Junior problems

J103. The numbers $1, 2, \ldots, 9$ are randomly arranged on a circle. Prove that there are adjacent numbers whose sum is at least 16.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Eldin Nisic, Sarajevo, Bosnia and Herzegovina

Assume on contrary, that it is possible to find an arrangement $a_1, a_2, \ldots, a_9$ of numbers $1, 2, \ldots, 9$ on a circle such that the sum of any three adjacent numbers is not greater that 15. Then:

$$\begin{aligned}
&a_1 + a_2 + a_3 \leq 15,\\
&a_1 + a_2 + a_3 \leq 15,\\
&a_2 + a_3 + a_4 \leq 15,\\
&\vdots\\
&a_9 + a_1 + a_2 \leq 15.
\end{aligned}$$

After summing up the above inequalities we deduce that $a_1 + a_2 + \cdots + a_9 \leq 45$. On the other hand $a_1 + a_2 + \cdots + a_9 = 1 + 2 + \cdots + 9 = 45$. Thus all 9 of the above inequalities are in fact equalities. This implies that $a_1 + a_2 + a + 3 = a_2 + a_3 + a_4 = 15$ and so $a_1 = a_4$, clearly a contradiction. Thus our assumption was wrong and as a consequence there must exist three adjacent numbers whose sum is greater than 15.

Also solved by John T. Robinson, Yorktown Heights, NY, USA; Roberto Bosch Cabrera, University of Havana, Cuba; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Oles Dobosevych, Ukraine.

J104. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a^2+b^2}{a^2+b^2+1}+\frac{b^2+c^2}{b^2+c^2+1}+\frac{c^2+a^2}{c^2+a^2+1} \geq \frac{a+b}{a^2+b^2+1}+\frac{b+c}{b^2+c^2+1}+\frac{c+a}{c^2+a^2+1}.$$

Proposed by Jingjun Han, Shanghai, China

First solution by Manh Dung Nguyen, Vietnam

Since

$$\frac{a^2+b^2}{a^2+b^2+1} = 1 - \frac{1}{a^2+b^2+1}$$

we may write the inequality as

$$\sum_{\text{cyc}} \frac{a+b+1}{a^2+b^2+1} \leq 3.$$

On the other hand, by the AM-GM inequality, we have

$$a^2+1 \geq 2a, b^2+1 \geq 2b, a^2+b^2 \geq 2ab$$

so

$$a^2+b^2+1 = \frac{a^2+b^2}{2} + \frac{a^2+b^2+2}{2} \geq ab+a+b.$$

It follows that

$$3(a^2+b^2+1) \geq a^2+b^2+1+2(ab+a+b) = (a+b+1)^2.$$

Thus, it suffices to show that

$$\sum_{\text{cyc}} \frac{1}{a+b+1} \leq 1.$$

Setting $a = x^3, b = y^3, c = z^3$ we have $x, y, z > 0$ and $xyz = 1$. We need to show that

$$\sum_{\text{cyc}} \frac{1}{x^3+y^3+1} \leq 1.$$

From the identity

$$x^3+y^3-xy(x+y) = (x+y)(x-y)^2 \geq 0$$

we have

$$x^3+y^3+1 \geq xy(x+y)+xyz = xy(x+y+z).$$

Therefore

$$\sum_{\text{cyc}} \frac{1}{x^3+y^3+1} \leq \sum_{\text{cyc}} \frac{1}{xy(x+y+z)} = 1$$

Equality holds when $x = y = z = 1$ or $a = b = c = 1$.

Second solution by Shamil Asgarli, Howard Ko, Burnaby, Canada

The given inequality is equivalent to proving that:

$$\frac{a+b+1}{a^2+b^2+1} + \frac{b+c+1}{b^2+c^2+1} + \frac{c+a+1}{c^2+a^2+1} \geq 3.$$

From the Cauchy-Schwarz inequality we have the following

$$3(a^2+b^2+1) \geq (a+b+1)^2$$

or equivalently

$$\frac{a+b+1}{a^2+b^2+1} \leq \frac{3}{a+b+1}.$$

Similarly, we have:

$$\frac{b+c+1}{b^2+c^2+1} \leq \frac{3}{b+c+1} \quad \text{and} \quad \frac{c+a+1}{c^2+a^2+1} \leq \frac{3}{c+a+1}.$$

Therefore, it is enough to prove:

$$\frac{3}{a+b+1} + \frac{3}{b+c+1} + \frac{3}{c+a+1} \geq 3$$

or

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \geq 1.$$

After clearing denominators and canceling like terms we have

$$a^2b + a^2c + b^2a + b^2c + c^2a + c^2b \geq 2(a+b+c)$$

or

$$ab + bc + ca - \frac{3}{a+b+c} \geq 2.$$

From the AM-GM inequality we have $a+b+c \geq 3\sqrt[3]{abc} = 3$ which implies $\frac{3}{a+b+c} \leq 1$. Back to our inequality: $ab+bc+ca-1 \geq 2$ or $ab+bc+ca \geq 3$. This last inequality is true by the AM-GM inequality and thus we are done.

Also solved by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Gheorghe Pupazan, Chisinau, Moldova; Roberto Bosch Cabrera, Havana, Cuba; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Magkos Athanasios, Kozani, Greece; Oles Dobosevych, Ukraine.

J105. Let $A_1A_2\dots A_n$ be a polygon that is inscribed in a circle $C(O,R)$ and at the same time circumscribed about a circle $\omega(I,r)$. The points of tangency of $A_1A_2\dots A_n$ with ω form another polygon $B_1B_2\dots B_n$. Prove that

$$\frac{P(A_1A_2\dots A_n)}{P(B_1B_2\dots B_n)} \le \frac{R}{r},$$

where $P(S)$ stands for the perimeter of figure S.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Cyclical notation (ie, $i = i+n$) will be used throughout the problem. Call $\alpha_i = \angle A_iA_jA_{i+1}$ where $j \ne i, i+1$. Clearly, $\angle A_{i-1}A_iA_{i+1} = \pi - \angle A_{i-1}A_{i+1}A_i - \angle A_iA_{i-1}A_{i+1} = \pi - \alpha_{i-1} - \alpha_i$. Denote by B_i the point of tangency of $\omega(I,r)$ with A_iA_{i+1}. Since $IB_{i-1} \perp A_{i-1}A_i$ and $IB_i \perp A_iA_{i+1}$, then $\angle B_{i-1}IB_i = \pi - \angle B_{i-1}A_iB_i = \alpha_{i-1}+\alpha_i$. Direct application of the Sine Law yields $A_iA_{i+1} = 2R\sin\alpha_i$, and $B_{i-1}B_i = 2r\sin\frac{\alpha_{i-1}+\alpha_i}{2}$, or the proposed inequality is equivalent to

$$\sum_{i=1}^{n}\sin\alpha_i \le \sum_{i=1}^{n}\sin\frac{\alpha_i+\alpha_{i+1}}{2}.$$

However,

$$\sin\alpha_i + \sin\alpha_{i+1} = 2\sin\frac{\alpha_i+\alpha_{i+1}}{2}\cos\frac{\alpha_i-\alpha_{i+1}}{2} \le 2\sin\frac{\alpha_i+\alpha_{i+1}}{2},$$

with equality if and only if $\alpha_i = \alpha_{i+1}$. Adding this inequality for $i = 1, 2, \dots, n$, the conclusion follows. Equality holds if and only if $A_1A_2\dots A_n$ is a regular n-gon.

Second solution by Oles Dobosevych, Lviv National University, Ukraine

Without lost of generality, let B_2 lie on the segment A_1A_2, B_3 lie on the segment A_2A_3,etc., B_1 lie on the segment A_1A_n. We have that

$$S_{A_1A_2\dots A_n} = S_{A_1IA_2} + S_{A_2IA_3} + \cdots + S_{A_nIA_1}. \tag{1}$$

Let us note that

$$S_{A_1IA_2} = \frac{1}{2}IB_2\cdot A_1A_2, S_{A_2IA_3} = \frac{1}{2}IB_3\cdot A_2A_3, \dots, S_{A_1IA_n} = \frac{1}{2}IB_1\cdot A_nA_1.$$

If we substitute this equation in (1) while keeping in mind that $IB_2 = IB_3 = \cdots = IB_n = IB_1 = r$ we get

$$S_{A_1A_2\dots A_n} = \frac{1}{2}r(A_1A_2 + A_2A_3 + \cdots + A_nA_1) = \frac{1}{2}rP(A_1A_2\dots A_n). \tag{2}$$

On the other hand

$$S_{A_1A_2\ldots A_n} = S_{A_1B_1OB_2} + S_{A_2B_2OB_3} + \cdots + S_{A_nB_nOA_1}. \tag{3}$$

Let ϕ_1 be the angle between lines OA_1 and B_1B_2, ϕ_2 the angle between lines OA_2 and B_2B_3,etc., ϕ_n the angle between lines OA_n and B_nB_1. Then

$$S_{A_1B_1OB_2} = \frac{1}{2}OA_1 \cdot B_1B_2 \sin\phi_1, \ldots, S_{A_nB_nOB_1} = \frac{1}{2}OA_n \cdot B_nB_1 \sin\phi_n.$$

If we substitute $OA_1 = OA_2 = \ldots = OA_n = R$ in (3) we get

$$S_{A_1A_2\ldots A_n} = \frac{1}{2}R(B_1B_2\sin\phi_1 + B_2B_3\sin\phi_2 + \cdots + B_nB_1\sin\phi_3). \tag{4}$$

We have that $\sin\phi_1 \leq 1, \sin\phi_2 \leq 1, \ldots, \sin\phi_n \leq 1$ and thus from (4) we get

$$\begin{aligned} S_{A_1A_2\ldots A_n} &= \frac{1}{2}R(B_1B_2\sin\phi_1 + B_2B_3\sin\phi_2 + \cdots + B_nB_1\sin\phi_3) \\ &\leq \frac{1}{2}R(B_1B_2 + B_2B_3 + \cdots + B_nB_1). \end{aligned}$$

The above inequality implies that

$$S_{A_1A_2\ldots A_n} \leq \frac{1}{2}RP(B_1B_2\ldots B_n). \tag{5}$$

From relations (2) and (5) we have

$$\frac{P(A_1A_2\ldots A_n)}{P(B_1B_2\ldots B_n)} \leq \frac{R}{r}.$$

Also solved by Roberto Bosch Cabrera, University of Havana, Cuba

J106. Prove that among any four positive real numbers there are two, say a and b, such that

$$ab + 1 \geq \frac{1}{\sqrt{3}}|a - b|.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Roberto Bosch Cabrera, University of Havana, Cuba

If two of the numbers are equal, say $a = b$, then $ab+1 = a^2+1 > 0 = \frac{1}{\sqrt{3}}|a-b|$. Assume without loss of generality that $a < b < c < d$. Assume by contradiction that the statement is false. In particular we have that

$$ab + 1 < \frac{1}{\sqrt{3}}(b - a), \tag{1}$$

$$bc + 1 < \frac{1}{\sqrt{3}}(c - b). \tag{2}$$

By (2) it follows that

$$b < \frac{c - \sqrt{3}}{\sqrt{3}c + 1} < \frac{1}{\sqrt{3}} < \frac{1}{\sqrt{3}} + a \Rightarrow \frac{1}{\sqrt{3}}(b - a) < \frac{1}{3}.$$

Combining this with (1) we have that

$$ab + 1 < \frac{1}{3} \Rightarrow ab < \frac{1}{3} - 1 = -\frac{2}{3}.$$

This is a contradiction and the result follows.

Second solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

If the minimum distance between two of the four numbers is less than or equal to $\sqrt{3}$, the inequality is proved. Let $x_1 < x_2 < x_3 < x_4$, be the four points, we have $x_3 > 2\sqrt{3}$ because $x_2 > x_1 + \sqrt{3}$ and $x_3 > x_2 + \sqrt{3}$. Squaring we prove the sufficient inequality

$$(x_3x_4)^2 + 1 + 2x_3^2 > \frac{1}{3}(x_3^2 + x_4^2)$$

and this follows immediately by $(x_3x_4)^2 > x_4^2 > \frac{x_4^2}{3}$.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Oles Dobosevych, Ukraine.

J107. Find all quadruples (a, b, c, d) of positive integers such that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)\left(1+\frac{1}{d}\right)=5.$$

Proposed by Shamil Asgarli, Burnaby, Canada

Solution by Manh Dung Nguyen, Hanoi University of Science, Vietnam

Without loss of generality assume that $a \geq b \geq c \geq d$. Then $5 \leq (1+\frac{1}{d})^4$ which implies that $d \leq 2$.

(a) If $d = 2$ then

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)=\frac{10}{3}.$$

A similar argument yields $a = b = c = 2$ which is impossible.

(b) If $d = 1$ then

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)=\frac{5}{2}.$$

By similar analysis we get that $c \leq 2$.

- If $c = 1$ then $\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right) = \frac{5}{4}$ or $(a-4)(b-4) = 20$, and we obtain $(a, b) = (24, 5), (14, 6), (9, 8)$
- If $c = 2$ then $(2a-3)(2b-3) = 15$ we obtain $(a, b) = (9, 2), (4, 3)$

In conclusion, the solutions are

$$(24, 5, 1, 1), (14, 6, 1, 1), (9, 8, 1, 1), (9, 2, 2, 1), (4, 3, 2, 1).$$

Also solved by Roberto Bosch Cabrera, University of Havana, Cuba; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Oles Dobosevych, Ukraine.

J108. Let n be a positive integer. Prove that the number of ordered pairs (a, b) of relatively prime positive divisors of n is equal to the number of divisors of n^2.

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

First solution by John T. Robinson, Yorktown Heights, NY, USA

A proof by induction on the number of distinct prime factors of n is as follows. We denote the cardinality of a set S by $\|S\|$, and the number of positive divisors of an integer n by $\tau(n)$. Noting that the case $n = 1$ is trivial, suppose n has only one prime factor, $n = p^k$. Then $\tau(n^2) = \tau(p^{2k}) = 2k + 1$, and

$$\|\{(a, b)|\ a \mid n, b \mid n, \gcd(a, b) = 1\}\| =$$
$$\left\|\{(p^i, 1)|\ 1 \le i \le k\}\right\| + \left\|\{(1, p^i)|\ 1 \le i \le k\}\right\| + \|\{(1, 1)\}\| =$$
$$2k + 1.$$

Next suppose $n = mp^k$, where the prime p is not a factor of m. Since τ is multiplicative and $\gcd(m^2, p^{2k}) = 1$, $\tau(n^2) = \tau(m^2)(2k + 1)$. By induction

$$\|\{(a, b)|\ a \mid m, b \mid m, \gcd(a, b) = 1\}\| = \tau(m^2).$$

We now see that

$$\begin{aligned}\|\{(c, d)|\ c \mid n, d \mid n, \gcd(c, d) = 1\}\| &= \|\{(a, b)|\ a \mid m, b \mid m, \gcd(a, b) = 1\}\| \\ &+ \left\|\{(ap^i, b)|\ a \mid m, b \mid m, \gcd(a, b) = 1, 1 \le i \le k\}\right\| \\ &+ \left\|\{(a, bp^i)|\ a \mid m, b \mid m, \gcd(a, b) = 1, 1 \le i \le k\}\right\| \\ &= \tau(m^2)(2k + 1)\end{aligned}$$

which completes the proof.

Second solution by Tarik Adnan Moon, Kushtia, Bangladesh.

Let

$$n = \prod_{i=1}^{k} p_i^{e_i},\ a = \prod_{p_i \ne p_j} p_i^{f_i},\ b = \prod_{p_i \ne p_j} p_j^{g_j}$$

be the canonical forms of a, b, and n, with $0 \le f_i \le e_i$ and $0 \le g_j \le e_j$. We know that the number of divisors of n^2, $\tau_{n^2} = \prod_{i=1}^{k}(2e_i + 1)$. So, it is enough to prove that the number of ordered pairs (a, b) satisfying the condition is also τ_{n^2} We consider the following expression,

$$S = \prod_{i=1}^{k}\left(p_i^{e_i} + p_i^{e_i-1} + \cdots + p_i^0 + \cdots + \frac{1}{p_i^{e_i-1}} + \frac{1}{p_i^{e_i}}\right).$$

When we expand S, we get all the required (a, b) in the numerator and denominator of every fraction. As the power of a prime p_i in numerator or denominator ranges from 0 to e_i and the primes in numerators and denominators are all distinct. We get, $\prod_{i=1}^{k}(2e_i + 1) = \tau_{n^2}$ distinct fractions and this is the number of ordered pairs (a, b).

Also solved by Roberto Bosch Cabrera, University of Havana, Cuba; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Oles Dobosevych, Ukraine.

Senior problems

S103. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$x_1 + x_2 + \cdots + x_n + \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \geq (n+1)\sqrt[n]{x_1 x_2 \cdots x_n}.$$

Proposed by Nica Cristina-Paula and Nica Nicolae, Romania

First solution by Marius Mainea, Romania

Using the AM-GM and Maclaurin's inequalities we have

$$\begin{aligned}
LHS &= x_1 + x_2 + \cdots + x_n + \frac{x_1 x_2 \cdots x_n}{\frac{\sum x_1 x_2 \cdots x_{n-1}}{n}} \\
&\geq \frac{x_1 + x_2 + \cdots + x_n}{n} + \frac{x_1 + x_2 + \cdots + x_n}{n} \\
&\underbrace{+ \cdots +}_{\text{n times}} \frac{x_1 + x_2 + \cdots + x_n}{n} + \frac{x_1 x_2 \cdots x_n}{\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^{n-1}} \\
&\geq (n+1)\sqrt[n+1]{\frac{x_1 + x_2 + \cdots + x_n}{n} x_1 x_2 \cdots x_n} \geq (n+1)\sqrt[n+1]{(\sqrt[n]{x_1 x_2 \cdots x_n})^{n+1}} \\
&= RHS
\end{aligned}$$

Second solution by Gheorghe Pupazan, Chisinau, Moldova

As the inequality is symmetric, without loss of generality, we may assume that $x_1 \geq x_2 \geq \cdots \geq x_n$ and that $x_1 x_2 \cdots x_n = 1$. This inequality is equivalent to $F(x_1, x_2, \ldots, x_n) \geq 0$, where

$$F(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n + \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} - n - 1.$$

We will show that F is minimal for $x_1 = x_n$, and hence for $x_1 = x_2 = \cdots = x_n$. To prove this, it is enough to show that $x_1 > x_n$ implies that:

$$F(x_1, x_2, \ldots, x_n) > F(\sqrt{x_1 x_n}, x_2, \ldots, x_{n-1}, \sqrt{x_1 x_n}).$$

Indeed, for this case we have that:

$$
\begin{aligned}
&F(x_1, x_2, \dots, x_n) - F(\sqrt{x_1 x_n}, x_2, \dots, x_{n-1}, \sqrt{x_1 x_n}) \\
&\quad = (\sqrt{x_1} - \sqrt{x_n})^2 + \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} - \frac{n}{\frac{2}{\sqrt{x_1 x_n}} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n-1}}} \\
&\quad = (\sqrt{x_1} - \sqrt{x_n})^2 - \frac{n\left(\frac{1}{\sqrt{x_n}} - \frac{1}{\sqrt{x_1}}\right)^2}{\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)\left(\frac{2}{\sqrt{x_1 x_n}} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n-1}}\right)} \\
&\quad = (\sqrt{x_1} - \sqrt{x_n})^2 \left(1 - \frac{n}{x_1 x_n \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)\left(\frac{2}{\sqrt{x_1 x_n}} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n-1}}\right)}\right) \\
&\quad > 0
\end{aligned}
$$

the last one being true, because

$$
\begin{aligned}
&x_1 x_n \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)\left(\frac{2}{\sqrt{x_1 x_n}} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n-1}}\right) \\
&\qquad = \left(\frac{x_n}{x_1} + \frac{x_n}{x_2} + \cdots + \frac{x_n}{x_{n-1}} + 1\right)\left(\frac{2x_1}{\sqrt{x_1 x_2}} + \cdots + \frac{x_1}{x_{n-1}} + \frac{x_1}{x_n}\right) > n
\end{aligned}
$$

as

$$\frac{x_n}{x_1} + \frac{x_n}{x_2} + \cdots + \frac{x_n}{x_{n-1}} + 1 > 1$$

and

$$\frac{2x_1}{\sqrt{x_1 x_2}} + \cdots + \frac{x_1}{x_{n-1}} + \frac{x_1}{x_n} \ge n$$

because of our assumption that $x_1 = \max\{x_1, x_2, \dots, x_n\}$. So we proved that F is minimal for $x_1 = x_2 = \cdots = x_n$ and for this case we have that $F = 0$, so the proof is complete.

Also solved by Manh Dung Nguyen, Hanoi University of Science, Vietnam; Roberto Bosch Cabrera, University of Havana, Cuba; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Arkady Alt, San Jose, California, USA; Oles Dobosevych, Ukraine.

S104. A set of four points in the plane is said to be "nice" if one can draw four circles centered at these points such that each circle is externally tangent to the other three. Given a triangle ABC with orthocenter H, incenter I, and excenters I_A, I_B, I_C, prove that $\{A, B, C, H\}$ and $\{I, I_a, I_b, I_c\}$ are nice if and only if triangle ABC is equilateral.

Proposed by Daniel Lasaosa, Universidad Publica de Navarra, Spain

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

If ABC is equilateral, H is the center of the triangle. Draw pairwise externally tangent circles with centers A, B, C passing respectively through the midpoints of CA and AB, through the midpoints of AB and BC, and through the midpoints of BC and CA. These three circles respectively intersect segments AH, BH, CH which are, by symmetry, equidistant from H, and the circle with center H passing through these points is externally tangent to the three previous circles, hence $\{A, B, C, H\}$ is nice. Consider now the homothecy with center $H = I$ that transforms A, B, C into I_A, I_B, I_C. Clearly, the four previous circles transform into four pairwise externally tangent circles with centers I, I_A, I_B, I_C, and $\{I, I_A, I_B, I_C\}$ is nice.

Assume now that $\{H, A, B, C\}$ is nice, and call $\rho, \rho_A, \rho_B, \rho_C$ the radii of the circles with respective centers H, A, B, C. Note that we may always assume that ABC is acute, since it cannot be rectangle (A, B, C, H would not be distinct), and if it is obtuse in A without loss of generality, then acute triangle HBC has orthocenter A, and we may invert without loss of generality the role of A and H. It is well known (or easily provable) that $HA = 2R\cos A$, $HB = 2R\cos B$ and $HC = 2R\cos C$, where R is the circumradius of ABC. Therefore, since $\rho_A + \rho_B = 2R\sin C$ and $(\rho_A + \rho) - (\rho_B + \rho) = AH - BH$, then

$$\frac{\rho_A}{R} = \frac{AH - BH + 2R\sin C}{2R} = \sin C + \cos A - \cos B.$$

Similarly, considering $\rho_A + \rho_C$ and $\rho_A - \rho_C$, we obtain $\frac{\rho_A}{R} = \sin B + \cos A - \cos C$. It follows that $\sin\left(B + \frac{\pi}{4}\right) = \sin\left(C + \frac{\pi}{4}\right)$, for either $B = C$, or $B + C = \frac{\pi}{2}$, the second option being impossible because $A \neq H$. Similarly, $A = B$, and ABC is equilateral.

Assume finally that $\{I, I_A, I_B, I_C\}$ is nice. The internal bisector II_A is perpendicular to the external bisector I_BI_C, and by cyclic permutation, $II_B \perp I_CI_A$ and $II_C \perp I_AI_B$, or I is the orthocenter of triangle $I_AI_BI_C$. By the previous result, $I_AI_BI_C$ is equilateral, and hence so is ABC.

Second solution by Roberto Bosch Cabrera, University of Havana, Cuba

Let a, b, c be the sides of $\triangle ABC$, and $\widehat{A}, \widehat{B}, \widehat{C}$ its angles, respectively. The solution proceed in two steps.

$\{A, B, C, H\}$ is "nice" $\Leftrightarrow \triangle ABC$ equilateral.
"$\Rightarrow$)" Let $C(A, r_1), C(B, r_2), C(C, r_3), C(H, r_4)$ be the circles centered in A, B, C, and H, respectively. We have that $HA - r_1 = HB - r_2 = HC - r_3 = r_4$ and I is the radical center of $C(A, r_1), C(B, r_2), C(C, r_3)$. From this we deduce that $(r_1, r_2, r_3) = (p - a, p - b, p - c)$ where p is the semiperimeter of $\triangle ABC$. On the other hand we have the well-known formulas

$$(HA, HB, HC) = (2R\cos\widehat{A}, 2R\cos\widehat{B}, 2R\cos\widehat{C})$$

and

$$(a, b, c) = (2R\sin\widehat{A}, 2R\sin\widehat{B}, 2R\sin\widehat{C})$$

where R is the circumradius of $\triangle ABC$. From the above we obtain

$$\begin{aligned} HA - HB &= (p-a) - (p-b) \\ \Rightarrow 2R(\cos\widehat{A} - \cos\widehat{B}) &= 2R(\sin\widehat{B} - \sin\widehat{A}) \\ \Rightarrow \sin\frac{\widehat{A} - \widehat{B}}{2}\left(\sin\frac{\widehat{A} + \widehat{B}}{2} - \cos\frac{\widehat{A} + \widehat{B}}{2}\right) &= 0 \\ \Rightarrow \widehat{A} &= \widehat{B} \quad or \quad \widehat{C} = \frac{\pi}{2}. \end{aligned}$$

Analogously

$$\begin{aligned} \widehat{B} = \widehat{C} \quad &or \quad \widehat{A} = \frac{\pi}{2} \\ \widehat{C} = \widehat{A} \quad &or \quad \widehat{B} = \frac{\pi}{2} \end{aligned}$$

and hence $\widehat{A} = \widehat{B} = \widehat{C}$.
"$\Leftarrow$"
Let a be the length of one of the sides in $\triangle ABC$. We draw three circles with radii $\frac{a}{2}$ and centers A, B, C, respectively. The fourth circle is centered at the center of $\triangle ABC$ and with radius $(\frac{2\sqrt{3}-3}{6})a$.
$\{I, I_a, I_b, I_c\}$ "nice" $\Leftrightarrow \triangle ABC$ equilateral.
"$\Rightarrow$"
Note that I is the orthocenter of $\triangle I_a I_b I_c$, so the latter is valid in this case too, we have
$\widehat{I_a} = \widehat{I_b} = \widehat{I_c} \Rightarrow \frac{\pi}{2} - \frac{\widehat{A}}{2} = \frac{\pi}{2} - \frac{\widehat{B}}{2} = \frac{\pi}{2} - \frac{\widehat{C}}{2} \Rightarrow \widehat{A} = \widehat{B} = \widehat{C}$.

"$\Leftarrow$"
We draw three circles with radii a and centers I_a, I_b, I_c respectively. Now we draw the fourth circle centered in the center of $\triangle ABC$ and with radius $(\frac{2\sqrt{3}-3}{3})a$.

S105. Let P be a point in the interior of a triangle ABC and let $d_a \geq d_b \geq d_c$ be distances from P to the triangle's sides. Prove that

$$\max(AP, BP, CP) \geq \sqrt{d_a^2 + d_b^2 + d_b d_c + d_c^2}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Assume that ABC is acute or rectangle. Call O, R its circumcenter and circumradius, and consider the three circles with centers A, B, C and radius R. Clearly these circles intersect at O, no point of the triangle being in the interior of more than two of these circles. Therefore, $\max(AP, BP, CP) \geq R$, with equality if and only if $P = O$. Note that if P is inside ABC, the inequality is strict for rectangle triangles, since O is the midpoint of the hypothenuse. Assume now that ABC is obtuse, and draw the perpendicular bisector of its longest side, which has length L. Clearly, if a point is at one side of this perpendicular bisector, its distance from the vertex which is at the opposite side exceeds $\frac{L}{2}$. Therefore, $\max(AP, BP, CP) > \frac{L}{2}$; equality could only be reached for the midpoint of the longest side of the triangle, which is not in the interior of ABC.

Assume now that d_a, d_b, d_c are known for a given triangle ABC and a point P in its interior. We may then take segments PX_A, PX_B, PX_C with respective lengths d_a, d_b, d_c, and draw perpendiculars r_a to PX_A at X_A, r_b to PX_B at X_B, and r_c to PX_C at X_C. If the orientation of PX_A, PX_B, PX_C is the same as the orientation of the perpendiculars from P to BC, CA, AB respectively, then the triangle formed by the intersection of lines r_a, r_b, r_c will be equal to ABC. Note however that, if without loss of generality $AP > BP$, we may change the orientation of d_c such that CP does not change since r_a and r_b remain unchanged, but BP increases at the expense of AP which decreases. In this way, we may decrease $\max(AP, BP, CP)$ until $AP = BP = CP$. At this point, triangle ABC will have changed from its original shape, but three facts are sure: 1) triangle ABC is now acute and P is its circumcenter, inside ABC, 2) the distances d_a, d_b, d_c are the same as at the beginning, and 3) $\max(AP, BP, CP)$ has decreased. Therefore, it suffices to show that the proposed result holds for the worst-case scenario, ie, for the acute triangle whose circumcenter is at distances d_a, d_b, d_c from its respective sides BC, CA, AB.

Clearly, in the worst-case scenario $d_a = 2R\cos A$, $d_b = 2R\cos B$, and $d_c = 2R\cos C$, while $\max(AP, BP, CP) \geq R$ with equality if and only if P is the circumcenter of ABC, and it suffices to show that

$$\sin^2 A = 1 - \cos^2 A \geq \cos^2 B + \cos B \cos C + \cos^2 C.$$

Since $\sin A = \sin B \cos C + \cos B \sin C$, the inequality may be rewritten as

$$0 \geq \cos B \cos C\,(1 + 2\cos B \cos C - 2\sin B \sin C) = \cos B \cos C(1 - 2\cos A).$$

Now, if $d_a \geq d_b \geq d_c$, then $A \geq B \geq C$, or $A \leq \frac{\pi}{3}$, and $\cos A \geq \frac{1}{2}$. Equality holds if and only if the smallest angle A equals $\frac{\pi}{3}$, i.e., if and only if ABC is equilateral. The proposed result follows, and equality holds in the if and only if ABC is equilateral and P is its circumcenter.

S106. Eight kids play two different games, A and B. At the beginning, they equally prefer the games. Each day starts with a random distribution of the kids in two groups of size 3 and 5. Every group plays the game preferred by the majority. However, each time a kid plays a game, he or she enjoys it so much, that it becomes his or her favorite game. Find the expected number of days after which all the kids will prefer the same game.

Proposed by Daniel Lasaosa, Universidad Publica de Navarra, Spain and Ivan Borsenco, MIT, USA

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Call $p_n(i)$, $n \geq 1$, the probability that, after the n-th day, exactly i kids prefer game A. By symmetry, since we may interchange the names of the games without altering the problem, $p_n(3) = p_n(5)$ and $p_n(0) = p_n(8)$, all other probabilities being 0. Note that 0 kids will prefer game A after day 1 if both groups have majority of kids that initially prefer B. This will happen:

1) In every case where 0 or 1 kid prefer A before playing the first day.
2) When 2 kids prefer A before playing the first day, unless they are both in the group of size 3.
3) when 3 kids prefer A before playing the first day, and one of them is in the group of size 3 and two of them in the group of size 5.

Note that if 4 or more kids prefer A before playing the first day, either at least three of them are in the group of size 5, or at least two of them are in the group of size 3, forcing a majority in either case. Now, given that two kids prefer A, the probability that both end up in the group size 3 is $\frac{3}{28}$, since the total number of ways in which the kids may be distributed is $\binom{8}{5} = 56$, while the number of favorable cases equals 6, the number of ways in which we may complete the group of size 3 with one kid that prefers B. Similarly, given that three kids prefer A, the probability that two of them end up in the group of size 5 and one of them ends up in the group of size 3 is $\frac{15}{28}$, since the total number of cases is again $\binom{8}{5} = 56$, and the number of favorable cases equals $\binom{3}{2} = 3$ ways to split the three kids that prefer A in one group of two and one group of one, multiplied by the $\binom{5}{3} = 10$ ways to complete the groups of sizes 3 and 5 with kids that prefer B. All taken into account,

$$p_1(0) = \frac{1}{2^8}\left(\binom{8}{0} + \binom{8}{1} + \binom{8}{2}\left(1 - \frac{3}{28}\right) + \binom{8}{3}\frac{15}{28}\right) = \frac{1}{4}.$$

The initial probabilities are then $p_1(0) = p_1(3) = p_1(5) = p_1(8) = \frac{1}{4}$. On any day, if exactly three kids prefer A before playing, the probability that all will end up preferring B is clearly $\frac{15}{28}$. In any other case, either three kids, or five kids, will end up preferring A, because three kids may force a majority in either group or in none, but never on both. Then, using the symmetry in the problem, and calling $P(n)$ the probability that not all kids prefer the same game (where clearly $P(n) = p_n(3) + p_n(5) = 1 - p_n(0) - p_n(8)$), we obtain

$$1 - P(n+1) = 1 - P(n) + \frac{15}{28}P(n), \qquad P(1) = \frac{1}{2}.$$

The recursive equation $P(n+1) = \frac{13}{28}P(n)$, with initial condition $P(1) = \frac{1}{2}$, clearly has solution $P(n) = \frac{1}{2}\left(\frac{13}{28}\right)^{n-1}$, as may be trivially verified by induction. The expected number of days where the kids will not all prefer the same game is then

$$\sum_{n=1}^{\infty} nP(n) = \frac{1}{2}\sum_{n=1}^{\infty} n\left(\frac{13}{28}\right)^{n-1}.$$

Now, for all $|x| < 1$,

$$\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} \frac{dx^n}{dx} = \frac{d}{dx}\sum_{n=1}^{\infty} x^n = \frac{d}{dx}\frac{x}{1-x} = \frac{1}{(1-x)^2}.$$

In our case, $x = \frac{13}{28}$, for a final result of $\frac{28^2}{15^2} = \frac{784}{225}$. Since this is the expected number of days where not all the kids will prefer the same game, it is also the expected number of days after which all the kids will prefer the same game.

Also solved by John T. Robinson, Yorktown Heights, NY, USA

S107. Prove that the number of sets of integers of range n that also contain n is equal to the number of triangulations of a regular $(n+3)$-gon in which every triangle of the triangulation contains at least one side of the polygon. (Range of a set is the difference between the greatest and the least element in the set.)

Proposed by Zoran Sunic, Texas A&M University, USA

First solution by John T. Robinson, Yorktown Heights, NY, USA

Let $F(n)$ be the number of sets of integers of range n that also contain n, and let $T(m)$ be the number of triangulations of a regular m-gon in which every triangle contains at least one side of the m-gon. First, $F(n)$ can be determined by enumerating the various cases. Since clearly $F(0) = 1$ and $F(1) = 2$, assume $n \geq 2$, and consider a set S with the given properties. If $\min(S) = 0$, this implies $\max(S) = n$, and there are $n-1$ integers between 0 and n that may or may not be present, giving 2^{n-1} sets for this case. Next if $\min(S) = i$, $0 < i < n$, there are $n-1$ possibilities for i, and in each case $\max(S) = n+i$; since in each such case S also contains n, there are $n-2$ integers between i and $n+i$ that may or may not be present (because n, which must be present, is excluded from this count), giving 2^{n-2} sets in each case, for a total of $(n-1)2^{n-2}$ sets for these cases. Finally, if $\min(S) = n$, this implies $\max(S) = 2n$, and there are $n-1$ integers between n and $2n$ that may or may not be present, giving 2^{n-1} additional sets. Adding these up, the result is that

$$F(n) = 2^{n-1} + (n-1)2^{n-2} + 2^{n-1} = 2^n + (n-1)2^{n-2}$$

(for $n \geq 2$). Factoring out 2^{n-2} gives a form for $F(n)$ that gives a possible clue as to a relationship to $T(m)$ for $m = n+3$:

$$F(n) = (4 + (n-1))2^{n-2} = (n+3)2^{n-2}\,.$$

Next, apparently $T(n+3) = (n+3)2^{n-2}$ is a known result; for example see

http://www.research.att.com/~njas/sequences/A045623

(although no proof is given there). A constructive proof is as follows. Any triangulation of a regular m-gon (for $m \geq 4$) in which all triangles contain at least one side of the m-gon will have two triangles each of which has two sides that are adjacent sides of the m-gon (this is easy to see using the well-known result that any triangulation of the m-gon has $m-2$ triangles, however if all of these have one side of the m-gon there are two m-gon sides left over which must each be an additional side for two of the triangles); the remaining $m-4$ triangles will contain exactly one side of the m-gon; call the two triangles with two m-gon sides caps. Starting from one of the caps, mark one side of the cap that is a side of the m-gon. Next, consider all paths that move starting from this cap to an adjacent triangle that has not been visited yet (where adjacent triangles are defined as those that share a side in the interior of the m-gon). For the first move, if the adjacent triangle has a side of the m-gon that is adjacent on the m-gon to the marked side, call this an "A" move; otherwise call this an "N" move. For subsequent moves, call the move "A" or "N" depending on whether the two triangles have adjacent sides on the m-gon or not. There will always be $m-3$ such moves, and the last move will be an "A" move, necessarily ending in the second cap.

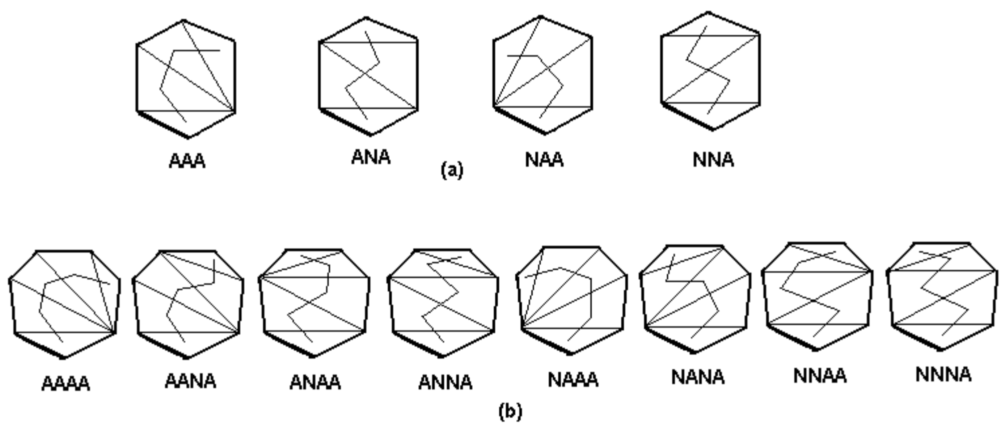

Figure for problem S107

For example, referring to case (a) of the figure, for a hexagon there are 4 possible sequences: AAA, ANA, NAA, and NNA. For a heptagon (7-gon), referring to case (b) of the figure, there are 8 possible sequences: AAAA, AANA, ANAA, ANNA, NAAA, NANA, NNAA, and NNNA. We see that in general there will be 2^{m-4} such sequences, and each represents a different triangulation under the constraint that all triangles share a side with the m-gon. Next note that the triangulations come in pairs, either pairs that are rotationally equivalent to each other, or (for m even) pairs that are reflections but not rotationally equivalent; in the latter case the triangulation rotates to itself with a rotation of 180° (instead of 360°). Therefore we do not get m different triangulations by rotating the ones enumerated so far, but rather half of that, which gives (for $m \geq 5$):

$$T(m) = \frac{1}{2}\, m\, 2^{m-4} = m\, 2^{m-5}\ .$$

We therefore have (for $n \geq 2$)

$$T(n+3) = (n+3)2^{n-2} = F(n),$$

and we also have (trivially) $T(3) = F(0) = 1$ and $T(4) = F(1) = 2$.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Note first that the triangulation of a regular n-gon is equivalent to the triangulation of a convex n-gon, which will help in an induction process. Take a regular n-gon $P_1P_2\dots P_n$, and consider its side $P_{n-1}P_n$, which is clearly contained in one of the triangles in the triangulation. We shall call a_n the number of triangulations such that $P_{n-1}P_n$ is the only side of the polygon contained in that triangle, and b_n the number of triangulations such that $P_{n-1}P_n$ is not the only side of the polygon contained in that triangle. In the first case, there exists another vertex P_j, with $j = 2, 3, \dots, n-3$ such that $P_nP_jP_{n-1}$ is a triangle of the triangulation. Consider now convex polygons $P_1P_2\dots P_jP_n$ and $P_{j+1}\dots P_{n-1}P_j$. Clearly, in each triangulation of $P_1P_2\dots P_n$ such that each triangle contains at least one side of the polygon, each one of these two polygons is triangulated in such a way that P_jP_n and $P_{n-1}P_j$ are not the only sides of the polygon contained in a given triangle of the triangulation, hence there are b_{j+1} possible triangulations of $P_1P_2\dots P_jP_n$, and b_{n-j} possible triangulations of $P_1P_2\dots P_jP_n$. The number of triangulations of $P_1P_2\dots P_n$ that we may find this way is then

$$a_n = \sum_{j=2}^{n-3} b_{j+1}b_{n-j}.$$

In the second case, there is a triangle of the triangulation that is either $P_{n-1}P_nP_1$, or $P_{n-2}P_{n-1}P_n$. In the $(n-1)$-gon resulting after eliminating this triangle, side $P_{n-1}P_1$ in the first case, and triangle $P_{n-2}P_n$ in the second case, needs to be a side of the polygon which is not the only one contained in a triangle of the triangulation, for a total of $2b_{n-1}$ possible such triangulations. Hence, $b_n = 2b_{n-1}$, and since the base case $n = 3$ yields $b_3 = 1$, then $b_n = 2^{n-3}$. Clearly, we then find

$$a_n = \sum_{j=2}^{n-3} 2^{n-5} = (n-4)2^{n-5}.$$

The total number of triangulations of a regular $(n+3)$-gon such that each triangle contains at least one side of the polygon is then $a_{n+3} + b_{n+3} = (n+3)2^{n-2}$.

Consider now a set of range n that contains n. Either n is its maximum, in which case the set is $\{0, \dots, n\}$, where each one of the $n-1$ integers $1, 2, \dots, n-1$ may appear or not, for a total of 2^{n-1} possible such sets, or n is its minimum, in which case the set is $\{n, \ldots, 2n\}$, where each one of the $n-1$ integers $n+1, n+2, \ldots, 2n-1$ may appear or not, for a total of 2^{n-1} possible such sets, or n is neither the maximum or the minimum, in which case the set is $\{i, \ldots, n \ldots, n+i\}$, with $n-1$ possible values for i $(1, 2, \ldots, n-1)$, and where each one of the $n-2$ integers from $i+1$ to $n+i-1$ that are not n may appear or not, for a total of $(n-1)2^{n-2}$ such sets. The total number of sets of range n that contain n is then clearly $2^{n-1} + 2^{n-1} + (n-1)2^{n-2} = (n+3)2^{n-2}$. The conclusion follows.

S108. In triangle ABC let D, E, F be the feet of the altitudes from vertices A, B, C. Denote by P and Q the feet of the perpendiculars from D onto AB and AC, respectively. Let $R = BE \cap DP, S = CF \cap DQ, M = BQ \cap CP$, and $RQ \cap PS$. Prove that M, N, and H are collinear, where H is the orthocenter of triangle ABC.

Proposed by Gabriel Alexander Chicas Reyes, Tokyo, Japan

First solution by Esteban Arreaga Ambiliz, Universidad de San Carlos, Guatemala

By construction we have that $D = RP \cap SQ$, $H = BR \cap CS$ and $A = PB \cap QC$ are collinear then, triangles BRP and CSQ are coaxial. By Desargues' Theorem it follows that BRP and CSQ are copolar. Then, by definition, triangles BRQ and CSP are also copolar. Thus, again by Desargues' Theorem, triangles BRQ and CSP are coaxial; consequently $H = BR \cap CS$, $N = RQ \cap SP$ and $M = QB \cap PC$ are collinear, as we wanted.

Second solution by Roberto Bosch Cabrera, Havana, Cuba

By the Menelaus theorem we need to prove that $\frac{PM}{MC}\frac{CH}{SH}\frac{SN}{NP} = 1$. Now we will find each term separately.

We use the following notation: $\mathbb{B}(A, B, C)$ indicates the relation of *betweenness*, i.e. that B is between A and C (this automatically means that A, B, C are different collinear points).

$\mathbb{B}(B, M, Q) \Rightarrow \frac{CQ}{QA}\frac{AB}{BP}\frac{PM}{MC} = 1$ by Menelaus theorem. So $\frac{PM}{MC} = \frac{QA}{CQ}\frac{BP}{AP}$. We have that $\cot C = \frac{CQ}{DQ}$ and $\tan C = \frac{QA}{DQ}$, hence $\frac{QA}{CQ} = \frac{\sin^2 C}{\cos^2 C}$. Note that $\frac{BP}{AB} = \frac{BP}{BP+PA} = \frac{1}{1+\frac{PA}{BP}}$. We have that $\tan B = \frac{PA}{DP}$ and $\cot B = \frac{BP}{DP}$, so $\frac{PA}{BP} = \frac{\sin^2 B}{\cos^2 B}$, from this we obtain that:

$$\frac{PM}{MC} = \frac{\cos^2 B \sin^2 C}{\cos^2 C} \tag{6}$$

We have that $\frac{CH}{SH} = \frac{CS+SH}{SH} = \frac{CS}{SH} + 1$. Now let T the foot of the perpendicular from S to BC, so holds that $\frac{CS}{SH} = \frac{CT}{TD}$. Besides $\angle CST = B$ and $\angle DST = C$, so $\tan B = \frac{CT}{TS}, \tan C = \frac{TD}{TS} \Rightarrow \frac{CT}{TD} = \frac{\tan B}{\tan C}$ and hence

$$\frac{CH}{SH} = \frac{\sin A}{\cos B \sin C} \tag{7}$$

$\mathbb{B}(Q, N, R) \Rightarrow \frac{PR}{RD}\frac{DQ}{SQ}\frac{SN}{NP} = 1$ by Menelaus theorem. So $\frac{SN}{NP} = \frac{RD}{PR}\frac{SQ}{DQ}$. We have that $\sin(\frac{\pi}{2} - A) = \cos A = \frac{PR}{BR} \Rightarrow PR = BR\cos A$. Now applying Sines

Law in $\triangle BRD$ we obtain $\frac{RD}{\sin(\frac{\pi}{2}-C)} = \frac{BR}{\sin(\frac{\pi}{2}-B)} \Rightarrow \frac{RD}{PR} = \frac{\cos C}{\cos A \cos B}$. Note that $\frac{SQ}{DQ} = \frac{SQ}{DS+SQ} = \frac{1}{\frac{DS}{SQ}+1}$. Besides holds that $\cos A = \frac{SQ}{SC}$ and by Sines Law in $\triangle DSC$ yields $\frac{DS}{\sin(\frac{\pi}{2}-B)} = \frac{SC}{\sin(\frac{\pi}{2}-C)} \Rightarrow \frac{DS}{SQ} = \frac{\cos B}{\cos A \cos C}$ and hence $\frac{SQ}{DQ} = \frac{\cos A \cos C}{\cos A \cos C + \cos B} = \frac{\cos A \cos C}{\sin A \sin C}$. Finally we have that:

$$\frac{SN}{NP} = \frac{\cos^2 C}{\sin A \sin C \cos B} \tag{8}$$

Multiplying $(1), (2), (3)$ we have:

$$\frac{PM}{MC}\frac{CH}{SH}\frac{SN}{NP} = 1$$

as we desired. We are done.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Applying Menelaus' theorem to triangles BQA and QRD, and since M, C, P are collinear, and so are N, P, S, we find

$$\frac{BM}{MQ}\frac{QC}{CA}\frac{AP}{PB} = 1, \qquad \frac{QN}{NR}\frac{RP}{PD}\frac{DS}{SQ} = 1.$$

Since $DSHR$ is a parallelogram because $DQ \parallel BE$ and $DP \parallel CF$, then $RH = SD$, or

$$\frac{BM}{MQ}\frac{QN}{NR}\frac{RH}{HB} = \frac{PB}{PR}\frac{QS}{QC}\frac{PD}{PA}\frac{AC}{HB}.$$

Now, triangles BPR and CQS are similar, since they are rectangle at P and Q respectively, and $\angle PBR = \angle EBA = \frac{\pi}{2} - \angle A = \angle ACF = \angle QCS$. Therefore, $\frac{PB}{PR} = \frac{QC}{QS}$. Moreover, it is well known (or easily found using straightforward trigonometry) that $HB = 2R\cos B$, where R is the circumradius of ABC. Since $AC = 2R\sin C$ because of the Sine Law, and triangle APD is rectangle at P with $\angle PAD = \frac{\pi}{2} - \angle B$, then $\frac{AC}{HB} = \tan B = \frac{AP}{PD}$. It follows clearly that $\frac{BM}{MQ}\frac{QN}{NR}\frac{RH}{HB} = 1$, and applying the reciprocal of Menelaus' theorem, M, N, H are collinear.

Undergraduate problems

U103. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n \leq n$. Prove that

$$a_1^{\frac{1}{a_1}} a_2^{\frac{1}{a_2}} \cdots a_n^{\frac{1}{a_n}} \leq 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Esteban Arreaga Ambiliz, Universidad de San Carlos, Guatemala

For $n = 1$ we have to prove that if $0 < a_1 \leq 1$ then $a_1^{\frac{1}{a_1}} \leq 1$, wich is true because $f(x) = x^a$ is an increasing function for positive x and a.

Now, for $n \geq 2$, let $s = \dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}$. By weighted AM-GM inequality we have that

$$\left(\prod_{k=1}^{n} a_k^{(a_k)^{-1}}\right)^{(s)^{-1}} \leq \sum_{k=1}^{n} \frac{1}{a_k s} \cdot a_k = \frac{n}{s} \tag{9}$$

But $\dfrac{n}{s}$ is just the harmonic mean of $a_1, a_2, \ldots, a_n$; then, by AM-HM inequality and $a_1 + a_2 + \cdots + a_n \leq n$ we have that

$$\frac{n}{s} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} \leq 1 \tag{10}$$

From (1) and (2) it follows that

$$a_1^{\frac{1}{a_1}} a_2^{\frac{1}{a_2}} \cdots a_n^{\frac{1}{a_n}} \leq 1$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$ and $a_1 + a_2 + \cdots + a_n = n$, that is, $a_1 = a_2 = \cdots = a_n = 1$.

Second solution by Vardan Verdiyan, Student, Yerevan, Armenia

Since $0 < a_1 + a_2 + ... + a_n \leq n \Rightarrow$ by the inequality $AM \geq GM$ we have:

$$n \geq a_1 + a_2 + ... + a_n \geq n\sqrt[n]{a_1 a_2 ... a_n} \Rightarrow 1 \geq a_1 a_2 ... a_n > 0$$

$$\Rightarrow \ln a_1 a_2 ... a_n \leq 0 \Leftrightarrow \ln a_1 + \ln a_2 + ... + \ln a_n \leq 0^{\,(*)}.$$

Without loss of generality suppose that $a_1 \geq a_2 \geq ... \geq a_n > 0 \Rightarrow$

$\ln a_1 \geq \ln a_2 \geq ... \geq \ln a_n$ and $\frac{1}{a_1} \leq \frac{1}{a_2} \leq ... \leq \frac{1}{a_n} \Rightarrow$ by the Chebyshev's inequality:

$$\frac{\ln a_1}{a_1} + \frac{\ln a_2}{a_2} + ... + \frac{\ln a_n}{a_n} \leq \frac{1}{n}\left(\ln a_1 + \ln a_2 + ... + \ln a_n\right)\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right).$$

On the other hand, using $^{(*)}$ and the fact that $a_1, a_2, ..., a_n$ are positive we have:

$$0 \geq \frac{1}{n}\left(\ln a_1 + \ln a_2 + ... + \ln a_n\right)\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right) \Rightarrow$$

$$0 \geq \frac{\ln a_1}{a_1} + \frac{\ln a_2}{a_2} + ... + \frac{\ln a_n}{a_n}$$

and hence:

$$1 = exp(0) \geq exp\left(\frac{\ln a_1}{a_1} + \frac{\ln a_2}{a_2} + ... + \frac{\ln a_n}{a_n}\right) = a_1^{\frac{1}{a_1}} a_2^{\frac{1}{a_2}} ... a_n^{\frac{1}{a_n}},$$

which completes our proof.

Also solved by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Nguyen Manh Dung, Hanoi University of Science, Vietnam; Roberto Bosch Cabrera, Cuba; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Arkady Alt, San Jose, California, USA; Magkos Athanasios, Kozani, Greece; Oles Dobosevych, Ukraine.

U104. Let x_0 be a fixed real number and let $f : \mathbb{R} \to \mathbb{R}$ be a function such that f is a derivative on the intervals $(-\infty, x_0), (x_0, \infty)$ and continuous at x_0. Prove that f is a derivative on $\mathbb{R}$.

Proposed by Mihai Piticari, Dragos Voda National College, Romania

First solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

It is a consequence of the Lagrange's theorem. Letting $f = F'$ we have $F(x) - F(x_0) = f(\xi_x)(x - x_0)$ and then $f(\xi_x) = \frac{F(x)-F(x_0)}{x-x_0}$. If $x \to 0$ also $\xi_x \to 0$ because $x_0 < \xi_x < x$ or $x < \xi_x < x_0$. The continuity of f at x_0 yields $\lim_{x\to x_0} f(\xi_x) = f(x_0) = F'(x_0)$ and we are done.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Consider functions $F_1 : (-\infty, x_0] \to \mathbb{R}$ and $F_2 : [x_0, \infty) \to \mathbb{R}$ such that $f(x) = F_1'(x)$ in $(-\infty, x_0]$ and $f(x) = F_2'(x)$ in (x_0, ∞), where F_1 and F_2 are continuous at $x = x_0$. Such functions always exist because we know that f has antiderivatives on the open intervals, which we may call F_1 and F_2, and since f is continuous on $x = x_0$, taking a real value, then F_1 and F_2 have real limits when $x \to x_0^-$ and $x \to x_0^+$, respectively, and we need only to define $F_1(x_0)$ and $F_2(x_0)$ equal to these respective limits.

Define now function $F : \mathbb{R} \to \mathbb{R}$ as follows: $F(x_0) = 0$, if $x > x_0$ then $F(x) = \int_{x_0}^{x} f(x)dx$, and if $x < x_0$, then $F(x) = \int_{x}^{x_0} f(x)dx$. Clearly, $F(x) = F_1(x) - F_1(x_0)$ for $x \le x_0$ and $F(x) = F_2(x) - F_2(x_0)$ for all $x \ge x_0$, therefore $F(x)$ is well defined, continuous in $\mathbb{R}$, and differentiable at least in $\mathbb{R}\backslash\{x_0\}$, its derivative being $f(x)$ in $\mathbb{R}\backslash\{x_0\}$. It suffices to show that $F'(x_0)$ exists and that it is equal to $f(x_0)$. Given non-negative δ, ϵ, call $M = \max_{x\in[x_0-\delta,x_0+\epsilon]} f(x)$ and $m = \min_{x\in[x_0-\delta,x_0+\epsilon]} f(x)$. Clearly,

$$m(\epsilon + \delta) = \int_{x_0-\delta}^{x_0+\epsilon} m dx \le F(x_0 + \epsilon) - F(x_0 - \delta) \le \int_{x_0-\delta}^{x_0+\epsilon} M dx = M(\epsilon + \delta).$$

Therefore,

$$m \le \lim \frac{F(x_0 + \epsilon) - F(x_0 - \delta)}{\epsilon + \delta} \le M,$$

this being true whether we take $\epsilon = 0$ and then take the limit when $\delta \to 0$, or we take $\delta = 0$ and then take the limit when $\epsilon \to 0$, or we take $\delta = \epsilon$ and then take the limit when $\epsilon \to 0$. For sufficiently small δ, ϵ, clearly m and M are arbitrarily close to each other and to $f(x_0)$ since f is continuous at x_0, and the derivatives of $F(x)$ at $x = x_0$ from the left and from the right are equal to each other, and equal to the symmetrically calculated derivative, and also equal to $f(x_0)$. The result follows.

U105. Find $\min\left(\frac{\operatorname{Im} z^5}{\operatorname{Im}^5 z}\right)$ over all z in $\mathbb{C}\backslash\mathbb{R}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Arkady Alt, San Jose, California, USA

Using polar form $z = r(\cos\varphi + i\sin\varphi)$,with $r > 0$ and $\varphi \neq k\pi, k \in \mathbb{Z}$ we obtain

$\frac{Imz^5}{Im^5 z} = \frac{\sin 5\varphi}{\sin^5\varphi} = \frac{16\sin^5\varphi - 20\sin^3\varphi + 5\sin\varphi}{\sin^5\varphi} = \frac{5}{\sin^4\varphi} - \frac{20}{\sin^2\varphi} + 16 =$

$5\left(\frac{1}{\sin^2\varphi} - 2\right)^2 - 4 \geq -4$ and, since lower bound -4 for $\frac{Imz^5}{Im^5 z}$ can be attained if and only if

$\sin^2\varphi = \frac{1}{2} \iff \varphi = \frac{(2n+1)\pi}{4}, n \in \mathbb{Z}$, then $\min\limits_{z\ i\in\ \mathbb{C}\setminus\mathbb{R}}\left(\frac{Imz^5}{Im^5 z}\right) = -4.$

Also solved by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; John T. Robinson, Yorktown Heights, NY, USA; Arin Chaudhuri; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Magkos Athanasios, Kozani, Greece; Brian Bradie, Newport News, USA; Roberto Bosch Cabrera, Cuba.

U106. Let x be a positive real number. Prove that

$$x^x - 1 \geq e^{x-1}(x-1).$$

Proposed by Vasile Cartoaje, University of Ploiesti, Romania

First solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

First we consider $x \geq 1$. Doing the derivative we get

$$f(x) \doteq x^x(1+\ln x) - xe^{x-1} \geq 0$$

or

$$g(y) = y^{ye}(2+\ln y) - y \geq 0 \qquad g(1/e) = 0 \qquad y = x/e \geq 1/e$$

$$g'(y) = y^{ye}\left(\frac{1}{y} + e(2+\ln y)(1+\ln y) - 1\right)$$

and we observe that $e(2+\ln y)(1+\ln y)$ increases for $y > e^{-3/2}$ and then for $y > 1/e$. Therefore we end this part of the proof by checking that $y^{ye} \geq y$ but this is obvious for $y > 1/e$ since the derivative of y^y is positive for $y > 1/e$ and $y^{ye} - y = 0$ for $y = 1/e$.

Now we consider $0 < x < 1$. We employ the following inequality $x^x \geq e^{x-1}$ for any $x > 0$ (we understand $x^x = 1$ for $x = 0$) obtaining $h(x) \doteq e^{x-1}(2-x) - 1 \geq 0$. $h(1) = 0$ and $h'(x) = e^{1-x}(1-x) \geq 0$ whence the conclusion.

The last step is to show $x^x \geq e^{x-1}$ for any $x > 0$. For $x > 1$ we write the inequality as $y^{ye} \geq 1/e$, $y = x/e > 1/e$ and observing that $(y^{ye})' = ey^{ye}(1 + \ln y) \geq 0$ for any $y \geq 1/e$ this first part is concluded. For $0 < x < 1$ the inequality is equivalent to $h(x) = e^{1/x} - e/x \geq 0$. We have $\lim_{x\to 0^+} h(x) = +\infty$, $h(1) = 0$ and $h'(x) = \frac{1}{x^2}(e - e^{1/x}) < 0$ so concluding that $x^x \geq e^{x-1}$ also for $0 < x < 1$. The proof is concluded.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

If $x \leq \frac{1}{e}$, then $1 + \ln x \leq 0$ with equality if and only if $x = \frac{1}{e}$, and $f'(x)$ is the sum of two terms, one non-positive, and the other one negative, or $f'(x) < 0$ for $x \leq \frac{1}{e}$. Consider now $x > \frac{1}{e}$, and call $x = e^y$, where clearly $y > -1$. Now, $f'(x) = 0$ is equivalent to $\left(e^{(y-1)(e^y-1)}(1+y) - 1\right)e^{y+e^y-1} = 0$, or $(y-1)(e^y-1) = -\ln(1+y)$. Call now $g(y) = (y-1)(e^y-1) + \ln(1+y)$. Clearly, the sign of $f'(x)$ is equal to the sign of $g(y)$, one of them being zero if and only if the other one is zero. Note first that $g(0) = 0$, ie $f'(1) = 0$. Now, $g'(y) = ye^y - 1 + \frac{1}{1+y} = y\left(e^y - \frac{1}{1+y}\right)$. If $y > 0$, then $e^y > 1 > \frac{1}{1+y}$, and

$g'(y) > 0$ for all $y > 0$, thus $g(y) > g(0) = 0$ for all $y > 0$, hence $f'(x) > 0$ for all $x > 1$. If $y < 0$, then $e^y < 1 < \frac{1}{1+y}$, and $g'(y) > 0$ again for all $0 > y > -1$, thus $g(y) < g(0) = 0$ for all $0 > y > -1$, hence $f'(x) < 0$ for all $\frac{1}{e} < x < 1$. The claim follows.

U107. Let $f : [0, \infty) \to [0, \infty)$ be a continuous function for which there is a positive integer a such that $f(f(x)) = x^a$ for all x. Prove that

$$\int_0^1 (f(x))^2 dx \geq \frac{2a-1}{a^2+6a-3}.$$

Proposed by Mihai Piticari, "Dragos Voda" National College, Romania

No solution has yet been received.

U108. Find all $n \geq 3$ such that there is a surjective homomorphism $\phi S_n \to S_{n-1}$, where S_n is the symmetric group of n elements.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

For all n, denote by I_n the identity permutation in S_n. Assume that a surjective homomorphism $\phi : S_n \to S_{n-1}$ exists, and call $U \subset S_n$ the kernel of ϕ, ie, the set of elements in S_n such that $\phi(u) = I_{n-1}$ iff $u \in U$. It is well known that U is a normal subgroup of S_n. Denote by $|X|$ the cardinal of set X. Since ϕ is surjective, for each $t \in S_{n-1}$ there is at least one $s \in S_n$ such that $\phi(s) = t$. Now, $sU = Us$ is the set of all elements in S_n whose image is $\phi(s) = t$, since if $s' \in sU$, $u \in U$ exists such that $s' = su$, or $\phi(s') = \phi(s)\phi(u) = \phi(s)$, while if $\phi(s) = \phi(s') = \phi(s(s^{-1}s')) = \phi(s)\phi(s^{-1}s')$, then $s^{-1}s' \in U$, or $s' = s(s^{-1}s') \in sU$. Therefore, there are exactly $|U|$ elements in S_n whose image is t for each $t \in S_{n-1}$, and $|S_n| = |U||S_{n-1}|$. Since $|S_n| = n!$ for all n, then $|U| = n$. Assume now that $n \geq 5$ and a surjective homomorphism $\phi : S_n \to S_{n-1}$ exists. It is well known that A_n, the alternating group on n elements, is the only proper nontrivial normal subgroup of S_n, ie, the only normal subgroup of S_n that is not either S_n or I_n. Therefore, $U = A_n$. But $|A_n| = \frac{n!}{2}$ and $|U| = n$, or $(n-1)! = 2$, and $n = 3$, absurd since $n \geq 5$. Therefore, surjective homomorphisms $\phi : S_n \to S_{n-1}$ may only exist for $n \leq 4$.

For $n = 3$, consider the surjective homomorphism $\phi : S_3 \to S_2$ defined by

$\phi(\{1,2,3\}) = \phi(\{2,3,1\}) = \phi(\{3,1,2\}) = \{1,2\}$,

$\phi(\{1,3,2\}) = \phi(\{2,1,3\}) = \phi(\{3,2,1\}) = \{2,1\}$.

For $n = 4$, consider the surjective homomorphism $\phi : S_4 \to S_3$ defined by

$\phi(\{1,2,3,4\}) = \phi(\{2,1,4,3\}) = \phi(\{3,4,1,2\}) = \phi(\{4,3,2,1\}) = \{1,2,3\}$,

$\phi(\{1,2,4,3\}) = \phi(\{2,1,3,4\}) = \phi(\{3,4,2,1\}) = \phi(\{4,3,1,2\}) = \{2,1,3\}$,

$\phi(\{1,3,2,4\}) = \phi(\{2,4,1,3\}) = \phi(\{3,1,4,2\}) = \phi(\{4,2,3,1\}) = \{1,3,2\}$,

$\phi(\{1,3,4,2\}) = \phi(\{2,4,3,1\}) = \phi(\{3,1,2,4\}) = \phi(\{4,2,1,3\}) = \{3,1,2\}$,

$\phi(\{1,4,2,3\}) = \phi(\{2,3,1,4\}) = \phi(\{3,2,4,1\}) = \phi(\{4,1,3,2\}) = \{2,3,1\}$,

$\phi(\{1,4,3,2\}) = \phi(\{2,3,4,1\}) = \phi(\{3,2,1,4\}) = \phi(\{4,1,2,3\}) = \{3,2,1\}$.

The claim follows.

Second solution by Roberto Bosch Cabrera, University of Havana, Cuba

We suppose the contrary, that is to say, exist $H \neq A_n$ normal in S_n. So we have that $A_n \cap H$ is normal in S_n, but $(A_n \cap H) \subseteq A_n \subset S_n$, and hence $(A_n \cap H)$ is normal in A_n. Now since A_n is simple for $n \geq 5$ we obtain that $(A_n \cap H) = \{e\}$ or $(A_n \cap H) = A_n$. From the last case we deduce that $A_n \subseteq H \Rightarrow |A_n| \leq |H|$, but by Lagrange's theorem $|H|$ is an divisor of $n!$, besides we have that $|A_n| = \frac{n!}{2}$ so $|H| = |A_n|$ and $H = A_n$. Now we consider the first case: $(A_n \cap H) = \{e\}$. It follows that H just contain odd permutations apart from e. But since the product of two odd permutations is an even permutation $H = \{e, h\}$ with $h^2 = e$. So $xH = \{x, xh\}$ and $Hx = \{x, hx\}$, but H is normal in S_n, and hence $xh = hx$ for all $x \in S_n$. In particular taking $x = (1, 2, 3, ..., n)$ since this element commutes only with its powers we obtain that h is equal to one of these permutations:

$$\begin{aligned} 123...n &\rightarrow 234...1 \\ 123...n &\rightarrow 345...2 \\ &\vdots \\ 123...n &\rightarrow n12...(n-1) \end{aligned}$$

but by inspection we confirm that none satisfie $h^2 = e$. So this case is avoid, and we are done.

■

Now we return to our problem. The homomorphism's theorem say that if ϕ is a homomorphism of G onto $\overline{G}$ with kernel K, then $G/K \approx \overline{G}$. It follows that homomorphic images of a given group must be expressible in the form G/K where K is normal in G. Setting $G = S_n$, $(n \geq 5)$, we obtain by lemma that the only (up to isomorphism) homomorphic images of S_n are $S_n/A_n \approx \{1, -1\}$, $S_n/S_n \approx \{e\}$, $S_n/\{e\} \approx S_n$. All distinct of S_{n-1}.
If $n = 3$ the statement of lemma remain true, for this see the lattice formed by the subgroups of S_3 and by inspection confirm this. So we have that the only homomorphic images of S_3 are $S_3/A_3 \approx \{1, -1\} \approx S_2$, $S_3/S_3 \approx \{e\}$, $S_3/\{e\} \approx S_3$. We define $\phi : S_3 \rightarrow S_2$ as follows: $\phi(A_3) = \phi(123, 231, 312) = 12$ and $\phi(132, 213, 321) = 21$.
Now just rest the case $n = 4$. If $\phi : S_4 \rightarrow S_3$ exist, then by homomorphism's theorem we have that $S_4/Ker\phi \approx S_3 \Rightarrow |Ker\phi| = 4$. So we need to find a normal subgroup of S_4 with 4 elements. Let see the permutations of S_4:

S_4

$p_1 = 1234$	$p_7 = 1342$	$p_{13} = 4231$	$p_{19} = 4123$
$p_2 = 2143$	$p_8 = 1423$	$p_{14} = 3142$	$p_{20} = 3421$
$p_3 = 3412$	$p_9 = 3241$	$p_{15} = 2413$	$p_{21} = 1243$
$p_4 = 4321$	$p_{10} = 4213$	$p_{16} = 1324$	$p_{22} = 3214$
$p_5 = 2314$	$p_{11} = 2431$	$p_{17} = 4312$	$p_{23} = 1432$
$p_6 = 3124$	$p_{12} = 4132$	$p_{18} = 2341$	$p_{24} = 2134$

By inspection we obtain that $\{p_1, p_2, p_3, p_4\}$ is the subgroup required, note that is isomorphic to K_4(Klein four-group). Now we find the quotient group S_4/K_4 and define the homomorphism $\phi : S_4 \to S_3$ as:

$$\begin{aligned}
\phi(p_1, p_2, p_3, p_4) &= 123 \\
\phi(p_5, p_8, p_9, p_{12}) &= 231 \\
\phi(p_6, p_7, p_{10}, p_{11}) &= 312 \\
\phi(p_{13}, p_2, p_3, p_4) &= 132 \\
\phi(p_{17}, p_{20}, p_{21}, p_{24}) &= 213 \\
\phi(p_{18}, p_{19}, p_{22}, p_{23}) &= 321
\end{aligned}$$

We are done.

Olympiad problems

O103. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\sqrt[3]{(1+a)(1+b)(1+c)} \geq \sqrt[4]{4(1+a+b+c)}.$$

Proposed by Pham Huu Duc, Ballajura, Australia

First solution by Manh Dung Nguyen, Hanoi University of Science, Vietnam

Firstly, we will prove a lemma

Lemma. If x, y, z are positive real numbers such that $xyz = 1$, then

$$1 + x + y + z \geq 2\sqrt{1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}}$$

Proof. By squaring, the inequality becomes

$$x^2 + y^2 + z^2 + 2(x + y + z) \geq 2\left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + 3$$

We have

$$\begin{aligned} & x^2 + y^2 + z^2 + 2(x + y + z) - 2\left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + 3 \\ & = x^2 + y^2 + z^2 + 2(x + y + z) - 2(xy + yz + zx) - 3 \\ & = (y - z)^2 + (x - 1)^2 + 2(x - 1)(y + z - 2) \geq 0, \end{aligned}$$

because the allowable assumption $x \leq y \leq z$ yields $1 - x \geq 0$ and

$$y + z - 2 \geq 2\sqrt{yx} - 2 = 2\left(\frac{1}{\sqrt{a}} - 1\right) \geq 0$$

Now we come back to the solution for the problem.
Using the substitution $x = \frac{1}{a}, y = \frac{1}{b}$ and $z = \frac{1}{c}$ and applying the **Lemma** we obtain

$$1 + ab + bc + ca \geq 2\sqrt{1 + a + b + c}$$

Thus

$$4(1 + a + b + c) \leq (1 + ab + bc + ca)^2$$

$$64(1 + a + b + c)^3 \leq 16(1 + ab + bc + ca)^2(1 + a + b + c)^2$$

Applying the inequality $4pq \leq (p + q)^2$ for all real numbers p, q we have

$$4(1+ab+bc+ca)(1+a+b+c) \leq (2+a+b+c+ab+bc+ca)^2 = (1+a)^2(1+b)^2(1+c)^2$$

Therefore

$$64(1+a+b+c)^3 \le (1+a)^4(1+b)^4(1+c)^4$$

or

$$\sqrt[4]{4(1+a+b+c)} \le \sqrt[3]{(1+a)(1+b)(1+c)}$$

So we are done.
Equality hold when $a=b=c=1$.

Second solution by Gheorghe Pupazan, Chisinau, Moldova

After we raise both sides to the 12-th power, the inequality becomes equivalent to

$$(1+a)^4(1+b)^4(1+c)^4 \ge 64(1+a+b+c)^3$$

Using that $(1+a)(1+b)(1+c) = 1+a+b+c+ab+bc+ca+abc = 1+a+b+c+1+ab+bc+ca$, we get that our inequality is equivalent to:

$$(1+a+b+c+1+ab+bc+ca)^4 \ge 64(1+a+b+c)^3$$

From AM-GM inequality we infer that:

$$\begin{aligned}(1+a+b+c+1&+ab+bc+ca)^4\\ &\ge \left(2\sqrt{(1+a+b+c)(1+ab+bc+ca)}\right)^4 =\\ &= 16(1+a+b+c)^2(1+ab+bc+ca)^2.\end{aligned}$$

So it suffices to prove that:

$16(1+a+b+c)^2(1+ab+bc+ca)^2 \ge 64(1+a+b+c)^3 \iff (1+ab+bc+ca)^2 \ge 4(1+a+b+c)$

Because $2abc(a+b+c) = 2(a+b+c)$ the inequality is equivalent to:

$$a^2b^2+b^2c^2+c^2a^2+2(ab+bc+ca) \ge 2(a+b+c)+3$$

Because from AM-GM inequality is known that $ab+bc+ca \ge 3\sqrt[3]{(abc)^2} = 3$, it suffices to show that:

$$a^2b^2+b^2c^2+c^2a^2+3 \ge 2(a+b+c)$$

Substitute $a=x^3, b=y^3$ and $c=z^3$. Using the fact that $xyz=1$, the inequality becomes equivalent to:

$$x^6y^6+y^6z^6+z^6x^6+3x^4y^4z^4 \ge 2(x^6y^3z^3+x^3y^6z^3+x^3y^3z^6)$$

From Schur's inequality, applied for the numbers x^2y^2, y^2z^2 and z^2x^2, we know that:

$$\begin{aligned} x^2y^2(x^2y^2 - y^2z^2)(x^2y^2 - z^2x^2) + y^2z^2(y^2z^2 - x^2y^2)(y^2z^2 - z^2x^2) \\ + z^2x^2(z^2x^2 - x^2y^2)(z^2x^2 - y^2z^2) \geq 0 \\ \iff x^6y^6 + y^6z^6 + z^6x^6 + 3x^4y^4z^4 \\ \geq x^6y^4z^2 + x^6y^2z^4 + x^2y^6z^4 + x^4y^6z^2 \\ + x^4y^2z^6 + x^2y^4z^6. \end{aligned}$$

So it suffices to prove that:

$$\begin{aligned} x^6y^4z^2 + x^6y^2z^4 + x^2y^6z^4 + x^4y^6z^2 + x^4y^2z^6 + x^2y^4z^6 \\ \geq 2(x^6y^3z^3 + x^3y^6z^3 + x^3y^3z^6). \end{aligned}$$

But the proof for this one is simple, as from Muirhead's inequality we know that $(6,4,2) \succ (6,3,3)$, so the conclusion follows.

Third solution by Vardan Verdiyan, Student, Yerevan, Armenia

Let us substitute $x = a + b + c$ and $y = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, hence since $abc = 1 \Rightarrow$ by the inequality $AM \geq GM$ we have: $y \geq 3 \Rightarrow y - 3 \geq 0$ $^{(*)}$.

Further, since $abc = 1$, note that our inequality is equivalent to:

$$\sqrt[3]{x + y + 2} \geq \sqrt[4]{4(x+1)} \Leftrightarrow (x + y + 2)^4 \geq 4^3(x+1)^3 \ ^{(1*)}.$$

By Shur's inequality we have: $\sum_{cyc} \frac{1}{a^3} + 3\frac{1}{abc} \geq \sum_{cyc} \frac{1}{bc}\left(\frac{1}{b} + \frac{1}{c}\right)$, which is equivalent to:

$$\left(\sum_{cyc} \frac{1}{a}\right)^3 + 9\frac{1}{abc} \geq 4\left(\sum_{cyc} \frac{1}{bc}\right)\left(\sum_{cyc} \frac{1}{a}\right).$$

After using the fact that $abc = 1$ we can rewrite the last inequality as:

$$\left(\sum_{cyc} \frac{1}{a}\right)^3 + 9 \geq 4\left(\sum_{cyc} a\right)\left(\sum_{cyc} \frac{1}{a}\right) \Leftrightarrow y^3 + 9 \geq 4xy \ ^{(2*)}.$$

By the inequality $AM \geq GM$ we have: $(x + y + 2)^4 \geq 2^4(x+1)^2(y+1)^2 \Rightarrow$ by $^{(1*)}$ it's enough to prove that

$$2^4(x+1)^2(y+1)^2 \geq 4^3(x+1)^3,$$

which is equivalent to:

$$(y+1)^2 \geq 4(x+1) \Leftrightarrow y(y+1)^2 \geq 4xy + 4y.$$

On the other hand, using $^{(2*)}$ it's enough to prove that $y(y+1)^2 \geq y^3 + 9 + 4y$

$$\Leftrightarrow 2y^2 \geq 3y + 9$$

$$\Leftrightarrow (y-3)^2 + (y-3)(y+3) + 3(y-3) \geq 0,$$

which is obviously true by $^{(*)}$. This completes our proof.

Also solved by Arkady Alt, San Jose, California, USA; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Daniel Lasaosa, Universidad Publica de Navarra, Spain.

O104. In a convex quadrilateral $ABCD$ let K, L, M, N be the midpoints of sides AB, BC, CD, DA, respectively. Line KM meets diagonals AC and BD at P and Q, respectively, and line LN meets diagonals AC and BD at R and S, respectively. Prove that if $AP \cdot PC = BQ \cdot QD$, then $AR \cdot RC = BS \cdot SD$.

Proposed by Nairi Sedrakian, Yerevan, Armenia

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

It is well known that $KLMN$ is a parallelogram, where $KL \parallel MN \parallel AC$ and $LM \parallel NK \parallel BD$. Furthermore, the distance from B to AC is double the distance between KL and AC, and similarly for the other vertices of the convex quadrilateral $ABCD$. Consider then a (not necessarily orthonormal) system of coordinates with origin O in the center of parallelogram $KLMN$, and unit vectors $\vec{u}, \vec{v}$ such that the respective coordinates of K, L, M, N are $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$. Assume now that $T = AC \cap BD$ has coordinates (x_0, y_0). Clearly, T is inside parallelogram $KLMN$, and since the distance from A to BD is double the distance from KN to BD, and A is on the parallel to KL through T, we find that A has coordinates $(-2-x_0, y_0)$. Similarly, the coordinates of B, C, D are respectively $(x_0, -2-y_0)$, $(2-x_0, y_0)$ and $(x_0, 2-y_0)$. Since line KM has, in this coordinate system, equation $y = x$, then P, Q have respective coordinates (y_0, y_0) and (x_0, x_0), because lines AC and BD have clearly respective equations $y = y_0$ and $x = x_0$. Therefore, $AP = (2+x_0+y_0)|u|$ and $PC = (2-x_0-y_0)|u|$. Similarly, $BQ = (2+x_0+y_0)|v|$ and $QD = (2-x_0-y_0)|v|$. Since T is in the interior of $KLMN$, then $-2 < x_0 + y_0 < 2$, and $AP \cdot PC = BQ \cdot QD$ iff $|u| = |v|$. In a completely analogous manner, we may also show that $AR \cdot RC = BS \cdot SD$ iff $|u| = |v|$, since line LN has equation $y = -x$, and R, S have respective coordinates $(-y_0, y_0)$, $(x_0, -x_0)$, for $AR \cdot RC = (2+x_0-y_0)(2+y_0-x_0)|u|^2$ and $BS \cdot SD = (2-x_0+y_0)(2+x_0-y_0)|v|^2$. The conclusion follows.

Note that condition $|u| = |v|$ is equivalent to saying that $KLMN$ is a rhombus, ie, since $AC = 2KL = 2MN$ by similarity of triangles ABC and KBC, then $AP \cdot PC = BQ \cdot QD$ if and only if $AR \cdot RC = BS \cdot SD$, and if and only if the diagonals AC and BD of $ABCD$ have equal length.

Second solution by Roberto Bosch Cabrera, University of Havana, Cuba

We draw the four segments KL, LM, MN, NK, the quadrilateral $KLMN$ is an parallelogram since $KL \| AC \| NM$ and $KN \| BD \| LM$. Denote by A', C' the points of intersection of AC with KN and LM respectively, by B', D' the points of intersection of BD with KL and NM respectively. Let $KA' = a$, $A'N = b$, $KB' = c$, $B'L = d$. See the figure. We will find the lengths of several segments as function of a, b, c, d. Note that $AP = AA' + A'P = c + \frac{a(c+d)}{a+b}$ since $\triangle KPA'$ is similar to $\triangle KMN$, hence

$$AP = \frac{bc + ad + 2ac}{a + b}$$

by analogy we obtain that

$$PC = \frac{bc + ad + 2bd}{a + b}$$
$$BQ = \frac{bc + ad + 2ac}{c + d}$$
$$QD = \frac{bc + ad + 2bd}{c + d}$$

It follows that $AP \cdot PC = BQ \cdot QD \Leftrightarrow a + b = c + d$ (i.e. $KLMN$ is an rhombus). Note that $AR = 2c + d - RC'$, but $RC' = \frac{a(c+d)}{a+b}$ since $\triangle LC'R$ is similar with $\triangle LMN$ and hence

$$AR = \frac{ac + bd + 2bc}{a + b}$$

analogously we have that

$$RC = \frac{ac + bd + 2ad}{a + b}$$
$$BS = \frac{ac + bd + 2ad}{c + d}$$
$$SD = \frac{ac + bd + 2bc}{c + d}$$

It follows that $AR \cdot RC = BS \cdot SD \Leftrightarrow a + b = c + d$ (i.e. $KLMN$ is an rhombus). Finally yields that

$$AP \cdot PC = BQ \cdot QD \Leftrightarrow AR \cdot RC = BS \cdot SD.$$

O105. Let $P(t)$ be a polynomial with integer coefficients such that $P(1) = P(-1)$. Prove that there is a polynomial $Q(x, y)$, with integer coefficients such that $P(t) = Q(t^2 - 1, t(t^2 - 1))$.

Proposed by Mircea Becheanu and Tiberiu Dumitrescu, University of Bucharest, Romania

First solution by John T. Robinson, Yorktown Heights, NY, USA

The proof is by induction on the degree of P, with even and odd degrees handled as separate cases. Since there is no P of degree 1 such that $P(1) = P(-1)$ (if $P(x) = ax + b$, $P(1) = P(-1)$ implies $a = 0$), we start with polynomials P of degrees 2 and 3.

If $P(x) = ax^2 + bx + c$, $P(1) = P(-1)$ implies that $b = 0$, so we have $P(x) = ax^2+c$. Taking $Q(x, y) = ax+a+c$, we see that $Q(t^2-1, t^3-t) = at^2-a+a+c = P(t)$.

If $P(x) = ax^3 + bx^2 + cx + d$, $P(1) = P(-1)$ implies $a + c = 0$, so we have $P(x) = -cx^3 + bx^2 + cx + d$. Taking $Q(x, y) = -cy + bx + b + d$, we see that $Q(t^2 - 1, t^3 - t) = -ct^3 + ct + bt^2 - b + b + d = P(t)$.

Now for the first inductive step, assume P is of degree $2n$ for $n > 1$:

$$P(x) = ax^{2n} + \text{(lower order terms)}.$$

Consider the polynomial $R(x) = P(x) - a(x^2 - 1)^n$: R is of degree less than $2n$, and satisfies $R(1) = R(-1)$. Therefore by induction there is a polynomial $S(x, y)$ such that $R(t) = S(t^2 - 1, t^3 - t)$. Taking $Q(x, y) = ax^n + S(x, y)$, we see that

$$\begin{aligned} Q(t^2 - 1, t^3 - t) &= a(t^2 - 1)^n + R(t) \\ &= a(t^2 - 1)^n + P(t) - a(t^2 - 1)^n \\ &= P(t). \end{aligned}$$

For the second inductive step, assume P is of degree $2n + 1$ for $n > 1$:

$$P(x) = ax^{2n+1} + \text{(lower order terms)}.$$

Consider the polynomial $R(x) = P(x) - a(x^2 - 1)^{n-1}(x^3 - x)$: R is of degree less than $2n + 1$, and satisfies $R(1) = R(-1)$. Therefore by induction there is a polynomial $S(x, y)$ such that $R(t) = S(t^2 - 1, t^3 - t)$. Taking

$$Q(x, y) = ax^{n-1}y + S(x, y),$$

we see that

$$\begin{aligned} Q(t^2-1, t^3-t) &= a(t^2-1)^{n-1}(t^3-t) + R(t) \\ &= a(t^2-1)^{n-1}(t^3-t) + P(t) - a(t^2-1)^{n-1}(t^3-t) \\ &= P(t), \end{aligned}$$

which completes the proof.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

By induction on the degree n of $P(t)$, the claim is trivially true for the base cases $n = 0$ and $n = 1$ with $S(x, y) = 0$. For the step, if the degree of $P(t)$ is $n = 2m$ even, and the coefficient of t^{2m} is $a_{2m} \neq 0$ (clearly integer), call $p(t) = P(t) - a_{2m}(t^2 - 1)^m$, which is obviously a polynomial with integer coefficients and degree at most $2m-1$. By hypothesis of induction, a polynomial $s(x, y)$ with integer coefficients exists such that $p(t) = s(t^2-1, t(t^2-1)) + r(t)$, where $r(t)$ is a polynomial with integer coefficients and degree at most 1. Then, $S(x, y) = s(x, y) + a_{2m}x^m$ clearly satisfies the claim with $R(t) = r(t)$. If the degree of $P(t)$ is $n = 2m + 1$ odd and at least 3, and the coefficient of t^{2m+1} is $a_{2m+1} \neq 0$ (clearly integer), call $p(t) = P(t) - a_{2m+1}t(t^2 - 1)^m$, which is obviously a polynomial with integer coefficients and degree at most $2m$. By hypothesis of induction, a polynomial $s(x, y)$ with integer coefficients exists such that $p(t) = s(t^2 - 1, t(t^2 - 1)) + r(t)$, where $r(t)$ is a polynomial with integer coefficients and degree at most 1. Then, $S(x, y) = s(x, y) + a_{2m+1}yx^{m-1}$ satisfies the claim with $R(t) = r(t)$. Note that $m - 1 \geq 0$ since $2m + 1 \geq 3$. The claim follows.

O106. A polynomial with integer coefficients is called "good" if it can be represented as a sum of cubes of several polynomials in x with integer coefficients. For example, $9x^3 - 3x^2 + 3x + 7 = (x-1)^3 + (2x)^3 + 2^3$ is good.

a) Is $3x^7 + 3x$ good?

b) Is $3x^{2008} + 3x^7 + 3x$ good?

Proposed by Nairi Sedrakian, Yerevan, Armenia

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

A polynomial with integer coefficients is called "nice" if the coefficients in the terms whose degree is not a multiple of 3, are multiples of 3, and their sum is multiple of 6. Clearly, the sum of nice polynomials is a nice polynomial, and the sum of good polynomials is a good polynomial.

Claim: A polynomial is good if and only if it is nice.

Proof: Consider the cube of a polynomial $p(x)$ with integer coefficients, where $p(x) = \sum_{i=0}^{n} a_i x^i$. All the terms of $p^3(x)$ are either of the form $a_i x^{3i}$, or of the form $3a_i^2 a_j x^{2i+j}$ where $i \neq j$, or of the form $6a_i a_j a_k x^{i+j+k}$, where i, j, k are distinct. Thus, all terms with degree which is not a multiple of 3 have coefficient which is a multiple of 3. Note that since $(2i+j)+(2j+i) = 3(i+j)$, then $2i+j$ is a multiple of 3 iff $2j+i$ is a multiple of 3, and if $a_i a_j$ is odd, then $a_i + a_j$ must be even. Therefore, $a_i a_j (a_i + a_j)$ is even and hence the sum of all coefficients in the terms which have a degree that is not a multiple of 3 must be a multiple of 6. The cube of a polynomial with integer coefficients is therefore always nice, and since the sum of nice polynomials is nice, then a good polynomial is always nice.

Reciprocally, consider a nice polynomial $P(x)$, and let us show by induction on n that it is also good, where n is the integer such that the degree of $P(x)$ does not exceed $3n$, but is larger than $3n-3$. For the base case $n = 1$, $P(x) = a_3 x^3 + 3a_2 x^2 + 3a_1 x + a_0$, where a_1, a_2 have the same parity. But then,

$$P(x) = \frac{a_1 + a_2}{2}(x+1)^3 + \frac{a_1 - a_2}{2}(x-1)^3 + (a_3 - a_1)x^3 + (a_0 - a_2)1^3,$$

and since $\frac{a_1+a_2}{2}$, $\frac{a_1-a_2}{2}$, $a_3 - a_1$ and $a_0 - a_2$ are integers, $P(x)$ is the sum of the cubes of polynomials of the form $\pm(x+1), \pm(x-1), \pm x, \pm 1$, and $P(x)$ is good. For the step, assume that the result is true for $n-1$. If $P(x) = a_{3n} x^{3n} + 3a_{3n-1} x^{3n-1} + 3a_{3n-2} x^{3n-2} + Q(x)$ is nice, where the degree of $Q(x)$

is at most $3n-3$, then either $a_{3n-1}+a_{3n-2}$ is even and $Q(x)$ is good, or $a_{3n-1}+a_{3n-2}$ is odd, and $Q(x)-3x^{3n-4}$ is good. In the first case,

$$P(x)=\frac{a_{3n-2}+a_{3n-1}}{2}(x^n+x^{n-1})^3+\frac{a_{3n-2}-a_{3n-1}}{2}(x^n-x^{n-1})^3+$$

$$+(a_{3n}-a_{3n-2})(x^n)^3+Q(x)-a_2(x^{n-1})^3,$$

and $P(x)$ is the sum of cubes of polynomials of the form $\pm(x^n+x^{n-1})$, $\pm(x^n-x^{n-1})$, $\pm x^n$, $\pm x^{n-1}$, and of good polynomial $Q(x)$, hence it is good. In the second case,

$$P(x)=\frac{a_{3n-2}+a_{3n-1}-1}{2}(x^n+x^{n-1})^3+\frac{a_{3n-2}-a_{3n-1}-1}{2}(x^n-x^{n-1})^3+$$

$$+(x^n+x^{n-2})^3+(a_{3n}-a_{3n-2})(x^n)^3+(Q(x)-3x^{3n-4})-a_2(x^{n-1})^3,$$

and $P(x)$ is the sum of the cubes of polynomials of the form $\pm(x^n+x^{n-1})$, $\pm(x^n-x^{n-1})$, $\pm x^n$, $\pm x^{n-1}$, x^n+x^{n-2}, and of good polynomial $Q(x)-3x^{3n-4}$, hence it is good. The claim follows.

Clearly, $3x^7+3x$ is nice, but $3x^{2008}+3x^7+3x$ is not, so $3x^7+3x$ is good, and $3x^{2008}+3x^7+3x$ is not good. In fact,

$$3x^7+3x=(x^3+x)^3+(-x^3)^3+(-x^2-x)^3+(x^2+1)^3+(x-1)^3+(-x)^3.$$

O107. Let p_1, p_2, p_3 be distinct primes and let n be a positive integer. Find the number of functions $f : \{1, 2, \ldots, 2n\} \to \{p_1, p_2, p_3\}$ for which $f(1)f(2)\cdots f(2n)$ is a perfect square.

Proposed by Dorin Andrica, "Babes-Bolyai" University and Mihai Piticari, "Dragos Voda" National College, Romania

First solution by Holden Lee, USA

Let a_1, a_2, a_3 be the number of values x such that $f(x) = p_1$, $f(x) = p_2$, $f(x) = p_3$, respectively. Then in order for $f(1)f(2)\cdots f(2n) = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3}$ to be a perfect square, we must have a_1, a_2, a_3 even. Given a_1, a_2, a_3 with $a_1 + a_2 + a_3 = 2n$, there are $\binom{2n}{a_1, a_2, a_3}$ ways to choose a_1, a_2, a_3 values of x whose images are p_1, p_2, p_3, respectively, and hence to define f. The number of possibilities for f is $\displaystyle\sum_{a_1+a_2+a_3=2n, a_1,a_2,a_3 even} \binom{2n}{a_1, a_2, a_3}$, or the sum of the coefficients of terms with all exponents even in the expansion of $(x + y + z)^{2n}$. Let

$$g(x, y, z) = \frac{1}{8} \sum_{(i,j,k)\in\{-1,1\}^3} (ix + jy + kz)^{2n}$$

Note that since g is an even polynomial, in all its variables, no term with an odd exponent will appear. Furthermore, the coefficients of the terms with only even exponents are equal to the corresponding coefficients in $(x+y+z)^{2n}$. Thus we would like the sum of coefficients of $g(x)$, which is equal to $g(1, 1, 1) = \dfrac{3^{2n} + 3}{4}$.

Second solution by John T. Robinson, Yorktown Heights, NY, USA

Let p_1, p_2, p_3 be distinct primes and let n be a positive integer. Find the number of functions $f : \{1, 2, ..., 2n\} \to \{p_1, p_2, p_3\}$ for which $f(1)f(2)\cdots f(2n)$ is a perfect square.

Solution - Let $F(2n)$ be the number of such functions for a given value of n, and suppose $p_1^{2i}p_2^{2j}p_3^{2n-2i-2j}$ is one such perfect square. Then considering $f(1)$ as a first bucket, $f(2)$ as a second, etc., the number of functions giving this square is the number of ways of placing $2i$ p_1s, $2j$ p_2s, and $2n - 2i - 2j$ p_3s into $2n$ buckets. Therefore

$$F(2n) = \sum_{0\le i\le n} \sum_{0\le j\le n-i} \binom{2n}{2i\ \ 2j\ \ 2n-2i-2j}.$$

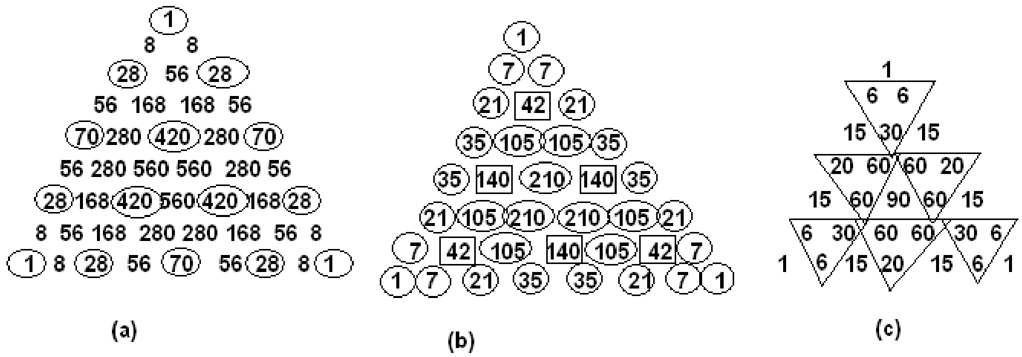

Figure for problem O107

Can this be simplified? Yes, although here we will just show a diagrammatic argument. Referring to the figure, $F(8)$ is the sum of the circled numbers in the $n = 8$ trinomial triangle of (a); these in turn (using the summing properties of trinomial coefficients, similar to those of binomial coefficients) are equal to the circled numbers in the $n = 7$ trinomial triangle of (b). Now the sum of all the numbers in the triangle of (b) is 3^7, so $F(8)$ is 3^7 minus the sum of the numbers enclosed in squares in (b), and the sum of the numbers in squares is the the sum of the numbers enclosed in triangles in the $n = 6$ trinomial triangle of (c). However note that the numbers not enclosed in triangles in (c) sum to $F(6)$, that is $F(8) = 3^7 - (3^6 - F(6))$. It is clear this pattern holds in general, and since $F(2) = 3$, we have

$$\begin{aligned} F(2n) &= 3^{2n-1} - 3^{2n-2} + 3^{2n-3} - ... + 3 \\ &= \frac{3^{2n} + 3}{4}. \end{aligned}$$

Also solved by Roberto Bosch Cabrera, University of Havana, Cuba; Daniel Lasaosa, Universidad Publica de Navarra, Spain.

O108. Prove that the set of positive integers that cannot be written as a sum of four nonzero squares has density zero.

Proposed by Iurie Boreico, Harvard University, USA

First solution by Iurie Boreico, Harvard University, USA

First, we prove that if a natural number r is of the form $2^k(8m+3)$ or $4^k(8m+6)$ for $k, m \in \mathbb{N}_0$, then r can be written as a sum of three non-zero perfect squares. Indeed, all numbers except numbers of form $4^j(8m+7)$ can be written as a sum of three perfect squares, and r is certainly not of this form. So r can be written as the sum of three perfect squares. If one of them would be 0, then actually r could be written as the sum of two perfect squares. However since $8m+3$ is 3 modulo 4, and $\frac{8m+2}{2} = 4m+3$ is 3 modulo 4 as well, r must have an odd prime divisor which is 3 modulo 4 and has an odd exponent in r. This contradicts the theorem that a number is the sum of two perfect squares if and only if every prime number congruent to 3 modulo 4 appears with an even exponent in the prime decomposition of the number.

Now let's come back to the problem. Choose a number n. If we can find a non-zero integer a such that $n - a^2$ would be of the form $2^k(8m+3)$ or $4^k(8m+6)$, then we can write n as a sum of four non-zero perfect squares, according to what was proven above. Now we seek for which n such an a exists. Say $n = 4^p \cdot q$, where q is not divisible by 4. We now investigate the residues of q modulo 8:

- if $q \equiv 1 \pmod 8$ then the numbers $q-1^2, q-3^2, q-5^2, q-7^2, \ldots, q-15^2$ are all divisible by 8 but give different residues modulo 64, on so we can pick up $b \le 15$ such that $q - b^2 = 8(8r+6)$ and thus if $q > 15^2$ we can set $a = 2^p \cdot b$.

- if $q \equiv 2 \pmod 8$ then $q - 4 \equiv 6 \pmod 8$ so we take $a = 2^{p+1}$.

- if $q \equiv 3 \pmod 8$ then $q - 16 \equiv 3 \pmod 8$ so $a = 2^{p+4}$

- if $q \equiv 5 \pmod 8$ then $q-1^2, q-3^2, q-5^2, q-7^2$ are all divisible by 4 but not by 8, and give different residues modulo 32, hence one of them is congruent to 12 modulo 32, and then we get $q - b^2 = 4(8m+3)$ so we set $a = 2^p \cdot b$

- if $q \equiv 6 \pmod 8$ then $q - 16 \equiv 6 \pmod 8$ so we get $a = 4^{p+2}$

Finally if $q \equiv 7 \pmod 8$ then $q - 4 \equiv 3 \pmod 8$ and so we set $a = 2^{p+1}$

Therefore, if $q > 225$ we can always find a non-zero a such that $n - a^2$ is positive, and is the sum of three non-zero squares.

Thus the numbers that cannot be written as a sum of four non-zero perfect squares can only be of form $q \cdot 4^p$ for $1 \le q \le 225$. For any N, there are at most $225 \cdot log_4 N$ such numbers that are less than N, as if $q \cdot 4^p \le N$ then $p \le log_4 N$. Since the limit $\lim_{N\to\infty} \frac{225 log_4 N}{N}$ is 0, the density of such numbers is 0, as desired.

Second solution by Roberto Bosch Cabrera, Cuba

Let A the set of positive integers that cannot be written as a sum of four nonzero squares. Can be shown that

$$A = \left\{1, 3, 5, 9, 11, 17, 29, 41, 2 \cdot 4^h, 6 \cdot 4^h, 14 \cdot 4^h\right\} \quad h = 0, 1, 2, ...$$

For this result see the following references

- Sierpiński, W. *Elementary theory of numbers.* 1964. Theorem 5-page 373.
- Pall, G. *On sums of squares.* Amer. Math. Monthly 40. 1933.

We denote the elements of A by $a_1, a_2, ...$ and so on. Hence

$$(a_1, a_2, ..., a_{15}) = (1, 2, 3, 5, 6, 8, 9, 11, 14, 17, 24, 29, 32, 41, 56)$$

Let $b_n = \frac{n}{a_n}$. We need to prove that $\lim b_n = 0$. Note that if $n \geq 16$ then b_n can be writen as

$$\begin{array}{rl} \dfrac{n}{14 \cdot 4^{\frac{n-12}{3}}}, & n \equiv 0(3) \\ \dfrac{n}{6 \cdot 4^{\frac{n-10}{3}}}, & n \equiv 1(3) \\ \dfrac{n}{2 \cdot 4^{\frac{n-8}{3}}}, & n \equiv 2(3) \end{array}$$

Since $4^m > m^2$ for all positive integer m we have

$$0 < b_n < \frac{n}{\left(\frac{n-12}{3}\right)^2} = \frac{9n}{n^2 - 24n + 144} \to 0$$

from this follows that $\lim b_n$ exist and is equal to 0. We are done.