Junior problems

J163. Let a, b, c be nonzero real numbers such that $ab + bc + ca \geq 0$. Prove that

$$\frac{ab}{a^2+b^2} + \frac{bc}{b^2+c^2} + \frac{ca}{c^2+a^2} \geq -\frac{1}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J164. If x and y are positive real numbers such that $\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right) = 2011$, find the minimum possible value of $x + y$.

Proposed by Neculai Stanciu, "George Emil Palade", Buzau, Romania

J165. Find all triples (x, y, z) of integers satisfying the system of equations

$$\begin{cases} \left(x^2+1\right)\left(y^2+1\right) + \frac{z^2}{10} = 2010 \\ (x+y)(xy-1) + 14z = 1985. \end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J166. Let P be a point inside triangle ABC and let d_a, d_b, d_c be the distances from point P to the sides of the triangle. Prove that

$$\frac{K}{d_a d_b d_c} \geq \frac{s}{Rr}$$

where K is the area of the pedal triangle of P and s, R, r are the semiperimeter, circumradius, and inradius of triangle ABC.

Proposed by Andrei Razvan Baleanu, "George Cosbuc", Motru, Romania

J167. Let a, b, c be real numbers greater than 1 such that

$$\frac{b+c}{a^2-1} + \frac{c+a}{b^2-1} + \frac{a+b}{c^2-1} \geq 1.$$

Prove that

$$\left(\frac{bc+1}{a^2-1}\right)^2 + \left(\frac{ca+1}{b^2-1}\right)^2 + \left(\frac{ab+1}{c^2-1}\right)^2 \geq \frac{10}{3}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J168. Let n be a positive integer. Find the least positive integer a such that the system

$$\begin{cases} x_1 + x_2 + \cdots + x_n = a \\ x_1^2 + x_2^2 + \cdots + x_n^2 = a \end{cases}$$

has no integer solutions.

Proposed by Dorin Andrica, "Babeş-Bolyai University", Cluj-Napoca, Romania

Senior problems

S163. (a) Prove that for each positive integer n there is a unique positive integer a_n such that
$$(1+\sqrt{5})^n = \sqrt{a_n} + \sqrt{a_n + 4^n}.$$
(b) When n is even, prove that a_n is divisible by $5 \cdot 4^{n-1}$ and find the quotient.

Proposed by Dorin Andrica, "Babeş-Bolyai University", Cluj-Napoca, Romania

S164. Let $ABCD$ be a cyclic quadrilateral whose diagonals are perpendicular to each other. For a point P on its circumscribed circle denote by ℓ_P the line tangent to the circle at P. Let $U = \ell_A \cap \ell_B, V = \ell_B \cap \ell_C, W = \ell_C \cap \ell_D, K = \ell_D \cap \ell_A$. Prove that $UVWK$ is a cyclic quadrilateral.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S165. Let I be the incenter of triangle ABC. Prove that
$$AI \cdot BI \cdot CI \geq 8r^3,$$
where r is the inradius of triangle ABC.

Proposed by Dorin Andrica, "Babeş-Bolyai University", Cluj-Napoca, Romania

S166. If $a_1, a_2, \ldots, a_k \in (0,1)$, and k, n are integers such that $k > n \geq 1$, prove that the following inequality holds
$$\min\{a_1(1-a_2)^n, a_2(1-a_3)^n, \ldots, a_k(1-a_1)^n\} \leq \frac{n^n}{(n+1)^{n+1}}.$$

Proposed by Marin Bancos, North University of Baia Mare, Romania

S167. Let I_a be the excenter corresnponding to the side BC of triangle ABC. Denote by A', B', C' the tangency points of the excircle of center I_a with the sides BC, CA, AB, respectively. Prove that the circumcircles of triangles AI_aA', BI_aB', CI_aC' have a common point, different from I_a, situated on the line G_aI_a, where G_a is the centroid of triangle $A'B'C'$.

Proposed by Dorin Andrica, "Babeş-Bolyai University", Cluj-Napoca, Romania

S168. Let $a_0 \geq 2$ and $a_{n+1} = a_n^2 - a_n + 1, n \geq 0$. Prove that
$$\log_{a_0}(a_n - 1)\log_{a_1}(a_n - 1)\cdots\log_{a_{n-1}}(a_n - 1) \geq n^n,$$
for all $n \geq 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Undergraduate problems

U163. Find the minimum of $f(x,y,z) = x^2 + y^2 + z^2 - xy - yz - zx$ over all triples (x,y,z) of positive integers for which 2010 divides $f(x,y,z)$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U164. Prove that $\varphi\Big(2^{2010!} - 1\Big)$ ends in at least 499 zeros.

Proposed by Dorin Andrica, "Babeş-Bolyai University", Cluj-Napoca, Romania

U165. Let $G = \{A_1, A_2, \ldots, A_m\} \subset M_n(\mathbb{R})$ such that $(G, \cdot)$ is a group. Prove that $Tr(A_1 + A_2 + \cdots + A_m)$ is an integers divisible by m.

Proposed by Mihai Piticari, "Dragos Voda" National College, Campulung Moldovenesc, Romania

U166. Find all functions $f : [0,\infty) \to [0,\infty)$ such that

(a) f is multiplicative

(b) $\lim_{x\to\infty}$ exists, is finite, and different from 0.

Proposed by Mihai Piticari, "Dragos Voda" National College, Campulung Moldovenesc, Romania

U167. Let $f : [0,1] \to \mathbb{R}$ be a continuously differentiable function such that $f(1) = 0$. Prove that

$$\left| \int_0^1 x f(x)dx \right| \le \frac{1}{6} \max_{x\in[0,1]} |f'(x)|.$$

Proposed by Duong Viet Thong, National Economics University, Ha Noi, Vietnam

U168. Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function on (a,b) and let $\max_{x\in[a,b]} |f''(x)| = M$. Prove that

$$\left| \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a) \right| \le \frac{(b-a)^3}{24} M.$$

Proposed by Duong Viet Thong, National Economics University, Ha Noi, Vietnam

Olympiad problems

O163. Prove that the equation

$$\frac{x^3+y^3}{x-y}=2010$$

is not solvable in positive integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, "Babeş-Bolyai University", Cluj-Napoca, Romania

O164. Let ABC be a triangle and let A_1 be a point on the side BC. Starting with A_1 construct reflections in one of the angle bisectors of triangle such that the next point lies on the other side of the triangle. The process is done in one direction: either clockwise or counterclockwise. Thus at the first step we construct an isosceles triangle A_1CB_1 with point B_1 lying on AC. At the second step we construct an isosceles triangle B_1AC_1 with point C_1 on AB. In fact we get a sequence of points $A_1, B_1, C_1, A_2, \ldots$.

(a) Prove that the process terminates in six steps, that is $A_1 \equiv A_3$

(b) Prove that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on the same circle.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O165. Let R and r be the circumradius and the inradius of a triangle ABC with the lengths of sides a, b, c. Prove that

$$2-2\sum_{cyc}\left(\frac{a}{b+c}\right)^2 \le \frac{r}{R}.$$

Proposed by Dorin Andrica, "Babeş-Bolyai University", Cluj-Napoca, Romania

O166. The incircle σ of triangle ABC with incenter I is tangent to sides BC and AC at points A_1 and B_1, respectively. Points A_2 and B_2 are diametrically opposite to A_1 and B_1 in σ. Let A_3 and B_3 be the intersection points of AA_2 with BC and BB_2 with AC, respectively. Let M be the midpoint of side AC and let N be the midpoint of A_1A_3. Line MI meets BB_1 in T and line AT meets BC in P. Let $Q \in (BC), R$ be the intersection of lines AB and QB_1 and $NR \cap AC = \{S\}$. Prove that $[AS] = 2[SM]$ if and only if $[BP] = [PQ]$.

Proposed by Andrei Razvan Baleanu, "George Cosbuc", Motru, Romania

O167. Prove that in any convex quadrilateral $ABCD$,

$$\cos\frac{A-B}{4}+\cos\frac{B-C}{4}+\cos\frac{C-D}{4}+\cos\frac{D-A}{4}\geq 2+\frac{1}{2}(\sin A+\sin B+\sin C+\sin D).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O168. Given a convex polygon $A_1A_2\ldots A_n, n\geq 4$, denote by R_i the radius of the circumcircle of triangle $A_{i-1}A_iA_{i+1}$, where $i=2,3,\ldots,n$ and A_{n+1} is the vertex A_1. Given that $R_2=R_3=\cdots=R_n$, prove that the polygon $A_1A_2\ldots A_n$ is cyclic.

Proposed by Nairi Sedrakyan, Armenia