Junior problems

J85. Let a and b be positive real numbers. Prove that

$$\sqrt[3]{\frac{(a+b)(a^2+b^2)}{4}} \geq \sqrt{\frac{a^2+ab+b^2}{3}}.$$

Proposed by Arkady Alt, San Jose, California, USA

First solution by Ivanov Andrey, Chisinau, Moldova

$$\sqrt[3]{\frac{(a+b)(a^2+b^2)}{4}} \geq \sqrt{\frac{a^2+ab+b^2}{3}}$$
$$\left(\frac{(a+b)(a^2+b^2)}{4}\right)^2 \geq \left(\frac{a^2+ab+b^2}{3}\right)^3$$
$$\frac{(a+b)^2(a^2+b^2)^2}{16} - \frac{(a^2+ab+b^2)^3}{27} \geq 0$$
$$27(a+b)^2(a^2+b^2)^2 - 16(a^2+ab+b^2)^3 \geq 0$$
$$27(a+b)^2(a^2+b^2)^2 - 16(a^2+ab+b^2)^3 \geq 0$$
$$27((a^2+b^2-2ab)+4ab)((a^2+b^2-2ab)+2ab)^2 - 16((a^2-2ab+b^2)+3ab)^3 \geq 0$$
$$27((a-b)^2+4ab)((a-b)^2+2ab)^2 - 16((a-b)^2+3ab)^3 \geq 0$$

Denote $(a-b)^2 = x \geq 0, ab = y \geq 0$. Then we have:
$$27(x+4y)(x+2y)^2 - 16(x+3y)^3 \geq 0$$
$$27(x+4y)(x^2+4xy+4y^2) - 16(x^3+9x^2y+27xy^2+27y^3) \geq 0$$
$$27x^3+108x^2y+108xy^2+108x^2y+432xy^2+432y^3-16x^3-144x^2y-432xy^2-432y^3 \geq 0$$
$$11x^3+72x^2y+108xy^2 \geq 0$$
$$x(11x^2+72xy+108y^2) \geq 0$$
Which is true because $x, y \geq 0$. Equality holds if $x = 0$ or $a = b$.

Second solution by Manh Dung Nguyen, Vietnam

The above inequality is equivalent to

$$\frac{3(a^2+b^2)}{2(a^2+ab+b^2)} \geq \frac{2\sqrt{a^2+ab+b^2}}{\sqrt{3}(a+b)}$$
$$\Leftrightarrow \frac{3(a^2+b^2)}{2(a^2+ab+b^2)} - 1 \geq \frac{2\sqrt{a^2+ab+b^2}}{\sqrt{3}(a+b)} - 1$$
$$\Leftrightarrow \frac{(a-b)^2}{2(a^2+ab+b^2)} \geq \frac{(a-b)^2}{\sqrt{3}(a+b)\left(2\sqrt{a^2+ab+b^2}+\sqrt{3}(a+b)\right)}.$$

So we need to prove that

$$\sqrt{3}(a+b)\left(2\sqrt{a^2+ab+b^2}+\sqrt{3}(a+b)\right) \geq \sqrt{3}(a+b)2\sqrt{3}(a+b)$$
$$= 6(a+b)^2 > 2(a^2+ab+b^2).$$

And thus we are done. Equality holds if and only if $a=b$.

Third solution by Oleh Faynshteyn, Leipzig, Germany

Raise the inequality to the sixth power to obtain the equivalent inequality

$$\frac{(a+b)^2(a^2+b^2)^2}{16} \geq \frac{(a^2+ab+b^2)^3}{27}$$

or after transformations

$$11a^6+6a^5b-15a^4b^2-4a^3b^3-15a^2b^4+6ab^5+11b^6 \geq 0,$$

or

$$11\left(\frac{a}{b}\right)^6+6\left(\frac{a}{b}\right)^5-15\left(\frac{a}{b}\right)^4-4\left(\frac{a}{b}\right)^3-15\left(\frac{a}{b}\right)^2+6\left(\frac{a}{b}\right)+11 \geq 0.$$

Substituting $\frac{a}{b}$ by t we obtain

$$11t^6+6t^5-15t^4-4t^3-15t^2+6t+11 = (t-1)^2(11t^4+28t^3+30t^2+28t+11) \geq 0$$

and thus we are done with equality when $t=1$ or $a=b$.

Fourth solution by Ovidiu Furdui, Cluj, Romania

The inequality to prove is equivalent to

$$\sqrt[3]{\frac{(a/b+1)(a^2/b^2+1)}{4}} \geq \sqrt{\frac{a^2/b^2+a/b+1}{3}}.$$

Thus, it suffices to prove that for all $x>0$ the following inequality holds

$$\sqrt[3]{\frac{(x+1)(x^2+1)}{4}} \geq \sqrt{\frac{x^2+x+1}{3}}.$$

Straight forward calculations show that the preceding inequality is equivalent to

$$(11x^4+28x^3+30x^2+28x+11)(x-1)^2 \geq 0,$$

which holds for all $x>0$, and the problem is solved.

Also solved by Andrea Munaro, Italy; Brian Bradie, Newport News, VA; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; G.R.A.20 Math Problems Group, Roma, Italy; Michel Bataille, France; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Roberto Bosch Cabrera, Cuba.

J86. A triangle is called α-angular if none of its angles exceeds α degrees. Find the least α for which each non α-angular triangle can be dissected into some α-angular triangles.

Proposed by Titu Andreescu, University of Texas at Dallas and Gregory Galperin, Eastern Illinois University, USA

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Without loss of generality let A be the smallest angle of a triangle ABC. Obviously, no matter how the decomposition of ABC is made, at least one of the triangles present in the decomposition will have A as one of its vertices, hence one of its angles will be no larger than A. This means that its other two angles add up to $\pi - A$, and at least one of its angles is at least $\frac{\pi}{2} - \frac{A}{2}$. Assume that $\alpha < \frac{\pi}{2}$. Then, a non-degenerate triangle ABC can be found with $0 < A < \pi - 2\alpha$, and triangle ABC cannot be decomposed in α-angular triangles, since at least one of the angles in the decomposition will be no smaller than $\frac{\pi}{2} - \frac{A}{2} > \alpha$. Hence $\alpha \geq \frac{\pi}{2}$. If we take $\alpha = \frac{\pi}{2}$, then a triangle is not α-angular iff it is obtuse. The altitude from the obtuse vertex decomposes the triangle in two right-angled (hence α-angular) triangles. So the minimum α is $\frac{\pi}{2}$.

Second solution by Vicente Vicario Garcia, Huelva, Spain

We use the habitual notations in a triangle. We will prove that $\alpha = 90°$. It is easy to prove, by contradiction, that $\alpha = 90°$. Assume that $\alpha < 90°$, then we can create an obtuse triangle with, for example $\angle A \to 180°$, and angles B, C arbitrarily close to 0. For this triangle the dissection is impossible because it cannot be dissected as desired. Without loss of generality, if the angle B is dissected by a straight line we have a contradiction, and the same happens in all of the other cases. We assume that ABC is an obtuse triangle with $A > 90°$. We will prove that there is a dissection into seven acute triangles, and this is the least possible number of triangles for the dissection. Let I be the incenter of the triangle ABC. Let J and J' be the points of intersection between the incircle and the interior bisectors CI and BI, respectively. Let MN ($M \in AC, n \in BC$) and PQ ($P \in AB, Q \in BC$) be interior segments in the triangle such that they are the intersection with the sides of the triangle and tangent to the incircle at points J and J', respectively. Then, the triangles NCM and PBQ are isosceles, and the angles of pentagon $APQNM$ are all greater than $90°$. Joining I with the vertices of pentagon we have that interior bisectors of these angles, determine five acute triangles, because the angles in the vertices A, P, Q, N, and M are acute and greater than $45°$. Then, the angles in I are acute. These five triangles, with the triangles NCM and PBQ, dissect the triangle ABC into seven acute triangles. On the other hand, a similar proof shows that every rectangular triangle is dissected into seven acute triangles too. In fact, we can prove that the least number of acute triangles in the dissection of one obtuse triangle, is seven. For which, we can see, obviously, that one segment must bisect the obtuse angle necesarily. Then, this segment must be end in an interior point of the triangle, for example, X. At least five segments must join in this vertex, because, otherwise, the angles in the vertex are not all acute. None of them must be ending in the interior of the triangle, because, in otherwise, the dissection is not minimal. Then, it is easy to see that the only possibility for the minimal dissection is five interior triangles (the pentagon) with the two exterior triangles making seven triangles for the minimal dissection.

J87. Prove that for any acute triangle ABC, the following inequality holds:

$$\frac{1}{-a^2+b^2+c^2}+\frac{1}{a^2-b^2+c^2}+\frac{1}{a^2+b^2-c^2}\geq\frac{1}{2Rr}.$$

Proposed by Mircea Becheanu, Bucharest, Romania

First solution by Brian Bradie, VA, USA

Using the Law of Cosines and the formula

$$R=\frac{abc}{4rs},$$

we can rewrite the original inequality as

$$\frac{a}{\cos\alpha}+\frac{b}{\cos\beta}+\frac{c}{\cos\gamma}\geq 4s=2(a+b+c), \tag{1}$$

where α, β and γ are the acute angles in the triangle. Using the Law of Sines, we can write

$$c=a\frac{\sin\gamma}{\sin\alpha}\quad\text{and}\quad b=a\frac{\sin\beta}{\sin\alpha}.$$

Substituting into (1) yields

$$\tan\alpha+\tan\beta+\tan\gamma\geq 2(\sin\alpha+\sin\beta+\sin\gamma). \tag{2}$$

On $(0,\frac{\pi}{2})$, $\tan x$ is convex and $\sin x$ is concave; it therefore follows from Jensen's inequality that

$$\begin{aligned}\tan\alpha+\tan\beta+\tan\gamma &\geq 3\tan\left(\frac{\alpha+\beta+\gamma}{3}\right)=3\tan\frac{\pi}{3}=3\sqrt{3},\ \text{and}\\ \sin\alpha+\sin\beta+\sin\gamma &\leq 3\sin\left(\frac{\alpha+\beta+\gamma}{3}\right)=3\sin\frac{\pi}{3}=\frac{3\sqrt{3}}{2}.\end{aligned}$$

Hence, (2) holds with equality if and only if $\alpha=\beta=\gamma$. Thus, the original inequality holds with equality if and only if the triangle is an equilateral triangle.

Second solution by Mihai Miculita, Oradea, Romania

Because $2Rr=2\frac{S}{p}\cdot\frac{abc}{4S}=\frac{abc}{2p}=\frac{abc}{a+b+c}$, the given inequality is equivalent to

$$\frac{1}{b^2+c^2-a^2}+\frac{1}{a^2+c^2-b^2}+\frac{1}{a^2+b^2-c^2}\geq\frac{abc}{a+b+c}. \quad (1)$$

Let us observe that since ABC is an acute triangle the following is true

$$\begin{aligned}
& b^2+c^2-a^2>0 \Rightarrow 2(b-c)^2(b^2+c^2-a^2)\ge 0\\
& \Leftrightarrow (b-c)^2(2b^2+2c^2-2a^2)\ge 0\\
& \Leftrightarrow (b-c)^2[(b+c)^2+(b-c)^2-2a^2]\ge 0\\
& \Leftrightarrow (b^2-c^2)^2+(b-c)^4-2a^2(b-c)^2\ge 0\\
& \Leftrightarrow (b-c)^4-2a^2(b-c)^2+a^4\ge a^4-(b^2-c^2)^2\\
& \Leftrightarrow [a^2-(b-c)^2]^2\ge (a^2+b^2-c^2)(a^2+c^2-b^2)\\
& \Leftrightarrow a^2-(b-c)^2\ge \sqrt{(a^2+b^2-c^2)(a^2+c^2-b^2)}\\
& \Leftrightarrow (a+b-c)(a+c-b)\ge \sqrt{(a^2+b^2-c^2)(a^2+c^2-b^2)}. \quad (2)
\end{aligned}$$

Thus, using the AM-GM inequality and using the result in (2) we have that:

$$\begin{aligned}
\frac{1}{2}\left(\frac{1}{b^2+c^2-a^2}+\frac{1}{a^2+c^2-b^2}\right) &\ge \frac{1}{\sqrt{(b^2+c^2-a^2)(a^2+c^2-b^2)}}\\
&\ge \frac{1}{(b+c-a)(a+c-b)}. \quad (3)
\end{aligned}$$

Summing up inequality (3) and the two obtained by a circular permutation of the letters we obtain

$$\begin{aligned}
\frac{1}{b^2+c^2-a^2}+\frac{1}{a^2+c^2-b^2}+\frac{1}{a^2+b^2-c^2} &= \frac{1}{2}\left(\frac{1}{b^2+c^2-a^2}+\frac{1}{a^2+c^2-b^2}\right)\\
&+\frac{1}{2}\left(\frac{1}{b^2+c^2-a^2}+\frac{1}{a^2+b^2-c^2}\right)+\frac{1}{2}\left(\frac{1}{a^2+c^2-b^2}+\frac{1}{a^2+b^2-c^2}\right)\\
&\ge \frac{1}{(b+c-a)(a+c-b)}+\frac{1}{(b+c-a)(a+b-c)}\\
&+\frac{1}{(a+c-b)(a+b-c)}\\
&= \frac{a+b+c}{(b+c-a)(a+c-b)(a+b-c)}\\
&\Rightarrow \frac{1}{b^2+c^2-a^2}+\frac{1}{a^2+c^2-b^2}+\frac{1}{a^2+b^2-c^2}\\
&\ge \frac{a+b+c}{(b+c-a)(a+c-b)(a+b-c)}. \quad (4)
\end{aligned}$$

It is known that

$$\sqrt{(b+c-a)(a+c-b)}\le \frac{(b+c-a)+(a+c-b)}{2}=c.$$

Multiplying the above inequality with its respective ones obtained by circular permutation of letters we obtain

$$(b+c-a)(a+c-b)(a+b-c)\le abc. \quad (5)$$

Using (4) and (5) we readily obtain the desired inequality (1).

Third solution by Ovidiu Furdui, Cluj, Romania

We will use the following standard trigonometric formulae

$$s = 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \quad \text{and} \quad r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

where s denotes the semiperimeter of triangle ABC. It is simply to check, by using the preceding formulas that $4sRr = abc$.

Let $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ be the function defined by $f(x) = \frac{x}{\cos x}$. A calculation shows that

$$f''(x) = \frac{x + x\sin^2 x + \sin(2x)}{\cos^3 x} > 0,$$

and hence, f is a convex function. Using the Law of Cosines combined with Jensen's inequality for convex functions we get that

$$\frac{1}{-a^2+b^2+c^2} + \frac{1}{a^2-b^2+c^2} + \frac{1}{a^2+b^2-c^2} = \sum_{cyclic} \frac{1}{2bc\cos A} = \frac{1}{2abc}\sum_{cyclic}\frac{a}{\cos A}$$

$$\geq \frac{1}{2abc} \cdot 3 \cdot \frac{\frac{a+b+c}{3}}{\cos\frac{A+B+C}{3}} = \frac{2s}{abc} = \frac{1}{2Rr},$$

and the problem is solved.

Fourth solution by Tarik Adnan Moon, Kushtia, Bangladesh

$$\sum_{cyc} \frac{1}{-a^2+b^2+c^2} \geq \frac{1}{2Rr}$$

We know that, $-a^2+b^2+c^2 = 2bc\cdot\cos A$. So,we need to prove that,

$$\sum_{cyc} \frac{1}{2bc\cdot\cos A} \geq \frac{1}{2Rr}$$

Lemma 1: We know that, $[ABC] = sr = \frac{abc}{4R} \Longrightarrow 4sr = \frac{abc}{R}$

After multiplying by $2abc$ we get,

$$\sum_{cyc} \frac{a}{\cos A} \geq \frac{abc}{Rr} = \frac{4sr}{r} = 4s$$

By Cauchy-Schwarz inequaltiy we get,

$$\left(\sum_{cyc} a\cdot\cos A\right)\left(\sum_{cyc}\frac{a}{\cos A}\right) \geq \left(\sum_{cyc} a\right)^2 = 4s^2 ...(1)$$

Lemma 2: We know that,

$$\left(\sum_{cyc} a \cdot \cos A\right) = \frac{2sr}{R}$$

So, it is left to prove that,

$$\left(\sum_{cyc} a \cdot \cos A\right) = \frac{2sr}{R} \leq s \Leftrightarrow R \geq 2r$$

And we are done.

Some words about the lemmas:
Lemma 1: Straightforward, just need to use extended law of sines.
Lemma 2: We know that, $a \cos A = 2R \sin A \cdot \cos A = R \cdot \sin 2A$
Then,we use the identity, $\sum \sin 2A = 4 \prod \sin A$
and using the extended law of sines we obtain, $4R^2 \prod \sin A = bc \sin A = 2[ABC]$
From these three we obtain, $\sum a \cos A = \frac{2[ABC]}{R} = \frac{2sr}{R}$

Also solved by Andrea Munaro, Italy; Arkady Alt, San Jose, California, USA; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; G.R.A.20 Math Problems Group, Roma, Italy; Ivanov Andrey, Chisinau, Moldova; Athanasios Magkos, Kozani, Greece; Michel Bataille, France; Ricardo Barroso Campos, Spain; Roberto Bosch Cabrera, Cuba; Samin Riasat, Notre Dame College, Dhaka, Bangladesh; Vicente Vicario Garcia, Huelva, Spain.

J88. Find the greatest n for which there are points $P_1, P_2, \ldots, P_n$ in the plane such that each triangle whose vertices are among $P_1, P_2, \ldots, P_n$, has a side less than 1 and a side greater than 1.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Assume that a point (P_1 wlog) is at a distance no smaller than 1 from at least three other points (P_2, P_3, P_4 wlog). Then, $P_2P_3 < 1$ since otherwise triangle $P_1P_2P_3$ would not satisfy the given condition. Similarly, $P_3P_4, P_4P_3 < 1$, and triangle $P_2P_3P_4$ does not satisfy the given condition. Therefore, each point in the set is at a distance not smaller than 1 from at most 2 other points. Reversing the sign of the previous inequalities, we prove similarly that each point of the set is at a distance not larger than 1 from at most 2 other points, and the set may not contain more than 5 points, such that each point has exactly two points at a distance larger than 1, and two points at a distance smaller than 1.

Consider now a regular pentagon whose sides are shorter than 1 but whose diagonals are longer than 1. This is possible since the diagonals of a regular pentagon are larger than its sides. Any triangle formed by 3 vertices of the pentagon will have at least one side equal to one of the sides of the pentagon, and at least one side equal to one of the diagonals of the pentagon. It follows that $n = 5$ is possible with this choice.

Second solution by Vicente Vicario Garcia, Huelva, Spain

We will prove that the greatest value of n is 5. It is readily noted that if $\{P_1, P_2, P_3, P_4, P_5\}$ are the vertices of a regular pentagon with side length L such that

$$\frac{2}{1+\sqrt{5}} < L < 1$$

then each triangle with vertices among $\{P_1, P_2, P_3, P_4, P_5\}$ has a side of length less than 1 and a side greater than 1.

On the other hand assume that $\{P_1, P_2, P_3, P_4, P_5, P_6\}$ are six points in the plane. We will prove by contradiction that this is impossible. Without loss of generality and by using the Pigeonhole principle, the segments P_1P_3, P_1P_4, P_1P_5 have all lengths either less than 1 or greater than 1. If, for example, the lengths are greater than 1, the the lengths of P_3P_4, P_4P_5, P_3P_5 cannot be all less than 1, meaning one of them must have length greater than 1, which is impossible. When P_1P_3, P_1P_4, P_1P_5 have all lengths less than 1 the contradiction is deducted in the same manner.

J89. Let A and B lie on circle $\mathcal{C}$ of center O and let C be the point on the small arc AB such that OA is the external angle bisector of $\angle BOC$. Denote by M the midpoint of BC and by N the intersection of AM and OC. Prove that the intersection of the angle bisector of $\angle BOC$ with the circle of center O and radius ON is the center of the circle tangent to lines OB and OC, and also internally tangent to $\mathcal{C}$.

Proposed by Francisco Javier Garcia Capitan, Spain

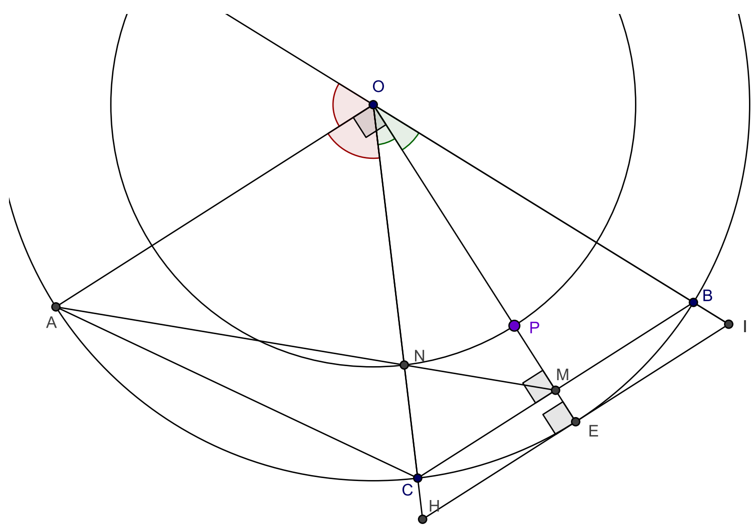
First solution by Ivanov Andrey, Chisinau, Moldova

Lemma. In triangle ABC, AD is bisector. Then $\frac{AI}{ID} = \frac{AB+AC}{BC}$.
Proof. By bisector theorem

$$\frac{BD}{CD} = \frac{AB}{AC} \Leftrightarrow \frac{BC}{CD} = \frac{AB+AC}{AC} \Rightarrow CD = \frac{AC \cdot BC}{AB+AC}.$$

By bisector theorem in triangle ACD we have

$$\frac{AI}{ID} = \frac{AC}{CD} \Leftrightarrow \frac{AI}{ID} = \frac{AC}{\frac{AC \cdot BC}{AB+AC}} \Leftrightarrow \frac{AI}{ID} = \frac{AB+AC}{BC}$$



Let P be the the intersection of the angle bisector of $\angle BOC = \alpha$ with the circle of center O and radius ON. It is clear that internal bisector of $\angle BOC$ passes through M. Let $E \in OM \cap \mathcal{C}$. Let d be a line tangent to $\mathcal{C}$ at E. Let $H \in OC \cap d$ and $I \in OB \cap d$. If circle with center P is tangent to $\mathcal{C}$ then it is tangent to IH. So we should prove that P is incenter of $\triangle OHI$. Because P lies on the bisector OE, by *Lemma* it is enough to prove that

$\frac{PE}{OP} = \frac{HI}{OH+OI} = \frac{2HE}{2OH} = \sin\alpha \Leftrightarrow \frac{R}{OP} = \sin\alpha + 1 \Leftrightarrow OP = \frac{R}{\sin\alpha+1}$. We have $OA = R$ and $OM = R\cos\alpha$ and $\angle AOM = 90°$.
So by Pythagora $AM = \sqrt{R^2 + R^2\cos^2\alpha} = R\sqrt{\cos^2\alpha + 1}$. Also note than $\sin\angle AMO = \frac{1}{\sqrt{\cos^2\alpha+1}}$.

By the law of sine in triangles ANO and ONM we have:
$\frac{AN}{\cos\alpha} = \frac{R}{\sin\angle ANO}$ and $\frac{MN}{\sin\alpha} = \frac{R\cos\alpha}{\sin\angle ANO}$ which implies $\frac{AM}{MN} = \frac{1}{\sin\alpha} \Leftrightarrow \frac{AM}{MN} = \frac{1+\sin\alpha}{\sin\alpha} \Leftrightarrow MN = \frac{R\sqrt{\cos^2\alpha+1}\sin\alpha}{1+\sin\alpha}$

Again by the law of sine in triangle OMN we have:
$\frac{ON}{\sin\angle AMO} = \frac{MN}{\sin\alpha} \Rightarrow ON = \frac{MN\sin\angle AMO}{\sin\alpha} = \frac{R\sqrt{\cos^2\alpha+1}\sin\alpha}{\sqrt{\cos^2\alpha+1}\sin\alpha(\sin\alpha+1)} = \frac{R}{\sin\alpha+1}$.

But P lies on the circle with center O and radius ON, so $OP = ON = \frac{R}{\sin\alpha+1}$ and we are done!

Second solution by Ricardo Barroso Campos, Spain

Let $\angle MOC = \alpha$ then $\angle COA = 90° - \alpha$. Let $OB = R$. We know that OA is perpendiclar on OM and ONA is similar to CNM. Thus

$$\frac{ON}{R - ON} = \frac{R}{R\sin\alpha} = \frac{1}{\sin\alpha} \Rightarrow ON = \frac{R}{1+\sin\alpha}.$$

Let Q be the intersection(inside the triangle) of the circle of radius ON with the OM, and let M be the intersection of OM with the circle of radius OC. Then we haev that

$$QU = R - \frac{R}{1+\sin\alpha} = \frac{R\sin\alpha}{1+\sin\alpha}.$$

Let S be the intersection of the perpendicular from Q to OC and T the intersection of the perpendicular from Q to OB. We have that $QS = QT$ and thus OM is the bisector of $\angle BOC$. In the right triangles OQS and OQT, we have that $QS = QT = OQ\sin\alpha = \frac{R\sin\alpha}{1+\sin\alpha}$ and so we are done.

Third solution by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

We have $\frac{AP}{PC} = \frac{AB}{CM}$, since $\triangle APB \sim \triangle CPM$. Again, $\frac{AQ}{QM} = \frac{AB}{DM}$ as $\triangle AQB \sim \triangle MQD$. Therefore $\frac{AQ}{QM} = \frac{AP}{PC}$, as $CM = DM$. This implies $PQ \parallel CM \parallel AB$.

Let P be the intersection of AB and OM. We'll be done if we can show that $CN = NP$, as this implies the center of the circle mentioned in the problem is equidistant from OB, OC and the midpoint of minor arc BC. Let BE be a diameter of $\mathcal{C}$.

Since $\angle COD + \angle COA = \frac{1}{2}(\angle BOC + \angle COE) = 90^\circ \implies OM \perp OA$ we have $\angle OCM = 90^\circ - \angle COD = \angle COA$. Therefore $BC \parallel OA$ and hence by the lemma stated above, $NP \parallel BC$. Thus $NP \perp OM$. But AB is the bisector of $\angle OBC$ and OM is the bisector of $\angle BOC$. Therefore $P = AB \cap OM$ is the incenter of $\triangle OBC$. Hence $\angle PCN = \angle PCD = \angle CPN$, as $NP \parallel CM$. Thus $CN = NP$.

Fourth solution by Tarik Adnan Moon, Kushtia, Bangladesh

Let $OM \cap \mathcal{C} = M'$. We draw tangent $B'C'$ of circle $\mathcal{C}$ at M. $\triangle OBC$ is icosceles, which implies, $OM \perp BC$ and OM is the bisector of $\angle BOC$.

Let, $OB \cap \mathcal{C} = A'$ and $AB \cap OM = I'$.

Since we have, $\angle A'OA = \angle AOC \implies \angle A'BA = \angle ABC$. Thus, AB is the bisector of $\angle OBC$. So, I' is the incenter of $\triangle OBC$

Let, the circle with center O and radius ON intersects OM at I.

So,We need to prove that,I is the incenter of $\triangle OC'B'$ As, $\triangle OCB$ and $\triangle OC'B'$ are homothetic with center O and ratio,

$$k = \frac{OB}{OB'} = \frac{OM}{OM'}$$

we shall prove that, Homothety with center O and ratio k sends I' to I. Which is equivalent to prove,

$$k = \frac{OI'}{OI} = \frac{OI'}{ON} = \frac{OM}{OM'}$$

Equivalently,

$$\frac{OI'}{OM} = \frac{ON}{OM'} = \frac{ON}{OC}$$

Let, $A'O = AO = CO = M'O = BO = b$ and $BC = a$. As AO and OM are the external and internal bisectors of $\angle BOC$, $\angle AOC = OMC = \frac{\pi}{2}$.

So, $AO \parallel BC$ and $\triangle AON \sim \triangle CMN$. Thus,

$$\frac{NC}{ON} = \frac{b}{\frac{a}{2}} \implies \frac{ON}{OC} = \frac{b}{b + \frac{a}{2}}$$

And,we know that,

$$OI = \frac{b - \frac{a}{2}}{\sin B} \quad \textbf{and} \quad OM = b \cdot \sin B \implies \frac{OI'}{OM} = \frac{b - \frac{a}{2}}{b \cdot \sin^2 B}$$

So, we left to prove that,

$$\frac{b - \frac{a}{2}}{b \cdot \sin^2 B} = \frac{b}{b + \frac{a}{2}} \implies b^2 - \left(\frac{a}{2}\right)^2 = b^2 \cdot \sin B \Leftrightarrow OM^2 = OM^2$$

Also solved by Andrea Munaro, Italy; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Vicente Vicario Garcia, Huelva, Spain.

J90. For a fixed positive integer n let $a_k = 2^{2^{k-n}} + k$, $k = 0, 1, \ldots, n$. Prove that

$$(a_1 - a_0) \cdots (a_n - a_{n-1}) = \frac{7}{a_1 + a_0}.$$

Proposed by Titu Andreescu, University of Texas at Dallas

First solution by Arkady Alt, San Jose, California, USA

Let $a := 2^{2^{-n}}$ then $a^{2^k} = \left(2^{2^{-n}}\right)^{2^k} = 2^{2^{k-n}}$ and, therefore,

$a_k - a_{k-1} = 2^{2^{k-n}} + k - 2^{2^{k-n-1}} - (k-1) = 2^{2^{k-n}} - 2^{2^{k-n-1}} + 1 = a^{2^k} - a^{2^{k-1}} + 1 = \dfrac{a^{2^{k+1}} + a^{2^k} + 1}{a^{2^k} + a^{2^{k-1}} + 1}, k = 0, 1, 2, ..., n$.

Thus, $\prod_{k=1}^{n} (a_k - a_{k-1}) = \prod_{k=1}^{n} \dfrac{a^{2^{k+1}} + a^{2^k} + 1}{a^{2^k} + a^{2^{k-1}} + 1} = \dfrac{a^{2^{n+1}} + a^{2^n} + 1}{a^{2^1} + a^{2^0} + 1} = \dfrac{2^{2^{n+1-n}} + 2^{2^{n-n}} + 1}{2^{2^{1-n}} + 1 + 2^{2^{0-n}} + 0} = \dfrac{7}{a_1 + a_0}$.

Second solution by Brian Bradie, VA, USA

We start with $a_n = n + 2$ and $a_{n-1} = n - 1 + \sqrt{2}$. Therefore

$$a_n - a_{n-1} = 3 - \sqrt{2} = \frac{7}{3 + \sqrt{2}} = \frac{7}{a_n + a_{n-1} + 2(n-1)}.$$

Now,

$$\begin{aligned}(a_k - a_{k-1})(a_k + a_{k-1} - 2(k-1)) &= \left(2^{2^{k-n}} + 1 - 2^{2^{k-1-n}}\right)\left(2^{2^{k-n}} + 1 + 2^{2^{k-1-n}}\right) \\ &= 2^{2^{k+1-n}} + 2 \cdot 2^{2^{k-n}} + 1 - 2^{2^{k-n}} \\ &= 2^{2^{k+1-n}} + k + 1 + 2^{2^{k-n}} + k - 2k \\ &= a_{k+1} + a_k - 2k,\end{aligned}$$

so

$$a_k - a_{k-1} = \frac{a_{k+1} + a_k - 2k}{a_k + a_{k-1} - 2(k-1)}.$$

Thus,

$$\begin{aligned}(a_{n-1} - a_{n-2})(a_n - a_{n-1}) &= \frac{7}{a_n + a_{n-1} + 2(n-1)} \cdot \frac{a_n + a_{n-1} + 2(n-1)}{a_{n-1} + a_{n-2} + 2(n-2)} \\ &= \frac{7}{a_{n-1} + a_{n-2} + 2(n-2)}\end{aligned}$$

$$
\begin{aligned}
(a_{n-2}-a_{n-3})(a_{n-1}-a_{n-2})(a_n-a_{n-1}) &= \frac{7}{a_{n-1}+a_{n-2}+2(n-2)} \\
&\quad\cdot \frac{a_{n-1}+a_{n-2}+2(n-2)}{a_{n-2}+a_{n-3}+2(n-3)} \\
&= \frac{7}{a_{n-2}+a_{n-3}+2(n-3)},
\end{aligned}
$$

and, continuing in this fashion, we find

$$(a_1-a_0)\cdots(a_n-a_{n-1}) = \frac{7}{a_1+a_0}.$$

Third solution by Daniel Campos Salas, Costa Rica

Let $P_n=(a_1-a_0)(a_2-a_1)...(a_n-a_{n-1})$. Note that for $k=1,...,n$, we have that

$$a_k-a_{k-1}=2^{2^{k-n}}-2^{2^{k-1-n}}+1=2^{2\cdot 2^{k-1-n}}-2^{2^{k-1-n}}+1=\frac{2^{3\cdot(2^{k-1-n})}+1}{2^{2^{k-1-n}}+1}.$$

In addition, we have that

$$\prod_{k=1}^{n}(2^{3\cdot(2^{k-1-n})}+1)=\prod_{k=1}^{n}\frac{2^{3\cdot(2^{k-n})}-1}{2^{3\cdot(2^{k-1-n})}-1}=\frac{7}{2^{3\cdot 2^{-n}}-1},$$

and

$$\prod_{k=1}^{n}(2^{2^{k-1-n}}+1)=\prod_{k=1}^{n}\frac{2^{2^{k-n}}-1}{2^{2^{k-1-n}}-1}=\frac{1}{2^{2^{-n}}-1}.$$

From these results we conclude that

$$P_n=\prod_{k=1}^{n}\frac{2^{3\cdot(2^{k-1-n})}+1}{2^{2^{k-1-n}}+1}=\frac{7\cdot(2^{2^{-n}}-1)}{2^{3\cdot 2^{-n}}-1}=\frac{7}{2^{2\cdot 2^{-n}}+2^{2^{-n}}+1}=\frac{7}{a_1+a_0},$$

as we wanted to prove.

Fourth solution by Athanasios Magkos, Kozani, Greece

For convenience set $x=\sqrt[2^n]{2}$, which implies $x^{2^n}=2$. Then we have $a_k=x^{2^k}+k$,hence

$$(a_1+a_0)(a_1-a_0)(a_2-a_1)...(a_n-a_{n-1})=$$

$$(x^2+x+1)(x^2-x+1)(x^4-x^2+1)...(x^{2^n}-x^{2^{n-1}}+1).$$

Observe now that $(x^2+x+1)(x^2-x+1) = x^4+x^2+1$, et.c. Then the above product is equal to $x^{2^{n+1}}+x^{2^n}+1 = \left(x^{2^n}\right)^2+x^{2^n}+1 = 2^2+2+1 = 7$ and the result follows.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Michel Bataille, France;Vicente Vicario Garcia, Huelva, Spain.

Senior problems

S85. Find the least real number r such that for each triangle with sidelength a, b, c,

$$\frac{\max(a,b,c)}{\sqrt[3]{a^3+b^3+c^3+3abc}} < r.$$

Proposed by Titu Andreescu, University of Texas at Dallas

First solution by Nguyen Tho Tung, Hanoi, Vietnam

If we let $a = b = n$ and $c = 1$ then

$$r > \frac{n}{\sqrt[3]{2n^3+3n^2+1}}.$$

But

$$\lim_{n\to\infty} \frac{n}{\sqrt[3]{2n^3+3n^2+1}} = \frac{1}{\sqrt[3]{2}}$$

thus

$$r \geq \frac{1}{\sqrt[3]{2}}.$$

We will prove that it satisfies the condition:

$$\frac{1}{\sqrt[3]{2}} > \frac{\max\{a,b,c\}}{\sqrt[3]{a^3+b^3+c^3+3abc}}.$$

Assume that $a = \max\{a, b, c\}$ then the inequality is equivalent to

$$b^3 + c^3 - a^3 + 3abc > 0 \Leftrightarrow (b + c - a)((b + a)^2 + (b - c)^2 + (c + a)^2) > 0.$$

This is true since $b+c-a > 0$. This concludes that the least number that satifies the condition is $\frac{1}{\sqrt[3]{2}}$.

Second solution by Perfetti Paolo, Dipartimento di matematica Universita degli studi di Tor Vergata, Italy

The case $r < 1$ is trivial. By the symmetry of the inequality and its homogeneity we set $a \geq b \geq c$, $a+b+c = 1$. Moreover we know that $(1/3, 1/3, 1/3) \prec (a, b, c) \prec (1/2, 1/2, 0)$ that is

$$0 \leq c \leq \frac{1}{3}, \quad \frac{1}{3} \leq a \leq \frac{1}{2}, \quad \frac{2}{3} \leq a + b \leq 1, \quad a + b + c = 1$$

We look for the minimum of the function $f(a, b, c) = a^3(r^3 - 1) + r^3b^3 + r^3c^3 + 3abc$, with $a, b, c \geq 0$ and $a + b + c = 1$. So we define the function $F(a, b, c, \lambda) = f(a, b, c) - \lambda(a + b + c - 1)$ and set its gradient equal to zero.

$$1)\ 3a^2(r^3 - 1) + 3bc - \lambda = 0$$

$$2)\ 3b^2r^3 + 3ac - \lambda = 0$$
$$3)\ 3c^2r^3 + 3ab - \lambda = 0$$
$$4)\ a + b + c = 1$$

The difference between 2) multiplied by b and 3) multiplied by c yields $b = c$, (condition (i)) or $\lambda = 3r^3(b^2 + c^2 + bc)$ (condition (ii)).

(i): $b = c$ in 1) yields $3a^3(r^3 - 1) - 3b^3r^3 = \lambda(a - b)$ which is impossible, unless $a = b = 0$, because $a \geq b$ and $3a^3(r^3 - 1) - 3b^3r^3 < 0$ by $r < 1$.

(ii): $b \neq c$ and $\lambda = 3r^3(b^2 + c^2 + bc)$. Using this value of λ in 2) and 3) we get $ac = ab$ whence $b = c$ (since $a \neq 0$), contradiction.

The conclusion is that there does not exist a critical point of $F(a, b, c, \lambda)$ namely a constrained critical point of $f(a, b, c)$. The minimum of f, which exists by the Weierstrass theorem, belongs to the boundary of set where $f(a, b, c)$ is defined.

$c = 0$ implies $a = b = 1/2$ whence $f(1/2, 1/2, 0) > 0$ if and only if $r^3 > 1/2$. Here is included also $a + b = 1$.

$c = 1/3$ implies $a = b = c = 1/3$ and then $f(1/3, 1/3, 1/3) = (3r^3 + 2)/27 > 0$ for any $r > 0$. Here are included also $a + b = 2/3$ and $a = 1/3$.

$a = 1/2$ implies $a = b + c = 1/2$

$$f(1/2, b, c) = \frac{3bc}{2}(1 - r^3) + \frac{2r^3 - 1}{8} \geq (2r^3 - 1)/8 > 0 \qquad \forall\ r > 2^{-1/3}$$

$b = c \leq 1/3$. By $a + 2b = 1$ we get $1/4 \leq b \leq 1/3$ and $a^3(1 - r^3) < r^3(3b^2 - 4b^3)$. We already know that $r^3 > 1/2$ so that the last inequality is implied by $\frac{a^3}{2} < \frac{1}{2}(3b^2 - 4b^3)$ or (employ $a = 1 - 2b$) $4b^3 - 9b^2 + 6b > 1$. The polynomial $P(b) = 4b^3 - 9b^2 + 6b$ decreases for $1/2 \leq b \leq 1$ and $P(1/4) = 1$, $P(1/3) = 31/27$. Thus for $r^3 > 1/2$ and $b = c$ the inequality holds true.

$a = b$. The inequality $f(a, b, c) > 0$ becomes $a^2(1 - 2r^3) < r^3(c^3 + 3a^2c)$ which is true for $r^3 > 1/2$.

The proof is complete

Third solution by Roberto Bosch Cabrera, University of Havana, Cuba

Without loss of generality let $\max(a, b, c) = c$ and let $\frac{a}{c} = x$, $\frac{b}{c} = y$ so we have

$$\frac{\max(a, b, c)}{\sqrt[3]{a^3 + b^3 + c^3 + 3abc}} = \frac{1}{\sqrt[3]{x^3 + y^3 + 1 + 3xy}} = f(x, y)$$

where $0 < x, y \leq 1$.

$a + b > c \Rightarrow y > 1 - x \Rightarrow x^3 + y^3 + 1 + 3xy > x^3 + (1-x)^3 + 1 + 3x(1-x) = 2$
$\Rightarrow f(x, y) < \frac{1}{\sqrt[3]{2}}$.

Let $K = \left\{(x, y) \in \mathbb{R}^2 : \ 0 < x, y \leq 1, \ x + y \geq 1\right\}$.

$f(x, y)$ is continuos on the compact K so exist $\max f(x, y)$ by Weierstrass's Theorem, to find this just we need consider when $x + y = 1$, in this case $f(x, y) = f(x, 1 - x) = \frac{1}{\sqrt[3]{2}}$, thus

$$\max_K f(x, y) = \frac{1}{\sqrt[3]{2}}$$

and finally

$$r = \sup_T f(x, y) = \frac{1}{\sqrt[3]{2}}$$

where $T = \left\{(x, y) \in \mathbb{R}^2 : \ 0 < x, y \leq 1, \ x + y > 1\right\}$.

Also solved by Vicente Vicario Garcia, Huelva, Spain; Michel Bataille, France.

S86. An equilateral triangle is dissected into n^2 equilateral triangles of side 1. How many regular hexagons appear?

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Consider unit vectors $\vec{u}$ and $\vec{v}$ along two sides of the triangle, designate by $(0,0)$ the vertex where these two sides meet, and designate each vertex of the equilateral triangles of side 1 by (i,j), such that the vector between $(0,0)$ and the designated vertex is $i\vec{u}+j\vec{v}$. All vertices inside the large original triangle or on its edges are then described by $i,j \geq 0$ and $i+j \leq n$. Assume that an hexagon of side k may have (i,j) as its center. Then, points $(i,j-k)$, $(i+k,j-k)$, $(i+k,j)$, $(i,j+k)$, $(i-k,j+k)$, $(i-k,j)$ must be inside the original triangle or on its edges, or $k \leq \min(i,j,n-i-j)$. This defines the vertices of unit triangles present inside an equilateral triangle of side $n-3k$, and there are therefore $1+2+...+(n+1-3k) = \frac{(n+1-3k)(n+2-3k)}{2}$ of them. Note that, by convexity, if the previous condition is met, then the hexagon is formed by some of the n^2 equilateral triangles of side 1, and this condition is necessary and sufficient. Note also that $k \leq \lfloor \frac{n}{3} \rfloor$, since otherwise $i,j,n-i-j$, whose sum is n, are each larger than $\frac{n}{3}$, absurd. Call $m = \lfloor \frac{n}{3} \rfloor$, and $r \in \{0,1,2\}$ such that $n = 3m+r$. The total number of hexagons is then

$$\sum_{k=1}^{m} \frac{(3m+r+1-3k)(3m+r+2-3k)}{2} =$$

$$= \frac{m(3m+r+1)(3m+r+2)}{2} - \frac{3m(6m+2r+3)(m+1)}{4} + \frac{9m(m+1)(2m+1)}{12} =$$

$$= \frac{m(3m^2+3mr+r^2-1)}{2},$$

where we have used the well known result $\sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}$. The total number of hexagons may also be expressed as a function only of n:

$$\frac{n(n^2-3)}{18} \quad \text{if} \quad n \equiv 0 \pmod 3,$$

$$\frac{(n-1)^2(n+2)}{18} \quad \text{if} \quad n \equiv 1 \pmod 3,$$

$$\frac{(n-2)(n+1)^2}{18} \quad \text{if} \quad n \equiv 2 \pmod 3.$$

Second solution by G.R.A.20 Math Problems Group, Roma, Italy

The grid can contain a regular hexagon of side a iff $1 \le a \le \lfloor n/3 \rfloor$. Its upper horizontal side can be placed along the k-th horizontal line, as soon as $k \ge a$. Then the lower horizontal side of such hexagon should be along the $k+2a$-th horizontal line and therefore $k + 2a \le n$. Hence the grid contains this hexagon iff $a \le k \le n - 2a$. Since this can be done in $(k - a + 1)$ ways then the total number of regular hexagons is

$$\begin{aligned} |H| &= \sum_{a=1}^{\lfloor n/3 \rfloor} \sum_{k=a}^{n-2a} (k - a + 1) \\ &= \sum_{a=1}^{\lfloor n/3 \rfloor} \sum_{k'=1}^{n-3a+1} k' \\ &= \frac{1}{2} \sum_{a=1}^{\lfloor n/3 \rfloor} (n - 3a + 1)(n - 3a + 2) \\ &= \left\lfloor \frac{n^3 - 3n + 2}{18} \right\rfloor \end{aligned}$$

Third solution by Raul A. Simon, Chile

When we divide an equilateral trinagle into smaller equilateral triangles(of side 1), we do it by means of parallels to its sides, dividing each side into n equal parts of length 1. Observe that the total number of resultin triangles is

$$\sum_{k=0}^{n} (2k + 1) = n^2. \quad (1)$$

We have then n^2 equilateral triangles of unit side. In order to form a regular hexagon, we need to join six neighboring "unit" triangles; that is, we must take them from two neighboring rows, three from each row. In other words, we must look - along the parallels to one side of the large triangle - for the "centers", that is, for the points where the six neighboring triangles meet. There is one center between the second and the third rows, two between the third and the fourth row, etc. Therefore the number C of centers is

$$C = H(1) = \sum_{k=0}^{n-1} \frac{n(n-1)}{2}. \quad (2)$$

This is also the number of hexagons of side 1. (Observe that the total number of rows is $n + 1$, but, since the first row - haveing only one unit triangle - is not involved in the process, the sum in (2) goes as far as $k = n - 1$.) For the hexagons of side 2 - which involve 4 rows - the involved rows are (2,3,4,5), (3,4,5,6), ... , $(n-2, n-2, n, n+1)$. The first group contains one hexagon, the second two hexagons, the thirds three hexagons, and the last row gives rise to $n-3$ hexagons. The total number of them is then

$$H(2) = \sum_{k=0}^{n-3} k = \frac{(n-2)(n-3)}{2}. \quad (3)$$

We may continue in the same fashion, forming even greater hexagons. The grand total is then

$$\left[\frac{n(n-1)}{2}\right] + \left[\frac{(n-2)(n-3)}{2}\right] + \cdots + \frac{2 \cdot 1}{3} \quad \text{if n is even, or} \quad (4a)$$

$$\left[\frac{n(n-1)}{2}\right] + \left[\frac{(n-2)(n-3)}{2}\right] + \cdots + \frac{3 \cdot 2}{3} \quad \text{if n is odd.} \quad (4b)$$

In general we have

$$H = \frac{1}{2}\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{6} \quad \text{if n is even} \quad (5a)$$

$$H = \frac{1}{2}\sum_{r=2}^{n} r(r+1) = \frac{n(n+1)(n+2)}{6} - 2 \quad \text{if n is odd.} \quad (5b)$$

Also solved by Johan Gunardi, Indonesia

S87. Let ABC be a triangle. The incircle $C(I,r)$ and the excicle $C(I_A, r_a)$ corresponding to the vertex A are tangent to AB at points D and E, respectively. Prove that the lines IE and I_aD intersect on BC if and only if $AB \perp BC$.

Proposed by Ciupan Andrei Laurentiu, Tudor Vianu High School, Romania

First solution by Andrea Munaro, Italy

In the following we'll use homogeneous barycentric coordinates.
Denote as usual by a, b, c the sides of the triangle opposite to vertexes A, B, C, respectively, and by s the semiperimeter of the triangle.
Then $I = (a : b : c)$, $I_a = (-a : b : c)$, $D = (s-b : s-a : 0)$ and $E = (-(s-c) : s : 0)$.
Then the equations of lines IE and I_aD are, respectively

$$(-cs)x + (c^2 - cs)y + (as + bs - bc)z = 0,$$

$$(ca - cs)x + (cs - bc)y + (a^2 + b^2 - as - bs)z = 0.$$

Since the side line BC has equation $x = 0$, then the three lines are concurrent iff

$$\begin{vmatrix} -cs & c^2 - cs & as + bs - bc \\ ca - cs & cs - bc & a^2 + b^2 - as - bs \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$$\Leftrightarrow ca^2 s + c^2 as - c^2 a^2 - cabs = 0$$

$$\Leftrightarrow s(a - b + c) = ca$$

$$\Leftrightarrow a^2 + c^2 = b^2.$$

Second solution by Nguyen Tho Tung, Hanoi, Vietnam

Let $T = AI \cap BC, M = I_aD \cap BC$ and $N = IE \cap BC$. By the Menelaus' theorem for the triangle ABT and points D, M, I_a we have

$$\frac{DA}{DB}\frac{MB}{MT}\frac{I_aT}{I_aA} = 1.$$

In a similar fashion we get

$$\frac{EA}{EB}\frac{NB}{NT}\frac{IT}{IA} = 1.$$

Now $M \equiv N$ if and only if

$$\frac{DA}{DB}\frac{I_aT}{I_aA} = \frac{EA}{EB}\frac{IT}{IA}$$

or the equivalent

$$\frac{\cot\frac{A}{2}}{\cot\frac{B}{2}}\frac{r}{R_a}=\frac{\cot\frac{A}{2}}{\tan\frac{B}{2}}\frac{IT}{I_aT}. \quad (1)$$

Applying the sine law we get that

$$IT=\frac{\sin\frac{B}{2}\cdot BI}{\sin\angle BTI}=\frac{r}{\sin\angle BTI}$$

and

$$TI_a=\frac{\cos\frac{B}{2}\cdot BI_a}{\sin\angle BTI}=\frac{R_a}{\sin\angle BTI}.$$

Thus we can conclude that

$$\frac{IT}{TI_a}=\frac{r}{R_a}. \quad (2)$$

From (1) and (2) we have that $\tan\frac{B}{2}=1$ which means that $\angle ABC=90^\circ$ or that $AB\perp BC$ and we are done.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Vicente Vicario Garcia, Huelva, Spain; Michel Bataille, France.

S88. Let a, b, c, d be non-negative real numbers. Prove that

$$a^2+b^2+c^2+d^2+1+abcd \geq ab+bc+cd+da+ac+bd.$$

Proposed by Alex Anderson, New Trier Township High School, Winnetka, USA

First solution by Arkady Alt, San Jose, California, USA

Let a, b, c, d be non-negative real numbers. Prove that

$a^2+b^2+c^2+d^2+1+abcd \geq ab+ac+ad+bc+bd+cd.$

Solution by Arkady Alt , San Jose ,California, USA.

Since by AM-GM inequality $1+abcd \geq 2\sqrt{abcd}$ then sufficies to prove

(1) $a^2+b^2+c^2+d^2+2\sqrt{abcd} \geq ab+ac+ad+bc+bd+cd.$

Due to homogeneity we assume $abcd=1$. Then inequality (1) becomes

(2) $a^2+b^2+c^2+d^2+2 \geq ab+ac+ad+bc+bd+cd \iff$

Also, due to the symmetry, we assume $d=\min\{a,b,c,d\}$.

Lemma.(Sharp Quadratic Mix Inequality)

For any non-negative a, b, c holds inequality

(3) $a^2+b^2+c^2-ab-bc-ca \geq \sqrt[3]{abc}\left(a+b+c-3\sqrt[3]{abc}\right).$

Proof.

Due to homogeneity assume that $a+b+c=1$ then, denoting

$p := ab+bc+ca, q := abc$, obtain (3) $\iff 1-3p \geq \sqrt[3]{q}\left(1-3\sqrt[3]{q}\right).$

Since $p \leq \dfrac{1+9q}{4} \iff 1 \geq 4p-9q$ (Schur Inequality $\sum_{cyc} a(a-b)(a-c) \geq 0$ in

p-q notation) and $q=abc \leq \left(\dfrac{a+b+c}{3}\right)^3 = \dfrac{1}{27}$ suffices to prove

$1-\dfrac{3(1+9q)}{4} \geq \sqrt[3]{q}\left(1-3\sqrt[3]{q}\right) \iff 1-27q \geq 4\sqrt[3]{q}\left(1-3\sqrt[3]{q}\right)$ for any $q \in \left[0, \dfrac{1}{27}\right]$.

We have $1-27q-4\sqrt[3]{q}+12\sqrt[3]{q^2} = \left(1-3\sqrt[3]{q}\right)\left(1+\sqrt[3]{q}+\sqrt[3]{q^2}\right)-4\sqrt[3]{q}\left(1-3\sqrt[3]{q}\right) =$

$\left(1-3\sqrt[3]{q}\right)\left(\left(1-3\sqrt[3]{q}+\sqrt[3]{q^2}\right)\right) = \left(1-3\sqrt[3]{q}\right)^2+\sqrt[3]{q^2}\left(1-3\sqrt[3]{q}\right) \geq 0.\blacksquare$

Since $d=\min\{a,b,c,d\}$ then $abc \geq d^3 \iff \sqrt[3]{abc} \geq d$ and (3) $\Longrightarrow$

(4) $\quad a^2+b^2+c^2-ab-bc-ca \ge d\left(a+b+c-3\sqrt[3]{abc}\right).$

Since (2) $\iff$ $a^2+b^2+c^2-ab-bc-ca \ge d(a+b+c)-d^2-2$ then, due to inequality (4), suffices to prove

$d\left(a+b+c-3\sqrt[3]{abc}\right) \ge d(a+b+c)-d^2-2 \iff d^2+2 \ge 3d\sqrt[3]{abc}.$

Since $abc=\dfrac{1}{d}$ we have

$d^2+2-3d\sqrt[3]{abc}=d^2+2-3\sqrt[3]{d^2}=\left(d^{\frac{2}{3}}-1\right)^2\left(d^{\frac{2}{3}}+1\right)\ge 0.$

Comment.

Up to notation inequality (2) is Turkevici's Inequality.

Indeed, substitution in (2) $a=x^2, b=y^2, c=z^2, d=t^2,$ where $x,y,z,t\ge 0$ gives us nequality

$x^4+y^4+z^4+t^4+2xyzt \ge x^2y^2+y^2z^2+z^2t^2+t^2x^2+x^2z^2+y^2t^2.$

Second solution by Brian Bradie, Christopher Newport University, USA

Let w, x, y, z be non-negative real numbers. By Turkevici's inequality

$$w^4+x^4+y^4+z^4+2wxyz \ge w^2x^2+x^2y^2+y^2z^2+z^2w^2+w^2y^2+x^2z^2, \quad (3)$$

with equality if and only if $w=x=y=z$ or one variable equal to zero and the other three equal to one another. Because w, x, y, z are non-negative real numbers, we can define the non-negative real numbers a, b, c, d according to

$$a=\sqrt{w}, \quad b=\sqrt{x}, \quad c=\sqrt{y}, \quad d=\sqrt{z}.$$

Substituting into (1) yields

$$a^2+b^2+c^2+d^2+2\sqrt{abcd} \ge ab+bc+cd+da+ac+bd, \qquad (4)$$

with equality if and only if $a=b=c=d$ or one variable equal to zero and the other three equal to one another. Moreover,

$$abcd-2\sqrt{abcd}+1=\left(\sqrt{abcd}-1\right)^2 \ge 0 \qquad (5)$$

with equality if and only if $abcd=1$. Adding (2) and (3) yields

$$a^2+b^2+c^2+d^2+1+abcd \ge ab+bc+cd+da+ac+bd$$

as desired, with equality if and only if $a=b=c=d$ and $abcd=1$; that is, if and only if $a=b=c=d=1$.

An alternate proof proceeds as follows. Because the inequality is symmetric in a, b, c, d, we may assume without loss of generality that $a \geq b \geq c \geq d$. Now

$$\begin{aligned}
&a^2+b^2+c^2+d^2+1+abcd-ab-bc-cd-da-ac-bd \\
&\qquad = (\sqrt{ab}+\sqrt{cd}-c-d)^2+2\sqrt{cd}(\sqrt{c}-\sqrt{d})^2+ \\
&\qquad\qquad (\sqrt{a}-\sqrt{b})^2((\sqrt{a}+\sqrt{b})^2-(c+d))+(\sqrt{abcd}-1)^2 \\
&\qquad \geq 0,
\end{aligned}$$

with equality if and only if $a=b=c=d$ and $abcd=1$; that is, if and only if $a=b=c=d=1$.

Third solution by Ganesh Ajjanagadde, Mysore, India

We are asked to prove that for all $a, b, c, d \geq 0$ the following inequality holds:

$$a^2+b^2+c^2+d^2+1+abcd \geq ab+bc+cd+da+ac+bd \tag{6}$$

By Turkevici's inequality, we know that
$w^4+x^4+y^4+z^4+2wxyz \geq w^2x^2+w^2y^2+w^2z^2+x^2y^2+x^2z^2+y^2z^2$.
Substituting $\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}$ for w, x, y, z respectively in this inequality, we get
$a^2+b^2+c^2+d^2+2\sqrt{abcd} \geq ab+ac+ad+bc+bd+cd$.
Thus in order to prove the given inequality, it suffices to show that
$a^2+b^2+c^2+d^2+1+abcd \geq a^2+b^2+c^2+d^2+2\sqrt{abcd}$, or, $1+abcd \geq 2\sqrt{abcd}$, which is clearly true by the AM-GM inequality.

Fourth solution by Perfetti Paolo, Dipartimento di Matematica Tor Vergata Roma, Italy

Proof j1) If $d=0$ the inequality becomes

$$a^2+b^2+c^2-ab-bc-ac+1 \geq 0$$

which holds true since $(x^2+y^2)/2 \geq |xy|$. By continuity the inequality is true also if $d \neq 0$ but sufficiently small, say $0 < d \leq \delta$, because

$$a^2+b^2+c^2-ab-bc-ac \geq 0$$

j2) If $a \to +\infty$ and $b, c, d \leq R$ the inequality (*) holds true because the l.h.s. goes to infinity quadratically while the r.h.s. at most linearly.

j3) By

$$\frac{a^2+b^2}{2}+c^2+d^2+1+abcd \geq bc+cd+da+ac+bd$$

it follows that if $a \to +\infty$, $b \to +\infty$ and $c, d \leq R$, (*) is true. $(x^2+y^2)/2 \geq |xy|$ has been used again

j4) Using three times $(x^2+y^2)/2 \geq |xy|$ we have

$$d^2 + 1 + abcd \geq cd + da + bd$$

and it follows by $a \to +\infty$, $b \to +\infty$, $c \to +\infty$, since

$$\frac{abc}{a+b+c} \geq \frac{abc}{3\max\{a,b,c\}} \to +\infty$$

j5) $a \to +\infty$, $b \to +\infty$, $c \to +\infty$, $d \to +\infty$, is like j4).

Given j1)–j5) we consider the function

$$f(a,b,c,d) = a^2 + b^2 + c^2 + d^2 + 1 + abcd - ab - bc - cd - da - ac - bd$$

on the compact set $D = \{\delta \leq a, b, c, d \leq R\}$. On ∂D the inequality holds true so we look for the minimum of f, which exists, at an internal point and the gradient of f yields

$$\begin{aligned}
&1) \quad 2a + bcd - b - c - d = 0,\\
&2) \quad 2b + acd - a - c - d = 0,\\
&3) \quad 2c + abd - a - b - d = 0,\\
&2d + abc - a - b - c = 0.
\end{aligned}$$

The difference between 1) and 2) yields $a = b$ (condition (i)) or $cd = 3$ (condition (ii)).

(i). Setting $a = b$ in 2) and 3) gives $a = c$ (condition (i.1)) or $ad = 3$ (condition (i.2)).

(i.1) $a = b = c$ in 2) gives $(a^2-1)d = 0$ and then $a = b = c = 1$. By 4) we get $2d = 2$ namely $d = 1$.

(i.2) $a = b \neq c$, $ad = 3$ hence $bd = 3 \neq bc$. The system between 1) and 4) is

$$2c + a - d = 0, \qquad 2d + a^2c - 2a - c = 0$$

or $3c + a^2c = 0$ which does not have any solution.

(ii). Inserting $cd = 3$ in 1) or 2) yields $a + b = (c+d)/2$. Taking into account of it, the system between 3) and 4) becomes

$$2c^2 + 3ab - cd - c(c+d)/2 = 0, \qquad 2d^2 + 3ab - cd - d(c+d)/2 = 0$$

that is $c = d$ but the case when two coordinates are equal has been already considered.

It follows that the only critical point is $a = b = c = d = 1$ and the relative hessian has the entries all zero except on the principal diagonal where they are equal to 2 (a minimum). The proof is completed.

Also solved by Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Manh Dung Nguyen, Vietnam; Nguyen Tho Tung, Hanoi, Vietnam; Vicente Vicario Garcia, Huelva, Spain.

S89. Let ABC be an acute triangle. Prove that the following conditions are equivalent:

(i) For any point $M \in (AB)$ and any point $N \in (AC)$ one may construct a triangle with sides CM, BN, MN.

(ii) $AB = AC$.

Proposed by Mircea Becheanu, Bucharest, Romania

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

(i) $\Rightarrow$ (ii) Assume wlog that $AC = b > c = AB$, but that a triangle may be constructed with sides BN, CM, MN. Consider M, N such that $\frac{AM}{AB} = \frac{AN}{AC} = x$. Using the similarity between triangles AMN and ABC, $MN = xa$. Using the triangular inequality, $BN < c + xb$, $CM > b - xc$. Therefore, it must be true for all $x \in (0, 1)$ that

$$b - xc = CM < BN + MN = c + x(b + a).$$

But this is not true when $0 < x < \frac{b-c}{a+b+c}$, contradiction.

(ii) $\Rightarrow$ (i) Assume wlog $AB = AC = 1$, then $BC^2 = a^2 = 2 - 2\cos A$, where obviously $0 < a^2 < 2$ because ABC is acute. Call $AM = x$, $AN = y$, where obviously $0 < x < 1$ and $0 < y < 1$, and assume wlog by symmetry in the problem that $CM \geq BN$. Using the cosine law yields $BN^2 = 1 + y^2 - 2y\cos A = (1-y)^2 + ya^2$, and similarly $CM^2 = (1-x)^2 + xa^2$, $MN^2 = (x-y)^2 + xya^2$. Trivially,

$$BN^2 + CM^2 - MN^2 = (a^2 + 2)(1 - x)(1 - y) + a^2 > 0,$$

so if the triangle cannot be defined, then $CM \geq BN + MN$, or equivalently, $CM^2 - BN^2 - MN^2 \geq 2BN \cdot MN$. Now, using the AM-GM inequality,

$$BN^2 \cdot MN^2 = xy^2a^4 + a^2\left(xy(1-y)^2 + y(x-y)^2\right) + (1-y)^2(x-y)^2 \geq$$

$$\geq xy^2a^4 + 2a^2y(1-y)|x-y|\sqrt{x} + (1-y)^2(x-y)^2 = \left(y\sqrt{x}a^2 + (1-y)|x-y|\right)^2,$$

leading to $2BN \cdot MN \geq 2a^2y\sqrt{x} + 2(1-y)|x-y|$. Now, this means that if the triangle cannot be constructed, there are values of x, y, a such that

$$CM^2 - BN^2 - MN^2 = a^2(x-y-xy) - 2(x-y)(1-y) \geq 2a^2y\sqrt{x} + 2(1-y)|x-y|.$$

Note however that the second term on the LHS of this inequality can be at most equal to the second term of the RHS, with equality iff $y \geq x$. Note also that the first term of the LHS will be no less than the first term in the RHS iff

$x - y - xy \geq 2y\sqrt{x}$, or iff $y \leq \frac{x}{(1+\sqrt{x})^2} < x$. Both conditions are then mutually exclusive, and $CM^2 - BN^2 - MN^2 < 2BN \cdot MN$, and a triangle may always be constructed with sides CM, BN, MN.

Second solution by Daniel Campos Salas, Costa Rica

Suppose first that (i) holds, and assume $AB \neq AC$. Let b, c be the sidelengths of AB, AC, and suppose without loss of generality that $c > b$. Take $AM = AN = \epsilon$, where $\epsilon = \min\left\{\dfrac{c-b}{2}, \dfrac{b+c}{14}\right\}$. We'll prove that $BM > CN + MN$, which is equivalent

$$BM^2 - CN^2 \geq MN(BM + CN).$$

From the triangle inequality we have that

$$b + c + 2\epsilon = (AB + AM) + (AC + AN) > BM + CN,$$

so it's enough to prove that

$$BM^2 - CN^2 \geq MN(b + c + 2\epsilon).$$

From the law of cosines in triangle ABM and ACN we have that

$$BM^2 - CN^2 = c^2 - 2c\epsilon\cos A - b^2 + 2b\epsilon\cos A = (c-b)(b + c - 2\epsilon\cos A),$$

and

$$MN = 2\epsilon\sin\frac{A}{2} < 2\epsilon \cdot \frac{\sqrt{2}}{2} \leq (c-b) \cdot \frac{\sqrt{2}}{2} \leq \frac{3(c-b)}{4}.$$

Then, it's enough to prove that

$$b + c - 2\epsilon\cos A \geq \frac{3}{4}(b + c + 2\epsilon),$$

or equivalently,

$$b + c \geq 2\epsilon(3 + 4\cos A).$$

This last inequality is satisfied by every $\epsilon \leq \dfrac{b+c}{14}$, and we're done. We conclude that $AB = AC$, as we wanted to prove.

Now suppose that $AB = AC$. Note that $\angle BCN < \angle BCA = \angle ABC$, so $BN < CN$. From the triangle inequality we have that $|BM - MN| < BN$, so we conclude that $|BM - MN| < CN$. This implies $BM + CN > MN$ and $MN + CN > BM$. The inequality $MN + BM > CN$ follows analogously, and this completes the proof.

S90. Prove that

$$\sum_{i=0}^{3n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{n-1+3i-10j}{n-1}=\frac{10^n+2}{3}.$$

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

Solution by G.R.A.20 Math Problems Group, Roma, Italy

Let

$$f(z)=\frac{1}{(1-z)^n}=\sum_{k=0}^{\infty}\binom{n-1+k}{n-1}z^k$$

and

$$g(z)=(1-z^{10})^n=\sum_{j=0}^{n}(-1)^j\binom{n}{j}z^{10j}.$$

Then

$$\sum_{i=0}^{3n}[z^{3i}]f(z)g(z)=\sum_{i=0}^{3n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{n-1+3i-10j}{n-1}[3i-10j\geq 0].$$

On the other hand, since

$$h(z)=\left(\frac{1-z^{10}}{1-z}\right)^n=(1+z+\cdots+z^9)^n=f(z)g(z)$$

then letting $\omega=e^{2\pi i/3}$ we have that

$$\sum_{i=0}^{3n}[z^{3i}]h(z)=\frac{1}{3}(h(1)+h(\omega)+h(\omega^2))=\frac{10^n+2}{3}.$$

Note that we have to consider only terms of the sum such that $3i-10j\geq 0$ otherwise it does not hold!

Undergraduate problems

U85. Evaluate

$$\text{a)} \sum_{k=1}^{\infty} \frac{1}{1^3+2^3+\cdots+k^3} \qquad\qquad \text{b)} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{1^3+2^3+\cdots+k^3}$$

Proposed by Brian Bradie, Christopher Newport University, USA

First solution by Jose Hernandez Santiago, Oaxaca, Mexico

a. From

$$\frac{1}{k^2(k+1)^2} = \left(\frac{2}{k+1}-\frac{2}{k}\right)+\frac{1}{k^2}+\frac{1}{(k+1)^2}$$

and the fact that $\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k}\right)=-1$ and $\sum_{k=1}^{\infty}\frac{1}{k^2}$ we conclude that

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{1}{1^3+2^3+\ldots+k^3} &= 2^2\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)^2}\\
&= 2^2\left\{2\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k}\right)+\sum_{k=1}^{\infty}\frac{1}{k^2}+\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}\right\}\\
&= 2^2\left(2(-1)+\frac{\pi^2}{6}+\left(\frac{\pi^2}{6}-1\right)\right)\\
&= 2^2\left(\frac{\pi^2}{3}-3\right)\\
&= \frac{4(\pi^2-9)}{3}.
\end{aligned}$$

b. The series development for $\ln 2$ is crucial here. Indeed,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{1^3+2^3+\ldots+k^3} &= 2^2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2(k+1)^2} \\
&= 2^2 \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{2}{k+1} - \frac{2}{k} + \frac{1}{k^2} + \frac{1}{(k+1)^2} \right) \\
&= 2^2 \cdot 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k+1} - 2^2 \cdot 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \\
&+ 2^2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} + 2^2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k+1)^2} \\
&= 2^2 \cdot 2(1-\ln 2) - 2^2 \cdot 2(\ln 2) \\
&+ 2^2 \left(\frac{\pi^2}{12} \right) + 2^2 \left(1 - \frac{\pi^2}{12} \right) \\
&= 4(3 - 4\ln 2).
\end{aligned}$$

Second solution by Arkady Alt, San Jose, California, USA

a) Since $1^3+2^3+...+k^3 = \dfrac{k^2(k+1)^2}{4}$ then $\dfrac{1}{1^3+2^3+...+k^3} = \dfrac{4}{k^2(k+1)^2} =$
$4\left(\dfrac{1}{k} - \dfrac{1}{k+1}\right)^2 = 4\left(\dfrac{1}{k^2} + \dfrac{1}{(k+1)^2} - \dfrac{2}{k(k+1)}\right) = \dfrac{4}{k^2} + \dfrac{4}{(k+1)^2} -$
$8\left(\dfrac{1}{k} - \dfrac{1}{k+1}\right)$. Hence, $\sum_{k=1}^{n} \dfrac{1}{1^3+2^3+...+k^3} = 4\left(\sum_{k=1}^{n} \dfrac{1}{k^2} + \sum_{k=1}^{n} \dfrac{1}{(k+1)^2}\right) -$
$8\sum_{k=1}^{n}\left(\dfrac{1}{k} - \dfrac{1}{k+1}\right) = 8\sum_{k=1}^{n} \dfrac{1}{k^2} - 4 - \dfrac{4}{(n+1)^2} + 8\left(1 - \dfrac{1}{n+1}\right) = 8\sum_{k=1}^{n} \dfrac{1}{k^2} +$
$4-$
$\dfrac{4}{(n+1)^2} - \dfrac{8}{n+1}$ and $\sum_{k=1}^{n} \dfrac{(-1)^{k-1}}{1^3+2^3+...+k^3} = 4\sum_{k=1}^{n}\left(\dfrac{(-1)^{k-1}}{k^2} - \dfrac{(-1)^k}{(k+1)^2}\right) -$
$8\left(\sum_{k=1}^{n} \dfrac{(-1)^{k-1}}{k} + \sum_{k=1}^{n} \dfrac{(-1)^k}{k+1}\right) = 4 - \dfrac{4(-1)^n}{(n+1)^2} - 8\left(\sum_{k=1}^{n} \dfrac{2(-1)^{k-1}}{k} - 1 + \dfrac{(-1)^n}{n+1}\right) =$
$12 - 16\sum_{k=1}^{n} \dfrac{(-1)^{k-1}}{k} - \dfrac{4(-1)^n}{(n+1)^2} - \dfrac{8(-1)^n}{n+1}$.

Since $\lim_{n\to\infty} \sum_{k=1}^{n} \dfrac{1}{k^2} = \sum_{k=1}^{\infty} \dfrac{1}{k^2} = \dfrac{\pi^2}{6}$, $\lim_{n\to\infty} \sum_{k=1}^{n} \dfrac{(-1)^{k-1}}{k} = \sum_{k=1}^{\infty} \dfrac{(-1)^{k-1}}{k} =$
$\ln 2$,

$\lim_{n\to\infty}\left(\frac{4}{(n+1)^2}-\frac{8}{n+1}\right) = 0, \ \lim_{n\to\infty}\left(\frac{4\,(-1)^n}{(n+1)^2}+\frac{8\,(-1)^n}{n+1}\right) = 0$ then
$\sum_{k=1}^{\infty}\frac{1}{1^3+2^3+...+k^3}=\lim_{n\to\infty}\left(8\sum_{k=1}^{n}\frac{1}{k^2}+4-\frac{4}{(n+1)^2}-\frac{8}{n+1}\right)=\frac{4\pi^2}{3}+$
4

and
$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{1^3+2^3+...+k^3}=\lim_{n\to\infty}\left(12-16\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}-\frac{4\,(-1)^n}{(n+1)^2}-\frac{8\,(-1)^n}{n+1}\right)=$
$12-16\ln 2.$

Third solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

a) $(4\pi^2-36)/3,$ b) $8-16\ln 2.$

Proof It is well known that $\sum_{j=1}^{k} j^3 = \frac{k^2(k+1)^2}{4}$, hence

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{1+2^3+\dots k^3} &= \sum_{k=1}^{n}\frac{4}{k^2(k+1)^2}\\ &= 4\sum_{k=1}^{n}\left(\frac{1}{k^2}+\frac{1}{(1+k)^2}+\frac{2}{1+k}-\frac{2}{k}\right)\\ &= 4\sum_{k=1}^{n}\left(2\frac{1}{k^2}-1\right)+8(\frac{2}{1+n}-1).\end{aligned}$$

The limit is equal to $(4\pi^2-36)/3$ since $\sum_{k=1}^{\infty}k^{-2}=\pi^2/6$

b)

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+k)^2}\\ &\qquad=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}+\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k^2}+1\\ &\qquad=1.\end{aligned}$$

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{1+k}-\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}$$

$$=\sum_{r=2}^{n+1}\frac{(-1)^{r-2}}{r}-\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}$$

$$=1-2\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}+\frac{(-1)^{n-1}}{n+1}\to 1-2\ln 2.$$

Combining the two sum and multiplying by 4 we get $8-16\ln 2$

The proof is completed.

Fourth solution by Roberto Bosch Cabrera, Cuba

a)

$$\begin{aligned}
S &= \sum_{k=1}^{\infty}\frac{1}{1^3+2^3+...+k^3}=\sum_{k=1}^{\infty}\frac{4}{k^2(k+1)^2}=4\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)^2\\
S &= 4\sum_{k=1}^{\infty}\frac{1}{k^2}-8\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)+4\left(-1+\sum_{k=1}^{\infty}\frac{1}{k^2}\right)\\
S &= 8\sum_{k=1}^{\infty}\frac{1}{k^2}-8\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)-4\\
S &= 8\cdot\frac{\pi^2}{6}-8\cdot 1-4=\frac{4}{3}(\pi^2-9)
\end{aligned}$$

b)

$$\begin{aligned}
S &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{1^3+2^3+...+k^3}=\sum_{k=1}^{\infty}\frac{4(-1)^{k-1}}{k^2(k+1)^2}=4\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)^2\\
S &= 4\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}-8\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)+4\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k+1)^2}\\
S &= 4\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}-8\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}+8\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k+1}+4\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k+1)^2}\\
S &= \frac{\pi^2}{3}-8\ln 2+8(1-\ln 2)+4-\frac{\pi^2}{3}=12-16\ln 2
\end{aligned}$$

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; G.R.A.20 Math Problems Group, Roma, Italy; Michel Bataille, France; Raul A. Simon, Chile; Vicente Vicario Garcia, Huelva, Spain.

U86. Determine all non-degenerate triangles with angles α, β, γ in radians and sides $\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma}$.

Proposed by Daniel Campos Salas, Costa Rica

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

The trivial solution is obviously $\alpha = \beta = \gamma = \frac{\pi}{3}$. Let us search non-trivial solutions, by considering that the sine law sets condition $\frac{\sin\alpha}{\sqrt{\alpha}} = \frac{\sin\beta}{\sqrt{\beta}} = \frac{\sin\gamma}{\sqrt{\gamma}}$. Considering the first derivative of function $f(x) = \frac{\sin x}{\sqrt{x}}$, we find $f'(x) = \frac{2x\cos x - \sin x}{2x\sqrt{x}}$, which is zero iff $\tan x = 2x$. Now, $\tan x$ is negative for $x \in (\frac{\pi}{2}, \pi)$, and $\tan x - 2x$ has second derivative $\frac{2\sin x}{\cos^3 x} > 0$ for $x \in (0, \frac{\pi}{2})$, so since $\tan x = 2x$ for $x = 0$, only one additional value of $x \in (0, \pi)$ such that $f'(x) = 0$ exists, and therefore each value of $f(x)$ occurs at most twice in $(0, \pi)$. Since the triangle has three angles, at least two of them must be equal, and the triangle is isosceles. Wlog, assume $\alpha = \pi - 2\beta = \pi - 2\gamma$, yielding condition $2\sqrt{\beta}\sin\beta\cos\beta = \sqrt{\alpha}\sin\beta$, or $4\beta\cos^2\beta = \alpha = \pi - 2\beta$. We need then to find solutions to equation $2\beta(1+2\cos^2\beta) = \pi$ for $\beta \in (0, \frac{\pi}{2})$. We may easily find that three different solutions are $\beta = \frac{\pi}{4}$, $\beta = \frac{\pi}{3}$, $\beta = \frac{\pi}{2}$, the second solution yielding the trivial case, and the last solution resulting in a degenerate triangle with two infinite parallel sides perpendicular to a finite side. Therefore, we find the additional solution $\alpha = \frac{\pi}{2}$, $\beta = \gamma = \frac{\pi}{4}$. We shall now show that no other solutions exist. In order to do that, consider the second derivative of $g(x) = 2x(1+2\cos^2 x) = 2x(2+\cos(2x))$, which is $g''(x) = -4\sin(2x) - 8\cos(2x)$. Obviously, $g''(x) = 0$ iff $\tan(2x) = -\frac{1}{2}$, which occurs only once for $x \in [0, \frac{\pi}{2}]$, and $g''(x)$ changes sign exactly once in this interval, $g'(x)$ may change signs at most twice in this interval, and $g(x) = k$ may have at most three solutions in $[0, \frac{\pi}{2}]$ for any real constant k. Since we have already found these three solutions, there are no others. The solutions are then equilateral triangle $\alpha = \beta = \gamma = \frac{\pi}{3}$ and isosceles triangle $\alpha = \frac{\pi}{2}$, $\beta = \gamma = \frac{\pi}{4}$ and its cyclic permutations.

U87. Let $f : (0, \infty) \to (0, \infty)$ be an unbounded function and let β be a positive real number. If for every $\alpha > 0$ we have

$$\lim_{x \to 0^+} (f(x) - \alpha f^{\beta}(\alpha x)) = 0,$$

prove that $\lim_{x \to 0^+} f(x) = 0$.

Proposed by Dorin Andrica, Babes-Bolyai University and Mihai Piticari, Campulung Moldovenesc, Romania

No solution has been yet recieved.

U88. Consider the sequence

$$a_n = \int_1^n \frac{dx}{(1+x^2)^n}.$$

Evaluate $\lim_{n\to\infty} n \cdot 2^n \cdot a_n$.

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College, Romania

First solution by Arin Chaudhuri, Cary, USA

We prove the stronger statement that: $\lim_{n\to\infty} \int_1^\infty \frac{n2^n}{(1+x^2)^n} dx = 1$

Let $b_n = \int_1^\infty \frac{n2^n}{(1+x^2)^n} dx$ and $w_n = 2^n n a_n = \int_1^n \frac{n2^n}{(1+x^2)^n} dx$

We will prove $\lim_{n\to\infty} b_n = 1$ and $\lim_{n\to\infty}(b_n - w_n) = 0$ and hence $\lim_{n\to\infty} w_n = 1$

Note $b_n = \int_1^\infty \frac{n}{(\frac{1+x^2}{2})^n} dx$

Fix an $n \geq 2$, make the substitution $y = n(x-1)$, i.e., $x = 1 + \frac{y}{n}$. When $x = 1$ then $y = 0$ and as $x \to \infty$: $y \to \infty$, also $ndx = dy$. Hence we have

$$b_n = \int_0^\infty \frac{dy}{\left(\frac{1+(1+y/n)^2}{2}\right)^n} = \int_0^\infty \frac{dy}{\left(1 + \frac{y}{n} + \frac{y^2}{2n^2}\right)^n}$$

Now

$$\exp(\frac{y}{n}) = 1 + \frac{y}{n} + \frac{1}{2!}(\frac{y}{n})^2 + \frac{1}{3!}(\frac{y}{n})^3 + \cdots$$

Hence for $y \geqq 0$

$$\exp(\frac{y}{n}) \geq 1 + \frac{y}{n} + \frac{y^2}{2n^2}$$

and hence

$$\exp(y) \geq (1 + \frac{y}{n} + \frac{y^2}{2n^2})^n$$

Also

$$(1 + \frac{y}{n} + \frac{y^2}{2n^2})^n \geq (1 + \frac{y}{n})^n$$

hence

$$(1 + \frac{y}{n})^n \leq (1 + \frac{y}{n} + \frac{y^2}{2n^2})^n \leq \exp(y)$$

or

$$\frac{1}{\exp(y)} \leq \frac{1}{(1 + \frac{y}{n} + \frac{y^2}{2n^2})^n} \leq \frac{1}{(1 + \frac{y}{n})^n}$$

Integrating all sides,

$$\int_0^\infty \frac{dy}{\exp(y)} \leq \int_0^\infty \frac{dy}{\left(1+\frac{y}{n}+\frac{y^2}{2n^2}\right)^n} \leq \int_0^\infty \frac{dy}{(1+\frac{y}{n})^n}$$

or

$$[-\exp(-y)]_0^\infty \leq \int_0^\infty \frac{dy}{\left(1+\frac{y}{n}+\frac{y^2}{2n}\right)^n} \leq [n\frac{(1+\frac{y}{n})^{-n+1}}{-n+1}]_0^\infty$$

or

$$1 \leq b_n \leq \frac{n}{n-1}$$

hence $\lim_{n\to\infty} b_n = 1$.

Now

$$0 \leq b_n - w_n = \int_n^\infty \frac{n2^n}{(1+x^2)^n}dx \leq \int_n^\infty \frac{n2^n}{x^{2n}}dx = \frac{n2^n}{(2n-1)n^{2n-1}} = \frac{2}{(2n-1)\left(\frac{n^2}{2}\right)^{n-1}}$$

Hence, $\lim_{n\to\infty}(b_n - w_n) = 0$, hence $\lim_{n\to\infty} w_n = \lim_{n\to\infty} 2^n n a_n = 1$

Second solution by Brian Bradie, Christopher Newport University, USA

Performing integration by parts twice, first with

$$u = \frac{1}{2x}, \quad dv = \frac{2x}{(1+x^2)^n},$$

and then with

$$u = \frac{1}{2x^3}, \quad dv = \frac{2x}{(1+x^2)^{n-1}},$$

yields

$$\begin{aligned}
\int \frac{dx}{(1+x^2)^n} &= -\frac{1}{2(n-1)}\frac{1}{x(1+x^2)^{n-1}} - \frac{1}{2(n-1)}\int \frac{dx}{x^2(1+x^2)^{n-1}} \\
&= -\frac{1}{2(n-1)}\frac{1}{x(1+x^2)^{n-1}} + \frac{1}{4(n-1)(n-2)}\frac{1}{x^3(1+x^2)^{n-2}} + \\
&\qquad \frac{3}{4(n-1)(n-2)}\int \frac{dx}{x^4(1+x^2)^{n-2}}.
\end{aligned}$$

Thus, as $n \to \infty$,

$$a_n \sim \frac{1}{(n-1)2^n} + o\left(\frac{1}{n2^n}\right),$$

and

$$\lim_{n\to\infty} n \cdot 2^n \cdot a_n = 1.$$

Third solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

Substituting $x = \tan t$ the integral becomes

$$\int_{\pi/4}^{\arctan n} \cos^{2n-2} t \, dt = \int_{\pi/4}^{\pi/2 - \arctan 1/n} \cos^{2n-2} t \, dt$$
$$= \int_{\pi/4}^{\pi/2} \cos^{2n-2} t \, dt - \int_{\pi/2-\arctan 1/n}^{\pi/2} \cos^{2n-2} t \, dt \doteq I_n + J_n$$

In J_n we set $z = \pi/2 - t$ so that

$$0 < J_n = \int_0^{\arctan 1/n} \sin^{2n-2} z \, dz \le \int_0^{\arctan 1/n} z^{2n-2} \, dz = \frac{1}{2n-1}\left(\arctan \frac{1}{n}\right)^{2n-1}$$

hence $\lim_{n\to\infty} n2^n J_n = 0$. Always by $z = \pi/2 - t$ and integrating by parts we get

$$I_n = \int_0^{\pi/4} \sin^{2n-2} t \, dt = -\frac{1}{(n-1)2^n} + \frac{2n-3}{2n-2} I_{n-1} \qquad n \ge 2$$

or

$$R_n = n2^n \int_0^{\pi/4} \sin^{2n-2} t \, dt = -\frac{n}{n-1} + \frac{n(2n-3)}{(n-1)^2} R_{n-1} \qquad n \ge 2$$

After proving that R_n is not increasing and bounded below, we can perform the limit $\lim_{n\to\infty} R_n$ getting the expected result. Changing variable $t = \arctan z$, $R_n \ge R_{n+1}$ yields

$$n\int_0^1 \frac{z^{2n-2}}{(1+z^2)^n} dz \ge 2(n+1)\int_0^1 \frac{z^{2n}}{(1+z^2)^{n+1}} dz = 2(n+1)\int_0^1 \frac{z^{2n-2}}{(1+z^2)^n}\frac{z^2}{1+z^2} dz$$

that is

$$n\int_0^1 \frac{z^{2n-2}}{(1+z^2)^n} dz \ge 2(n+1)\int_0^1 \frac{z^{2n-2}}{(1+z^2)^n} dz - 2(n+1)\int_0^1 \frac{z^{2n-2}}{(1+z^2)^{n+1}} dz$$

or $\displaystyle\int_0^1 \frac{z^{2n-2}}{(1+z^2)^{n+1}}(n - z^2(2+n)) dz \ge 0$ or $(z = 1/u)$ $\displaystyle\int_1^{+\infty} \frac{u^2 n - 2 - n}{(1+u^2)^{n+2}} du \ge 0$

$$\int_1^{+\infty} \frac{u^2 n - 2 - n}{(1+u^2)^{n+2}} du \ge \frac{1}{2^{n+2}}\int_1^{\sqrt{1+\frac{2}{n}}} \frac{u^2 n - 2 - n}{u^{n+2}} du + \frac{1}{2^{n+2}}\int_{\sqrt{1+\frac{2}{n}}}^{+\infty} \frac{u^2 n - 2 - n}{u^{2n+4}} du$$
$$= \frac{1}{2^{n+2}}\frac{n}{1-n}\left((1+\frac{2}{n})^{\frac{1-n}{2}} - 1\right) + \frac{2+n}{1+n}\left((1+\frac{2}{n})^{\frac{-1-n}{2}} - 1\right)$$
$$+ \frac{n}{1+2n}(1+\frac{2}{n})^{\frac{-1-2n}{2}} + \frac{2+n}{3+2n}(1+\frac{2}{n})^{\frac{-3-2n}{2}} \doteq A_n.$$

By elementary calculus we have $A_n = 2^{-n-2}\left(\frac{1}{e^2} - \frac{2}{ne} + O(n^{-2})\right) > 0$ that is R_n is not increasing at least for n large. That R_n is bounded below by 0 is obvious.

Fourth solution by Ovidiu Furdui, University of Toledo, USA

We solve the following generalization of this problem.

Generalization. Let $k \geq 1$ be a real number and let $a_n = \int\limits_1^n \frac{dx}{(1+x^k)^n}$. Then $\lim\limits_{n\to\infty} n \cdot 2^n \cdot a_n = \frac{2}{k}$. Let $I_n = \int\limits_1^\infty \frac{dx}{(1+x^k)^n}$. Then $a_n = I_n - \int\limits_n^\infty \frac{dx}{(1+x^k)^n}$. We have,

$$0 \leq \int\limits_n^\infty \frac{dx}{(1+x^k)^n} \leq \int\limits_n^\infty \frac{dx}{x^{kn}} = \frac{1}{(kn-1)n^{kn-1}}.$$

It follows that

$$0 \leq n \cdot 2^n \cdot \int\limits_n^\infty \frac{dx}{(1+x^k)^n} \leq \frac{n \cdot 2^n}{(kn-1)\cdot n^{kn-1}},$$

and hence, $\lim\limits_{n\to\infty} n \cdot 2^n \int\limits_n^\infty \frac{dx}{(1+x^k)^n} = 0$. Thus, it suffices to calculate $L = \lim\limits_{n\to\infty} n \cdot 2^n \cdot I_n$. Integrating by parts we get that

$$\begin{aligned}
I_n = \int\limits_1^\infty \frac{dx}{(1+x^k)^n} &= \frac{x}{(1+x^k)^n}\Bigg|_1^\infty + \int\limits_1^\infty \frac{knx^2dx}{(1+x^k)^{n+1}} \\
&= -\frac{1}{2^n} + kn\int\limits_1^\infty \left(\frac{1}{(1+x^k)^n} - \frac{1}{(1+x^k)^{n+1}}\right)dx \\
&= -\frac{1}{2^n} + knI_n - knI_{n+1}.
\end{aligned}$$

It follows that, $knI_{n+1} = -\frac{1}{2^n} + (kn-1)I_n$. We multiply the preceding recurrence relation by 2^{n+1} and we get that

$$\frac{kn}{n+1}\cdot(n+1)\cdot 2^{n+1}\cdot I_{n+1} = -2 + 2\cdot\frac{kn-1}{n}\cdot n\cdot 2^n \cdot I_n. \tag{7}$$

Letting n converge to ∞ in (7) we get that $kL = -2 + 2kL$ from which it follows that $L = 2/k$, and the problem is solved.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain

U89. Let $f:[0,\infty)\to[0,a]$ a continuous function on $(0,\infty)$ with a Darboux property on $[0,\infty)$ and $f(0)=0$. Prove that if

$$xf(x)\geq\int_0^x f(t)dt,$$

for every $x\in[0,\infty)$, then f has an antiderivative.

Proposed by Dorin Andrica, Babes-Bolyai University and Mihai Piticari, Campulung Moldovenesc, Romania

Solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

The existence of the antiderivative on $(0,+\infty)$ follows by the continuity of f on $(0,+\infty)$ via the Torricelli–Barrow theorem. If f would be continuous also at $x=0$, by the Torricelli–Barrow theorem the antiderivative would be $int_0^x f(t)dt$ for any $x\geq 0$ so we suppose that f is not continuous at $x=0$. The Darboux property on $[0,+\infty)$ forbids f to have a jump for $x\to 0^+$ namely we cannot have $\lim_{x\to 0^+}f(x)=l\neq 0$. Also by the Darboux property on $[0,+\infty)$, there exists a sequence $\{x_k\}\in(0,+\infty)$ tending to zero such that $\lim_{k\to+\infty}f(x_k)=0$ and then by (1), $\lim_{k\to+\infty}\frac{1}{x_k}\int_0^{x_k}f(t)dt=0$. Moreover we may suppose by the nonnegativity of f and the discontinuity at $x=0$, that $\frac{1}{x_k}\int_0^{x_k}f(t)dt\neq 0$. We claim that for any sequence $\{y_k\}\in(0,+\infty)$, $y_k\to 0^+$, there exists a sequence $\{x_k\}$ satisfying $\lim_{k\to+\infty}f(x_k)=0$, $\int_0^{x_k}f(t)dt\neq 0$, and a constant C such that

$$0\leq\frac{1}{y_k}\int_0^{y_k}f(t)dt\leq C\frac{1}{x_k}\int_0^{x_k}f(t)dt$$

whence

$$\lim_{k\to+\infty}\frac{1}{y_k}\int_0^{y_k}f(t)dt=0=f(0)$$

and this implies the existence of the antiderivative of f over the whole $[0,+\infty)$.

To prove the claim we suppose it false namely that there exists a sequence $\tilde{y}_k\to 0^+$ such that for any sequence $x_k\to 0^+$ satisfying $\lim_{k\to+\infty}f(x_k)=0$, $\int_0^{x_k}f(t)dt\neq 0$, for any $C>0$, there exists $\tilde{k}$ such that

$$\frac{1}{\tilde{y}_{\tilde{k}}}\int_0^{\tilde{y}_{\tilde{k}}}f(t)dt\geq C\frac{1}{\tilde{x}_{\tilde{k}}}\int_0^{\tilde{x}_{\tilde{k}}}f(t)dt\neq 0$$

This is absurd because we can take C as large as we need but $\frac{1}{\tilde{y}_{\tilde{k}}}\int_0^{\tilde{y}_{\tilde{k}}}f(t)dt\leq a$. The proof is complete.

U90. Let α be a real number greater than 2. Evaluate

$$\sum_{n=1}^{\infty}\left(\zeta(\alpha)-\frac{1}{1^\alpha}-\frac{1}{2^\alpha}-\cdots-\frac{1}{n^\alpha}\right),$$

where ζ denotes the Riemann-Zeta function.

Proposed by Ovidiu Furdui, University of Toledo, USA

First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

It is well known that for $p>1$, the power series or p-series $\sum_{k=1}^{\infty}\frac{1}{k^p}$ equals $\zeta(p)$. Then, the proposed sum equals

$$\sum_{n=1}^{\infty}\sum_{k=n+1}^{\infty}\frac{1}{k^\alpha}=\sum_{k=2}^{\infty}\sum_{n=1}^{k-1}\frac{1}{k^\alpha}=\sum_{k=2}^{\infty}\frac{k-1}{k^\alpha}=\sum_{k=1}^{\infty}\frac{1}{k^{\alpha-1}}-\sum_{k=1}^{\infty}\frac{1}{k^\alpha}=\zeta(\alpha-1)-\zeta(\alpha).$$

Second solution by G.R.A.20 Math Problems Group, Roma, Italy

Since the terms of this double sum are positive, we can reorder them:

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\zeta(\alpha)-\sum_{k=1}^{n}\frac{1}{k^\alpha}\right) &= \sum_{n=1}^{\infty}\sum_{k=n+1}^{\infty}\frac{1}{k^\alpha}=\sum_{k=2}^{\infty}\sum_{n=1}^{k-1}\frac{1}{k^\alpha}\\ &= \sum_{k=2}^{\infty}\frac{k-1}{k^\alpha}=\zeta(\alpha-1)-\zeta(\alpha).\end{aligned}$$

Also solved by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Brian Bradie, Christopher Newport University, USA; Michel Bataille, France.

Olympiad problems

O85. Let a, b, c be non-negative real numbers such that $ab+bc+ca=1$. Prove that

$$4 \leq \left(\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}}\right)(a+b+c-abc).$$

Proposed by Arkady Alt, San Jose, California, USA

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Calling $x=b+c$, $y=c+a$, $z=a+b$, we have $a+b+c=\frac{x+y+z}{2}$. Moreover, we may write $1+a^2=ab+bc+ca+a^2=(a+b)(c+a)=yz$ and similarly $1+b^2=zx$, $1+c^2=xy$. Furthermore, $(1-ab)=bc+ca=cz$, leading to $c-abc=c^2z=z(xy-1)=xyz-z$. Similarly, $a-abc=xyz-x$ and $b-abc=xyz-y$. Then,

$$a+b+c-abc=\frac{2(a+b+c)+(a-abc)+(b-abc)+(c-abc)}{3}=xyz,$$

and we may rewrite the proposed inequality as

$$4 \leq \left(\frac{1}{\sqrt{yz}}+\frac{1}{\sqrt{zx}}+\frac{1}{\sqrt{xy}}\right)xyz=\sqrt{xy}\sqrt{yz}+\sqrt{yz}\sqrt{zx}+\sqrt{zx}\sqrt{xy}.$$

Now, using the inequality between arithmetic and quadratic means, we have

$$\sqrt{yz}=\sqrt{1+a^2}=\frac{2}{\sqrt{3}}\sqrt{\frac{1+1+1+3a^2}{4}} \geq \frac{2}{\sqrt{3}}\frac{3+a\sqrt{3}}{4}=\frac{a+\sqrt{3}}{2},$$

with equality iff $a=\frac{1}{\sqrt{3}}$, and similarly for the other two products, leading to $\sqrt{zx}\sqrt{xy} \geq \frac{3+\sqrt{3}(b+c)+bc}{4}$, and similarly for the other two combinations. It then suffices to prove that

$$4 \leq \frac{9+2\sqrt{3}(a+b+c)+1}{4},$$

or equivalently, that $a+b+c \geq \sqrt{3}$. But this is true since, using the scalar product (Cauchy-Schwarz) inequality, we have

$$2=2(ab+bc+ca)=(a+b+c)^2-(a^2+b^2+c^2) \leq (a+b+c)^2-(ab+bc+ca),$$

and $(a+b+c)^2 \geq 2+(ab+bc+ca)=3$, with equality iff $a=b=c$. The result follows, and equality is reached if and only if $a=b=c=\frac{1}{\sqrt{3}}$.

Also solved by Ngoc Quy Nguyen, Da Nang, VietNam; Manh Dung Nguyen, Vietnam; Magkos Athanasios, Kozani, Greece; Andrea Munaro, Italy; Daniel Campos Salas, Costa Rica; Michel Bataille, France; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Oleh Faynshteyn, Leipzig, Germany; Samin Riasat, Notre Dame College, Dhaka, Bangladesh; Vicente Vicario Garcia, Huelva, Spain.

O86. The sequence $\{x_n\}$ is defined by $x_1 = 1$, $x_2 = 3$ and $x_{n+1} = 6x_n - x_{n-1}$ for all $n \geq 1$. Prove that $x_n + (-1)^n$ is a perfect square for all $n \geq 1$.

Proposed by Brian Bradie, Christopher Newport University, USA

First solution by Arkady Alt, San Jose, California, USA

First we will find solution of recurrence equation $a_{n+1} - 6a_n + a_{n-1} = 0, n \in \mathbb{N}$ with $a_0 = 3, a_1 = 1$ as initial conditions $(a_0 = 6a_1 - a_2 = 3)$.

Since $a_n = c_1 x_1^n + c_2 x_2^n$, $n \in \mathbb{N} \cup \{0\}$, where $x_1 = 3 + 2\sqrt{2} = \left(1 + \sqrt{2}\right)^2$ and $x_2 = 3 - 2\sqrt{2} = \left(1 - \sqrt{2}\right)^2$ are solutions of characteristic equation $x^2 - 6x + 1 = 0,$ and $c_1 = \dfrac{3 - 2\sqrt{2}}{2}, c_2 = \dfrac{3 + 2\sqrt{2}}{2}$ are solution of the system

$$\begin{cases} a_0 = 3 = c_1 + c_2 \\ a_1 = 1 = c_1 x_1 + c_2 x_2 \end{cases}.$$

then $a_n = \dfrac{\left(3 + 2\sqrt{2}\right)^{n-1} + \left(3 - 2\sqrt{2}\right)^{n-1}}{2} = \dfrac{\left(\sqrt{2} + 1\right)^{2(n-1)} + \left(1 - \sqrt{2}\right)^{2(n-1)}}{2}$.

Thus, $a_n + (-1)^n = \dfrac{\left(1 + \sqrt{2}\right)^{2(n-1)} + \left(1 - \sqrt{2}\right)^{2(n-1)} + 2\,(-1)^n}{2} =$

$\dfrac{\left(1 + \sqrt{2}\right)^{2(n-1)} + \left(1 - \sqrt{2}\right)^{2(n-1)} - 2\left(1 + \sqrt{2}\right)^{n-1}\left(1 - \sqrt{2}\right)^{n-1}}{2} = t_n^2,$ where

$t_n = \dfrac{\left(1 + \sqrt{2}\right)^{n-1} - \left(1 - \sqrt{2}\right)^{n-1}}{\sqrt{2}}$ all t_n are non-negative integers because satisfy $t_{n+1} - 2t_n - t_{n-1} = 0$ and $t_0 = 2, t_1 = 0$.

Second solution by Roberto Bosch Cabrera, Cuba

Let $y_n = x_n + (-1)^n$ for $n \geq 1$. Then $y_1 = 0,\ y_2 = 4,\ y_3 = 16$ and

$$y_{n+1} = 5y_n + 5y_{n-1} - y_{n-2}$$

for $n \geq 3$. Also, let $z_1 = 0,\ z_2 = 2$, and for $n \geq 3$, set $z_n = 2z_{n-1} + z_{n-2}$. We will show that $y_n = z_n^2$ for all n. This holds for $n = 1, 2, 3$ and assuming the claim for $y_1, \ldots, y_n$ we have

$$\begin{aligned} y_{n+1} &= 5z_n^2 + 5z_{n-1}^2 - z_{n-2}^2 \\ &= 5z_n^2 + 5z_{n-1}^2 - (z_n - 2z_{n-1})^2 \\ &= 4z_n^2 + 4z_{n-1}z_n + z_{n-1}^2 \\ &= (2z_n + z_{n-1})^2 = z_{n+1}^2 \end{aligned}$$

Thus the claim follows by induction, and the proof is complete.

Third solution by Daniel Campos Salas, Costa Rica

The roots of the characteristic equation of the sequence $\{x_n\}$ are

$$a = 3 + 2\sqrt{2} \text{ and } b = 3 - 2\sqrt{2},$$

and it's easy to prove that

$$x_n = \frac{3 - 2\sqrt{2}}{2}a^n + \frac{3 + 2\sqrt{2}}{2}b^n = \frac{a^{n-1} + b^{n-1}}{2}.$$

Note that $a = (1 + \sqrt{2})^2$ and $b = (1 - \sqrt{2})^2$, therefore,

$$\begin{aligned} x_n + (-1)^n &= \frac{(1 + \sqrt{2})^{2(n-1)} - 2(-1)^{n-1} + (1 - \sqrt{2})^{2(n-1)}}{2} \\ &= \frac{\left((1 + \sqrt{2})^{n-1} - (1 - \sqrt{2})^{n-1}\right)^2}{2} \\ &= \left(\frac{\sqrt{2}}{2}(1 + \sqrt{2})^{n-1} - \frac{\sqrt{2}}{2}(1 - \sqrt{2})^{n-1}\right)^2. \end{aligned}$$

If we define the sequence $\{y_n\}$ by $y_n = \frac{\sqrt{2}}{2}(1 + \sqrt{2})^{n-1} - \frac{\sqrt{2}}{2}(1 - \sqrt{2})^{n-1}$ for all $n \geq 1$, it follows that $y_1 = 0, y_1 = 2$ and $y_{n+2} = 2y_{n+1} + y_n$ for all $n \geq 1$. These results imply that every term of the sequence $\{y_n\}$ is an integer, from where we conclude that $x_n + (-1)^n = (y_n)^2$ is a perfect square, as we wanted to prove.

Also solved by G.R.A.20 Math Problems Group, Roma, Italy; Michel Bataille, France; Nguyen Tho Tung, Vietnam; Raul A. Simon, Chile; Vicente Vicario Garcia, Huelva, Spain.

O87. Let G be a graph with n vertices, $n \geq 5$. The edges of a graph are colored in two colors such that there are no monochromatic cycles of length $3, 4$, and 5. Prove that there are no more than $\left\lfloor \frac{n^2}{3} \right\rfloor$ edges in the graph.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

First solution by Daniel Lasaosa, Universidad Publica de Navarra

We will show first the following

Lemma: in a monochromatic graph (all edges have the same color) with n vertices, a cycle of length at most 5 appears when there are exactly $\lfloor \frac{n^2}{6} \rfloor + 1$ edges in the graph.

Proof: For $n = 5$, $\lfloor \frac{5^2}{6} \rfloor + 1 = 5$. If each vertex has exactly 2 edges, walk the graph starting from any vertex, and upon arrival at one vertex through one of its edges, leave it through its other edge. After at most 5 moves, we arrive at an already visited vertex, for a cycle of at most 5 vertices. If one vertex has less than 2 edges, remove this vertex and its corresponding edges, leaving a graph with 4 vertices and at least 4 edges. We eliminate edges if necessary, trimming their number down to 4. Again, if each vertex has exactly 2 edges, a cycle of length at most 4 may be found. Otherwise, one of the vertices has 3 edges joining it to the remaining 3 vertices, and any additional edge between any two of these 3 vertices results in a cycle of length 3. For $n = 6$, assume that the graph has 7 edges. If no vertex has less than 3 edges, the total number of edges is at least $\frac{3 \cdot 6}{2} = 9$, absurd. So at least one vertex has at most 2 edges. Eliminate this vertex and its corresponding edges, and a graph with 5 vertices and at least 5 edges results, guaranteeing at least one cycle of length at most 5. For $n > 6$, assume that the result is true for $n - 1$. Assume that no vertex in the graph has less than $\frac{n+1}{3}$ edges. The total number of edges is then at least $\frac{n^2+n}{6} > \lfloor \frac{n^2}{6} \rfloor + 1$, absurd. So there is at least one vertex with at most $\lfloor \frac{n}{3} \rfloor$ edges. Eliminate this vertex and its corresponding edges, and a graph with $n-1$ vertices and at least $\lfloor \frac{n^2}{6} \rfloor - \lfloor \frac{n}{3} \rfloor + 1$ edges appears. Analyzing the different cases for remainder of n modulus 6, we see that this number is at least $\lfloor \frac{(n-1)^2}{6} \rfloor + 1$, except when $n \equiv 5 \pmod 6$, where this number is actually $\lfloor \frac{(n-1)^2}{6} \rfloor + 2$. In any case, a cycle of length at most 5 is guaranteed to appear in the resulting graph since the result is true for $n - 1$, and the lemma is proved by induction.

Assume now that there are more than $\lfloor \frac{n^2}{3} \rfloor$ edges painted in two colors in a graph with n vertices, but no monochromatic cycle of length at most 5 appears. By the previous lemma, there cannot be more than $\lfloor \frac{n^2}{6} \rfloor$ edges of each color in the graph, for a total of $2\lfloor \frac{n^2}{6} \rfloor \leq \lfloor \frac{n^2}{3} \rfloor$ edges, absurd. The result follows.

Second solution by G.R.A.20 Math Problems Group, Roma, Italy

It's easy to verify (we made about ten diagrams) that if the edges of the graph $K_5 \setminus \{e\}$ are colored in two colors then there is a monochromatic cycles of length 3, 4, or 5. Hence G does not contain a copy of $K_5 \setminus \{e\}$. By a result due to Ivan Borsenco (see K_k *versus* $K_{k+1} \setminus \{e\}$ published by MR) we have that since G does not contain a copy of $K_5 \setminus \{e\}$ then it does not contain neither a copy of K_4 and by Turan's Theorem it follows that there are no more than

$$\left\lfloor \left(1 - \frac{1}{4-1}\right) \frac{n^2}{2} \right\rfloor = \left\lfloor \frac{n^2}{3} \right\rfloor$$

edges.

O88. Determine all pairs (z, n) such that

$$z + z^2 + \cdots + z^n = n|z|,$$

where $z \in C$ and $|z| \in \mathbb{Z}_+$.

Proposed by Dorin Andrica, Babes-Bolyai University and Mihai Piticari, Campulung Moldovenesc, Romania

Solution by Daniel Lasaosa, Universidad Publica de Navarra

For $n = 1$, we find $z = |z|$, and $(z, 1)$ is a solution iff $z \in \mathbb{Z}^+$. For $|z| = 1$, we find $n = |z + z^2 + ... + z^n| \leq |z| + |z|^2 + ... + |z|^n = n$, with equality iff all z^k are collinear, ie, iff $z \in \mathbb{R}$, and $(1, n)$ is the only possible solution with $|z| = 1$, but it is valid for any positive integer n. These solutions may be considered "trivial". Let us now look for nontrivial solutions.

For $n = 2$, the equation becomes $z + z^2 = 2|z|$, which after expressing $z = |z|e^{i\theta}$ and separating into real and imaginary parts yields $\cos\theta + |z|\cos(2\theta) = 2$ and $\sin\theta + |z|\sin(2\theta) = 0$. The latter results in either $\sin\theta = 0$ or $\cos\theta = -\frac{1}{2|z|}$, the second option yielding $\cos(2\theta) = 2\cos^2\theta - 1 = \frac{1-2|z|^2}{2|z|^2}$. Insertion in the former yields $|z| = -2$, obviously absurd. So $\sin\theta = 0$, resulting in $|z| = 2 + 1 = 3$ when $\cos\theta = -1$, and $|z| = 2 - 1 = 1$ when $\cos\theta = 1$, for trivial solution $(1, 2)$ and additional solution $(-3, 2)$.

If $|z| > 1$, consider $n|z|(z - 1) = z^{n+1} - z$, which after expressing $z = |z|e^{i\theta}$ and separating real and imaginary parts yields

$$|z|^n \cos((n + 1)\theta) = (n|z| + 1)\cos\theta - n,$$

$$|z|^n \sin((n + 1)\theta) = (n|z| + 1)\sin\theta.$$

Squaring both equations and adding them, we find

$$|z|^n = \sqrt{(n|z| + 1)^2 + n^2 - 2n(n|z| + 1)\cos\theta} \leq n|z| + n + 1,$$

with equality iff $\cos\theta = -1$. The derivatives of $|z|^n$ and $n|z|+n+1$ with respect to n are respectively $|z|^n \ln|z| > \frac{|z|^n}{2} \geq |z|^{n-1}$, and $|z| + 1$, where we have used that $|z| \geq 2$ and $\ln 2 > \frac{1}{2}$ since $4 > e$. Note that, for $n \geq 3$ and $|z| \geq 2$, $|z|^{n-1} \geq 2|z| > |z| + 1$, and since for $n = 3$, $|z| = 3$ we obtain $|z|^n = 27 > 13 = n|z| + n + 1$, and for $n = 4$, $|z| = 2$ we obtain $|z|^n = 16 > 13 = n|z| + n + 1$, we cannot have solutions for $n \geq 4$ when $|z| = 2$, or for $n \geq 3$ when $|z| \geq 3$. Since the cases $n = 1, 2$ for any $|z|$, and $|z| = 1$ for any n, have already been discussed, we need to find only whether solutions exist for $|z| = 2$ and $n = 3$. In this case, the equations become $8\cos(4\theta) = 7\cos\theta - 3$ and $8\sin(4\theta) = 7\sin\theta$, for $64 = 49 + 9 - 42\cos\theta$, or $\cos\theta = -\frac{1}{7}$. But this is absurd, since substitution in the previous equations yield $\cos(4\theta) = -\frac{1}{2}$, but direct calculation yields $\cos(2\theta) = 2\cos^2\theta - 1 = -\frac{47}{49}$, and similarly $\cos(4\theta) = \frac{2\cdot 47^2 - 49^2}{49^2} \neq -\frac{1}{2}$, and the possible solution found is actually artificially introduced in the squaring-and-adding process. Hence no solutions exist for $n = 3$ and $|z| = 3$, and the only possible solutions are $(-3, 2)$, and the trivial solutions $(1, n)$ for all positive integer n and $(z, 1)$ for all positive integer z.

O89. Let P be an arbitrary point in the interior of a triangle ABC and let P' be its isogonal conjugate. Let I be the incenter of triangle ABC and let X, Y, Z be the midpoints of the small arcs BC, CA, AB. Denote by A_1, B_1, C_1 the intersections of lines AP, BP, CP with sides BC, CA, AB, respectively, and let A_2, B_2, C_2 be the midpoints of the segments IA_1, IB_1, IC_1. Prove that lines XA_2, YB_2, ZC_2 are concurrent on line IP'.

Proposed by Cosmin Pohoata, Tudor Vianu National College, Romania

First solution by Andrea Munaro, University of Trento, Italy

We use barycentric coordinates. Denote as usual by a, b, c the sides of the triangle opposite to vertexes A, B, C, respectively, and by s the semiperimeter of the triangle.
Clearly $I = (a : b : c)$ and suppose $P = (p : q : r)$. Then $P' = (a^2qr : b^2rp : c^2pq)$, $A_1 = (0 : q : r)$ and cyclic.
To find the coordinates of X, Y, Z we consider the intersection between the circumcircle, that has equation $a^2yz+b^2zx+c^2xy = 0$, with the angle bisectors, obtaining $X = (-a^2 : b(b + c) : c(b + c))$ and cyclic.
To find midpoint of IA_1 we may use absolute barycentric coordinates of I and A_1, obtaining $A_2 = \left(\frac{a}{a+b+c} : \frac{q}{q+r} + \frac{b}{a+b+c} : \frac{r}{q+r} + \frac{c}{a+b+c}\right)$ and cyclic.
Then the equations of lines XA_2, YB_2, ZC_2 and IP' are, respectively

$$(bcr + b^2r - bcq - c^2q)x + (car + caq + a^2r)y + (-a^2q - abr - abq)z = 0$$

$$(bcr + b^2r + bcp)x + (car - cap - c^2p + a^2r)y + (-b^2p - abr - abp)z = 0$$

$$(c^2q + bcp + bcq)x + (-c^2p - cap - caq)y + (-b^2p + a^2q - abp + abq)z = 0$$

$$(bc^2pq - b^2crp)x + (ca^2qr - ac^2pq)y + (ab^2rp - a^2bqr)z = 0$$

Finally, calculating determinants, we see that, for example, XA_2, YB_2, IP' concur and YB_2, ZC_2, IP' concur.

Second solution by Daniel Campos Salas, Costa Rica

Let

$$\overrightarrow{P} = \frac{\overrightarrow{A}ax + \overrightarrow{B}by + \overrightarrow{C}cz}{ax + by + cz} \text{ and } \overrightarrow{P'} = \frac{\overrightarrow{A}ayz + \overrightarrow{B}bzx + \overrightarrow{C}cxy}{ayz + bzx + cxy}.$$

Note that

$$\overrightarrow{I} = \frac{\overrightarrow{A}a + \overrightarrow{B}b + \overrightarrow{C}c}{2s}, \ \overrightarrow{X} = \frac{-\overrightarrow{A}a^2 + \overrightarrow{B}b(b + c) + \overrightarrow{C}c(b + c)}{4s(s - a)},$$

$$\overrightarrow{A_1}=\frac{\overrightarrow{B}by+\overrightarrow{C}cz}{by+cz},\ \overrightarrow{A_2}=\frac{\overrightarrow{A}a+\overrightarrow{B}b+\overrightarrow{C}c}{4s}+\frac{\overrightarrow{B}by+\overrightarrow{C}cz}{2(by+cz)}.$$

To prove the result it's enough to prove that XA_2 passes through a fixed point in IP'. If $y=z$ then lines XA_2 and IP' coincide, and we're done. So suppose that $y\neq z$ and let Q be the intersection of XA_2 and IP'. It follows that

$$\overrightarrow{Q}=m\overrightarrow{A_2}+(1-m)\overrightarrow{X}=n\overrightarrow{P'}+(1-n)\overrightarrow{I},$$

for some real numbers m,n. Then Q is a fixed point if and only if n is symmetric with respect to a,b,c and x,y,z. Comparing the coefficient of $\overrightarrow{A}$ in the expression above it follows that

$$\frac{am}{4s}-\frac{a^2(1-m)}{4s(s-a)}=\frac{anyz}{ayz+bxz+cxy}+\frac{a(1-n)}{2s}.$$

Note that

$$\frac{1}{4s}+\frac{a}{4s(s-a)}=\frac{1}{4(s-a)}\quad\text{and}\quad\frac{a}{4s(s-a)}+\frac{a}{2s}=\frac{a(b+c)}{4s(s-a)},$$

then, that equation is equivalent to

$$\frac{m}{4(s-a)}=\frac{nyz}{ayz+bzx+cxy}-\frac{n}{2s}+\frac{b+c}{4s(s-a)}.\tag{1}$$

Comparing the coefficient of $\overrightarrow{B}$ it results that

$$\frac{bm}{4s}+\frac{bmy}{2(by+cz)}+\frac{b(b+c)(1-m)}{4s(s-a)}=\frac{bnxz}{ayz+bxz+cxy}+\frac{b(1-n)}{2s}.$$

Note that

$$\frac{-(b+c)}{4s(s-a)}+\frac{1}{4s}=\frac{-1}{4(s-a)}\quad\text{and}\quad\frac{1}{2s}-\frac{b+c}{4s(s-a)}=\frac{-a}{4s(s-a)},$$

so it follows that the last equation is equivalent to

$$\frac{-m}{4(s-a)}+\frac{my}{2(by+cz)}=\frac{nxz}{axy+bzx+cxy}-\frac{n}{2s}-\frac{a}{4s(s-a)}.\tag{2}$$

Analogously, after comparing the coefficient of $\overrightarrow{C}$ we obtain that

$$\frac{-m}{4(s-a)}+\frac{mz}{2(by+cz)}=\frac{nxy}{axy+bzx+cxy}-\frac{n}{2s}-\frac{a}{4s(s-a)}.\tag{3}$$

From (2) and (3) it follows that

$$\frac{m(y-z)}{2(by+cz)} = \frac{-nx(y-z)}{ayz+bzx+cxy}.$$

Since we assume that $y \neq z$ it follows that $m = \dfrac{-2nx(by+cz)}{ayz+bzx+cxy}$. Substituting this in (1), multiplying by $(a+b+c)(-a+b+c)(ayz+bzx+cxy)$, and rearranging some terms it follows that

$$\begin{aligned}
& -(b+c)(ayz+bzx+cxy) \\
= \; & n(x(by+cz)(a+b+c)+yz(a+b+c)(-a+b+c) \\
& -(-a+b+c)(ayz+bzx+cxy)) \\
= \; & n(b+c)\left(yz(-a+b+c)+zx(a-b+c)+xy(a+b-c)\right).
\end{aligned}$$

If follows that

$$n = \frac{-(ayz+bzx+cxy)}{yz(-a+b+c)+zx(a-b+c)+xy(a+b-c)},$$

which is symmetric as we wanted to prove, and we're done.

Third solution by Michel Bataille, France

We shall use areal (barycentric) coordinates relatively to (A, B, C). Let A' be the mid-point of BC and O be the circumcentre of ΔABC. Then

$$I(a,b,c), \quad O(a\cos A, b\cos B, c\cos C), \quad A'(0,1,1).$$

The lines AI and OA' have respective equations $cy-bz=0, \quad x(b^2-c^2)+a^2(y-z)=0$ [using the law of cosines, the relation $(a\cos A)(b^2-c^2)+a^2(b\cos B - c\cos C)=0$ is easily verified], and it follows that $X(a^2, -b(b+c), -c(b+c))$. Now, let $P(\alpha,\beta,\gamma)$ where $\alpha,\beta,\gamma > 0$ and $\alpha+\beta+\gamma=1$. It is known that $P'(\frac{a^2}{\alpha}, \frac{b^2}{\beta}, \frac{c^2}{\gamma})$ and easily found that $A_1(0,\beta,\gamma)$ and $A_2(a(\beta+\gamma), 2\beta s + b(\beta+\gamma), 2\gamma s + c(\beta+\gamma))$ (as usual s denotes the semiperimeter of ΔABC). Let us seek the point of intersection Q of XA_2 with IP' as $Q(\frac{a^2}{\alpha}+\lambda a, \frac{b^2}{\beta}+\lambda b, \frac{c^2}{\gamma}+\lambda c)$ where $\lambda \in \mathbb{R}$. The equation of the line XA_2 is

$$\begin{vmatrix} x & a^2 & a(\beta+\gamma) \\ y & -b(b+c) & 2\beta s + b(\beta+\gamma) \\ z & -c(b+c) & 2\gamma s + c(\beta+\gamma) \end{vmatrix} = 0$$

or after some easy transformations: $(\beta c - \gamma b)[x(b+c)-a(y+z)] - 2as(\gamma y - \beta z) = 0$.

Substituting the coordinates of Q for x, y, z gives

$$(\beta c - \gamma b)\left[\frac{a^2 b}{\alpha} - \frac{ab^2}{\beta} + \frac{a^2 c}{\alpha} - \frac{ac^2}{\gamma}\right] - 2sa\left(\frac{\gamma b^2}{\beta} - \frac{\beta c^2}{\gamma}\right) + 2sa\lambda(\beta c - \gamma b) = 0.$$

Assuming $P \notin AI$ that is, $\beta c - \gamma b \neq 0$, we deduce

$$\lambda = -\frac{\gamma b + \beta c}{\gamma\beta} - \frac{b(\beta a - \alpha b)}{2s\alpha\beta} - \frac{c(\gamma a - \alpha c)}{2s\alpha\gamma}$$

or, recalling $\alpha + \beta + \gamma = 1$,

$$\lambda = -\frac{abc}{\alpha\beta\gamma(a+b+c)}\left(\frac{\alpha(1-\alpha)}{a} + \frac{\beta(1-\beta)}{b} + \frac{\gamma(1-\gamma)}{c}\right).$$

The symmetric form of this result shows that the same point Q will also be found as the intersections $YB_2 \cap IP'$ and $ZC_2 \cap IP'$ and the desired result follows. If P lies on AI (with $P \neq I$) then the lines XA_2 and IP' coincide but we can show as above that $YB_2 \cap IP'$ and $ZC_2 \cap IP'$ coincide, hence the conclusion still holds.

Also solved by Mihai Miculita, Oradea, Romania; Daniel Lasaosa, Universidad Publica de Navarra, Spain.

O90. Find all positive integers n having at most four distinct prime divisors such that

$$n \mid 2^{\phi(n)} + 3^{\phi(n)} + \cdots + n^{\phi(n)}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Gabriel Dospinescu, Ecole Normale Superieure, France

Solution by Daniel Lasaosa, Universidad Publica de Navarra

For some prime p, let p divide n with multiplicity a. Obviously, p must divide $2^{\phi(n)} + 3^{\phi(n)} + ... + n^{\phi(n)}$, and calling $n = p^a q$, $\phi(n) = p^{a-1}(p-1)\phi(q)$. Now, if k is prime with p, and since $k^{p-1} \equiv 1 \pmod{p}$ by Fermat's little theorem, we find $k^{\phi(n)} \equiv 1 \pmod{p}$. In $\{2, 3, ..., n\}$, there are $\frac{n}{p}$ multiples of p, and $n - \frac{n}{p} - 1$ numbers prime with p. Therefore, $2^{\phi(n)} + 3^{\phi(n)} + ... + n^{\phi(n)} \equiv -\frac{n}{p} - 1 \pmod{p}$, and the given condition is equivalent to the following "simpler" condition: n is the product of distinct prime factors, and p divides $\frac{n}{p} + 1$ for any prime factor p of n. We have included in this condition the fact that, since $\frac{n}{p} \equiv -1 \pmod{p}$, p cannot divide $\frac{n}{p}$.

Assume that $n = p$ is itself prime. Then $p|1 + 1 = 2$, yielding $n = p = 2$.

Assume that n is the product of two distinct primes $p < q$. Then, $q|p + 1$, and $q \leq p + 1 \leq q$, or p, q are consecutive integers. This is only possible if $p = 2$, $q = 3$, and since $2|3 + 1$ and $3|2 + 1$, then $n = 6$ is the only solution in this case.

Assume that n is the product of three distinct primes $p < q < r$. A positive integer a exists such that $ar = pq + 1 \leq p(r-2) + 1 < pr$, or $a < p < q$. Since q divides $a(pr+1) = p^2q + p + a$, then $q|p + a < 2q$, and $p + a = q$. Similarly, p divides $a(qr+1) = pq^2 + q + a = pq^2 + p + 2a$, and $p|2a < 2p$. If $p|a$, then $p|q$ absurd, or $a = 1$ and $p = 2$, yielding $q = 3$ and $r = 7$, for $n = 42$. Note that $7|\frac{42}{7} + 1 = 7$, $3|\frac{42}{3} + 1 = 15$ and $2|\frac{42}{2} + 1 = 22$, so $n = 42$ is indeed the only solution in this case.

Assume finally that n is the product of four distinct primes $p < q < r < s$. A positive integer a exists such that $as = pqr + 1 \leq pq(s-2) + 1 < pqs$, or $a < pq$. Since $r|a(pqs+1) = p^2q^2r + pq + a$, then $r|pq + a$, and a positive integer b exists such that $br = pq + a < 2pq < 2pr$, or $b < 2p$. Similarly, $p|qr + a$, and $p|b(qr + a) = pq^2 + a(q + b)$; if $p|a$, then $p|r$ absurd, so $p|q + b$. Similarly, $q|p + b < 3p$, and either $p + b = q$ or $p + b = 2q$, resulting respectively in $p|2b$ and $p|3b$. Since again p cannot divide b (otherwise $p|q$), we have respectively $p = 2$ and $p = 3$. We distinguish then two possible cases:

a) If $p = 2$, since $b < 2p = 4$ and p cannot divide b, we have either $b = 1$ or $b = 3$. The latter results in $q = 5$, and $3r = br = pq + a < 2pq = 20$, absurd since then $r \leq 5$. So $b = 1$, $q = 3$, and $r = 6 + a < 12$, with possible solutions $r = 7, a = 1$ and $r = 11, a = 5$. The latter results in $5s = pqr + 1 = 67$, absurd, so $r = 7$, $a = 1$, and $s = 43$. Since $43|\frac{n}{43} + 1 = 43$, $7|\frac{n}{7} + 1 = 259 = 7 \cdot 37$, $3|\frac{n}{3} + 1 = 603$ and $2|\frac{n}{2} + 1 = 904$, then $n = 2 \cdot 3 \cdot 7 \cdot 43 = 1806$ is a solution.

b) If $p = 3$, since $b < 2p = 6$ and p cannot divide b, we have $b = 1, 2, 4, 5$, resulting respectively in $q = 2, 3, 5, 7$, the first two options being absurd since $q \leq p$. If $q = 5$, then $4r = 16$ absurd, and if $q = 7$, $5r = 22$, also absurd.

So the only solutions where n has at most 4 distinct prime divisors are $n = 2$, $n = 6$, $n = 42$ and $n = 1806$. Note that a trend seems to appear where each value of n with an additional prime divisor is found by taking $n' = n(n + 1)$, but this trend breaks down for n divisible by 5 primes, since $1807 = 13 \cdot 139$, and taking $n = 2 \cdot 3 \cdot 7 \cdot 13 \cdot 43 \cdot 139$, then $\frac{n}{139} + 1$ is not divisible by 139.

Also solved by Nguyen Tho Tung, Hanoi, Vietnam