On a geometric inequality involving medians

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In this note we give two proofs to the inequality

$$m_a + m_b + m_c \leq 4R + r, \tag{1}$$

where m_a, m_b, m_c are the medians of a triangle ABC and R and r are its inradius and circumradius, respectively. Also, we extend inequality (1) to a special class of cevians of triangle ABC.

1 First Proof

We use the following known inequalities:

$$a^2 + b^2 + c^2 \leq 8R^2 + 4r^2 \quad \text{(Gerretsen)} \tag{2}$$

$$s\sqrt{3} \leq 4R + r \quad \text{(Mitrinović)} \tag{3}$$

$$a^2 + b^2 + c^2 \leq 4\sqrt{3}K + 3\sum (a-b)^2 \quad \text{(Finsler-Hadwiger)} \tag{4}$$

where a, b, c are the sidelengths, $s = \dfrac{1}{2}(a+b+c)$ is the semiperimeter, and K is the area of triangle ABC. Proofs for these classical inequalities can be found in [2].

It is well-known that we can construct a triangle Δ_1 having the side-lengths m_a, m_b, m_c. For the area K_1 of this new triangle we have

$$K_1 = \frac{3}{4}K. \tag{5}$$

Applying inequality (4) to triangle Δ_1 we obtain

$$\sum m_a^2 \leq 4\sqrt{3}K_1 + 3\sum (m_a - m_b)^2,$$

hence

$$6\sum m_a m_b \leq 4\sqrt{3}K_1 + 5\sum m_a^2. \tag{6}$$

Using the median formula $m_a = \dfrac{1}{4}[2(b^2+c^2) - a^2]$ it follows that

$$\sum m_a^2 = \frac{3}{4}\sum a^2.$$

Inequalities (5) and (6) yield

$$2\sum m_a m_b \leq \sqrt{3}K + \frac{5}{4}\sum a^2. \tag{7}$$

But $K = s \cdot r$ and if we apply (2) and (3) we get

$$2\sum m_a m_b \leq r(4R + r) + \frac{5}{4}(8R^2 + 4r^2) = 10R^2 + 4Rr + 6r^2.$$

Therefore

$$\begin{aligned}\left(\sum m_a\right)^2 &= \sum m_a^2 + 2\sum m_a m_b = \frac{3}{4}\sum a^2 + 2\sum m_a m_b \\ &\leq \frac{3}{4}(8R^2 + 4r^2) + 10R^2 + 4Rr + 6r^2 = 16R^2 + 4Rr + 9r^2 \\ &= 16R^2 + 4Rr + r^2 + 8r^2 \leq 16R^2 + 8Rr + r^2 = (4R + r)^2.\end{aligned}$$

(Here we used the well-known Euler inequality $8r^2 \leq 4Rr$.)

2 Second Proof

This is a simple geometric argument. We consider two cases:

Case I. Triangle ABC is acute. Let O be the circumcenter of ABC and let A_1 be the middpoint of side BC. Let d_a, d_b, and d_c be the distances from O to sides BC, CA, and AB, respectively. Then $m_a \leq d_a + R$ (see Figure 1). Using Carnot's relation $d_a + d_b + d_c = R + r,$ (8) we get

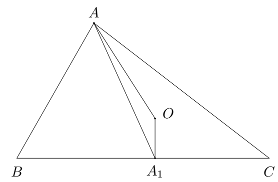

Figure 1:

$$\sum m_a \leq d_a + d_b + d_c + 3R = R + r + 3R = 4R + r.$$

For a simple proof to (8) let us use Figure 2.

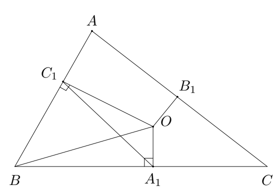

Figure 2:

Quadrilateral A_1BC_1O is inscribed in the circle of diameter OB. From Ptolemey's Theorem,

$$d_a\frac{c}{2} + d_c\frac{a}{2} = R\frac{b}{2},$$

and two other similar relations. Summing up these three relations we get

$$\sum d_a(b+c) = R(a+b+c). \tag{9}$$

On the other hand,

$$\sum d_a \frac{a}{2} = K = rs = r\frac{a+b+c}{2},$$

hence

$$\sum d_a a = r(a+b+c). \tag{10}$$

Adding (9) and (10) yields

$$\sum d_a(a+b+c) = (R+r)(a+b+c),$$

and (8) is proved.

Case II. Triangle ABC is obtuse. Assume, for instance, that $\widehat{A} > 90°$. Then $m_b < \dfrac{a+c}{2}$ and $m_c < \dfrac{a+b}{2}$. So

$$m_b + m_c < a + \frac{b+c}{2}. \tag{11}$$

Let $\widehat{A}_1$ be the angle $\widehat{BAA_1}$ and let $\widehat{A}_2$ be the angle $\widehat{CAA_1}$ (Figure 3).

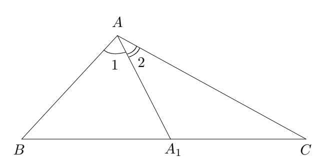

Figure 3:

In addition, $\widehat{A} > 90°$ implies $\widehat{A} > \widehat{B} + \widehat{C}$, hence $\widehat{A}_1 > B$ or $\widehat{A}_2 > C$. Without loss of generality assume that $\widehat{A}_1 > B$. Then $m_a < \dfrac{a}{2}$ and using

(11) we get

$$m_a + m_b + m_c < \frac{3}{2}a + \frac{b+c}{2} = 2a + \frac{b+c-a}{2} < 4R + r,$$

since $a < 2R$ and $\dfrac{b+c-a}{2} = s - a < r$.

3 An extension of inequality (1)

Let AA_1, BB_1, and CC_1 be three cevians in a triangle ABC and let

$$\frac{A_1B}{A_1C} = \alpha_1, \quad \frac{B_1C}{B_1A} = \alpha_2, \quad \frac{C_1A}{C_1B} = \alpha_3. \tag{12}$$

The following theorem is the main result of this section.

Theorem 1 *1. Segments AA_1, BB_1, and CC_1 are sides of a triangle if and only if $\alpha_1 = \alpha_2 = \alpha_3$.*

2. Let $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, Δ_α be the triangle having sides AA_1, BB_1, and CC_1, and K_α be the area of Δ_α. Then

$$K_\alpha = \frac{\alpha^2 + \alpha + 1}{(\alpha+1)^2} K. \tag{13}$$

3. If $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, then

$$AA_1 + BB_1 + CC_1 \leq \frac{2}{\alpha+1}\sqrt{\frac{\alpha^2+\alpha+1}{3}}(4R + r), \tag{14}$$

where R and r are the circumradius and inradius of ABC, respectively.

Proof. 1) We have

$$\begin{cases} \overrightarrow{OA_1} = \dfrac{1}{\alpha_1+1}\overrightarrow{OB} + \dfrac{\alpha_1}{\alpha_1+1}\overrightarrow{OC} \\ \\ \overrightarrow{OB_1} = \dfrac{1}{\alpha_2+1}\overrightarrow{OC} + \dfrac{\alpha_2}{\alpha_2+1}\overrightarrow{OA} \\ \\ \overrightarrow{OC_1} = \dfrac{1}{\alpha_3+1}\overrightarrow{OA} + \dfrac{\alpha_3}{\alpha_3+1}\overrightarrow{OB}, \end{cases} \tag{15}$$

hence

$$\begin{aligned} \sum \overrightarrow{AA_1} = \sum(\overrightarrow{OA_1} - \overrightarrow{OA}) &= \sum \left(\frac{1}{\alpha_1+1}\overrightarrow{OB} + \frac{\alpha_1}{\alpha_1+1}\overrightarrow{OC} - \overrightarrow{OA} \right) \\ &= \sum \left(\frac{1}{\alpha_3+1} + \frac{\alpha_2}{\alpha_2+1} - 1 \right) \overrightarrow{OA}. \end{aligned}$$

It follows that $\sum \overrightarrow{AA_1} = \overrightarrow{0}$ if and only if

$$\begin{cases} \dfrac{1}{\alpha_3+1} + \dfrac{\alpha_2}{\alpha_2+1} = 1 \\ \\ \dfrac{1}{\alpha_1+1} + \dfrac{\alpha_3}{\alpha_3+1} = 1 \\ \\ \dfrac{1}{\alpha_2+1} + \dfrac{\alpha_1}{\alpha_1+1} = 1 \end{cases}$$

hence $\alpha_1 = \alpha_2 = \alpha_3$.

2) We can write

$$
\begin{aligned}
K_\alpha &= \frac{1}{2}(\overrightarrow{AA_1} \times \overrightarrow{BB_1}) = \frac{1}{2}(\overrightarrow{OA_1} - \overrightarrow{OA}) \times (\overrightarrow{OB_1} - \overrightarrow{OB}) \\
&= \frac{1}{2}\left(\frac{1}{\alpha+1}\overrightarrow{OB} + \frac{\alpha}{\alpha+1}\overrightarrow{OC} - \overrightarrow{OA}\right)\times\left(\frac{1}{\alpha+1}\overrightarrow{OC} + \frac{\alpha}{\alpha+1}\overrightarrow{OA} - \overrightarrow{OB}\right) \\
&= \frac{1}{2}\left[\frac{1}{(\alpha+1)^2}\overrightarrow{OB} \times \overrightarrow{OC} + \frac{\alpha}{(\alpha+1)^2}\overrightarrow{OB} \times \overrightarrow{OA} + \frac{\alpha^2}{(\alpha+1)^2}\overrightarrow{OC} \times \overrightarrow{OA}\right. \\
&\left. -\frac{\alpha}{\alpha+1}\overrightarrow{OC} \times \overrightarrow{OB} - \frac{1}{\alpha+1}\overrightarrow{OA} \times \overrightarrow{OC} + \overrightarrow{OA} \times \overrightarrow{OB}\right] \\
&= \frac{1}{2} \cdot \frac{\alpha^2+\alpha+1}{(\alpha+1)^2}(\overrightarrow{OA} \times \overrightarrow{OB} + \overrightarrow{OB} \times \overrightarrow{OC} + \overrightarrow{OC} \times \overrightarrow{OA}) \\
&= \frac{\alpha^2+\alpha+1}{(\alpha+1)^2}K.
\end{aligned}
$$

3) From (12) we obtain $A_1B = \dfrac{\alpha a}{\alpha+1}$. Applying the Law of Cosines in triangle ABA_1 we get

$$AA_1^2 = c^2 + \frac{\alpha^2}{(\alpha+1)^2}a^2 - \frac{\alpha}{\alpha+1}(a^2+c^2-b^2),$$

hence

$$\sum AA_1^2 = \frac{\alpha^2+\alpha+1}{(\alpha+1)^2}\sum a^2. \tag{16}$$

Applying inequality (4) in triangle Δ_α we obtain

$$\sum AA_1^2 \le 4\sqrt{3}K_\alpha + 3\sum(AA_1 - BB_1)^2,$$

hence

$$6\sum AA_1 \cdot BB_1 \le 4\sqrt{3}K_\alpha + 5\sum AA_1^2.$$

It follows, via (13) and (16), that

$$\begin{aligned}
6\sum AA_1\cdot BB_1 &\le 4\sqrt{3}\frac{\alpha^2+\alpha+1}{(\alpha+1)^2}K+5\frac{\alpha^2+\alpha+1}{(\alpha+1)^2}\sum a^2\\
&=4\frac{\alpha^2+\alpha+1}{(\alpha+1)^2}\sqrt{3}s\cdot r+5\frac{\alpha^2+\alpha+1}{(\alpha+1)^2}\sum a^2\\
&\le 4\frac{\alpha^2+\alpha+1}{(\alpha+1)^2}\sqrt{3}s\cdot r+5\frac{\alpha^2+\alpha+1}{(\alpha+1)^2}(8R^2+4r^2)\\
&=4\frac{\alpha^2+\alpha+1}{(\alpha+1)^2}[\sqrt{3}s\cdot r+5(2R^2+r^2)]\\
&\le 4\frac{\alpha^2+\alpha+1}{(\alpha+1)^2}[r(4R+r)+10R^2+5r^2]\\
&=4\frac{\alpha^2+\alpha+1}{(\alpha+1)^2}(10R^2+4Rr+6r^2).
\end{aligned}$$

We can write

$$\begin{aligned}
&\sum AA_1^2+2\sum AA_1\cdot BB_1\le \frac{\alpha^2+\alpha+1}{(\alpha+1)^2}\sum a^2\\
&+\frac{4}{3}\cdot\frac{\alpha^2+\alpha+1}{(\alpha+1)^2}(10R^2+4Rr+6r^2)\\
&\le \frac{4}{3}\cdot\frac{\alpha^2+\alpha+1}{(\alpha+1)^2}\left[\frac{3}{4}(8R^2+4r^2)+10R^2+4Rr+6r^2\right]\\
&=\frac{4}{3}\cdot\frac{\alpha^2+\alpha+1}{(\alpha+1)^2}(4R+r)^2,
\end{aligned}$$

and the conclusion follows.

Remarks. 1) The construction of triangle Δ_α is the following (Figure 4). Let $A_1M \| BB_1$ such that BA_1MB_1 is a parallelogram. Then triangle AA_1M is the desired triangle Δ_α since is not difficult to prove that $AMCC_1$ is also a parallelogram.

2) For $\alpha=1$, from (14) we get inequality (1).

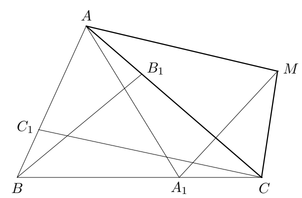

Figure 4:

3) For any triangle Δ_α, where $\alpha > 0$, we have

$$AA_1 + BB_1 + CC_1 < \frac{2\sqrt{3}}{3}(4R + r).$$

Indeed, we can write the coefficient in (14) as follows

$$\frac{2}{\alpha+1}\sqrt{\frac{\alpha^2+\alpha+1}{3}} = \frac{2\sqrt{3}}{3}\sqrt{\frac{\alpha^2+\alpha+1}{(\alpha+1)^2}} = \frac{2\sqrt{3}}{3}\sqrt{1-\frac{\alpha}{(\alpha+1)^2}} < \frac{2\sqrt{3}}{3}.$$

References

[1] Andrica, D., Varga, Cs., Văcăreţu, D., *Selected Topics and Problems on Geometry* (Romanian), PLUS, Bucharest, 2002.

[2] Mitrinović, D.S., Pecarić, J., Volonec, V., *Recent Advances in Geometric Inequalities*, Kluwer, 1989.

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