Unrigorously Jensen

Delong Meng

Abstract

This article provides four examples of how Jensen's Inequality can be applied indirectly to combinatorics.

The cornerstone idea of the article is if f is a convex function and $x_1+x_2+\cdots+x_n=s$ where x_i are positive integers, then

$$f(x_1)+f(x_2)+\cdots+f(x_n)\geq r\cdot f(k+1)+(n-r)f(k)$$

where $s=nk+r$ and $0\leq r<n$. In other words, if the sum of the integers is given, then the sum of the functions is the least when the integers are closest to each other. This is not quiet Jensen's Inequality, but the smoothing argument can be very useful in many Olympiad problems.

Consider a function $f(x)=\binom{x}{2}=\frac{x^2-x}{2}$. What is is the concavity of $f(x)$? It is not difficult to find that is convex because $f''(x)=1>0$. The following two examples are beautiful applications of this fact and Jensen's inequality.

Example 1. There are a girls and b boys in the class, where $b\geq 3$ is an odd number. Each boy rates a girls as either beautiful or smart. Suppose that any two boys' ratings coincide for at most k girls. Prove that $\frac{k}{a}\geq\frac{b-1}{2b}$.

(International Mathematical Olympiad, 1998)

Solution. Let us count the number of pairs of agreement. Since any two boys agree for at most k girls, the total number of pairs of agreement is at most $k\binom{b}{2}$. Now let us count this number in a different way. Each girl is rated by b boys, and suppose p of them rated beautifulness and q of them rated smartness with $p+q=b$. For each girl, there are $\binom{p}{2}+\binom{q}{2}$ pairs of agreement.

Now we want to use convexity. Consider the function $f(x)=\binom{x}{2}=\frac{x^2-x}{2}$.

By Jensen's Inequality, $f(p)+f(q)$ would be minimized when $p=\frac{b+1}{2}$ and $q=\frac{b-1}{2}$. This is because b is odd and the best way to bring two integers closest is to have two integers differ by 1. Now

$$\binom{p}{2}+\binom{q}{2}\geq\binom{\frac{b+1}{2}}{2}+\binom{\frac{b-1}{2}}{2}=\frac{(b-1)^2}{4}.$$

So the total pairs of agreement is at least $\dfrac{a(b-1)^2}{4}$ and thus $\dfrac{a(b-1)^2}{4} \leq \dfrac{kb(b-1)}{2}$ which is equivalent to $\frac{k}{a} \geq \frac{b-1}{2b}$.

Example 2. Suppose in the previous problem, these a girls are either friendly or mean to each other. Justin is going to receive 1 bonus point for every set of three girls who are either mutually friendly or mutually mean to each other in his rating. What is the minimum number of bonus points Justin will get?

Solution. This problem also use the idea of counting in two ways. If three girls are either mutually friendly or mutually mean to each other, we call them a monochromatic triangle. If a girl is either mean or friendly to two other girls, then we call this a monochromatic angle. Since each monochromatic triangle has three monochromatic angles, and all other triangles have only one monochromatic angle, the total number of monochromatic angles is

$$3m + \binom{a}{3} - m = 2m + \binom{a}{3}$$

Where m is the minimum number of bonus points Justin must get. On the other hand, suppose a girl is mean to p people and friendly to q people, then there are $\dbinom{p}{2} + \dbinom{q}{2}$ monochromatic angles from this girl where $p + q = a - 1$. Does this look familiar? Yes, it is the exact same idea as last problem!

If a is even, then $\dbinom{p}{2} + \dbinom{q}{2} \geq \dbinom{\frac{a}{2}}{2} + \dbinom{\frac{a-2}{2}}{2} = \dfrac{(a-2)^2}{4}$. And since there are a girls, the total number of monochromatic angles is at least $\dfrac{a(a-2)^2}{4}$. Therefore

$$2m + \binom{a}{3} \geq \frac{a(a-2)^2}{4}$$

After simplifying the above, we obtain $m \geq \dfrac{a(a-2)(a-4)}{24}$.

If a is odd, then $\dbinom{p}{2} + \dbinom{q}{2} \geq 2\dbinom{\frac{a-1}{2}}{2}$. And similarly we obtain that

$$2m + \binom{a}{3} \geq 2a\binom{\frac{a-1}{2}}{2}$$

This time after simplifying, we obtain $m \geq \dfrac{a(a-1)(a-5)}{24}$.

Remark. We can even try to further generalize this Ramsey type problem of more colors, but it is much more complicated.

Consider another famous function $f(x) = \ln x$. What is the concavity of $f(x)$? The second derivative, $f''(x) = -\dfrac{1}{x^2} \leq 0$, is negative, so the function is concave.

Example 3. In a class a guy assigns an integer value for the prettiness of a girls. If three girls' prettiness are three consecutive integers, then he calls them triadic. What is the maximum number of triadics?

(IMO Shortlist, 2001)

Solution. Actually, we can find a solution for a more generalized problem! Instead of triadics, let us count the maximum number of k consecutive prettiness. Let us sort the girls by prettiness. Now if two girls' prettiness in a sorted array differ by more than 1, then increase the prettiness of the last one by 1 would only increase the number of triadics. Therefore, we can assume all the numbers are consecutive and increasing. Without loss of generality, let $x_1, x_2, \cdots, x_s$ be the number of girls with prettiness as $1, 2, \cdots, s$.

We want to maximize the number

$$x_1x_2\cdots x_k + x_2x_3\cdots x_{k+1} + \cdots + x_{s-k+1}x_{s-k+2}\cdots x_s,$$

where $x_1 + x_2 + \cdots + x_s = a$. If $s \geq k+1$, then replacing x_{k+1} with $x_1 + x_{k+1}$ and delete x_1 does not reduce the value of the expression. So we can assume there are only k prettinesses.

Given $x_1 + x_2 + \cdots + x_k = a$, we want to maximize $x_1x_2\cdots x_k$. Let us look at the function $f(x) = \ln(x)$, which is concave. Note that

$$\ln(x_1x_2\cdots x_k) = \ln x_1 + \ln x_2 + \cdots + \ln x_k.$$

By Jensen's Inequality, we want to bring the numbers as close as we can. Let $a = ks + r$ where $0 \leq r < k$. Then we would have r values of $(s+1)$ and $n - r$ values of s. Therefore, the maximum is $s^{n-r}(s+1)^r$.

Example 4. Prove that for all integers $n \geq 2$, the product

$$\prod_{1 \leq i < j \leq n} (a_j - a_i)$$

is divisible by the product

$$\prod_{1 \leq i < j \leq n} (j - i)$$

(Red MOP Homework, 2006)

Solution. Consider any prime or powers of a prime p. We claim that the number of terms divisible by p is more on the top than on the bottom by considering the residues of $a_i \pmod p$. Let $b_0, b_1, \cdots, b_{p-1}$ be sequence such that b_j represents the number of $a_i \equiv j \pmod p$. So the number of terms of the first product that's divisible by p is

$$\binom{b_0}{2} + \binom{b_1}{2} + \cdots + \binom{b_{p-1}}{2}.$$

Since $b_0 + b_1 + \cdots + b_{p-1} = n$ and the function $f(x) = \binom{x}{2}$ is convex, the above expression is minimized when there are r values of $k+1$ and $n-r$ values of k where $n = kp + r$ and $0 \le r < p$. This is exactly what we have in the product $\prod_{1 \le i < j \le n} (j - i)$.

Therefore, for any powers of a prime, the first product always have more terms than the second product, so the second product divides the first product.

Delong Meng
Baton Rouge Magnet High School, USA