## Junior problems

J55. Let $a_0 = 1$ and $a_{n+1} = a_0 \cdot ... \cdot a_n + 4, n \geq 0$. Prove that $a_n - \sqrt{a_{n+1}} = 2$ for all $n \geq 1$.

*Proposed by Dr. Titu Andreescu, University of Texas at Dallas*

J56. Two players, $A$ and $B$, play the following game: player $A$ divides an $9 \times 9$ square into strips of unit width and various lengths. After that player $B$ picks an integer $k$, $1 \leq k \leq 9$, and removes all strips of length $k$. Find the largest area $K$ that $B$ can remove, regardless the way $A$ divides the square into strips.

*Proposed by Iurie Boreico, Harvard University*

J57. Let $a, b, c$ be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 \geq 16.$$

*Proposed by Mircea Becheanu, Bucharest, Romania*

J58. Let $ABC$ be a triangle and let $AA_1, BB_1, CC_1$ be the cevians that pass through point $P$. Denote by $X, Y, Z$ the midpoints of $B_1C_1, A_1C_1, A_1B_1$, respectively. Prove that $AX, BY, CZ$ are concurrent.

*Proposed by Ivan Borsenco, University of Texas at Dallas*

J59. Consider an $n \times n$ square grid. We color some of the squares in black. Prove that we can find a connected black figure consisting of three squares if

(a) $\frac{n^2}{2} + 1$ squares are colored for even $n$,

(b) $\frac{n(n+1)}{2}$ squares are colored for odd $n$.

*Proposed by Ivan Borsenco, University of Texas at Dallas*

J60. Let $a, b, c$ be positive real numbers. Prove that

$$\frac{bc}{a^2 + bc} + \frac{ca}{b^2 + ca} + \frac{ab}{c^2 + ab} \leq \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b}.$$

*Proposed by Pham Huu Duc, Ballajura, Australia*

## Senior problems

S55. Let $X = \{x_1, x_2, ..., x_n\}$ be a set of positive real numbers. Prove that there exist no more than $\frac{2^n}{\sqrt{n}}$ subsets of $X$, whose sum of elements is equal to 1.

*Proposed by Iurie Boreico, Harvard University*

S56. Let $G$ be the centroid of triangle $ABC$. Prove that

$$\sin\angle GBC + \sin\angle GCA + \sin\angle GAB \leq \frac{3}{2}.$$

*Proposed by Tran Quang Hung, Ha Noi National University, Vietnam*

S57. Suppose we have a graph with six vertices. The edges of a graph are colored in two colors. Prove that one can always find three different monochromatic cycles in it.

*Proposed by Ivan Borsenco, University of Texas at Dallas*

S58. Let $M, N$ be the midpoints of $AB$ and $CD$ of a cyclic quadrilateral $ABCD$. The circumscircles of triangles $BAN$ and $CMD$ intersect $CD$ and $AB$ at points $P$ and $Q$, respectively. Prove that $PQ$ passes through the intersection of the diagonals $AC$ and $BD$.

*Proposed by Ciupan Andrei, Bucharest, Romania*

S59. Consider the family of those subsets of $\{1, 2, ..., 3n\}$ whose sum of the elements is a multiple of 3. For each subset of this family compute the square of the sum of its elements. Find the sum of the numbers obtained in this way.

*Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris*

S60. Consider triangle $ABC$ and let $\alpha(I_a)$, $\beta(I_b), \gamma(I_c)$ be the excircles corresponding to the vertices $A, B, C$, respectively. Let $P$ a point in the interior of the triangle $ABC$ and consider its cevian $AA_1, BB_1, CC_1$. Denote by $X, Y, Z$ the tangents from $A', B', C'$ to the excircles $\alpha(I_a), \beta(I_b), \gamma(I_c)$, respectively, such that ($X \notin BC$, $Y \notin CA$, $Z \notin AB$). Prove that the lines $AX$, $BY$, $CZ$ are concurrent.

*Proposed by Cosmin Pohoata, Bucharest, Romania*

## Undergraduate problems

U55. Let $f : [0,1] \to \mathbb{R}$ be a bijective differentiable function. Prove that there exist $c \in (0,1)$ such that

$$\int_{f(0)}^{f(1)} f^{-1}(x)dx = \frac{1}{2} \cdot f'(c).$$

*Proposed by Cezar Lupu, University of Bucharest, Romania*

U56. Let $x, y, z$ be positive real numbers. Prove that

$$\frac{3\sqrt{3}}{2} \leq \sqrt{x+y+z}\left(\frac{\sqrt{x}}{y+z} + \frac{\sqrt{y}}{x+z} + \frac{\sqrt{z}}{x+y}\right).$$

*Proposed by Byron Schmuland, University of Alberta, Canada*

U57. Solve in positive integers the following equation: $x^3 - y^2 = 2$.

*Proposed by Juan Ignacio Restrepo, Columbia*

U58. Let $n \in \mathbb{N}$ and denote by $u(n)$ the number of ones in the binary representation of $n$. Example: $u(10) = u(1010_2) = 2$. Let $k, m, n \geq 0$ be integers such that $k \geq mn$.

Express $\displaystyle\sum_{i=0}^{2^k-1} (-1)^{u(i)} \binom{\binom{i}{m}}{n}$ in closed form.

*Proposed by Josh Nichols-Barrer, Massachusetts Institute of Technology*

U59. Let $\phi$ be Euler's totient function, where $\phi(1) = 1$. Prove that for all positive integers $n$ we have

$$1 > \sum_{k=1}^{n} \frac{\phi(k)}{k} \ln\left(\frac{2^k}{2^k - 1}\right) > 1 - \frac{1}{2^n}.$$

*Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris*

U60. Let $P_n$ and $Q_n$ be the number of connected and disconnected graphs with $n$ vertices, respectively.

a) Prove that $\displaystyle\lim_{n\to\infty} \frac{P_n}{2^{\frac{n(n-1)}{2}}} = 1, \lim_{n\to\infty} \frac{Q_n}{2^{\frac{n(n-3)}{2}}} = 1$ for labeled graph.

b) Prove that $\displaystyle\lim_{n\to\infty} \frac{P_n}{2^{\frac{n(n-1)}{2}}} = 1, \lim_{n\to\infty} \frac{Q_n}{n \cdot 2^{\frac{n(n-3)}{2}}} = 1$ for unlabeled graph.

*Proposed by Iurie Boreico, Harvard University*

## Olympiad problems

O55. For each positive integer $k$, let $f(k) = 4^k + 6^k + 9^k$. Prove that for all nonnegative integers $m$ and $n$, $f(2^m)$ divides $f(2^n)$ whenever $m$ is less than or equal to $n$.

*Proposed by Dr. Titu Andreescu, University of Texas at Dallas*

O56. We have $k$ hedgehogs in the upper-left unit square of a $m \times n$ grid. Each of them moves towards the lower-right unit square of the grid, by moving each minute either one unit to the right or one unit down. What is the least possible number of grid squares that are not visited by any of the hedgehogs?

*Proposed by Iurie Boreico, Harvard University*

O57. Consider a triangle $ABC$ with the orthocenter $H$, incenter $I$, and circumcenter $O$. Denote by $D$ the point of tangency of its circle with the side $BC$. Suppose that $H_a$ is the midpoint of $AH$ and $M$ is the midpoint of $BC$. If $I \in H_aM$, prove that $AO||HD$.

*Proposed by Liubomir Chiriac, Princeton University*

O58. Let $a, b$ be positive integers such that $\gcd(a, b) = 1$. Find all pairs $(m, n)$ of positive integers such that $a^m + b^m$ divides $a^n + b^n$.

*Proposed by Dorin Andrica and Dorian Popa, Cluj-Napoca, Romania*

O59. Let $P_n$ and $Q_n$ be the number of connected and disconnected unlabeled graphs in the graph with $n$ vertices. Prove that

$$P_n - Q_n \geq 2(P_{n-1} - Q_{n-1}).$$

*Proposed by Ivan Borsenco, University of Texas at Dallas*

O60. Let $A_1, A_2, ..., A_n$ and $B_1, B_2, ..., B_n$ be the subsets of the set $\{1, 2, ..., n\}$, such that for all $i$ and $j$, $A_i$ and $B_j$ have exactly one common element and for all nonempty subsets $T$ of $\{1, 2, ..., n\}$, there exists $i$ such that the intersection of $A_i$ and $T$ has an odd number of elements. Prove or disprove that $B_1 = B_2 = ... = B_n$.

*Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris*