Junior problems

J109. Let a, b, c be positive real numbers. Prove that

$$\frac{(a+b)^2}{c} + \frac{c^2}{a} \geq 4b.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J110. Let $\tau(n)$ and $\phi(n)$ denote the number of divisors of n and the number of positive integers less than or equal to n that are relatively prime to n, respectively. Find all n such that $\tau(n) = 6$ and $3\phi(n) = 7!$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J111. Prove that there is no n for which $\displaystyle\prod_{k=1}^{n}(k^4 + k^2 + 1)$ is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J112. Let a, b, c be integers such that $\gcd(a, b, c) = 1$ $ab + bc + ca = 0$ and. Prove that $|a + b + c|$ can be expressed in the form $x^2 + xy + y^2$, where x and y are integers.

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

J113. Call *penta*-sequence a sequence of consecutive positive integers such that each of them can be written as a sum of five nonzero perfect squares. Prove that there are infinitely many penta-sequences of length 7.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J114. Let p be a prime. Find all solutions to the equation $a + b - c - d = p$, where a, b, c, d are positive integers such that $ab = cd$.

Proposed by Iurie Boreico, Harvard University, USA

Senior problems

S109. Solve the system of equations

$$\sqrt{x}-\frac{1}{y}=\sqrt{y}-\frac{1}{z}=\sqrt{z}-\frac{1}{x}=\frac{7}{4}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S110. Let X be a point on the side BC of a triangle ABC. The parallel through X to AB meets CA at V and the parallel through X to AC meets AB at W. Let $D = BV \cap XW$ and $E = CW \cap XV$. Prove that DE is parallel to BC and

$$\frac{1}{DE}=\frac{1}{BX}+\frac{1}{CX}.$$

Proposed by Francisco Javier García Capitán, Spain

S111. Prove that there are infinitely many positive integers n that can be expressed as $a^4+b^4+c^4+d^4-4abcd$, where a, b, c, d are positive integers, such that n is divisible by the sum of its digits.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S112. Let a, b, c be the side lengths and let s be the semiperimeter of a triangle ABC. Prove that

$$(s-c)^a(s-a)^b(s-b)^c \leq \left(\frac{a}{2}\right)^a\left(\frac{b}{2}\right)^b\left(\frac{c}{2}\right)^c.$$

Proposed by Johan Gunardi, Jakarta, Indonesia

S113. Prove that for different choices of signs $+$ and $-$ the expresion

$$\pm 1 \pm 2 \pm 3 \pm \cdots \pm (4n+1),$$

yields all odd positive integers less than or equal to $(2n+1)(4n+1)$.

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

S114. Consider triangle ABC with angle bisectors AA_1, BB_1, CC_1. Denote by U the intersection of AA_1 and B_1C_1. Let V be the projection from U onto BC. Let W be the intersection of the angle bisectors of $\angle BC_1V$ and $\angle CB_1V$. Prove that A, V, W are collinear.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Undergraduate problems

U109. Find all pairs (m,n) of integers such that $m^2+2mn-n^2=1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania

U110. Let $a_1,a_2,\ldots,a_n$ be real numbers with $a_n,\ a_0\neq 0$ such that the polynomial $P(X)=(-1)^n a_n X^n+(-1)^{n-1}a_{n-1}X^{n-1}+\cdots+a_2X^2-a_1X+a_0$ has all of its zeros in the interval $(0,\infty)$, and let $f:\mathbb{R}\to\mathbb{R}$ be an n-time differentiable function. Prove that if

$$\lim_{x\to\infty}\left(a_n f^{(n)}(x)+a_{n-1}f^{(n-1)}(x)+\cdots+a_2 f^{''}(x)+a_1 f^{'}(x)+a_0 f(x)\right)=L\in\overline{\mathbb{R}},$$

then $\lim\limits_{x\to\infty} f(x)$ exists and $\lim\limits_{x\to\infty} f(x)=\frac{L}{a_0}$.

Proposed by Radu Ţiţiu, Targu-Mures, Romania

U111. Let n be a given positive integer and let $a_k=2\cos\frac{\pi}{2^{n-k}}, k=0,1,\ldots,n-1$. Prove that

$$\prod_{k=0}^{n-1}(1-a_k)=\frac{(-1)^{n-1}}{1+a_0}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U112. Let x,y,z be real numbers greater than 1. Prove that

$$x^{x^3+2xyz}\cdot y^{y^3+2xyz}\cdot z^{z^3+2xyz}\geq (x^x y^y z^z)^{xy+yz+zx}.$$

Proposed by Cezar Lupu, University of Bucharest, Romania and Valentin Vornicu, San Diego, USA

U113. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that f is a periodic function that does not have a least period.

Proposed by Radu Ţiţiu, Targu-Mures, Romania

U114. Let a,b,c be nonnegative real numbers. Evaluate

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i,j=1}^{n}\frac{1}{\sqrt{i^2+j^2+ai+bj+c}}.$$

Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania

Olympiad problems

O109. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a+b+1}{a+b^2+c^3}+\frac{b+c+1}{b+c^2+a^3}+\frac{c+a+1}{c+a^2+b^3} \leq \frac{(a+1)(b+1)(c+1)+1}{a+b+c}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O110. Hexagon $A_1A_2A_3A_4A_5A_6$ is inscribed in a circle $C(O, R)$ and at the same time circumscribed about a circle $\omega(I, r)$. Prove that if

$$\frac{1}{A_1A_2}+\frac{1}{A_3A_4}+\frac{1}{A_5A_6}=\frac{1}{A_2A_3}+\frac{1}{A_4A_5}+\frac{1}{A_6A_1},$$

then one of its diagonals coincides with OI.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O111. Prove that for each positive integer n the number

$$\left(\binom{n}{0}+2\binom{n}{2}+2^2\binom{n}{4}+\cdots\right)^2\left(\binom{n}{1}+2\binom{n}{3}+2^2\binom{n}{5}+\cdots\right)^2$$

is triangular.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O112. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3+abc}{(b+c)^2}+\frac{b^3+abc}{(c+a)^2}+\frac{c^3+abc}{(a+b)^2} \geq \frac{3}{2}\cdot\frac{a^3+b^3+c^3}{a^2+b^2+c^2}.$$

Proposed by Cezar Lupu, University of Bucharest, Romania and Pham Huu Duc, Ballajura, Australia

O113. Let P be a point on the circumcircle Γ of a triangle ABC. The tangents from P to the incircle of ABC meet again the circumcircle at X and Y, respectively. Prove that line XY is parallel to a side of triangle ABC if and only if P is the tangency point of Γ with some mixtilinear incircle of triangle ABC.

Proposed by Cosmin Pohoata, Bucharest, Romania

O114. Prove that for all real numbers x, y, z the following inequality holds

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) \geq 3(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)$$

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France