Junior problems

J193. Let $ABCD$ be a square of center O. The parallel through O to AD intersects AB and CD at M and N and a parallel to AB intersects diagonal AC at P. Prove that

$$OP^4 + \left(\frac{MN}{2}\right)^4 = MP^2 \cdot NP^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Mihai Stoenescu, Bischwiller, France

The parallel to AB intersects MN at T. It is clear that AMO and OTP are right isosceles triangles. So, we can denote $AM = MO = x$ and $OT = TP = t$. Let $MP = y$, $NP = z$, and $OP = u$ for convenience. So we have $TN = x - t$. So we must prove that $u^4 + x^4 = y^2 \cdot z^2$. By Pythagoras, we have

$$\begin{aligned} y^2 \cdot z^2 &= [(x+t)^2 + t^2] \cdot [(x-t)^2 + t^2] = (x^2 + 2t^2 + 2xt) \cdot (x^2 + 2t^2 - 2xt) \\ &= (x^2 + 2t^2)^2 - 4x^2t^2 = x^4 + 4t^4. \end{aligned}$$

From triangle OTP we have $2t^2 = u^2$ then $4t^4 = u^4$ and we are done.

Second solution by Christopher Wiriawan, Jakarta, Indonesia

Denote the intersection between line MN and the parallel to AB as Q and let the perpendicular from P to AB as R.

It is easy to see that $QN = RB$, thus by Pythagorean Theorem we have,

$$OP^4 + \left(\frac{MN}{2}\right)^4 = (OQ^2 + QP^2)^2 + \left(\frac{MN}{2}\right)^2 = \left(\left(\frac{MN}{2} - QN\right)^2 + \left(\frac{MN}{2} - RB^2\right)\right)^2 + \left(\frac{MN}{2}\right)^4$$

which is also equivalent to

$$\left(2\left(\frac{MN}{2} - QN\right)^2\right)^2 + \left(\frac{MN}{2}\right)^4 = 4\left(\frac{MN}{2} - QN\right)^4 + \left(\frac{MN}{2}\right)^4$$

Thus, we want to prove that this last expression equals the RHS. Now, by using Pythagorean Theorem we have,

$$NP^2 = QN^2 + QP^2 = QN^2 + \left(\frac{MN}{2} - QN\right)^2 = 2QN^2 - MN \cdot QN + \left(\frac{MN}{2}\right)^2$$

$$MP^2 = QP^2 + MQ^2 = \left(\frac{MN}{2} - QN\right)^2 + (MN - QN)^2 = 5\left(\frac{MN}{2}\right)^2 - 3MN \cdot QN + 2QN^2$$

Thus,

$$NP^2 \cdot MP^2 = \left(2QN^2 - MN \cdot QN + \left(\frac{MN}{2}\right)^2\right)\left(5\left(\frac{MN}{2}\right)^2 - 3MN \cdot QN + 2QN^2\right)$$

which is equivalent to

$$4QN^4 - 16QN^3\left(\frac{MN}{2}\right) + 24QN^2\left(\frac{MN}{2}\right)^2 - 16QN\left(\frac{MN}{2}\right)^3 + 4\left(\frac{MN}{2}\right)^4 + \left(\frac{MN}{2}\right)^4$$

which is $4\left(\frac{MN}{2} - QN\right)^4 + \left(\frac{MN}{2}\right)^4$, thus proving the desired expression.

Also solved by Arber Selimi, Bedri Pejani - Peje, Kosovo; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Gabriel Alexander Chicas Reyes, El Salvador; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Roberto Bosch Cabrera, Havana, Cuba; Kannappan Sampath, Tiruchirappalli, India.

J194. Let a, b, c be the side-lenghts of a triangle with the largest side c. Prove that

$$\frac{ab\left(2c+a+b\right)}{\left(a+c\right)\left(b+c\right)} \le \frac{a+b+c}{3}.$$

Proposed by Arkady Alt, San Jose, California, USA

First solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Without loss of generality assume that $c \ge a \ge b$. Consider the function

$$f(x) = \frac{a+b+x}{3} - ab\left(\frac{1}{a+x} + \frac{1}{b+x}\right).$$

We have

$$f(c) - f(a) = (c-a)\left(\frac{1}{3} + \frac{ab}{2a(a+c)} + \frac{1}{(b+a)(b+c)}\right) \ge 0 \tag{1}$$

and

$$f(a) = \frac{2a+b}{3} - ab(\frac{1}{2a} + \frac{1}{a+b}) \ge \frac{2a+b}{3} - \frac{b}{2} - \frac{ab}{4}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{5}{12}(a-b) \ge 0.$$

Note that, the given inequality holds for any positive a, b, c with the largest c.

Second solution by Hoang Quoc Viet, University of Auckland, New Zealand

Assume that $c = \max\{a, b, c\}$, hence

$$a(c-b) + b(c-a) \ge 0. \tag{1}$$

The original inequality can be written as

$$\frac{3abc}{a+b+c} \le c^2 + a(c-b) + b(c-a).$$

We have

$$\frac{3abc}{a+b+c} \le c\left(\frac{c^2 + c(a+b)}{a+b+c}\right) = c^2. \tag{2}$$

Combining (1) and (2) the inequality is proved.

Also solved by Arber Selimi, Bedri Pejani - Peje, Kosovo; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Henry Ricardo New York, USA; Mihai Stoenescu, Bischwiller, France; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Roberto Bosch Cabrera, Havana, Cuba; Christopher Wiriawan, Jakarta, Indonesia.

J195. Find all primes p and q such that both $pq - 555p$ and $pq + 555q$ are perfect squares.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Ercole Suppa, Teramo, Italy

Since $pq - 555p = p(q - 555)$ is a perfect square, p divides $q - 555$ and $q > 555$. Therefore there exists an integer $a \geq 1$ such that

$$q - 555 = ap \tag{1}$$

Likewise q divides $p + 555$, so there exists an integer $b \geq 1$ such that

$$p + 555 = bq \tag{2}$$

From (1) and (2) it follows that

$$\begin{aligned} & p + 555 = b(555 + ap) \quad \Rightarrow \\ & (1 - ab)p = 555(b - 1) \geq 0 \quad \Rightarrow \\ & 1 - ab \geq 0 \quad \Rightarrow \quad a = 1,\ b = 1 \end{aligned}$$

Therefore $q - p = 555$, so $p = 2$ (otherwise $q - p$ would be an even number) and $q = 557$.

Second solution by Roberto Bosch Cabrera, Florida, USA

We have $pq - 555p = p(q - 555)$ and $pq + 555q = q(p + 555)$, hence $p \mid q - 555$ and $q \mid p + 555$, so $pq \mid (q - 555)(p + 555) = pq + 555q - 555p - 555^2$. Thus $pq \mid 3 \cdot 5 \cdot 37(p + 555 - q)$. But note that $p + 555 = qk$ hence $p \mid 3 \cdot 5 \cdot 37 \cdot (k - 1)$. If $p = 3 \Rightarrow 3 \mid q - 555 \Rightarrow 3 \mid q \Rightarrow q = 3$ contradiction since $q > 555$. Analogously for $p = 5$ and $p = 37$. Thus we arrive to $p \mid k - 1$, hence $k = ph + 1$. But $q - 555 = pr$ and hence

$$p + 555 = q(ph + 1) \Rightarrow p = pqh + pr \Rightarrow 1 = qh + r \Rightarrow h = 0, r = 1 \Rightarrow k = 1$$

Hence $q = p + 555$, if $p \geq 3$ we obtain that $p + 555$ is even, contradiction, so $p = 2$ and $q = 557$ is the solution to our problem.

Also solved by Arber Selimi, Bedri Pejani - Peje, Kosovo; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Prasanna Ramakrishnan, Trinidad and Tobago; Christopher Wiriawan, Jakarta, Indonesia.

J196. Let I be the incenter of triangle ABC and let A', B', C' be the feet of altitudes from vertices A, B, C. If $IA' = IB' = IC'$, then prove that triangle ABC is equilateral.

Proposed by Dorin Andrica and Liana Topan, Babes-Bolyai University, Romania

Solution by Anthony Erb Lugo

We start by noting that $IA' = IB' = IC'$ implies that I is the circumcenter of triangle $A'B'C'$ whose circumcircle is of course the nine-point circle. Thus, I is the center of the nine-point circle. Furthermore, Feuerbach's Theorem tells us that the nine-point circle is internally tangent to the incircle. Since the nine-point circle and the incircle have the same center and are internally tangent we can conclude that these circles have the same radius. Moreover, if we let r be inradius and R be the circumradius of triangle ABC then the condition that the radius of the nine-point circle, $\frac{R}{2}$, is the same as the radius of the incircle tells us that

$$\frac{R}{2} = r \implies R = 2r$$

which is the equality case of Euler's Triangle Inequality. But Euler's Triangle Inequality, $R \geq 2r$, has equality if and only if triangle ABC is equilateral and so we're done.

Also solved by Arber Selimi, Bedri Pejani - Peje, Kosovo; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Mihai Stoenescu, Bischwiller, France.

J197. Let x, y, z be positive real numbers. Prove that

$$\sqrt{2\left(x^2y^2+y^2z^2+z^2x^2\right)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right)} \geq x\sqrt{\frac{1}{y}+\frac{1}{z}}+y\sqrt{\frac{1}{z}+\frac{1}{x}}+z\sqrt{\frac{1}{x}+\frac{1}{y}}.$$

Proposed by Vazgen Mikayelyan, Yerevan, Armenia

Solution by Arkady Alt, San Jose, California, USA

Note that by the Cauchy-Schwarz Inequality we have that

$$\sum_{cyc} x\sqrt{\frac{1}{y}+\frac{1}{z}} \leq \sqrt{\sum_{cyc} x^2 \sum_{cyc}\left(\frac{1}{y}+\frac{1}{z}\right)} = \sqrt{2\sum_{cyc} x^2 \sum_{cyc}\frac{1}{x}},$$

so it would suffice to prove that

$$\left(x^2y^2+y^2z^2+z^2x^2\right)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right) \geq \left(x^2+y^2+z^2\right)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$$

Now, let $a := \frac{1}{x}, b := \frac{1}{y}, c := \frac{1}{z}$; then the inequality to be proven can be rewritten as

$$\left(\frac{1}{a^2b^2}+\frac{1}{b^2c^2}+\frac{1}{c^2a^2}\right)\left(a^3+b^3+c^3\right) \geq \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)(a+b+c),$$

which is equivalent with

$$\left(a^2+b^2+c^2\right)\left(a^3+b^3+c^3\right) \geq \left(a^2b^2+b^2c^2+c^2a^2\right)(a+b+c),$$

i.e.

$$\sum_{cyc} a^5 + \sum_{cyc} a^3\left(b^2+c^2\right) \geq \sum_{cyc} a^3\left(b^2+c^2\right) + \sum_{cyc} ab^2c^2,$$

which turns out to be just the immediate

$$\sum_{cyc} a^5 \geq \sum_{cyc} ab^2c^2 = abc\,(ab+bc+ca),$$

which can be seen for example as a consequence of the AM-GM Inequality.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Anthony Erb Lugo; Arber Selimi, Bedri Pejani - Peje, Kosovo; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Henry Ricardo New York, USA; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

J198. Find all pairs (x, y) for which $x! + y! + 3$ is a perfect cube.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ivan Borsenco, MIT, USA and Roberto Bosch Cabrera, Florida, USA

Let $x! + y! + 3 = z^3$ with $x, y \geq 7$. We have that $z^3 \equiv 3 \mod 7$, so this gives a contradiction since the cubic residues modulo 7 are only $0, 1, 6$. Thus, we can suppose without loss of generality that $x \leq 6$.

Now let us do the case work:
A) $x = 1$
$y! + 4 = z^3$ and $y \geq 7 \Rightarrow z^3 \equiv 4 \mod 7$; impossible, so $y \leq 6$, but for any value of y we obtain a cube.
B) $x = 2$
$y! + 5 = z^3$ and $y \geq 7 \Rightarrow z^3 \equiv 5 \mod 7$; impossible, so $y \leq 6$, and we obtain the solution $y = 5$.
C) $x = 3$
$y! + 9 = z^3$ and $y \geq 7 \Rightarrow z^3 \equiv 2 \mod 7$; impossible, so $y \leq 6$, and we obtain the solution $y = 6$.
D) $x = 4$
$y! + 27 = z^3$; we postpone this for now.
E) $x = 5$
$y! + 123 = z^3$ and $y \geq 7 \Rightarrow z^3 \equiv 4 \mod 7$; impossible, so $y \leq 6$, and we obtain the solution $y = 2$.
F) $x = 6$
$y! + 723 = z^3$ and $y \geq 7 \Rightarrow z^3 \equiv 2 \mod 7$; impossible, so $y \leq 6$, and we obtain the solution $y = 3$.

Now, let's turn back to D). First, note that the case $y = 1, 2, 3, ..., 8$ can be easily checked manually.

For$y > 8$ we have $81 \mid y!$; so $z = 3z'$ and $y!/27 = z'^3 - 1$. Now, $3 \mid z'^3 - 1$, so $z' \equiv 1 \mod 3$. But if $z' = 3k + 1$; then

$$\frac{y!}{27} = (3k+1)^3 - 1 = 27k^3 + 27k^2 + 9k = 9k(3k^2 + 3k + 1).$$

Thus $y! = 243k(3k^2 + 3k + 1)$. Furthermore, let $f(n)$ to be equal to the power of 3 dividing n and note that $f(n) \leq \log_3 n$, so $f(y!) = [243k(3k^2+3k+1)/3] + [243k(3k^2+3k+1)/9] + [243k(3k^2+3k+1)/27] + \ldots$, where $[\cdot]$ denotes as usual the floor function.

But $f(y!) = f(243k(3k^2 + 3k + 1)) = 5 + f(k) \leq 5 + \log_3 k$, so we can see that for $k \geq 1$, $[243k(3k^2 + 3k + 1)/3] = 81k(3k^2 + k + 1) > 5 + \log_3 k$ and hence the equation has no solution. Finally, we get that the solutions to original equation are $(x, y) = (2, 5); (x, y) = (5, 2); (x, y) = (3, 6); (x, y) = (6, 3)$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Kannappan Sampath, Tiruchirappalli, India.

Senior problems

S193. Find all pairs (x, y) of positive integers such that $x^2 + y^2 = p^6 + q^6 + 1$, for some primes p and q.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy

Suppose that $3 \neq q$ and $3 \neq p$, then 3 is coprime with both p and q (because they are prime). So 9 is coprime with both p and q. Then, being $\varphi(9) = 6$ we have that $p^6 + q^6 + 1 \equiv 3 \pmod 9$ by Euler theorem. But then we should have $x^2 + y^2 \equiv 0 \pmod 3$ which implies that $x \equiv 0 \pmod 3$ and $y \equiv 0 \pmod 3$. Hence $x^2 + y^2 \equiv 0 \pmod 9$ and $3 \equiv 0 \pmod 9$, contradiction. Then we should have that without loss of generality $p = 3$. Now $3^6 + q^6 + 1 \equiv 2 + q^6 \pmod 4$. So if q is odd, then $x^2 + y^2 \equiv 3 \pmod 4$, contradiction. Then $q = 2$. So we have to solve $x^2 + y^2 = 794 = 2 \pmod{397}$. But we have that $397 \equiv 1 \pmod 4$ and it is prime. So we get this factorization in primes of $\mathbb{Z}[i]$:

$$794 = (1 + i)(1 - i)(6 + 19i)(6 - 19i).$$

Therefore the unique solutions are $(25, 13)$ and $(13, 25)$.

Also solved by Arber Selimi, Bedri Pejani - Peje, Kosovo; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Prasanna Ramakrishnan, Trinidad and Tobago; Roberto Bosch Cabrera, Havana, Cuba.

S194. Let p be a prime of the form $4k+3$ and let n be a positive integer. Prove that for each integer m there are integers a and b such that $a^{2^n}+b^{2^n}\equiv m \pmod{p}$.

Proposed by Tigran Hakobyan, Yerevan, Armenia

Solution by Omran Kouba, Damascus, Syria

Let $\mathbb{F}_p=\mathbb{Z}/p\mathbb{Z}$, and let $Q_p=\{x^2 : x\in\mathbb{F}_p\}$ be the set quadratic residues modulo p. It is well-known that $|Q_p|=2k+2=(p+1)/2$. Also, note that the function $\sigma : Q_p\longrightarrow Q_p$ defined by $\sigma(x)=x^2$ is a bijection. Indeed, suppose that $\sigma(x)=\sigma(x')$.

- If $x'\equiv 0 \mod p$ then $p|x^2$, thus $p|x$ and $x\equiv x'(\mod p)$.
- If $x'\not\equiv 0 \mod p$, then $(x-x')(x+x')\equiv 0 \mod p$.

But $x+x'\not\equiv 0 \mod p$ since, otherwise, we would have a quadratic residue x equal to a non-quadratic residue $-x'$, (here we used the fact that -1 is not a quadratic residue modulo p as $p\equiv 3 \mod 4$,) and since $\mathbb{F}_p$ is an integral domain, we conclude that $x-x'\equiv 0 \mod p$. So $x\equiv x' \mod p$ and thus it follows that σ is injective. Furthermore, we get that $\sigma : Q_p\longrightarrow Q_p$ is also surjective since the set Q_p is finite.

Now, in what is about to follow we define $\rho=\sigma^{-1}$, that is, the square root function defined on Q_p. Furthermore, consider an integer m, and the sets $A=Q_p$ and $B=\{(m-x) \mod p : x\in Q_p\}$, which as a matter of fact are two subsets of $\mathbb{F}_p$ of cardinality $|A|=|B|=2k+2>p/2$. Therefore, A and B can not be disjoint and so there must be a common element $\alpha\in A\cap B$. Hence, there exist $(\alpha,\beta)\in Q_p\times Q_p$ such that $m\equiv\alpha+\beta \mod p$. But now we can set $a=\rho^n\alpha$ and $b=\rho^n\beta$, and obtain precisely that $m\equiv a^{2^n}+b^{2^n} \mod p$, which is the desired conclusion.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy.

S195. Let ABC be a triangle with incenter I and circumcenter O and let M be the midpoint of BC. The bisector of angle A intersects lines BC and OM at L and Q, respectively. Prove that

$$AI \cdot LQ = IL \cdot IQ.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Gabriel Alexander Chicas Reyes, El Salvador

It is a well-known that the bisector of $\angle CAB$ and the perpendicular bisector OM of BC intersect on the circumcircle of ABC. In other words, Q is the midpoint of the arc BC not containing A.

Observe that $BQ = IQ$, since by angle chasing $\angle QBI = \angle QIB = \pi/2 - \angle BCA/2$ (in fact Q is the circumcenter of BIC). On the other hand, since the triangles BLQ and ALC are similar we can write $\dfrac{BQ}{LQ} = \dfrac{AC}{LC}$. Now repeated use of the angle bisector theorem in the triangles ABC and ABL yields

$$\frac{IQ}{LQ} = \frac{BQ}{LQ} = \frac{AC}{LC} = \frac{AB}{BL} = \frac{AI}{IL}.$$

Therefore $AI \cdot LQ = IQ \cdot IL$, as desired.

Second solution by Roberto Bosch Cabrera, Florida, USA

Let D the foot of perpendicular from I to BC. We need to prove that $\frac{AI}{IL} = \frac{IQ}{LQ} = \frac{IL}{LQ} + 1$. We have that $\frac{AI}{IL} = \frac{AB}{BL} = \frac{\sin(A/2+C)}{\sin A/2}$ because I is incenter and by Sines Law. Besides $\frac{IL}{LQ} = \frac{DL}{LM} = \frac{BL-BD}{BM-BL} = \frac{BL-(s-b)}{a/2-BL}$ because triangles IDL and LQM are similar. Where $a = BC, b = AC, c = AB$ and s is the semiperimeter. We have that $BL = \frac{c\sin A/2}{\sin(A/2+C)} = \frac{2R\sin C\sin A/2}{\sin(A/2+C)}$ and $s - b = \frac{a+c-b}{2} = R(\sin A + \sin C - \sin B)$, where R is the circumradius. So we need to prove that

$$\frac{\sin(A/2+C)}{\sin A/2} = \frac{\frac{2\sin C\sin A/2}{\sin(A/2+C)} + \sin B - \sin A - \sin C}{\sin A - \frac{2\sin C\sin A/2}{\sin(A/2+C)}} + 1$$

which after several transformations is equivalent to prove

$$\sin B + \sin C = 2\cos A/2 \sin(A/2 + C)$$

but

$$\sin B + \sin C = 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) = 2\cos A/2\cos(90^\circ - A/2 - C) = 2\cos A/2\sin(A/2+C).$$

Also solved by Arber Selimi, Bedri Pejani - Peje, Kosovo; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Leandro Remolina Ardila, Santander, Colombia; Omran Kouba, Damascus, Syria.

S196. Find the least prime that can be written as $\frac{a^3+b^3}{2011}$ for some positive integers a and b.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Since 2011 is prime (it is not divisible by any prime up to 43, and it is less than $47^2 = 2209$), then $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ is the product of two primes. Since $a, b \geq 1$, clearly $a + b > 1$. If $a^2 - ab + b^2 = 1$, then $ab \leq (a - b)^2 + ab = 1$, or $a = b = 1$, and $a^3 + b^3 = 2$ is clearly not a multiple of 2011, hence either $a + b$ is the lowest prime and $a^2 - ab + b^2 = 2011$, or $a + b = 2011$ and $a^2 - ab + b^2$ is the lowest prime. Note however that

$$a^2 - ab + b^2 = \frac{3(a-b)^2 + (a+b)^2}{4} \geq \frac{(a+b)^2}{4},$$

or if $a + b = 2011$, then $a^2 - ab + b^2 > 2011 \cdot 500$, while if $a^2 - ab + b^2 = 2011$, then $a + b \leq \sqrt{4 \cdot 2011}$ and $a + b < 90$ because $2011 < 45^2 = 2025$.

Let us therefore look for values a, b such that $a + b = p$ is prime and $a^2 - ab + b^2 = 2011$. Clearly, $3ab = (a + b)^2 - (a^2 - ab + b^2) = p^2 - 2011$, or $p \geq 47$. Note $\frac{p^2-2011}{3}$ is always an integer since $p^2 \equiv 2011 \equiv 1 \pmod 3$ for primes $p \neq 3$. Now, if $p \equiv \pm 3 \pmod{50}$, then $p^2 \equiv 9 \pmod{100}$, and $p^2 - 2011 \equiv -2 \pmod{100}$, or $ab = \frac{p^2-2011}{3} \equiv 66 \pmod{100}$. Then $(a - b)^2 = p^2 - 4ab \equiv 9 - 64 \equiv 45 \pmod{100}$ cannot be a perfect square, or $p \neq 47, 53$, and $p \geq 59$.

Note however that for (a, b) a permutation of $(10, 49)$, we have

$$\frac{49^3 + 10^3}{2011} = \frac{118649}{2011} = 59,$$

and this is therefore the lowest prime of this form that can be found.

Also solved by Arber Selimi, Bedri Pejani - Peje, Kosovo; Prasanna Ramakrishnan, Trinidad and Tobago; Roberto Bosch Cabrera, Havana, Cuba.

S197. Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence. Prove that for any prime $p \geq 3$, p divides $F_{2p} - F_p$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

The Fibonacci sequence for $n \geq 0$ is $0, 1, 1, 2, 3, 5, \ldots$. Note that for $p = 2$, $F_{2p} - F_p = F_4 - F_2 = 3 - 1 = 2$ is a multiple of 2, and the result is clearly true in this case. Let now p be an odd prime. The general form of the terms in the sequence is well known to be

$$F_n = \frac{\varphi_+{}^n - \varphi_-{}^n}{\sqrt{5}}, \qquad \text{where} \qquad \varphi_+ = \frac{1+\sqrt{5}}{2}, \qquad \varphi_- = \frac{1-\sqrt{5}}{2},$$

are the roots of the characteristic equation $x^2 - x - 1 = 0$. Define now $(G_n)_{n\geq 0}$ as $G_0 = 2$, $G_1 = 1$, and for all $n \geq 0$, $G_{n+2} = G_{n+1} + G_n$. It is easy to show (it is done in the same way as for the Fibonacci sequence), that the general form of the terms in this sequence is $G_n = \varphi_+{}^n + \varphi_-{}^n$, or $F_{2n} = F_n G_n$. It therefore suffices to show that $G_p \equiv 1 \pmod{p}$. Now, by Fermat's little theorem, $2^p \equiv 2 \cdot 2^{p-1} \equiv 2 \pmod{p}$ because p is an odd prime, or

$$G_p = \varphi_+{}^p + \varphi_-{}^p \equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{p}{2k} 5^k \equiv \binom{p}{0} 5^0 \equiv 1 \pmod{p},$$

since for any $k = 1, 2, \ldots, \frac{p-1}{2}$, both $2k, p-2k$ are positive integers less than p, and p does not appear as a factor in $(2k)!$ or $(p-2k)!$, but it does appear as a factor in $p!$. The conclusion follows.

Second solution by Roberto Bosch Cabrera, Havana, Cuba

We have the well-known identity $F_{m+n} = F_{m-1}F_n + F_m F_{n+1}$, so $F_{2p} = F_{p-1}F_p + F_p F_{p+1}$ and hence $F_{2p} - F_p = F_p(F_{p-1} + F_{p+1} - 1) = F_p(L_p - 1)$ where L_p is the $p - th$ Lucas number. We will prove that $p \mid L_p - 1$.

$$\begin{aligned} L_p &= \left(\frac{1+\sqrt{5}}{2}\right)^p + \left(\frac{1-\sqrt{5}}{2}\right)^p \\ &= \frac{\sum_{k=0}^{p} \binom{p}{k} (\sqrt{5})^k + \sum_{k=0}^{p} \binom{p}{k} (-1)^k (\sqrt{5})^k}{2^p} \\ &= \frac{\sum_{k=0}^{p} \binom{p}{k} (\sqrt{5})^k [1 + (-1)^k]}{2^p} \end{aligned}$$

Hence

$$2^p(L_p - 1) = \binom{p}{2} \cdot 5 \cdot 2 + \binom{p}{4} \cdot 5^2 \cdot 2 + \cdots + \binom{p}{p-1} \cdot 5^{\frac{p-1}{2}} \cdot 2 - (2^p - 2)$$

But $\binom{p}{2}, \cdots, \binom{p}{p-1}$ are divisible by p and $2^p - 2$ is divisible by p by Fermat's theorem, thus $p \mid 2^p(L_p - 1)$ and finally $p \mid L_p - 1$.

Remark: There exists n composite such that $n \mid L_n - 1$, this numbers are called Lucas pseudoprimes, the first few of these are $705, 2465, 2737, 3745, 4181, 5777, 6721, \ldots$(Sloane's $A005845$).

S198. Let x, y, z be positive real numbers such that $(x-2)(y-2)(z-2) \geq xyz - 2$. Prove that

$$\frac{x}{\sqrt{x^5+y^3+z}} + \frac{y}{\sqrt{y^5+z^3+x}} + \frac{z}{\sqrt{z^5+x^3+y}} \leq \frac{3}{\sqrt{x+y+z}}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Hoang Quoc Viet, University of Auckland, New Zealand

I shall demonstrate the new method that I temporarily call "Partition". To see how it works, we compare the following evaluations (Holder's inequality is applied for 3 variables in this case)

$$(x^5+y^3+z)\left(\frac{1}{x}+1+z\right)\left(\frac{1}{x}+1+z\right) \geq (x+y+z)^3 \tag{3}$$

and

$$3(x^5+y^3+z)\left(\frac{1}{x^2}+1+z^2\right) \geq (x+y+z)^3 \tag{4}$$

Clearly, by comparing the above estimations, we can recognize that destination is unchanged $(x+y+z)^3$ except for the multiplying factors that really mattter. Which one is better? (3) or (4)

By comparing the multiplying factors, we have

$$3\left(\frac{1}{x^2}+1+z^2\right) \geq \left(\frac{1}{x}+1+z\right)^2$$

which is a direct result of Cauchy- Schwarz inequality for 3 variables.

Hence, (3) is better.

Now, we may use (3) to prove the original inequality. (3) is equivalent to

$$\begin{aligned}\sum_{cyc} \frac{x}{\sqrt{x^5+y^3+z}} &\leq \sum_{cyc} \frac{1+x+xz}{(x+y+z)\sqrt{x+y+z}} \\ &\leq \frac{1}{\sqrt{x+y+z}}\left(\frac{3+xy+yz+zx+x+y+z}{x+y+z}\right)\end{aligned}$$

From the given condition, it implies that

$$2(x+y+z) \geq xy+yz+zx+3$$

Thus, the above inequality is proven completely.

Also solved by Arber Selimi, Bedri Pejani - Peje, Kosovo.

Undergraduate problems

U193. Let n be a positive integer. Find the largest constant $c_n > 0$ such that, for all positive real numbers $x_1, \ldots, x_n$,

$$\frac{1}{x_1^2} + \cdots + \frac{1}{x_n^2} + \frac{1}{(x_1 + \cdots + x_n)^2} \geq c_n \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 + ... + x_n}\right)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania

First solution by N.J. Buitrago A., Universidad Nacional, Colombia

Let n be a positive integer. Find the largest constant $c_n > 0$ such that, for all positive real numbers $x_1, \ldots, x_n$,

$$\frac{1}{x_1^2} + \cdots + \frac{1}{x_n^2} + \frac{1}{(x_1 + \cdots + x_n)^2} \geq c_n \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 + \cdots + x_n}\right)^2.$$

Solution by N.J. Buitrago A., Universidad Nacional, Colombia.

Using the Arithmetic-Quadratic Mean Inequality to the $n+1$ numbers $\frac{1}{x_1}, \ldots, \frac{1}{x_n}, \frac{1}{x_1+\cdots+x_n}$, we find

$$\sqrt{\frac{\frac{1}{x_1^2} + \cdots + \frac{1}{x_n^2} + \frac{1}{(x_1+\cdots+x_n)^2}}{n+1}} \geq \frac{\frac{1}{x_1} + \cdots + \frac{1}{x_n} + \frac{1}{x_1+\cdots+x_n}}{n+1}.$$

Therefore

$$\frac{1}{x_1^2} + \cdots + \frac{1}{x_n^2} + \frac{1}{(x_1 + \cdots + x_n)^2} \geq \frac{1}{n+1} \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 + \cdots + x_n}\right)^2.$$

Then is enough to take $c_n = \frac{1}{n+1}$.

Second solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

By the homogeneity of the inequality we set $x_1 + \ldots + x_n = 1$. If $x_k = 1/n$ for any k we get $\tilde{c}_n = (n^3+1)/(n^2+1)^2$ thus the actual value of c_n cannot be greater than $\tilde{c}_n$.

Moreover power–means-inequality yields if $a_k \geq 0$ $(a_1^2 + \ldots + a_n^2) \geq (a_1 + \ldots + a_n)^2/n$. thus the inequality is implied by $(S = 1/x_1 + \ldots + 1/x_n)$

$$S^2(nc_n - 1) + 2nc_nS + nc_n - n \leq 0, \qquad S \geq n^2$$

The lower bound on S follows by

$$1 = x_1 + \ldots + x_n \geq n\left(\prod_{k=1}^n x_k\right)^{1/n}, \quad S \geq n\left(\prod_{k=1}^n x_k\right)^{-1/n}$$

whence $S \geq n^2$. We observe that $nc_n < 1$ since

$$c_n \leq \tilde{c}_n = \frac{n^3+1}{(n^2+1)^2} < \frac{1}{n}$$

and this follows by $2n^2 + 1 > n$. Thus the parabola $P(S) \doteq S^2(nc_n - 1) + 2nc_nS + nc_n - n$ is concave and we have two cases.

First case. $\Delta = nc_n + n^2c_n - n \le 0$. If $c_n \le 1/(n+1)$ the maximum has ordinate $-\frac{\Delta}{4(nc_n-1)} \ge 0$ evidently

Second case. $\Delta = nc_n + n^2c_n - n > 0$ but the zeroes of $P(S)$ are less than n^2. In such a case $P(S) \le P(n^2) \le 0$ if and only if $c_n \le \tilde{c}_n$. Indeed the zeroes of $P(S)$ are

$$S_{1,2}\frac{-nc_n \pm \sqrt{nc_n + n^2c_n - n}}{nc_n - 1}$$

and we must show that

$$\frac{nc_n + \sqrt{nc_n + n^2c_n - n}}{-nc_n + 1} \le n^2$$

If $c_n \ge 1/(n+1)$ but $c_n \le \tilde{c}_n$ we get clearing the denominators and squaring

$$\begin{aligned} n^4 - 2n^5c_n - 2n^3c_n + n^6c_n^2 + 2n^4c_n^2 + c_n^2n^2 - c_nn - n^2c_n + n = \\ n(c_n(n^2+1)^2 - n^3 - 1)(c_nn - 1) \ge 0 \end{aligned}$$

which clearly holds in the range of c_n since $c_n \le \tilde{c}_n < 1/n$.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Roberto Bosch Cabrera, Florida, USA.

U194. Prove that the set of positive integers n for which n divides $2^{n^2+1}+3^n$ has density 0.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Solution by Gabriel Dospinescu, Ecole Polytechnique, France

Let A_x be the set of those positive integers $n \le x$ such that n divides $2^{n^2+1}+3^n$. Let $n \in A_x$ and observe that n is odd. Moreover, if p is a prime factor of n, then 2^{n^2+1} is a perfect square mod p. We deduce that -3^n is a square mod p and so -3 is a square mod p. Using the quadratic reciprocity law, it follows that $p = 1 \pmod 3$. Thus A_x is included in the set of those $n \le x$ all of whose prime factors are $1 \pmod 3$. An inclusion-exclusion argument combined with the fact that

$$\sum_{p=1 \pmod 3} \frac{1}{p} = \sum_{p=2 \pmod 3} \frac{1}{p} = \infty$$

(as it follows from basic estimates used in the proof of Dirichlet's theorem) easily shows that the last set has density 0. The result follows.

U195. Given a positive integer n, let $f(n)$ be the square of the number of its digits. For example $f(2) = 1$ and $f(123) = 9$. Show that $\sum_{n=1}^{\infty} \frac{1}{nf(n)}$ is convergent.

Proposed by Roberto Bosch Cabrera, Florida, USA

Solution by Jose Hernandez Santiago, Oaxaca, Mexico

Let k be a fixed but arbitrary natural number. We know that, if m_k is the number of digits in the decimal expansion of k, then $10^{m_k-1} \leq k < 10^{m_k}$. It follows that

$$\begin{aligned}
\sum_{n=1}^{k} \frac{1}{nf(n)} &= \sum_{n=1}^{10-1} \frac{1}{nf(n)} + \sum_{n=10}^{10^2-1} \frac{1}{nf(n)} + \ldots + \sum_{n=10^{m_k-1}}^{k} \frac{1}{nf(n)} \\
&= \sum_{n=1}^{10-1} \frac{1}{n \cdot 1^2} + \sum_{n=10}^{10^2-1} \frac{1}{n \cdot 2^2} + \ldots + \sum_{n=10^{m_k-1}}^{k} \frac{1}{n \cdot m_k^2} \\
&\leq \sum_{n=1}^{10-1} \frac{1}{1 \cdot 1^2} + \sum_{n=10}^{10^2-1} \frac{1}{10 \cdot 2^2} + \ldots + \sum_{n=10^{m_k-1}}^{k} \frac{1}{10^{m_k-1} \cdot m_k^2} \\
&< \frac{10}{1 \cdot 1^2} + \frac{10^2}{10 \cdot 2^2} + \ldots + \frac{10^{m_k}}{10^{m_k-1} \cdot m_k^2} \\
&= 10 \times \left(\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{m_k^2} \right) \\
&\leq 10 \cdot \zeta(2)
\end{aligned}$$

and we are done.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Henry Ricardo New York, USA; N.J. Buitrago A., Universidad Nacional, Colombia; Luigino Capone, University of Rome Tor Vergata, Italy; Moubinool Omarjee, Paris , France; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Emanuele Natale, Università di Roma "Tor Vergata", Roma, Italy.

U196. Let $A, B \in M_2(\mathbb{Z})$ be commuting matrices such that for any positive integer n there exists $C \in M_2(\mathbb{Z})$ such that $A^n + B^n = C^n$. Prove that $A^2 = 0$ or $B^2 = 0$ or $AB = 0$.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Solution by Gabriel Dospinescu, Ecole Polytechnique, France

We begin with the following preliminary result:

Lemma. Let $z_1, z_2, ..., z_k$ be complex numbers such that $z_1^n + z_2^n + ... + z_k^n$ is the nth power of an integer for any n. Then at most one of the numbers $z_1, z_2, ..., z_k$ is nonzero.

Proof. We may assume that all z_i's are nonzero and we will prove that $k = 1$. If all z_i are real numbers, this can be proved easily with analytic tools, but the case of complex numbers seems to be more delicate. First, since $z_1^n + ... + z_k^n$ is an integer for all n, the Newton's formulae show that all symmetric sums in the z_i's are rational numbers and so z_i are all algebraic numbers. Pick a number field K containing all z_i's and pick a prime ideal I of K lying above a prime p sufficiently large. We may ensure that I is relatively prime to all z_i's. Let n be the norm of I. Then $x^n = 1 \pmod{I}$ for any $x \in O_K$ prime to I. We deduce that $z_1^n + ... + z_k^n = k \pmod{I}$. On the other hand, we know that $z_1^n + ... + z_k^n = a^n = 1 \pmod{I}$ for some integer a, which is necessarily prime to I (otherwise p would divide k, but we may choose $p > k$ from the very beginning). We deduce that $k = 1 \pmod{I}$ and then $k = 1 \pmod{p}$. As p was arbitrarily large, we obtain $k = 1$ and we are done.

Coming back to the solution of the problem, we recall that a standard argument shows that if A is nonscalar and if B commutes with A, then $B = aA + bI_2$ for some numbers a, b. If both A, B are scalar, the result is immediate from the lemma (we are even in a very easy case of that lemma). So, assume that A is not scalar, let z_1, z_2 be its eigenvalues. Pick a, b such that $B = aA + bI_2$. Thus the eigenvalues of B are $az_1 + b$ and $az_2 + b$. By assumption $\det(A^n + B^n)$ is the nth power of an integer for all n. But

$$\det(A^n + B^n) = (\det A)^n + (\det B)^n + (a \det A + bz_1)^n + (a \det A + bz_2)^n.$$

Using the lemma, we deduce that at most one of the numbers $\det A$, $\det B$, $a \det A + bz_1$ and $a \det A + bz_2$ is nonzero. A small case analysis easily yields the result.

U197. Let $n \geq 2$ be an integer. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x_1, x_2, ..., x_n \in \mathbb{R}$,

$$\sum_{i=1}^{n} f(x_i) - \sum_{1 \leq i < j \leq n} f(x_i + x_j) + \cdots + (-1)^{n-1} f(x_1 + ... + x_n) = 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain;

Since

$$\binom{n}{1} - \binom{n}{2} + \cdots + (-1)^{n-1}\binom{n}{n} = \binom{n}{0} - (1 + (-1))^n = 1,$$

taking $x_1 = \cdots = x_n = 0$ yields $f(0) = 0$. For $n = 2$, the problem becomes the Cauchy functional equation $f(x + y) = f(x) + f(y)$, whose continuous solutions are $f(x) = kx$ for any real constant k. We shall now show by induction that, for all $n \geq 2$, $f(x)$ is necessarily an $n - 1$-degree polynomial in x without a constant term; this is clearly true for the base $n = 2$.

For the step, define first $g(x, y) = f(x+y)-f(x)-f(y)$, and fix y. Taking $x_n = y$, note that substitution in the proposed equation yields

$$\sum_{i=1}^{n-1} g(x_i, y) - \sum_{1 \leq i < j \leq n-1} g(x_i + x_j, y) + \cdots + (-1)^{n-2} g(x_1 + \cdots + x_{n-1}, y) = 0,$$

ie, for a fixed y, $g(x, y)$ is a solution of the problem for $n - 1$, hence by hypothesis of induction, an $n-2$-degree polynomial in x without a constant term, whose coefficients will depend on the value of y. Since $g(x, y)$ must be invariant upon exchange of x, y, $g(x, y)$ is also an $n - 2$-degree polynomial in y without a constant term, whose coefficients will depend on the value of x. We conclude that $g(x, y)$ is a polynomial in x, y, where the degree of each variable is at most $n-2$, and where all terms are multiples of xy.

Define $h(x) = g(x, x) = f(2x) - 2f(x)$, which is clearly a polynomial in x, whose least-degree term is x^2. Therefore,

$$\lim_{x \to 0} \frac{h(x)}{x^2} = \lim_{x \to 0} \frac{f(2x) - 2f(x) + f(0)}{x^2}$$

exists and is finite, ie $f(x)$ is twice differentiable at $x = 0$. Therefore, for all x and $y \to 0$, we find

$$\lim_{y=0} \frac{f(x + y) - f(x)}{y} = \lim_{y \to 0} \frac{f(y)}{y} + \lim_{y \to 0} \frac{g(x, y)}{y},$$

which clearly exists and is finite for all real x, since the first limit in the RHS is clearly $f'(0)$, which exists and is finite since f is twice differentiable at 0, and the second limit is the coefficient of y (which is a function of x) in $g(x, y)$. It follows that, since the coefficients of each term in $g(x, y)$ (considered as a function of y with x as a parameter) are clearly $n - 2$-degree polynomials in x, then $f'(x)$ exists and is an $n - 2$-degree polynomial in x, hence $f(x)$ is an $n - 1$-degree polynomial in x, without a constant term since $f(0) = 0$.

We now show that any $n - 1$-degree polynomial in x without a constant term is indeed a solution of the proposed equation. By linearity in the functional equation, note that it suffices to show that x^k is a solution for all $1 \leq k \leq n - 1$. Now, if $f(x) = x^k$, then

$$\sum_{i=1}^{n} f(x_i) - \sum_{1 \leq i < j \leq n} f(x_i + x_j) + \cdots + (-1)^{n-1} f(x_1 + \cdots + x_n)$$

is the sum of terms of the form $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_p}^{\alpha_p}$, where $p \leq n$, and $\alpha_1, \alpha_2, \dots, \alpha_p$ are positive integers with sum k. For a specific set of indices $i_1, i_2, \dots, i_p$, and specific values of $\alpha_1, \alpha_2, \dots, \alpha_p$, let us calculate the coefficient of this term. Clearly, this term will appear exactly in all terms of the form $(-1)^{u-1} f(s)$, where s is the sum of u of the x_i, such that $x_{i_1}, x_{i_2}, \dots, x_{i_p}$ are part of these u x_i. Clearly $p \leq u \leq n$, and exactly $\binom{n-p}{u-p}$ such terms will exist for each u (as many as possible forms to choose $u - p$ other x_i out of the remaining $n - p$). The coefficient of $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_p}^{\alpha_p}$ in each term will clearly be

$$K = \frac{k!}{(\alpha_1)!(\alpha_2)! \dots (\alpha_p)!},$$

clearly independent on u. The total coefficient will then be

$$\sum_{u=p}^{n} (-1)^{u-1} K \binom{n-p}{u-p} = (-1)^{p-1} K \sum_{v=0}^{n-p} (-1)^v \binom{n-p}{v} = (-1)^{p-1} K (1 + (-1))^{n-p} = 0,$$

where we have made the variable exchange $v = u - p$. It follows that all terms have 0 coefficient, and indeed $f(x) = x^k$ satisfies the proposed equation for all $1 \leq k \leq n - 1$. We conclude that, for each n, $f(x)$ is a solution iff it is an $n - 1$-degree polynomial with no constant term.

Also solved by Omran Kouba, Damascus, Syria.

U198. Define a sequence $(x_n)_n$ by $x_0 = 1$ and $x_{n+1} = 1 + x_n + \frac{1}{x_n}$ for $n \geq 0$. Prove that there is a real number a such that

$$\lim_{n\to\infty} \frac{n}{\log n} \cdot (a + n + \log n - x_n) = 1.$$

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

We will prove the assertion

$$x_n = n + \ln n + 1 - \frac{\ln n}{n} + O(\frac{\ln(n+1)}{n^2})$$

which yields the desired limit. To prove the assertion we proceed by induction For $n = 1$ clearly holds. Let's suppose it holds for $1 \leq n \leq r$. For $n = r + 1$ we need to show that $(q_{n+1} \doteq O(\frac{\ln(n+2)}{(n+1)^2}))$

$$x_{r+1} = r + 1 + \ln(r+1) + 1 - \frac{\ln(r+1)}{r+1} + q_{r+1} =$$
$$1 + \left(r + \ln r - \frac{\ln r}{r} + 1 + q_r\right) + \frac{1}{r + \ln r + 1 - \frac{\ln r}{r} + q_r} = 1 + x_r + \frac{1}{x_r}$$

whence

$$\ln(r+1) - \ln r = \left(\frac{\ln(r+1)}{r+1} - \frac{\ln r}{r}\right) + q_r - q_{r+1} + \frac{1}{r + \ln r + 1 - \frac{\ln r}{r} + q_r} \tag{1}$$

By employing $\ln(r+1) = \ln r + \ln(1 + \frac{1}{r}) = \ln r + \frac{1}{r} - \frac{1}{2r^2} + o(\frac{1}{r^2})$ we get

$$\frac{\ln(r+1)}{r+1} - \frac{\ln r}{r} = \frac{1}{r^2} - \frac{\ln r}{r^2} + o(\frac{1}{r^2})$$

and

$$\frac{1}{r + \ln r + 1 - \frac{\ln r}{r} + q_r} = \frac{1}{r} - \frac{\ln r}{r^2} - \frac{1}{r^2} + o(\frac{1}{r^2})$$

Inserting in (1) we have

$$\frac{1}{r} - \frac{1}{2r^2} + o(\frac{1}{r^2}) = \left(\frac{1}{r^2} - \frac{\ln r}{r^2} + o(\frac{1}{r^2})\right) + q_r - q_{r+1} + \frac{1}{r} - \frac{\ln r}{r^2} - \frac{1}{r^2} + o(\frac{1}{r^2})$$

that is

$$q_r = q_{r+1} + \frac{2\ln r}{r^2} - \frac{1}{2r^2} + o(\frac{1}{r^2})$$

and the right hand side is $O(\frac{\ln(r+1)}{r^2})$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Moubinool Omarjee, Paris , France; Omran Kouba, Damascus, Syria; Adrian Neacsu, Pitesti, Romania.

Olympiad problems

O193. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a+b+\frac{1}{abc}+1}+\frac{1}{b+c+\frac{1}{abc}+1}+\frac{1}{c+a+\frac{1}{abc}+1} \leq \frac{a+b+c}{a+b+c+1}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote $d=\frac{1}{abc}$, $a+b+c+d+1=s$, or the inequality is equivalent to

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}+\frac{1}{s-d} \leq 1,$$

where $abcd=1$, $a+b+c+d=s-1$. After some algebra, the proposed inequality is equivalent to

$$\sigma s(s-2) \geq 3s^2+(abc+bcd+cda+dab)(s-1)-1,$$

where we have defined $\sigma=ab+bc+cd+da+ac+bd$, $q=a^2+b^2+c^2+d^2$, and clearly $(s-1)^2=q+2\sigma$. Now,

$$\frac{\sigma^2-(a^2b^2+b^2c^2+c^2d^2+d^2a^2+a^2c^2+b^2d^2)}{2}=(s-1)(abc+bcd+cda+dab)-1,$$

or the problem is equivalent to showing that

$$2\sigma(s-1)^2+(a^2b^2+b^2c^2+c^2d^2+d^2a^2+a^2c^2+b^2d^2) \geq 6(s-1)^2+\sigma^2+2\sigma+12(s-1)+6.$$

Now, by the inequality between arithmetic and quadratic means,

$$a^2b^2+b^2c^2+c^2d^2+d^2a^2+a^2c^2+b^2d^2 \geq \frac{\sigma^2}{6}, \qquad 3(s-1)^2 \geq 8\sigma,$$

whereas by the AM-GM inequality, $s-1 \geq 4$ and $\sigma \geq 6$, with equality in all cases iff $a=b=c=d=1$. Therefore,

$$2\sigma(s-1)^2-6(s-1)^2 \geq \sigma(s-1)^2 \geq \frac{5\sigma}{16}\cdot\frac{8\sigma}{3}+\frac{\sigma}{8}\cdot 4^2+\frac{s-1}{2}\cdot 6\cdot 4+\frac{1}{16}\cdot 6\cdot 4^2 \geq$$

$$\geq \sigma^2+2\sigma+12(s-1)+6-(a^2b^2+b^2c^2+c^2d^2+d^2a^2+a^2c^2+b^2d^2).$$

The conclusion follows, equality holds iff $a=b=c=1$.

O194. Let A be a set of nonnegative integers, containing 0 and let a_n be the number of solutions of the equation $x_1 + x_2 + \cdots + x_n = n$, with $x_1, \ldots, x_n \in A,\ a_0 = 1$. Find A, if for all $n \geq 0$,

$$\sum_{k=0}^{n} a_k a_{n-k} = \frac{3^{n+1} + (-1)^n}{4}.$$

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

The number of solutions of the equation $x_1 + x_2 + \cdots + x_n = n$ with $x_1, \ldots, x_n \in 0, 1, 2$ is given by

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{k}$$

because there are $\binom{n}{k}$ ways to assign k times the value 0 and $\binom{n-k}{k}$ ways to assign k times the value 2 to the remaining $n-k$ components. Moreover

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{k} = \sum_{k=0}^{n} \binom{2k}{k} \binom{n}{2k} = [x^n] \frac{1}{1-x} \cdot \frac{1}{\sqrt{1 - \frac{4x^2}{1-x^2}}} = [x^n] \frac{1}{\sqrt{1-2x-3x^2}}$$

and

$$\sum_{k=0}^{n} a_k a_{n-k} = [x^n] \frac{1}{(\sqrt{1-2x-3x^2})^2} = [x^n] \frac{1}{1-2x-3x^2} = \frac{3^{n+1} + (-1)^n}{4}.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Emanuele Tron, Marie Curie High School, Pinerolo, Italy; Omran Kouba, Damascus, Syria.

O195. Let O, I, H be the circumcenter, incenter, and orthocenter of a triangle ABC, and let D be an interior point to triangle ABC such that $BC \cdot DA = CA \cdot DB = AB \cdot DC$. Prove that A, B, D, O, I, H are concyclic if and only if $\angle C = 60°$.

Proposed by Titu Andreescu, Dorin Andrica, and Catalin Barbu

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Claim: Denote by U, V, W the projections of D onto BC, CA, AB. Then, UVW is equilateral, and $\angle ADB = \angle C + 60°$.

Proof: Since $\angle AWD = \angle AVD = 90°$, $AVDW$ is cyclic with diameter DA, or $VW = AD \sin \angle A = \frac{BC \cdot AD}{2R}$, and by cyclic permutation of A, B, C, this quantity equals UV, WU. Moreover, $\angle ADW = \angle AVW = 180° - \angle A - \angle AWV$, and

$$\angle ADB = \angle ADW + \angle BDW = 360° - \angle A - \angle B - \angle AWV - \angle BWU = \angle C + 60°.$$

Incidentally, D is called the first isodynamic point, and it is inside ABC iff no angle of ABC exceeds 120°.

It is well known that $\angle AIB = 90° + \frac{1}{2}\angle C$ and $\angle AHB = 180° - \angle C$; if ABC is obtuse at C, then O, C are on opposite sides of AB and $\angle AOB = 360° - 2\angle C$, while if ABC is not obtuse at C, then O, C are on the same side of AB and $\angle AOB = 2\angle C$. Note therefore that A, B, O, I can only be concyclic iff ABC is acute at C, since otherwise we would need $\angle AOB + \angle AIB = 180°$, or equivalently $270° = \frac{3}{2}\angle C$, ie $\angle C = 180°$, absurd since ABC would be degenerate and O, I could not be defined. We may thus assume that ABC is acute at C.

But if ABC is acute at C, then A, B, and any two of D, O, I, H are concyclic iff the corresponding pair from the following four angles are equal:

$$\angle AIB = 90° + \frac{1}{2}\angle C, \qquad \angle AHB = 180° - \angle C,$$

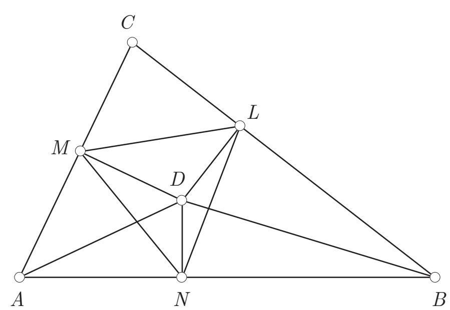
$$\angle AOB = 2\angle C, \qquad \angle ADB = \angle C + 60°,$$

ie iff $\angle C = 60°$. The conclusion follows.

Second solution by Henry Ricardo New York, USA

First of all, observe that D lies on the A-Apollonian circle of $\triangle ABC$ since $\frac{DB}{DC} = \frac{AB}{AC}$. Likewise D lies on the B-Apollonian circle. Hence D is the first point isodynamic, being interior to $\triangle ABC$.

Let $\triangle LMN$ be the pedal triangle of D and let R be the circumradius of $\triangle ABC$ (see figure).



From cyclic quadrilaterals $ANDM$, $BLDN$ we have

$$MN = DA \cdot \sin A, \quad NL = DB \cdot \sin B \quad \Rightarrow$$

$$\frac{MN}{NL} = \frac{DA}{DB} \cdot \frac{\sin A}{\sin B} = \frac{CA}{CB} \cdot \frac{\sin A}{\sin B} = \frac{2R \sin B}{2R \sin A} \cdot \frac{\sin A}{\sin B} = 1$$

Therefore $MN = NL$ and similarly we obtain $NL = LM$, so $\triangle LMN$ is an equilateral triangle. Thus we have

$$\begin{aligned}
\angle ADB = 360^\circ - \angle MDA - \angle LDM - \angle BDL = \\
&= 360^\circ - \angle MNA - (180^\circ - C) - \angle BNL = \\
&= 180^\circ + C - (180^\circ - \angle MNL) = \\
&= 180^\circ + C - 120^\circ = 60^\circ + C
\end{aligned}$$

Now it is easy to prove the result.

If A, B, D, O, I, H are concyclic then

$$\angle AIB = \angle ADB \quad \Rightarrow \quad 90^\circ + \frac{C}{2} = 60^\circ + C \quad \Rightarrow \quad C = 60^\circ$$

Conversely if $C = 60^\circ$ we have

$$\begin{aligned}
\angle ADB &= 60^\circ + C = 120^\circ \\
\angle AOB &= 2C = 120^\circ \\
\angle AIB &= 90^\circ + \frac{C}{2} = 120^\circ \\
\angle AHB &= A + B = 180^\circ - C = 120^\circ
\end{aligned}$$

so A, B, D, O, I, H are concyclic, and we are done.

O196. Let ABC be a triangle such that $\angle ABC > \angle ACB$ and let P be an exterior point in its plane such that

$$\frac{PB}{PC} = \frac{AB}{AC}$$

Prove that

$$\angle ACB + \angle APB + \angle APC = \angle ABC.$$

Proposed by Mircea Becheanu, Bucharest, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note that the relation $\frac{PB}{PC} = \frac{AB}{AC}$ clearly defines an Apollonius circle γ with center on line BC, passes through A and through the point D where the internal bisector of angle A intersects BC, leaving B inside γ and C outside γ because $\angle ABC > \angle ACB$. Note that the powers of B, C with respect to γ are respectively p_B, p_C, such that

$$\frac{p_B}{p_C} = \frac{BD \cdot BD'}{CD \cdot CD'} = \frac{BA^2}{CA^2} = \frac{c^2}{b^2},$$

where D' is the point diametrally opposite D in γ. Let now T, U be the second points where PB, PC meet γ (the first one being clearly P in both cases). Note therefore that

$$\frac{CT}{CU} = \frac{b \cdot BT}{c \cdot CU} = \frac{b \cdot p_B}{c \cdot CU \cdot PB} = \frac{b^2 \cdot p_B}{c^2 \cdot CU \cdot PC} = \frac{b^2 \cdot p_B}{c^2 \cdot p_C} = 1,$$

or $CT = CU$, and similarly $BT = BU$, ie BC is the perpendicular bisector of TU, which are therefore symmetric with respect to BC. Therefore, if P is on the same half plane as A,

$$\angle APB = \angle APT = 180^\circ - \angle ADT = 180^\circ - \angle ADB - \angle BDT =$$

$$= 180^\circ - \angle ADB - \angle BDU = 180^\circ - 2\angle ADB - \angle ADU = 180^\circ - 2\angle ADB - \angle APU =$$

$$= 180^\circ - 2\angle ADB - \angle APC,$$

and similarly we find the same result if P is on the opposite half plane. In either case, we find

$$\angle APB + \angle APC = 180^\circ - 2\angle ADB = 180^\circ - 2\left(180^\circ - B - \frac{A}{2}\right) = 2B + A - 180^\circ = B - C.$$

The conclusion follows.

O197. Let x, y, z be integers such that $3xyz$ is a perfect cube. Prove that $(x+y+z)^3$ is a sum of four cubes of nonzero integers.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

If $xyz, x+y-z, y+z-x, z+x-y \neq 0$, note that

$$(x+y+z)^3 = (x+y-z)^3 + (y+z-x)^3 + (z+x-y)^3 + 24xyz,$$

where the first three terms in the RHS are clearly nonzero cubes, and $24xyz = 2^3(3xyz)$ is also a nonzero cube. This is however not a solution when at least one of $xyz, x+y-z, y+z-x, z+x-y$ is zero. In that case, we can use that $7^3 = 343 = 216+125+1+1 = 6^3+5^3+1^3+1^3$, or if $x+y+z = s \neq 0$, then

$$(x+y+z)^3 = s^3 = (7s)^3 + (-6s)^3 + (-5s)^3 + (-s)^3$$

is the sum of four nonzero cubes. Finally, if $x+y+z=0$. Then for any $a, b \neq 0$,

$$(x+y+z)^3 = 0 = a^3 + b^3 + (-a)^3 + (-b)^3$$

is the sum of four nonzero cubes. Note therefore that for any three integers x, y, z, $(x+y+z)^3$ may be written as the sum of four nonzero cubes, regardless of whether $3xyz$ is a perfect cube or not.

Second solution by Roberto Bosch Cabrera, Florida, USA.

Let $3xyz = w^3$. We have the identity

$$(x+y+z)^3 = (x+y-z)^3 + (y+z-x)^3 + (z+x-y)^3 + (2w)^3$$

which can be proved expanding the cubes in both sides. Now if $x+y-z=0$ then $(x+y+z)^3 = 8z^3$ but $1 = (-1)^3 + (7)^3 + (-5)^3 + (-6)^3$ and hence $8z^3 = (-2z)^3 + (14z)^3 + (-10z)^3 + (-12z)^3$, analogously if $y+z-x=0$ or $z+x-y=0$.

O198. Let a, b, c be positive real numbers such that

$$(a^2+1)(b^2+1)(c^2+1)\left(\frac{1}{a^2b^2c^2}+1\right)=2011.$$

Find the greatest possible value of $\max(a(b+c), b(c+a), c(a+b))$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and
Gabriel Dospinescu, Ecole Polytechnique, France

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Assume that when $a(b+c)$ reaches its maximum possible value (which by symmetry in the variables will also be the maximum that we are looking for), the values of $p = abc$ and $s = b + c$ are known. It remains thus to find the maximum value of a under these two constraints; it would seem at first sight that bc would need to be minimized, since $p = abc$ is known, and a needs to be maximized; let us see that it is otherwise. Note first that

$$(b^2+1)(c^2+1) = s^2 + b^2c^2 - 2bc + 1 = s^2 + (bc-1)^2,$$

$$a^2+1 = \frac{2011}{s^2+(bc-1)^2}\cdot\frac{p^2}{p^2+1}.$$

Thus the maximum of the RHS, hence of a^2 and of a, is obtained when $bc = 1$, since p, s are known and fixed. We may thus perform substitution $bc = 1$, and the relation given in the problem statement becomes

$$(a^2+1)^2s^2 = 2011a^2.$$

Under this constraint, we need to maximize as, or equivalently, $a^2s^2 = x$. Now, the previous relation between a and s may be rewritten as

$$x^2 - (2011-2s^2)x + s^4 = 0,$$

wherefrom the largest of both (clearly positive real) roots is

$$x = \frac{2011-2s^2+\sqrt{(2011-2s^2)^2-4s^4}}{2} = \frac{2011-2s^2+\sqrt{2011}\sqrt{2011-4s^2}}{2}.$$

It finally follows that, since $bc = 1$, and s needs to be minimized so that $x = a^2s^2$ is maximized, and $s^2 \geq 4bc = 4$ by the AM-GM inequality, then the maximum is achieved when $b = c = 1$, and the square of this maximum is

$$x = \frac{2003+\sqrt{2011}\sqrt{1995}}{2},$$

and finally,

$$\max\{a(b+c), b(c+a), c(a+b)\} = \sqrt{\frac{2003+\sqrt{2011}\sqrt{1995}}{2}} = \frac{\sqrt{2011}+\sqrt{1995}}{2}$$

with equality iff two of a, b, c are 1, and the other one is $\frac{\sqrt{2011}+\sqrt{1995}}{4}$. These values can be easily shown to satisfy the relation given in the problem statement.