Solutions for Admission Test C

1. Find all integers n for which $4n+9$ and $9n+1$ are both perfect squares.

Solution: Let $4n+9 = a^2$ and $9n+1 = b^2$ with $a, b \geq 0$. Then $9a^2-4b^2 = 9(4n+9)-4(9n+1) = 77$. It follows that $(3a-2b)(3a+2b) = 7 \cdot 11$. Because $3a-2b < 3a+2b$, it is enough to consider the following cases:

$$3a-2b = 1, 3a+2b = 77; \quad 3a-2b = 7, 3a+2b = 11.$$

Solving these systems of equations we get $(a, b) = (13, 19)$ and $(a, b) = (3, 1)$. Thus the only n that satisfy our property are $n = \mathbf{0}$ and $n = \mathbf{40}$.

2. Time is displayed on an electronic clock from $00:00:00$ to $23:59:59$. How many times in 24 hours the display shows exactly four 4's?

Solution: Observe that on the first position only three numbers $\{0, 1, 2\}$ can appear. It follows that four $4'$s appear exactly at five of the remaining positions. Consider the following three cases:

1^{st} *case:* 4 does not appear on the second position. There are $24-2$ possibilities. Because the last four positions display 4 and from 00 to 23 only 04, 14 have 4 on the second position.

2^{nd} *case:* 4 does not appear on the third or fifth position. We have $2 \cdot 5$ possibilities. We have two numbers $\{0, 1\}$ to be displayed on the first position and five numbers $\{0, 1, 2, 3, 5\}$ on the third/fifth place.

3^{rd} *case:* 4 does not appear on the fourth or sixth position. There are $2 \cdot 9$ possibilities. Again we have two numbers $\{0, 1\}$ to be displayed on the first position and nine numbers $\{0, 1, 2, 3, 5, 6, 7, 8, 9\}$ on the fourth/sixth position.

Thus there are $22+2\cdot10+2\cdot18 = \mathbf{78}$ times when the display shows exactly four $4'$s.

3. Let $s_1, s_2, ..., s_{25}$ be the squares of some 25 consecutive integers. Prove that

$$\frac{s_1 + s_2 + ... + s_{25}}{25} - 52$$

is also the square of an integer.

Solution: Let $s_1 = n^2, s_2 = (n+1)^2, ..., s_{25} = (n+24)^2$. Then

$$\frac{s_1 + s_2 + ... + s_{25}}{25} - 52 = \frac{n^2 + (n+1)^2 + ... + (n+24)^2}{25} - 52 =$$

$$= \frac{25n^2 + 24 \cdot 25n + \frac{1}{6} \cdot 24 \cdot 25 \cdot 49}{25} - 52 = n^2 + 24n + 196 - 52 = (n+12)^2,$$

and the problem is solved.

4. Consider a cyclic quadrilateral $ABCD$. Let P be the point of intersection of its diagonals. Denote by A_1, B_1, C_1, D_1 the projections of P onto quadrilateral's sides. Prove that quadrilateral $A_1B_1C_1D_1$ is circumscribed about a circle.

Solution: Suppose $A_1 \in AB, B_1 \in BC, C_1 \in CD, D_1 \in DA$. Observe that quadrilaterals AA_1PD_1 and A_1BB_1P are cyclic. Hence $\angle D_1AP = \angle D_1A_1P$ and $\angle B_1BP = \angle B_1A_1P$. Also, because $ABCD$ is cyclic, $\angle D_1AP = \angle DAC = \angle DBC = \angle PBB_1$. It follows that $\angle D_1A_1P = \angle B_1A_1P$ and A_1P is the angle bisector of $\angle D_1A_1B_1$. Analogously, B_1P, C_1P, D_1P are angle bisectors of the corresponding angles. Because the angle bisector is the locus of points that are equidistant from triangle's sides, P is the center of the circle that is inscribed in $A_1B_1C_1D_1$, and we are done.

5. Find all triples (x, y, z) of integers, solutions to the system of equations

$$\begin{cases} xy + z = 100 \\ x + yz = 101. \end{cases}$$

Solution: Taking the difference we have $x(1-y) + z(y-1) = 1$ or $(y-1)(z-x) = 1$. We have two systems of equations to solve:

$z - x = 1, y - 1 = 1$ and $z - x = -1, y - 1 = -1$.

Solutions to the systems are $y = 0, z = 100, x = 101$ and $y = 2, x = 33, y = 34$. Thus the triples are $(x, y, z) = \mathbf{(101,0,100)}$ and $(x, y, z) = \mathbf{(33,2,34)}$.

6. Let $ABCD$ be a trapezoid such that $AB \parallel CD$. Let P be the point of intersection of diagonals AC and BD. If $\text{area}_{PAB} = 16$ and $\text{area}_{PCD} = 25$, find area_{ABCD}.

Solution: Observe that the areas of triangles ACD and BCD are equal. This follows from the fact these triangles have the same base and the corresponding altitudes are equal because $AB \parallel CD$. Subtracting the common area_{CPD} we get $\text{area}_{APD} = \text{area}_{BPC}$. The next step is to observe that

$$\text{area}_{APB} \cdot \text{area}_{CPD} = \text{area}_{APD} \cdot \text{area}_{BPC}.$$

Note that triangles APB and CPD are similar, therefore $AP \cdot PC = BP \cdot PD$. Also $\sin \angle APB = \sin \angle BPC = \sin \angle CPD = \sin \angle APD$. Let $S = \text{area}_{APD} = \text{area}_{BPC}$. We have $S^2 = \text{area}_{APB} \cdot \text{area}_{CPD} = 16 \cdot 25$, hence $S = 20$. Then

$$\text{area}_{ABCD} = \text{area}_{APB} + \text{area}_{CPD} + \text{area}_{APD} + \text{area}_{BPC} = 16+20+20+25 = \mathbf{81}.$$

7. An electronic board initially displays number 36. Each minute the number shown is multiplied or divided by either 2 or 3 and the new number is displayed. Can the number shown after one hour be 12?

Solution: Let n be the number displayed on the electronic board. Note that n can be represented as $n = 2^\alpha \cdot 3^\beta$, where $\alpha, \beta \in \mathbb{Z}$. Observe the following: after an even number of moves the parity of $\alpha + \beta$ is invariant. At the beginning we have $36 = 2^2 \cdot 3^2$ with $\alpha + \beta = 2 + 2 = 4$, but after one hour, namely 60 minutes we have $12 = 2^2 \cdot 3$ with $\alpha + \beta = 2 + 1 = 3$. Because they do not have the same parity, number 12 cannot be shown.

8. Consider an equilateral triangle ABC and a point P on the small arc BC of its circumcircle. Let A' be the point of intersection of PA and BC. Prove that

$$\frac{1}{PA'} = \frac{1}{PB} + \frac{1}{PC}.$$

Solution: In triangle BPC, PA_1 is the angle bisector of $\angle BPC$, because $\angle PBA = \angle CPA = \frac{120^\circ}{2} = 60^\circ$. Let $BC = a$, $PB = x$, and $PC = y$. Then from the Law of Cosines, $a^2 = x^2 + y^2 - xy \cos 120^\circ = x^2 + y^2 + xy \qquad (1)$

From Stewart's Theorem in triangle PBC,

$$x^2 \cdot A'C + y^2 \cdot A'B - PA'^2 \cdot a = A'B \cdot A'C \cdot a.$$

But $\frac{x}{y} = \frac{A'B}{A'C}$, yielding $A'B = \frac{ax}{x+y}$ and $A'C = \frac{ay}{x+y}$. It follows that

$$x^2 \frac{ay}{x+y} + y^2 \frac{ax}{x+y} - PA'^2 \cdot a = \frac{ax}{x+y} \cdot \frac{ay}{x+y} \cdot a.$$

Hence

$$PA'^2 = xy \left(1 - \frac{a^2}{(x+y)^2}\right) \qquad (2).$$

From (1) and (2), $PA'^2 = xy \cdot \dfrac{xy}{(x+y)^2}$, so

$$\frac{1}{PA'} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y} = \frac{1}{PB} + \frac{1}{PC}.$$

9. What is maximum number of knights you can place on a chessboard such that none is attacking another?

Solution: The key to the solution is two consider a 2×4 chessboard. Clearly we can place four knights and we cannot place five knights that will not attack each other. That is why we cannot have more than 32 knights on the chessboard. Assume there are at least 33 knights on the chessboard. Divide the chessboard into eight 2×4 subboards. By the Pigeonhole Principle there will exist a 2×4 board that has five knights on it, contradiction. It follows that there are at most 32 knights and they can be placed to not attack each other: place all knights on the white cells.

10. What is the least number of congruent triangles that dissect an 8×8 square with one of its 1×1 corners removed?

Solution: Let us prove that there are at least 18 triangles. It follows that that the area of such a triangle is at least 3.5. Consider a triangle A that has its side lying on segment a. If the base of A lies on a then its length is at most 1. But the height on that base is at most 7, therefore area$_A \leq \frac{1}{2} \cdot 7 \cdot 1 = 3.5$. If the base of triangle A exceeds segment a, then the base of triangle B is entirely in b, hence not greater than 1. The height on b in triangle B does not exceed 7. Therefore area$_B \leq 3.5$. It follows that the area of the triangle does not exceed 3.5. Because the area of the figure does not exceed $64 - 1$, there are at least 18 triangles. An example of such construction is the following: