Junior problems

J175. Let $a, b \in (0, \frac{\pi}{2})$ such that $\sin^2 a + \cos 2b \geq \frac{1}{2}\sec a$ and $\sin^2 b + \cos 2a \geq \frac{1}{2}\sec b$. Prove that

$$\cos^6 a + \cos^6 b \geq \frac{1}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J176. Solve in positive real numbers the system of equations

$$\begin{cases} x_1 + x_2 + \cdots + x_n = 1 \\ \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 x_2 \cdots x_n} = n^3 + 1. \end{cases}$$

Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzau, Romania

J177. Let x, y, z be nonnegative real numbers such that $ax + by + cz \leq 3abc$ for some positive real numbers a, b, c. Prove that

$$\sqrt{\frac{x+y}{2}} + \sqrt{\frac{y+z}{2}} + \sqrt{\frac{z+x}{2}} + \sqrt[4]{xyz} \leq \frac{1}{4}(abc + 5a + 5b + 5c).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J178. Find the sequences of integers $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ such that

$$(2+\sqrt{5})^n = a_n + b_n \frac{1+\sqrt{5}}{2}$$

for each $n \geq 0$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

J179. Solve in real numbers the system of equations

$$\begin{cases} (x+y)(y^3 - z^3) = 3(z-x)(z^3 + x^3) \\ (y+z)(z^3 - x^3) = 3(x-y)(x^3 + y^3) \\ (z+x)(x^3 - y^3) = 3(y-z)(y^3 + z^3) \end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J180. Let a, b, c, d be distinct real numbers such that

$$\frac{1}{\sqrt[3]{a-b}} + \frac{1}{\sqrt[3]{b-c}} + \frac{1}{\sqrt[3]{c-d}} + \frac{1}{\sqrt[3]{d-a}} \neq 0.$$

Prove that $\sqrt[3]{a-b} + \sqrt[3]{b-c} + \sqrt[3]{c-d} + \sqrt[3]{d-a} \neq 0$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Senior problems

S175. Let p be a prime. Find all integers $a_1, \ldots, a_n$ such that $a_1 + \cdots + a_n = p^2 - p$ and all solutions to the equation $px^n + a_1x^{n-1} + \cdots + a_n = 0$ are nonzero integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

S176. Let ABC be a triangle and let AA_1, BB_1, CC_1 be cevians intersecting at P. Denote by $K_a = K_{AB_1C_1}$, $K_b = K_{BC_1A_1}$, $K_c = K_{CA_1B_1}$. Prove that $K_{A_1B_1C_1}$ is a root of the equation

$$x^3 + (K_a + K_b + K_c)x^2 - 4K_aK_bK_c = 0.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S177. Prove that in any acute triangle ABC,

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \geq \frac{5R+2r}{4R}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S178. Prove that there are sequences $(x_k)_{k\geq 1}$ and $(y_k)_{k\geq 1}$ of positive rational numbers such that for all positive integers n and k,

$$(x_k + y_k\sqrt{5})^n = F_{kn-1} + F_{kn}\frac{1+\sqrt{5}}{2},$$

where $(F_m)_{m\geq 1}$ is the Fibonacci sequence.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

S179. Find all positive integers a and b for which $\frac{(a^2+1)^2}{ab-1}$ is a positive integer.

Proposed by Valcho Milchev, Petko Rachov Slaveikov Secondary School, Bulgaria

S180. Solve in nonzero real numbers the system of equations

$$\begin{cases} x^4 - y^4 = \frac{121x-122y}{4xy} \\ x^4 + 14x^2y^2 + y^4 = \frac{122x+121y}{x^2+y^2}. \end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Undergraduate problems

U175. What is the maximum number of points of intersection that can appear after drawing in a plane l lines, c circles, and e ellipses?

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

U176. In the space, consider the set of points (a, b, c) where $a, b, c \in \{0, 1, 2\}$. Find the maximum number of non-collinear points contained in the set.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U177. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be integers greater than 1. Prove that there are infinitely many primes p such that p divides $b_i^{\frac{p-1}{a_i}} - 1$ for all $i = 1, 2, \ldots, n$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

U178. Let k be a fixed positive integer and let $S_n^{(j)} = \binom{n}{j} + \binom{n}{j+k} + \binom{n}{j+2k} + \cdots$, $j = 0, 1, \ldots, k-1$. Prove that

$$\left(S_n^{(0)} + S_n^{(1)} \cos\frac{2\pi}{k} + \cdots + S_n^{(k-1)} \cos\frac{2(k-1)\pi}{k}\right)^2$$
$$+ \left(S_n^{(1)} \sin\frac{2\pi}{k} + S_n^{(2)} \sin\frac{4\pi}{k} + \cdots + S_n^{(k-1)} \sin\frac{2(k-1)\pi}{k}\right)^2 = \left(2\cos\frac{\pi}{k}\right)^{2n}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

U179. Let $f : [0, \infty] \to R$ be a continuous function such that $f(0) = 0$ and $f(2x) \leq f(x) + x$ for all $x \geq 0$. Prove that $f(x) < x$ for all $x \in [0, \infty]$.

Proposed by Samin Riasat, University of Dhaka, Bangladesh

U180. Let $a_1, \ldots, a_k, b_1, \ldots, b_k, n_1, \ldots, n_k$ be positive real numbers and $a = a_1 + \cdots + a_k, b = b_1 + \cdots + b_k, n = n_1 + \cdots + n_k, k \geq 2$. Prove that

$$\int_0^1 (a_1 + b_1 x)^{n_1} \cdots (a_k + b_k x)^{n_k} dx \leq \frac{(a+b)^{n+1} - a^{n+1}}{(n+1)b}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Olympiad problems

O175. Find all pairs (x, y) of positive integers such that $x^3 - y^3 = 2010(x^2 + y^2)$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O176. Let $P(n)$ be the following statement: for all positive real numbers $x_1, x_2, \ldots, x_n$ such that $x_1 + x_2 + \cdots + x_n = n$,

$$\frac{x_2}{\sqrt{x_1 + 2x_3}} + \frac{x_3}{\sqrt{x_2 + 2x_4}} + \cdots + \frac{x_1}{\sqrt{x_n + 2x_2}} \geq \frac{n}{\sqrt{3}}.$$

Prove that $P(n)$ is true for $n \leq 4$ and false for $n \geq 9$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

O177. Let P be point situated in the interior of a circle. Two variable perpendicular lines through P intersect the circle at A and B. Find the locus of the midpoint of the segment AB.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

O178. Let m and n be positive integers. Prove that for each odd positive integer b there are infinitely many primes p such that $p^n \equiv 1 \pmod{b}^m$ implies $b^{m-1} \mid n$.

Proposed by Vahagn Aslanyan, Yerevan, Armenia

O179. Prove that any convex quadrilateral can be dissected into $n \geq 6$ cyclic quadrilaterals.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

O180. Let p be a prime. Prove that each positive integer $n \geq p$, p^2 divides $\binom{n+p}{p}^2 - \binom{n+2p}{2p} - \binom{n+p}{2p}$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania