Solutions for Mathematical Reflections 4

Juniors

J19. Let a, b, c, d be positive integers such that $3(a+b) \geq 2|ab+1|$. Prove that

$$9(a^3+b^3) \geq |a^3b^3+1|$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

First solution by Ivan Borsenco, University of Texas at Dallas

First solution. Our inequality can can be rewritten as

$$\frac{3(a+b)}{2|ab+1|} \geq \frac{a^2b^2-ab+1}{6(a^2-ab+b^2)},$$

because $a^2b^2-ab+1 \geq 0$ and $a^2-ab+b^2 \geq 0$. We will prove that

$$\frac{3(a+b)}{2|ab+1|} \geq 1 \geq \frac{a^2b^2-ab+1}{6(a^2-ab+b^2)}$$

Left side of the inequality is true from the problem statement, thus it remains to prove $6(a^2-ab+b^2) \geq a^2b^2-ab+1$ or $6a^2-3ab+6b^2 \geq (ab+1)^2$. But $\dfrac{9(a+b)^2}{4} \geq (ab+1)^2$ and it is enough to show that $4(2a^2-ab+2b^2) \geq 3(a+b)^2$ which is equivalent to $5a^2+5b^2 \geq 10ab$ or $5(a-b)^2 \geq 0$ and the problem is solved.

Second solution by Ashay Burungale

Second solution. Note that $a+b \geq 0$. First of all let us prove

$$(a^2-ab+b^2) \geq \frac{1}{6} \cdot |\frac{9}{4}(a+b)^2 - 3ab|$$

This is equivalent $8(a^2-ab+b^2) \geq |3(a+b)^2-4ab|$ or

$$8(a^2-ab+b^2) \geq 8(a-b)^2 + 8|ab| \geq |3(a-b)^2+8ab| = |3(a+b)^2-4ab|.$$

Using $3(a+b) \geq 2|ab+1|$ we get

$$\frac{1}{6} \cdot |\frac{9}{4}(a+b)^2 - 3ab| \geq \frac{1}{6}|(ab+1)^2 - 3ab)|$$

Thus

$$(a^2 - ab + b^2) \geq \frac{1}{6}|(ab+1)^2 - 3ab)|.$$

Multiplying the right side with $3(a+b)$ and the left side with $2|ab+1|$ our inequality becomes

$$9(a^3 + b^3) \geq |a^3b^3 + 1|$$

and we are done.

Also solved by Daniel Campos Salas, Costa Rica.

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J20. Let a, b, c, d be positive integers such that $ab+cd = (a+b)(c+d)$.

a) Prove that equation has infinitely many of solutions such that a, b, c, d are pairwise not equal.

b) Prove that $\max(a, b, c, d) \geq \frac{\sqrt{3}+1}{4\sqrt{3}}(a+b+c+d)$

Proposed by Ivan Borsenco, University of Texas at Dallas

Solution by Ivan Borsenco, University of Texas at Dallas

Solution. a) Our equation can be rewritten as

$$(a-(c+d))(b-(c+d)) = c^2+cd+d^2$$

So it is enough to consider $a-(c+d) = c^2+cd+d^2$ and $b-(c+d) = 1$. Then we obtain quadruple $(c^2+cd+d^2+c+d, c+d+1, c, d)$ which generates infinitely number of of pairwise different solutions.

b) Using inequalities $(a+b)^2 \geq 4ab, (c+d)^2 \geq 4cd$ we have

$$(a+b)^2 + (c+d)^2 \geq 4(a+b)(c+d).$$

Without loss of generality we can suppose $a+b \geq c+d$. From upper equation we have

$$((a+b)-(2+\sqrt{3})(c+d))((a+b)-(2-\sqrt{3})(c+d)) \geq 0.$$

Knowing that $(a+b)-(2-\sqrt{3})(c+d) \geq 0$ we deduce that

$(a+b)-(2+\sqrt{3})(c+d) \geq 0$ or $\frac{1}{2+\sqrt{3}}(a+b) \geq c+d$.

Thus $\frac{3+\sqrt{3}}{2+\sqrt{3}}(a+b) \geq (a+b+c+d)$ and using $2 \cdot \max(a, b, c, d) \geq a+b$ we get $\frac{2\sqrt{3}(\sqrt{3}+1)}{2+\sqrt{3}} \max(a, b, c, d) \geq (a+b+c+d)$.

Rewriting what we got: $\max(a, b, c, d) \geq \frac{4+2\sqrt{3}}{4\sqrt{3}(\sqrt{3}+1)}(a+b+c+d) =$
$= \frac{(\sqrt{3}+1)^2}{4\sqrt{3}(\sqrt{3}+1}(a+b+c+d) = \frac{\sqrt{3}+1}{4\sqrt{3}}(a+b+c+d)$

and the problem is solved.

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J21. A $(2m+1)\times(2n+1)$ grid is colored with two colors. A 1×1 square is called row-dominant if there are at least $n+1$ squares of its color in its row. Define column-dominant squares in the same way. Prove that there at least $m+n+1$ both column-dominant and row-dominant squares.

Proposed by Iurie Boreico, Moldova

Solution by Iurie Boreico, Moldova

Solution. In every row, there are at least $n+1$ row-dominant squares, so at least $(2m+1)(n+1)$ row-dominant squares in total. Analogously there are at least $(2n+1)(m+1)$ column-dominant squares in total. We thus get at least $(2m+1)(n+1)+(2n+1)(m+1)=(2m+1)(2n+1)+m+n+1$ row-dominant or column-dominant squares. There is a total of $(2m+1)(2n+1)$ squares and the exceed (of at least $m+n+1$) can come only from squares which are both row and column dominant. Hence the conclusion follows.

Also solved by Ashay Burungale

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J22. There are n 1's written on a board. At each step we can select two of the numbers in the board and replace them with $\sqrt[3]{\frac{a^2b^2}{(a+b)}}$. We keep applying this operation until there is only one number left. Prove that this number is not less than $\dfrac{1}{\sqrt[3]{n}}$.

Proposed by Liubomir Chiriac, Princeton University

Solution by Iurie Boreico, Moldova

Solution. Instead of any number x on the blackboard write $\dfrac{1}{x^3}$.

Now we are working with this set of numbers. The transformation

$(a,b)\to\sqrt[3]{\dfrac{a^2b^2}{a+b}}$ turns into $(x,y)\to\sqrt[3]{x^2y}+\sqrt[3]{xy^2}\le x+y$.

So the sum of the numbers does not increase. Initially we have the sum of numbers is n so in the end we get a number which is at most n. Thus after reversing our numbers back the number left on the original blackboard is then at least $1/\sqrt[3]{n}$.

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J23. Let $ABCDEF$ be a hexagon with parallel opposite sides, and let $FC \cap AB = X_1$, $FC \cap ED = X_2$, $AD \cap EF = Y_1$, $AD \cap BC = Y_2$, $BE \cap CD = Z_1$, $BE \cap AF = Z_2$. Prove that

a) If X_1, Y_1, Z_1 are collinear then X_2, Y_2, Z_2 are also collinear.

b) The lines $X_1Y_1Z_1$ and $X_2Y_2Z_2$ are parallel.

Proposed by Santiago Cuellar

Solution by Ivan Borsenco, University of Texas at Dallas

Solution. Denote $AD \cap BE = U$, $BE \cap CF = V$, $CF \cap AD = W$. Suppose X_1, Y_1, Z_1 are collinear, then by Menelaos theorem applied to the triangle UVW and the line through points X_1, Y_1, Z_1 we have

$$\frac{X_1V}{X_1W} \cdot \frac{Y_1W}{Y_1U} \cdot \frac{Z_1U}{Z_1V} = 1$$

Also because $ABCDEF$ is a hexagon with parallel opposite sides we can use similarity of triangles and get

$$\frac{X_1V}{X_2V} = \frac{BV}{VE} = \frac{BC}{EF}, \frac{X_1W}{X_2W} = \frac{AW}{WD} = \frac{AF}{FD},$$

$$\frac{Y_1W}{Y_2W} = \frac{FW}{WC} = \frac{AF}{CD}, \frac{Y_1U}{Y_2U} = \frac{EU}{BU} = \frac{DE}{AB},$$

$$\frac{Z_1U}{Z_2U} = \frac{DU}{AU} = \frac{DE}{AB}, \frac{Z_1V}{Z_2V} = \frac{CV}{VF} = \frac{BC}{EF}.$$

Replacing the values into first relation we easily obtain

$$\frac{X_2V}{X_2W} \cdot \frac{Y_2W}{Y_2U} \cdot \frac{Z_2U}{Z_2V} = 1$$

Thus we proved that if X_1, Y_1, Z_1 are collinear then X_2, Y_2, Z_2 are also collinear.

b) It enough to prove $\frac{X_1W}{X_2W} = \frac{Y_1W}{Y_2W}$. Using similarity we have

$$\frac{X_1W}{X_2W} = \frac{AW}{WD} = \frac{AF}{CD}, \frac{Y_1W}{Y_2W} = \frac{FW}{CW} = \frac{AF}{CD}.$$

and we are done.

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J24. Consider a triangle ABC and a point M inside the triangle. Denote d_a, d_b, d_c the distances from the point M to the triangle's sides. Prove that

$$2S(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{R}) \geq d_a+d_b+d_c$$

where S, R are triangle's area and circumradius.

Proposed by Ivan Borsenco, University of Texas at Dallas

First solution by Iurie Boreico, Moldova

First solution. Suppose $a \leq b \leq c$. Summing up areas for triangles AMB, BMC, CMA we get $2S = ad_a + bd_b + cd_c \geq a(d_a+d_b+d_c)$ or $\frac{2S}{a} \geq d_a+d_b+d_c$. Thus it is enough to prove
$2S(\frac{1}{b}+\frac{1}{c}-\frac{1}{R}) \geq 0$ or $\frac{1}{b}+\frac{1}{c} \geq \frac{1}{R}$. But this is easy
$\frac{1}{2R\sin b}+\frac{1}{2R\sin c} \geq \frac{1}{2R}+\frac{1}{2R}=\frac{1}{R}$ and we are done.

Second solution by Ivan Borsenco, University of Texas at Dallas

Second solution. First of all we recall Erdos theorem:

$$AM+BM+CM \geq 2(d_a+d_b+d_c)$$

Denote projections as A_1, B_1, C_1, then $\frac{A_1B_1}{sin\gamma}=\frac{A_1B_1 \cdot 2R}{c}=CM$.
Also using triangle inequality for the triangle A_1MB_1 we get
$d_a+d_b \geq A_1B_1$ or $\frac{d_a+d_b}{c} \geq \frac{CM}{2R}$.

Summing for AM, BM, CM:

$$\frac{d_a+d_b}{c}+\frac{d_b+d_c}{a}+\frac{d_a+d_c}{b} \geq \frac{AM+BM+CM}{2R} \geq \frac{d_a+d_b+d_c}{R}$$

which we rewrite as $(d_a+d_b+d_c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{R}) \geq (\frac{d_a}{a}+\frac{d_b}{b}+\frac{d_c}{c})$

Using $ad_a + bd_b + cd_c = 2S$ and Cauchy-Schwartz inequality we get

$$(ad_a + bd_b + cd_c)(\frac{d_a}{a} + \frac{d_b}{b} + \frac{d_c}{c}) \geq (d_a + d_b + d_c)^2$$

Putting in the previous one

$$(d_a + d_b + d_c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{R}) \geq \frac{(d_a + d_b + d_c)^2}{2S}$$

$$2S(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{R}) \geq (d_a + d_b + d_c)$$

and our problem is solved.

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Seniors

S19. Let ABC be a scalene triangle. A point P is called nice if AD, BE, CF are concurrent, where D, E, F are the projections of P on the respective sides of ABC. Find how many nice points lie on the line OI.

Proposed by Iurie Boreico, Moldova and Ivan Borsenco, UTD

Solution by Iurie Boreico, Moldova

Solution. Let x_1, x_2 be the length of the projectiones of $\overline{BP}$ onto BC and $\overline{CP}$ onto BC respectively; being positive iff D, C are on the same side of B. Define analogously y_1, y_2, z_1, z_2. Then we have

$$x_1 + x_2 = a, y_1 + y_2 = b, z_1 + z_2 = c$$

A characteristic property of the line OI is that

$$x_1 + y_1 + z_1 = x_2 + y_2 + z_2 = u$$

for any point on OI (the converse is also true): this trivially holds for O, I hence for any point on line OI satisfies this property by linearity of projection. Also from Carnot Lemma we have $x_1^2 + y_1^2 + z_1^2 = x_2^2 + y_2^2 + z_2^2$ which since $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$ is equivalent to

$$x_1y_1 + y_1z_1 + z_1x_1 = x_2y_2 + y_2z_2 + z_2x_2 = v$$

Now by Ceva Theorem AD, BE, CF concur if and only if

$$x_1y_1z_1 = x_2y_2z_2 = w$$

Both triples (x_1, y_1, z_1) and (x_2, y_2, z_2) are solutions for polynomial of degree 3: $x^3 - ux^2 + vx - w = 0$. Thus x_1, y_1, z_1 is a permutation of x_2, y_2, z_2. If $x_1 = x_2$ then P is the intersection of the perpendicular bisector of BC and OI hence $P = O$. Analogously if $y_1 = y_2$ or $z_1 = z_2$. If $x_1 = y_2$ $x_2 \neq y_1$ since $a = x_1 + x_2 \neq y_1 + y_2 = b$ so $x_2 = z_1$ and $z_2 = y_1$. Analogously if $x_2 = y_1$ we get $x_1 = z_2$ and $z_1 = y_2$. This gives us two more points, which are I and J, where J is the intersection of the perpendiculars from the centers of the exinscribed circles of ABC onto the corresponding sides. Hence, the total number of nice points is 3.

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S20. Let ABC be an acute triangle and P a point inside this triangle. Prove that if $a = BC$, $b = CA$ and $c = AB$, then the following inequality holds:

$$(AP + BP + CP)^2 \geq \sqrt{3}(aPA + bPB + cPC)$$

Proposed by Khoa Lu Nguyen, M.I.T

Solution by Hung Quang Tran, Hanoi National University

Solution. Let $\angle BPC = x, \angle CPA = y, \angle APB = z, x + y + z = 2\pi$. Using Cosine Theorem we have $a^2 = PB^2 + PC^2 - 2PBPC \cos x$

$$a^2 PA = PA(PB^2 + PC^2) - 2PAPBPC \cos x$$

Analogously we express $b^2 PB$ and $c^2 PC$. Summing them and using famous inequality $\cos x + \cos y + \cos z \geq -\dfrac{3}{2}$ for $x + y + z = 2\pi$ we get

$$a^2 PA + b^2 PB + c^2 PC = PA(PB^2 + PC^2) + PB(PA^2 + PC^2) +$$

$$+ PC(PA^2 + PB^2) - 2PAPBPC(\cos x + \cos y + \cos z) \leq$$

$$PA(PB^2 + PC^2) + PB(PA^2 + PC^2) + PC(PA^2 + PB^2) + 3PAPBPC =$$

$$= (PA + PB + PC)(PA \cdot PB + PB \cdot PC + PA \cdot PC) \leq \frac{1}{3}(PA + PB + PC)^3$$

Thus we have $(PA + PB + PC)^3 \geq 3(a^2 PA + b^2 PB + c^2 PC)$.

Multiplying by $(PA + PB + PC)$ and using Cauchy-Schwartz inequality we obtain

$$(PA + PB + PC)^4 \geq 3(a^2 PA + b^2 PB + c^2 PC)(PA + PB + PC) \geq$$

$$\geq 3(aPA + bPB + cPC)^2$$

which is equivalent to $(PA + PB + PC)^2 \geq \sqrt{3}(aPA + bPB + cPC)^2$ and the problem is solved.

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S21. Let p be a prime number and let $a_1, a_2, \ldots, a_n$ be distinct positive integers not exceeding $p-1$. Suppose that

$$p | a_1^k + a_2^k + \ldots + a_n^k$$

for $k = 1, 2, \ldots, p-2$. Find $\{a_1, a_2, \ldots, a_n\}$.

Proposed by Pascual Restrepo Mesa, Universidad de los Andes, Colombia

First solution by David E. Narvaez, Technological University of Panama

First solution. Let g be a primitive root modulo p. Since $1 \leq a_i \leq p-1$, $p \not| a_i$, for $i = 1, 2, \ldots, n$ and there is a positive integer α_i such that $a_i \equiv g^{\alpha_i} \pmod p$. Therefore, the condition

$$p | a_1^k + a_2^k + \ldots + a_n^k$$

can then be written as

$$\begin{aligned} a_1^k + a_2^k + \ldots + a_n^k &\equiv (g^{\alpha_1})^k + (g^{\alpha_2})^k + \ldots + (g^{\alpha_n})^k \pmod p \\ 0 &\equiv \left(g^k\right)^{\alpha_1} + \left(g^k\right)^{\alpha_2} + \ldots + \left(g^k\right)^{\alpha_n} \pmod p \end{aligned}$$

Notice that g^k, for $k = 1, 2, \ldots, p-2$ generates the set $\{2, 3, \ldots, p-1\}$, therefore the polynomial

$$P(x) = x^{\alpha_1} + x^{\alpha_2} + \ldots + x^{\alpha_n}$$

has $p-2$ distinct roots modulo p, which are $2, 3, \ldots, p-1$, thus we can write

$$\begin{aligned} P(x) &\equiv (x-2)(x-3)\cdots(x-(p-1)) \pmod p \\ x^{\alpha_1} + x^{\alpha_2} + \ldots + x^{\alpha_n} &\equiv \frac{x^{p-1}-1}{x-1} \pmod p \\ x^{\alpha_1} + x^{\alpha_2} + \ldots + x^{\alpha_n} &\equiv x^{p-2} + x^{p-3} + \ldots + 1 \end{aligned}$$

From this we can easily deduce that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} = \{0, 1, \ldots, p-2\}$, and $\{a_1, a_2, \ldots, a_n\} = \{g^{\alpha_1}, g^{\alpha_2}, \ldots, g^{\alpha_n}\} = \{1, 2, \ldots, p-1\}$.

Second solution by Ashay Burungale

Second solution. First of all we need the following lemma:

Lemma. Let $S_k = k^1 + k^2 + ... + k^{p-2}$, then
$p|S_k + 1$ if $k \not\equiv 0, 1(\mod p)$ and $p|S_k + 2$ if $k \equiv 1(\mod p)$.

Proof. Clearly if $k \equiv 1(\mod p)$ then

$$S_k = k^1 + k^2 + ... + k^{p-2} \equiv p - 2 \equiv -2(\mod p).$$

Suppose $k \not\equiv 1(\mod p)$ then $S_k = k(k^{p-2} - 1)/k - 1$, or $S_k = \dfrac{k^{p-1} - k}{k - 1}$.
From the Fermat Theorem we have $k^{p-1} \equiv 1(\mod p)$, so

$$S_k = \frac{k^{p-1} - k}{k - 1} \equiv \frac{-(k-1)}{k - 1} \equiv -1(\mod p).$$

Thus $S_k \equiv -1(\mod p)$ for $k \not\equiv 0, 1(\mod p)$, lemma is proven.

Solution. From the problem statement we have $p|\sum\limits_{i=1}^{p-2} a_1^i + a_2^i + \ldots + a_n^i$

Since they all are distinct and positive integers we have at most one of them is equal to 1 and all other $a_i \not\equiv 0, 1(\mod p)$, thus using our lemma

$\sum\limits_{i=1}^{p-2} a_1^i + a_2^i + \ldots + a_n^i \equiv (-1) \cdot n(\mod p) = -n(\text{mod } p)$ or

$\sum\limits_{i=1}^{p-2} a_1^i + a_2^i + \ldots + a_n^i \equiv (-2) + (-1) \cdot (n - 1) = -n - 1(\text{mod } p).$

It follows that $p| - n$ or $p| - (n + 1)$ and we know from the problem statement that $n \le p - 1$. Hence the only one possibility is $n = p - 1$ which gives us the solution $\{a_1, a_2, ..., a_n\}$ $=\{1, 2, ..., p - 1\}$, which clearly satisfies the condition.

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S22. Let n, k be positive integers. Eve gives to Adam k apples. However she can first give him some bitter apples, at most n. The procedure goes as follows: Eve gives to Adam the apples one by one, and he can either eat it (and find out whether it's sweet or not), or throw it. Adam knows that the bitter apples come first, and the sweet come last. Find, in terms of n, the least value of k for which Adam can ensure he eats more sweet apples than bitter ones.

Proposed by Iurie Boreico, Moldova

Solution by Iurie Boreico, Moldova

Solution. It's clear that if an apple is sweet, then all the apples coming after it are sweet, so Adam can surely eat all the apples he takes after encountering one sweet one. If the apple is bitter, then all the k sweet apples are yet to come.

We prove that Adam can ensure he eats more sweet apples then bitter ones if and only if $n < \frac{k(k+3)}{2}$. For these values of n we can show the following strategy: let Adam eat the apple k. If it's sweet, he won. Otherwise let him it the apple $2k-1$. If it's sweet, then there is at least one more sweet apple coming next, as there are just $k-2$ apples between k and $2k-1$. So he has eaten two sweet apples and one bitter. At the next step Adam will eat the apple $k+(k-1)+(k-2)$ and so on. Reasoning analogously, we see that if Adam eats at least one sweet apple he won. However he will eat at least one sweet apple: if not, then he will have eaten the apples $k, k+(k-1), \ldots, \frac{k(k+1)}{2}$, and the last apple for sure is sweet otherwise there will be at least $\frac{k(k+3)}{2} > n$ apples.

Assume that Adam decides to eat the apple numbered a_1. Then $a_1 \leq k$ otherwise it could be possible that the first k apples were sweet so he would miss all of them. If $n > 2k-1 \geq a_1+k-1$ then the apple numbered a_1 could be bitter, so Adam could be forced to eat another apple, numbered a_2. Then $a_2 \leq a_1+k-1$ because otherwise it could be possible that the apples numbered $a_2-k+1, \ldots, a_2$ were sweet, and Adam would eat only at most one of them (namely a_2), and with already one bitter apple eaten, he would fail. So $a_2 \leq 2k-1$. The continuation is now by induction: we prove that if $n >\geq frack(k+3)2$ for $l < k$ then Adam is forced to eat at least l possibly bitter apples $a_1, a_2, \ldots, a_l$ with $a_{i+1}-a_i \leq k-i$ (so $a_l < n-k+l$). The base was just proven, and for the induction step notice that if $n \geq \frac{k(k+3)}{2} \geq a_l+k$ and Adam ate the bitter apples $a_1, \ldots, a_l$ then he would be forced to eat one more apple a_{l+1}. Then $a_{l+1} \leq a_l+k-l$, otherwise it could be possible that the apples $a_{l+1}-k+l, \ldots, a_{l+1}+l-1$ were the last and sweet apples, and Adam could only eat l of them, starting from a_2 which is not more than the number of bitter ones already consumed. The induction step is complete. Now by setting $l = k$ Adam has to eat at least k apples which can all be bitter. Since he can eat at most k sweet apples, he fails. Therefore $n \leq \frac{k(k+1)}{2}$.

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S23. Let a, b, c, d be positive numbers, prove the following inequality

$$3(a^2 - ab + b^2)(c^2 - cd + d^2) \geq (a^2c^2 - abcd + b^2d^2)$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by Bin Zhao

Solution. We observe that

$$(a^2c^2 - abcd + b^2d^2) - (a^2 - ab + b^2)(c^2 - cd + d^2) = (ad + bc)(a - b)(c - d)$$

Thus the inequality is equivalent to:

$$(a^2 - ab + b^2)(c^2 - cd + d^2) \geq 2(ad + bc)(a - b)(c - d)$$

If $a = b = 0$ or $c = d = 0$ the inequality is obviously true. Now let us ignore this case. Without loss of generality suppose $a \neq 0, c \neq 0$ (analogously for $a \neq 0, d \neq 0$). Dividing the above inequality by a^2c^2 both sides and substituting $x = \frac{b}{a}, y = \frac{d}{c}$, we get the following inequality

$$(x^2 - x + 1)(y^2 - y + 1) \geq 2(x + y)(x - 1)(y - 1)$$

which is equivalent to

$$(y^2 - 3y + 3)x^2 - (3y^2 - 5y + 3)x + 3y^2 - 3y + 1 \geq 0$$

The left side is a binomial of x,we have the discriminant:

$$\Delta = (3y^2 - 5y + 3)^2 - 4(y^2 - 3y + 3)(3y^2 - 3y + 1) =$$

$$= -3y^4 + 18y^3 - 33y^2 + 18y - 3 = -3(y^2 - 3y + 1)^2 \leq 0$$

So we have:

$$(y^2 - 3y + 3)x^2 - (3y^2 - 5y + 3)x + 3y^2 - 3y + 1 \geq 0$$

Thus we prove the original inequality. And the inequality hold when: $a = \frac{3+\sqrt{5}}{2}b$, $c = \frac{3+\sqrt{5}}{2}d$ or $b = \frac{3+\sqrt{5}}{2}a$, $d = \frac{3+\sqrt{5}}{2}c$.

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S24. Let ABC be an acute-angled triangle inscribed in a circle $\mathcal{C}$. One considers all equilateral triangles DEF with vertices on $\mathcal{C}$. The Simpson lines of D, E, F with respect to the triangle ABC form a triangle T. Find the greatest possible area of the triangle T.

Proposed by Iurie Boreico, Moldova and Ivan Borsenco, UTD

Solution by Iurie Boreico, Moldova

Solution. We will firstly prove that $\mathbb{T}$ is equilateral. Indeed, if $D \in \overline{BC}$ let D_A, D_B, D_C be its projections on the sides of ABC. WLOG $D_B \in (AC), B \in (AD_C)$. Then the angle between the Simpson line of D and BC is $m(\angle D_C D_A B) = m(D_C DB) = \frac{1}{2}m(\overline{DBA}) - 90 = \frac{1}{2}m(\overline{DC}) + B - 90$. Proceeding analogously for E, F we can conclude that the angle between the Simpson lines of two points is actually half of the measure of arc between these two points. Particularly, since DEF is equilateral, so is T.

Another fact that we will use the well-known fact that the Simpson line l_D of the point D passes through the midpoint M od the segment $[DH]$, where H is the orthocenter of ABC. Indeed, using the notations and the situation analyzed above let H' be the symmetric of H wrt BC that lies on $\mathcal{C}$ from Hamilton Theorem and let D'_A be the symmetric of D wrt BC. Since l_D passes through D_A, to prove it passes through M is equivalent to proving that the angle between l_D and DD_A equals the angle $\angle MD_AD \equiv \angle HD'_AD \equiv \angle HH'D$ as $HH'DD'_A$ is an isosceles trapezoid. However this results immediately from angle chasing, since we know the angle between l_D and BC.

Now, let's pass to the solution of the problem. Since $\mathbb{T}$ is equilateral, we must find the largest possible value of the inradius of $\mathbb{T}$. However the Euler circle of ABC clearly contains the midpoints M, N, P of segments $[HD], [HE], [HF]$ and so intersects the sides of the triangle $\mathbb{T}$. It easily follows now that its radius is not less than the inradius of $\mathbb{T}$. Indeed, if $\mathbb{T} = UVW$ and J is the midpoint of $[OH]$ then we get $d(J, UV) \leq \frac{R}{2})$, where R is the radius of $\mathcal{C}$ so $[JUV] \leq \frac{UVR}{4}$. Summing this with the analogous relations for VW, UW we get

$$p_{UVW} r_{UVW} = S[UVW] \leq S[JUV] + S[JUW] + S[JVW] \leq$$

$$\frac{R(UV + VW + UW)}{4} = p_{UVW} \frac{R}{2}$$

so $r_{UVW} \leq \frac{R}{2}$.

Now we shall find a triangle UVW with $r_{UVW} = \frac{R}{2}$. To do this, we must ensure that the Euler circle of ABC touches the Simpson lines of l_D, l_E, l_F. Let's effectively compute the positions of D, E, F that satisfy this condition. Suppose D belongs to the small arc $\overline{BC}$. As l_D passes through M, it is tangent iff $l_D \perp MJ$ or $l_D \perp DO$. The angle between DO and BC is $180 - m(\angle ODC) - m(\angle BCD) = 180 - (90 - \frac{1}{2}m(\overline{CD})) - \frac{1}{2}m(\overline{BD}) = 90 + \frac{1}{2}(m(\overline{CD}) - m(\overline{BD}))$. However the angle between l_D and BC is, as computed above $\frac{1}{2}m(\overline{DC}) + B - 90$. So $DO \perp l_D$ iff $m(\overline{CD}) - \frac{1}{2}m(\overline{BD}) + B = 90$. As $m(\overline{BD}) = 2A - m(\overline{CD})$, this is possible if and only if $m(\overline{CD}) = \frac{2(90+A-B)}{3}$ and so the position of D on the arc $\overline{BC}$ is uniquely determined (and since the triangle is acute-angled, we can check that D indeed lies on the arc). Analogously if we let $E \in \overline{AC}$ we get $m(\overline{CE}) = \frac{2(90+B-A)}{3}$, thus $m(\overline{DE}) = m(arcCD) + m(\overline{CE} = 120)$. So we conclude that the triangle DEF that we seek is indeed equilateral. Its inradius is $\dfrac{R}{2}$ so its area is $\dfrac{3\sqrt{3}R^2}{4}$, and this is the maximal possible area.

★

Undergraduate

U19. Let f_0 be a real-valued function, continuous on the interval $[0, 1]$ and for each integer $n \geq 0$ let $f_{n+1}(x) = \displaystyle\int_0^x f_n(t)dt$. Suppose that there is a positive integer k with the property that $f_k(1) = \dfrac{1}{(k+1)!}$. Prove that there exists x_0 such that $f_0(x_0) = x_0$.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

No solutions proposed.

★

U20. Prove that one cannot find an entire function f such that $f(f(x)) = e^x$ for all real number x, but can find an infinitely many times differentiable function with this property.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure,Paris

No solutions proposed.

★

U21. Evaluate
$$\int_0^1 \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\} dx$$
where $\{x\}$ is the fractional part of x.

Proposed by Ovidiu Furdui, Western Michigan University

Solution by Bin Zhao

Solution. The answer is $2\gamma - 1$,where γ is the Euler's Constant.

First we will prove:
$$\int_0^{\frac{1}{2}} \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\} dx = \gamma - \frac{1}{2}$$
This is because
$$\int_0^{\frac{1}{2}} \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\} dx = \sum_{n=2}^{+\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\} dx$$
Set
$$a_n = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\} dx$$
because for $n \geq 2$, we have
$$\left\{\frac{1}{x}\right\} = \frac{1}{x} - n, \left\{\frac{1}{1-x}\right\} = \frac{1}{1-x} - 1 = \frac{x}{1-x},$$
when $\dfrac{1}{n+1} < x < \dfrac{1}{n}$. From here
$$a_n = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\} dx = \int_{\frac{1}{n+1}}^{\frac{1}{n}} (\frac{1}{x} - n)\frac{x}{1-x} dx = \int_{\frac{1}{n+1}}^{\frac{1}{n}} n - \left(\frac{n-1}{1-x}\right) dx$$
$$= \frac{1}{n+1} - (n-1)\left(\ln(1 - \frac{1}{n+1}) - \ln(1 - \frac{1}{n})\right) =$$
$$= \frac{1}{n+1} - (n-1)\ln\left(\frac{n^2}{(n-1)(n+1)}\right).$$

Thus

$$\int_0^{\frac{1}{2}} \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\} dx = \sum_{n=2}^{+\infty}\left(\frac{1}{n+1} - (n-1)\ln\left(\frac{n^2}{(n-1)(n+1)}\right)\right)$$

but

$$S_N = \sum_{n=2}^{N}\frac{1}{n+1} - \left((n-1)\ln\left(\frac{n^2}{(n-1)(n+1)}\right)\right) =$$

$$= \sum_{n=2}^{N}\frac{1}{n+1} - \sum_{n=2}^{N}(n-1)\ln\left(\frac{n^2}{(n-1)(n+1)}\right) =$$

$$= \sum_{n=2}^{N}\frac{1}{n+1} - \sum_{n=2}^{N}\ln\left(\frac{n^{2n-2}}{(n-1)^{n-1}(n+1)^{n-1}}\right) =$$

$$= \sum_{n=2}^{N}\frac{1}{n+1} - \ln\frac{2^2\cdot 3^4\cdots N^{2N-2}}{(1\cdot 3)(2^2\cdot 4^2)\cdots((N-1)^{N-1}(N+1)^{N-1})}$$

$$= \sum_{n=2}^{N}\frac{1}{n+1} - \ln\frac{N^N}{(N+1)^{N-1}} = \sum_{n=2}^{N}\frac{1}{n+1} - \ln\left(\frac{N}{N+1}\right)^{N-1} - \ln N$$

With the well-known formula

$$\lim_{n\to+\infty}(1+\frac{1}{n})^n = e, \lim_{n\to+\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n} - \ln n\right) = \gamma$$

We have

$$\int_0^{\frac{1}{2}} \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\} dx = \lim_{N\to+\infty} S_N = \gamma - \frac{1}{2}.$$

But set $t = 1-x$,we have

$$\int_{\frac{1}{2}}^{1} \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\} dx = \int_0^{\frac{1}{2}} \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\} dx$$

Finally

$$\int_0^{1} \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\} dx = 2\gamma - 1$$

★

U22. Let $||.||$ be a norm in $\mathbb{C}^n$ and define

$$|||A||| = \sup_{||x||\leq 1} ||Ax||$$

for any complex matrix A in $M_n(\mathbb{C})$. Let $a < 2$ and G be a subgroup of $GL_n(\mathbb{C})$ such that for $A \in G$ one has $|||A - I_n||| \leq a$. Prove that G is finite.

Proposed by Gabriel Dospinescu and Alexander Thiery

Solution by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Solution. Consider $A \in G$. Then for all integers k we know that $|||A^k - I_n||| \leq a$. Let λ be an eigenvalue of A and let x be a proper vector. Then $(A^k - I_n)x = (\lambda^k - 1)x$ and so $|\lambda^k - 1| \leq a < 2$ for all integers k. Thus $|\lambda|^k < 3$ for all integers k, which implies that $|\lambda| = 1$. Thus $\lambda = e^{i\pi r}$ for some real number r. It easily follows from the previous inequality that $\cos(\pi kr) \geq 1 - \frac{a^2}{2} > -1$ for all integers k. Using Kronecker's theorem, we deduce that r must be rational and then a simple argument allows to prove that the denominator of r is uniformly bounded in terms of a only. This shows the existence of a number N not depending on $A \in G$ such that $\lambda^N = 1$ for all $A \in G$ and all λ an eigenvalue of A.

Fix now $A \in G$ and write $A^N = I_n + B$, where B is nilpotent(by the previous arguments, all eigenvalues of B are zero). We know that $|||A^{Np} - I_n||| < 2$ for all p. But using the binomial formula, this implies $|||\binom{p}{1}B + \binom{p}{2}B^2 + ... + \binom{p}{n-1}B^{n-1}||| < 2$ for all p (we used here the fact that $B^n = O_n$ because B is nilpotent). Suppose that B is not zero and take $j \geq 1$ the least positive integer for which $B^j \neq O$. Then $2 > \binom{p}{j}^{|}||B^j||| - \binom{p}{j-1}|||B^{j-1}||| - ... - \binom{p}{1}|||B|||$, which is impossible for all positive integers p (just divide by p^j and take the limit for $p \to \infty$). Thus $B = 0$.

We have established the existence of a positive integer N such that $A^N = I_n$ for all $A \in G$. By Burnside's theorem, any subgroup of $GL_n(\mathbb{C})$ of finite exponent is finite. Therefore the group is finite. However, we will present a proof of Burnside's theorem: consider $A_1, A_2, ..., A_r \in G$ a basis of the linear space spanned by the elements of G, with $r \leq n^2$. We claim that the function $f(X) = (tr(A_1X), tr(A_2X), ..., tr(A_rX))$ is injective when restricted to G. Supposing the contrary, let $A, B \in G$ such that $f(A) = f(B)$. Because $A_1, ..., A_r$ generate the linear space spanned by G, it follows that $tr(AX) = tr(BX)$ for all $X \in G$. Thus if

$U = AB^{-1}$ we have $tr(UX) = tr(X)$ for all $X \in G$. Thus $tr(U) = n$ and also $U^N = I_n$. This implies that $U = I_n$, because $tr(U)$ is the sum of the eigenvalues of U, which are roots of unity. If their sum if n, it means that all of them are 1, so U is unipotent. Because U is also diagonalisable in $M_n(\mathbb{C})$, it follows that $U = I_n$, which proves the claim. Now, the proof is finished, because clearly the image of f is finite: all $tr(X)$ with $X \in G$ are among the collection of sums of n roots of unity of order N, thus in a finite set. This shows that G is finite.

Remark

If $a = 2$, the statement is false, because for the euclidean norm we can take the set of unitary matrices, which is an infinite group with all desired properties. However, even in this case some strong restrictions are imposed. First of all, it is clear that such subgroups must be bounded. Thus their closure is also bounded. It follows that the closure of any such subgroup is a compact subgroup of $GL_n(\mathbb{C})$. It can be shown (the proof is not trivial at all) that such subgroups are conjugated to subgroups of the unitary group.

★

U23. Find the following sum

$$\sum_{k=0}^{n-1} \frac{1}{1 + 8sin^2(\frac{k\pi}{n})}$$

Proposed by Dorin Andrica and Mihai Piticari

No solutions proposed.

★

U24. Find all linear maps $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that for any $A \in M_n(\mathbb{C})$ we have $f(A^k) = f^k(A)$ for some integer $k > 1$

Proposed by Gabriel Dospinescu, Ecole Normale Superieure,Paris

Solution by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Solution. First of all, let us prove that $f(A^r) = f^r(A)$ for all positive integers r and all A. Fix an A and observe that $f((A + xI_n)^k) = f(A + xI_n)^k = (f(A) + xI_n)^k$ for all complex x. Using the binomial formula and the linearity of f, we obtain

$$f(A^k) + \binom{k}{1} x f(A^{k-1}) + ... + \binom{k}{k-1} x^{k-1} f(A) + x^k I_n$$
$$= f(A)^k + \binom{k}{1} x f(A)^{k-1} + ... + x^k I_n.$$

By identifying coefficients in the last equality we deduce that $f(A^r) = f^r(A)$ for all $r \le k$. In particular, $f(A^2) = f(A)^2$. Thus $f(A^2 + AB + BA + B^2) = f(A)^2 + f(A)f(B) + f(B)f(A) + f(B)^2$. This implies

$$f(AB) + f(BA) = f(A)f(B) + f(B)f(A).$$

By taking $B = A^p$ in this last relation, we find $2f(A^{p+1}) = f(A)f(A^p) + f(A^p)f(A)$, which allows an immediate proof by induction of the equality $f(A^r) = f(A)^r$ for all r. Observe that f is continuous (being linear in a finite dimensional space), so we can write

$$f(e^A) = f(I_n + A + \frac{A^2}{2!} + ...) = I_n + f(A) + \frac{f(A)^2}{2!} + ... = e^{f(A)}$$

by the previous result. Now, take A any invertible matrix with complex entries. A classical result asserts that there exists a matrix B such that $A = e^B$. Thus $f(A) = e^{f(B)}$, that is $f(A)$ is also invertible.

Despite all our efforts, we could not find an elementary continuation of these arguments, so we are obliged to use a deep result of Marcus and Purves (1959) saying that any linear map preserving invertibility and such that $f(I_n) = I_n$ is of the form $f(X) = BXB^{-1}$ or $f(X) = BX^TB^{-1}$ for some constant invertible matrix B. For a proof we recommend the excellent book **Problems and theorems in linear algebra** by V.Prasolov. Using this result, it is easy to conclude: the above maps are all solutions of the problem.

★

Olympiad

O19. Let a, b, c be positive numbers, prove the following inequalities
a) $(a^3+b^3+c^3)^2 \geq (a^4+b^4+c^4)(ab+bc+ac)$
b) $9(a^4+b^4+c^4)^2 \geq (a^5+b^5+c^5)(a+b+c)^3$
Proposed by Ivan Borsenco, University of Texas at Dallas

First solution by Daniel Campos Salas, Costa Rica.
First solution. From Lagrange's identity we have that

$$(a^3+b^3+c^3)^2 = (a^4+b^4+c^4)(a^2+b^2+c^2) - \sum_{cyc} a^2b^2(a-b)^2.$$

Therefore,

$$\begin{aligned} & (a^3+b^3+c^3)^2 - (a^4+b^4+c^4)(ab+bc+ac) \\ = \ & (a^4+b^4+c^4)(a^2+b^2+c^2-ab-bc-ac) - \sum_{cyc} a^2b^2(a-b)^2 \\ = \ & \sum_{cyc} \frac{1}{2}(a^4+b^4+c^4)(a-b)^2 - a^2b^2(a-b)^2 \\ = \ & \frac{1}{2}\sum_{cyc}\big((a^2-b^2)^2+c^4\big)(a-b)^2 \geq 0, \end{aligned}$$

and we have proved the first inequality.

Again, from Lagrange's identity we have that

$$(a^4+b^4+c^4)^2 = (a^5+b^5+c^5)(a^3+b^3+c^3) - \sum_{cyc} a^3b^3(a-b)^2.$$

Note also the identity

$$9(a^3+b^3+c^3) - (a+b+c)^3 = \sum_{cyc}(4a+4b+c)(a-b)^2.$$

Then,

$$
\begin{aligned}
& 9(a^4+b^4+c^4)^2-(a^5+b^5+c^5)(a+b+c)^3 \\
= \;& (a^5+b^5+c^5)(9(a^3+b^3+c^3)-(a+b+c)^3)-9\sum_{cyc} a^3b^3(a-b)^2 \\
= \;& \sum_{cyc}\big((a^5+b^5+c^5)(4a+4b+c)-9a^3b^3\big)(a-b)^2 \\
\geq \;& \sum_{cyc}\big(4(a^5+b^5)(a+b)-9a^3b^3\big)(a-b)^2 \\
\geq \;& \sum_{cyc}(16a^3b^3-9a^3b^3)(a-b)^2 \geq 0,
\end{aligned}
$$

and we're done.

Second solution by Tung Nguyen Trong, Natural Science, Hanoi, Vietnam

Second solution. a) We consider the following inequality:

$\sum(a^3+b^3-ab^2-a^2b)^2 \geq 0 \leftrightarrow$

$\sum a^6+b^6+(a^2b^4+a^4b^2)+4a^3b^3-2(a^4b^2+a^2b^4)-2(a^5b+b^5a) \geq 0 \leftrightarrow$

$\sum a^6+b^6+4a^3b^3 \geq \sum(a^4b^2+a^2b^4)-2\sum(a^5b+b^5a)$

Dividing both two sides by 2:

$(a^3+b^3+c^3)^2 \geq \frac{1}{2}\sum(a^4b^2+a^2b^4)+\sum(a^5b+b^5a)$

By AM-GM we have:

$\sum(a^4b^2+a^2b^4)=\sum(a^4b^2+a^4c^2) \geq 2\sum a^4bc$

So $(a^3+b^3+c^3)^2 \geq \sum a^4bc+\sum(a^5b+b^5a)=(a^4+b^4+c^4)(ab+bc+ca)$ and we are done.

b) This time let us look at this inequality:

$\sum(2a^4+2b^4-2a^2b^2-ab^3-a^3b)^{\geq}0 \leftrightarrow$

$\sum 4(a^8+b^8)+14a^4b^4+(a^2b^6+a^6b^2)-8(a^6b^2+a^2b^6)-4(a^5b^3+a^3b^5)$
$-4(a^7b+ab^7)+4(a^5b^3+a^3b^5) \geq 0 \leftrightarrow$

$\sum 4(a^8+b^8)+14a^4b^4 \geq 7\sum(a^6b^2+a^2b^6)+\sum 4(a^7b+ab^7)$

It's easy to see that:

$\sum 4(a^8+b^8)+18a^4b^4 = 9(a^4+b^4+c^4)^2-(a^8+b^8+c^8)$

From here we will use the notation (.) inspire of $\sum_{cyclic}$ for short! It suffices to show that:

$7(a^6b^2+a^2b^6)+4(a^7b+ab^7)+4(a^4b^4) \geq (a^5+b^5+c^5)(a+b+c)^3-(a^8+b^8+c^8)(*)$

By some calculations we get:

$RHS = (a^5b^3+a^3b^5)+3(a^7b+ab^7)+3(a^6b^2+a^2b^6)+3a^5(b^2c+c^2b)+6(a^5abc)$

So (*) is equivalent to:

$4(a^6b^2+a^2b^6)+(a^7b+ab^7)+4(a^4b^4) \geq (a^5b^3+a^3b^5)+3a^5(b^2c+c^2b)+6(a^6bc) \leftrightarrow 8(a^6b^2+a^2b^6)+2(a^7b+ab^7)+8(a^4b^4) \geq 2(a^5b^3+a^3b^5)+6a^5(b^2c+c^2b)+12(a^6bc)$

By AM-GM we obtain:

$\sum a^6b^2+a^2b^6+2a^4b^4 \geq 2\sum(a^5b^3+a^3b^5) \leftrightarrow 4(a^6b^2+a^2b^6)+8(a^4b^4) \geq 8(a^5b^3+a^3b^5) \geq 2(a^5b^3+a^3b^5)+6a^5(b^2c+c^2b)$

According Muirhead's Inequality:$\sum_{sym} a^5b^3 \geq \sum_{sym} a^5b^2c$. It remains to prove that:

$4(a^6b^2+a^2b^6)+2(a^7b+ab^7) \geq 12a^6bc$

But it's also true by Muirhead:

$$\sum_{sym} a^6b^2 \geq \sum_{sym} a^6bc \text{ and } \sum_{sym} a^7b \geq \sum_{sym} a^6bc$$

and we are done.

Third solution by Ashay Burungale

Third solution. According to EV-Theorem, it suffices to consider the case $b = c = 1$.

First inequality becomes $(a^3 + 2)^2 \geq (a^4 + 2)(2a + 1)$ or

$$a^6 - 2a^5 - a^4 + 4a^3 - 4a + 2 \geq 0$$

$$(a - 1)^2(a^4 - 2a^2 + 2) \geq 0$$

Second inequality becomes $9(a^4 + 2)^2 \geq (a^5 + 2)(a + 2)^3$ which transforms after some work into

$$(a - 1)^2(4a^6 + 5a^5 - 9a^3 + 8a + 10) \geq 0$$

and we are done.

Finally we want to say that EV-Theorem is a powerful method which we advice to find and use in proving symmetric inequalities.

★

O20. In triangle ABC the incircle touches AC at E and BC at D. The excircle(corresponding to A) touches the side BC at A_1 and the extensions of sides AB, AC at C_1 and B_1, respectively. If $DE \cap A_1B_1 = \{L\}$, prove that L lies on the circumcircle of ΔA_1BC_1.

Proposed by Liubomir Chiriac, Princeton University

First solution by David E. Narvaez, Technological University of Panama

First solution. If $L = A_1$, then the result is trivial. Otherwise, let a, b and c be the lenghts of sides BC, AC and AB respectively, $\gamma = \angle ACB$, $\beta = \angle ABC$, $s = \frac{a+b+c}{2}$, I_A the center of the excircle corresponding to A. Finally, let l be the line perpendicular to BC passing through B and let $A' = DE \cap l$. We claim that A_1A' is a diameter of the circumcircle of triangle A_1BC_1.

Since $CA_1 = CB_1 = s - b$, $\angle CB_1A_1 = \frac{\pi - \angle A_1CB_1}{2} = \frac{\pi-(\pi-\gamma)}{2} = \frac{\gamma}{2}$, and B_1A_1 is then parallel to the bisector of angle $\angle ACB$, which is perpendicular to line DE. Therefore $\angle A_1LA' = 90$, and since AA' is a diameter of the circumcircle of A_1BC_1, L belongs to the circumference.

To prove our claim, consider that l is parallel to line I_AA_1, and

$$BA' = BD \cdot \tan\frac{\pi-\gamma}{2} = \frac{s-b}{\tan\frac{\gamma}{2}} = \frac{CA_1}{\tan\frac{\gamma}{2}} = I_AA_1$$

$BA'I_AA_1$ is a rectangle, which proves our claim.

Second solution by Daniel Campos Salas, Costa Rica.

Second solution.

Let α, β, θ be the angles A, B, C, respectively, and let $F = A_1B_1 \cap AB$. Note that L can lie inside or outside ABC. In the first case, for proving that C_1BLA_1 is cyclic, we should prove that $\angle BLF = \angle A_1C_1B = \dfrac{\beta}{2}$. For the second case, proving that C_1BA_1L is cyclic is equivalent to prove that $\angle BLF = \angle A_1C_1B = \dfrac{\beta}{2}$. In both cases we have to prove that $\angle BLF = \dfrac{\beta}{2}$. Since $\triangle AB_1C_1$ is isosceles, and $\angle A_1B_1C = \dfrac{\theta}{2}$ it follows that $\angle C_1B_1F = \dfrac{\beta}{2}$. Then, the problem reduces to prove that $BL \parallel B_1C_1$.

Note that $\angle BA_1F = \frac{\theta}{2}$, then by the sine law applied to $\triangle A_1BF$ implies that $\frac{FB}{BA_1} = \frac{\sin\frac{\theta}{2}}{\sin\left(\frac{\theta}{2}+\alpha\right)}$. From the Mollweide's formulas we have that $\frac{\sin\frac{\theta}{2}}{\sin\left(\frac{\theta}{2}+\alpha\right)} = \frac{c}{a+b}$, and since $BA_1 = BC_1$ it follows that $\frac{FB}{BC_1} = \frac{c}{a+b}$, from where we conclude that

$$\frac{FC_1}{BC_1} = 1 + \frac{FB}{BC_1} = \frac{2s}{a+b}. \tag{1}$$

Since $AB_1 = s$, from the sine law on $\triangle AB_1F$ it follows that

$$FB_1 = \frac{s\sin\alpha}{\sin\left(180 - \left(\alpha + \frac{\theta}{2}\right)\right)} = \frac{s\sin\alpha}{\sin\left(\alpha+\frac{\theta}{2}\right)}. \tag{2}$$

Since $\angle B_1EL + \angle EB_1L = \left(90 - \frac{\theta}{2}\right) + \frac{\theta}{2} = 90$, then DE is perpendicular to B_1L. Since $B_1E = a$, it follows that $LB_1 = a\cos\frac{\theta}{2}$.

$$\tag{3}$$

From (2) and (3), the Mollweide's formulas, the double-angle formula, and the sine law it follows that

$$\frac{FB_1}{LB_1} = \frac{s\sin\alpha}{a\cos\frac{\theta}{2}\sin\left(\alpha+\frac{\theta}{2}\right)} = \frac{2sc\sin\alpha}{a(a+b)\sin\theta} = \frac{2s}{a+b}. \tag{4}$$

From (1) and (4) it follows that BL is parallel to B_1C_1, and the conclusion follows.

Also solved by Ivan Borsenco

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O21. Let p be a prime number. Find the smallest degree of a polynomial f with integer coefficients such that the numbers $f(0), f(1)...f(p-1)$ are perfect $(p-1)$th powers.

Proposed by Pascual Restrepo Mesa

Solution by Iurie Boreico, Moldova

Solution. Assume that the degree of the polynomial is less than $p-1$. Let's work in $\mathbb{Z}[p]$. We know that $0^k + 1^k + \ldots + (p-1)^k = 0$, therefore $f(0) + \ldots + f(p-1) = 0$. However $f(x) = 0, 1$ in $\mathbb{Z}[p]$, and this implies either $f(x) = 0$ for all $x = 0, 1, \ldots, p-1$ or $f(x) = 1$ for all $x = 0, 1, \ldots, p-1$. So either $f(x)$ or $f(x) - 1$ is zero in Z_p impossible.

Therefore the degree of f is at least $p-1$ and it's clear it can be $p-1$, so this is the answer.

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O22. Consider a triangle ABC and two points P, Q in it's plane. Let A_1, B_1, C_1 and A_2, B_2, C_2 be the cevians in our triangle. Denote U, V, W as second intersections of circles $(AA_1A_2), (BB_1B_2), (CC_1C_2)$ with the circumcircle of the triangle ABC. Let X be the intersection of AU with BC, similarly define Y for BV and AC and Z for CW and AB. Prove that X, Y, Z are collinear.

Proposed by Khoa Lu Nguyen, MIT and Ivan Borsenco, University of Texas at Dallas

Solution by Khoa Lu Nguyen, MIT and Ivan Borsenco, University of Texas at Dallas

Solution. From the Power of a Point Theorem applied to the circles (ABC) and (AA_1A_2) and point X we get

$$XB \cdot XC = XA \cdot XU = XA_1 \cdot XA_2$$

Suppose without loss of generality that B is between X and C, A_1 is between B and A_2. Let us denote $XB = x, BA_1 = u, A_1A_2 = v, A_2C = w$ then the upper relationship is equivalent to

$$x(x + u + v + w) = (x + u)(x + u + v).$$

From where we get $x = \dfrac{u(u+v)}{w-u}$. Now let us calculate $\dfrac{XB}{XC}$.

$$\frac{XB}{XC} = \frac{x}{x+u+v+w} = \frac{u(u+v)}{w(v+w)} = \frac{BA_1 \cdot BA_2}{BA_2 \cdot CA_2}$$

Thus we can use Menelaos Theorem for points X, Y, Z with respect to the triangle ABC to prove that they are collinear.

$$\prod \frac{XB}{XC} = \prod \frac{BA_1 \cdot BA_2}{BA_2 \cdot CA_2} = 1$$

Last equality is due Ceva theorem applied for P and Q, and we are done.

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O23. Let ABC be a triangle and let A_1, B_1, C_1 be the points where the angle bisectors of A, B and C meet the circumcircle of triangle ABC respectively. Let M_a be the midpoint of the segment connecting the intersections of segments A_1B_1 and A_1C_1 with segment BC. Define M_b and M_c analogously. Prove that AM_a, BM_b, and CM_c are concurrent if and only if ABC is isosceles.

Proposed by Dr.Zuming Feng, Philips Exeter Academy

Solution by Daniel Campos Salas, Costa Rica.

Solution. Let I be the incenter of triangle ABC and let $P = BC \cap A_1C_1$, $Q = BC \cap A_1B_1$, $R = AB \cap A_1C_1$, and $S = CC_1 \cap AB$. Note that $\angle RPB = \angle CPA_1 = \dfrac{\angle A + \angle C}{2}$ and $\angle PRB = \angle ARC_1 = \dfrac{\angle A + \angle C}{2}$. Then, $\triangle BPR$ is isosceles and since BI is the angle bisector it follows that B_1I is perpendicular to A_1C_1 and the segment PR is bisected by BI. Analogously, A_1I and C_1I are perpendicular to B_1C_1 and A_1B_1 respectively. Then, I is the orthocenter of $A_1B_1C_1$. Using the well-known fact that the reflection of the orthocenter on the sides of the triangles lies on its circumcircle and that BI is perpendicular to $A'C'$, it follows that PR bisects BI. Since PR and BI bisect each other, it follows that $BPIR$ is a parallelogram, then IP is parallel to AB.

It is well-known that $\dfrac{CS}{IS} = \dfrac{2s}{c}$, and since IP is parallel to AB it follows that $\dfrac{BC}{PB} = \dfrac{a}{PB} = \dfrac{2s}{c}$. It follows that $PB = \dfrac{ac}{2s}$. Analogously, $QC = \dfrac{ab}{2s}$. Then, $PQ = \dfrac{a^2}{2s}$. It follows that $BM_a = \dfrac{ac}{2s} + \dfrac{a^2}{4s} = \dfrac{a(a+2c)}{4s}$. Analogously, $CM_a = \dfrac{a(a+2b)}{4s}$, then $\dfrac{BM_a}{CM_a} = \dfrac{a+2c}{a+2b}$. Analogously, $\dfrac{CM_b}{AM_b} = \dfrac{b+2a}{b+2c}$ and $\dfrac{AM_c}{BM_c} = \dfrac{c+2b}{c+2a}$.

It follows that $\dfrac{BM_a \cdot CM_b \cdot AM_c}{CM_a \cdot AM_b \cdot BM_c} = \dfrac{(a+2c)(b+2a)(c+2b)}{(a+2b)(b+2c)(c+2a)}$. By Ceva's theorem we have that AM_a, BM_b, and CM_c are concurrent if and only if $\dfrac{(a+2c)(b+2a)(c+2b)}{(a+2b)(b+2c)(c+2a)} = 1$, which holds if and only if $a^2b + b^2c + c^2a - ab^2 + bc^2 + ca^2 = (a-b)(a-c)(b-c) = 0$, which implies the desired result.

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O24. Find all integers a, b, c such that

$$2^n a + b | c^n + 1$$

for every positive integer n.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Solution by Iurie Boreico, Moldova

Solution. We need the following lemma:

Lemma: Let $u_1, u_2, \dots u_n$ be different non-zero reals. If $z_m = \sum_{i=1}^{n} c_i u_i^m = 0$ for all sufficiently big m then $c_i = 0$.

Proof: Consider the generating function $Z(X) = \sum_{i=0}^{\infty} z_i x^i$. If $z_m = 0$ for all sufficient big n then Z is a polynomial. However $Z(X) = \sum_{i=0}^{n} \frac{c_i}{X - u_i}$ therefore $Z(X) \prod (X - u_i) = \sum c_i \prod_{j \neq i} (X - u_j)$. And if $c_k \neq 0$ then the left-hand side, as a polynomial, is divisible by $X - u_k$ while all terms in the RHS except $c_k \prod_{i \neq k} (X - u_i)$ not. Contradiction.

Firstly we prove that c is a power of 2. Let $x_n = \dfrac{c^n + 1}{a 2^n + b}$.

Assume that $2^{m-1} < c < 2^m$. Set

$$y_n = (c^n + 1) \frac{a}{b} (1 - \frac{1}{2^n \frac{a}{b}} + \frac{1}{2^{2n} \frac{a^2}{b^2}} \pm \dots \pm \frac{1}{2^{mn} \frac{a^m}{b^m}}).$$

By using the identity $\frac{1}{1+x} - (\frac{1}{x} - \frac{1}{x^2} \pm \dots \pm \frac{1}{x^{m-1}}) = \frac{(-1)^m}{x^m (1+x)}$ we deduce that $|x_n - y_n| = \dfrac{c^n + 1}{2^{mn} \frac{a^m}{b^m}} (2^m a + b) \to 0$ when $n \to \infty$.

Now y_n satisfy a polynomial recurrence with rational coefficient, so there are rational constants $c_1, c_2, \dots, c_r$ with $\sum c_i y_{n+i} = 0$. We may assume that c_i are integers or elese multiply by a suitable integer. Therefore $\sum c_i x_{n+i} = \sum c_i (x_{n+i} - y_{n+i}) \to 0$. As this number is certainly an integer, we conclude that it's eventually zero, so x_i starting from some point satify the same recurrence. We deduce then that $x_n = \sum_{i=0}^{r} k_i \frac{1}{2^{ni}} + \sum_{i=0}^{r} l_i \frac{c^n}{2^{ni}}$ for all sufficiently big n. Hence $c^n + 1 = x_n (2^n a + b)$, so $c^n + 1 - x_n (2^n a + b) = -a l_0 (2c)^n + c^n (1 - b l_0 - a l_1) - \sum_{i=1}^{r} (b l_i + a l_{i+1}) (\frac{1}{c}^{2^i})^n + (1 - a k_0) - \sum_{i=0}^{r} (b k_i + a k_{i+1}) (\frac{1}{2^i})^n = 0$. As $2c, c, \frac{c}{2}, \dots, 1, \frac{1}{2}, \dots$ are distinct, applying the lemma we deduce that all the coefficients computed above are zero. Thus $b l_r = b l_{r-1} + a l_r = \dots = b l_1 + a l_2 = b l_0 + a l_1 - 10$ thus we deduce that $l_1 = l_2 = \dots = l_r = 0$. However $a l_0 = 0$ so $l_0 = 1$. But then $1 - b l_0 - a l_1 \neq 0$ contradiction.

This shows that c is a power of 2. Let $c = 2^m$. Then

$$2^n a + b | a^m(2^{mn} + 1) = (2^n a)^m - (-b)^m + (a^m + (-b)^m).$$

Since $2^n a + b | (2^n a)^m - (-b)^m$, we conclude that $2^n a + b | (a^m - (-b)^m)$. This is possible only when $a = 0$ or $a^m - (-b)^m = 0$. As $a > 0$ we have $a^m + (-b)^m = 0$ so m is odd and $a = b$. Then $a | 2^{mn} + 1$, so a is odd. If $n = \phi(a)$ $a | 2^{mn} - 1$ so $a | 2$ hence $a = 1$. Therefore $a = b = 1, c = 2^m$ for some odd integer m, and it's obvious that they satisfy the requirements.

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