On a Turan's graph Theorem generalization using equivalence relations

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Abstract

In this paper, we prove the fundamental theorem of equivalence relations and use it to prove a generalization of Turan's Theorem.

1 Introdution:

Consider a binary relation $\sim$ defined on a non-empty set S. The relation $\sim$ is an equivalence relation if and only if three conditions are satisfied:

- **Reflexivity:** $x \sim x$, for all $x \in S$
- **Simetry:** $x \sim y$, where $x, y \in S$, implies $y \sim x$
- **Transitivity:** $x \sim y, y \sim z$, where $x, y, z \in S$, implies $x \sim z$

The equivalence class of an element $x \in S$ is the subset of all $y \in S$ such that y is related to x. We will write this as:

$$[x] = \{y \in S \mid y \sim x\} = \{y \in S \mid x \sim y\}$$

The following theorem shows that there is a partition of S for every equivalence relation defined on it.

Theorem 1 (Fundamental Theorem of Equivalence Relations):

1. Given a non-empty set S and an equivalence relation $\sim$ on it, it is possible to partition $S = S_1 \cup S_2 \cup \ldots \cup S_m$ such that any two elements in the same S_i are related and any two elements in different subsets S_i are not related.

2. A partition $S = S_1 \cup S_2 \cup \ldots \cup S_m$ defines an equivalence relation $\sim$ on S.

Proof: The second part is the easier to verify. In fact, it is enough to consider a relation $\sim$ such that two elements $x, y \in S$ are related $(x \sim y)$ if and only if they belong to the same S_i. The relation is obviously reflexive and symetric. Moreover, by the definition of $\sim$: $x \sim y$ and $x \in S_j$ implies that $y \in S_j$. To show transitivity, consider $x, y, z \in S$ such that $x \sim y, y \sim z$ and say $x \in S_i$, then we get $y \in S_i$ and $z \in S_i$, again by definition of $\sim$ we have $x \sim z$.

We will prove the first part using mathematical induction on $|S|$. The base case $|S| = 1$ is trivial. In order to prove the induction step, we will consider the statement 1 of the above theorem true for all S such that $|S| \leq n \;\; (n \geq 1)$ and for any equivalence relation $\sim$ defined on S.

Now, let's prove the result for some equivalence relation $\sim$ on S, such that $|S| = n+1$. Take any element a of S and consider $[a] = \{x \in S | x \sim a\} = \{x \in S | a \sim x\}$

Claim 1: Any two elements $b, c \in [a]$ are related.

In fact, if $a \in \{b, c\}$, by the definition of $[a]$ we get $b \sim c$. If $a \notin \{b, c\}$, then we have: $b \sim a$ and $a \sim c$, implying that $b \sim c$ because $\sim$ is transitive.

Claim 2: There are no $b, c \in S$ such that $b \notin [a], c \in [a]$ and $c \sim b$.

From $b \notin [a]$ we have that $a \sim b$ is false. Now, suppose there are $b, c \in S$ such that $b \notin [a], c \in [a]$ and $c \sim b$, thus $a \sim c$ and $c \sim b$ imply that: $a \sim b$ because of the transitivity of $\sim$ ($\Rightarrow\Leftarrow$). These claims are equivalent to saying that:

$$b, c \in [a] \Rightarrow [b] = [c] = [a]; b \notin [a], c \in [a] \Rightarrow [b] \cap [c] = \emptyset. \tag{1}$$

Notice that $[a] \neq \emptyset$ $(a \in [a])$ implies that $|S \backslash [a]| \prec |S|$. In consequence, we can apply the induction hypothesis to the set $S \backslash [a]$ and relation $\sim$, concluding that there is a partition $S \backslash [a] = S_1 \cup S_2 \cup \ldots \cup S_t$ for which every two elements in the same S_i are related and every two elements in different S_i's are not. This result and the result from (1), yield: $S = (S \backslash [a]) \cup [a] = S_1 \cup S_2 \cup \ldots \cup S_t \cup [a]$ is a partition as desired in the first part of the theorem.

2 Turan's generalized Theorem

Definition 1: A k-partite graph is a graph whose vertices can be partitioned into k disjoint sets so that no two vertices within the same set are adjacent.

Definition 2: A complete k-partite graph is a k-partite graph for which every pair of vertices not belonging to the same set are joined by an edge.

Now, consider a graph with n vertices which is partitioned into k sets such that the sizes of any two sets differ by 0 or 1. The resulting graph is called Turan's graph $T(n, k)$. Denote by $W(n, k, r)$ $(2 \leq r \leq k)$ the set K_r in the Turan's graph. In particular $W(n, k, 2) = E(n, k)$ is the set of edges in the Turan's graph.

Theorem 2 (Turan's Graph Theorem): Let $G = (V, E)$ a graph with n vertices without any K_{k+1}. Then

$$|E| \leq |E(n, k)|.$$

Turan's Graph Theorem is a widely known result in extremal graph theory with several proofs. See Reference 1 for an example.

There is a proof of a slighty stronger theorem using Theorem 1.

Theorem 3: Let $G = (V, E)$ a graph with n vertices without any K_{k+1} and the set of K_r $(1 \leq r \leq k)$ within G is W. Then

$$|W| \leq |W(n, k, r)|.$$

Proof: First, define a *lonely triangle* as a graph with 3 vertices and only 1 edge. Let M be the maximum number of K_r's for a n-vertex graph without K_{k+1}. From the graphs with no K_{k+1} and with exactly M K_rs, take the one $G = (V, E)$ with the minimum number of lonely triangles within G, call T this minimum. In addition, for every $v \in V$ define $r(v)$ as the number of K_r's with the condition $v \in K_r$ and $t(v)$ as the number of lonely triangles containing v.

Lemma: For any two vertices not joined by an edge $a, b \in V$ the relation $r(a) = r(b)$ is true.

Proof: Assume without the loss of generality that $r(a) \succ r(b)$. Now construct another graph G_1 by replacing b with b', don't join ab' and add the edge vb' if and only if va is an edge in G (b' is a copy of a). Note that there is no K_t including both a and b', therefore there is no K_{k+1} in G_1. Now, notice that because b' is a copy of a, the new number of K_r will be $M + r(b') - r(b) = M + r(a) - r(b) \succ M$ $(\Rightarrow\Leftarrow)$

Now define a binary relation $\sim$ on V in such a way that two vertices are related if and only if they are not joined by an edge. The relation $\sim$ is obviously reflexive and symmetric. We will show that $\sim$ is transitive.

Hypothesis: Suppose it's not. Then, there are $a, b, c \in V$ such that ab is an edge but bc and ca are not (abc is a lonely triangle). By the lemma we have that $r(b) = r(c)$ and $r(c) = r(a)$ therefore

$$r(b) = r(c) = r(a). \tag{2}$$

Let $r(xy)$ be the number of K_r such that $x, y \in K_r$. We will prove that $r(ab) = 0$. First, note that the number of K_r which includes a or b is $r(a) + r(b) - r(ab)$ and that any K_r including c includes neither a nor b. We will consider another graph G_2 that results from G after removing a, b and replacing them by a', b' in such a way that both are copies of c. Furthermore we we will not consider the edges

ab, bc, ca and we will keep unchanged to the configuration of the rest of the graph. We will count the number of K_rs in G_2 including a' or b'. In graph G_1, there are exactly $r(c)$ K_rs including c but neither a nor b. Because a', b' are copies of c, the number of K_r including b' but neither a' nor c is $r(c)$. Similarly for a'. Moreover, there is no K_r including both a' and b'. Therefore, using (2), we conclude that the number of K_r in the new graph is $M + [r(c) + r(c) - 0] - [r(a) + r(b) - r(ab)] = M + [2r(c)] - [2r(c) - r(ab)] = M + r(ab) \geq M$. Because M is maximal, the new graph G_2 cannot have more than M K_rs, therefore it has exactly M; this equality is reached when $r(ab) = 0$.

So far we have: $r(a) = r(b) = r(c)$ and $r(ab) = 0 = r(bc) = r(ca)$

We will construct another graph Ψ_1; first erase vertices b and c and replace them by b' and c'. For any vertex $v \in V$ different from a, add the edge vb' iff va is an edge in G. Similarly for c' (b' and c' are copies of a). Don't consider edges ab, bc, ca and keep unchanged the configuration of the rest of the graph. Call this graph Ψ_1. In a similar way as before we can show that the number of K_r is still M. We will now count the number of lonely triangles that contain at least one vertex from $\{a, b, c\}$ in G_1 and $\{a, b', c'\}$ in Ψ_1 (Note that the only dissapearings or appearings of lonely triangles occur when they contain at least one vertex from $\{a, b, c\}$ or $\{a, b', c'\}$). In G_1, we can calculate the number of lonely triangles including at least one vertex from $\{a, b', c'\}$ using the Inclusion-Exclusion principle, this number is $t(a) + t(b) + t(c) - t(ab) - t(bc) - t(ca) + 1$.

In the new graph, we'll calculate for parts. First, we'll find the number of lonely triangles with exactly one vertex from $\{a, b', c'\}$. In G_1, the number of graphs including a but neither b nor c is $t(a) - t(ab) - t(ac) + 1$. In the new graph, the number of graphs including a but neither b' nor c' would also be $t(a) - t(ab) - t(ac) + 1$ and because b', c' are copies of a, the number we're looking for is $3(t(a) - t(ab) - t(ac) + 1)$.

Second, we'll find the number of lonely triangles with exactly two vertices from $\{a, b', c'\}$. It's zero! This is because a vertex v different from a, b', c' will be either joined with all 3 vertices a, b', c' or will be joined with none of them. Therefore, any triangle with exactly 2 vertices from $\{a, b', c'\}$ will have either 2 or 0 edges. Third, the number of lonely triangles with all three vertices $\{a, b', c'\}$ is obviously 0.

Summarizing, in the new graph Ψ_1 the number of lonely triangles with at least one vertex among $\{a, b', c'\}$ is $3(t(a) - t(ab) - t(ac) + 1) + 0 + 0 = 3(t(a) - t(ab) - t(ac) + 1)$.
Because of the minimality of the number of lonely triangles in G_1, we get: $t(a) + t(b) + t(c) - t(ab) - t(bc) - t(ca) + 1 \leq 3(t(a) - t(ab) - t(ac) + 1)$. Or equivalently:

$$t(b) + t(c) + 2t(ab) + 2t(ac) \leq 2t(a) + t(bc) + 2 \tag{3}$$

Similarly we can get similar inequalities by constructing Ψ_2, Ψ_3, which result from erasing a, c and copying twice b and from erasing a, b and copying twice c, respectively.

$$t(c) + t(a) + 2t(bc) + 2t(ba) \leq 2t(b) + t(ca) + 2 \tag{4}$$

$$t(a) + t(b) + 2t(ca) + 2t(cb) \leq 2t(c) + t(ab) + 2 \tag{5}$$

Summing $(3), (4), (5)$, we get $6 \geq 3(t(ab) + t(bc) + t(ca)) \geq 3(1 + 1 + 1) = 9$ ($t(ab), t(bc), t(ca) \geq 1$ because abc is a lonely triangle). $(\Rightarrow\Leftarrow)$ This contradiction shows that there can't be any lonely triangle within G. Thus, the relation $\sim$ is transitive.

Therefore, the relation $\sim$ is an equivalence relation. After applying theorem 1, we conclude that the set V can be partitioned as $V = V_1 \cup V_2 \cup \ldots \cup V_t$ in such a way that any two vertices are related (not joined by an edge) iff they belong to the same V_j. In other words, G is a complete t-partite graph for some t. Note that $t \leq k$, because the graph does not contain any K_{k+1}. Thus, we'd better consider $V = V_1 \cup V_2 \cup \ldots \cup V_k$, with possibly some $V_i = \emptyset$. To show that G is the Turan's graph $T(n, k)$, we only need to show that the cardinal of the sets V_j ($|V_j| = a_j$) differ by at most 1. Suppose not, w.l.o.g assume $a_1 \geq a_2 + 2$. It is easy to see that the number of K_r in k-partite graph G is:

$$K(G) = \sum a_{i_1} \cdot a_{i_2} \ldots a_{i_r}$$

where the sum is taken from all subsets $\{i_1, i_2, \ldots i_r\} \subset \{1, 2, \ldots, k\}$. Finally, note that by changing one vertex from V_1 to V_2, we get the new sets V_1' and V_2' with $|V_1'| = a_1' = |V_1| - 1 = a_1 - 1$ and $|V_2'| = a_2' = |V_2| + 1 = a_2 + 1$. The number of K_r would be

$$K(G') = \sum a_{i_1}' \cdot a_{i_2}' \cdot a_{i_3} \ldots a_{i_r}$$

But, notice: $K(G') - K(G) = a_1' \cdot \sum a_{\rho_1} \cdot a_{\rho_2} \ldots a_{\rho_{r-1}} + a_2' \cdot \sum a_{\rho_1} \cdot a_{\rho_2} \ldots a_{\rho_{r-1}} + a_1' \cdot a_2' \cdot \sum a_{\pi_1} a_{\pi_2} \ldots a_{\pi_{r-2}} - a_1 \cdot \sum a_{\rho_1} \cdot a_{\rho_2} \ldots a_{\rho_{r-1}} - a_2 \cdot \sum a_{\rho_1} \cdot a_{\rho_2} \ldots a_{\rho_{r-1}} - a_1 \cdot a_2 \cdot \sum a_{\pi_1} a_{\pi_2} \ldots a_{\pi_{r-2}}$

where the sums are taken from all subsets $\{\rho_1, \rho_2 \ldots \rho_{r-1}\} \subset \{3, 4, \ldots, k\}$ and $\{\pi_1, \pi_2, \ldots \pi_{r-2}\} \subset \{3, 4, \ldots, k\}$.

Remember that $a_1' = a_1 - 1$ and $a_2' = a_2 + 1$, then: $a_1' + a_2' - a_1 - a_2 = 0$ and $a_1' \cdot a_2' - a_1 \cdot a_2 = a_1 - a_2 - 1$. Replacing this into the above expression: $K(G') - K(G) = (a_1 - a_2 - 1) \sum a_{\pi_1} a_{\pi_2} \ldots a_{\pi_{r-2}} \succ 0$ $(\Rightarrow\Leftarrow)$

Therefore, the maximum is reached in a complete k-partit graph where any two parts differ by at most 1. This is the Turan's graph: $M = W(n, k, r)$ ■.

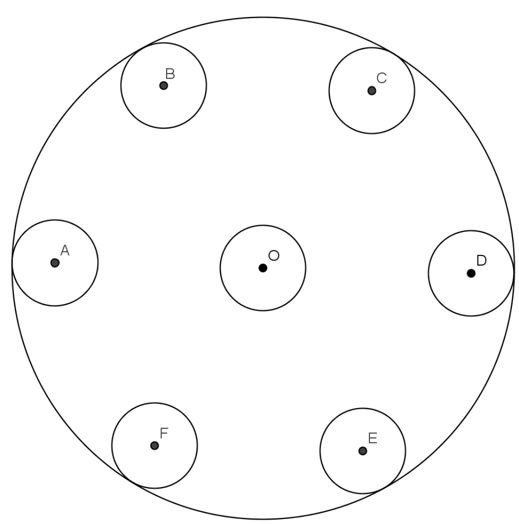
Remark: Note that the expression $\sum a_{\pi_1} a_{\pi_2} \ldots a_{\pi_{r-2}}$ will be positive if there are at least some $a_{\pi_1}, a_{\pi_2}, \ldots, a_{\pi_{r-2}} \succ 0$, but if there aren't even $r-2$ parts greater than 0, when adding the parts V_1, V_2 there isn't any K_r in the graph. Note finally that if the maximum number of K_r in the graph is 0, then the graph has w vertices, where $w \leq r-1$, and even in that case the maximum occurs for the Turan's graph with w parts of size 1 and $k-w$ parts of size 0.

We will now use Theorem 3 to tackle a tough problem.

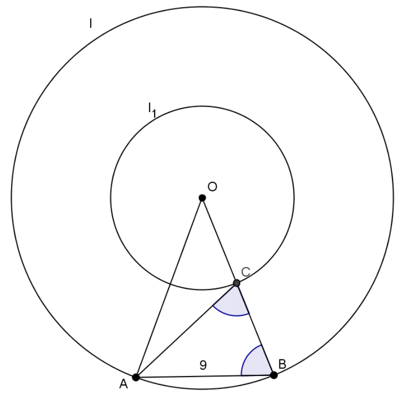
Problem: There are 63 points arbitrarily on the circle C with its diameter being 20. Let S denote the number of triangles whose vertices are three of the 63 points and the lenght of its sides is not less than 9. Find the maximum of S.

China TST 2007, problem 3

Solution: It is natural to think of a graph where the points are the vertices and an edge joins two vertices if the distance between them is at least 9. If we showed that there is no 8-clique, by using Theorem 3, we would conclude that the maximum number of 3-cliques is $S = |W(63,7,3)| = \binom{7}{3} \cdot 9^3 = 25515$, and the maximum of S is reached for Turan's graph $T(63,7)$. In order to give an example, let us call O to the center of C. Construct a regular hexagon $ABCDEF$ with sidelenght $9+\frac{2}{3}$ with O as its center of simetry. Consider the circles of radius $\frac{1}{3}$ with centers in the 7 points. This configuration has the property that the distance between any two points on different circles is at least 9. Hence, any configuration of 63 points (with 9 points on each circle) with no 3 points in the same line reaches the maximum of S.



Now, suppose there is a 8-clique, call it H. Consider a chord AB on Γ of lenght 9, a point C on OB such that $AC = AB$ and a circle Γ_1 with center O and radius OC. It is easy to note: $AB = 2OB\cos\angle OBA$; $BC = 2AB\cos\angle CBA = 2AB\cos\angle OBA$. Then:



$$\cos\angle OBA = \frac{AB}{2OB} = \frac{BC}{2AB} \Rightarrow BC = \frac{AB^2}{OB} = \frac{9^2}{10} = \frac{81}{10} \tag{6}$$

$$OC = OB - BC = 10 - \frac{81}{10} = \frac{19}{10} \tag{7}$$

Equation (7) shows that there cannot be more than 1 point in the circle Γ, because if there were 2, the distance between them would be no more than the lenght of the diameter of Γ_1, which is $\frac{19}{10}\cdot 2 \prec 9$. Contradiction.
Now, suppose there are 7 points on $\Gamma\backslash\Gamma_1$ that belong to H, let us say A and B are two of them. Define $\vec{OA}\cap\Gamma_1 = A_1; \vec{OA}\cap\Gamma = A_2$ and similarly B_1, B_2.

Lemma:$AB \le A_2B_2$.

Proof: Call A_3 and B_3 to the foots of the perpendiculars from A to $\vec{OB}$ and from B to $\vec{OA}$ respectively, then at least one of the following is true: $A_3 \in OB; B_3 \in OA$, without loss of generality let us assume $B_3 \in OA$, then $BA \le BA_2$. *Case 1:* If the foot of the perpendicular from A_2 to OB_2 lies on OB, we have: $A_2B \le A_2B_2$, which leads to $BA \le A_2B \le A_2B_2$. *Case 2:* If the foot of the perpendicular from A_2 to OB_2 lies on BB_2, we have $A_2B \le A_2B_1$ and thus $BA \le A_2B \le A_2B_1$. It will be enough to show that $A_2B_1 \le A_2B_2$. Due to the cosines law and using equation (6), we have:

$$\begin{aligned} A_2B_1^2 &= A_2B_2^2 + B_1B_2^2 - 2A_2B_2 \cdot B_1B_2 \cdot cos\angle A_2B_2B_1 \\ &= A_2B_2^2 + B_1B_2^2 - 2A_2B_2 \cdot B_1B_2 \cdot cos\angle A_2B_2O \\ &= A_2B_2^2 + B_1B_2^2 - 2A_2B_2 \cdot B_1B_2 \cdot \frac{A_2B_2}{2OB_2} \qquad (8) \\ &= A_2B_2^2 + \left(\frac{81}{10}\right)^2 - 2A_2B_2 \cdot \frac{81}{10} \cdot \frac{A_2B_2}{20} \\ &= \frac{19}{100} \cdot A_2B_2^2 + \frac{81^2}{100} \qquad (9) \end{aligned}$$

From $A_2B_1 \geq AB \geq 9$ and equation (9) we get $A_2B_2 \geq 9$. Moreover, note that using the equation (8), the relation $A_2B_1 \leq A_2B_2$ or equivalently $A_2B_1^2 \leq A_2B_2^2$, becomes $B_1B_2^2 \leq 2A_2B_2 \cdot B_1B_2 \cdot \dfrac{A_2B_2}{2OB_2}$ or

$$B_1B_2 \cdot OB_2 \leq A_2B_2^2$$

However, this is true because $A_2B_2 \geq 9$, $B_1B_2 = \frac{81}{10}$ and $OB_2 = 10$. ■

To finish the problem, call A, B, C, D, E, F, G the 7 points on $\Gamma \backslash \Gamma_1$ and define $A_1 = \vec{OA} \cap \Gamma$, similarly B_1, C_1, D_1, E_1, F_1 and G_1 such that the points $A_1, B_1, C_1, D_1, E_1, F_1$ and G_1 are in that order. Therefore we have: $A_1B_1 \geq AB \geq 9$ and similarly the other 7 relations. But we also have that the measure of the arc AB is greater than the lenght of the segment A_1B_1, then: $\widehat{A_1B_1} \geq A_1B_1 \geq 9$, summing up these 7 inequalities we get:

$$63 \leq \widehat{A_1B_1} + \widehat{B_1C_1} + \widehat{C_1D_1} + \widehat{D_1E_1} + \widehat{E_1F_1} + \widehat{F_1G_1} + \widehat{G_1A_1} = 20\pi \approx 62.83 (\Rightarrow\Leftarrow)$$

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