Junior problems

J181. Let a, b, c, d be positive real numbers. Prove that

$$\left(\frac{a+b}{2}\right)^3 + \left(\frac{c+d}{2}\right)^3 \leq \left(\frac{a^2+d^2}{a+d}\right)^3 + \left(\frac{b^2+c^2}{b+c}\right)^3$$

Proposed by Pedro H. O. Pantoja, Natal-RN, Brazil

J182. Circles $C_1(O_1, r)$ and $C_2(O_2, R)$ are externally tangent. Tangent lines from O_1 to C_2 intersect C_2 at A and B, while tangent lines from O_2 to C_1 intersect C_1 at C and D. Let $O_1A \cap O_2C = \{E\}$ and $O_1B \cap O_2D = \{F\}$. Prove that $EF \cap O_1O_2 = AD \cap BC$.

Proposed by Roberto Bosch Cabrera, Florida, USA

J183. Let x, y, z be real numbers. Prove that

$$(x^2+y^2+z^2)^2 + xyz(x+y+z) \geq \frac{2}{3}(xy+yz+zx)^2 + (x^2y^2+y^2z^2+z^2x^2).$$

Proposed by Neculai Stanciu, George Emil Palade, Buzau, Romania

J184. Find all quadruples (x, y, z, w) of integers satisfying the system of equations

$$x+y+z+w = xy+yz+zx+w^2-w = xyz-w^3 = -1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J185. Let $H(x, y) = \frac{2xy}{x+y}$ be the harmonic mean of the positive real numbers x and y. For $n \geq 2$, find the greatest constant C such that for any positive real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ the following inequality holds

$$\frac{C}{H(a_1+\cdots+a_n, b_1+\cdots+b_n)} \leq \frac{1}{H(a_1, b_1)} + \cdots + \frac{1}{H(a_n, b_n)}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

J186. Let ABC be a right triangle with $AC = 3$ and $BC = 4$ and let the median AA_1 and the angle bisector BB_1 intersect at O. A line through O intersects hypotenuse AB at M and AC at N. Prove that

$$\frac{MB}{MA} \cdot \frac{NC}{NA} \leq \frac{4}{9}.$$

Proposed by Valcho Milchev, Kardzhali, Bulgaria

Senior problems

S181. Let a and b be positive real numbers such that

$$|a-2b| \le \frac{1}{\sqrt{a}} \quad \text{and} \quad |2a-b| \le \frac{1}{\sqrt{b}}.$$

Prove that $a+b \le 2$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S182. Let a, b, c be real numbers such that $a > b > c$. Prove that for each real number x the following inequality holds

$$\sum_{\text{cyc}} (x-a)^4(b-c) \ge \frac{1}{6}(a-b)(b-c)(a-c)[(a-b)^2+(b-c)^2+(c-a)^2].$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

S183. Let $a_0 \in (0,1)$ and $a_n = a_{n-1} - \frac{a_{n-1}^2}{2}$, $n \ge 1$. Prove that for all n,

$$\frac{n}{2} \le \frac{1}{a_n} - \frac{1}{a_0} < \frac{n+1+\sqrt{n}}{2}.$$

Proposed by Arkady Alt, San Jose, California, USA

S184. Let $H_n = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, $n \ge 2$. Prove that

$$e^{H_n} > \sqrt[n]{n!} \ge 2^{H_n}.$$

Proposed by Tigran Hakobyan, Vanadzor, Armenia

S185. Let A_1, A_2, A_3 be non-collinear points on parabola $x^2 = 4py$, $p > 0$, and let $B_1 = l_2 \cap l_3, B_2 = l_3 \cap l_1, B_3 = l_1 \cap l_2$ where l_1, l_2, l_3 are tangents to the parabola at points A_1, A_2, A_3, respectively. Prove that $\dfrac{[A_1A_2A_3]}{[B_1B_2B_3]}$ is a constant and find its value.

Proposed by Arkady Alt, San Jose, California, USA

S186. We wish to assign probabilities p_k, $k = 0, 1, 2, 3$, to random variables X_1, X_2, and X_3 taking values in the set $\{0, 1, 2, 3\}$ (some of them possibly with probability 0), such that the X_i, $i = 1, 2, 3$, will be identically distributed with $P(X_i = k) = p_k$, $k = 0, 1, 2, 3$, and $X_1 + X_2 + X_3 = 3$. Prove that this is possible if and only if $p_2 + p_3 \le 1/3$, $p_1 = 1 - 2p_2 - 3p_3$, and $p_0 = p_2 + 2p_3$.

Proposed by Shai Covo, Kiryat-Ono, Israel

Undergraduate problems

U181. Consider sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, where $a_0 = b_0 = 1$, $a_{n+1} = a_n + b_n$, and $b_{n+1} = (n^2+n+1)a_n + b_n,\ n \geq 1$. Evaluate $\lim_{n\to\infty} B_n$, where

$$B_n = \frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}}.$$

Proposed by Neculai Stanciu, George Emil Palade, Buzau, Romania

U182. Find all continuous functions f on $[0,1]$ such that $f(x) = c$ if $x \in \left[0, \frac{1}{2}\right]$ and $f(x) = f(2x-1)$ if $x \in \left(\frac{1}{2}, 1\right]$, where c is a given constant.

Proposed by Arkady Alt, San Jose, California, USA

U183. Let m and n be positive integers. Prove that

$$\sum_{k=0}^{n} \frac{1}{k+m+1}\binom{n}{k} \leq \frac{(m+2n)^{m+n+1} - n^{m+n+1}}{(m+n+1)(m+n)^{m+n+1}}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

U184. Let $f, g : [a,b] \to \mathbb{R}$ be differentiable functions such that $\int\limits_a^b f(x)dx = 0$. Prove that there is some $c \in (a,b)$ satisfying

$$f'(c)\int\limits_c^b g(x)dx + g'(c)\int\limits_c^b f(x)dx = 2f(c)g(c).$$

Proposed by Duong Viet Thong, National Economics University, Vietnam

U185. Determine if there is a non-constant complex analytic function satisfying the conditions:

(i) $f(f(z)) = f(z)$ for all complex numbers z

(ii) there is a complex number z_0, such that $f(z_0) \neq z_0$.

Proposed by Harun Immanuel, Airlangga University, Indonesia

U186. Let $(A,+,\cdot)$ be a finite ring of characteristic ≥ 3 such that $1+x \in U(A)\bigcup\{0\}$ for each $x \in U(A)$. Prove that A is a field.

Proposed by Sorin Radulescu, Aurel Vlaicu College, Bucharest and Mihai Piticari, Dragos Voda College, Campulung Moldovenesc, Romania

Olympiad problems

O181. Let a, b, c be the sidelengths of a triangle. Prove that

$$\sqrt{\frac{abc}{-a+b+c}}+\sqrt{\frac{abc}{a-b+c}}+\sqrt{\frac{abc}{a+b-c}} \geq a+b+c.$$

Proposed by Titu Andreescu, Unversity of Texas at Dallas, USA and Gabriel Dospinescu, Ecole Normale Superieure, France

O182. On side BC of triangle ABC consider m points, on CA n points, and on AB s points. Join the points from sides AB and AC with the points on side BC. Determine the maximum number of the points of intersection situated in the interior of triangle ABC.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

O183. Evaluate

$$\sum_{k=1}^{2010} \tan^4\left(\frac{k\pi}{2011}\right).$$

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

O184. Points A, B, C, D lie on a line in this order. Using a straight edge and a compas construct parallel lines a and b through A and B, and parallel lines c and d through C and D, such that their points of intersection are vertices of a rhombus.

Proposed by Mihai Miculita, Oradea, Romania

O185. Find the least integer $n \geq 2011$ for which the equation

$$x^4+y^4+z^4+w^4-4xyzw=n$$

is solvable in positive integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O186. Let n be a positive integer. Prove that each odd common divisor of

$$\binom{2n}{n}, \binom{2n-1}{n}, \ldots, \binom{n+1}{n}$$

is a divisor of 2^n-1.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania