On an Algebraic Identity

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Roberto Bosch Cabrera

Abstract

We solve problem U23 from Mathematical Reflections (which remained open for a while.) We use a simple algebraic identity concerning polynomials and their derivatives, demonstrating its usefulness in solving problems.

U23. Evaluate the sum

$$\sum_{k=0}^{n-1} \frac{1}{1+8\sin^2(\frac{k\pi}{n})}$$

Dorin Andrica and Mihai Piticari

Solution:
It is well known that

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

so

$$1+8\sin^2\left(\frac{k\pi}{n}\right) = 5-2\left(e^{i\frac{2k\pi}{n}}+e^{-i\frac{2k\pi}{n}}\right).$$

Let

$$e^{i\frac{2k\pi}{n}} = \xi_k$$

then we need to find the following sum:

$$\sum_{k=0}^{n-1} \frac{\xi_k}{-2\xi_k^2+5\xi_k-2}.$$

But we know the following

$$\frac{x}{-2x^2+5x-2} = \frac{-x}{(2-x)(1-2x)} = \frac{2}{3}\cdot\frac{1}{2-x} - \frac{1}{3}\cdot\frac{1}{1-2x}$$

so it follows that

$$\sum_{k=0}^{n-1} \frac{\xi_k}{-2\xi_k^2 + 5\xi_k - 2} = \frac{2}{3} \cdot \sum_{k=0}^{n-1} \frac{1}{2 - \xi_k} - \frac{1}{6} \cdot \sum_{k=0}^{n-1} \frac{1}{\frac{1}{2} - \xi_k}.$$

Observe that $\xi_k,\ k = 0, ..., n-1$ are the n roots of the polynomial $p(x) = x^n - 1$.

So we have to solve the following subproblem:

Let a_k $(k = 1, ..., n)$ be the roots of the polynomial $p(x)$ of degree n. Find the sum:

$$\sum_{k=1}^{n} \frac{1}{x - a_k}$$

This question is well known and it was the motivation to write this article. The answer is quite simple

$$\sum_{k=1}^{n} \frac{1}{x - a_k} = \frac{p'(x)}{p(x)}. \tag{1}$$

We continue with the proof to the above statement. Because $p(x) = (x - a_1) \cdots (x - a_n)$ then the product rule of differentiation yields

$$p'(x) = (x - a_2) \cdots (x - a_n) + \cdots + (x - a_1) \cdots (x - a_{n-1})$$

and we are done.

$\square$

Now we turn to the solution of U23:

In this case $p(x) = x^n - 1$ so $\frac{p'(x)}{p(x)} = \frac{nx^{n-1}}{x^n - 1}$. We know the following equalities are true

$$\sum_{k=0}^{n-1} \frac{1}{2 - \xi_k} = \frac{p'(2)}{p(2)} = \frac{n2^{n-1}}{2^n - 1}$$

and

$$\sum_{k=0}^{n-1} \frac{1}{\frac{1}{2} - \xi_k} = \frac{p'(\frac{1}{2})}{p(\frac{1}{2})} = \frac{\frac{n}{2^{n-1}}}{\frac{1}{2^n} - 1} = \frac{2n}{1 - 2^n}.$$

Finally

$$\sum_{k=0}^{n-1} \frac{1}{1 + 8\sin^2(\frac{k\pi}{n})} = \frac{2}{3} \cdot \frac{n2^{n-1}}{2^n - 1} - \frac{1}{6} \cdot \frac{2n}{1 - 2^n} = \frac{n}{3} \cdot \frac{2^n + 1}{2^n - 1}.$$

$\square$

We now discuss some problems and leave others for the reader to solve.

1. Let $p(x)$ be a non-constant polynomial with real roots. Prove that

$$p'(x)^2 \geq p(x)p''(x) \text{ for all } x \in \mathbb{R}.$$

(Martin Aigner, Gűnter M. Ziegler, Proofs from The Book, Third edition)

Solution:
If $x = a_i$ is a root of $p(x)$, then there is nothing to show. Assume that x is not a root, then

$$\frac{p'(x)}{p(x)} = \sum_{k=1}^{n} \frac{1}{x - a_k}.$$

Differentiating this expression we obtain

$$\frac{p''(x)p(x) - p'(x)^2}{p(x)^2} = -\sum_{k=1}^{n} \frac{1}{(x - a_k)^2} < 0$$

and this concludes the proof.

$\square$

2. Let P be an algebraic polynomial of degree n having only real zeros and real coefficients.
a) Prove that for every real x the following inequality holds:

$$(n-1)P'(x)^2 \geq nP(x)P''(x) \tag{2}$$

b) Examine the cases of equality.

(IMC 1998)

Solution:
Observe that both sides of (2) are equal to zero if $n = 1$. Suppose that $n > 1$ and let $x_1, ..., x_n$ be the zeros of P. Clearly (2) is true when $x = x_i$, $i \in \{1, ..., n\}$, and equality is possible only if $P'(x_i) = 0$, i.e if x_i is a multiple zero of P. Now suppose that x is not a zero of P. Using the identities

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i}, \quad \frac{P''(x)}{P(x)} = \sum_{1 \leq i < j \leq n} \frac{2}{(x - x_i)(x - x_j)}.$$

we find

$$(n-1)\left(\frac{P'(x)}{P(x)}\right)^2 - n\frac{P''(x)}{P(x)} = \sum_{i=1}^{n} \frac{n-1}{(x - x_i)^2} - \sum_{1 \leq i < j \leq n} \frac{2}{(x - x_i)(x - x_j)}.$$

But this last expression is simply

$$\sum_{1\le i<j\le n}\left(\frac{1}{x-x_i}-\frac{1}{x-x_j}\right)^2$$

and therefore is positive. The inequality is proved. In order for (2) to hold with equality sign for every real x it is necessary that $x_1 = x_2 = \ldots = x_n$. A direct verification shows that indeed, if $P(x) = c(x-x_1)^n$, then (2) becomes an identity.

□

3. Suppose that every zero of the polynomial $f(x)$ is simple. Find the sum of the reciprocals of the differences between the roots of the equation $f'(x) = 0$ and the roots of the equation $f(x) = 0$.

(E.J.Barbeau. Polynomials.)

Solution:
Let r_i be the zeros of $f(x)$ and let s_j be the zeros of $f'(x)$. Then, since

$$f'(x) = f(x)\sum(x-r_i)^{-1},$$

for each j we have:

$$0 = f'(s_j) = f(s_j)\sum(s_j-r_i)^{-1}.$$

Because $f(s_j)\neq 0$ then we can conclude that $\sum(s_j-r_i)^{-1} = 0$.

□

4. Determine those values of the real number a and positive integer n exceeding 1 for wich

$$\sum_{k=1}^{n}\frac{x_k+2}{x_k-1} = n-3$$

where $x_1, ..., x_n$ are the zeros of $x^n + ax^{n-1} + a^{n-1}x + 1$.

(E.J.Barbeau. Polynomials.)

Solution: Let $p(x) = x^n + ax^{n-1} + a^{n-1}x + 1$ then $p'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^{n-1}$. We have

$$\frac{x_k+2}{x_k-1} = 1 + \frac{3}{x_k-1}$$

therefore

$$\sum_{k=1}^{n}\frac{x_k+2}{x_k-1} = n - 3\sum_{k=1}^{n}\frac{1}{1-x_k} = n - 3\frac{p'(1)}{p(1)}$$

and we have

$$n - 3 \cdot \frac{n + an - a + a^{n-1}}{a^{n-1} + a + 2} = n - 3$$

or equivalently $(a + 1)(n - 2) = 0$, so $a = -1$ or $n = 2$. The case $a = -1$ yields the polynomial $x^n - x^{n-1} - x + 1 = (x - 1)(x^{n-1} - 1)$. But in this case, one of the zeros is 1 and the left side of the given equation is undefined. Hence $a \neq -1$. The case $n = 2$ yields the polynomial $x^2 + +2ax + 1$, whose zeros can be verified to satisfy the condition, provided $a \neq -1$.

$\square$

5. Let $-1 < x < 1$. Show that

$$\sum_{k=0}^{6} \frac{1 - x^2}{1 - 2x\cos(\frac{2k\pi}{7}) + x^2} = 7 \cdot \frac{1 + x^7}{1 - x^7} \tag{3}$$

(Longlist IMO 1988)

Solution:
We will use the follwing fact

$$\cos\left(\frac{2k\pi}{7}\right) = \frac{e^{i\frac{2k\pi}{7}} + e^{-i\frac{2k\pi}{7}}}{2}.$$

Let

$$e^{i\frac{2k\pi}{7}} = \xi_k$$

so

$$\frac{1 - x^2}{1 - 2x\cos(\frac{2k\pi}{7}) + x^2} = \frac{2(1 - x^2)\xi_k}{-2x\xi_k^2 + (2 + 2x^2)\xi_k - 2x}.$$

The discriminant of the quadratic $-2x\xi_k^2 + (2 + 2x^2)\xi_k - 2x$ is $D = 4(1 - x^2)^2$. Therefore $\sqrt{D} = 2\left|1 - x^2\right| = 2(1 - x^2)$ and the zeros are x and $\frac{1}{x}$. We have $-2x\xi_k^2 + (2 + 2x^2)\xi_k - 2x = -2x(\xi_k - x)(\xi_k - \frac{1}{x})$.
Finally,

$$\frac{1 - x^2}{1 - 2x\cos(\frac{2k\pi}{7}) + x^2} = \frac{(x - \frac{1}{x})\xi_k}{(\xi_k - x)(\xi_k - \frac{1}{x})} = \frac{\frac{1}{x}}{\frac{1}{x} - \xi_k} - \frac{x}{x - \xi_k}$$

and

$$\sum_{k=0}^{6} \frac{1 - x^2}{1 - 2x\cos(\frac{2k\pi}{7}) + x^2} = \frac{1}{x}\sum_{k=0}^{6} \frac{1}{\frac{1}{x} - \xi_k} - x\sum_{k=0}^{6} \frac{1}{x - \xi_k} = \frac{1}{x} \cdot \frac{p'(\frac{1}{x})}{p(\frac{1}{x})} - x \cdot \frac{p'(x)}{p(x)} = 7 \cdot \frac{1 + x^7}{1 - x^7}$$

where $p(x) = x^7 - 1$.

$\square$

6. Let $x_1, x_2, ..., x_n$ be real numbers that satisfy $0 < x_1 < x_2 < \cdots < x_n < 1$ and let $x_0 = 0,\ x_{n+1} = 1$. Given:

$$\sum_{j=0, j\neq i}^{n+1} \frac{1}{x_i - x_j} = 0 \qquad i = 1, 2, ..., n$$

prove that $x_{n+1-i} = 1 - x_i$ for $i = 1, 2, ..., n$.

(Shortlist IMO 1986)

Solution:
Let $P(x) = (x - x_0)(x - x_1)\cdots(x - x_n)(x - x_{n+1})$. Then

$$\begin{aligned} P'(x) &= \sum_{j=0}^{n+1} \frac{P(x)}{x - x_j} \\ P''(x) &= \sum_{j=0}^{n+1} \sum_{k\neq j} \frac{P(x)}{(x - x_j)(x - x_k)}. \end{aligned}$$

Therefore

$$P''(x_i) = 2P'(x_i) \sum_{j\neq i} \frac{1}{x_i - x_j}$$

for $i = 0, 1, ..., n + 1$ and the given condition implies $P''(x_i) = 0$ for $i = 1, 2, ..., n$. Consequently

$$x(x - 1)P''(x) = (n + 2)(n + 1)P(x). \tag{4}$$

It is easy to observe that there is a unique monic polynomial of degree $n+2$ satisfying the differential equation (4). On the other hand, the polynomial $Q(x) = (-1)^n P(1-x)$ also satisfies this equation, is monic, and $\deg Q = n + 2$. Therefore $(-1)^n P(1 - x) = P(x)$, and the result follows.

$\square$

7. Let $p(z)$ be a polynomial of degree n with complex coefficients. Its roots (in the complex plane) can be covered by a disk of radius r. Show that for any complex number k, the roots of the polynomial $np(z) - kp'(z)$ can be covered by a disk of radius $r + |k|$.

(Putnam 1957)

Solution:
Let the roots of $p(z)$ be $a_1, a_2, ..., a_n$. Suppose they all lie in a disk with center c and radius r, then $|c - a_n| \leq r$. Suppose that $|c - w| > r + |k|$. We will show that w is not a root of $np(z) - kp'(z)$. Indeed, we have that $|w - a_i| \geq |w - c| - |c - a_i| > r + |k| - r = |k|$, thus $\frac{p'(z)}{p(z)} = \sum \frac{1}{(z-a_i)}$ (note that this is still true if we have repeated roots), so $|\frac{p'(w)}{p(w)}| < \frac{n}{|k|}$ and hence $|k\frac{p'(w)}{p(w)}| < n$. Thus $|n - k\frac{p'(w)}{p(w)}| > 0$, but $|p(w)| > 0$ (since w lies outside the disk containing all the roots of $p(z)$), so $|np(w) - kp'(w)| = |p(w)||n - k\frac{p'(w)}{p(w)}| > 0$.

□

8. Let $p(z)$ be a polynomial of degree n, all of whose zeros have absolute value 1 in the complex plane and let $g(z) = \frac{p(z)}{z^{\frac{n}{2}}}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

(Putnam 2005)

Solution:
We have $z^{\frac{n}{2}} g(z) = p(z)$, therefore $\frac{g'(z)}{g(z)} = \frac{p'(z)}{p(z)} - \frac{n}{2z}$. Moreover

$$\frac{g'(z)}{g(z)} = \sum_{j=1}^{n} \left(\frac{1}{z - r_j} - \frac{1}{2z} \right) = \frac{1}{2z} \sum_{j=1}^{n} \frac{z + r_j}{z - r_j}$$

where $r_1, ..., r_n$ are the zeros of $p(z)$. Now if $z \neq r_j$ for all j, then

$$\frac{z + r_j}{z - r_j} = \frac{(z + r_j)(\overline{z} - \overline{r_j})}{|z - r_j|^2} = \frac{|z|^2 - 1 + 2Im(\overline{z} r_j)i}{|z - r_j|^2}$$

and so

$$Re\left(\frac{zg'(z)}{g(z)}\right) = \frac{|z|^2 - 1}{2} \cdot \left(\sum_{j=1}^{n} \frac{1}{|z - r_j|^2} \right).$$

Since the quantity in parentheses is positive $g'(z)$ can be 0 only if $|z| = 1$. If on the other hand $z = r_j$ for some j, then $|z| = 1$ anyway.

□

9. For any polynomial g, denote by $d(g)$ the minimun distance of any two of its real zeros ($d(g) = \infty$ if g has at most one real zero). Assume that g and $g + g'$ both are of degree $k \geq 2$ and have k distinct real zeros. Then $d(g + g') \geq d(g)$.

(IMC 2007)

Solution:
Let $x_1 < x_2 < \cdots < x_k$ be the roots of g. Suppose that a and b are roots of $g+g'$ satisfying $0 < b-a < d(g)$. Then a and b cannot be roots of g and

$$\frac{g'(a)}{g(a)} = \frac{g'(b)}{g(b)} = -1. \tag{5}$$

Since $\frac{g'}{g}$ is strictly decreasing between consecutive zeros of g (see problem **1.**), we must have $a < x_j < b$ for some j. For all $i = 1, 2, ..., k-1$ we have $x_{i+1} - x_i > b - a$ hence $a - x_i > b - x_{i+1}$. If $i < j$ both sides of this inequality are negative, if $i \geq j$ both sides are positive. In any case $\frac{1}{a-x_i} < \frac{1}{b-x_{i+1}}$ and hence:

$$\frac{g'(a)}{g(a)} = \sum_{i=1}^{k-1} \frac{1}{a-x_i} + \frac{1}{\underbrace{a-x_k}_{<0}} < \sum_{i=1}^{k-1} \frac{1}{b-x_{i+1}} + \frac{1}{\underbrace{b-x_1}_{>0}} = \frac{g'(b)}{g(b)}$$

This contradicts (5).

$\square$

Problems for independent study

1. From problem **5.** deduce that: $\csc^2(\frac{\pi}{7}) + \csc^2(\frac{2\pi}{7}) + \csc^2(\frac{3\pi}{7}) = 8.$

(Longlist IMO 1988)

2. Let the roots of an nth degree polynomial $P(z)$, with complex coefficients, lie on the unit circle in the complex plane. Prove that the roots of the polynomial

$$2zP'(z) - nP(z)$$

lie on the same circle.

(IMC 1995)

3. If all zeros of a polynomial $P(z)$ lie in the same half plane, then all zeros of the derivative $P'(z)$ lie in the same half plane.

(In a sharper formulation the problem tells us that the smallest convex polygon that contains the zeros of $P(z)$ also contains the zeros of $P'(z)$).

(Gauss-Lucas Theorem)

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