{i=1}^{n}a_i\right)^2-\sum_{i=1}^{n}a_i^2$$

Therefore

$$\sum_{1\le i<j\le n}a_ia_j\le\frac{n}{2}$$

Equality holds when $a_1=a_2=\cdots=a_n=\frac{1}{\sqrt{n-1}}$.

Also solved by Arkady Alt, San Jose, California, USA; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Michel Bataille, France; Christophe Debry.

J128. Consider the sequences are given

(a) $(a_n)_{n\in\mathbb{N}^*} : 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots, 1, 2, 3, \ldots, p-1, p, \ldots$

(b) $(b_n)_{n\in\mathbb{N}^*} : 1, 2, 1, 3, 2, 1, 4, 3, 2, 1, \ldots, p, p-1, p-2, \ldots, 2, 1, \ldots$

How many of the first 2009 terms of these sequences are equal?

Proposed by Marian Teler, Costesti, Romania and Marin Ionescu, Pitesti, Romania

Solution by Christophe Debry

For every $k \in \mathbb{N}^*$, there exists a unique $n_k \in \mathbb{N}^*$ such that $\frac{1}{2}n_k(n_k - 1) < k \leq \frac{1}{2}n_k(n_k + 1)$. It's not hard to prove that $a_k = k - \frac{1}{2}n_k(n_k - 1)$ and $b_k = \frac{1}{2}n_k(n_k + 1) + 1 - k$, for all $k \in \mathbb{N}^*$. Equal terms of the sequences satisfy $a_k = b_k$ for a $k \in \mathbb{N}^*$. Using the above formulae about a_k and b_k, this is equivalent with $2k = n_k^2 + 1$. We conclude that $a_k = b_k$ if and only if there exists a positive integer n such that $2k = n^2 + 1$. Clearly, n must be odd, and so there should exist a m such that $2k = (2m+1)^2+1$, and thus $k = 2m^2+2m+1$. We find that the only equal terms are those with index $2 \cdot 1^2 + 2 \cdot 1 + 1 = 5$, 13, 25, …, $2 \cdot 31^2 + 2 \cdot 31 + 1 = 1985$. The total number of equal terms is therefore 31.

J129. Given a nondegenerate triangle ABC, consider circles $\Gamma_a, \Gamma_b, \Gamma_c$ with diameters BC, CA, and AB, respectively. For which triangles ABC are $\Gamma_a, \Gamma_b, \Gamma_c$ concurrent?

Proposed by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Solution by Christophe Debry, Belgium

Let C and D be the intersections of Γ_a and Γ_b. As $\angle BDC = \angle CDA = 90^\circ$ (as D lies on circles with diameter BC and CA), we get that $\angle BDA = 90^\circ + 90^\circ = 180^\circ$, and so $D \in AB$. So suppose that Γ_a, Γ_b and Γ_c are concurrent. Then either $C \in \Gamma_c$ or $D \in \Gamma_c$. The first case gives a right triangle with right angle at C. The second case gives either $D = A$ or $D = B$, because $D \in AB \cap \Gamma_c = \{A, B\}$. For $D = A$, we get a right angle at A, and for $D = B$ we get a right angle at B. The only triangles for which Γ_a, Γ_b and Γ_c are concurrent, are therefore the right triangles. (One easily checks that these triangles satisfy the conditions from the problem.)

J130. Consider a triangle ABC. Let D the orthogonal projection of A onto BC and let E and F be points on lines AB and AC respectively such that $\angle ADE = \angle ADF$. Prove that the lines AD, BF, and CE are concurrent.

Proposed by Francisco Javier García Capitán, Spain

Solution by Ercole Suppa, Teramo, Italy

We will prove a lemma first:

Lemma. If P is a point on the side BC of a triangle $\triangle ABC$ we have

$$\frac{PB}{PC} = \frac{AB}{AC} \cdot \frac{\sin \angle PAB}{\sin \angle PAC}$$

Proof.

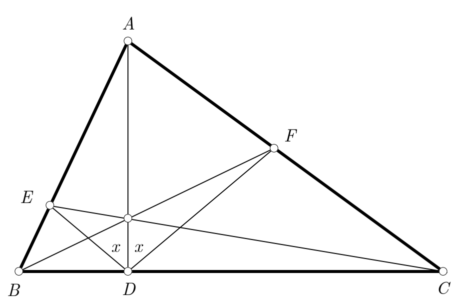
In triangles $\triangle PAB$ and $\triangle PAC$, the law of sines gives

$$\frac{PB}{\sin \angle PAB} = \frac{AB}{\sin \angle APB}$$

$$\frac{PC}{\sin \angle PAC} = \frac{AC}{\sin(180^\circ - \angle APB)} = \frac{AC}{\sin \angle APB}$$

Dividing the above relations we get the desired result. ■

Coming back to the problem, let us denote $x = \angle ADE = \angle ADF$, as shown in figure.



By the Lemma we have

$$\frac{AE}{EB} = \frac{AD}{BD} \cdot \frac{\sin x}{\sin(90^\circ - x)}$$

$$\frac{CF}{FA} = \frac{DC}{AD} \cdot \frac{\sin(90^\circ - x)}{\sin x}$$

Therefore

$$\frac{AE}{EB} \cdot \frac{BD}{DC} \cdot \frac{CF}{FA} = \frac{AD}{BD} \cdot \frac{\sin x}{\sin(90^\circ - x)} \cdot \frac{BD}{DC} \cdot \frac{DC}{AD} \cdot \frac{\sin(90^\circ - x)}{\sin x} = 1$$

so, by Ceva's theorem the lines AD, BF, and CE are concurrent and we are done. $\square$

Also solved by Gennaro Rispoli, Italy; Michel Bataille, France; Salem Malikic, Bosnia and Herzegovina.

J131. Let P be a point inside a triangle ABC and let d_a, d_b, d_c be the distances from point P to the triangle's sides. Prove that

$$d_a \cdot h_a^2 + d_b \cdot h_b^2 + d_c \cdot h_c^2 \geq (d_a + d_b + d_c)^3,$$

where h_a, h_b, h_c are the altitudes of the triangle.

Proposed by Magkos Athanasios, Kozani, Greece

First solution by Magkos Athanasios, Kozani, Greece

First need the following

Lemma: If $x, y, z, a, b, c > 0$ we have

$$\frac{x^3}{a^2} + \frac{y^3}{b^2} + \frac{z^3}{c^2} \geq \frac{(x+y+z)^3}{(a+b+c)^2}.$$

The proof follows from Hölder's Inequality. Indeed, we have

$$(a+b+c)^{\frac{2}{3}} (\frac{x^3}{a^2} + \frac{y^3}{b^2} + \frac{z^3}{c^2})^{\frac{1}{3}} \geq x + y + z$$

and the result follows. We now prove the initial inequality. Using the above lemma we have

$$\frac{d_a}{a^2} + \frac{d_b}{b^2} + \frac{d_c}{c^2} = \frac{d_a^3}{(ad_a)^2} + \frac{d_b^3}{(bd_b)^2} + \frac{d_c^3}{(cd_c)^2} \geq \frac{(d_a + d_b + d_c)^3}{(ad_a + bd_b + cd_c)^2} = \frac{(d_a + d_b + d_c)^3}{4S^2},$$

since $ad_a + bd_b + cd_c = 2S$, where S denotes the area of the triangle ABC. The result follows upon substituting $a = \frac{2S}{h_a}, b = \frac{2S}{h_b}, c = \frac{2S}{h_c}$.

Second solution by Ercole Suppa, Teramo, Italy

By denoting with Δ be the area of $\triangle ABC$, we have

$$\frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c} = \frac{d_a}{\frac{2\Delta}{a}} + \frac{d_a}{\frac{2\Delta}{b}} + \frac{d_a}{\frac{2\Delta}{c}} = \frac{ad_a + bd_b + cd_c}{2\Delta} = 1 \qquad (1)$$

By using the Hölder's inequality

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} {x_i}^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} {y_i}^q \right)^{\frac{1}{q}}$$

with $p = 3$, $q = \frac{3}{2}$ and

$$x_1 = \left(d_a h_a^2\right)^{\frac{1}{3}}, \qquad x_2 = \left(d_b h_b^2\right)^{\frac{1}{3}}, \qquad x_3 = \left(d_c h_c^2\right)^{\frac{1}{3}}$$

$$y_1 = \left(\frac{d_a}{h_a}\right)^{\frac{2}{3}}, \qquad y_2 = \left(\frac{d_b}{h_b}\right)^{\frac{2}{3}}, \qquad y_3 = \left(\frac{d_c}{h_c}\right)^{\frac{2}{3}}$$

we get

$$\begin{aligned} d_a + d_b + d_c &= \sum \left(d_a h_a^2\right)^{\frac{1}{3}} \left(\frac{d_a}{h_a}\right)^{\frac{2}{3}} \le \left(\sum d_a h_a^2\right)^{\frac{1}{3}} \left(\sum \frac{d_a}{h_a}\right)^{\frac{2}{3}} = \\ &= \left(d_a h_a^2 + d_b h_b^2 + d_c h_c^2\right)^{\frac{1}{3}} \left(\frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c}\right)^{\frac{2}{3}} \qquad (2) \end{aligned}$$

From (1) and (2) it follows that

$$\begin{aligned} d_a + d_b + d_c &\le \left(d_a h_a^2 + d_b h_b^2 + d_c h_c^2\right)^{\frac{1}{3}} \qquad \Rightarrow \\ (d_a + d_b + d_c)^3 &\le d_a h_a^2 + d_b h_b^2 + d_c h_c^2 \end{aligned}$$

and we are done.

Also solved by Arkady Alt, San Jose, California, USA; Michel Bataille, France; Christophe Debry, Belgium.

J132. Consider a regular hexagon $A_1A_2A_3A_4A_5A_6$ with center O. In how many different ways up to rotation can one color regions A_iOA_{i+1} (take i mod 6) in n colors?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Ivan Borsenco, Massachusetts Institute of Technology, USA

Let us calculate the number of colorings stay the same when we rotate with angle

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}.$$

For simplicity, denote the six regions of the hexagon H by $S_1 = A_1OA_2, \ldots, S_6 = A_6OA_1$. By saying $S_i = S_j$ we mean that regions S_i and S_j are colored in the same color.

• $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$, then we must have $S_1 = S_2 = \cdots = S_6$. There are only n colorings of the hexagons that stay same under these rotations, when the hexagon is colored in one color.

• $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$, then $S_1 = S_3 = S_5$ and $S_2 = S_4 = S_6$, so there are n^2 colorings such that hexagons are invariant under rotation $\frac{2\pi}{3}, \frac{4\pi}{3}$. Also, there are exactly $n^2 - n$ coloring that are invariant only under these rotations and not by any other considered.

• $\theta = \pi$, then $S_1 = S_4$, $S_2 = S_5$, $S_3 = S_6$ and there are n^3 such hexagons. Note that there are exactly $n^3 - n$ hexagons that stay the same only when we rotate under angle $180°$ and are different under all other rotations.

To sum up the inequivalent colorings, we observe that the number of colorings which don't equal to any other rotation is $n^6 - (n^3 - n) - (n^2 - n) - n$. Only one hexagon out of the six counted in the above sum must be considered. Finally, using the same idea, we get that the total number of inequivalent colorings is given by

$$\frac{1}{6}\left(n^6 - (n^3 - n) - (n^2 - n) - n\right) + \frac{1}{3}(n^3 - n) + \frac{1}{2}\left(n^2 - n\right) + n = \frac{1}{6}(n^6 + n^3 + 2n^2 + 2n).$$

Senior problems

S127. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 \geq 3$. Prove that

$$\frac{x^3}{\sqrt{y^2+z^2+7}} + \frac{y^3}{\sqrt{z^2+x^2+7}} + \frac{z^3}{\sqrt{x^2+y^2+7}} \geq 1.$$

Proposed by Orif Ibrogimov, Samarqand State University, Uzbekistan

Solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

By Cauchy–Schwarz the l.h.s. of the inequality is greater than or equal to

$$\frac{(x^2+y^2+z^2)^2}{\sqrt{x^2y^2+x^2z^2+7x^2}+\sqrt{y^2z^2+y^2x^2+7y^2}+\sqrt{z^2x^2+z^2y^2+7x^2}} \qquad (1)$$

Moreover by the concavity of $\sqrt{x}$, the denominator of (1) is less than or equal to

$$\sqrt{3}\sqrt{7(x^2+y^2+z^2)+2(x^2y^2+y^2z^2+z^2x^2)}$$

thus we are left with proving that

$$(x^2+y^2+z^2)^4 \geq 21(x^2+y^2+z^2)+6(x^2y^2+y^2z^2+z^2x^2)$$

or

$$(x^2+y^2+z^2)^4 + 3(x^4+y^4+z^4) \geq 21(x^2+y^2+z^2)+3(x^2+y^2+z^2)^2$$

By power–means $3(x^4+y^4+z^4) \geq (x^2+y^2+z^2)^2$ so that by defining $S \doteq x^2+y^2+z^2$ the inequality reads as

$$S^4 - 2S^2 - 21S = S(S-3)(S^2+3S+7) \geq 0$$

which holds since $S \geq 3$ and we are done.

Also solved by Hoang Quoc Viet, Auckland University; Arkady Alt, San Jose, California, USA; N.J. Buitrago A., Universidad Nacional, Colombia; Manh Dung Nguyen, Hanoi, Vietnam; Christophe Debry, Belgium; Magkos Athanasios, Kozani, Greece; Salem Malikic, Sarajevo , Bosnia and Herzegovina; Tuan Le, Anaheim,USA.

S128. Let $A_1, A_2, \ldots, A_n$ be a regular $n-$gon inscribed in a circle of center O and radius R. Prove that for each point M in the plane of the $n-$gon the following inequality holds:

$$\prod_{k=1}^{n} MA_k \leq (OM^2 + R^2)^{\frac{n}{2}}.$$

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania

Solution by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

Let us work in the complex plane with O as the origin and without loss of generality, $R = 1$. Let $\omega = \exp(2\pi i/n)$ and let the complex numbers $\omega, \omega^2, \ldots, \omega^n, x$ correspond to the points $A_1, A_2, \ldots, A_n, M$ respectively. Then our inequality is equivalent to

$$\prod_{k=1}^{n} |x - \omega^k| \leq \sqrt{(|x|^2 + 1)^n}.$$

Since $\omega, \omega^2, \ldots, \omega^n$ are the roots of $z^n - 1 = 0$, we have

$$\prod_{k=1}^{n} |x - \omega^k| = |x^n - 1| \leq |x|^n + 1,$$

by the triangle inequality. Hence it remains to show that

$$(|x|^n + 1)^2 \leq (|x|^2 + 1)^n \Leftrightarrow 2|x|^n \leq \sum_{k=1}^{n-1} \binom{n}{k} |x|^{2k},$$

which follows from AM-GM inequality since $n \geq 3$ and

$$\sum_{k=1}^{n-1} \binom{n}{k} |x|^{2k} \geq n|x|^2 + n|x|^{2n-2} \geq 2n|x|^n \geq 2|x|^n.$$

Equality holds iff $|x| = 0$ i.e. when $M \equiv O$.

Also solved by N.J. Buitrago A., Universidad Nacional, Colombia; Arkady Alt, San Jose, California, USA; Michel Bataille, France; Christophe Debry, Belgium.

S129. Let $a_1, a_2, \ldots, a_n \in [0,1]$ and λ be real numbers such that $a_1 + a_2 + \cdots + a_n = n + 1 - \lambda$. For any permutation $(b_i)_{i=1}^n$ of $(a_i)_{i=1}^n$ prove that

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq n + 1 - \lambda^2.$$

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

First solution by Christophe Debry, Belgium

As $a_1, \ldots, a_n \in [0,1]$, we have $\lambda = n + 1 - (a_1 + \ldots + a_n) \geq n + 1 - n = 1$. For every $i = 1, \ldots, n$ this implies that $a_i, b_i \leq 1 \leq \lambda$ (where (b_i) is a permutation of (a_i)), and therefore $(\lambda - a_i)(1 - b_i) \geq 0$. Summing these inequalities for every $i = 1, \ldots, n$, we find

$$0 \leq n\lambda - \lambda \sum_{i=1}^n b_i - \sum_{i=1}^n a_i + \sum_{i=1}^n a_i b_i.$$

Because $a_1 + \ldots + a_n = b_1 + \ldots + b_n = n + 1 - \lambda$, this yields $a_1 b_1 + \ldots + a_n b_n \geq -n\lambda + \lambda(n + 1 - \lambda) + (n + 1 - \lambda) = n + 1 - \lambda^2$, as desired.

Second solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

The problem is symmetric respect to any permutation of $(a_1, \ldots, a_n)$ since $a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \geq n + 1 - \lambda^2$ must be true *for any* permutation $(b_1, \ldots, b_n)$. Hence we may suppose $a_1 \geq a_1 \geq \ldots \geq a_n$ and by the rearrangement inequality we may write

$$a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \geq a_1 a_n + a_2 a_{n-1} + \ldots + a_n a_1$$

Let n even. We prove

$$a_1 a_n + a_2 a_{n-1} + \ldots + a_n a_1 = 2 \sum_{k=1}^{n/2} a_k a_{n+1-k} \geq n + 1 - \lambda^2$$

that is

$$(a_1 + a_n)^2 + \ldots + (a_{\frac{n}{2}} + a_{\frac{n}{2}+1})^2 \geq n + 1 - \lambda^2 + \sum_{k=1}^n a_k^2$$

By power means we have

$$(a_1 + a_n)^2 + \ldots + (a_{\frac{n}{2}} + a_{\frac{n}{2}+1})^2 \geq \frac{2}{n} (\sum_{k=1}^n a_k)^2$$

and then

$$\frac{2}{n} (\sum_{k=1}^n a_k)^2 \geq \sum_{k=1}^n a_k^2 + n + 1 - \lambda^2$$

If we define $S \doteq \sum_{k=1}^{n} a_k$ and $S_2 \doteq \sum_{k=1}^{n} a_k^2$, by $\lambda = S - n - 1$ we have

$$S^2(n+2) + n^2(n+1) \geq 2n(n+1)S + nS_2$$

Since $a_k \in 0,1]$ it follows $S_2 \leq S$ hence

$$S^2(n+2)+n^2(n+1) \geq 2n(n+1)S+nS \qquad \text{or} \qquad S^2(n+2)-Sn(2n+3)+n^2(n+1) \geq 0$$

that is $S \leq 2n\frac{n+1}{n+2}$ or $S \geq 2n$. Since $S \leq n < 2n\frac{n+1}{n+2}$ we get the result for n even.

Let n odd.

$$a_1a_n + a_2a_{n-1} + \ldots + a_na_1 = (a_{\frac{n+1}{2}})^2 + 2\sum_{k=1}^{\frac{n-1}{2}} a_ka_{n+1-k}$$

that is

$$(a_{\frac{n+1}{2}})^2 + \Big((a_1+a_n)^2 + \ldots + (a_{\frac{n-1}{2}}a_{\frac{n+3}{2}})^2\Big) - \sum_{k=1,k\neq\frac{n+1}{2}}^{n} a_k^2$$

By Power–means

$$(a_{\frac{n+1}{2}})^2 + \Big((a_1+a_n)^2 + \ldots + (a_{\frac{n-1}{2}}a_{\frac{n+3}{2}})^2\Big) \geq \left(\sum_{k=1}^{n} a_k\right)^2 \frac{2}{n+1}$$

so we get

$$\left(\sum_{k=1}^{n} a_k\right)^2 \frac{2}{n+1} - \sum_{k=1,k\neq\frac{n+1}{2}}^{n} a_k^2 \geq n+1-\lambda^2$$

As for the case n even we define $S \doteq \sum_{k=1}^{n} a_k$, $S_2 \doteq \sum_{k=1}^{n} a_k^2$, $\lambda = S - n - 1$ so that

$$2S^2 - S_2(n+1) + (n+1)(a_{\frac{n+1}{2}})^2 \geq (n+1)(-n^2 - S^2 + 2Sn + 2S - n)$$

which is implied by (remember $S_2 \leq S$ and $a_k \geq 0$)

$$2S^2 - S(n+1) \geq (n+1)(-n^2 - S^2 + 2Sn + 2S - n)$$

or

$$S^2(3+n) - S(2n^2+5n+3) + n^3 + 2n^2 + n \geq 0$$

and then $S \leq n+1$, $S \geq 2n\frac{n+1}{n+3}$ for $n \geq 3$. If $n = 2$ we have $S \leq 12/5$. In both cases the inequality of the problem holds. The proof is complete.

S130. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} \cos[(n-k)x+ky] = \left(2\cos\frac{x-y}{2}\right)^{n} \cos n\frac{x+y}{2}$$

for all positive integers n and all real numbers x and y.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, "Babes-Bolyai" University, Romania

Solution by Michel Bataille, France

The real number $\sum_{k=0}^{n} \binom{n}{k} \cos[(n-k)x+ky]$ is the real part of the complex number

$$Z = \sum_{k=0}^{n} \binom{n}{k} e^{i((n-k)x+ky)} = \sum_{k=0}^{n} \binom{n}{k} (e^{ix})^{n-k}(e^{iy})^{k}.$$

From the binomial theorem, $Z = (e^{ix}+e^{iy})^n$ which rewrites as

$$Z = \left(e^{i\frac{x+y}{2}}\left(e^{i\frac{x-y}{2}} + e^{-i\frac{x-y}{2}}\right)\right)^{n} = \left(2\cos\frac{x-y}{2}\right)^{n} e^{ni\frac{x+y}{2}}.$$

Thus, the real part of Z is also

$$\left(2\cos\frac{x-y}{2}\right)^{n} \cos n\frac{x+y}{2}$$

and the result follows.

Also solved by El Alami Idrissi Anass, Maroc; N.J. Buitrago A., Universidad Nacional, Colombia; Arkady Alt, San Jose, California, USA; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; G. C. Greubel, Newport News, VA; Christophe Debry, Belgium.

S131. Let P be a point in the interior of a triangle ABC and let P_a, P_b, and P_c be the symmetrical points of A, B, and C with respect to P. Parallels to PB and PC drawn through P_a intersect lines AB and AC at A_b and A_c, respectively. In the same way we define points B_a, B_c, C_a, and C_b. Prove that points A_b, A_c, B_a, B_c, C_a, and C_b are on an ellipse.

Proposed by Catalin Barbu, Bacau, Romania

Solution by Christophe Debry, Belgium

Because P is the midpoint of $[AP_A]$ and PB is parallel to A_bP_a, we get that B is the midpoint of $[AA_b]$. In the same way, we find that C is the midpoint of $[BB_c]$ and $[AA_c]$, A is the midpoint of $[BB_a]$ and $[CC_a]$ and B is the midpoint of $[AA_b]$ and $[CC_b]$. The conclusion is that A_bA_c and B_aC_a are both parallel to BC; B_cB_a and C_bA_b are parallel to CA; and C_aC_b and A_cB_c are parallel to AB.

S132. Let G be a 4-partite graph on n vertices. Prove that the number of k-cliques, $k \geq 3$, in G is less than or equal to $\frac{n^4+16n^3}{256}$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Christophe Debry, Belgium

Let A, B, C and D be the 4 disjoint sets of vertices such that no two vertices in A, B, C or D are adjacent. Clearly, there are no k-cliques with $k \geq 5$, for if there were any, by the pigeonhole principle we would get a set (A, B, C or D) with at least 2 vertices in the clique. This is impossible because no two vertices in those sets are adjacent. Therefore, we should only count the number of 3-cliques and 4-cliques. Let $|A| = a$, $|B| = b$, $|C| = c$ and $|D| = d$. The number of 4-cliques is less than or equal to $abcd$ and the number of 3-cliques is less than or equal to $abc + bcd + cda + dab$. (This number is the maximal number of 3-cliques with one vertex in A, one in B and one in C, plus...) It therefore suffices to prove that

$$abcd + abc + bcd + cda + dab \leq \frac{n^4 + 16n^3}{256}.$$

AM-GM yields

$$256abcd = \left(4\sqrt[4]{abcd}\right)^4 \leq (a+b+c+d)^4 = n^4.$$

AM-GM for 60 (2+2+2+9+9+9+9+9+9) variables gives

$$2(a^3+b^3+c^3) + 9(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) \geq 60abc.$$

Adding the same inequalities for the three other triples, yields

$$(a^3+b^3+c^3+d^3) + 3f(a,b,c,d) \geq 10(abc+bcd+cda+dab),$$

where $f(a,b,c,d)$ is the symmetric sum of a^2b over $\{a,b,c,d\}$. One easily checks that this is equivalent with

$$n^3 = (a+b+c+d)^3 \geq 16(abc+bcd+cda+dab).$$

Adding this last inequality 16 times to the inequality $256abcd \leq n^4$ gives the desired result.

Undergraduate problems

U127. Let $(a_n)_{n\geq 1}$ be a convergent sequence. Evaluate

$$\lim_{n\to\infty}\left(\frac{a_1}{n+1}+\frac{a_2}{n+2}+\cdots+\frac{a_n}{2n}\right).$$

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania and Mihai Piticari, "Dragos-Voda" National College, Romania

Solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

Let $A=\lim_{n\to\infty}a_n$ and $H_n\doteq\sum_{k=1}^n 1/k=\ln n+\gamma+o(1)$ and γ is the Euler constant. It is a standard result $\lim_{n\to\infty}\sum_{k=1}^n\frac{1}{n+k}=\ln 2$. Now we employ Abel's "summation by parts"

$$\sum_{k=1}^n a_kb_k=a_nB_n+\sum_{k=1}^{n-1}(a_k-a_{k+1})B_k,\qquad B_k\doteq\sum_{i=1}^k b_i$$

and write

$$\sum_{k=1}^n\frac{a_k}{n+k}=a_n\sum_{k=1}^n\frac{1}{n+k}+\sum_{k=1}^{n-1}(a_k-a_{k+1})\sum_{j=1}^k\frac{1}{n+j}$$

that is

$$\sum_{k=1}^n\frac{a_k}{n+k}=a_n\sum_{k=1}^n\frac{1}{n+k}+\sum_{k=1}^{n-1}(a_k-a_{k+1})(H_{n+k}-H_n)\tag{1}$$

Trivially $\lim_{n\to\infty}a_n\sum_{k=1}^n\frac{1}{n+k}=A\ln 2$ and $-H_n\sum_{k=1}^{n-1}(a_k-a_{k+1})=-H_n(a_1-a_n)$.

Now we reapply Abel's "summation by parts" to $\sum_{k=1}^{n-1}(a_k-a_{k+1})H_{n+k}$ and obtain

$$H_{2n-1}\sum_{k=1}^{n-1}(a_k-a_{k+1})+\sum_{k=1}^{n-2}(H_{n+k}-H_{n+k+1})\sum_{j=1}^k(a_j-a_{j+1})$$

Telescoping we have

$$H_{2n-1}\sum_{k=1}^{n-1}(a_k-a_{k+1})=H_{2n-1}(a_1-a_n),\qquad \sum_{j=1}^k(a_j-a_{j+1})=a_1-a_{k+1}$$

and

$$a_1\sum_{k=1}^{n-2}(H_{n+k}-H_{n+k+1})=a_1(H_{n+1}-H_{2n-1})$$

Now (1) is the sum of

$$a_n(H_{2n}-H_n)-H_n(a_1-a_n)+H_{2n-1}(a_1-a_n)+a_1(H_{n+1}-H_{2n-1}) \qquad (2)$$

and

$$-\sum_{k=1}^{n-2}(H_{n+k}-H_{n+k+1})(a_{k+1}-A)-A\sum_{k=1}^{n-2}(H_{n+k}-H_{n+k+1}) \qquad (3)$$

The limit $n\to\infty$ of (2) is zero so we are left with (3).

$$-A\sum_{k=1}^{n-2}(H_{n+k}-H_{n+k+1})=-A(H_{n+1}-H_{2n-1})\to A\ln 2$$

$$\sum_{k=1}^{n-2}(H_{n+k}-H_{n+k+1})(a_{k+1}-A)=\sum_{k=1}^{n-2}\frac{1}{n+k+1}(a_{k+1}-A)$$

tend to zero as $n\to\infty$ because by definition $|a_k-A|<\varepsilon$ for any $k>k_\varepsilon$ so we break

$$\sum_{k=1}^{n-2}\frac{a_{k+1}-A}{n+k+1}=\sum_{k=1}^{k_\varepsilon}\frac{a_{k+1}-A}{n+k+1}+\sum_{k=k_\varepsilon}^{n-2}\frac{a_{k+1}-A}{n+k+1}$$

and

$$\left|\sum_{k=1}^{n-2}\frac{a_{k+1}-A}{n+k+1}\right|\le k_\varepsilon\max_{1\le k\le k_\varepsilon}|a_{k+1}-A|/(n+2)\to 0$$

$$\left|\sum_{k=k_\varepsilon}^{n-2}\frac{a_{k+1}-A}{n+k+1}\right|\le\varepsilon\sum_{k=k_\varepsilon}^{n-2}\frac{1}{n+k+1}\le\varepsilon\frac{n-1-k_\varepsilon}{n+1+k_\varepsilon}$$

and we are done

Also solved by Arkady Alt, San Jose, California, USA; Michel Bataille, France; Christophe Debry, Belgium.

U128. Let f be a twice differentiable continuous real-valued function defined on $[0,1]$ such that $f(0)=f(1)=f'(1)=0$ and $f'(0)=1$. Prove that

$$\int_0^1 \left(f''(x)\right)^2 dx \geq 4.$$

Proposed by Duong Viet Thong, Nam Dinh University of Technology Education, Vietnam

Solution by Arkady Alt, San Jose, California, USA

By Cauchy Inequality we have

$$\int_0^1 (3x-2)^2\,dx \cdot \int_0^1 \left(f''(x)\right)^2 dx \geq \left(\int_0^1 (3x-2)\,f''(x)\,dx\right)^2$$

and since

$$\int_0^1 (3x-2)^2\,dx = \left(\frac{(3x-2)^3}{9}\right)_0^1 = \frac{1}{9}+\frac{8}{9} = 1,$$

$$\int_0^1 (3x-2)\,f''(x)\,dx = \left[\begin{array}{c} u' = f''(x)\,;u = f'(x) \\ v = 3x-2; v' = 3 \end{array}\right] = \left((3x-2)\,f'(x)\right)_0^1 - 3\int_0^1 f'(x)\,dx =$$

$$\left(1\cdot f'(1) - (-2)\cdot f'(0)\right) - 3\left(f(1)-f(0)\right) = 2$$

then $\int_0^1 \left(f''(x)\right)^2 dx \geq 4$.Equality occurs if $f(x) = x(x-1)^2$.

Also solved by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Christophe Debry, Belgium.

U129. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n > 0$ such that

$$a_1^x + a_2^x + \cdots + a_n^x \geq b_1^x + b_2^x + \cdots + b_n^x,$$

for all x in $\mathbb{R}$. Prove that the function $f : \mathbb{R} \to (0, \infty)$,

$$f(x) = \left(\frac{a_1}{b_1}\right)^x + \left(\frac{a_2}{b_2}\right)^x + \cdots + \left(\frac{a_n}{b_n}\right)^x$$

is increasing.

Proposed by Cezar Lupu, University of Bucharest, Romania

Solution by Michel Bataille, France

First suppose $n = 1$. The hypothesis is $a_1^x \geq b_1^x$ for all real x. Taking successively $x = 1$ and $x = -1$, it follows that $a_1 = b_1$ and so f is the constant function $x \mapsto 1$.
If $n = 2$, taking $a_1 = e, a_2 = e^2, b_1 = e^2, b_2 = e$, we have $a_1^x + a_2^x = b_1^x + b_2^x$ for all x and $f(x) = 2\cosh x$. Therefore f is decreasing on $(\infty, 0]$ and increasing on $[0, \infty)$ and the stated result does not hold. Actually, we show that the result just found about the variations of f is the general one.
The first two derivatives of f are given by

$$f'(x) = \left(\frac{a_1}{b_1}\right)^x \ln\left(\frac{a_1}{b_1}\right) + \left(\frac{a_2}{b_2}\right)^x \ln\left(\frac{a_2}{b_2}\right) + \cdots + \left(\frac{a_n}{b_n}\right)^x \ln\left(\frac{a_n}{b_n}\right)$$

$$f''(x) = \left(\frac{a_1}{b_1}\right)^x \left[\ln\left(\frac{a_1}{b_1}\right)\right]^2 + \left(\frac{a_2}{b_2}\right)^x \left[\ln\left(\frac{a_2}{b_2}\right)\right]^2 + \cdots + \left(\frac{a_n}{b_n}\right)^x \left[\ln\left(\frac{a_n}{b_n}\right)\right]^2.$$

Since $f''(x) \geq 0$ for all x, we see that the function f' is increasing on $\mathbb{R}$. We will prove that $f'(0) = 0$. It will follow that $f'(x) \leq 0$ for $x \leq 0$ and $f'(x) \geq 0$ for $x \geq 0$ and so f is decreasing on $(\infty, 0]$ and increasing on $[0, \infty)$, as announced.
From the hypothesis, we deduce that for all x,

$$b_1^x\left(\left(\frac{a_1}{b_1}\right)^x - 1\right) + b_2^x\left(\left(\frac{a_2}{b_2}\right)^x - 1\right) + \cdots + b_n^x\left(\left(\frac{a_n}{b_n}\right)^x - 1\right) \geq 0 \qquad (1).$$

For x in a neighborhood of 0, we have

$$\left(\frac{a_k}{b_k}\right)^x - 1 = x\ln\left(\frac{a_k}{b_k}\right) + x\varepsilon_k(x)$$

where $\lim_{x\to 0}\varepsilon_k(x) = 0$ and $k = 1, 2, \ldots n$. Substituting into (1) and dividing by x, we obtain if $x > 0$,

$$b_1^x \ln\left(\frac{a_1}{b_1}\right) + \varepsilon_1(x) + b_2^x \ln\left(\frac{a_2}{b_2}\right) + \varepsilon_2(x) + \cdots + b_n^x \ln\left(\frac{a_n}{b_n}\right) + \varepsilon_n(x) \geq 0$$

and if $x < 0$,

$$b_1^x \ln\left(\frac{a_1}{b_1}\right) + \varepsilon_1(x) + b_2^x \ln\left(\frac{a_2}{b_2}\right) + \varepsilon_2(x) + \cdots + b_n^x \ln\left(\frac{a_n}{b_n}\right) + \varepsilon_n(x) \leq 0.$$

Letting x tend to 0 in the two cases, we see that

$$\ln\left(\frac{a_1}{b_1}\right) + \ln\left(\frac{a_2}{b_2}\right) + \cdots + \ln\left(\frac{a_n}{b_n}\right)$$

must be ≥ 0 and ≤ 0, hence equals 0. But this means that $f'(0) = 0$, as desired.

U130. Let f be a three times differentiable real-valued function defined on $(0,1)$ such that $|f'''(x)| \geq 1$ for all $0 < x < 1$. Consider the set

$$M = \{x \in (0,1) : |f'(x)| \leq 2\}.$$

Prove that for the measure of the set M the following inequality holds:

$$\mu(M) \leq 4\sqrt{2}.$$

Proposed by Orif Ibrogimov, Samarqand State University, Uzbekistan

Solution by Christophe Debry, Belgium

As $M \subset (0,1)$, the properties of measures yield

$$\mu(M) \leq \mu((0,1)) = 1 < 4\sqrt{2}.$$

U131. Prove that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\arctan\frac{k}{n}}{n+k}\cdot\frac{\varphi(k)}{k}=\frac{3\log 2}{4\pi},$$

where φ denotes the Euler totient function.

Proposed by Cezar Lupu, University of Bucharest, Romania

No solution has yet been received.

U132. Let $P \in \mathbb{R}[X]$ be a nonconstant polynomial and let $f : \mathbb{R} \to \mathbb{R}$ be a function with the intermediate value property such that $P \circ f$ is continuous. Prove that f is continuous.

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania and Gabriel Dospinescu, Ecole Normale Superieure, Paris, France

Solution by Gabriel Dospinescu, Ecole Normale Superieure, Paris, France

Let x be a real number and consider a sequence x_n that converges to x. Then the continuity of $P \circ f$ implies that $P(f(x_n))$ converges to $P(f(x))$ and so $f(x_n)$ is bounded (as a nonconstant polynomial is a proper map). We claim that $f(x_n)$ converges. If not, there are two real numbers $a < b$ and two subsequences y_n and z_n of x_n such that $f(y_n)$ converges to a and $f(z_n)$ converges to b. Now, take any $c \in (a, b)$. Then for large enough n we have $f(y_n) < c < f(z_n)$ and, since f has the intermediate value property, there exists t_n between y_n and z_n such that $c = f(t_n)$. But since y_n and z_n converge to x, so does t_n. Moreover, $P(c) = P(f(t_n))$ must converge to $P(f(x))$ by continuity of $P \circ f$. Therefore we find that $P(c) = P(f(x))$ for all $c \in (a, b)$, which implies that P is constant, a contradiction.

So, the sequence $f(x_n)$ converges for all choices of a convergent sequence x_n. This immediately implies that if x_n tends to x, then $f(x_n)$ tends to $f(x)$, since one can consider the sequence $x_1, x, x_2, x, \ldots$, which still converges to x (and for which the sequence obtained after applying f has infinitely many terms equal to $f(x)$). This obviously implies the continuity of f.

Remark All we used about P is that it is a proper map and it is non constant on any nontrivial interval. So, by replacing P with any map g satisfying these two properties, the conclusion still holds. An interesting question is to find all maps g with the following property: if f has the intermediate value property and $g \circ f$ is continuous, then f is continuous.

Olympiad problems

O127. Let n be an integer greater than 1. A set A is called stable if there is at least one positive real number in A and whenever $x_1, x_2, \ldots, x_n$ are elements of A, not necessarily distinct and such that $x_1^2 + x_2^2 + \cdots + x_n^2 \in A$, so does $x_1 + x_2 + \cdots + x_n$. Find all subsets A of R such that A is stable for any stable subset B of R we have $A \subseteq B$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris, France

Solution by Gabriel Dospinescu, Ecole Normale Superieure, Paris, France

First , observe that the set $X = (-n, n)$ is stable, as an obvious consequence of the Cauchy-Schwartz inequality. Thus, any solution A of the problem is a subset of X. Next, we claim that if $a^2 \leq nb$, then we can find real numbers $x_1, x_2, \ldots, x_n$ with $x_1 + x_2 + \cdots + x_n = a$ and $x_1^2 + x_2^2 + \cdots + x_n^2 = b$. This is not difficult: it is enough to search $(x_1, x_2, \ldots, x_n)$ in the form $(x, y, \ldots, y)$ and to note that in this case everything comes down to the solvability in real numbers of the equation

$$n(n_1)x^2 - 2(n-1)ax + a^2 - b = 0,$$

which is clear because $a^2 \leq nb$. Now, consider A a solution of the problem and choose $u \in A$ a positive number. By the previous observation and the stability of A, the interval $[-\sqrt{nu}, \sqrt{nu}]$ is a subset of A. Working with $\sqrt{nu}$ instead of u, we infer that $[-\sqrt{n\sqrt{nu}}, \sqrt{n\sqrt{nu}}]$ is also a subset of A. Continue this provess and an obvious induction gives that

$$I_k = [-n^{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^k}} u^{\frac{1}{2^k}}, n^{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^k}} u^{\frac{1}{2^k}}]$$

is a subset of A and this is for all k. Because any $x \in (-n, n)$ there exists k such that $x \in I_k$, it follows that $(-n, n)$ is a subset of A and thus $A = (-n, n)$ is the only solution of the problem.

O128. Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be real numbers that sum up to 1. Let $b_k = \sqrt{1 - \frac{1}{4^k}}\sqrt{a_1^2 + a_2^2 + \cdots + a_k^2}$. Find the minimum value of

$$b_1 + b_2 + \cdots + b_{n-1} + 2b_n.$$

as a function of n.

Proposed by Alex Anderson, Washington University in St. Louis, USA

Solution by Michel Bataille, France

We assume $n \geq 2$ and observe that

$$1 - \frac{1}{4^k} = \frac{3}{4}\left(\frac{1}{4^{k-1}} + \frac{1}{4^{k-2}} + \cdots + \frac{1}{4} + 1\right).$$

From Cauchy-Schwarz inequality

$$b_k = \sqrt{\frac{3}{4}}\sqrt{\frac{1}{4^{k-1}} + \frac{1}{4^{k-2}} + \cdots + \frac{1}{4} + 1}\sqrt{a_1^2 + a_2^2 + \cdots + a_k^2} \geq \sqrt{\frac{3}{4}}\left(\frac{a_1}{2^{k-1}} + \frac{a_2}{2^{k-2}} + \cdots + \frac{a_{k-1}}{2} + a_k\right)$$

so that

$$\begin{aligned}
\sum_{k=1}^{n} b_k &\geq \sqrt{\frac{3}{4}}\left(a_1 \cdot \sum_{k=1}^{n} \frac{1}{2^{k-1}} + a_2 \cdot \sum_{k=1}^{n-1} \frac{1}{2^{k-1}} + \cdots + a_{n-1}(1 + \frac{1}{2}) + a_n\right) \\
&= \sqrt{\frac{3}{4}}\left(a_1 \cdot 2(1 - \frac{1}{2^n}) + a_2 \cdot 2(1 - \frac{1}{2^{n-1}}) \cdots + a_{n-1} \cdot 2(1 - \frac{1}{2^2}) + a_n \cdot 2(1 - \frac{1}{2})\right) \\
&= \sqrt{\frac{3}{4}}(2(a_1 + a_2 + \cdots + a_n)) - \sqrt{\frac{3}{4}}\left(\frac{a_1}{2^{n-1}} + \frac{a_2}{2^{n-2}} + \cdots + \frac{a_{n-1}}{2} + a_n\right) \\
&\geq \sqrt{3} - b_n.
\end{aligned}$$

It follows that $b_1 + b_2 + \cdots + b_{n-1} + 2b_n \geq \sqrt{3}$.
In addition, if we take $a_1 = \frac{1}{2^n - 1}$, $a_2 = \frac{2}{2^n - 1}$, $a_3 = \frac{2^2}{2^n - 1}, \ldots, a_n = \frac{2^{n-1}}{2^n - 1}$, then we have $a_1 + a_2 + \cdots + a_n = 1$ and an easy calculation yields first $b_k = \left(2^k - \frac{1}{2^k}\right)\frac{1}{(2^n - 1)\sqrt{3}}$, $\quad k = 1, 2, \ldots, n$ and then $b_1 + b_2 + \cdots + b_{n-1} + 2b_n = \sqrt{3}$.
We can conclude that the required minimum is $\sqrt{3}$ for all n.

O129. Let ABC be a triangle and let points P and Q lie on sides AB and AC, respectively. Let M and N be the midpoints of BP and CQ, respectively. Prove that the centers of the nine-point circles of triangles ABC, APQ, and AMN are collinear.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Michel Bataille, France

Let O, H, N (resp. O_1, H_1, N_1 ; resp. O_2, H_2, N_2) denote the circumcentre, orthocentre, centre of the nine-point circle of ΔABC (resp. ΔAPQ ; resp. ΔAMN).
Let $P = \alpha A + (1-\alpha)B$ and $Q = \beta A + (1-\beta)C$ where $\alpha, \beta \in [0,1]$.
From the well known relation $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, we deduce

$$\overrightarrow{OA} + \overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OH} + \alpha\overrightarrow{BA} + \beta\overrightarrow{CA}.$$

Since we also have $\overrightarrow{OA} + \overrightarrow{OP} + \overrightarrow{OQ} = 3\overrightarrow{OO_1} + \overrightarrow{O_1H_1}$, it follows that

$$2\overrightarrow{OO_1} + \overrightarrow{HH_1} = \overrightarrow{U} \qquad (1)$$

where $\overrightarrow{U} = \alpha\overrightarrow{BA} + \beta\overrightarrow{CA}$.
In a similar way, since $M = \frac{\alpha}{2}A + (1-\frac{\alpha}{2})B$ and $N = \frac{\beta}{2}A + (1-\frac{\beta}{2})B$, we obtain

$$2\overrightarrow{OO_2} + \overrightarrow{HH_2} = \frac{1}{2}\overrightarrow{U} \qquad (2).$$

Using the well-known fact that N is the midpoint of OH (and N_1 the midpoint of O_1H_1), we deduce from (1), $2\overrightarrow{NN_1} + \overrightarrow{OO_1} = \overrightarrow{U}$. Similarly, $2\overrightarrow{NN_2} + \overrightarrow{OO_2} = \frac{1}{2}\overrightarrow{U}$ and so

$$2\overrightarrow{NN_2} - \overrightarrow{NN_1} = \frac{1}{2}\overrightarrow{OO_1} - \overrightarrow{OO_2} \qquad (3).$$

Now, let J, J_1, J_2 be the midpoints of AB, AP, AM, rspectively, and K, K_1, K_2 the midpoints of AC, AQ, AN. Observing that the lines OJ, O_1J_1, O_2J_2 as well as OK, O_1K_1, O_2K_2 are parallel and that $\overrightarrow{JJ_2} = \frac{1}{2}\overrightarrow{JJ_1}$, $\overrightarrow{KK_2} = \frac{1}{2}\overrightarrow{KK_1}$, we deduce that the homothety with centre O and factor $\frac{1}{2}$ must transform the lines O_1J_1 and O_1K_1 into O_2J_2 and O_2K_2, respectively, hence O_1 into O_2. As a result, $\overrightarrow{OO_2} = \frac{1}{2}\overrightarrow{OO_1}$ and (3) gives $2\overrightarrow{NN_2} = \overrightarrow{NN_1}$. The result follows.

O130. Let $a_1, a_2, \ldots, a_{2009}$ be distinct positive integers not exceeding 10^6. Prove that there are indices i, j such that $|\sqrt{ia_i} - \sqrt{ja_j}| \geq 1$.

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

Solution by Li Zhou, Polk State College, USA

Let

$$f(x) = \frac{(\sqrt{x}+1)^2}{2006} - \frac{(\sqrt{x}-1)^2}{2009} = \frac{3(x+1) + 8030\sqrt{x}}{2006 \cdot 2009}.$$

Then f is an increasing function of $x > 0$. Hence $f(a_1) \leq f(10^6) < 3$, and therefore the interval $I = \left[(\sqrt{a_1}-1)^2/2009, (\sqrt{a_1}+1)^2/2006\right]$ contains at most three integers. By the pigeonhole principle, there is an $i \in \{2006, \ldots, 2009\}$ such that $a_i \notin I$, that is, $|\sqrt{ia_i} - \sqrt{a_1}| > 1$, completing the proof.

Stronger result. Assume first that $a_1 \geq 174^2$. Let

$$f(x) = \frac{(\sqrt{x}+174)^2}{1201} - \frac{(\sqrt{x}-174)^2}{2009}.$$

Then f is an increasing function of $x > 0$. Hence $f(a_1) \leq f(10^6) < 808$, and therefore the interval $I = \left[(\sqrt{a_1}-174)^2/2009, (\sqrt{a_1}+174)^2/1201\right]$ contains at most 808 integers. By the pigeonhole principle, there is an $i \in \{1201, 1202, \ldots, 2009\}$ such that $a_i \notin I$, which implies that $|\sqrt{ia_i} - \sqrt{a_1}| > 174$.

Consider next that $a_1 < 174^2$, then there is a $j \in \{1836, 1837, \ldots, 2009\}$ such that $a_j \geq 174$. Thus

$$\sqrt{ja_j} - \sqrt{a_1} > \sqrt{1836 \cdot 174} - \sqrt{174^2} > 174.$$

O131. Let G be a graph on n vertices such that there are no K_4 subgraphs in it. Prove that G contains at most $\left(\frac{n}{3}\right)^3$ triangles.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Ivan Borsenco, Massachusetts Institute of Technology, USA

For $n = 2$, $n = 3$, and $n = 4$, the result is clear. Let G be a graph with n vertices, having no 4-clique. Observe that we can add edges to G until it contains a triangle ABC. Let $G \backslash \Delta ABC$ be the subgraph formed by the remaining $n - 3$ vertices.

Using Turan's Theorem $G \backslash \Delta ABC$ containing no K_4 subgraphs has at most $\frac{(n-3)^2}{3}$ edges. By induction hypothesis it contains at most $\left(\frac{n-3}{3}\right)^3$ triangles. The remaining triangles in G are formed as a union of one vertex belonging to ABC and an edge from $G \backslash \Delta ABC$, or by a union of one edge from ABC and one vertex belonging to $G \backslash \Delta ABC$.

Every edge from $G \backslash \Delta ABC$ forms a triangle with at most one vertex belonging to ABC. Each vertex from $G \backslash \Delta ABC$ can form a triangle with at most one pair of vertices that belongs to ABC. Otherwise, in both cases, we get a K_4 subgraph formed. Hence the total number of triangles is at most

$$\left(\frac{n-3}{3}\right)^3 + \frac{(n-3)^2}{3} + (n-3) + 1 = \left(\frac{n-3}{3} + 1\right)^3 = \left(\frac{n}{3}\right)^3,$$

and our induction is complete. The equality case holds in the case of a 3-partite graph $K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$.

O132. Let m and n be integers greater than 1. Prove that

$$\sum_{k_1+k_2+\cdots+k_n=m,\ k_1,k_2,\ldots,k_n\geq 0} \frac{1}{k_1!k_2!\cdots k_n!}\cos(k_1+2k_2+\cdots+nk_n)\frac{2\pi}{n}=0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, "Babes-Bolyai" University, Romania

Solution by Michel Bataille, France

Let L denote the left-hand side of the proposed identity. We observe that L is the real part of the complex number

$$Z=\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_1,k_2,\ldots,k_n\geq 0}} \frac{e^{\frac{2\pi i}{n}(k_1+2k_2+\cdots+nk_n)}}{k_1!k_2!\cdots k_n!}=\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_1,k_2,\ldots,k_n\geq 0}} \frac{\omega^{k_1}(\omega^2)^{k_2}\cdots(\omega^n)^{k_n}}{k_1!k_2!\cdots k_n!}$$

where $\omega=e^{\frac{2\pi i}{n}}$.

Now, using the multinomial theorem,

$$Z=(\omega+\omega^2+\cdots+\omega^{n-1}+1)^m=\left(\frac{\omega^n-1}{\omega-1}\right)^m=0.$$

(since $\omega^n=1$). Thus, $L=\mathrm{Re}(Z)=0$.