Solutions for Admission Test A

Problem 1. In a *magic square*, the sum of the three entries in each row, column, or diagonal has the same value. The figure shows four of the entries of a *magic square*. What is x?

		3
X	4	5

Solution by Jacob Emmert-Aronson

Solution. We have the matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, where we know that

$$a+b+c = d+e+f = g+h+i = a+d+g = b+e+h = c+f+i = a+e+i = c+e+g$$

and $c = 3, g = x, h = 4$, and $i = 5$. We will define $S = a+b+c$. Now, the center value, e is equal to $S-3-x$, which, due to the fact that $x = S-4-5 = S-9$, can be rewritten as $S-3-S+9 = 6$. Because $S = x+9, f = x+1$ and $b = x-1$, we have

$$\begin{pmatrix} a & x-1 & 3 \\ d & 6 & x+1 \\ x & 4 & 5 \end{pmatrix}.$$

Now, $a = S-(x-1) = S-(S-10)-3 = 7$ and $d = S-6-(S-8) = 2$, thus we get

$$\begin{pmatrix} 7 & x-1 & 3 \\ 2 & 6 & x+1 \\ x & 4 & 5 \end{pmatrix}.$$

We have a diagonal of the matrix in which we know all the terms. $S = 7+6+5 = 18$, and $x = 18-9 = 9$. The matrix with all values filled in is

$$\begin{pmatrix} 7 & 8 & 3 \\ 2 & 6 & 10 \\ 9 & 4 & 5 \end{pmatrix}.$$

Indeed, all rows, columns, and diagonals do add up to the same sum 18.

Problem 2. Find the least positive integer whose product of digits is 10!.

Solution by Jessica Oehrlein

Solution. Observe that $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 3628800$. In order to get the least positive integer we need to find the least number of digits that should be used in the construction. The process that gives us such number is step by step picking up the largest possible digit from the product $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$. Thus we we have a first digit 9 that comes from 9 and the second 9 comes from the 3 and the 3 in 6. This leaves us with $10 \cdot 8 \cdot 7 \cdot 2 \cdot 5 \cdot 4 \cdot 2$. Now we cannot extract 9 and therefore we try to extract 8. The first 8 comes from the 8 and the second 8 comes from 2 and 4. Continuing the idea of the process, the 7 comes and leaves us with $10 \cdot 5 \cdot 2$. Even though we have a 2 left, we have no 3, so we cannot divide by 6. The first 5 comes from the 5. The second 5 comes from the 5 in 10. So we have $2, 2, 1$ left; from here we extract 4. It follows that we have 8 digits formed: $9, 9, 8, 8, 7, 5, 5, 4$. We order the digits from the least to greatest to get the least number whose product of digits is 10! and this is

$$45578899.$$

Problem 3. The numbers $d_1, d_2, ..., d_6$ are distinct digits of the decimal system and are different from 6. Prove that $d_1 + d_2 + ... + d_6 = 36$ if and only if $(d_1 - 6)(d_2 - 6) \cdot ... \cdot (d_6 - 6) = -36$.

Solution by Kevin Sackel

Solution. The first step of our proof is to show that if the sum of d_1, d_2, d_3, d_4, d_5, and d_6 equals 36 and each is a distinct digit of the decimal system and different from 6, then the equation $(d_1 - 6)(d_2 - 6)(d_3 - 6)(d_4 - 6)(d_5 - 6)(d_6 - 6) = -36$ holds. Assume the contrary.

Observe that if we add all of the digits from 0 to 9 not including 6, we get

$$0 + 1 + 2 + 3 + 4 + 5 + 7 + 8 + 9 = 39.$$

Now we must take away three of the added digits to get a new sum of 36. In other words, these three digits must add up to 3. The only way to do this is to subtract the $0, 1$, and 2. So the digits d_1 through d_6 must be $3, 4, 5, 7, 8$, and 9 in some order. This satisfies the equation $(d_1 - 6)(d_2 - 6)(d_3 - 6)(d_4 - 6)(d_5 - 6)(d_6 - 6) = -36$, and the first step is complete.

The second step is to prove the backward statement. Each factor $d_k - 6$ is either $-6, -5, -4, -3, -2, -1, 1, 2$, or 3. The only six factors that multiply to -36 are $-3, -2, -1, -1, 1, 2$, and 3. Thus d_1 through d_6 are $3, 4, 5, 7, 8$, and 9 in some order, which is exactly what we found before.

Problem 4. In the standard 8×8 chessboard (with squares colored alternatively black and white) there are 64 1×1 squares, 49 2×2 squares, etc. How many of those squares have more than half of their areas colored black?

Solution by Eric Guan

Solution. The total number of squares is

$$\underbrace{64}_{1\times1} + \underbrace{49}_{2\times2} + \underbrace{36}_{3\times3} + \underbrace{25}_{4\times4} + \underbrace{16}_{5\times5} + \underbrace{9}_{6\times6} + \underbrace{4}_{7\times7} + \underbrace{1}_{8\times8} = 204.$$

Every square with even side length will have an equal number of black and white squares, so we only must consider the odd cases.

In a square with odd side length, there is one more 1×1 square of the color in one of the corners. As the chessboard is symmetric and number of white colors is equal to black, we get that the number of square with more black 1×1 squares is half of all squares with odd side:

$$\frac{1}{2} \cdot (\underbrace{64}_{1\times1} + \underbrace{36}_{3\times3} + \underbrace{16}_{5\times5} + \underbrace{4}_{7\times7}) = 60.$$

Problem 5. In a chess (round-robin) tournament, five participants withdrew after having played two games each. If 100 games were played in all, what was the initial number of participants?

Solution by Ivan Tolkachev

Solution. Let n be the number of players that played till the end of the tournament. The number of games played by them is $\binom{n}{2}$. Since the total number of games played was 100 and 5 players left after playing 2 games each, let the number of games they played equal to t. Observe that $5 \leq t \leq 10$, because the maximum is attained when these 5 players played 10 games with somebody who played till the end. Minimum is attained when 5 players played between them, because then every game between them counts as for the first as for the second. Thus using that $\frac{(n-1)n}{2} = 100 - t$ we get

$$90 \leq \frac{(n-1)n}{2} \leq 95.$$

The only integer that fits this range is $n = 14$, because $\frac{12 \cdot 13}{2} < 90 < 95 < \frac{14 \cdot 15}{2}$. It follows that the number of players that played till the end of tournament is 14, and therefore totally there were 19 participants.

Problem 6. Find all quadruples (x, y, z, w) of positive integers such that

$$x^2 + y^2 + z^2 + w^2 = 3(x + y + z + w).$$

Solution by Jean Feng

Solution. Completing the square we get

$$(x - \frac{3}{2})^2 + (y - \frac{3}{2})^2 + (z - \frac{3}{2})^2 = 4 \cdot \frac{9}{4}$$

or

$$(2x - 3)^2 + (2y - 3)^2 + (2z - 3)^2 + (2w - 3)^2 = 36.$$

All squares are odd numbers. Observe that 36 can be written as the sum of four odd squares in two ways

$$1 + 1 + 9 + 25 = 36 \text{ and } 9 + 9 + 9 + 9 = 36.$$

From here we deduce that (x, y, z, w) is a permutation of

$$(1, 1, 3, 4),\ (2, 2, 3, 4),\ (1, 2, 3, 4),\ \text{and } (3, 3, 3, 3).$$

Problem 7. The www.awesomemath.org homepage displays the triangle "Forum", the equilateral triangles "Summer Program", "Year-round" and "Math Reflections", as well as three "white" lines that appear to pass through the same point. Prove that these lines are indeed concurrent.

Solution. Let us take a triangle ABC and construct outside equilateral triangles ABZ, ACY, BCX. Using the Law of Sines we get

$$\frac{BX}{\sin\angle BAX} = \frac{AX}{\sin\angle ABX} \text{ and } \frac{CX}{\sin\angle CAX} = \frac{AX}{\sin\angle ACX}.$$

Denoting the angles of the triangle ABC by α, β, γ, we get $\angle ABX = 60^\circ + \beta, \angle ACX = 60^\circ + \gamma$. Therefore

$$\frac{BX}{\sin\angle BAX} = \frac{AX}{\sin(60^\circ + \beta)} \text{ and } \frac{CX}{\sin\angle CAX} = \frac{AX}{\sin(60^\circ + \gamma)}.$$

It follows that

$$\frac{\sin\angle BAX}{\sin\angle CAX} = \frac{\sin(60^\circ + \beta)}{\sin(60^\circ + \gamma)}.$$

Analogously,

$$\frac{\sin\angle ACZ}{\sin\angle BCZ} = \frac{\sin(60^\circ + \alpha)}{\sin(60^\circ + \beta)}\ ;\ \frac{\sin\angle CBY}{\sin\angle ABY} = \frac{\sin(60^\circ + \gamma)}{\sin(60^\circ + \alpha)}.$$

Using the converse of Ceva's Theorem in trigonometric form we get

$$\frac{\sin\angle BAX}{\sin\angle CAX}\cdot\frac{\sin\angle ACZ}{\sin\angle BCZ}\cdot\frac{\sin\angle CBY}{\sin\angle ABY} = \frac{\sin(60^\circ + \beta)}{\sin(60^\circ + \gamma)}\cdot\frac{\sin(60^\circ + \alpha)}{\sin(60^\circ + \beta)}\cdot\frac{\sin(60^\circ + \gamma)}{\sin(60^\circ + \alpha)} = 1,$$

so the lines AX, BY, CZ are concurrent.

Problem 8. The average age of the 2006 AwesomeMath Summer Program participants (students and assistants) would have increased by one month either if three additional 18 year olds had enrolled or if three 12 year olds had not participated. What was the number of participants?

Solution by Rohit Agrawal

Solution. We let the number of participants be n and the sum of the ages in years of all n participants be k. The average age of participants is $\frac{k}{n}$ and an 1 month average increase is equal to $+\frac{1}{12}$. With the given information, we setup the following equations

$$\frac{k-12\cdot 3}{n-3}=\frac{k}{n}+\frac{1}{12}$$

$$\frac{k+18\cdot 3}{n+3}=\frac{k}{n}+\frac{1}{12}$$

From here, we start solving:

$$\frac{k-12\cdot 3}{n-3}=\frac{k+18\cdot 3}{n+3}$$

$$(k-36)(n+3)=(k+54)(n-3)$$

$$nk-36n+3k-108=nk+54n-3k-162$$

$$6k=90k-54,\ \ k=15n-9.$$

We substitute this value k into the first equation:

$$\frac{15n-9-36}{n-3}=\frac{15n-9}{n}+\frac{1}{12}$$

$$12n(15n-45)=12(15n-9)(n-3)+n(n-3)$$

$$180n^2-540n=180n^2-540n-108n+324+n^2-3n$$

$$0=n^2-111n+324$$

$$(n-108)(n-3)=0$$

$$n=108 \text{ or } n=3.$$

As $n=3$ gives us division by zero, we get that $n=108$. Therefore we know that the answer and the number of participants of the AwesomeMath 2006 Math Program is 108.

Problem 9. Find the least positive integer n such that each n-element subset of the set $\{1, 2, ..., 2007\}$ contains two elements, not necessarily distinct, such that their sum is a power of 2.

Solution by Wenyo Cao

Solution. We claim that the least possible n is 1002. We will prove that if n is any less, the condition will not be satisfied.

If $n = 1001$, let

$A = \{1025, 1026, ..., 2007\}$, $|A| = 2007 - 1025 + 1 = 983$

$B = \{33, 34, ..., 40\}$, $|B| = 40 - 33 + 1 = 8$

$C = \{17, 18, ..., 23\}$, $|C| = 23 - 17 + 1$

$D = \{5, 6, 7\}$, $|D| = 7 - 5 + 1 = 3.$

If $S = A \cup B \cup C \cup D$ is the subset, then the condition is not satisfied and $|S| = 983 + 8 + 7 + 3 = 1001$. For values of n less than 1001, we just keep removing elements from S.

To prove that the conditions are satisfied when $n = 1002$, we partition the set $\{1, 2, ..., 2007\}$ into 1001 two-element sets

$$\{1, 7\}, \{9, 23\}, \{24, 40\}, \{41, 2007\}$$

$$\{2, 6\}, \{10, 22\}, \{25, 39\}, \{42, 2006\}$$

$$\{3, 5\}, \quad ..... \quad , \quad ..... \quad , \quad .....$$

$$\{15, 17\}, \{31, 33\}, \{1023, 1025\}$$

and

$$\{4, 8, 16, 32, 1024\} \ .$$

The five-element set only contains powers of two. We cannot have a power of two in our chosen subset because we can just chose the power of two twice and their sum would be

$$2^x + 2^x = 2 \cdot (2^x) = 2^{x+1},$$

which is also a power of two. Thus our chosen subset must contain elements from the 1001 two-element sets. Therefore, by the Pigeonhole Principle, it must contain two elements from the same set, which sum up to a power of two.

Problem 10. Let I be the incenter of triangle ABC and let the line passing through I and perpendicular to AI intersect BC at A'. Points B' and C' are defined similarly. Prove that A', B', C' lie on a line perpendicular to OI, where O is the circumcenter of triangle ABC.

Solution. Assume without loss of generality that B lies between A' and C. Let $\angle BAC = \alpha, \angle ABC = \beta, ACB = \gamma$. Then $\angle AIB = 180^\circ - \alpha/2 - \beta/2 = 90^\circ + \gamma/2$ and therefore $\angle A'IB = \angle AIA' - 90^\circ = \gamma/2$. It follows that $\angle A'IB = \angle ICB$ and $\triangle A'IB \sim \triangle A'CI$. Thus we have $A'I^2 = A'B \cdot A'C$.

Let k be the circumcircle of triangle ABC, and let p be the "zero circle" with center I (this is the circle with center I and radius 0). Then, the power of the point A' with respect to the circle k equals $A'B \cdot A'C$, and the power of the point A' with respect to the circle p equals $A'I^2$. It follows that A' has equal powers with respect to the circles k and p; thus, the point A' lies on the radical axis of the circles k and p. Similarly, the points B' and C' also lie on this radical axis. Hence, all three points A', B', C' lie on one line, namely on the radical axis of the circles k and p. This radical axis is clearly perpendicular to the line OI, since O and I are the respective centers of the circles k and p, and the radical axis of two circles is always perpendicular to the line joining their centers.