Junior problems

J43. In triangle ABC the median AM intersects the internal bisector BN at P. Denote by Q the point of intersection of lines CP and AB. Prove that triangle BNQ is isosceles.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

J44. Consider a triangle ABC and let $g_a,\ g_b,\ g_c$ and $n_a\ n_b,\ n_c$ be the Gergonne cevians and the Nagel cevians, respectively. Prove that

$$g_a + g_b + g_c + 2\max(a, b, c) \geq n_a + n_b + n_c + 2\min(a, b, c).$$

Proposed by Mircea Lascu, Zalau, Romania

J45. Let a and b be real numbers. Find all pairs (x, y) of real numbers solutions to the system

$$\begin{cases} x + y = \sqrt[3]{a+b} \\ x^4 - y^4 = ax - by \end{cases}$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas.

J46. A quadrilateral is called bicentric if it is both inscribed in a circle and circumscribed to a circle. Construct with a ruler and compass a bicentric quadrilateral with all of its sidelengths distinct.

Proposed by Ivan Borsenco, University of Texas at Dallas

J47. In triangle ABC let m_a and l_a be the median and the angle bisector from the vertex A, respectively. Prove that

$$0 \leq m_a^2 - l_a^2 \leq \frac{(b-c)^2}{2}.$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

J48. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b(b+c)^2} + \frac{b}{c(c+a)^2} + \frac{c}{a(a+b)^2} \geq \frac{9}{4(ab+bc+ca)}.$$

Proposed by Ho Phu Thai, Da Nang, Vietnam

Senior problems

S43. Consider an acute triangle ABC and let Γ be its circumcircle, centered at O. Denote by D, E, and F the midpoints of the minor arcs BC, CA, and AB, respectively. Let Γ_A be the circle through O which is tangent to Γ at D. Define analogously Γ_B and Γ_C. Let O_A be the intersection of Γ_B and Γ_C, different from O. Define analogously O_B and O_C. Prove that triangles ABC and $O_AO_BO_C$ are similar if and only if ABC is equilateral.

Proposed by Daniel Campos, Costa Rica

S44. Let $C(O)$ be a circle and let P be a point outside of C. Tangents from P intersect the circle at A and B. Let M be the midpoint of AP and let $N = BM \cap C(O)$. Prove that $PN = 2MN$.

Proposed by Pohoata Cosmin, Bucharest, Romania

S45. Consider two sequences of integers, (a_n) and (b_n) such that $|a_{n+2} - a_n| \leq 2$ for all n in $\mathbb{Z}$ and $a_m + a_n = b_{m^2+n^2}$, for all m, n in $\mathbb{Z}$. Prove that there exist at most 6 distinct numbers in the sequence a_n.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

S46. Let ABC be a triangle and let D, E, F be the points of tangency of the incircle with the sides of the triangle. Prove that the centroid of triangle DEF and the centroid of triangle ABC are isogonal if and only if triangle ABC is equilateral.

Proposed by Pohoata Cosmin, Bucharest, Romania

S47. Consider an $n \times n$ grid filled with ones. A move consists of taking a square with numbers (a, b, c, d) and rewriting the entries in one of the two following ways:

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 0 & 2a+b \\ \hline 0 & 2c+d \\ \hline\end{array} \quad ; \quad \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 0 & c \\ \hline b & 4a+d \\ \hline\end{array}$$

Prove that no matter how one makes moves, at one point there is only one nonzero entry on the table. Also prove that the value of this entry is unique.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

S48. Consider an equilateral triangle divided into 16 congruent equilateral triangles. Prove that no matter how we label these triangles with the numbers 1 through 16, there will be two adjacent triangles whose difference of the labels is at least 4.

Proposed by Ivan Borsenco, University of Texas at Dallas

Undergraduate problems

U43. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Prove that for any positive integer n there exists $c \in [0,1]$ such that

$$f(c) = f(c + \frac{1}{n}).$$

Proposed by Jose Luis Diaz-Barrero, Barcelona, Spain

U44. Let x, y be positive real numbers such that $x^y + y = y^x + x$. Prove that $x + y \leq 1 + xy$.

Proposed by Cezar Lupu, University of Bucharest, Romania

U45. Let $A \in M_n(R)$ be a matrix that has zeros on the main diagonal and all other entries are from the set $\{-1, 1\}$. Is it possible that $\det A = 0$ for $n = 2007$? What about for $n = 2008$?

Proposed by Aleksandar Ilic, Serbia

U46. Let k be a positive integer and let

$$a_n = \left\lfloor \left(k + \sqrt{k^2+1}\right)^n + \left(\frac{1}{2}\right)^n \right\rfloor,\ n \geq 0.$$

Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_{n-1}a_{n+1}} = \frac{1}{8k^2}.$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

U47. Let P be arbitrary point inside equilateral triangle ABC. Find the minimum value of

$$\frac{1}{PA} + \frac{1}{PB} + \frac{1}{PC}.$$

Proposed by Hung Quang Tran, Ha Noi National University, Vietnam

U48. Let n an integer greater than 1 and let $k \geq 1$ be a real number. For an n dimensional simplex $X_1X_2 \ldots X_{n+1}$ define its k-perimeter by $\sum_{1 \leq i < j \leq n+1} |X_iX_j|^k$. Take now a regular simplex $A_1A_2 \ldots A_{n+1}$ and consider all simplexes $B_1B_2 \ldots B_{n+1}$ where B_i lies on the face $A_1 \ldots A_{i-1}A_{i+1} \ldots A_n$. Find, in terms of the k-perimeter of $A_1A_2 \ldots A_{n+1}$, the minimal possible k-perimeter of $B_1B_2 \ldots B_{n+1}$.

Proposed by Iurie Boreico, Moldova

Olympiad problems

O43. Let a, b, c be positive real numbers. Prove that

$$\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}}+\sqrt{\frac{a+b}{c}} \geq \sqrt{\frac{16(a+b+c)^3}{3(a+b)(b+c)(c+a)}}.$$

Proposed by Vo Quoc Ba Can, Can Tho University, Vietnam

O44. Let $ABCD$ be a cyclic quadrilateral and let O be the intersection of its diagonals. Denote by I_{ab} and I_{cd} the centers of incircles of triangles OAB and OCD, respectively. Prove that the perpendiculars from O, I_{ab}, I_{cd} to lines AD, BD, AC, respectively, are concurrent.

Proposed by Mihai Miculita, Oradea, Romania

O45. Consider a positive real number t. A grasshopper has a finite number of pointwise nests. It can add new nests as follows: from two nests A and B it can jump to point C with $\dfrac{\overrightarrow{AC}}{\overrightarrow{AB}} = t$, and make C its nest. Prove that there are points in the plane that cannot be made nests.

Proposed by Iurie Boreico, Moldova

O46. Let O and I be the circumcenter and the incenter of triangle ABC, respectively. Denote by D the intersection of the incircle of ABC with BC and by E and F the intersections of AI and AO with the circumcircle of ABC, respectively. Let S be the intersection of FI and ED, M the intersection of SC and BE, and N the intersection of AC and BF. Prove that M, I and N are collinear.

Proposed by Pohoata Cosmin, Bucharest, Romania

O47. Consider the Fibonacci sequence $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Prove that

$$\sum_{i=0}^{n} \frac{(-1)^{n-i} F_i}{n+1-i}\binom{n}{i} = \begin{cases} \dfrac{2F_{n+1}}{n+1} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

Proposed by Gabriel Alexander Reyes, San Salvador, El Salvador

O48. Let $f \in \mathbb{Z}[X]$ be a monic irreducible polynomial of degree n whose zeros $x_1, x_2, ..., x_n$ are all real numbers. Let $S_k = x_1^{2k} + x_2^{2k} + ... + x_n^{2k}$. Prove that there exist a universal constant $c > 0$, such that

$$S_1 \cdot S_2 \cdot ... \cdot S_{n-1} \geq c \cdot \frac{e^{2n}}{n^2}$$

holds for all n.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris