On a Turan Theorem for cyclic graphs

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Abstract. This article is a summary of a few results about Turan type theorem on cyclic graphs C_m.

1 Introduction

For a graph H, the Turan number, denoted by $ex(n; H)$, is defined as the maximum number of edges in a graph on n vertices that does not contain the given graph H.

The famous Turan Theorem states:

Theorem 1. Let n and k be positive integers with $k \geq 2$. Then

$$ex(n; K_k) \leq \frac{(k-2)n^2}{2(k-1)}.$$

The author of this article proved in [2] the following fact:

Theorem 2. Let n and k be positive integers with $k \geq 2$. Then

$$ex(n; K_{k+1} \backslash \{e\}) \leq \frac{(k-2)n^2}{2(k-1)},$$

where $K_{k+1} \backslash \{e\}$ is the complete graph on $k+1$ vertices with one edge removed. In this article we are interested in finding bounds for $ex(n; C_m)$, where C_m is a cyclic graph on m vertices.

2 Bounds for $ex(n;\ C_{2m+1})$, $m \geq 1$

The fact that every bipartite graph does not contain an odd cycle implies that $\lfloor \frac{n^2}{4} \rfloor$ edges does not ensure the existence of a C_{2m+1} cycle. Turan Theorem solves the problem for $m = 1$, because $K_3 \equiv C_3$, in this case $ex(n; C_3) \leq \lfloor \frac{n^2}{4} \rfloor$. Let us prove the same bound for $m = 2$.

Theorem 3. Let G be a graph on $n \geq 7$ vertices with more than $\frac{n^2}{4}$ edges. Then there exists a C_5 subgraph in G.

Proof. Let us prove the base case $n = 7$. Using Theorem 2, there is a $K_4 \backslash \{e\}$ subgraph in G, which we denote by the quadrilateral $ABCD$ with the edge AC removed. Denote the remaining three vertices by X, Y, Z. If G has at least 13 edges we show that G contains a 5-cycle.

If the edge AC is present in the graph, then having one of the vertices X, Y, Z connected to some two of the vertices $\{A, B, C, D\}$ yields a C_5 subgraph. Thus in total we have at most six edges in quadrilateral $ABCD$, three edges in the triangle XYZ, and three edges between them, in total twelve edges, a contradiction.

Assume that the edge AC is not present in the graph. Note that each of the vertices X, Y, Z is connected to at most one of the vertices of quadrilateral $ABCD$, or connected to both B and D. Otherwise, we get a C_5 subgraph: for example, if X is connected to A and C, then $XADBC$ is a C_5 subgraph. There are five edges in the quadrilateral $ABCD$ and at most three edges in triangle XYZ. Hence there are at least five edges connecting vertices of $ABCD$ with vertices of XYZ. Then there are at least two vertices, say X and Y, that are connected to B and D. Hence $XBADY$ would be a cycle if the edge XY was in G. Therefore we can assume that there are at least six edges connecting vertices of $ABCD$ with vertices of XYZ, yielding X, Y, Z are connected to B and D. Thus triangle XYZ contains no edges in G, giving at most $5 + 6 = 11$ edges, a contradiction. The base case for $n = 7$ is proved and analogously, using case checking, one can prove the statement for $n = 8, 9, 10$.

In order to prove the induction step we use again Theorem 2: there is a $K_4 \backslash \{e\}$ subgraph in G, with the same notation $ABCD$. If $ABCD$ is a complete K_4 subgraph, then the remaining $n-4$ vertices are connected to at most one of the vertices of $ABCD$. By the induction hypothesis there are at most $\lfloor \frac{(n-4)^2}{4} \rfloor$ edges among the remaining $n-4$ vertices. Thus in total the number of edges is at most

$$6 + (n-4) + \left\lfloor \frac{(n-4)^2}{4} \right\rfloor \leq \left\lfloor \frac{n^2}{4} \right\rfloor .$$

Assume that the edge AC is not present in the graph. The remaining $n-4$ vertices are connected to at most one of the vertices of quadrilateral $ABCD$, or connected to both B and D. Hence, totally we get at most

$$5 + 2(n-4) + \left\lfloor \frac{(n-4)^2}{4} \right\rfloor - 1 = \left\lfloor \frac{n^2}{4} \right\rfloor$$

edges. The -1 appears from the impossibility of the following case: if all remaining $n-4$ vertices are connected to B and D with two edges, then these vertices among them are connected by no edge. The last equality proves the induction step and the theorem.

Remark. The author of this paper does not know any results for $H = C_{2m+1}$, $m \geq 3$. Since in the subgraph C_{2m+1} the density of the edges is less than in C_3, it is very probable that for all $n \geq N$ we have $ex(n, C_{2m+1}) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$.

3 Bounds for $ex(n; C_{2m})$

The bounds that prove the existence of an even cycle C_{2m} are much more subtle. The following result is due to Erdös (1965), who published it without proof, and due to Bondy and Simonovits (1974), who published a proof.

Theorem 4. Let $m \geq 2$ be a fixed integer. Then

$$ex(n, C_{2m}) \leq 10mn^{1+\frac{1}{m}}$$

for $n \geq 10^{m^2}$.

The proof of this result can be found in [1]. It follows that asymptotically $ex(n, C_{2m}) = O\left(n^{1+\frac{1}{m}}\right)$, in contrast to $ex(n, C_{2m+1}) = O\left(n^2\right)$. Let us present the proof for the upper bound when $m = 2$, which can be found in [1] or in [3].

Theorem 5. If a graph G on n vertices contains no C_4 subgraphs, then

$$|E| \leq \left\lfloor \frac{n}{4}(1 + \sqrt{4n-3}) \right\rfloor,$$

where E is the set of edges in the graph G.

Proof. Denote by $d(u)$ the degree of vertex u. Let S be the set of pairs $(u, \{v, w\})$, where u is adjacent to v and w, with $v \neq w$. We count the cardinality of S in two ways. Summing over u, we find $|S| = \sum_{u \in V} \binom{d(u)}{2}$. On the other hand, every two vertices v, w have at most one common neighbor, otherwise G contains a 4-cycle. Hence $|S| \leq \binom{n}{2}$, and we conclude

$$\sum_{u \in V} \binom{d(u)}{2} \leq \binom{n}{2},$$

or

$$\sum_{u \in V} d(u)^2 \leq n(n-1) + \sum_{u \in V} d(u).$$

Using the AM-GM inequality we have

$$\left(\sum_{u \in V} d(u)\right)^2 \leq n \sum_{u \in V} d(u)^2,$$

therefore

$$n \sum_{u \in V} d(u)^2 \leq n^2(n-1) + n \sum_{u \in V} d(u)$$

and $2|E| = \sum_{u \in V} d(u)$ imply

$$4|E|^2 \leq n^2(n-1) + 2n|E|,$$

or

$$|E|^2 - \frac{n}{2}|E| - \frac{n^2(n-1)}{4} \leq 0.$$

Solving this quadratic equation we obtain the desired result.

In [1] it is proved that the upper bound has the right order for $m = 2, 3, 5$. Namely, there exists some constant $c = c(m) > 0$ such that $ex(n, C_{2m}) \geq cn^{1+1/m}$ for $m = 2, 3, 5$. However, for $m = 4$ it is not known whether upper bound for C_8 has the right order.

4 Expected number of C_n subgraphs

Further we consider another problem: if it is known that the graph on n vertices has $|E|$ edges, what would be the expected number of C_n subgraphs in it? This problem is very hard and only recently, for $n = 3$, Alexander Razborov found an asymptotically explicit formula for the minimal possible density of triangles in a graph with edge density ρ where ρ is fixed (see [4]). In this article we present less tight bounds on the expected number of C_3 and C_4 subgraphs. The following result is due to Mc Kay (1963).

Theorem 6. Let G be a graph on n vertices with $|E|$ number of edges. If $|T|$ is the number of triangles in G, then

$$\frac{(4|E| - n^2)|E|}{3n} \leq |T| \leq \frac{(n-2)|E|}{3}.$$

Proof. Firstly we prove the right hand side of the inequality. Consider an arbitrary edge from E. The remaining $n - 2$ vertices can form at most $(n - 2)$ triangles that would contain this edge. Hence the greatest number of edges that triangles in G possess is $(n-2)|E|$. But each triangle contributes three edges to the sum, so $3|T| \leq (n-2)|E|$.

For the left side of the inequality, consider the set of pairs $(u, \{v, w\})$, where u is adjacent to v and w, with $v \neq w$. The number of elements in this set is $\sum_{i=1}^{n} \binom{d_i}{2}$, where $d_1, d_2, \ldots, d_n$ are the vertices' degrees in G. Again pick an arbitrary edge $\{v, w\}$ from the set E of edges. Each of the remaining $(n-2)$ vertices form at most $(n-2)$ pairs of the form $(u, \{v, w\})$, when the vertex u from the remaining $n-2$ vertices is connected to either v or w. If the vertex u is connected to both of them, then a triangle is formed, yielding two pairs $(u, \{v, w\})$. In this case one pair is accounted for the above mentioned sum, but the other is accounted for the triangle formed. Counting the number of such pairs, in total we get at most $(n-2)|E| + 3|T|$

pairs, since every triangle contributes three pairs to the sum. But each pair was counted twice, hence

$$2\sum_{i=1}^{n}\binom{d_i}{2} \le (n-2)|E| + 3|T|.$$

Using the AM-GM inequality we have

$$\sum_{i=1}^{n}\binom{d_i}{2} \ge \frac{1}{2n}\left(\sum_{i=1}^{n} d_i\right)^2 - \frac{1}{2}\sum_{i=1}^{n} d_i = \frac{2|E|^2 - n|E|}{n}.$$

Plugging this result in the upper inequality we get

$$2 \cdot \frac{2|E|^2 - n|E|}{n} \le (n-2)|E| + 3|T|,$$

yielding $\frac{4|E|^2 - n^2|E|}{n} \le 3|T|$, as desired.

The next theorem gives a lower bound for the number of C_4 subgraphs in a graph on n vertices with enough number of edges to always contain it.

Theorem 7. Let G be a graph on n vertices and $|E| > \left\lfloor \frac{n}{4}(1 + \sqrt{4n-3}) \right\rfloor$ edges. Then the number of C_4 subgraphs in G is at least

$$\frac{|E|(2|E| - n)(4|E|^2 - 2|E|n - n^2(n-1))}{2n^3(n-1)}.$$

Proof. Let $f_{ij} = |N_i \cap N_j|$, where N_i denotes the neighbors of the vertex v_i. Note that

$$\sum_{i=1}^{n}\binom{d_i}{2} = \sum_{1\le i<j\le n} f_{ij},$$

this follows from counting the set of pairs $(u, \{v, w\})$, where u is adjacent to v and w, with $v \ne w$. Note that S, the number of C_4 subgraphs in G, is equal to

$$\sum_{1\le i<j\le n}\binom{f_{ij}}{2}.$$

Using the AM-GM inequality we have

$$\sum_{1\le i<j\le n}\binom{f_{ij}}{2} \ge \frac{1}{2\binom{n}{2}}\left(\sum_{1\le i<j\le n} f_{ij}\right)^2 - \frac{1}{2}\sum_{1\le i<j\le n} f_{ij}.$$

Similarly,

$$\sum_{i=1}^{n}\binom{d_i}{2} \ge \frac{1}{2n}\left(\sum_{i=1}^{n} d_i\right)^2 - \frac{1}{2}\sum_{i=1}^{n} d_i = \frac{2|E|^2 - n|E|}{n}.$$

Thus

$$S \geq \frac{1}{\binom{n}{2}} \left(\frac{2|E|^2 - n|E|}{n} \right)^2 - \frac{1}{2} \left(\frac{2|E|^2 - n|E|}{n} \right),$$

which is equivalent to

$$S \geq \frac{|E|(2|E| - n)(4|E|^2 - 2|E|n - n^2(n-1))}{2n^3(n-1)}.$$

5 Bibliography

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