AwesomeMath Summer Program UC Santa Cruz 2009 Teaching Schedule

Subject area	Morning lecture and problem session Instructors and teaching assistants	Afternoon lecture and problem session Instructors and teaching assistants
Algebra		***Algebra 1.5*** *Juan Restrepo* (weeks 1, 2, 3)
	Algebra 2.5 *Josh Nichols-Barrer* (weeks 1, 2, 3)	
	Algebra 3.5 *Mircea Becheanu* (weeks 1, 2, 3)	
Combinatorics	***Math Counts with Proofs*** *Chengde Feng* (weeks 1, 2, 3)	
		Counting Strategies *Ivan Matić* (weeks 1, 2, 3)
		Combinatorial Arguments *Bogdan Enescu* (weeks 1, 2, 3)
Geometry	***Elements of Geometry*** *Yunhua Feng* (weeks 1, 2, 3)	
	Computational Geometry *Ivan Borsenco* (weeks 1, 2, 3)	
		Geometric Proofs *Oleg Mushkarov* (week 1) *Dorin Andrica* (weeks 2, 3)
Number Theory		***Number Sense*** *Mircea Becheanu* (week 1) *Naoki Sato* (week 2) *Josh Nichols-Barrer* (week 3)
		Modular Arithmetic *Dorin Andrica* (week 1) *Oleg Mushkarov* (weeks 2, 3)
	Number Theory *Gabriel Dospinescu* (week 1, 2, 3)	

Titu Andreescu and *Zuming Feng* will be guest lecturing.

Daily Schedule

Classes will be held August 1, 3–8, 11–15, and 18–20, following a schedule like the one below. There will be two Mondays with no classes, August 10 and 17, for special trips and activities. The academic load for weekends will be adjusted accordingly. Academic team contests will be held on Sundays.

7:30 AM – 8:00 AM	Morning routine
8:00 AM – 9:00 AM	Breakfast
9:00 AM – 10:30 AM	Morning lectures
10:45 am – 12:15 pm	Morning problem sessions
12:15 PM – 2:00 PM	Lunch
2:00 PM – 3:30 PM	Afternoon lectures
3:45 PM – 5:15 PM	Afternoon problem sessions
5:15 PM – 7:00 PM	Dinner
7:00 PM – 10:00 PM	Recreational activities (optional)
7:00 PM – 9:00 PM	Homework office hours (optional)
8:00 PM – 9:30 PM	Mathematics forum (optional)
10:00 PM – 11:00 PM	Check-in and nightly routine
11:00 PM	Lights out

Academic Curriculum

There are four main subject areas covered in math competitions: Algebra, Combinatorics, Geometry, and Number Theory. Our goal at the AwesomeMath Summer Program is to build and hone students' problem solving skills in these four fields. We offer 12 courses, each of which lasts for the duration of the camp and covers one of the four subjects mentioned above (though there will be some overlap between disciplines, many techniques are helpful for different kinds of problems.) Each course meets 5 times per week, with a lecture and a problem session taught by one or two instructors. The instructor(s) of each course will teach the 90-minute lecture, and supervise the 90-minute problem sessions following the lecture. Because communication plays an important role in developing problem solving skills, students will be asked to participate actively in the problem sessions, i.e., to ask questions and present and defend solutions. Each student will select two courses, one in the morning and one in the afternoon, based on the following criteria:

- Students' personal choices and interests (please see below for course selection tips);
- Students' mathematical background and ability as reflected in the personal application, recommendation letters, and achievements on AMC8 (2007 and 2008), AMC10/12 (2007, 2008, 2009), AIME (2007, 2008, 2009), ARML (2007, 2008, 2009), USAMO, and AwesomeMath Summer Camp (2006, 2007, 2008).

When you arrive, we will go over your selections with you to ensure that you are in the courses that suit you best. Switching between courses will be allowed only in exceptional cases and only during the first week. Permission to change will be granted based on the recommendations of the academic teams.

How Students Should Select Courses

Students should complete the course selection survey and contact AwesomeMath staff via email about the course selections as soon as possible. To keep the teaching quality high and ensure individual attention to each student, each class has a maximum size. (Popular courses might have multiple classes, but it is important for us to know this in advance to make such arrangements.) **We grant students' course selections, following our guidelines, on a first-come-first-serve basis.**

Course Selection Tips

- Select the areas in which the student is most interested, or in which the student needs the most work.
- Both the content and difficulty of each course are very important. Please read the course descriptions carefully.
- All of our courses are challenging and beyond the scope of regular/accelerated/honor classes in school settings. There is an **entry level** course in each area. Each of these entry level courses will be challenging for most able young minds because these are **contest mathematics** courses. In particular, *Algebra 1.5* is harder than any algebra course (including Algebra 1 and 2) taught in high schools, and *Math Counts with Proofs* is beyond the requirements for the MathCounts competitions (at state and national levels).

Course Selection Guidelines

- *If a student only has MathCounts state level experience, what courses should he/she take?*

  Please choose **two level 1** courses. We recommend a combination of the student's strong subject and his/her weak subject.

- *If a student's (9th grader or below) AMC10/12 scores are below the qualifying line for AIME, what courses should he/she take?*

  Please choose **two level 1** courses. We recommend a combination of the student's strong subject and his/her weak subject.

- *If a student's (10th grader or above) AMC10/12 scores are below the qualifying line for AIME, what course should he/she take?*

  Please choose **one level 1** course, with this being the student's weak subject, and **one level 2 course**, with this being his/her strong subject.

- *If a student has an AIME score between 1 and 3, what courses should he/she take?*

  Please choose either

  (1) **two level 1** courses. We recommend a combination of the student's strong subject and his/her weak subject; **OR**

  (2) **one level 1** course, with this being the student's weak subject, and **one level 2** course, with this being his/her strong subject.

- *If a student has an AIME score between 4 and 7, what courses should he/she take?*

  Please choose **two level 2** courses. We recommend a combination of the student's strong subject and his/her weak subject.

- *If a student has an AIME score between 8 and 11, what courses should he/she take?*

  Please either

  (1) **two level 2** courses. We recommend a combination of the student's strong subject and his/her weak subject; **OR**

  (2) **one level 2** course, with this being the student's weak subject, and **one level 3** course, with this being his/her strong subject.

- *If a student has an AIME score of 12 or above, what courses should he/she take?*

  Please choose **two level 3** courses. We recommend a combination of the student's strong subject and his/her weak subject.

- *If a student's background does not fall into any of the above, what courses should he/she take?*

  Please contact AwesomeMath staff as soon as possible about the student's background and interests, and we will make a recommendation promptly.

- *If a student's background falls into to the above categories, but he/she wants to choose the courses not following the guidelines, what should he/she do?*

  Please contact AwesomeMath staff via email as soon as possible about the student's background and interests, and we will make a recommendation promptly. **If there are still questions about our recommendation, the student might be required to take a placement test in the subjects for which he/she is not following our recommendation.**

Course Descriptions

Subject	Beginning (level 1) courses	Intermediate (level 2) courses	Advanced (level 3) courses
Algebra	Algebra 1.5	Algebra 2.5	Algebra 3.5
Combinatorics	Math Counts with Proofs	Counting Strategies	Combinatorial Argument
Geometry	Elements of Geometry	Computational Geometry	Geometry Proofs
Number Theory	Number Sense	Modular Arithmetic	Number Theory
	These courses are computationally oriented with a touch on proofs. They are suited for most USA math competitions (MathCounts National level, AMC10, AMC12, ARML, and the entry level of AIME).	These courses are about half computational problems and half proofs. They are well suited for the hard end of AIME and the entry level of Math Olympiad contests.	These courses are proof oriented. They are well suited for students who can easily pass AIME and are seriously preparing for Math Olympiad contests.

Algebra courses

- *Algebra 1.5*

  Develops essential skills such as factoring, grouping, recognizing roots, telescoping sums/products, and rationalizing. Solving (systems of) equations/inequalities (linear, absolute value, quadratic, rational, radical) is the main theme of the course. Discriminants, Viète's relations, and symmetric polynomials also play a central role. **This is the entry level algebra course. It covers all AMC levels and easy end of AIME and ARML. This course is a good fit for students with MathCounts state level experience, AMC10/12 scores approaching AIME qualifying cuts, or an AIME score between 1 and 3.**

- *Algebra 2.5*

  Studies special systems of equations, discriminants, Viète's relations, symmetric polynomials, functional properties. Introduces (weighted) AM-GM-HM and Cauchy-Schwartz inequalities. **This is the intermediate level algebra course. It covers the hard end of AMC12, and the medium to hard end of ARML and AIME. A student with an AIME score between 4 and 7 should be a good fit for this course.**

- *Algebra 3.5*

  Discusses functional equations, classical inequalities such as AM-GM-HM, Cauchy-Schwarz, Power-mean, and Jensen's inequalities, as well as Muirhead's and Schur's inequalities, and inequalities related to symmetric polynomials. **This is the advanced level algebra course. It covers the hard end of AIME and all levels of USAMO. A student with a strong algebra background and an AIME score of 8 or above should consider this course.**

Combinatorics courses

- *Math Counts with Proofs*

  Studies the addition and multiplication principles, permutations and combinations, and probability. Teaches how to deal with over-counting and many useful properties of integer divisors. It also introduces mathematical proofs using pigeonhole principle, well-ordering, etc. **This is the entry level combinatorics course. It covers MathCounts, all the AMC levels, and the easy end of AIME and ARML. This course is a good fit for students with MathCounts state level experience, AMC10/12 scores approaching AIME qualifying cuts, or AIME scores between 1 and 3.**

- *Counting Strategies*

  Discusses counting strategies such as the addition and multiplication principles, permutations and combinations, properties of the binomial coefficients, bijections, recursions, and the inclusion- exclusion principle. **This is the intermediate level combinatorics course. It covers the hard end of AMC12, the medium to hard end of AIME and ARML, as well as the beginning USAMO level. A student with an AIME score between 4 and 7 should be a good fit for this course.**

- *Combinatorial Arguments*

  Introduces methods of mathematical proofs, including induction, proofs by contradiction, the Pigeonhole Principle, the well-ordering principle, colorings, assigning weights, bijections/mappings, recursion, calculating in two ways, and combinatorial constructions. Topics may include graph theory and combinatorial geometry. A focal point of the course is combinatorial number theory. **This is the advanced level combinatorics course. It covers the hard end of AIME and the medium to hard end of USAMO. A student who is familiar with mathematics proofs and has an AIME score of 8 or above should consider this course.**

Geometry courses

- *Elements of Geometry*

  Deals with computational geometry in two dimensions using Euclidean methods, including manipulation of angles and lengths, as well as the basic properties of polygons, circles, and the relations between figures. Analytic geometry is also a focal point. **This is the entry level geometry course. It covers MathCounts, all AMC levels, and the easy end of AIME and ARML. This course is a good fit for students with MathCounts state level experience, AMC10/12 scores approaching AIME qualifying cuts, or AIME scores between 1 and 3.**

- *Computational Geometry*

  Studies non-synthetic techniques in solving geometry problems: coordinate geometry, vectors (2- and 3-dimensional), planes, spheres, trigonometry, and complex numbers. Features many important geometric themes: The Law of Sines and the Law of Cosines, Ptolemy's theorem, Ceva's theorem, Menelaus's theorem, Stewart's theorem, Heron's and Brahmagupta's formulas, Brocard points, dot product and the vector form of the Law of Cosines, the Cauchy-Schwarz inequality, 3-dimensional coordinate systems, as well as linear representation and traveling on the earth (sphere). **This is the intermediate level geometry course. It covers the hard end of AMC12, the medium to hard end of AIME and ARML. A student with an AIME score between 4 and 7 should consider this course.**

- *Geometric Proofs*

  Focuses on classical topics such as concurrency, collinearity, cyclic quadrilaterals, special centers/points of triangles, and geometric constructions. Introduces important transformations – translation, reflections, and spiral similarities, with a touch on projective and inversive geometry. **This is the advanced level geometry course. It covers the hard end of AIME and the medium to hard end of USAMO. A student with a strong background in geometry and an AIME score of 8 or above should consider this course.**

Number Theory courses

- *Number Sense*

  Studies divisibility, factoring, numerical systems, divisors and arithmetic functions of divisors. Setting-up and solving linear Diophantine equations is also a focal point of the course. **This is the entry level number theory course. It covers MathCounts, all AMC levels, and the easy end of AIME and ARML. This course is a good fit for students with MathCounts state level experience, AMC10/12 scores approaching AIME qualifying cuts, or AIME scores between 1 and 3.**

- *Modular Arithmetic*

  Develops essential skills in number theory: divisibility, the division algorithm, prime numbers, the Fundamental Theorem of Arithmetic, GCD, LCM, Bézout's identity, the Euclidean algorithm, modular arithmetic, and divisibility criteria in the decimal system. Studies numerical functions such as the number of divisors or the sum of divisors of integers. **This is the intermediate level number theory course. It covers the hard end of AMC12 and the medium to hard end of AIME and ARML. A student qualified for AIME with a score between 4 and 7 should be a good fit for this course.**

- *Number Theory*

  Focuses on in-depth discussions of Diophantine equations, residue classes, quadratic reciprocity, Fermat's little theorem, Euler's theorem, primitive roots, and Euler's totient function, etc. **This is the advanced level number theory course. It covers the hard end of AIME and the medium to hard end of USAMO. A student with a strong background in number theory and an AIME score of 8 or above should consider this course.**

Sample Problems

Algebra

Algebra 1.5

1. If $a+b=1$ and $a^2+b^2=2$, compute a^4+b^4.

2. Simplify
$$\frac{(1+ax)^2-(a+x)^2}{(1+bx)^2-(b+x)^2} \div \frac{(1+ay)^2-(a+y)^2}{(1+by)^2-(b+y)^2}.$$

3. Let a, b and c be distinct real numbers such that
$$a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}.$$
Show that $|abc|=1$.

4. Compute
$$\sum_{n=0}^{\infty}\frac{n}{n^4+n^2+1}.$$

Algebra 2.5

1. Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be real numbers. Prove that
$$(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2)\geq(a_1b_1+a_2b_2+\cdots+a_nb_n)^2.$$

2. Find all positive integers a, b, c such that the equations
$$x^2-ax+b=0, \qquad x^2-bx+c=0, \qquad x^2-cx+a=0$$
have integer roots.

3. Let $f(x)=ax^2+bx+c$ be a quadratic function with integer coefficients with the property that for every positive integer n there is an integer c_n such that n divides $f(c_n)$. Prove that f has rational zeros.

4. Let a, b integer numbers. Solve the equation
$$(ax-b)^2+(bx-a)^2=x$$
when it is known that it has an integer root.

Algebra 3.5

1. Find all polynomials with complex coefficients such that $P(x^2)=[P(x)]^2$ is identically true.

2. Let $a, b, c \geq 0$. Prove that
$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}+\frac{1}{a+b+c+1}\geq 1$$

3. (IMO 2005 Shortlist) Let a, b, c, d, e, f be positive integers such that $S=a+b+c+d+e+f$ is a divisor of $ab+bc+ca-(de+ef+fd)$ and $abc+def$. Show that S is a composite number.

4. (IMO 2004) Find all real polynomials with real coefficients $P(x)$ which satisfy the equality
$$P(a-b)+P(b-c)+P(c-a)=2P(a+b+c)$$
for all triples a, b, c of real numbers such that $ab+bc+ca=0$.

Combinatorics

Math Counts with Proofs

1. (AIME 1993) How many even integers between 4000 and 7000 have four different digits?

2. How many ordered triples (x, y, z) of non-negative integers have the property that $x + y + z = 8$?

3. (Purple Comet 2008) There are three men and eleven women taking a dance class. In how many different ways can each man be paired with a woman partner and then have the eight remaining women be paired into four pairs of two?

4. (Purple Comet 2004) We want to paint some identically-sized cubes so that each face of each cube is painted a solid color and each cube is painted with six different colors. If we have seven different colors to choose from, how many distinguishable cubes can we produce?

Counting Strategies

1. How many positive integers less than 5000 are multiples or 3, 5 or 7, but not multiples of 35?

2. Let m be a positive integer, and let $n = 2^m$. Prove that the numbers
$$\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}$$
are all even. How many of these numbers are divisible by 4?

3. A number of n tennis players take part in a tournament in which each of them plays exactly one game with each of the others. If x_i and y_i denote the number of wins and losses, respectively, of the ith player, prove that
$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2$$

4. Let S be a set with 9 elements and let $A_1, A_2, \ldots, A_{13}$ be distinct subsets of S, each having 3 elements. Prove that among these subsets there exist two, A_i and A_j, such that $|A_i \cap A_j| = 2$.

Combinatorial Arguments

1. The numbers $1, 2, \ldots, 49$ are placed in a 7×7 table. We then add the numbers in each row and each column. Among these 14 sums we have a even numbers and b odd numbers. Is it possible that $a = b$?

2. The numbers $a_1, a_2, \ldots, a_{108}$ are written on a circle such that the sum of any 20 consecutive numbers equals 1000. If $a_1 = 1$, $a_{19} = 19$, and $a_{50} = 50$, find a_{100}.

3. An even number, $2n$, of knights arrive at King Arthur's court, each one of them having at most $n - 1$ enemies. Prove that Merlin the wizard can assign places for them at a round table in such a way that every knight is sitting only next to friends.

4. On an 8×8 chessboard whose squares are colored black and white in an arbitrary way we are allowed to simultaneously switch the colors of all squares in any 3×3 and 4×4 region. Can we transform any coloring of the board into one where all the squares are black?

Geometry

Elements of Geometry

1. Let $ABCD$ be a parallelogram, and let M and N be the midpoints of sides BC and CD, respectively. Suppose $AM = 2$, $AN = 1$, and $m\angle MAN = 60°$. Compute AB.

2. How large an equilateral triangle can one fit inside a square with side length 2?

3. Charlyn walks completely around the boundary of a square whose sides are each 5 km long. From any point on her path, she can see exactly 1 km horizontally in all directions. What is the area of the region consisting of all points Charlyn can see during her walk?

4. Points A and B lie on a circle centered at O, and $\angle AOB = 60°$. A second circle is internally tangent to the first and tangent to both OA and OB. What is the ratio of the area of the smaller circle to that of the larger circle?

Computational Geometry

1. (AIME 2005) In quadrilateral $ABCD$, $BC = 8$, $CD = 12$, $AD = 10$, and $\angle A = \angle B = 60°$. Given that $AB = p + \sqrt{q}$, where p and q are positive integers, find $p + q$.

2. (a) Let G be the centroid of triangle ABC. Prove that for any point M,

$$MA^2 + MB^2 + MC^2 = 3MG^2 + AG^2 + BG^2 + CG^2.$$

   (b) Let I be the incenter of triangle ABC. Prove that for any point X,

$$a \cdot AX^2 + b \cdot BX^2 + c \cdot CX^2 = (a + b + c) \cdot IX^2 + a \cdot IA^2 + b \cdot IB^2 + c \cdot IC^2.$$

3. (a) Prove that in any triangle

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{1}{8}.$$

   (b) Prove that in any triangle

$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} \leq \frac{9}{4}.$$

4. Let M, N, P, Q, R, S be the midpoints of the sides AB, BC, CD, DE, EF, FA of a hexagon. Prove that $RN^2 = MQ^2 + PS^2$ if and only if $MQ \perp PS$. (Hint: use complex numbers.)

Geometric Proofs

1. Prove that for any point M inside parallelogram $ABCD$, the following relation holds:

$$MA \cdot MC + MB \cdot MD \geq AB \cdot BC$$

2. (Blanchet's Theorem) Let AD be an altitude in $\triangle ABC$. Point P is on segment AD. Let E bet the intersection of BP and AC. Let F be the intersection of CP and AB. Prove that $\angle ADE = \angle ADF$.

3. (Alex Anderson) Let w_1 be a circle smaller than and internally tangent to the circle w_2 at T. A tangent to w_1 (at T'), intersects w_2 at A and B. If A, T', and B are fixed, what is the locus of T?

4. (USAMO 2006) Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $AE : ED = BF : FC$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

Number Theory

Number Sense

1. Show that the number 101010 cannot be a difference of two squares of integers.
2. Let a unit step be the diagonal of a unit square. Starting from the origin, go one step to $(1, 1)$. The turn 90° counterclockwise (to the left) and go two steps to $(-1, 3)$. Then turn 90° counterclockwise (to the left) and go three steps to $(-4, 0)$. At each step you continue to turn 90° counterclockwise and increase the length of the movement by one at each step. What is the final position after 100 moves?
3. Find all positive integers a and b such that $a^2 + b^2 = lcm(a, b) + 7\gcd(a, b)$.
4. Compute the sum of the greatest odd divisor of each of the numbers $2006, 2007, \ldots, 4012$.

Modular Arithmetic

1. Show that $\frac{1}{9}(10^n + 3 \cdot 4^n + 5)$ is an integer for all $n \geq 1$.
2. Show that if $a^5 \pm 2b^5$ is divisible by 11, then both a and b are divisible by 11.
3. If $\{a_1, a_2, \ldots, a_{p-1}\}$ and $\{b_1, b_2, \ldots, b_{p-1}\}$ are complete sets of nonzero residue classes modulo some odd prime p, show that $\{a_1b_1, a_2b_2, \ldots, a_{p-1}b_{p-1}\}$ is not a set of complete residue classes modulo p.
4. Given that $a + b\sqrt[3]{2} + c\sqrt[3]{4} = 0$ where a, b, and c are integers, show that $a = b = c = 0$.

Number Theory

1. Find all solutions to $2^k = 9^m + 7^n$.
2. Let p be a prime, and let k be a nonnegative integer. Calculate
$$\sum_{n=1}^{p-1} n^k \bmod p.$$
3. Prove that the equation $x^2 + 7xy - y^2 = 401$ has no integer solutions.
4. Determine all positive integers n for which there exists an integer m such that $2^n - 1$ is a divisor of $m^2 + 9$.