Centroids and Tiling Problems

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Abstract. The most common technique in solving tiling problems is coloring. In this article we present another approach, based on the use of centroids (centers of mass) and their properties. The only appearance of this technique in the problem-solving literature the author is aware of was in Prasolov's *Problems in Plane Geometry* (problem 6 in this paper). The author assumes this is the first systematical introduction of it.

1 Technique outline

The technique is based on the following theorem:

Theorem 1. Let O be the center of mass of points $X_1, \ldots, X_n$ with masses $m_1, \ldots, m_n$, then the center of mass is unique and it satisfies

$$m_1\overrightarrow{OX_1} + \cdots + m_n\overrightarrow{OX_n} = 0. \tag{1}$$

A special case of the previous theorem implies that if $m_1 = m_2 = \cdots = m_n$, then the center of mass satisfies the following relation:

$$\overrightarrow{OX_1} + \overrightarrow{OX_2} + \cdots + \overrightarrow{OX_n} = \overrightarrow{0}. \tag{2}$$

In order to solve tiling problems we are going to use the following technique:

At the first step we find the center of mass of the given tiles and the figure being tiled. Then we divide the tiles into smaller squares such that their centers of mass are located in one the vertices of smaller squares (see Fig. 2 to 4). After "rescaling", the figure to be tiled is divided into smaller squares in the same manner (Fig. 1 shows such a transformation from a 10×10 square to a 20×20 square). We place a coordinate system such that its origin is at the figure's center of mass, as shown in Fig. 1. The unit length of the coordinate system will be the sidelength of the "small" square. After these trasformations are conducted, using (2), the problem transforms into a number theory problem. If the tiling of the figure exists, then the sums of both abscissae and ordinates of tiles masspoints have to be equal to zero.

In the technique described above formula (2) will be the primary tool to tackle the problem. However, relation (1) as a general formula can have a wider range of applications, and the reader is encouraged to seek for those in other tiling problems.

2 Solved problems

Problem 1. (Engel) A rectangular floor is covered by 2×2 and 1×4 tiles. One tile got smashed. There is a tile of the other kind available. Prove that the floor cannot be covered by rearranging the tiles.

Solution. Location of tiles' centroids is shown in Fig. 2.

Since the straight (1×4) tetromino has to be divided in squares as shown on Fig. 2 so its centroid can have integer coordinates, same division is conducted on the square (2×2) tetromino. Finally, the same division is applied on the rectangle being tiled. As we already pointed out, the sums of abscissae and ordinates in the given coordinate system have to be zeros. Note that, using original square division of the rectangle being tiled, it's dimensions can be odd $\times$ even or even $\times$ even. In the first case, if we take the odd dimension parallel to the abscissa. It is not difficult to see that abscissae for the centroids of the ordinate-parallel tetrominoes in the new coordinate system are even, while for the centroids of the square tetrominoes and the abscissa-parallel tetrominoes they are odd. Sum of odd and even numbers can be zero only if the number of odds is even. Therefore the number of square and abscissa-parallel tetrominoes is even. When the ordinates of the centroids are concerned, same reasoning implies that the number of abscissa-parallel tetrominoes is even, since their ordinates are odd. Hence the number of square tetraminoes must be even. However, If we replace one of the square tetrominoes with a straight one, the parity conditions is not satisfied, so such tiling is not possible. Analogous conclusion can be reached in the second (even$\times$even) case.

Problem 2. (Engel) Prove that a 10×10 chessboard cannot be covered by 25 T-tetrominoes.

Solution. Location of the tile's centroid is shown in the Fig. 3.

The new division requires us to tile a 40×40 square. Notice that for each T-tetromino, coordinates of its centroid have different parity. Since 25 is odd, number of A and B, or C and D tiles is odd. Since they have an odd abscissa, or ordinate, respectively, the sum of abscissae or ordinates is nonzero, a contradiction. Thus this tiling does not exist.

Problem 3. (Engel) Prove that a 10×10 board cannot be covered by 25 straight tetrominoes.

Solution. The centroid's location of our tiles was shown in the Fig. 2. Since the abscissae of ordinate-paralell tiles and ordinates of abscissa-parallel tiles are odd, number of such tiles has to be even. Note that this implies that the total number of tiles has to be even, and since 25 is odd, we conclude such tiling is impossible.

Problem 4. (Engel) Prove that an 8×8 chessboard cannot be covered by 15 T-tetrominoes and one square tetromino.

Solution. Using Fig. 2a and 3, we come to a conclusion that the square tetromino doesn't affect the parity of abscissa and ordinate sum, so the sums of coordinates for tetrominoes have to be even. As we already shown in Problem 2, for an odd number of T-tetrominoes, exactly one of these sums is odd. Hence, the tiling is impossible.

Problem 5. (Engel) Consider an $n \times n$ chessboard with the four corners removed. For which values of n can you cover the board with L-tetrominoes?

Solution. Fig. 4a shows us the position of L-tile's centroid. Notice that both of its coordinates are odd, so the total number of tiles has to be even. Our board is consisted of $n^2 - 4$ original squares, and since we have shown the number of tiles is even, dividing the total number of squares with four gives an even quotient, i.e. $8 \mid n^2 - 4$. It is clear that n is even, so for $n = 2m$ we have $2 \mid m^2 - 1$, so m is odd, i.e. $m = 2k + 1$, therefore the necessary condition is $n = 4k + 2$. Construction shows this is also a sufficient condition.

Problem 6. (Prasolov) A centrally symmetric figure consists of n L-tetrominoes and k straight tetrominoes of size 1×4. Prove that n is even.

Solution. It is easy to verify, using Fig. 4a and 4b, that the number of L-tetrominoes has to be even, because an odd number of L-tetrominoes would imply that one of the coordinate sums is also odd, hence nonzero.

Problem 7. (Dobosevych, MR) Consider an $n \times n$ board tiled with T-tetrominos. Let a, b, c, d be the number of tetrominos of types A, B, C, D, respectively. Prove that $4 \mid (a + b - c - d)$.

Solution. Using Fig. 3. and the results from the examples above help us to solve this problem in a fast and elegant manner. First of all, we prove that $4 \mid n$.

It is clear that n is even, and if we assume it is even, but not divisible by 4, then for $n = 4m + 2$, $a + b + c + d = \frac{(4m+2)^2}{4} = (2m + 1)^2$ tiles are needed (which is odd). Generalizing our analysis in Problem 2, it is clear that both $a + b$ and $c + d$ have to be even. Since $a + b + c + d$ in this case is odd, the tiling is not possible. Thus $a + b + c + d$ must be even, i.e. $4 \mid n$.

It follows that $a + b + c + d = \frac{(4m)^2}{4} = 4m^2$, so $4 \mid (a + b + c + d)$. Since both $a + b$ and $c + d$ are even, and their sum is divisible by 4, we conclude that $4 \mid (a + b - c - d)$, as desired.

3 Afterthoughts

These few examples are just the tip of the iceberg, taking into account the numerous possible applications of this technique in tiling problems. These parity arguments used in previous problems generally can be substituted with other divisibility arguments, Diophantine equations, translating the tiling problem into a number theory one. One should also explore the possibilites of the "weighted tiles", i.e. giving different squares of the tile different weight, moving the centroid that way. Of course, the method can be effectively used in higher dimensions.

References

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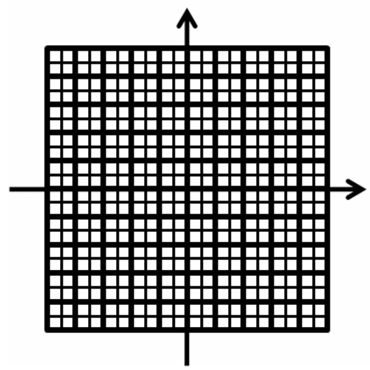
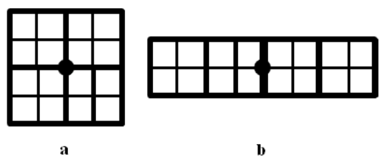


Figure 1:



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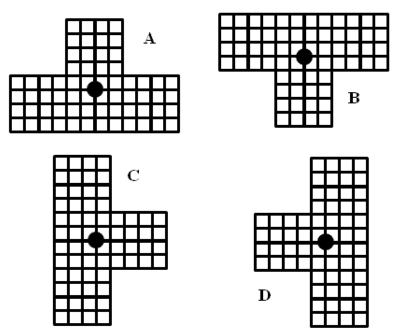

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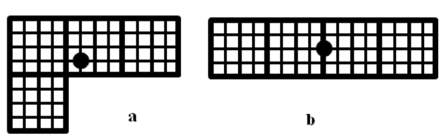

Figure 4: