Solutions

J13. Prove that for any positive integer n, the system of equations

$$\begin{aligned} x + y + 2z &= 4n \quad (1) \\ x^3 + y^3 - 2z^3 &= 6n \quad (2) \end{aligned}$$

is solvable in nonnegative integers x, y, z.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by Daniel Campos Salas

It is enough to verify that for any positive integer n, the triplet $(x, y, z) = (n + 1, n - 1, n)$ is a valid solution.

Also solved by the proposer.

★

J14. Let a, b, c be positive numbers such that $abc = 1$. Prove that:

$$a\left(b^2 - \sqrt{b}\right) + b\left(c^2 - \sqrt{c}\right) + c\left(a^2 - \sqrt{a}\right) \geq 0$$

Proposed by Zdravko F. Starc, Vrsac, Serbia and Montenegro

Solution by José Alvarado, Colegio San Ignacio de Loyola, Perú

Lemma: If p, q, r are real numbers, then

$$p^2 + q^2 + r^2 \geq pq + qr + rp$$

Substituting $a = \frac{x^2}{y^2}, b = \frac{y^2}{z^2}, c = \frac{z^2}{x^2}$ the inequality becomes

$$\frac{x^2y^2}{z^4} + \frac{y^2z^2}{x^4} + \frac{z^2x^2}{y^4} \geq \frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy}$$

Dividing by $x^2y^2z^2$ we see that the inequality is equivalent to the lemma with the numbers $p = \frac{1}{x^3}, q = \frac{1}{y^3}, r = p = \frac{1}{z^3}$, and we are done.

Also solved by Daniel Campos Salas and the proposer.

★

J15. Find the least positive number α with the following property: in every triangle, one can choose two side of lengths a, b such that

$$\alpha > \frac{a}{b} \geq 1.$$

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College,Romania

Solution by Daniel Campos Salas

Let a, b, c be the sides of the triangle with $a \geq b \geq c$. Without loss of generality suppose that $c = 1$.

Let $\phi = \dfrac{1+\sqrt{5}}{2}$. Note that $\phi(\phi - 1) = 1$. Construct a triangle with $b = \phi$ and a be arbitrarily close to ϕ^2, such that $\phi^2 > a$. Then, a, b, c satisfy the triangle inequality.

Then, $\dfrac{a}{b}$ is a number least than ϕ, but arbitrarily close to it, $\dfrac{a}{c}$ is a number least than ϕ^2, but arbitrarily close to it, and $\dfrac{b}{c} = \phi$, which implies that α is greater than or equal to ϕ.

Let us prove that for $\alpha = \phi$ the result holds in any triangle. Suppose there exists a triangle that doesn't satisfy the inequality condition. Let $b = 1 + x$ and $a = 1 + x + y$, with x, y positive reals, and with $1 > y$ because of the triangle inequality. Because of our assumption we have that

$$\frac{1+x+y}{1+x} \geq \phi$$

and

$$\frac{1+x}{1} = 1 + x \geq \phi. \tag{1}$$

It is easy to verify that the inequality

$$\frac{1+x+y}{1+x} \geq \phi \text{ holds if and only if } \; y \geq (1+x)(\phi - 1). \tag{2}$$

Multiplying (1) and (2) it follows that

$$y(1+x) \geq \phi(\phi - 1)(1+x) = 1 + x,$$

that implies $y \geq 1$ which contradicts the fact that $1 > y$.

We conclude that the least value for α is $\phi = \dfrac{1+\sqrt{5}}{2}$.

Also solved by the proposer.

★

J16. Consider a scalene triangle ABC and let $X \in (AB)$ and $Y \in (AC)$ be two variable points such that $(BX) = (CY)$. Prove that the circumcircle of triangle AXY passes through a fixed point (different from A).

Proposed by Liubomir Chiriac, student, Chisinau, Moldova

Solution by Daniel Campos Salas

Let M be the midpoint of the major arc BC. We'll prove that the circumcircle of AXY passes trough M, which is equivalent to prove that A, M, X, and Y lie on a circle.

Suppose without loss of generality that the quadrilateral $AMXY$ is convex, and let E and F be the intersections of MX and MY with the circumcircle of triangle ABC respectively. Because of our assumption it follows that E lies in the minor arc BC and C lies in the minor arc EF.

Since $AMBC$ is cyclic we have that $\angle YCM = \angle ACM = \angle ABM = \angle XBM$, and because $BX = CY$ and $BM = CM$ (since M lies in the perpendicular bisector of BC) it follows that triangles MXB and MYC are congruent. From this, we conclude that $\angle YMC = \angle XMB$, which implies that the minor arcs BE and CF are equal, which implies that the minor arcs BC and EF are equal. It follows that $\angle EMF = \angle BAC$, and this implies that $\angle XMY = \angle XAY$ which completes our proof.

Also solved by the proposer.

★

J17. Let a, b, c be positive numbers. Prove the following inequality:

$$(ab+bc+ca)^3 \leq 3(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)$$

Proposed by Ivan Borsenco, student, Chisinau, Moldova

Solution by Daniel Campos Salas

From Hölder's inequality we have that

$$(a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3)^3 \le (a_1^3 + a_2^3 + a_3^3)(b_1^3 + b_2^3 + b_3^3)(c_1^3 + c_2^3 + c_3^3).$$

Setting

$$(a_1, a_2, a_3) = (1, 1, 1),$$

$$(b_1, b_2, b_3) = (\sqrt[3]{a^2b}, \sqrt[3]{b^2c}, \sqrt[3]{c^2a})$$

and

$$(c_1, c_2, c_3) = (\sqrt[3]{ab^2}, \sqrt[3]{bc^2}, \sqrt[3]{ca^2}),$$

we obtain

$$(ab + bc + ca)^3 \le 3(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2)$$

Also solved by the proposer.

★

J18. Let n be an integer greater than 2. Prove that

$$2^{2^{n+1}} + 2^{2^n} + 1$$

is the product of three integers greater than 1.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by Mario Inocente Castro, Colegio San Ignacio de Loyola. Perú

Recall the identity $a^4 + a^2 + 1 = (a^2 - a + 1)(a^2 + a + 1)$.

Applying this identity twice to our number, we obtain:

$$\begin{aligned}
\left(2^{2^{n-1}}\right)^4 + \left(2^{2^{n-1}}\right)^2 + 1 &= \left(\left(2^{2^{n-1}}\right)^2 - \left(2^{2^{n-1}}\right) + 1\right)\left(\left(2^{2^{n-1}}\right)^2 + \left(2^{2^{n-1}}\right) + 1\right) \\
&= \left(2^{2^n} - 2^{2^{n-1}} + 1\right)\left(\left(2^{2^{n-2}}\right)^4 + \left(2^{2^{n-2}}\right)^2 + 1\right) \\
&= \left(2^{2^n} - 2^{2^{n-1}} + 1\right)\left(2^{2^{n-1}} - 2^{2^{n-2}} + 1\right)\left(2^{2^{n-1}} + 2^{2^{n-2}} + 1\right)
\end{aligned}$$

So it is the product of three positive integers for $n > 2$.

Also solved by Daniel Campos Salas and the proposer

★

S13. Let k be an integer and let $n = \sqrt[3]{k+\sqrt{k^2-1}} + \sqrt[3]{k-\sqrt{k^2-1}} + 1$. Prove that $n^3 - 3n$ is an integer.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by Pascual Restrepo Mesa, Universidad de los Andes, Colombia

Recall the identity: $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$, Plugging in $a = \sqrt[3]{k+\sqrt{k^2+1}}, b = \sqrt[3]{k-\sqrt{k^2+1}}, c = 1-n$ we obtain that since $a+b+c=0$, then $a^3+b^3+c^3-3abc=0$.

However this is equivalent to $2k+(1-n)^3-3(1-n)=0$, since $ab=1$. Therefore we obtain $n^3-3n^2 = 2k-2$ which is an integer. This completes the solution.

Note: The original problem had $n = \sqrt[3]{k+\sqrt{k^2+1}} + \sqrt[3]{k-\sqrt{k^2+1}} + 1$, however, using the same argument of the previous solution we obtain $n^3-3n^2 = 2k+4-6n$ which is not an integer always, some readers realized this and even suggested the correct form of the problem. However all readers attempts of solution to the problem published were correct and changing this 1 by a -1 would gave them the proposed solution.

Also solved by Jose Luis Díaz-Barrero, Daniel Campos Salas, Mario Inocente Castro, Ovidiu Furdui and the proposer

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S14. Let a, b, c be the sides of a scalene triangle ABC and let S be its area, prove that:

$$\frac{2a+b+c}{a(a-b)(a-c)}\frac{2b+c+a}{b(b-c)(b-a)}\frac{2c+a+b}{c(c-a)(c-b)} < \frac{3\sqrt{3}}{4S}$$

Proposed by Jose Luis Díaz-Barrero, Universitat Politécnica de Catalunya, Spain

Solution by Iurie Boreico, student, Chisinau, Moldova

Multiply by $|(a-b)(b-c)(c-a)|$. By taking the absolute value of the RHS it suffice to prove that

$$\frac{2a+b+c}{a}(b-c)+\frac{2b+c+a}{b}(c-a)+\frac{2c+a+b}{c}(a-b) < \frac{3\sqrt{3}(a-b)(b-c)(c-a)}{4S}.$$

Now the LHS equals

$$2(b-c)+2(c-a)+2(a-b)+\frac{(b+c)(b-c)}{a}+\frac{(c+a)(c-a)}{b}+\frac{(a+b)(a-b)}{c}$$

Which can be simplified as

$$\frac{b^2-c^2}{a}+\frac{c^2-a^2}{b}+\frac{a^2-b^2}{c}=\frac{(b^2-c^2)bc+(c^2-a^2)ca+(a^2-b^2)ab}{abc}$$

But this is obviously equal to

$$\frac{(a+b+c)(a-b)(b-c)(c-a)}{abc}.$$

Now by canceling the common factor $|(a-b)(b-c)(c-a)|$ we reduce the inequality to $\frac{a+b+c}{abc}<\frac{3\sqrt{3}}{4S}$, or equivalently that $S<\frac{3\sqrt{3}abc}{a+b+c}$ or $\frac{abc}{4S}\geq\frac{a+b+c}{3\sqrt{3}}$. But $\frac{abc}{4S}=R$, where R is the circumradius of the triangle. As we know the famous inequality $R\geq\frac{\sqrt{a^2+b^2+c^2}}{3}$ and $\sqrt{a^2+b^2+c^2}\geq\frac{a+b+c}{\sqrt{3}}$, the conclusion follows.

The equality can hold for equilateral triangle, but this is not the case of our problem.

Also solved by the proposer

★

S15. Consider a scalene triangle ABC and let $X\in\overline{AB}$ and $Y\in\overline{AC}$ be two variable points such that $BX=CY$. Let $Z=BY\cap CX$ and the circuncircles of $\triangle AYB,\triangle AXC$ meet in K and A. Prove that the reflection of K across the midpoint of AZ belongs to a fixed line.

Proposed by Liubomir Chiriac, student, Chisinau, Moldova

Solution by iurie Boreico, student, Chisinau, Moldova

Let $BX=CY=x, AB=c, BC=a, AC=b$.

Let's compute the position of Z. As $\frac{XA}{XB}=\frac{c-x}{x},\frac{YA}{YC}=\frac{b-x}{x}$, we deduce that Z is the gravity center of the system of points $A(x),B(c-x),C(b-x)$. Thus $\overline{AX}=\frac{c-x}{b+c-x}\overline{AB}+\frac{b-x}{b+c-x}\overline{AC}=\overline{AB}+\overline{AC}-\frac{b\overline{AB}+c\overline{AC}}{b+c-x}=\overline{AB}+\overline{AC}-\frac{b+c}{b+c-x}\overline{AS}$ where S is the foot of the bisector from A.

Now let's compute the position of Y. Let P, Q be the circumcenters of ACX and ABY, and O the circumcenter of ABC. As P is the intersection of the perpendicular bisectors of AX and AC, while O is the intersection of the perpendicular bisectors of AB and AC we conclude that the vector $\overline{OP}$ is perpendicular to AC and has magnitude $\frac{BX}{2\sin A} = \frac{x}{2\sin A}$. Analogously the vector $\overline{OQ}$ is perpendicular to AB and has the same magnitude. So the triangle OPQ is isosceles in O. Now the midpoint M of AZ is actually the projection of A on the line PQ. As OPQ is isosceles, AM is parallel to the bisector of $\angle POQ$, which is parallel to the bisector of $\angle BAC$ (this is because the right angles AXZ and AYZ are congruent). So AM is actually the bisector of angle A. Now let N be the midpoint of PQ and R be the projection of O onto the bisector of A. Then $MNOR$ is a rectangle so $ON = MR$ so $AM = AR - ON = AR - OP\cos\frac{A}{2} = AR - x\frac{\cos\frac{A}{2}}{2\sin A} = AR - \frac{x}{4\sin\frac{A}{2}}$. If we denote by u the vector parallel to $\overline{AR}$ with length $\frac{x}{4\sin\frac{A}{2}}$, we have $\overline{AM} = \overline{AP} - xu$ so $\overline{AK} = 2\overline{AP} - x$. We also have $\overline{AS} = ku$ where $k = 4AS\sin\frac{A}{2}$.

Therefore the reflection J of K w.r.t the midpoint of AZ satisfies $\overline{AJ} = \overline{AZ} - \overline{AK} = \overline{AB} + \overline{AC} - 2\overline{AR} - u(\frac{k(b+x)}{b+c-x} + 2)$. If we denote by W the point that satisfies $\overline{AW} = \overline{AB} + \overline{AC} - 2\overline{AR}$ then J lies on the parallel through W to the bisectors of angle BAC and we are done.

Also solved by the proposer

★

S16. Let M_1 be a point inside the triangle ABC and M_2 it's isogonal conjugate. Prove that

$$4R^2r^2 \geq (R^2 - OM_1^2)(R^2 - OM_2^2).$$

Where R, r are the circunradius and inradius of ABC respectively.

Proposed by Ivan Borsenco, student, Chisinau, Moldova

Solution by Ivan Borsenco, student, Chisinau, Moldova

Let (x, y, z) be the barycentric coordinates for point M_1, then due to Lagrange theorem we have that for any point P in the triangle's plane the following equality holds

$$x \cdot PA^2 + y \cdot PB^2 + z \cdot PC^2 = (x+y+z)PM^2 + \frac{yza^2 + xzb^2 + xyc^2}{x+y+z}.$$

Using O instead of P we have $(R^2 - OM_1^2) = \dfrac{yza^2 + xzb^2 + xyc^2}{(x+y+z)^2}$.

The isogonal conjugate M_2 has barycentric coordinates $(\frac{a^2}{x}, \frac{b^2}{y}, \frac{c^2}{z})$ and thus

$$(R^2 - OM_2^2) = \frac{\frac{a^2b^2c^2}{yz} + \frac{a^2b^2c^2}{xz} + \frac{a^2b^2c^2}{xy}}{(\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z})^2} = \frac{a^2b^2c^2 \cdot xyz(x+y+z)}{(yza^2 + xzb^2 + xyc^2)^2}.$$

Multiplying the two relations

$$(R^2 - OM_1^2)(R^2 - OM_2^2) = \frac{yza^2 + xzb^2 + xyc^2}{(x+y+z)^2} \cdot \frac{a^2b^2c^2xyz(x+y+z)}{(yza^2 + xzb^2 + xyc^2)^2} =$$

$$\frac{a^2b^2c^2xyz}{(x+y+z)(yza^2 + xzb^2 + xyc^2)} \leq \frac{a^2b^2c^2}{(a+b+c)^2} = 4R^2r^2.$$

Last step is Cauchy-Schwartz inequality, and we are done.

Also solved by Iurie Boreico

★

S17. We have m numbers. Let $m > n > 1$. We are allowed to take n of them $a_1, a_2, \cdots, a_n$ and ask: is it true that $a_1 < a_2 < \cdots < a_n$. Prove that we can find the relative order of all them by using at most

$$n! - n^2 + 2n - 2 + (n-1)([log_n m] + 1)(m-1) - m$$

questions.

(the published question asked to find a number k such that we could find the order in at most k questions, however the author submitted this version, were he asks to prove that $k = n! - n^2 + 2n - 2 + (n-1)([log_n m] + 1)(m-1) - m$ is enough)

Proposed by Iurie Boreico, student, Chisinau, Moldova

Solution by Iurie Boreico, student, Chisinau, Moldova

We can find the order of first n of the numbers by $n! - 1$ questions, looking at all possible orderings. Now, suppose we've found the relative order of first k numbers and let's find the relative order of first $k+1$ numbers. Suppose we have $a_1 < a_2 < \cdots < a_k$ and let's find where does a_{k+1} fit. We use for this a sort of "binary search": pick up $n-1$ numbers among $1, 2, \cdots, k$ that divide the interval $[1, k]$ most equally possible (this is achieved by taking the numbers $a_{[\frac{k}{n}]}, a_{[\frac{2k}{n}]}, \cdots a_{[\frac{(n-1)k}{n}]}$). We can the relative order of a_{k+1} and these numbers by at most n questions(indeed, ask $n-1$ questions,question q_i being is it true that $a_{[\frac{k}{n}]} < \cdots < a_{[\frac{ik}{n}]} < a_{k+1} < a_{[\frac{(i+1)k}{n}]} < \cdots < a_{[\frac{(n-1)k}{n}]}$). Then we find an i such that $a_{[\frac{ik}{n}]} < a_{k+1} < a_{[\frac{(i+1)k}{n}]}$. Therefore, by at most n questions we reduced the length of the interval of searching from k to at most $< \frac{k}{n} >$(($< x >$ is the least integer number not less than x)). We repeat this "binary search" until we find exactly the position of a_{k+1} (that is, the interval of searching is 1 or 0). Now if $k \leq n^j$ then it's clear that after i steps the interval will be at most n^{j-i}, so we need at most $j =< log_n k >$ steps to insert a_{k+1} into the sequence. Therefore, the number of questions needed is at most $n!-1+(n-1)(< log_n(n+1) > + \cdots + < log_n(m-1) >)$. All we need is just to evaluate this number: suppose that $n^k \leq m < n^{k+1}$. Then there are $n^- n$ numbers r for which $< log_n r >= 2$, $n^3 - n^2$ numbers for which $< log_n r >= 3$ and so on, finally we have $m-1-n^k$ numbers r for which $< log_n r >= k+1$. Therefore the sum is $n!-1+(n-1)(2(n^2-n)+3(n^3-n^2)+\cdots+k(n^k-n^{k-1})+(k+1)(m-1-n^k)) = n!-1+(n-1)((k+1)(m-1)-n^k-n^{k-1}-\cdots-n^2-2n)$. Since $n^{k+1} > m$, $n^k+n^{k-1}+\cdots+n^2+2n = \frac{n^{k+1}-1}{n-1}+n-1 \geq \frac{m}{n-1}+n-1$, and so our sum is at most $n!-1+(n-1)((k+1)(m-1)-\frac{m}{n-1}-n+1) = n!-n^2+2n-2+(n-1)([log_n m]+1)m-m$, as desired.

★

S18. Find the least positive integer n for which the polynomial

$$P(x) = x^{n-4} + 4n$$

can be written as a product of four non constant polynomials with integer coefficients.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by Pascual Restrepo Mesa, Universidad de los Andes, Colombia

We will show that the least number is 16. First we have to check numbers from 1 to 15 doesn't work. This is not difficult, for $n = 10, 11, 12, 13, 14, 15$ we get by Eissenstein criterion that the polynomial is irreducible (just pick respectively the primes 5,11,3,13,7,5 and check they divide all coefficients except the first one and it's square doesn't divide the last term). For $n = 9$, we get $P(x) = x^5+36$. If we could factor it in the desired way then one of the factors would be linear (just compare degrees) so this equation would indeed have an integer solution, but it is clear that is not the case. Similarly in the case $n = 8$ we get $P(x) = x^4 + 32$ and same reasoning as before shows we need an integer root but thats not the case neither.

Now we factor it for $n = 16$, we have

$$\begin{aligned} x^{12}+64 &= x^12+16x^6+64-16x^6 \\ &= (x^6+8)^2-(4x^3)^2 \\ &= (x^6-4x^3+8)(x^6+4x^3+8) \end{aligned}$$

But on the other hand we can also get:

$$\begin{aligned} x^{12}+64 &= (x^4+4)(x^8-4x^4+16) \\ &= (x^4+4x^2+4-4x^2)(x^8-4x^4+16) \\ &= ((x^2+2)^2-(2x)^2)(x^8-4x^4+16) \\ &= (x^2-2x+2)(x^2+2x+2)(x^8-4x^4+16) \end{aligned}$$

Since the factors x^2-2x+2 and x^2+2x+2 are irreducibles (again by Eissenstein criteria with the prime 2), they must divide the factors x^6-4x^3+8 and x^6+4x^3+8 in some order. In fact we have:

$$x^6-4x^3+8=(x^2+2x+2)(x^4-2x^3+2x^2-4x+4)$$

$$x^6+4x^3+8=(x^2-2x+2)(x^4+2x^3+2x^2+4x+4)$$

Furthermore we observe both identities are equivalent (just switch x to $-x$), so it was enough to find one and the other will come up naturally.

So in this case our polynomial is the product of four integer polynomials

Also solved by the proposer.

★

U14. Calculate:
$$\int_0^1 \frac{\ln(x)\ln(1-x)}{(1+x)^2}dx$$

Proposed by Ovidiu Furdui, Western Michigan University

Solution by Ovidiu Furdui, Western Michigan University

Solution: The integral equals: $\frac{\pi^2}{24}-\frac{\ln^2(2)}{2}$.

We need the following two results:

$$\int_0^1 \frac{\ln(x)}{1-x}\,dx = -\frac{\pi^2}{6}. \qquad (1)$$

and

$$\int_0^1 \frac{\ln(1-x)}{1+x}\,dx = \frac{\ln^2(2)}{2} - \frac{\pi^2}{12}. \qquad (2)$$

These two integrals can be calculated by using symbolic calculations as for instance Maple or Mathematica, however for the sake of completeness the proofs of these results will be included at the end of the solution of the problem.

Integrating by parts with

$f(x) = \ln(x)\ln(1-x) \qquad f'(x) = \frac{\ln(1-x)}{x} - \frac{\ln(x)}{1-x}$

and

$g'(x) = \frac{1}{(1+x)^2} \qquad g'(x) = -\frac{1}{1+x}$

we simply get that

$I = \int_0^1 \frac{\ln(x)\ln(1-x)}{(1+x)^2}\,dx = \int_0^1 \frac{\ln(1-x)}{x(1+x)}\,dx - \int_0^1 \frac{\ln(x)}{1-x^2}\,dx = \int_0^1 \left(\frac{\ln(1-x)}{x} - \frac{\ln(1-x)}{1+x}\right) - \int_0^1 \frac{\ln(x)}{1-x^2}\,dx$

Therefore $I = \int_0^1 \frac{\ln(1-x)}{x}\,dx - \int_0^1 \frac{\ln x}{1-x^2}\,dx - \frac{\ln^2(2)}{2} + \frac{\pi^2}{12}$. in view of (2).

Making the change of variables $1-x=y$ in the first integral above and after simple calculations we simply get that:

$I = \int_0^1 \frac{\ln y}{1-y}\,dy - \int_0^1 \frac{\ln x}{1-x^2}\,dx - \frac{\ln^2(2)}{2} + \frac{\pi^2}{12} = \int_0^1 \frac{x\ln x}{1-x^2}\,dx - \frac{\ln^2(2)}{2} + \frac{\pi^2}{12}.$

Making the change of variable $x = \sqrt(t)$ in the above integral we get that:

$I = \frac{1}{4}\int_0^1 \frac{\ln t}{1-t}\,dt - \frac{\ln^2(2)}{2} + \frac{\pi^2}{12} = \frac{\pi^2}{24} - \frac{\ln^2 2}{2}$ in view of (1).

Proof of (1)

$\int_0^1 \frac{\ln(x)}{1-x}\,dx = \int_0^1 \ln x \left(\sum_{k=0}^\infty x^k\right)\,dx = \sum_{k=0}^\infty \int_0^1 x^k \ln x\,dx = \sum_0^\infty \frac{-1}{(k+1)^2} = -\frac{\pi^2}{6}$ since elementary calculations show that

$\int_0^1 x^k \ln x\,dx = \frac{-1}{(k+1)^2}$

Proof of (2)

Let $H_n = \sum_{k=1}^n \frac{1}{k}$ be the n^{th} harmonic number. The following expansion can be proven by elementary calculations, [for example multiplying the power series expansions of $\ln(1-x)$]:

$$\tfrac{1}{2}\ln^2(1-z) = \textstyle\sum_{n=1}^{\infty} \tfrac{H_n}{n+1} z^{n+1}, \qquad |z| < 1$$

In view of Abel's theorem since the series is convergent for $z = -1$ we get that:

$$\tfrac{1}{2}\ln^2(2) = \textstyle\sum_{n=1}^{\infty} (-1)^{n+1} \tfrac{H_n}{n+1}$$

It simply follows after some rearrangements that:

$$\textstyle\sum_{k=1}^{\infty} (-1)^k \tfrac{H_k}{k} = \tfrac{\ln^2(2)}{2} - \tfrac{\pi^2}{12}$$

We observe that

$-\int_0^1 x^n \ln(1-x)\, dx = \int_0^1 x^n\, dx \int_0^x \frac{1}{1-t}\, dt = \int_0^1 \frac{dt}{1-t} \int_t^1 x^n\, dx = \frac{1}{n+1} \int_0^1 \frac{1-t^{n+1}}{1-t}\, dt = \frac{1}{n+1} \int_0^1 \left(1 + t + t^2 + \cdots + t^n\right) dt = \frac{H_{n+1}}{n+1}$

Therefore we get that: $\int_0^1 x^n \ln(1-x)\, dx = -\frac{H_{n+1}}{n+1}$

Now we have that:

$\int_0^1 \frac{\ln(1-x)}{1+x}\, dx = \int_0^1 \ln(1-x) \sum_{n=0}^{\infty} (-x)^n\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{-H_{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^n \frac{H_n}{n} = \frac{\ln^2(2)}{2} - \frac{\pi^2}{12}.$

Observation: Notice that in view of the transformation $x = 1 - y$ the following equality also holds:

$$\textstyle\int_0^1 \frac{\ln(x)\ln(1-x)}{(1+x)^2}\, dx = \int_0^1 \frac{\ln(x)\ln(1-x)}{(2-x)^2}\, dx = \frac{\pi^2}{24} - \frac{\ln^2(2)}{2}.$$

★

U15. Let $f : [a, b] \to \mathbb{R}$ be a continuous and convex function. Prove that

$$\int_a^b f(x)dx \geq 2 \int_{\frac{3a+b}{4}}^{\frac{3b+a}{4}} f(x)dx \geq (b-a) f\left(\frac{a+b}{2}\right)$$

Proposed by Cezar Lupu, University of Bucharest, and Tudorel Lupu, Decebal Highschool, Constanta

Solution by Pascual Restrepo Mesa, Universidad de los Andes, Colombia

Since f is convex we have $tf(x)+(1-t)f(y) \geq f(tx+(1-t)y)$ for all $x, y \in [a, b]$ and $t \in [0, 1]$. Therefore, if we write $p = \frac{a+b}{2}$ we obtain:

$$\frac{1}{4}f(p+x) + \frac{3}{4}f(p-x) \geq f(p - \frac{1}{2}x)$$

For all $x \in [\frac{a-b}{2}, \frac{b-a}{2}]$, So integrating this inequality in that interval, we get:

$$\int_{\frac{a-b}{2}}^{\frac{b-a}{2}} (\frac{1}{4}f(p+x) + \frac{3}{4}f(p-x))dx \geq \int_{\frac{a-b}{2}}^{\frac{b-a}{2}} f(p - \frac{1}{2}x)dx = 2\int_{\frac{a-b}{2}}^{\frac{b-a}{2}} f(p-t)dt$$

But this is just

$$\int_a^b f(x)dx \geq 2\int_{\frac{3a+b}{4}}^{\frac{3b+a}{4}} f(x)dx$$

Which proves the first inequality.

For the second one observe that for all $x \in [\frac{a-b}{4}, \frac{b-a}{4}]$ we have:

$$\frac{1}{2}f(p+x) + \frac{1}{2}f(p-x) \geq f(p)$$

Integrating this relation from $\frac{a-b}{4}$ to $\frac{b-a}{4}$, we obtain

$$\int_{\frac{a-b}{4}}^{\frac{b-a}{4}} (\frac{1}{2}f(p+x) + \frac{1}{2}f(p-x))dx \geq \int_{\frac{a-b}{4}}^{\frac{b-a}{4}} f(p)dx$$

Which is equivalent to

$$\int_{\frac{3a+b}{4}}^{\frac{3b+a}{4}} f(x)dx \geq \frac{b-a}{2} f\left(\frac{a+b}{2}\right)$$

And proves the second inequality.

Also solved by the proposer

★

U16. Let $n \geq 1$ be a natural number. Prove that:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\prod_{j=0}^{n}(k+j)} = -\frac{2^n \ln(2)}{n!} + \frac{1}{n!}\sum_{k=1}^{n} \frac{2^{n-k}}{k}$$

Proposed by Ovidiu Furdui, Western Michigan University

Solution by Karsten Bohlen

Lemma
For all $n \in \mathbb{N}_0$ we have:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k+n} = (-1)^{n+1} \ln(2) + (-1)^{n+1} \sum_{k=1}^{n} \frac{(-1)^k}{k}$$

Proof
It can be easily seen that the series converges for every n with Leibnitz-Criterion. Then the cases $n = 0, n = 1$ yield $-\ln(2)$ and $\ln(2) - 1$. So we proceed with Induction

$$\begin{aligned}
&(-1)^{n+2} \ln(2) + (-1)^{n+2} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \\
&= -\left((-1)^{n+1} \ln(2) + (-1)^{n+1} \sum_{k=1}^{n} \frac{(-1)^k}{k} + \frac{1}{n+1} \right) \\
&= -\sum_{k=1}^{\infty} \frac{(-1)^k}{k+n} - \frac{1}{n+1} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+n} + \frac{(-1)}{n+1} \qquad (\textit{ by Hypothesis }) \\
&= \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k+n} = \sum_{k=1}^{\infty} \frac{(-1)^{k+2}}{k+n+1} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k}{k+n+1} \quad \square
\end{aligned}$$

Lemma

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+1} = \frac{1}{n+1}$$

Proof

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+1}\\
&=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k\\
&=\frac{1}{n+1}\sum_{k=1}^{n+1}\binom{n+1}{k}(-1)^{k+1}\\
&=\frac{1}{n+1}(-1)\sum_{k=1}^{n+1}\binom{n+1}{k}(-1)^k\\
&=\frac{1}{n+1}(-1)\left(0-\binom{n+1}{0}(-1)^0\right)\\
&=\frac{1}{n+1}\quad\square
\end{aligned}
$$

Now we have:

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{(-1)^k}{\prod_{j=0}^{n}(k+j)}&=\frac{1}{n!}\sum_{k=1}^{n}\frac{2^{n-k}}{k}-\frac{2^n\ln(2)}{n!}\\
\Leftrightarrow n!\sum_{k=1}^{\infty}\frac{(-1)^k}{\prod_{j=0}^{n}(k+j)}&=\sum_{k=1}^{n}\frac{2^{n-k}}{k}-2^n\ln(2)
\end{aligned}
$$

For the left hand side we get:

$$
\begin{aligned}
n!\sum_{k=1}^{\infty}\frac{(-1)^k}{\prod_{j=0}^{n}(k+j)}&=\sum_{k=1}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^{k+j}}{k+j} && \text{(\textit{Partial Fractions})}\\
&=\sum_{j=0}^{n}\sum_{k=1}^{\infty}\binom{n}{j}\frac{(-1)^{k+j}}{k+j} && \text{(\textit{by Convergence})}\\
&=\sum_{j=0}^{n}\binom{n}{j}(-1)^j\left((-1)^{j+1}\ln(2)+(-1)^{j+1}\sum_{k=1}^{j}\frac{(-1)^k}{k}\right) && \text{(\textit{with Lemma})}\\
&=-2^n\ln(2)-\sum_{j=1}^{n}\binom{n}{j}\sum_{k=1}^{j}\frac{(-1)^k}{k} && \text{(\textit{Binomial Sum})}
\end{aligned}
$$

Adding $2^n\ln(2)$ on both sides we only have to show that:

$$\sum_{k=1}^{n}\binom{n}{k}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}=2^n\sum_{k=1}^{n}2^{-k}k^{-1}$$

Which is true for $n=1, n=2$. By Induction:

$$\begin{aligned}
\sum_{k=1}^{n+1}\binom{n+1}{k}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j} &= \sum_{k=1}^{n+1}\left(\binom{n}{k}+\binom{n}{k-1}\right)\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}\\
&= 2^n\sum_{k=1}^{n}2^{-k}k^{-1}+\sum_{k=1}^{n+1}\binom{n}{k-1}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j} && (\textit{ by Hypothesis })\\
&= 2^n\sum_{k=1}^{n}2^{-k}k^{-1}+\sum_{k=0}^{n}\binom{n}{k}\sum_{j=1}^{k+1}\frac{(-1)^{j+1}}{j}\\
&= 2^n\sum_{k=1}^{n}2^{-k}k^{-1}+\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{k}\frac{(-1)^{j}}{j+1}\\
&= 2^n\sum_{k=1}^{n}2^{-k}k^{-1}+\sum_{k=1}^{n}\binom{n}{k}\sum_{j=0}^{k-1}\frac{(-1)^{j}}{j+1}+\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+1}\\
&= 2^{n+1}\sum_{k=1}^{n}2^{-k}k^{-1}+\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+1} && (\textit{ by Hypothesis })\\
&= 2^{n+1}\sum_{k=1}^{n}2^{-k}k^{-1}+\frac{1}{n+1}=2^{n+1}\sum_{k=1}^{n+1}2^{-k}k^{-1} && (\textit{ with Lemma })
\end{aligned}$$

So it's true for all natural n and we are done.

Also solved by the proposer

★

U17. Find all real numbers a such that the sequence $x_n = n\{an!\}$ converges.

Proposed by Gabriel Dospinescu, "Louis le Grand" College, Paris, France

Solution by Gabriel Dospinescu, "Louis le Grand College", Paris, France

Let $x_n = n\{an!\}$ tend to $l \in \mathbf{R}$. Observe that

$$\frac{x_n}{n} = an! - \lfloor an! \rfloor$$

$$\frac{x_{n+1}}{n+1} = an!(n+1) - \lfloor a(n+1)! \rfloor = (n+1)(\frac{x_n}{n} + \lfloor an! \rfloor) - \lfloor a(n+1)! \rfloor$$

and we get:

$$\frac{x_{n+1}}{n+1} = \frac{x_n}{n} + x_n + (n+1)\lfloor an! \rfloor - \lfloor a(n+1)! \rfloor \, (*)$$

and since $\frac{x_n}{n} \to 0$, then $(n+1)\lfloor an! \rfloor - \lfloor a(n+1)! \rfloor \to l$, but this means that for some $n \geq n_0$ we must have

$$(n+1)\lfloor an! \rfloor - \lfloor a(n+1)! \rfloor = l(**)$$

since it is the limit of an integer sequence, so in fact, l is an integer.

So for $n \geq n_0$ we have that $\frac{x_{n+1}}{n+1} = (n+1)\frac{x_n}{n} - l$ so by putting $v_n = \frac{x_n}{n}$ we have:

$$v_{n+1} = (n+1)v_n - l$$

And dividing this expression by $(n+1)!$, we obtain:

$$\frac{v_{n+1}}{(n+1)!} - \frac{v_n}{n!} = \frac{-l}{n!}$$

Summing this relation for all numbers bigger than n_0 to n we get a telescopic sum equal to:

$$\frac{v_n}{n!} = \frac{v_{n_0}}{n_0!} - \sum_{k=n_0+1}^{n} \frac{l}{k!}$$

But since $0 \leq v_n < 1$, we obtain $0 \leq \frac{v_{n_0}}{n_0!} - \sum_{k=n_0+1}^{n} \frac{l}{k!} < 1$ and taking limits we obtain:

$$\frac{v_{n_0}}{n_0!} = \sum_{k=n_0+1}^{\infty} \frac{l}{k!}$$

And in fact we get for $n \geq n_0$ that

$$\frac{v_n}{n_!} = \sum_{k=n+1}^{\infty} \frac{l}{k!}$$

So

$$a = \frac{\lfloor an! \rfloor}{n!} + \sum_{k=n+1}^{\infty} \frac{l}{k!}$$

and therefore a belongs to $\mathbb{Q}+e\mathbb{Z}$ (since l is an integer and $\sum_{k=n+1}^{\infty}\frac{1}{k!}=e-\sum_{k=0}^{n}\frac{1}{n!}$)

Lets write $a=\frac{p}{q}+ke$ with p,q,k integers, and lets prove this number works. For big n we have:

$$\{an!\}=\{kn!e\}=\{k(n!+\frac{n!}{1!}+\frac{n!}{2!}...+\frac{n!}{(n-1)!}+\frac{n!}{n!}+\frac{n!}{(n+1)!}...)\}=\{\frac{k}{n+1}+\frac{k}{(n+1)(n+2)}+...\}$$

But if n is large enough then:

$$\{an!\}=\frac{k}{n+1}+\frac{k}{(n+1)(n+2)}+...$$

and then

$$n\{an!\}=k(\frac{n}{n+1}+\frac{n}{(n+1)(n+2)}+...)$$

but since

$$\frac{n}{n+1}\leq\frac{n}{n+1}+\frac{n}{(n+1)(n+2)}+...<\frac{n}{n+1}+\frac{n}{(n+1))^2}...=\frac{n}{n+1}\frac{1}{1-\frac{1}{n+1}}$$

We find that actually $x_n\to k$ and therefore the sequence converges to k.

So finally, we have that the answer is $a\in\mathbb{Q}+\mathbb{Z}e$.

★

U18. Let a and b be two positive real numbers. Evaluate

$$\int_a^b\frac{e^{\frac{x}{a}}-e^{\frac{b}{x}}}{x}\,dx$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by Karsten Bohlen

We have

$$\exp\left(\frac{x}{a}\right) = \sum_{n=0}^{\infty} \frac{x^n}{a^n n!}$$
$$\exp\left(\frac{b}{x}\right) = \sum_{n=0}^{\infty} \frac{b^n}{x^n n!}$$

Thus

$$\int_a^b \frac{e^{\frac{x}{a}}}{x}\,dx = \int_a^b \sum_{n=0}^{\infty} \frac{x^{n-1}}{a^n n!}\,dx = \sum_{n=0}^{\infty} \int_a^b \frac{x^{n-1}}{a^n n!}\,dx \tag{3}$$
$$-\int_a^b \frac{e^{\frac{b}{x}}}{x}\,dx = -\int_a^b \sum_{n=0}^{\infty} \frac{b^n}{x^{n+1} n!}\,dx = -\sum_{n=0}^{\infty} \int_a^b \frac{b^n}{x^{n+1} n!}\,dx \tag{4}$$

For (3) we get

$$\begin{aligned}\int_a^b \frac{x^{n-1}}{a^n n!}\,dx &= \left[\frac{x^n}{a^n n! n}\right]_a^b \\ &= \frac{b^n - a^n}{a^n n! n} \text{ for } n > 0 \\ \int_a^b \frac{1}{x}\,dx &= [\ln(x)]_a^b \\ &= \ln(b) - \ln(a) \text{ for } n = 0\end{aligned}$$

And for (4) we get

$$\begin{aligned}-\int_a^b \frac{b^n}{x^{n+1} n!}\,dx &= -\left[-\frac{b^n}{x^n n! n}\right]_a^b \\ &= \frac{a^n - b^n}{a^n n! n} \text{ for } n > 0 \\ &= -\int_a^b \frac{1}{x}\,dx = -[\ln(x)]_a^b \\ &= \ln(a) - \ln(b) \text{ for } n = 0\end{aligned}$$

Putting everything in we get

$$\begin{aligned}\int_a^b \frac{e^{\frac{x}{a}} - e^{\frac{b}{x}}}{x}\,dx &= \sum_{n=0}^{\infty} \ln(b) - \ln(a) + \ln(a) - \ln(b) + \frac{b^n - a^n}{a^n n! n} + \frac{a^n - b^n}{a^n n! n} \\ &= 0\end{aligned}$$

Solution 2 by Ovidiu Furdui

The integral is 0. If we make the change of variable $x = at$, we get that

$$I = \int_a^b \frac{e^{\frac{x}{a}} - e^{\frac{b}{x}}}{x} dx = \int_1^\alpha \frac{e^t - e^{\frac{\alpha}{t}}}{t} dt,$$

where $\alpha = \dfrac{b}{a}$. The substitution $t = \dfrac{\alpha}{x}$ transforms the preceding integral to,

$$I = \int_1^\alpha \frac{e^t - e^{\frac{\alpha}{t}}}{t} dt = -I.$$

Therefore $I = 0$.

Also solved by the proposer

★

O13. Let ABC be a triangle and P be an arbitrary point inside the triangle. Let A', B', C' be respectively the intersections of AP and BC, BP and CA, CP and AB. Through P, we draw a line perpendicular to PA that intersect BC at A_1. We define B_1 and C_1 analogously. Let P' be the isogonal conjugate of the point P with respect to triangle $A'B'C'$. Show that A_1, B_1, C_1 all lie on a line l that is perpendicular to PP'.

Proposed by Khoa Lu Nguyen Sam Houston High School, Houston, Texas

Solution by Khoa Lu Nguyen Sam Houston High School, Houston, Texas

Given four collinear points X, Y, Z, T, by denoting $(XYZT)$, we mean the cross-ratio of four points X, Y, Z, T. Given four concurrent lines x, y, z, t, by denoting (x, y, z, t), we mean the cross-ratio of four lines x, y, z, t.

We first introduce a useful lemma.

Lemma

Let ABC be a triangle and P' be the isogonal conjugate of an arbitrary point P with respect to ABC. Then the six projections from P and P' to the sides of triangle ABC lie on a circle with center the midpoint of PP'.

Proof

Let P_a, P_b, P_c be the projections from P to the sides BC, CA, AB. Similarly, let P'_a, P'_b, P'_c be the projections from P' to the sides BC, CA, AB. Call O the midpoint of PP'. We need to show that $P_a, P_b, P_c, P'_a, P'_b, P'_c$ lie on a circle center O.

Consider the trapezoid $PP'P'_aP_a$ that has $\angle PP_aP'_a = \angle P'P'_aP_a = 900$and O the midpoint of PP'. Hence, O must lie on the perpendicular bisector of $P_aP'_a$. By similar argument, we obtain that O also lies on the perpendicular bisector of $P_bP'_b$ and $P_cP'_c$.

Since P' is the isogonal conjugate of P with respect to ABC, we have $\angle BAP = \angle P'AC$, or $\angle P_cAP = \angle P'AP'_b$. Hence, we obtain $\angle APP_c = \angle P'_bP'A$. On the other hand, since quadrilaterals AP_cPP_b and $AP'_cP'P'_b$ are cyclic, it follows that $\angle APP_c = \angle AP_bP_c$ and $\angle P'_bP'A = \angle P'_bP'_cA = \angle P'_bP'_cP_c$. Thus, $\angle AP_bP_c = \angle P'_bP'_cP_c$. This means that $P_bP_cP'_cP'_b$ is inscribed in a circle. We notice that the center of this circle is the intersection of the perpendicular bisectors of $P_bP'_b$ and $P_cP'_c$ which is O. In the same manner, we obtain $P_cP_aP'_cP'_a$ is inscribed in a circle with center O. Thus these two circles are congruent as they have the same center and pass through a common point P_c. Therefore, these six projections all lie on a circle with center O.

Back to the problem, let P_a, P_b, P_c be the projections from P to the sides $B'C', C'A', A'B'$ and O be the midpoint of PP'. Then (O) is the circumcircle of triangle $P_aP_bP_c$. We will show a stricter result. In fact, l is the polar of P with respect to the circle (O). We will prove that A_1 lies on the polar of P with respect to the circle (O), and by similar argument so do B_1 and C_1.

Let B_2 and C_2 be the intersections of the line PA_1 with $A'C'$ and $A'B'$ respectively. Let M and N be the intersections of the line PA_1 and the circle (O). It is sufficient to show that $(A1PMN) = -1$.

Let X be the projection from P to BC. Then five points P, P_b, P_c, X, A' lie on the circle (a) with diameter PA'. Since $(A'A1, A'A, A'C', A'B') = -1$, we obtain that PP_cXP_b is a harmonic quadrilateral. This yields that P_bP_c and the tangents at P and X of the circle (a) are concurrent at a point U. Since the tangent at P of the circle (a) is PA_1, it follows that U is the concurrent point of P_bP_c, PA_1 and the perpendicular bisector of PX. Consider the right triangle PXA_1 at X. Since U lies on PA_1 and the perpendicular bisector of PX, we obtain U is the midpoint of PA_1. Therefore the midpoint U of PA_1 lies on the line P_bP_c.

Since $A'P_b \cdot A'B_2 = A'P_c \cdot A'C_2 = A'P^2$, it follows that $P_bP_cC_2B_2$ is cyclic. Hence, $UB_2 \cdot UC_2 = UP_b \cdot UP_c$. On the other hand, we have $UP_b \cdot UP_c = UM.UN$ since P_bP_cMN is cyclic. Thus, we obtain $UB_2 \cdot UC_2 = UM.UN$. (*)

Since $(A'A_1, A'A, A'C', A'B') = -1$, we have $(A_1PB_2C_2) = -1$. Since U is the midpoint of A_1P, it follows that $UB_2 \cdot UC_2 = UA_1^2 = UP^2$ according to Newton's formula. From (*), we obtain $UA_1^2 = UP^2 = UM.UN$. Thus $(A_1PMN) = -1$. And the problem is solved.

Also solved by Ivan Borsenco

★

O14. We have a planar graph G with vertices' degrees $3,4$ or 5, such that vertices with the same degree are not connected. Suppose that number of 5 sided faces is greater than number of 3 sided faces. Denote v_3 the number of vertices with degree 3 and v the total number of vertices.

Prove that $v_3 \geq \dfrac{v+23}{4}$.

Proposed by Ivan Borsenco, Student, Chisinau, Moldova

Solution by Ivan Borsenco, student, Chisinau, Moldova

Let g be minimal number of vertices in G then $2m = gf_g + (g+1)f_{g+1} + ... \geq g(\sum f_i) = gf$. In our case $2m = 3f_3 + 4f_4 + 5f_5 + ... \geq 4(\sum f_i) + f_5 - f_3 \geq 4f + 1$, hence looking mod 2 we have $2m \geq 4f + 2$ (*). Also we observe that $v = v_3 + v_4 + v_5$ and $2m = 3v_3 + 4v_4 + 5v_5 = 4v + v_5 - v_3$ (**). Adding two relationships $4m \geq 4f + 2 + 4v + v_5 - v_3$ (***). At the next step we use Euler equality for planar graph: $v + f = m + 2$; we multiply it by 4 and put it in (***), the result is $v_3 \geq v_5 + 10$ (1).

From the condition of the problem that vertices with the same degree are not connected we can deduce that edges outgoing from vertices of v_3 with v_5 are greater or equal then edges from vertices of v_4, in other words $3v_3 + 5v_5 \geq 4v_4$. Using upper deduced inequality (1) we have $8v_3 \geq 4v_4 + 50$, working mod 4: $2v_3 \geq v_4 + 13$ (2).

Finally using (1) and (2) we get $4v_3 \geq v_3 + v_4 + v_5 + 23 = v + 23$ and we are done.

Also solved by Iurie Boreico

★

O15. The cells of a $(n^2 - n + 1) \times (n^2 - n + 1)$ matrix are colored in n colors. A color is called dominant in a row or column if there are at least n cells of this color on this row or column. A cell if called extremal if its color is dominant both on its row and on its column. Find all $n \geq 2$ for which there is a coloring with no extremal cells.

Proposed by Iurie Boreico, Student, Chisinau, Moldova

Solution by Iurie Boreico, Student, Chisinau, Moldova

We shall prove that all such n are $n \geq 4$. If $n = 2$ the proof is analogous to b).

Let's pass now to the harder case $n = 3$. So have a 7×7 matrix colored in 3 colors.

So we call a color dominating in a row or column if this row or column contains at least 3 cells of this color. Call a cell horizontal if its color is dominating on its row and vertical if its color is dominating on its column. We must show that there is a cell which is both horizontal and vertical.

It's clear that on every row or column there is at least one dominating color from Dirichlet Principle. Assume that no cell is both vertical and horizontal. A color C cannot be dominant on five or more rows (the same with colors) because otherwise there are at lest 3×5 horizontal cells of color C on these rows, therefore by Dirichlet Principle some three of them lie on the same column and are therefore vertical.

Again by Dirichlet Principle, there is a color C dominant on at least three rows. Denote by B, C the other two colors and by H the horizontal cells of color A. Now we distinguish two cases with some subcases:

i) A is dominant on three rows, let them be $1, 2, 3$. Then $|H| \geq 9$. As no three cells from H lie on the same column, H lies on at least 5 columns, so we distinguish three subcases:

a)H lies on 5 columns, let them be the first 5 columns. Consider the submatrix T formed by the first five columns and the last four rows. As on each of the first five columns there are at most two cells from A and $|H| \geq 9$, there is at most one cell of color A in T. Call a color dominant in a row in T if there are at least three cells of this color on this row. Assume that both B and C are dominant on some row from T. Let X be the set of horizontal cells of color B from T and Y be the set of horizontal cells of color C from T. Then X lies on at least three columns and so does Y. Since there are just five columns in T, there is a column containing cells from X and Y(and H). Then in this column no color can meet three times, which contradicts the fact that the column (in the big matrix this time) has seven cells. We conclude that only cell of one color (say B) are dominant on rows from T. As there are at least five rows in T containing five cells of color B an C (we must drop one of the four rows if it's "spoiled" by one cell of color A), B must be dominant on all these three rows, and we deduce that X contains at least nine cells and lies on all the five rows of T. Hence every of the first five columns contains not more than two cells of color A and at most two cells of color B so at least three cells of color C. So C dominates five columns, which was shown above that is impossible.

b) H lies on the first six columns. Define T, X, Y as above, just now for the first six columns. There can be at most three cells of color A in T, co one of the rows consists of cells of just colors B, C. If, for example, there are four cells of color B on this row the one the corresponding 4 columns C is dominant and we go to ii). So we may assume that 3 cells are of color B and 3 of color C. We've already seen that no two cells from X and Y can be on the same column, so X lies on three column (WLOG the first three ones) and Y on the other three columns. One more column contains at most one cell of color A so contains at least three vertices of the same color, say B. The they all three lie in the first three columns. One more column contains at most one cell of color A so contains at least three vertices of the same color. They can't be of color B because then some of them would be horizontal and vertical, so they are of color C and lie on the columns $4, 5, 6$. Hence all the other cells in the first half of T are of color A or C and in the second half of color A or B. One of this halves (WLOG the first) contains at most one cell of color A, so it contains one more row with all three cells of color B or C. No such cell can be of color B because we would get a horizontal and vertical cell form X, hence they all three are of color C and hence from Y, so there is a column containing vertices from both X and Y and this is the desired contradiction.

ii)A is dominant on the first four rows. Then $|H| \geq 12$, so H lies on the first six or on all the seven columns:

a) H lies on the first six columns. Consider T, X, Y analogously. This time T contains only cells of colors B or C. If one of X, Y is empty, say Y then there are at least four (horizontal) cells of color B on each of the last three rows, and then by we deduce that there are exactly two cells from X on every of the six columns in T. Then C is dominant on the first six columns - contradiction. If X, Y are non-empty, like in the previous case we assume X lies on the first three rows and Y on the last three rows. Then every row contains three cells of color X and three cells of color Y so $|X| = |Y| = 9$ and there are three cells from X in a column, which is a contradiction.

b)H lies on all the columns. Consider T, X, Y. At most two cells of color A lie in X hence one of the rows (the last one) of T contains just cells of color B or C. We may assume the first three are of color B and the last four of color C. Then C is dominant on the last four columns but there are no cells of color C on the last row and one of the last four columns, so for color C we are in the investigated case ii a).

For $n = 4$ we have the following example:

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 4 & 4 & 3 & 3 & 3 & 4 & 2 & 2 & 2 \\
4 & 2 & 2 & 2 & 2 & 4 & 3 & 3 & 3 & 1 & 1 & 4 & \\
2 & 4 & 3 & 3 & 3 & 3 & 4 & 4 & 2 & 1 & 1 & 1 & 2 \\
2 & 4 & 4 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & 4 & 2 & 2 \\
3 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 3 & 1 & 1 & 1 & 3 \\
2 & 4 & 4 & 4 & 3 & 1 & 1 & 1 & 1 & 3 & 3 & 2 & 2 \\
2 & 3 & 3 & 3 & 2 & 1 & 4 & 4 & 4 & 4 & 1 & 1 & 2 \\
1 & 4 & 4 & 4 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 1 & 1 \\
2 & 3 & 3 & 1 & 1 & 2 & 3 & 1 & 4 & 4 & 4 & 4 & 2 \\
2 & 4 & 4 & 2 & 4 & 2 & 1 & 1 & 1 & 3 & 3 & 3 & 3 \\
4 & 1 & 2 & 2 & 1 & 2 & 3 & 3 & 1 & 4 & 4 & 4 & \\
3 & 3 & 4 & 1 & 4 & 4 & 2 & 2 & 2 & 1 & 1 & 3 & 3 \\
1 & 1 & 1 & 4 & 4 & 4 & 3 & 3 & 3 & 2 & 2 & 2 & 1
\end{pmatrix}$$

Finally, let's handle the case $n \geq 5$ by solving the more general problem:

Let $m \geq 5, k, l$ be positive integers. The cells of a $(km+1) \times (lm+1)$ matrix are colored in one of m colors. A color is called dominant in a row if at least $l+1$ of the cells in the row have this color, and analogously for columns. A cell is called extremal if its color is dominant in its row and its column. Then, there exists a coloring with the following property:

there is no extremal cell, in every row or column there is exactly one dominant color, all colors are dominant for k rows except one that is dominant for $k+1$ rows, and analogously for columns.

We prove this by induction on $k+l$. The base $k = l = 1$ is performed by induction on m:

If $m = 5$ then we have the example:

$$\begin{pmatrix} 1 & 1 & 4 & 5 & 3 & 2 \\ 4 & 2 & 2 & 5 & 3 & 1 \\ 4 & 5 & 3 & 3 & 2 & 1 \\ 4 & 3 & 5 & 1 & 1 & 2 \\ 2 & 3 & 1 & 5 & 4 & 4 \\ 5 & 4 & 1 & 2 & 3 & 5 \end{pmatrix}$$

We can also pass from m to $m+1$ by adding a row and a column and coloring all the new cell in color $m+1$, except the corner cell.

Now let's perform the step from l to $l+1$, the step from k to $k+1$ being analogous.

So, we must add a 'little' $(km+1) \times m$ matrix to the the $(km+1) \times (lm+1)$ matrix. We can ensure that every color meets exactly once in every row of the 'little' matrix. The every row will contain exactly one horizontal cell. WLOG assume the horizontal cells in the first $k+1$ rows are of color 1(first class rows), in next k rows of color 2(second class rows) and so on. We order the first class rows as $1, 2, \ldots, m$ (this are the color of the cells). The second class rows are ordered as $2, 1, 4, 5, \ldots, m, 3$, the third class rows as $3, 4, \ldots, m, 1, 2$ and so on(all the rows are shifts of each other, except a little irregularity from the second class rows second class). Finally, pick up a first class row and exchange the cell of color 1 with the cell of color 3. It's easy to see that the induction step is thus fulfilled.

For $k = l = n-1$ it's our problem.

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O16. Let ABC be an acute angled-triangle. Let W be the center of the nine point circle and let G be its centroid. Let $A', B'C'$ and A'', B'', C'' be the projections of W and G on the sides BC, CA, AB respectively. Prove that the perimeter of $A''B''C''$ is not less than the perimeter of $A'B'C'$.

Proposed by Iurie Boreico, Student, Chisinau, Moldova

Solution by Li Zhou, Polk Community College, Florida

We start with a nice lemma

Lemma

Let A, B, C, D be four points and P, Q the midpoints of AD and BC, then $AB + CD \geq 2PQ$.

Proof

Denote by $\vec{XY}$ the vector from X to Y, we have:

$$2\,\vec{PQ} = (\vec{PA} + \vec{AB} + \vec{BQ}) + (\vec{PD} + \vec{DC} + \vec{CQ}) = AB + DC$$

And the triangle inequality yields $AB + CD \geq 2PQ$.

Now we use this lemma to prove a theorem that will be of big importance in our proof.

Theorem

let l be a line and P a point in the line moving across the line but inside a triangle ABC, let $f(P)$ be the perimeter of the pedal triangle of P, then $f(P)$ is a convex function as P moves across the line.

Proof

Let P, Q be two points and M the midpoint of PQ. Let X_a, X_b, X_c be the projections on the sides BC, CA, AB of a point X. Then by the lemma we obtain:

$$P_aP_b + Q_aQ_b \geq 2M_aM_b$$

$$P_bP_c + Q_bQ_c \geq 2M_bM_c$$

$$P_cP_a + Q_cQ_a \geq 2M_cM_a$$

And adding these inequalities we obtain $f(P) + f(Q) \geq 2f(M)$ so f is convex.

Now lets see what happens when we apply this to the Euler line:

Let l be the line passing trough the orthocenter H of triangle ABC and its centroid G (Euler Line), then $f(P)$ is convex as P moves on this line and f achieves its minimum in H (by the first lemma and because a well known result regarding the orthic triangle), so since it is well known that W also lies on this line between G and H we deduce by convexity that $f(H) \leq f(W) \leq f(G)$ which is exactly what we wanted to show.

Comment by the solver: In fact we can see that as P moves inside the triangle, the points (f, P) generate a convex surface with minimun at H

Also solved by the proposer

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O17. Let α be a root of the polynomial $P(x) = x^n + a_{n-1}x^{n-1}... + a_1x + a_0$ where $a_i \in [0, 1]$. Prove that:

$$Re\alpha < \frac{1+\sqrt{5}}{2}.$$

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College,Romania

Solution by Iurie Boreico, student, Chisinau, Moldova

If $Re\alpha \leq 0$, then this is true. Assume now that $Re\alpha > 0$. Let $|t| = |\alpha|$. Write the condition $P(\alpha) = 0$ as

$$1 + a_{n-1}\frac{1}{\alpha} + a_2\frac{1}{\alpha^2} + \ldots + a_0\frac{1}{\alpha^n}.$$

Take now the real part,

$$1 + a_{n-1}Re\frac{1}{\alpha} = a_{n-2}Re\frac{1}{\alpha^2} + \ldots.$$

Now as $\frac{1}{\alpha} = \frac{\overline{\alpha}}{t^2}$ we conclude that $a_{n-1}Re\frac{1}{\alpha} \geq 0$. We also have $a_kRe\frac{1}{\alpha^{n-k}} \leq a_k|\frac{1}{\alpha^{n-k}}|\frac{1}{t^{n-k}}$, and so we deduce that

$$1 \leq \frac{1}{t^2} + \frac{1}{t^3} + \ldots + \frac{1}{t^n}.$$

Now if $t \leq 1$ then $Re\alpha \leq t < \frac{1+\sqrt{5}}{2}$ otherwise we have

$$1 \leq \frac{1}{t^2} + \frac{1}{t^3} + \ldots + \frac{1}{t^n} < \frac{1}{t^2} + \frac{1}{t^3} + \frac{1}{t^4} \ldots = \frac{1}{t(t-1)}$$

so $t(t-1) < 1$ which implies $t < \frac{1+\sqrt{5}}{2}$, and hence $Re\alpha < \frac{1+\sqrt{5}}{2}$.

Also solved by the proposer

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