Junior problems

J7. Prove that:

$$\sum_{n=1}^{9999} \frac{1}{\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt[4]{n}+\sqrt[4]{n+1}\right)} = 9.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Titu Andreescu, University of Texas at Dallas

Observe the following two facts

$$\frac{1}{\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt[4]{n}+\sqrt[4]{n+1}\right)} = \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt[4]{n+1}-\sqrt[4]{n}\right)}{(n+1-n)\left(\sqrt{n+1}-\sqrt{n}\right)},$$

and

$$\frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt[4]{n+1}-\sqrt[4]{n}\right)}{(n+1-n)\left(\sqrt{n+1}-\sqrt{n}\right)} = \sqrt[4]{n+1}-\sqrt[4]{n}.$$

Thus the sum telescopes,

$$\sum_{n=1}^{9999} \frac{1}{\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt[4]{n}+\sqrt[4]{n+1}\right)} = \sum_{n=1}^{9999} \sqrt[4]{n+1}-\sqrt[4]{n} = \sqrt[4]{10000}-\sqrt[4]{1} = 9.$$

Also solved by Iurie Boreico, Moldova; Adeel Khan

J8. Let a, b be distinct real numbers such that

$$|a-1|+|b-1|=|a|+|b|=|a+1|+|b+1|.$$

Find the minimal possible value of $|a-b|$.

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College, Romania

First solution by Bogdan Enescu, "B.P.Hasdeu" National College, Romania

The case $a=b$ can be easily discarded. Suppose that $a<b$ and consider the function $f:\mathbb{R}\longrightarrow\mathbb{R}, f(x)=|x-a|+|x-b|$. It is not difficult to see that f is decreasing on $(-\infty, a]$, constant on $[a, b]$, and increasing on $[b,+\infty)$. The condition in the hypothesis is equivalent to $f(-1)=f(0)=f(1)$, hence -1 and 1 must belong to the interval $[a, b]$. We conclude that $|a-b|\leq 2$.

Second solution by Daniel Campos Salas, Costa Rica

Squaring all the given identities and canceling repeating terms, we get

$$|(a-1)(b+1)|=|ab|=|(a+1)(b-1)|.$$

Since, $a\neq b$, it follows that

$$(a-1)(b+1)\neq(a+1)(b-1)\Rightarrow(a-1)(b+1)=-(a+1)(b-1)\Rightarrow ab=1.$$

Using the above we have that $|a-b|=|(a-1)(b+1)|=|ab|=1$.

Squaring the above and adding $4ab$, it follows that

$$(a+b)^2=(a-b)^2+4ab=1+4=5\Rightarrow|a+b|=\sqrt{5},$$

which concludes that the minimal(and only) possible value of $|a+b|$ is $\sqrt{5}$

J9. Show that the following equation does not have integral solutions

$$(x-y)^2 - 5(x+y) + 25 = 0.$$

Proposed by Ovidiu Pop, Satu Mare, Romania

First solution by Ovidiu Pop, Satu Mare, Romania

Suppose the contrary. Solving as a quadratic in x gives the discriminant $40y-75$ which has to be a square, hence y is divisible by 5. From the original equation we find that x must be divisible by 5 as well. Replacing $x = 5x', y = 5y'$ gives $(x'-y')^2 - (x'+y') + 1 = 0$. Solving again as a quadratic in x gives the discriminant $8y-3$ which is never a square.

Second solution by Daniel Campos Salas, student, Costa Rica

We shall prove a lemma first:

Lemma: *The equation $(x-y)^2 - p(x+y) + p^2 = 0$ has no solutions in integers, when p is an odd prime.*

Proof: Suppose for the sake of contradiction the lemma is false and such integers exist.

Note that the equation can be rewritten as

$$(x-y)^2 = p\big((x+y)-p\big),$$

this implies that $p|(x-y)^2$, but since p is a prime it follows that

$$p|x-y \tag{1}$$

From (1) it follows that the equation can be rearranged as

$$p^2\left(\left(\frac{x-y}{p}\right)^2 + 1\right) = p(x+y),$$

from where we conclude that

$$p|x+y \tag{2}$$

From (1) and (2) we have that

$$p|2x, 2y,$$

but since $(p, 2) = 1$, we conclude that

$$p|x, y$$

Let $x = pm, y = pn$, this implies the original equation turns to

$$(m-n)^2 - (m+n) + 1 = 0, \text{ or}$$

$$(m^2 - m) + (n^2 - n) - (2mn) = -1,$$

but each term inside the brackets is even, then this implies that 1 is even, which is a contradiction, thus our lemma is true.

It remains just to set $p = 5$ to obtain the desired result $\blacksquare$

Also solved by Iurie Boreico, Moldova; Adeel Khan

J10. Let $A = 1! \cdot 2! \cdot \ldots \cdot 1002!$ and $B = 1004! \cdot 1005! \cdot \ldots \cdot 2006!$. Prove that $2AB$ is a square and that $A + B$ is not a square.

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College,Romania

First Solution by Iurie Boreico, Moldova

Write, $a \sim b$, if $\frac{a}{b}$ is a perfect square in $\mathbb{Q}$. The relation is clearly reflexive, symmetric, multiplicative and transitive. Let us show that $2AB \sim 1$. We have the relation $k!(k+1)! \sim (k+1)$(their ratio is $(k!)^2$). Therefore,

$$A \sim 2 \cdot 4 \cdot \ldots \cdot 1002, B \sim 1004 \cdot 1006 \cdot 1008 \cdot \ldots \cdot 2006.$$

So,

$$AB \sim \frac{1}{1004} 2 \cdot 4 \cdot \ldots \cdot 2006 \cdot 1004! = \frac{1}{1004} \cdot 2^{1003} \cdot 1003! \cdot 1004! \sim \frac{1}{1004} \cdot 2 \cdot 1004 = 2.$$

So

$$2AB \sim 4 \sim 1,$$

as desired.

For the second part, observe that the exponent of 499 in the prime factorization of A is odd, because

$$A \sim 2 \cdot 4 \cdot \ldots \cdot 1004,$$

and the exponent of 499 in $2 \cdot 4 \cdot \ldots \cdot 1004$ is 1. The exponent of 499 in B is strictly greater then in A, therefore, the exponent of 499 in $A + B$ equals the exponent of 499 in A, which is an odd number, so $A+B$ is not a perfect square.

J11. Consider an arbitray parallelogram $ABCD$ with center O and let P be a point different from O, that satisfies $PA \cdot PC = OA \cdot OC$ and $PB \cdot PD = OB \cdot OD$. Show that the sum of lenghts of two of the segments PA, PB, PC, PD equals the sum of lenghts of the other two.

Proposed by Iurie Boreico, Moldova

Solution by Iurie Boreico, Moldova

In out proof we will use the following lemma:

Lemma:
Consider a triangle ABC is which $AM \cdot BM = AB \cdot AC$ where M is the midpoint of BC. Then $AM = |AB - AC|$

Proof:
Let a, b, c be the sides of the triangle ABC. Then our relation turns into $a^2 = 4bc$, hence

$$\frac{2b^2 + 2c^2 - a^2}{4} = \frac{2b^2 + 2c^2 - 4bc}{4} = \frac{(b-c)^2}{2}.$$

thus, $AM = \frac{1}{\sqrt{2}}|AB - AC|$, as desired.

By the lemma we get that $|PA - PC| = |PB - PD| = \sqrt{2}OP$ and so $|PA - PC| = |PB - PD|$ and this means that the largest of PA, PC plus the smallest of PB, PD equals the largest of PB, PD plus the smallest of PA, PC.

J12. Let $ABCD$ be a convex quadrilateral. A square is called inscribed if its vertices lie on different sides of $ABCD$. If there are two different squares inscribed in $ABCD$, prove that there are infinitely many squares inscribed in $ABCD$.

Proposed by Iurie Boreico, Moldova

Solution by Iurie Boreico, Moldova

Let $X \in (AB), Y \in (AD)$, the perpendiculars from X, Y to XY intersect BC, CD in Z, T. Then $XYTZ$ is an inscribed square if and only if $XZ = YT = XY$(it may not be inscribed if, for example, the perpendicular from X meets CD before BC, but we shall deal with this case later). Set $XY = l$ and $\angle AYX = x$. From the law of sines we get:

$$AD = AY + DY = XY\frac{\sin x}{\sin A} + YT\frac{\cos x}{\sin B}.$$

Thus $YT = l$ if and only if $a\sin x + b\cos x = \frac{AD}{l}$(*) where a, b are non-zero positive constants (they equal $\frac{1}{\sin A}$ and $\frac{1}{\sin B}$).

Analogously we get hat $XT = l$ if and only if

$$a' \sin 180 - A - x + b' \cos 180 - A - x = \frac{AB}{l},$$

which from basic trigonometric formulae is equivalent to $c\sin x + d\cos x = \frac{AB}{l}$(**). Particularly from (*),(**) we deduce

$$AB(c\sin x + d\cos x) = AD(a\sin x + b\cos x). \qquad (***)$$

Now we know that we have two different inscribed squares. Their sides can not be parallel (this is very easy to prove), so there are two different angles x_1, x_2 that satisfy (***).

However (***) can be written as $(ABc - ADa)\sin x + (ABd - ADb)cosx = 0$. If $(ABc - ADa) \neq 0$ we can write this condition as $\sin(x+t) = 0$ which has a single solution between 0 and 180, and this is not the case as there are two solutions. So $ABc - ADa = 0$. If $ABd - ADb \neq 0$ then (***) transforms to $\cos x = 0$ which has again a single solution and this is not the case. So $Abd - ADb = 0$ which means (***) holds for all x. Then for any x we can find an l for which (*) holds. (**) will hold immediately from dividing (*) and (***). So this will mean the existence of a squares $XYTZ$ with corners on the sides of $ABCD$ with side-length l and with angle $\angle AYX = x$. Taking x close enough to x_1 the square $XYZT$ will be inscribed, and so we get infinitely many squares as we can take infinitely many x.

Senior problems

S7. Let $x_1, x_2, \ldots, x_n$ be real numbers greater than or equal to $\frac{1}{2}$. Prove that

$$\prod_{i=1}^{n}\left(1+\frac{2x_i}{3}\right)^{x_i} \geq \left(\frac{4}{3}\right)^{n} \sqrt[4]{(x_1+x_2)(x_2+x_3)\ldots(x_{n-1}+x_n)\,(x_n+x_1)}.$$

Proposed by Iurie Boreico and Marcel Teleucă, Moldova

Solution by Iurie Boreico and Marcel Teleucă, Moldova

Using the Cauchy-Schwartz inequality, we have

$$\sqrt{\left(x^2+\frac{3}{4}\right)\left(y^2+\frac{3}{4}\right)} = \sqrt{\left(x^2+\frac{1}{4}+\frac{1}{2}\right)\left(\frac{1}{4}+y^2+\frac{1}{2}\right)} \geq \frac{x+y+1}{2} \geq \sqrt{x+y}. \quad (*)$$

However, since $2x_i \geq 1$ we have

$$\left(1+\frac{2x_i}{3}\right)^{2x_i} \geq 1+\frac{4}{3}x_i^2 = \frac{4}{3}\left(x_i^2+\frac{3}{4}\right). \quad (**)$$

Now applying (*) for x_i, x_{i+1} and multiplying them, using (**) we deduce the desired inequality.

S8. Let O, I, and r be the circumcenter, incenter, and inradius of a triangle ABC. Let M be a point inside the triangle, and let d_a, d_b, d_c, be the distances from M to the sides BC, AC, AB. Prove that if $d_a \cdot d_b \cdot d_c \geq r^3$, then M lies inside the circle with center O and radius OI.

Proposed by Ivan Borsenco, University of Texas at Dallas

Solution by Ivan Borsenco, University of Texas at Dallas

Let A_1, B_1, C_1 be the projections of M onto the sides BC, AC, AB, triangle the $A_1B_1C_1$ is called the podar triangle og M in triangle ABC. We have $\angle B_1MC_1 = 180 - \alpha$, $\angle A_1MC_1 = 180 - \beta$, $\angle A_1MB_1 = 180 - \gamma$, then

$$2 \cdot A_{ABC} = 2 \cdot S = a \cdot d_a + b \cdot d_b + c \cdot d_c,$$

$$2 \cdot A_{A_1B_1C_1} = 2 \cdot S_1 = d_b \cdot d_c \cdot sin\alpha + d_a \cdot d_c \cdot sin\beta + d_a \cdot d_b \cdot sin\gamma$$

or

$$2 \cdot S_1 = \frac{d_a \cdot d_b \cdot d_c}{2 \cdot R} \cdot (\frac{a}{d_a} + \frac{b}{d_b} + \frac{c}{d_c}).$$

Applying the Cauchy-Schwartz inequality we get:

$$4 \cdot S \cdot S_1 = \frac{d_a \cdot d_b \cdot d_c}{2 \cdot R}(a \cdot d_a + b \cdot d_b + c \cdot d_c)(\frac{a}{d_a} + \frac{b}{d_b} + \frac{c}{d_c}) \geq \frac{d_a \cdot d_b \cdot d_c (a+b+c)^2}{2 \cdot R},$$

and by the podar triangle theorem $\frac{S_1}{S} = \frac{R^2 - OM^2}{4 \cdot R^2}$,

so, $\frac{4 \cdot S^2 (R^2 - OM^2)}{4 \cdot R^2} \geq \frac{d_a \cdot d_b \cdot d_c (a+b+c)^2}{2 \cdot R}$.

Finally let $S = p \cdot r$ and $a + b + c = 2 \cdot p$: $\frac{r^2 (R^2 - OM^2)}{2 \cdot R} \geq d_a \cdot d_b \cdot d_c$.

Returning to our initial problem, let us consider the set of points M such that $d_a \cdot d_b \cdot d_c \geq r^3$, then they satisfy one necessary condition: $\frac{r^2 \cdot (R^2 - OM^2)}{2 \cdot R} \geq r^3$. This means $R^2 - OM^2 \geq 2 \cdot R \cdot r$ or $OI^2 = R^2 - 2 \cdot R \cdot r \geq OM^2$, thus for every point M in the defined set we proved that $OM \leq OI$, and so we are done.

S9. Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Prove that

$$\prod_{k=1}^{n}\left(\sum_{k=1}^{n} a_k^{T_k}\right) \geq \left(\sum_{k=1}^{n} a_k^{\frac{T_{n+1}}{3}}\right)^n,$$

where $T_k = \frac{k(k+1)}{2}$ is the k^{th} triangular number.

Proposed by José Luis Díaz-Barrero, Spain

Solution by José Luis Díaz-Barrero, Spain

We consider the function $f(x) = \ln(a_1^x + a_2^x + \cdots + a_n^x)$ that is convex in $\mathbb{R}$, as can be easily proved. Applying Jensen's inequality to $f(x)$, we obtain

$$\sum_{k=1}^{n} p_k \ln\left(a_1^{x_k} + \cdots + a_n^{x_k}\right) \geq \ln\left(a_1^{\sum_{k=1}^{n} p_k x_k} + \cdots + a_n^{\sum_{k=1}^{n} p_k x_k}\right),$$

where p_k are positive numbers of sum one and $x_1, x_2, \cdots, x_n \in \mathbb{R}$. Taking into account that $f(x) = \ln(x)$ is injective, then the preceding expression becomes

$$\ln\left(\prod_{k=1}^{n}\left(\sum_{k=1}^{n} a_k^{x_k}\right)^{p_k}\right) \geq \ln\left(a_1^{\sum_{k=1}^{n} p_k x_k} + \cdots + a_n^{\sum_{k=1}^{n} p_k x_k}\right),$$

or equivalently,

$$\prod_{k=1}^{n}\left(\sum_{k=1}^{n} a_k^{x_k}\right)^{p_k} \geq \left(a_1^{\sum_{k=1}^{n} p_k x_k} + \cdots + a_n^{\sum_{k=1}^{n} p_k x_k}\right).$$

Setting $p_k = \dfrac{1}{n},\ 1 \leq k \leq n$ and $x_k = T_k,\ 1 \leq k \leq n$, and taking into account that $\displaystyle\sum_{k=1}^{n} T_k = \frac{n}{3} T_{n+1}$, we have

$$\prod_{k=1}^{n}\left(\sum_{k=1}^{n} a_k^{T_k}\right)^{1/n} \geq \left(\sum_{k=1}^{n} a_k^{\frac{T_{n+1}}{3}}\right),$$

and we are done.

Also solved by Iurie Boreico, Moldova

S10. Let $(a_n)_{n\geq 1}$ be a sequence of positive numbers such as $a_{n+1} = a_n^2 - 2$ for all $n \geq 1$. Show that for all $n \geq 1$ we have $a_n \geq 2$.

Proposed by Dr. Laurenţiu Panaitopol, University of Bucharest, Romania

Solution by Bogdan Enescu, "B.P.Hasdeu" National College,Romania

Define the sequence $(b_n)_{n\geq 1}$ as follows:

$$b_1 = \sqrt{2}, b_{n+1} = \sqrt{2 + b_n},$$

for all n. Observe that $a_n > b_m$ is equivalent to:

$$a_n^2 - 2 > b_m^2 - 2,$$

or

$$a_{n+1} > b_{m-1}.$$

Now, since $a_{n+1} > 0$, it follows that $a_n > \sqrt{2}$, that is, $a_n > b_1$. We deduce succesively,

$$a_{n-1} > b_2, a_{n-2} > b_3, \ldots, a_1 > b_n,$$

for all n. It is easy to prove inductively that

$$b_n = 2\cos\frac{\pi}{2^{n+1}},$$

hence

$$\lim_{n\to\infty} b_n = 2.$$

We obtained that $a_1 \geq 2$. An obvious inductive argument shows that $a_n \geq 2$, for all n.

S11. Consider the sequences given by

$$a_0 = 1, \quad a_{n+1} = \frac{3a_n + \sqrt{5a_n^2 - 4}}{2}, \quad n \geq 1,$$
$$b_0 = 0, \quad b_{n+1} = a_n - b_n, \quad n \geq 1.$$

Prove that $(a_n)^2 = b_{2n+1}$.

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Bogdan Enescu, "B.P.Hasdeu" National College,Romania

Squaring the first equation yields

$$a_{n+1}^2 + a_n^2 - 3a_n a_{n+1} + 1 = 0.$$

Substitute n by $n+1$ gives

$$a_{n+2}^2 + a_{n+1}^2 - 3a_{n+1}a_{n+2} + 1 = 0.$$

Subtracting the above equalities we obtain

$$(a_{n+2} - a_n)(a_{n+2} - 3a_{n+1} + a_n) = 0.$$

It is easy to see that (a_n) is increasing, hence $a_{n+2} - a_n \neq 0$. We deduce that the sequence satisfies the following recursive equation:

$$a_{n+2} = 3a_{n+1} - a_n,$$

for all n. Furthermore, we have $a_0 = 1, a_1 = 2$, hence all the terms are positive integers. We prove the requested equality by induction. The base case is easy to check, so let us suppose that $b_{2n-1} = (a_{n-1})^2$, and prove that $b_{2n+1} = (a_n)^2$. We have

$$b_{2n+1} = a_{2n} - b_{2n} = a_{2n} - a_{2n-1} + b_{2n-1}.$$

Therefore, the equality we want to prove is equivalent to

$$(a_n)^2 - (a_{n-1})^2 = a_{2n} - a_{2n-1}.$$

Observe that for $n \geq 2, k \geq 1$ we have

$$a_k a_n - a_{k-1}a_{n-1} = a_k(3a_{n-1} - a_{n-2}) - a_{k-1}a_{n-1} = a_{n-1}(3a_k - a_{k-1}) - a_k a_{n-2} = a_{k+1}a_{n-1} - a_k a_{n-2}.$$

Using the latter, we obtain

$$(a_n)^2 - (a_{n-1})^2 = a_n a_n - a_{n-1}a_{n-1} = a_{n+1}a_{n-1} - a_n a_{n-2} = \ldots = a_{2n-1}a_1 - a_{2n-2}a_0.$$

Finally, observe that

$$a_{2n-1}a_1 - a_{2n-2}a_0 = 2a_{2n-1} - a_{2n-2} = 3a_{2n-1} - a_{2n-2} - a_{2n-1} = a_{2n} - a_{2n-1},$$

as desired.

S12. Let a be a real number. Prove that

$$5(\sin^3 a + \cos^3 a) + 3\sin a \cos a = .04$$

if and only if

$$5(\sin a + \cos a) + 2\sin a \cos a = .04.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Titu Andreescu, University of Texas at Dallas

The first equality can be written as

$$\sin^3 a + \cos^3 a + \left(-\frac{1}{5}\right)^3 - 3(\sin a)(\cos a)\left(-\frac{1}{5}\right) = 0.$$

We have seen that the expression $x^3 + y^3 + z^3 - 3xyz$ factors as

$$(x+y+z)\frac{1}{2}[(x-y)^2 + (y-z)^2 + (z-x)^2].$$

Here $x = \sin a$, $y = \cos a$, $z = -\frac{1}{5}$, and $x^3 + y^3 + z^3 - 3xyz = 0$. It follows that $x + y + z = 0$ or $x = y = z$. The latter would imply $\sin a = \cos a = -\frac{1}{5}$, which violates the identity $\sin^2 a + \cos^2 a = 1$. Hence $x + y + z = 0$, implying $\sin a + \cos a = \frac{1}{5}$. This yields $5(\sin a + \cos a) = 1$ and

$$\sin^2 a + 2\sin a \cos a + \cos^2 a = \frac{1}{25}.$$

It follows that $1 + 2\sin a \cos a = .04$, hence

$$5(\sin a + \cos a) + 2\sin a \cos a = .04,$$

as desired.

Conversely,

$$5(\sin a + \cos a) + 2\sin a \cos a = .04$$

implies

$$125(\sin a + \cos a) = 1 - 50\sin a \cos a.$$

Squaring both sides and setting $2\sin a \cos a = b$, yields

$$125^2 + 125^2 b = 1 - 50b + 25^2 b^2,$$

which simplifies to

$$(25b + 24)(25b - 651) = 0.$$

We obtain $2\sin a\cos a = -\frac{24}{25}$, or $2\sin a\cos a = \frac{651}{25}$. The latter is impossible because $\sin 2a \leq 1$. Hence $2\sin a\cos a = -.96$, and we obtain $\sin a + \cos a = .2$. Then

$$\begin{aligned}
&5(\sin^3 a + \cos^3 a) + 3\sin a\cos a \\
&\quad = 5(\sin a + \cos a)(\sin^2 a - \sin a\cos a + \cos^2 a) + 3\sin a\cos a \\
&\quad = \sin^2 a - \sin a\cos a + \cos^2 a + 3\sin a\cos a \\
&\quad = (\sin a + \cos a)^2 = (.2)^2 = .04,
\end{aligned}$$

as desired.

Undergraduate problems

U7. Evaluate
$$\int_1^e \frac{1+x^2\ln x}{x+x^2\ln x}dx.$$

Proposed by Zdravko Starc, Vrsac, Serbia and Montenegro

Solution by Ovidiu Furdui, Western Michigan University

The integral equals: $e-\ln(1+e)$. To see this we observe that the following fraction decomposition holds:

$$\frac{1+x^2\ln x}{x+x^2\ln x}=\frac{1}{x}+1-\frac{1+\ln x}{1+x\ln x}$$

Since
$$\frac{d}{dx}\ln(1+x\ln x)=\frac{1+\ln x}{1+x\ln x}$$

this implies that:

$$\int_1^e \frac{1+x\ln x}{x+x^2\ln x}\,dx=\int_1^e \frac{1}{x}\,dx+\int_1^e dx-\int_1^e \frac{1+\ln x}{1+x\ln x}\,dx.$$

Hence

$$\int_1^e \frac{1+x\ln x}{x+x^2\ln x}\,dx=e-\ln(1+e).$$

Also solved by José Luis Díaz-Barrero, Spain

U8. Let ω_n be an n-th primitive root of the unity and let

$$H_n = \prod_{i=0}^{n-1}(1 + \omega_n^i - \omega_n^{2i}).$$

Show that H_n is an integer and give a formula for H_n which uses only integers.

Proposed by Mietek Dabkowski, The University of Texas at Dallas; and by Josef Przytycki, George Washington University

Solution by Patrick, Paris, France

Let ϕ_1 and ϕ_2 the two real roots of equation $x^2 - x - 1 = 0$. Then

$$H_n = \prod_{k=0}^{k=n-1} (-1)(\phi_1 - \omega_n^k)(\phi_2 - \omega_n^k).$$

But, since ω_n is a primitive root of unity, then $\{\omega_n^k\}_{k=0,n-1}$ is the set A_n of all n-th roots of unity. So

$H_n = \prod_{r \in A_n}(-1)(\phi_1 - r_k)(\phi_2 - r_k)$

$H_n = (-1)^n \prod_{r \in A_n}(\phi_1 - r_k) \prod_{r \in A_n}(\phi_2 - r_k)$

$H_n = (-1)^n(\phi_1^n - 1)(\phi_2^n - 1)$

$H_n = (-1)^n((-1)^n + 1 - \phi_1^n - \phi_2^n)$

$H_n = (-1)^{n+1}F_n + (-1)^n + 1$, where F_n is the n-th element of the Fibonacci sequence.

U9. Let $\|\cdot\|$ be a norm on $\mathcal{M}_n(\mathbb{C})$ and let $A_1, A_2, \ldots, A_p$ be complex matrices of order n. Prove that for every $x > 0$ there exists $z \in \mathbb{C}$, with $|z| < x$, such that

$$\left\|(I_n - zA_1)^{-1} + (I_n - zA_2)^{-1} + \cdots + (I_n - zA_p)^{-1}\right\| \geq p.$$

Proposed by Gabriel Dospinescu, "Louis le Grand" College, Paris

Solution by Gabriel Dospinescu, "Louis le Grand" College, Paris

First Solution

The problem is a consequence of the maximum modulus principle applied to the analytic function $f : V \to \mathcal{M}_n(\mathbb{C}), f(z) = \sum_{i=1}^{p}(I_n - zA_i)^{-1}$, defined on a neighborhood V of 0_n.

Second Solution

Let us prove first that for any matrix A we have

$$I_n = \frac{1}{2\pi}\int_0^{2\pi}(I_n - re^{-it}A)^{-1}dt$$

if r is a sufficiently small. Indeed, take $\|\cdot\|$ an algebra norm on $\mathcal{M}_n(\mathbb{C})$ and take $R > \|A\|$. Then $\|R^{-j}e^{-ijt}A^j\| \leq \|R^{-1}A\|^j$ and ,therefore,

$$\sum_{j=0}^{\infty} R^{-j}e^{-ijt}A^j,$$

converges absolutely. Clearly, its sum is

$$(I_n - \frac{1}{R}e^{-it}A)^{-1}.$$

Moreover, we have

$$Re^{it}(Re^{it}I_n - A)^{-1} = \sum_{i=0}^{\infty} R^{-j}e^{-ijt}A^j,$$

and by normal convergence of the series we can write

$$\frac{1}{2\pi}\int_0^{2\pi} Re^{it}(Re^{it}I_n - A)^{-1} = I_n.$$

Therefore,

$$pI_n = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{j=1}^{p}(I_n - re^{-it}A_j)^{-1}\right)dt,$$

for sufficiently small p. By taking norms, the conclusion follows.

U10. Find all functions $f : [0, +\infty) \to [0, +\infty)$, differentiable at $x = 1$ and satisfying

$$f(x^3) + f(x^2) + f(x) = x^3 + x^2 + x,$$

for all $x \geq 0$.

Proposed by Mihai Piticari, Campulung, Romania

First solution by Iurie Boreico, Moldova

Let $g(x) = f(x) - x$, then g is differentiable in 1, so by substituting $x = 1$ and then taking the derivative we get $g(1) = g'(1) = 0$. We will prove that $g(x) = 0$.

Indeed suppose that $a = g(t) \neq 0, t \neq 1$. Consider $x_0 = t$ and construct x_k inductively as follows: Let

$$u^3 = x^{k-1},$$

then

$$g(u) + g(u^2) + g(u^3) = 0,$$

so,

$$|g(u) + g(u^2)| \geq |g(u^3)|.$$

Then, either

$$g(u) \geq \frac{1}{3} g(u^3,)$$

or,

$$g(u^2) \geq \frac{2}{3} g(u^3)$$

. In any case we can select $x_k \in \{u, u^2\}$ with

$$|g(x_k)| \geq \frac{\ln x_k}{\ln x_{k-1}} |g(x_{k-1})|,$$

and hence by induction

$$|g(x_k)| \geq \ln(x_k) \frac{|g(t)|}{\ln |}.$$

It's pretty clear that x_k converges to 1 and we have

$$\left| \frac{g(x_k) - 1}{x_k - 1} \right| \geq \left| \frac{\ln(x_k)}{|x_k - 1|} \right| \left| \frac{a}{\ln t} \right|.$$

But as $x_k \to 1$, $\frac{\ln x_k}{x_k - 1} \to 1$ and this contradicts the fact that

$$\lim_{x \to 1} \frac{g(x) - g(1)}{x - 1} = 0.$$

Second solution by Li Zhou, Winter Haven FL

Clearly, $f(x) = x$ is a solution. We show that it is the only solution. Let $g(x) = f(x) - x$ for $x \geq 0$. Then g is differentiable at $x = 1$ and

$$g(x^3) + g(x^2) + g(x) = 0, \tag{3}$$

for all $x \geq 0$. Letting $x = 0$ and $x = 1$ in (1) we see that $g(0) = 0 = g(1)$. Differentiating (1) at $x = 1$ we get $g'(1) = 0$.

Now take any $a \in (0, \infty)$ and $a \neq 1$. Letting $x = a^{1/3}$ in (3) we get $g(a) = -\left(g(a^{1/3}) + g(a^{2/3})\right)$. By an easy induction we find

$$g(a) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} g(a^{2^k/3^n})$$

for all $n \in$. Since $g'(1) = 0$, for any $\epsilon > 0$, there exists $\delta \in (0,1)$ such that $|g(x)| < \epsilon |x - 1|$ whenever $0 < |x - 1| < \delta$. Moreover, $a^x = 1 + x \ln a + O(x^2)$ as $x \to 0$. Hence, for sufficiently large n,

$$\begin{aligned} |g(a)| &\leq \sum_{k=0}^{n} \binom{n}{k} |g(a^{2^k/3^n})| < \epsilon \sum_{k=0}^{n} \binom{n}{k} |a^{2^k/3^n} - 1| \\ &\leq \epsilon \left(\frac{|\ln a|}{3^n} \sum_{k=0}^{n} \binom{n}{k} 2^k + \frac{C}{3^{2n}} \sum_{k=0}^{n} \binom{n}{k} 2^{2k} \right) = \epsilon \left(|\ln a| + C \left(\frac{5}{9}\right)^n \right), \end{aligned}$$

where C is some fixed constant. Since $\epsilon > 0$ is arbitrary, we conclude that $g(a) = 0$. Hence $g \equiv 0$, completing the proof. $\square$

Comment. In fact, for any $p - 1 \geq q > 1$, we reach the same conclusion if we change the functional equation into

$$f(x^p) + f(x^q) + f(x) = x^p + x^q + x.$$

U11. Two players, A and B, play the following game: player A divides an $n \times n$ square into strips of unit width (and various lenghts). After that, player B picks an integer k, $1 \leq k \leq n$, and removes all strips of lenght k. Let $l(n)$ be the largest area B can remove, regardless the way A divides the square into strips.

Proposed by Iurie Boreico, student, Moldova

Solution by Iurie Boreico, Moldova

Set $m = l(n)$. Then there are at most $[\frac{m}{k}]$ strips of length k, therefore the total area of the parcel n^2 is at most $\sum_{k=1}^{n} k[\frac{m}{k}]$. So

$$\sum_{k=1}^{n} k[\frac{m}{k}] = mn - \sum_{k=1}^{n}(m \bmod k) \geq n^2.$$

Now, let's estimate $S = \sum_{k=1}^{n}(m \bmod k)$. Clearly $m \geq n$ from Pigeonhole principle. However $m < 2n$ because player A can divide the parcel trivially: one strip $1 \times n$, one strip $1 \times (n-1)$, one strip $(n-1) \times 1$ and so on (one strip $1 \times k$ and one $k \times 1$ for any $1 \leq k \leq n-1$).

Now,

$$S = \sum_{n \geq k > \frac{m}{2}} (m \bmod k) + T = \sum_{k \leq \frac{m}{2}} (n \bmod k)$$
$$= \sum_{n \geq k > \frac{m}{2}} (n-k) + T = O(m) + (n - \frac{m}{2})(\frac{m}{2} - \frac{m}{4}) + T.$$

Now, if $\frac{m}{k+1} < h \leq \frac{m}{k}$ then $(m \bmod h) = m - kh$. So

$$\sum_{\frac{m}{k+1} < h \leq \frac{m}{k}} (m \bmod h) = \sum_{\frac{m}{k+1} < h \leq \frac{m}{k}} m - kh.$$

Now as the number of integers h in the interval $(\frac{m}{k+1}; \frac{m}{k}]$ is $\frac{m}{k(k+1)} + O(1)$, and the numbers $m - kh$ are an arithmetic progression with first term between 0 and k and difference k we can compute $\sum_{\frac{m}{k+1} < h \leq \frac{m}{k}} m - kh$ to be

$$\frac{m}{k(k+1)} \times \frac{m}{2(k+1)} + mO(\frac{1}{k+1}).$$

Now summing this for all k from 2 to m we deduce that T is

$$\begin{aligned}\frac{1}{2}m^2(\sum_{k=2}^{m}\frac{1}{k(k+1)^2})+mO(\frac{1}{2}+\ldots+\frac{1}{m+1}) &= \frac{1}{2}m^2(\sum_{k=2}^{m}\frac{1}{k(k+1)^2})+O(m\ln m)\\ &= \frac{1}{2}m^2(\sum_{k=2}^{m}\frac{1}{k(k+1)^2})+o(m^2).\end{aligned}$$

The sum

$$\sum_{k=2}^{m}\frac{1}{k(k+1)^2}=\sum_{k=2}(\frac{1}{k}-\frac{1}{k+1}-\frac{1}{(k+1)^2})$$

tends to

$$\frac{1}{2}-(\frac{\pi^2}{6}-1-\frac{1}{4})=\frac{7}{4}-\frac{\pi^2}{6}$$

so

$$\sum_{k=2}^{m}\frac{1}{k(k+1)^2}=\frac{7}{4}-\frac{\pi^2}{6}+o(\frac{1}{m}),$$

which gives us for T the estimate $(\frac{7}{8}-\frac{\pi^2}{12})m^2+o(m^2)$.

Now return to the relation we obtained before

$$mn-O(m)-(n-\frac{m}{2})(\frac{m}{2}-\frac{m}{4})-T\geq n^2$$

which means

$$\frac{3}{8}m^2+\frac{1}{2}n^2-T\geq n^2$$

or

$$o(n^2)+(\frac{\pi^2}{12}-\frac{1}{2})m^2\geq\frac{n^2}{2}.$$

This, in turn, means that

$$\frac{m}{n}\geq\sqrt{\frac{6}{\pi^2-6}}+o(1)$$

thus the limit we seek is at least $\sqrt{\frac{6}{\pi^2-6}}$. We shall prove that this is the value we need.

From the results above we can easily deduce that we can find an m with $\frac{m}{n} = \sqrt{\frac{6}{\pi^2-6}} + O(1)$ such that $\sum_{k=1}^{n} k[\frac{m}{k}] > n^2$.

Let's show how to divide the parcel into strips such that there are at most $[\frac{m}{k}]$ strips of length k and thus proving that $l(n) \leq m$.

Indeed, consider a collection of strips containing $[\frac{n}{k}]$ strips of length k. Let's place them horizontally one by one, at each moment placing the largest strip possible (of course that strips must not overlap and must fit inside the parcel) in the rightmost allowed position. We prove that at some moment the whole parcel will be covered.

Note that we shall place some strips of length k and then we shall stop. There are two possible reasons for stopping:

a)all strips of length k are already used

b)We can not fit a strip of length k satisfying the required conditions anymore. This means that in any row there are at most $k-1$ unoccupied squares (clearly the unoccupied squares in each row are a few leftmost ones).

At some time, b) will occur (as the total area of the strips from the collection exceeds the area of the parcel). Let k be the largest integer such that we encounter b) when trying to place a strip of length k. Then there are at most $n(k-1)$ unoccupied squares, so

$$\sum_{i=k}^{n} i[\frac{m}{i}] + n(k-1) \geq n^2 + k.$$

If $i \geq \frac{n}{2}$ then $[\frac{m}{i}] \leq 2$ so if $k \geq \frac{n}{2}$ we can prove at most $n-1$ strips are already placed on the table so we can put one more string (as there is one completely free row). So, $k < \frac{n}{2}$, so $n-k \geq k$

However, in some rows there are even less then $k-1$ squares left: in the row containing the strip of length n there is no square left, in the row containing the strip of length 1 there is one square left and so on. More generally, pick up one string of length n, one of length $n-1$ etc., one string of length $n-k+2$. Let $r \leq n-k+1$ be the number of rows containing these strips.

The number of occupied squares is then at least

$$\begin{aligned}
&(n-r)(n-k+1) + n + (n-1) + \ldots + (n-k+2) \\
&\geq \frac{(2n-k+2)(k-1)}{2} + (n-k+1)(n-k+1) = n(n-k+1) + \frac{k^2+k}{2}.
\end{aligned}$$

Let us prove now that $k < m - n$. Indeed, suppose not. Let $l = [\frac{m+1}{2}] - k$. Then we have

$$n + (n-1) + \ldots + [\frac{m}{2}] + 1 + \sum_{j=k}^{[\frac{m}{2}]} j[\frac{m}{j}] \geq n(n-k+1) + \frac{k^2+k}{2} + k,$$

(the last term k is because one strip of length k is unused).

Since $j[\frac{m}{j}] \leq m$, we deduce

$$\frac{n^2 - [\frac{m}{2}]^2 + n - [\frac{m}{2}]}{2} + m([\frac{m}{2}] - k + 1) \geq n(n-k+1) + \frac{k^2+k}{2} + k,$$

thus

$$\frac{n^2 - \frac{m^2}{4}}{2} + n - (\frac{m}{2})^2 + m(\frac{m}{2} - k + 1) \geq n(n-k+1) + \frac{k^2+3k}{2})),$$

so,

$$\frac{3m^2}{8} \geq \frac{n^2}{2} + (m-n)k + \frac{k^2+3k}{2}.$$

As $k > (m-n)$ we deduce

$$\frac{3m^2}{8} > \frac{n^2}{2} + (m-n)^2 + (m-n) + \frac{(m-n)^2 + 3(m-n)}{2},$$

then

$$\frac{3m^2}{4} + m > n^2 + 3(m-n)^2 + 5(m-n),$$

thus

$$\frac{9}{4}m^2 - 6mn + 4n^2 + 3(m-n) \geq m,$$

hence

$$(\frac{3m}{2} - 2n)^2 \leq 5n - 4m,$$

or

$$(3m-4n)^2 \leq 4n - 3m - (m-n) < 4n - 3m - \frac{1}{4},$$

implying that $(3m - 4n - \frac{1}{2})^2 < 0$.

This contradiction shows us that $k < m - n$. Now let's prove by induction on $0 \le j \le k - 1$ that after we have placed all strips of length $k - j$ we could, there are at most $n(k - j - 1)$ squares uncovered. The basis is obvious, as the uncovered area is at most $n(k-1)$. Now let's do the induction step: from $j - 1$ to j: If we encountered b) after placing the strips of length $k - j$, then in every row there are at most $k - j - 1$ squares left so there are at most $n(k - j - 1)$ clear fields left. Finally, if we encountered situation a), we have used all strips of length $k - j$, so $(k - j)[\frac{m}{k-j}]$. As $k - j \le n - m$ we have

$$(k-j)[\frac{m}{k-j}] > (k-j)(\frac{m}{k-j} - 1) = m - (k-j) \ge m - (m-n) = n,$$

so these steps occupy at least n squares. Since there were at most $n(n - j)$ squares left before by the induction step, after placing the strips of length $k - j$ we have at most $n(n - j) - n = n(n - j + 1)$ free squares, and the induction is finished. Now by setting $j = k - 1$ we see that after using the strips 1×1 there are at most $n(1 - 1) = 0$ squares left, so the whole parcel is filled. This finishes the proof that $\lim_{n\to\infty} \frac{l(n)}{n} = \sqrt{\frac{6}{\pi^2-6}}$.

Moreover, we have proven above that $l(n)$ is actually the least integer m that satisfies $\sum_{k=1}^{n} k[\frac{m}{k}] \ge n^2$. Searching for such a m in the neighborhood of $10 \times \sqrt{\frac{6}{\pi^2-6}}$ we find that $l(10) = 13$.

U12. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that $e^{a_n} + na_n = 2$, for all positive integers n. Evaluate

$$\lim_{n\to\infty} n\left(1 - na_n\right).$$

Proposed by Teodora Liliana Rădulescu,Craiova, Romania

Solution by Teodora Liliana Rădulescu, Craiova, Romania

For any integer $n \geq 1$, denote $f_n(x) = e^x + nx - 2$. Since $f_n'(x) = e^x + n > 0$, we deduce that f_n increases on $\mathbb{R}$. So, by $f_n(0) = -1 < 0$ and $f_n(\ln 2) = n \ln 2 > 0$, we conclude that for all integer $n \geq 1$, there exists a unique $a_n \in (0, \ln 2)$ such that $f_n(a_n) = 0$. Next, we observe that

$$f_{n+1}(a_n) = e^{a_n} + na_n - 2 + a_n = f_n(a_n) + a_n = a_n > 0.$$

Since $f_{n+1}(a_{n+1}) = 0$ and f_n increases for any $n \geq 1$, we deduce that $a_n > a_{n+1}$, for all $n \geq 1$. So, there exists $\ell := \lim_{n\to\infty} a_n \in [0, \ln 2)$. If $\ell > 0$ then the recurrence relation $e^{a_n} + na_n = 2$ yields the contradiction $+\infty = 2$. Thus, $\ell = 0$ and $\lim_{n\to\infty} na_n = 1$, that is $a_n \sim n^{-1}$ as $n \to \infty$. Using again the recurrence relation we obtain

$$1 - na_n = e^{a_n} - 1 \sim e^{1/n} - 1 \qquad \text{as } n \to \infty.$$

Since $e^x - 1 = x + o(x)$ as $x \to 0$ we deduce that

$$1 - na_n \sim \frac{1}{n} \qquad \text{as } n \to \infty.$$

It follows that $\lim_{n\to\infty} n(1 - na_n) = 1$.

Remarks. (i) If (a_n) and (b_n) are sequences of positive real numbers, we have used the notation

$$a_n \sim b_n \qquad \text{as } n \to \infty$$

if $\lim_{n\to\infty} a_n/b_n = 1$.

(ii) With similar arguments as above (which are based on the asymptotic expansion of e^x around $x = 0$) we can prove that $\lim_{n\to\infty} n[n(1 - na_n) - 1] = 1/2$.

Olympiad problems

O7. n the convex hexagon $ABCDEF$ the following equalities hold:

$$AD = BC + EF, BE = AF + CD, CF = AB + DE.$$

Prove that

$$\frac{AB}{DE} = \frac{CD}{AF} = \frac{EF}{BC}.$$

Proposed by Nairi Sedrakyan, Armenia

Solution by Ivan Borsenco, University of Texas at Dallas

First we will prove the following lemma:

Lemma: Let $ABCD$ be an arbitrary quadrilateral. Denote by a, b, c, d it's sides and e, f diagonals, then $a^2 + c^2 + 2bd \geq e^2 + f^2$.

Proof: Denote by M, N, P the midpoints of BC, AD, BD, by the triangle inequality $MN \leq MP + NP$ or $2MN \leq b + d$. Using Euler's theorem for quadrilaterals and midpoints of opposite sides or diagonals we get:

$$b^2 + d^2 + e^2 + f^2 = a^2 + c^2 + 4MN^2 \leq a^2 + c^2 + (b + d)^2.$$

The equality holds when $BC||AD$ and our lemma is proved.

Now we will solve the problem.

Let $AB = a, BC = b, CD = c, DE = d, EF = e, AF = f, AC = x_1, BD = x_2, CE = x_3, DF = x_4, EA = x_5, FB = x_6, AD = l_1, BE = l_2, CF = l_3$. We have $l_1 = b + e, l_2 = c + f, l_3 = a + f$. Apply our lemma for $ABCD$ and $DEFA$ to get $a^2 + c^2 + 2bl_1 \geq x_1^2 + x_2^2$ and $d^2 + f^2 + 2el_1 \geq x_4^2 + x_5^2$. Add them together to get:

$$a^2 + c^2 + d^2 + f^2 + 2l_1^2 \geq x_1^2 + x_2^2 + x_4^2 + x_5^2.$$

Summing all such quadrilaterals we get

$$2(a^2 + b^2 + c^2 + d^2 + e^2 + f^2) + 2(l_1^2 + l_2^2 + l_3^2) \geq 2(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2).$$

Next step is to apply our lemma for quadrilateral $BCEF$: $x_3^2 + x_6^2 + 2be \geq l_2^2 + l_3^2$. Summing with inequalities for $CDAF$, $DEAB$:

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) + 2ad + 2be + 2cf \geq 2(l_1^2 + l_2^2 + l_3^2).$$

Multiplying the second inequality by 2 and adding with the first one we get:

$$2(a^2 + b^2 + c^2 + d^2 + e^2 + f^2) + 4(ad + be + cf) \geq 2(l_1^2 + l_2^2 + l_3^2),$$

or just

$$2(a + d)^2 + 2(b + e)^2 + 2(c + f)^2 \geq 2(l_1^2 + l_2^2 + l_3^2).$$

As one can see the equality holds and thus be our lemma's case of equality we have that $BC||AD||EF$; and similarly other lines are parallel too. Let a line through C, parallel to AB, intersect AD at P, then $AP = b$ and thus $PD = e$ and $PD||EF$. Hence $PDEF$ is a parallelogram and C, P, F are collinear and so are A, P, D. So, triangles APF and DPC are similar, therefore,

$$\frac{AP}{DP} = \frac{PF}{PC} = \frac{AF}{CD}.$$

Rewriting in terms of hexagon's sides:

$$\frac{BC}{EF} = \frac{DE}{AB} = \frac{AF}{CD}.$$

O8. Let a, b, c, x, y, z be real numbers and let $A = ax + by + cz$, $B = ay + bz + cx$ and $C = az + bx + cy$. Suppose that $|A - B| \geq 1$, $|B - C| \geq 1$ and $|C - A| \geq 1$. Find the smallest possible value of the product

$$\left(a^2 + b^2 + c^2\right)\left(x^2 + y^2 + z^2\right).$$

Proposed by Adrian Zahariuc, student, Bacău, Romania

Solution by Adrian Zahariuc, student, Bacău, Romania

Answer: 4/3. First, let us prove that

$$\left(a^2 + b^2 + c^2\right)\left(x^2 + y^2 + z^2\right) \geq \frac{4}{3}$$

for all such $a, b, c, x, y, z \in \mathbb{R}$. We have

$$\left(a^2 + b^2 + c^2\right)\left(x^2 + y^2 + z^2\right) = (ax + by + cz)^2 + (ay - bx)^2 + (bz - cy)^2 + (cx - az)^2 \geq$$

$$(ax + by + cz)^2 + \frac{(ay - bx + bz - cy + cx - az)^2}{3} = A^2 + \frac{|B - C|^2}{3}$$

Of course, this means that

$$\left(a^2 + b^2 + c^2\right)\left(x^2 + y^2 + z^2\right) \geq \max\left\{A^2 + \frac{|B - C|^2}{3}, B^2 + \frac{|C - A|^2}{3}, C^2 + \frac{|A - B|^2}{3}\right\} \geq \frac{4}{3}$$

because this set contains at least one (or in fact, at least two) numbers greater then or equal to 4/3. This can be easily checked.

Now we have to find an example of $a, b, c, x, y, z \in \mathbb{R}$ for which equality occurs. It is very natural to assume that $A = 1$, $B = 0$ and $C = -1$. Because for these particular values of A, B, C we have

$$A^2 + \frac{|B - C|^2}{3} = B^2 + \frac{|C - A|^2}{3} = C^2 + \frac{|A - B|^2}{3} = \frac{4}{3},$$

in all three cases we must have equality, which means that

$$ay - bx = bz - cy = cx - az \Rightarrow ay - bx = bz - cy = cx - az = \frac{1}{3} \qquad (1)$$

$$by - az = cz - bx = ax - cy \Rightarrow by - az = cz - bx = ax - cy = \frac{2}{3} \qquad (2)$$

$$ax - bz = cz - ay = by - cx \Rightarrow ax - bz = cz - ay = by - cx = \frac{1}{3} \qquad (3)$$

If $ay = \alpha$, $bz = \beta$, $cx = \gamma$, then the previous equalities can be written as

$$\begin{pmatrix} ax & by & cz \\ ay & bz & cx \\ az & bx & cy \end{pmatrix} = \begin{pmatrix} \beta + \frac{1}{3} & \gamma + \frac{1}{3} & \alpha + \frac{1}{3} \\ \alpha & \beta & \gamma \\ \gamma - \frac{1}{3} & \alpha - \frac{1}{3} & \beta - \frac{1}{3} \end{pmatrix}$$

This should be regarded as a table, not as a matrix, since it has nothing to do with. Now, it is easy to see that

$$\alpha\left(\alpha \pm \frac{1}{3}\right) = \left(\beta \mp \frac{1}{3}\right)\left(\gamma \mp \frac{1}{3}\right) \Rightarrow \alpha^2 = \beta\gamma + \frac{1}{9} \Rightarrow \beta^2 + \beta\gamma + \gamma^2 = \frac{1}{9}.$$

Hence, by symmetry,

$$\alpha^2 + \alpha\beta + \beta^2 = \beta^2 + \beta\gamma + \gamma^2 = \gamma^2 + \gamma\alpha + \alpha^2 = \frac{1}{9}.$$

Remember that $\alpha + \beta + \gamma = 0$. Let us try to find $u, v, w \in \mathbb{R}$ such that $u + v + w = 0$ and

$$u^2 + uv + v^2 = v^2 + vw + w^2 = w^2 + wu + u^2.$$

Experimentation with small cases reveals that $u = 1$, $v = 0$ and $w = -1$ works, giving the common value 1. Now, back to the problem, we could select $\alpha = \frac{1}{3}$, $\beta = 0$ and $\gamma = -\frac{1}{3}$. In this case, we should have

$$\begin{pmatrix} ax & by & cz \\ ay & bz & cx \\ az & bx & cy \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{2}{3} & 0 & -\frac{1}{3} \end{pmatrix}$$

It is not hard at all to see that $(a, b, c, x, y, z) = \left(\frac{1}{3}, 0, -\frac{1}{3}, 1, 1, -2\right)$ satisfies this last condition. As expected, these numbers indeed have all the desired properties.

O9. Let a be a positive integer such that for any positive integer n, the number $a+n^2$ can be written as a sum of two squares. Prove that a is a square.

Proposed by Iurie Boreico, Moldova

Let $p_1, p_2, \ldots, p_k$ be the primes dividing a whose exponent in the prime decomposition of a are odd. If $k=1, p_1=2$ we have a counterexample $n=\sqrt{8a}$. So, we may assume that one of p_i is not 2. Consider the numbers t_i not divisible by p_i. The equations $x \equiv t_i(\pmod{p_i})$ and the equation $x \equiv 1(\pmod 4)$ have a solution by the Chinese Remainder Theorem. Then by Dirichlet Theorem we can find p with this property, moreover we can choose p not dividing a. Then $\binom{a}{p} = \prod_{i=1}^{k} \binom{p_i}{p}$. Now using Gauss Reciprocity Law we get that this quantity directly depends only on $\binom{t_i}{p_i}$. We can take t_i such that $\binom{t_i}{p_i}$ has any sign we wish, so we can ensure that $\binom{a}{p} = 1$. This means there is n with $p|a+n^2$. But as $p=4k+3$ $a+n^2$ cannot be then written as a sum of two squares, contradiction. (By $\binom{x}{y}$ we meant the quadratic residues)

So the set $p_1, p_2, \ldots, p_k$ is empty and a is a perfect square.

O10. Let P be an integer polynomial such that $P(2^m)$ is a n-th power of a integer number for any positive integer m. Prove that P itself is a n-th power of an integer polynomial.

Proposed by Iurie Boreico, Moldova

Solution by Iurie Boreico, Moldova

Firstly, a particular case:

Lemma. If $P \in \mathbb{Z}[X]$ is a polynomial with degree nk and leading coefficient a^n, and $P(k)$ is a n th power of a integer number for infinitely many integers the $P(X) = Q^k(X)$ for some $Q(X) \in Z[X]$.

We shall construct a polynomial $R(X) \in \mathbb{Q}[X]$ such that the leading k coefficients of $R^n(X)$ coincide with the leading k coefficients of $P(X)$. The coefficients of R can be chosen by induction: firstly choose the leading coefficient of R be a. If we have already selected the leading h coefficients of R to be $a_1, a_2, \cdots, a_h$ we can find the next coefficient a_{h+1} from the relation $[x^{nk-h}](a_1x^k + \cdots + a_hx^{n+1-h} + a_{h+1}x^{n-h}) = [x^{nk-h}](P(X))$. This expression is linear in a_{h+1} (the only unknown coefficient in the left-hand side is obtained by taking $n-1$ times a_1x^k and one time $a_{h+1}x^{k-h}$), thus having a unique solution. Next, let's prove that $P(X) = R^n(X)$ (this will give that $R(X) \in \mathbb{Z}[X]$). If not, then $P(x) = R^n(x)$ is true for finitely many x so there are infinitely many integers r for which $R(r) \neq \sqrt[n]{P(r)} \in \mathbb{Z}$. If M is the least common multiple of the denominators of the coefficients of R then $MR(R) \in \mathbb{Z}$ hence —$R(r) - \sqrt[n]{P(r)}| \geq \frac{1}{M}$. This cannot be true, because $R^n(X) - P(X)$ has degree at most $n - k - k1$ so $R^n(r) - P(r) < \frac{1}{2m} r^{nk-k-1}$ for sufficiently big $|r|$. As $|R^n(r) - P(r)| > |R(r) - \sqrt[n]{P(r)}||R(r)|^{n-1}$ and $|R(r)|^{n-1} > \frac{(}{r}^{nk}2$, we deduce $|R(r) - \sqrt[n]{P(r)}| \leq \frac{1}{M}$ for sufficiently big $|r|$ and this is the desired contradiction.

To proceed, we need a second lemma:

If $P(X) \in \mathbb{Z}[X]$ and $P(X)P(a_1X)P(a_2X)\cdots P(a_{n-1}X) = R^n(X)$ for integers $a_1 < a_2 < \cdots < a_{n-1}$ then $P(X) = x^mQ^n(X)$ for some $m \in \mathbb{N}, Q \in \mathbb{Z}[X]$.

To prove this, we show that every root except 0 has multiplicity divisible by n. Indeed, suppose not and let w be a root of P with multiplicity not divisible by n and maximal absolute value. Then $R^n(X)$ has $a_{n-1}w$ as a root, and clearly its multiplicity is exactly the multiplicity of w in P, but this is impossible since its multiplicity in R^n must be divisible by n. This concludes the proof.

To prove the problem, we see that the polynomial

$$P(X)P(2^nX)P(2^{2n}X)\cdots P(2^{(n-1)n}X)$$

satisfies the conditions in the first lemma hence it is a n-th power of a polynomial with integer coefficients. Then from second lemma we see that $P(X) = X^r Q^n(X), Q \in \mathbb{Z}[X]$. If $P \neq 0$ then we can choose such a k that $Q(2^{kn+1}) \neq 0$ and plotting $X = 2^{kn+1}$ gives us that$2^{(kn+1)r}$ is a perfect nth power hence $n|r$ and this concludes the proof.

O11. Let a, b, c be distinct positive integers. Prove the following inequality:

$$\frac{a^2b + a^2c + b^2a + b^2c + c^2a + c^2b - 6abc}{a^2 + b^2 + c^2 - ab - bc - ac} \geq \frac{16abc}{(a+b+c)^2}.$$

Proposed by Iurie Boreico and Ivan Borsenco , Moldova

Solution by Iurie Boreico and Ivan Borsenco, Moldova

We will make some useful transformations

$$a^2b + a^2c + b^2a + b^2c + c^2a + c^2b - 6abc = c(a-b)^2 + b(a-c)^2 + a(b-c)^2,$$

and

$$a^2 + b^2 + c^2 - ab - bc - ac = \frac{1}{2}((a-b)^2 + (a-c)^2 + (b-c)^2).$$

Then our inequality becomes

$$(c(a-b)^2 + b(a-c)^2 + a(b-c)^2)(a+b+c)^2 \geq 8abc((a-b)^2 + (a-c)^2 + (b-c)^2),$$

$$(a-b)^2\left(\frac{(a+b+c)^2}{ab} - 8\right) + (a-c)^2\left(\frac{(a+b+c)^2}{ac} - 8\right) + (b-c)^2\left(\frac{(a+b+c)^2}{bc} - 8\right) \geq 0$$

Note that the following inequality is true:

$$\frac{(x+2y)^2}{xy} - 8 \geq 0$$

or

$$(x+2y)^2 \geq 8xy, (x-2y)^2 \geq 0.$$

Without loss of generality we can assume that $a \geq b \geq c$.

Using the above observation we have that the expression

$$(a-b)^2\left(\frac{(a+2b)^2}{ab} - 8\right) + (a-c)^2\left(\frac{(a+2c)^2}{ac} - 8\right) + (b-c)^2\left(\frac{(b+2c)^2}{bc} - 8\right) \geq 0$$

Our final step is to prove that the first expression is greater theh the second one:

$$(a-b)^2\left(\frac{(a+b+c)^2}{ab}-8\right)+(a-c)^2\left(\frac{(a+b+c)^2}{ac}-8\right)+(b-c)^2\left(\frac{(a+b+c)^2}{bc}-8\right)\geq$$

$$(a-b)^2\left(\frac{(a+2b)^2}{ab}-8\right)+(a-c)^2\left(\frac{(a+2c)^2}{ac}-8\right)+(b-c)^2\left(\frac{(b+2c)^2}{bc}-8\right).$$

Grouping:

$$(a-b)^2\left(\frac{(a+b+c)^2}{ab}-\frac{(a+2b)^2}{ab}\right)+(a-c)^2\left(\frac{(a+b+c)^2}{ac}-\frac{(a+2c)^2}{ac}\right)$$
$$+(b-c)^2\left(\frac{(a+b+c)^2}{bc}-\frac{(b+2c)^2}{bc}\right)\geq 0.$$

$$(a-b)^2\frac{(c-b)(2a+3b+c)}{ab}+(a-c)^2\frac{(b-c)(2a+b+3c)}{ac}+(b-c)^2\frac{(a-c)(a+2b+3c)}{bc}\geq 0.$$

Note that $(b-c)^2\dfrac{(a-c)(a+2b+3c)}{bc}\geq 0$, because $a\geq b\geq c$. It is left to prove that to prove:

$$(a-b)^2\frac{(c-b)(2a+3b+c)}{ab}+(a-c)^2\frac{(b-c)(2a+b+3c)}{ac}\geq 0$$

or

$$(a-c)^2(b-c)\frac{2a+b+3c}{ac}\geq(a-b)^2(b-c)\frac{2a+3b+c}{ab}$$

But $(a-c)^2\geq(a-b)^2$ and $\dfrac{2a+b+3c}{ac}\geq\dfrac{2a+3b+c}{ab}$,
because $(2a+b+3c)b\geq(2a+3b+c)c$, $(b-c)(2a+b+c)\geq 0$.

Equality takes place in case $(a,b,c)=(2x,x,x)$, this can be verified directly and the problem is solved.

Also solved by Yudong Wu, P.R.China.

O12. Consider the system

$$\begin{cases} x_1x_2x_3 - x_4 &= a_1 \\ x_2x_3x_4 - x_1 &= a_2 \\ x_3x_4x_1 - x_2 &= a_3 \\ x_4x_1x_2 - x_3 &= a_4, \end{cases}$$

where $x_i \in [\sqrt{2}-1, 1]$ are not all equal. Prove that

$$a_1 + a_2 + a_3 + a_4 \neq 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Ivan Borsenco, University of Texas at Dallas

Assume the contrary, that $a_1 + a_2 + a_3 + a_4 = 0$. Then

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = x_1 + x_2 + x_3 + x_4.$$

Using Mac-Lauren inequality we have

$$\left(\frac{x_1+x_2+x_3+x_4}{4}\right)^3 \geq \left(\frac{x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4}{4}\right), \Rightarrow$$
$$\left(\frac{x_1+x_2+x_3+x_4}{4}\right)^3 \geq \left(\frac{x_1+x_2+x_3+x_4}{4}\right).$$

Thus $x_1 + x_2 + x_3 + x_4 \geq 4$, with the equality if and only if $x_1 = x_2 = x_3 = x_4 = 1$, contradiction.