About a nice inequality

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In [1.](page 75), Gabriel Dospinescu, Mircea Lascu and Marian Tetiva gave two solutions to the following inequality:

Let a, b, c be three nonnegative real numbers. Show that:

$$a^2 + b^2 + c^2 + 2abc + 3 \geq (1 + a)(1 + b)(1 + c). \tag{1}$$

On the Mathlinks Forum, Darij Grinberg proved a sharper inequality:

For any a, b, c positive real numbers, the following inequality holds:

$$a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab + bc + ca). \tag{2}$$

The solution given by Darij on the Mathlinks Forum is based on Schur's inequality written in the form:

$$2(ab + bc + ca) - (a^2 + b^2 + c^2) \leq \frac{9abc}{a + b + c} \tag{3}$$

and AM-GM,

$$2abc + 1 = abc + abc + 1 \geq 3\sqrt[3]{a^2b^2c^2} \tag{4}$$

From (3) and (4), the inequality (2) reduces to:

$$3\sqrt[3]{a^2b^2c^2} \geq \frac{9abc}{a + b + c}, \tag{5}$$

which is equivalent to $a + b + c \geq 3\sqrt[3]{abc}$, which is plain AM-GM.
Vasile Cârtoaje gave another solution to (2). (see [2], page 17).
In this paper, we shall give another solution to (2) via convex functions using Popoviciu's inequality.

Theorem 1. *(Tiberiu Popoviciu, 1965)*
Let I be an interval and $f : I \to \mathbb{R}$ a convex function. Then for any $x, y, z \in I$ the following inequality is valid:

$$f(x)+f(y)+f(z)+3f\left(\frac{x+y+z}{3}\right) \geq 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right].$$

A nice proof of this result can be found in [3].
Now, let us consider the function $f : \mathbb{R} \to (0, \infty)$, $f(t) = \exp(2t)$. A simple calculation of derivatives shows that f is convex. By Popoviciu's inequality, we obtain:

$$\exp(2x) + \exp(2y) + \exp(2z) + 3\exp\left(\frac{2(x+y+z)}{3}\right)$$
$$\geq 2[\exp(x+y) + \exp(y+z) + \exp(z+x)].$$

which is equivalent to:

$$\exp^2(x) + \exp^2(y) + \exp^2(z) + 3\sqrt[3]{\exp^2(x) \cdot \exp^2(y) \cdot \exp^2(z)}$$
$$\geq 2[\exp(x) \cdot \exp(y) + \exp(y) \cdot \exp(z) + \exp(z) \cdot \exp(x)].$$

Now, denote $\exp(x) = a > 0$, $\exp(y) = b > 0$, $\exp(z) = c > 0$. We get the inequality:

$$a^2 + b^2 + c^2 + 3\sqrt[3]{a^2b^2c^2} \geq 2(ab + bc + ca). \tag{6}$$

Now, again from AM-GM,

$$2abc + 1 = abc + abc + 1 \geq 3\sqrt[3]{a^2b^2c^2}$$

and (6), the conclusin follows.
Now we give a few applications of (6). The first preoblem has been given at the first IMO Romanian test in 2001 (proposed by Mircea Becheanu) .

A1. *Let a, b, c three positive reals. Prove that*

$$\sum_{cyc}(b + c - a)(c + a - b) \leq \sqrt{abc}(\sqrt{a} + \sqrt{b} + \sqrt{c}). \tag{7}$$

Solution. One can verify by a simple calculation, that:

$$\sum_{cyc}(b + c - a)(c + a - b) = 2(ab + bc + ca) - (a^2 + b^2 + c^2).$$

Now, by (6), we only need to see that

$$3\sqrt[3]{a^2b^2c^2} \leq \sqrt{abc}(\sqrt{a} + \sqrt{b} + \sqrt{c}) \tag{8}$$

which follows from AM-GM, $\sqrt{a} + \sqrt{b} + \sqrt{b} \geq \sqrt[3]{\sqrt{abc}} = 3\sqrt[6]{abc}$.

The next problem was one of the shortlisted problems of Romanian National Olympiad, 2004 (proposed by Valentin Vornicu).

A2. *Let a, b, c three positive reals. Show that*

$$\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})+(a+b+c)^2 \geq 4\sqrt{3abc(a+b+c)}. \tag{9}$$

Solution. By A1, we get:

$$\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c}) \geq 2(ab+bc+ca)-(a^2+b^2+c^2). \tag{10}$$

Now, we only need to prove that:

$$ab+bc+ca \geq \sqrt{3abc(a+b+c)}, \tag{11}$$

which is evident.

The next application is from the Romanian Junior Selection test for Balkan Olympiad 2005 and it was proposed by the first author of this paper.

A3. *Let x, y, z three positive reals such that*

$$(x+y)(y+z)(z+x)=1.$$

Show that

$$xy+yz+zx \leq \frac{3}{4}. \tag{12}$$

Solution. We put $x+y=c, z+x=b$ and $y+z=a$. It follows that $x=\dfrac{b+c-a}{2}, y=\dfrac{c+a-b}{2}$ and $z=\dfrac{a+b-c}{2}$. Now, we only need to prove that

$$\sum_{cyc}(b+c-a)(c+a-b) \leq 3. \tag{13}$$

which is equivalent to

$$2(ab+bc+ca) \leq a^2+b^2+c^2+3.$$

But, since $abc=1$ this last inequality is a simple consequence of (6).

The following problem is quite hard. It was proposed by Hojoo Lee at the Asia-Pacific Olympiad in 2004.

A4. *Let a, b, c three positive reals. Show that*

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca). \tag{14}$$

Solution. Expanding everything, the problem reduces to

$$(abc)^2+2(a^2b^2+b^2c^2+c^2a^2)+4(a^2+b^2+c^2)+8 \geq 9(ab+bc+ca). \tag{15}$$

From the evident inequalities $3(a^2+b^2+c^2) \geq 3(ab+bc+ca)$ and $2(a^2b^2+b^2c^2+c^2a^2)+6 \geq 4(ab+bc+ca)$, we only need to prove that

$$(abc)^2 + a^2 + b^2 + c^2 \geq 2(ab+bc+ca). \tag{16}$$

Now, by (6), we only need to see that $(abc)^2+2 \geq \sqrt[3]{a^2b^2c^2}$, which is AM-GM.

The following problem was given by Titu Andreescu at the USA Selections test for IMO in 2000.

A5. *Prove that for any positive real numbers a, b, c the following inequality holds*

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \leq \max\{(\sqrt{a}-\sqrt{b})^2, (\sqrt{b}-\sqrt{c})^2, (\sqrt{c}-\sqrt{a})^2\} \tag{17}$$

Solution. It is clear that

$$\frac{(\sqrt{a}-\sqrt{b})^2 + (\sqrt{b}-\sqrt{c})^2 + (\sqrt{b}-\sqrt{c})^2}{3}$$
$$\leq \max\{(\sqrt{a}-\sqrt{b})^2, (\sqrt{b}-\sqrt{c})^2, (\sqrt{c}-\sqrt{a})^2\}$$

So, we are left to prove that

$$a+b+c+3\sqrt[3]{abc} \geq 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}) \tag{18}$$

which is (13). Thus the problem is solved.

The next problem is a geometric inequality due to George Polya and Gabor Szego.

A6. *Prove that if a, b, c are the sidelenghts of a triangle and S its area, then*

$$abc \geq \left(\frac{4S}{\sqrt{3}}\right)^{3/2} \tag{19}$$

Solution. The given inequality is equivalent to

$$3\sqrt[3]{a^2b^2c^2} \geq 4S\sqrt{3}. \tag{20}$$

Now, using (6) we only need to prove that

$$2(ab+bc+ca) - (a^2+b^2+c^2) \geq 4S\sqrt{3}. \tag{21}$$

But, this inequality is equivalent to Hadwiger-Finsler inequality (see [4.])

$$a^2+b^2+c^2 \geq 4S\sqrt{3} + (a-b)^2 + (b-c)^2 + (c-a)^2 \tag{22}$$

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