On a Method of Proving Symmetric Inequalities

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Abstract

In this article we develop a method for proving a special class of inequalities.

1 Introduction

The following problem was proposed at the International Mathematical Olympiad in 2001:

Problem 1. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1. \tag{1}$$

Solution. One of the solutions to this problem is based on the inequality

$$\frac{a}{\sqrt{a^2+8bc}} \geq \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}} \tag{2}$$

and the two similar ones. Inequality (1) is then obtained by summing up these three inequalities.

In what follows, we show that a similar approach applies to a larger class of inequalities. In particular, we explain how the number $\frac{4}{3}$ in (2) comes about.

2 Preliminaries

Given a polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

we let $\mu(P) = na_n + (n-1)a_{n-1} + \cdots + a_0$.

Lemma 1. *If $P(x) = (x-1)^2(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0)$, then $\mu(P) = 0$.*

Proof. The lemma follows directly from the definition of μ after opening the brackets. ∎

Lemma 2. *Let $P(x)$ be a polynomial with real coefficients. Suppose $P(x) \geq 0$ for all $x \geq 0$. If $a > 0$ is a root of P, then it has multiplicity at least 2.*

Proof. Note that P has a local minimum at a. Using Taylor's series and the fact that $P(a) = P'(a) = 0$, we get

$$P(x) = (x-a)^2 \left(\frac{P''(a)}{2!} + \frac{P^{(3)}(a)}{3!}(x-a) + \cdots + \frac{P^{(n)}(a)}{n!}(x-a)^{n-2} \right).$$

∎

The next lemma provides us with a useful tool.

Lemma 3 (Main Lemma). *Let $f(x) = \beta_n x^{\alpha_n} + \beta_{n-1} x^{\alpha_{n-1}} + \cdots + \beta_1 x^{\alpha_1} + \beta_0$ with real β_i's and nonnegative rational α_i's. Suppose that $f(x) \geq 0$ for all $x \geq 0$ and $f(1) = 0$. Then*

$$\beta_n \alpha_n + \beta_{n-1}\alpha_{n-1} + \cdots + \beta_1 \alpha_1 = 0.$$

Proof. Choose a positive integer l so that each product $l\alpha_i$ is integer. Note that $P(x) = f(x^l)$ is a polynomial with $P(1) = 0$ and $P(x) \geq 0$ for all $x \geq 0$. By Lemma 2 we have $P(x) = (x-1)^2 Q(x)$. From Lemma 1 it follows that $\mu(P) = 0$. On the other hand, we have $\mu(P) = l(\beta_n\alpha_n + \beta_{n-1}\alpha_{n-1} + \cdots + \beta_1\alpha_1)$, which finishes the proof. ∎

3 Solving Problems by Using the Main Lemma

Problem 2 (Nesbitt [1]). *Prove that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

for all positive real a, b and c.

Solution. It suffices to find a positive rational α such that for all positive real numbers a, b, and c,

$$\frac{a}{b+c} \geq \frac{3a^\alpha}{2(a^\alpha + b^\alpha + c^\alpha)}. \tag{3}$$

Summing up (3) with its two similar inequalities will yield the desired inequality. Substituting $b = c = 1$ in (3), we obtain the following condition on α: $f(a) \geq 0$ for all $a > 0$, where $f(a) = a^{\alpha+1} + 2a - 3a^\alpha$. Note that $f(1) = 0$. By Lemma 3, we have $\alpha + 1 + 2 - 3\alpha = 0$ and hence $\alpha = \frac{3}{2}$. Let us now prove that (3) with $\alpha = \frac{3}{2}$ is true. This inequality can be rewritten as

$$a^{\frac{5}{2}} + 2b^{\frac{3}{2}}a + c^{\frac{3}{2}}a \geq 3a^{\frac{3}{2}}b + 3a^{\frac{3}{2}}c$$

which is the sum of the following two inequalities

$$a^{\frac{5}{2}} + b^{\frac{3}{2}}a + b^{\frac{3}{2}}a \geq 3a^{\frac{3}{2}}b \quad \text{and} \quad a^{\frac{5}{2}} + c^{\frac{3}{2}}a + c^{\frac{3}{2}}a \geq 3a^{\frac{3}{2}}c.$$

These two hold by the AM-GM inequality.

Problem 1 can be solved similarly. The details are left to the interested reader.

Problem 3 (Austria, 1970)**.** *Prove that*

$$\frac{a+b+c}{2} \geq \frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} \tag{4}$$

for all positive real numbers $a, b,$ and c.

Solution. It suffices to find a positive rational α such that for all positive real numbers a, b, and c,

$$\frac{b^\alpha + c^\alpha}{2(a^\alpha + b^\alpha + c^\alpha)} \geq \frac{2bc}{(b+c)(a+b+c)} \tag{5}$$

and then use the same method as in the solution to Problem 1. Setting $a = b = 1$ in (5) we obtain the inequality:

$$c^{\alpha+2} - c^{\alpha+1} + 2c^\alpha + c^2 - 5c + 2 \geq 2$$

for all $c \geq 0$. By using Lemma 3 we conclude that $\alpha = 1$. It is easy to prove that (5) with $\alpha = 1$ is indeed, because in this case we can rewrite it as

$$(b-c)^2 \geq 0.$$

Problem 4. *Prove that*

$$a^3b + b^3c + c^3a \geq abc(a+b+c) \tag{6}$$

for all nonnegative real numbers a, b, and c.

Solution. It suffices to find three nonnegative real numbers k, l, and m such that $k + l + m = 1$ and

$$ka^3b + lb^3c + mc^3a \geq a^2bc \tag{7}$$

for all nonnegative real numbers a, b, c. Once this is done, we can use the familiar method: write the two similar inequalities

$$ma^3b + kb^3c + lc^3a \geq b^2ac \tag{8}$$

and

$$la^3b + mb^3c + kc^3a \geq c^2ab \tag{9}$$

and then sum up (7), (8), and (9) to arrive at (6). We know that $ka^3b + lb^3c + (1 - k - l)c^3a \geq a^2bc$. Substituting $y = z = 1$ and $m = 1 - k - l$ in (7), we obtain:

$$ka^3 + l + (1 - k - l)a \geq a^2$$

for all $a \geq 0$. Consider the function $f(a) = ka^3 - a^2 + (1 - k - l)a + l$. Because $f(1) = 0$ and $f(a) \geq 0$ for all $a \geq 0$, by Lemma 3 we conclude that

$$2k - l = 1. \tag{10}$$

In the same manner we take $a = c = 1$ in (6) and conclude that

$$k + 3l = 1. \tag{11}$$

Solving the system of equations (10)–(11), we get $k = \frac{4}{7}$, $l = \frac{1}{7}$, and hence $m = \frac{2}{7}$. Let us prove that inequality (7) is true for these values, that is

$$\frac{4}{7}a^3b + \frac{1}{7}b^3c + \frac{2}{7}c^3a \geq a^2bc. \tag{12}$$

It can be proven by using the AM-GM inequality, because inequality (12) can be writen as

$$a^3b + a^3b + a^3b + a^3b + b^3c + c^3a + c^3a \geq 7a^2bc$$

4 Independent Study Problems

Problem 5. *Prove that*

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \geq 2$$

for all positive real numbers a, b, and c.

Problem 6. *Let ABC be a triangle. Prove that*

$$\frac{\sin B \sin C}{\sin^2 \frac{A}{2}} + \frac{\sin C \sin A}{\sin^2 \frac{B}{2}} + \frac{\sin A \sin B}{\sin^2 \frac{C}{2}} \geq 9.$$

Problem 7 (Ukraine 2006)**.** *Prove that*

$$3(a^3 + b^3 + c^3 + abc) \geq 4(a^2b + b^2c + c^2a)$$

for all positive real numbers a, b, and c.

Problem 8 (IMO 2001/2, generalization)**.** *Prove that*

$$\frac{a}{\sqrt{a^2 + \lambda bc}} + \frac{b}{\sqrt{b^2 + \lambda ca}} + \frac{c}{\sqrt{c^2 + \lambda ab}} \geq \frac{3}{\sqrt{1 + \lambda}}$$

for all positive real numbers a, b, c and any real number $\lambda \geq 8$.

References

[1] A. M. Nesbitt, Problem 15114, *Educational Times*, 3(1903), 37-38.

[2] H. Lee, T. Lovering, and C. Pohoata, *Infinity*, `http://www.cpohoata.com/wp-content/uploads/2008/10/inf081019.pdf`.

[3] Mathlinks, $x/(y+z)^2 + y/(z+x)^2 + z/(x+y)^2 >= 9/(4(x+y+z))$, `http://www.mathlinks.ro/viewtopic.php?t=5084`.