Junior problems

J187. Let $m \geq 1$ and $f : [m, \infty) \to [1, \infty),\ f(x) = x^2 - 2mx + m^2 + 1$.

(a) Prove that f is bijective;

(b) Solve the equation $f(x) = f^{-1}(x)$;

(c) Solve the equation $x^2 - 2mx + m^2 + 1 = m + \sqrt{x-1}$.

Proposed by Bogdan Enescu, B.P. Hasdeu National College, Romania

Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

(a) The parabola $f(x) = (x-m)^2 + 1$ is symmetric with respect to the axis $x = m$ thus it is bijective on its domain $[m, \infty)$. A more detailed proof about injectivity is the following

$$x' > x' \implies f(x') > f(x) \quad \text{yields} \quad (x'-m)^2 + 1 > (x-m)^2 + 1$$
$$\iff \ |x'-m| > |x-m| \ \iff \ x' - m > x - m \ \iff x' > x$$

From the calculation above we deduce that the function is increasing as well thus its image is a subset of the interval $[f(m), \infty) = [1, \infty)$. As for the surjectivity we must solve for any $y \geq 1$ the equation

$$(x-m)^2 + 1 = y \qquad \iff \quad |x-m| = \sqrt{y-1} \quad \iff \quad x = \sqrt{y-1} + m$$

(b) The equation $f(x) = f^{-1}(x)$ may occur only on the bisector of the first and third quadrant $y = x$ thus we must solve

$$(x-m)^2 + 1 = x \quad \iff \quad x^2 - x(1+2m) + m^2 + 1 = 0 \implies x = \frac{1 + 2m \pm \sqrt{4m-3}}{2}$$

The root $x = \frac{1+2m+\sqrt{4m-3}}{2}$ satisfies $x > 1$ for any $m \geq 1$

(c) $x^2 - 2mx + m^2 + 1 = m + \sqrt{x-1}$ is the equation $f(x) = f^{-1}(x)$ whose solution is precisely that found in (b). Indeed the inversion of $y = (x-m)^2 + 1$ yields $x = m + \sqrt{y-1}$

Also solved by Arkady Alt, San Jose, USA; Bedri Hajrizi, Albania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Tigran Hakobyan, Armenia.

J188. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{10a+11b+11c}+\frac{1}{11a+10b+11c}+\frac{1}{11a+11b+10c}\leq\frac{1}{32a}+\frac{1}{32b}+\frac{1}{32c}.$$

Proposed by Tigran Hakobyan, Armenia

Solution by Andrea Fanchini, Cantú, Italy

Recall the well-known inequality,

$$\sum_{i=1}^{n}\frac{1}{a_i}\geq\frac{n^2}{\sum_{i=1}^{n}a_i}$$

we have

$$\underbrace{\frac{1}{a}+\cdots+\frac{1}{a}}_{10}+\underbrace{\frac{1}{b}+\cdots+\frac{1}{b}}_{11}+\underbrace{\frac{1}{c}+\cdots+\frac{1}{c}}_{11}\geq\frac{32^2}{10a+11b+11c}$$

which is equivalent to

$$\frac{10}{32a}+\frac{11}{32b}+\frac{11}{32c}\geq\frac{32}{10a+11b+11c}.$$

Similarly

$$\frac{11}{32a}+\frac{10}{32b}+\frac{11}{32c}\geq\frac{32}{11a+10b+11c}$$

and

$$\frac{11}{32a}+\frac{11}{32b}+\frac{10}{32c}\geq\frac{32}{11a+11b+10c}.$$

Adding the three inequalities we get the desired result.

Also solved by Arkady Alt, San Jose, USA; Anthony Erb Lugo, San Juan, Puerto Rico; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Henry Ricardo, NY, USA; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Prasanna Ramakrishnan; Prithwijit De, HBCSE, India; Zolbayar Shagdar, Ulaanbaatar, Mongolia.

J189. Find all primes q_1, q_2, q_3, q_4, q_5 such that $q_1^4 + q_2^4 + q_3^4 + q_4^4 + q_5^4$ is the product of two consecutive even integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note first that an even number of q_1, q_2, q_3, q_4, q_5 must be odd, otherwise $q_1^4 + q_2^4 + q_3^4 + q_4^4 + q_5^4$ would be odd, and hence not the product of two even numbers. Furthermore,

$$(4a \pm 1)^4 = 4^4 a^4 \pm 4^4 a^3 + 6 \cdot 4^2 a^2 \pm 4^2 a + 1 \equiv 1 \pmod{16},$$

or for any odd prime q_i, we have $q_i^4 \equiv 1 \pmod{16}$, while for even q_i (clearly $q_i = 2$ in this case since no other even prime exists), we have $q_i^4 \equiv 0 \pmod{16}$. Thus, if $0, 2, 4$ of the q_i are odd, then respectively $q_1^4 + q_2^4 + q_3^4 + q_4^4 + q_5^4 \equiv 0, 2, 4 \pmod{16}$. Now, the product of two consecutive even integers is a multiple of 8, since exactly one of the two even integers is a multiple of 4 (though it may be a multiple of a higher power of 2), while the other one is a multiple of 2 but not of 4. It follows that $q_1^4 + q_2^4 + q_3^4 + q_4^4 + q_5$ must leave remainders 0 or 8 modulus 16, or we conclude that all q_i are even, and $q_1 = q_2 = q_3 = q_4 = q_5 = 2$, yielding $q_1^4 + q_2^4 + q_3^4 + q_4^4 + q_5^4 = 5 \cdot 2^4 = 80 = 8 \cdot 10$, where clearly $8, 10$ are consecutive even numbers. There is thus a unique solution, $q_1 = q_2 = q_3 = q_4 = q_5 = 2$.

Also solved by Bedri Hajrizi, Albania; Henry Ricardo, NY, USA; Prasanna Ramakrishnan; Tigran Hakobyan, Armenia.

J190. Points A', B', C' are chosen on sides BC, CA, AB of triangle ABC such that lines AA', BB', CC' are concurrent at M and

$$\frac{AM}{MA'} \cdot \frac{BM}{MB'} \cdot \frac{CM}{MC'} = 2011.$$

Evaluate

$$\frac{AM}{MA'} + \frac{BM}{MB'} + \frac{CM}{MC'}$$

Proposed by Bogdan Enescu, B.P. Hasdeu National College, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote $x = \frac{BA'}{BC}$, $y = \frac{CB'}{CA}$, $z = \frac{AC'}{AB}$, hence $CA' = (1-x)a$, $AB' = (1-y)b$, $BC' = (1-z)c$. By Ceva's Theorem, $xyz = (1-x)(1-y)(1-z)$, while applying Menelaus' theorem to triangle $AA'C$ cut by line BB', we obtain $\frac{AM}{MA'} = \frac{1-y}{xy}$, and analogously we find $\frac{BM}{MB'} = \frac{1-z}{yz}$, $\frac{CM}{MC'} = \frac{1-x}{zx}$. Thus, $xyz = (1-x)(1-y)(1-z) = \frac{1}{2011}$. We conclude that

$$\frac{AM}{MA'} + \frac{BM}{MB'} + \frac{CM}{MC'} = \frac{x+y+z-(xy+yz+zx)}{xyz} =$$

$$= \frac{1 - xyz - (1-x)(1-y)(1-z)}{xyz} = 2011 - 1 - 1 = 2009.$$

J191. Find all positive integers n for which $(n-2)! + (n+2)!$ is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Anthony Erb Lugo, San Juan, Puerto Rico

Note the identity

$$(n-2)! + (n+2)! = (n-2)! \cdot (n^2+n-1)^2$$

Thus, if $(n-2)! + (n+2)!$ is a perfect square, then $(n-2)!$ must also be a perfect square. For $n = 2, 3$ we have that $(n-2)! = 1$, so $n = 2$ and $n = 3$ are solutions. In the case of $n \geq 4$, we note that the largest prime number smaller than or equal to $n-2$, which we will denote by p, can only appear once in the prime factorization of $(n-2)!$, implying that $(n-2)!$ is never a perfect square for $n \geq 4$. If it were to appear at least twice in the factorization, then it is necessary to have $n-2 \geq 2p$. But, by Bertrand's postulate, we know that there is a prime number in between p and $2p$, contradicting p's maximality. Hence, $n = 2$ and $n = 3$ are the only solutions.

Also solved by Arkady Alt, San Jose, USA; Bedri Hajrizi, Albania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Gabriel Alexander Chicas Reyes, El Salvador; Jose Hernandez Santiago, Oaxaca, Mexico; Prithwijit De, HBCSE, India; Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy; Tigran Hakobyan, Armenia.

J192. Consider an acute triangle ABC. Let $X \in AB$ and $Y \in AC$ such that quadrilateral $BXYC$ is cyclic and let R_1, R_2, R_3 be the circumradii of triangles AXY, BXY, ABC, respectively. Prove that if $R_1^2 + R_2^2 = R_3^2$, then BC is the diameter of the circle $(BXYC)$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Gabriel Alexander Chicas Reyes, El Salvador

Applying the law of sines to the triangles BXY, AXY and ABC yields

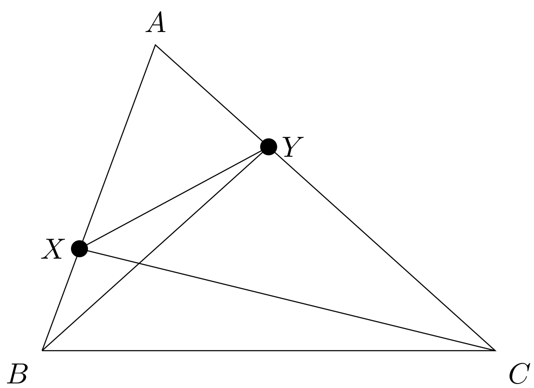
$$\frac{BY}{\sin\angle BXY} = 2R_2, \ \frac{AY}{\sin\angle AXY} = 2R_1, \frac{AB}{\sin\angle ACB} = 2R_3.$$

From the first two equalities it is clear that $\dfrac{AY}{BY} = \dfrac{R_1}{R_2}$; moreover $\angle AXY = \angle ACB$ because the quadrilateral $BXYC$ is cyclic, so from the second and third equalities we have $\dfrac{AY}{AB} = \dfrac{R_1}{R_3}$. Using these relations we compute

$$AY^2 + BY^2 = AY^2 + \frac{R_2^2}{R_1^2}AY^2 = \frac{R_1^2 + R_2^2}{R_1^2}AY^2 = \frac{R_3^2}{R_1^2}AY^2 = AB^2.$$

It follows that ABY is a right triangle, with $\angle AYB = \angle BYC = \pi/2$. Since $BXYC$ is a cyclic quadriteral, this readily implies that BC is a diameter of the circle $(BXYC)$, as claimed.

Second solution by Prithwijit De, Mumbai, India;



Let $\angle BYC = \alpha$. Then in triangle ABY, $\angle ABY = \angle BYC - \angle BAC = \alpha - A$. As $BXYC$ is a cyclic quadrilateral $\angle AXY = \angle YCB = C$ and $\angle AYX = \angle XBC = B$. Therefore the triangles AXY and ABC are similar. Hence, there exists some positive real number λ such that

$$\frac{AY}{AB} = \lambda. \ \ldots(1)$$

Also by sine rule $AY = 2R_1 \sin C$ and $AB = 2R_3 \sin C$, which gives upon division

$$\frac{AY}{AB} = \frac{R_1}{R_3}. \ \ldots(2)$$

From (1) and (2), $\lambda = \dfrac{R_1}{R_3}$.

Since $BXYC$ is cyclic the circumcircle of triangle BXY is same as the circumcircle of triangle BYC. Therefore by sine rule,

$$\frac{BC}{\sin\alpha} = 2R_2 \Rightarrow \frac{2R_3 \sin A}{\sin\alpha} = 2R_2 \Rightarrow \frac{R_2}{R_3} = \frac{\sin A}{\sin\alpha}. \ \ldots (3)$$

Again by sine rule in triangle ABY we get

$$\frac{AY}{\sin\angle ABY} = \frac{AB}{\sin\angle AYB} \Rightarrow \frac{AY}{AB} = \frac{\sin(\alpha - A)}{\sin(\pi-\alpha)} \Rightarrow \lambda = \frac{R_1}{R_3} = \frac{\sin(\alpha - A)}{\sin\alpha} \ldots (4)$$

We know that $R_1^2 + R_2^2 = R_3^2$. Dividing both sides of this equation by R_3^2 and using (3) and (4) we obtain

$$\left(\frac{\sin(\alpha - A)}{\sin\alpha}\right)^2 + \left(\frac{\sin A}{\sin\alpha}\right)^2 = 1 \ldots (5)$$

Simplifying (5) gives us

$$2\cos\alpha \sin A = 0. \ \ldots (6)$$

Thus $\cos\alpha = 0 \Rightarrow \alpha = 90^\circ$ and BC is the diameter of the circle $(BXYC)$.

Also solved by Anthony Erb Lugo, San Juan, Puerto Rico; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Saturnino Campo Ruiz, Fray Luis de Len de Salamanca, Spain.

Senior problems

S187. Find all positive integers n for which the interval

$$\left(\frac{1+\sqrt{5+4\sqrt{24n-23}}}{2},\frac{1+\sqrt{5+4\sqrt{24n+25}}}{2}\right)$$

contains at least one integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy

Let x be a real number such that, for some a,

$$x \leqslant \frac{1+\sqrt{5+4\sqrt{24n+a}}}{2}$$

then

$$n \leqslant \frac{\left(\frac{(2x-1)^2-5}{4}\right)^2-a}{24} = \frac{(x(x-1)-1)^2-a}{24} = \frac{(x+1)x(x-1)(x-2)-a+1}{24}.$$

Therefore, by taking $a=-23$ and $a=25$, we find that

$$x \in \left(\frac{1+\sqrt{5+4\sqrt{24n-23}}}{2},\frac{1+\sqrt{5+4\sqrt{24n+25}}}{2}\right)$$

if and only if

$$n \in \left(\frac{(x+1)x(x-1)(x-2)}{24}-1,\frac{(x+1)x(x-1)(x-2)}{24}+1\right).$$

If $x=m\geq 4$ is an integer then the above open interval contains only one integer:

$$n=\frac{(m+1)m(m-1)(m-2)}{24}=\binom{m+1}{4}=1,5,15,35,70,126,210,330,\ldots.$$

Also solved by Omran Kouba, Institute for Applied Sciences and Technology, Syria; Arkady Alt, San Jose, USA; Bedri Hajrizi, Albania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Prithwijit De, HBCSE, India; Tigran Hakobyan, Armenia.

S188. Let $a \geq b \geq c$ be the side-lengths of a triangle ABC in which $b + c \geq \frac{7}{4}a$. Denote by O and I the circumcenter and the incenter of this triangle, respectively. Prove that circle centered at O and having radius OI lies entirely inside triangle ABC.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology and Roberto Bosch Cabrera, Florida, USA

Solution by by the authors

Remark: The problem has been slightly changed because the previous statement was incorrect.

Note that $b^2+c^2 \geq \frac{(b+c)^2}{2} \geq \frac{49}{32}a^2 > a^2$. Hence our triangle is acute-angled. The shortest distance from O to triangle's sides is equal to $R\cos A$. The condition that a circle centered at O and radius OI lies entirely inside triangle ABC holds if and only if $R\cos A \geq OI$, which is equivalent to $R^2 - R^2\sin^2 A \geq R^2 - 2Rr$ or $4bc \geq a(a+b+c)$. So we need to prove that:

$$a \geq b \geq c, \quad b + c \geq \frac{7}{4}a \Rightarrow 4bc \geq a(a+b+c)$$

Doing $x = \frac{b}{a}, y = \frac{c}{a}$ we can rewrite the above as:

$$0 < y \leq x \leq 1, \quad x + y \geq \frac{7}{4} \Rightarrow 4xy \geq x + y + 1$$

But

$$4xy + \frac{4}{3} \geq \frac{4}{3}x^2 + 4xy \geq \frac{4}{3}(x+y)^2 \geq (x+y) + \frac{7}{3}$$

the first inequality follows from $x \leq 1$, the second one from $x \geq y$ and the third from $x + y \geq \frac{7}{4}$. So by transitivity we obtain $4xy \geq x + y + 1$ as desired.

Remark: We can correct the statement of problem in the other direction, that is to say: Let $a \geq b \geq c$ be the side-lengths of a triangle ABC in which the circle $C(O, OI)$ lies entirely inside. Show that $b + c \geq 2a\cos(\frac{\pi}{5})$.

Solution: We arrive again to condition $4bc \geq a(a+b+c)$ which imply that $a^2 + a(b+c) - (b+c)^2 \leq 0$. It follows that $a \leq \frac{\sqrt{5}-1}{2}(b+c)$, so $\frac{b+c}{2a} \geq \frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4} = \cos\frac{\pi}{5}$.

Also solved by Omran Kouba, Institute for Applied Sciences and Technology, Syria.

S189. Let a, b, c be real numbers such that $a < 3$ and all zeros of the polynomial $x^3 + ax^2 + bx + c$ are negative real numbers. Prove that $b + c \neq 4$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by G.R.A.20 Math Problems Group, Roma, Italy

Since all zeros of $P(x)$ are negative real numbers, it follows that

$$P(x) = x^3 + ax^2 + bx + c = (x + \alpha)(x + \beta)(x + \gamma)$$

for some positive real numbers α, β, γ, and

$$a = \alpha + \beta + \gamma, \quad b = \alpha\beta + \beta\gamma + \alpha\gamma, \quad c = \alpha\beta\gamma.$$

By AM-GM inequality

$$\frac{a}{3} \geq \left(\frac{b}{3}\right)^{1/2} \geq c^{1/3},$$

hence, $a < 3$ implies that $b < 3$ and $c < 1$, that is $b + c < 4$.

Second solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy;

Let p, q, r the absolute values of the zeroes of the polynomial. Viete's formula yields

$$a = p + q + r, \quad b = pq + qr + pr, \quad c = pqr$$

$a < 3$ implies by the AGM $3 > p + q + r \geq 3(pqr)^{1/3}$ and then $pqr < 1$. Moreover

$$pq + qr + pr \leq (p + q + r)^2/3 < 3$$

If $b + c = 4$ we would have

$$4 = pq + qr + pr + pqr < 3 + pqr < 4$$

contradiction.

Third solution by Omran Kouba, Institute for Applied Sciences and Technology, Syria

Since all zeros of the polynomial $p(x) = x^3 + ax^2 + bx + c$ are negative real numbers, then by Roll's theorem, (or simply by multiplicity for multiple roots,) we see that $p'(x) = 3x^2 + 2ax + b$, has also negative real zeros. In particular, its discriminant must be positive, hence $3b \leq a^2 < 9$, or $b < 3$.

On the other hand, if $-\lambda$, $-\mu$ and $-\nu$ are the negative real zeros of p, then $c = \lambda\mu\nu$ and $a = \lambda + \mu + \nu$. Thus, by the AM-GM inequality we see that $c \leq \left(\frac{a}{3}\right)^3 < 1$. From $c < 1$ and $b < 3$ we conclude that $b + c < 4$.

Also solved by Arkady Alt, San Jose, USA; Anthony Erb Lugo, San Juan, Puerto Rico; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Prasanna Ramakrishnan; Tigran Hakobyan, Armenia.

S190. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\leq\frac{1}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2.$$

Proposed by Arkady Alt, San Jose, California, USA

Solution by Anthony Erb Lugo, San Juan, Puerto Rico

By the AM-GM Inequality, we have

$$\frac{1}{2a^2+bc}=\frac{1}{a^2+a^2+bc}\leq\frac{1}{3a\sqrt[3]{abc}}$$

This implies that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\leq\frac{1}{3\sqrt[3]{abc}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Thus, it is sufficient to prove that

$$\frac{1}{3\sqrt[3]{abc}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\leq\frac{1}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$$

But this last inequality is equivalent to

$$\frac{3}{\sqrt[3]{abc}}\leq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

which follows directly from the AM-GM Inequality, so we are done.

Also solved by Andrea Fanchini, Cantú, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Henry Ricardo, NY, USA; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Prithwijit De, Mumbai, India.

S191. Prove that for any positive integer k the sequence $(\tau(k+n^2))_{n\geq 1}$ is unbounded, where $\tau(m)$ denotes the number of divisors of m.

Proposed by Al-Yazeed Ibrahim Basyoni, Saudi Arabia

Solution by Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy

Claim 1. Given a non constant polynomial $P(x) \in \mathbb{Z}[x]$, then there are infinitely many primes that divides some term of the sequence $P(n)$, with $n \in \mathbb{N}$. If $P(0) = 0$ then the thesis is trivial since this implies that n divides $P(n)$ for any positive integer n. Now assume that $P(0) \neq 0$ and let m be the maximum exponent in the prime factorization of $P(0)$. With this notation, we prove that infinitely many primes are divisors of some terms in the sequence. Assume by contradiction that they are finite, say $q_1, \ldots, q_s$, and let us consider the unbounded sequence

$$\left\{P\left(\prod_{i=1}^{s} q_i^{m+n}\right)\right\}_{n\in\mathbb{N}}.$$

Each term of this sequence has, by our assumptions, the prime q_i in the factorization with an exponent at most m. But there are just a finite number of such values, which is a contradiction because the sequence unbounded. So we have infinitely many primes that divides some terms in the original sequence.

Claim 2. Now for every $g \in \mathbb{N}$ from Claim 1) we have $p_1, \ldots, p_g$ distinct primes so that there are $x_1, \ldots, x_g \in \mathbb{N}$ so that p_i divides $P(x_i)$ for all $i \in \{1, \ldots, g\}$. Then, by the Chinese Remainder Theorem, there is $x \in \mathbb{N}$ such that $x \equiv x_i \pmod{p_i}$ forall $i \in \{1, \ldots, g\}$. Finally we have that p_i divides $P(x)$ forall $i \in \{1, \ldots, g\}$ which implies that $\tau(P(x)) \geq 2^g$. Since g is arbitrary large, we are done.

Also solved by Omran Kouba, Institute for Applied Sciences and Technology, Syria; Daniel Lasaosa, Universidad Pública de Navarra, Spain; G.R.A.20 Problem Solving Group, Roma, Italy; Tigran Hakobyan, Armenia.

S192. Let s, R, r and r_a, r_b, r_c be the semiperimeter, circumradius, inradius, and exradii of a triangle ABC. Prove that

$$s\sqrt{\frac{2}{R}} \leq \sqrt{r_a} + \sqrt{r_b} + \sqrt{r_c} \leq \frac{s}{\sqrt{r}}.$$

Proposed by Arkady Alt, San Jose, California, USA

First solution by Anthony Erb Lugo, San Juan, Puerto Rico

Let K denote the area of triangle ABC (with sides a, b and c). We have that

$$K = r_a(s-a) = rs = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R}$$

We start by proving the right hand side of the inequality, note that

$$K^2 = (r_a(s-a))(rs) = s(s-a)(s-b)(s-c)$$

or

$$r_a r = (s-b)(s-c) \implies \sqrt{r_a r} = \sqrt{(s-b)(s-c)}$$

Next, by the AM-GM inequality, we have

$$\sqrt{r_a r} = \sqrt{(s-b)(s-c)} \leq \frac{(s-b)+(s-c)}{2} = \frac{a}{2}$$

Applying this cyclically

$$\sqrt{r_a r} + \sqrt{r_b r} + \sqrt{r_c r} \leq \frac{a+b+c}{2} = s$$

Next, we divide by $\sqrt{r}$ on both sides, this ends the proof of the right hand side. Now we need to prove the left hand side

$$s\sqrt{\frac{2}{R}} \leq \sqrt{r_a} + \sqrt{r_b} + \sqrt{r_c}$$

We multiply both sides by $\sqrt{2R}$ so that the inequality is equivalent with

$$a+b+c \leq \sqrt{2r_aR} + \sqrt{2r_bR} + \sqrt{2r_aR}$$

Next, we recall the equality

$$r_a(s-a) = \frac{abc}{4R}$$

which is equivalent to

$$2r_aR = \frac{abc}{2(s-a)} = \frac{abc}{b+c-a}$$

Thus, applying the last equality cyclically, it is sufficient to prove that

$$a+b+c \leq \sqrt{\frac{abc}{b+c-a}} + \sqrt{\frac{abc}{a+c-b}} + \sqrt{\frac{abc}{a+b-c}}$$

which is the same as problem O181 from issue 1 of 2011.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Consider the triangles whose sides are the internal bisector of A, line AB, and the respective perpendiculars to AB through the incenter and the excenter opposite vertex A. It is well known that he feet of these perpendiculars are respectively at distances $\frac{b+c-a}{2}, \frac{b+c+a}{2}$ from A, or since both triangles are right-angled, and the angle opposite sides with lengths r, r_a is $\frac{A}{2}$, using the Cosine Law we have

$$\tan^2\frac{A}{2} = \frac{4rr_a}{(b+c-a)(b+c+a)} = \frac{2rr_a}{bc(1+\cos A)} = \frac{rr_a}{bc\cos^2\frac{A}{2}},$$

$$4rr_a = 4bc\sin^2\frac{A}{2} = 2bc(1-\cos A) = 2bc + a^2 - b^2 - c^2,$$

or

$$4r(r_a+r_b+r_c) = 2ab+2bc+2ca-(a^2+b^2+c^2) \le \frac{(a+b+c)^2}{3} = \frac{4s^2}{3},$$

the last inequality being equivalent to $a^2+b^2+c^2 \ge ab+bc+ca$, clearly true because of the scalar product inequality, with equality iff $a=b=c$. We have thus proved that

$$r_a+r_b+r_c \le \frac{s^2}{3r},$$

which is stronger than the proposed inequality, since the inequality between arithmetic and quadratic means allows us to write

$$\sqrt{r_a}+\sqrt{r_b}+\sqrt{r_c} \le 3\sqrt{\frac{r_a+r_b+r_c}{3}} \le 3\sqrt{\frac{s^2}{9r}} = \frac{s}{\sqrt{r}}.$$

Equality holds iff $a=b=c$ because this condition is also equivalent to $r_a=r_b=r_c$.

Note also that $\frac{r_a}{s} = \tan\frac{A}{2}$, and

$$\frac{s}{R} = \sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

while

$$4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 2\sin^2\frac{A}{2} + 2\cos\frac{B+C}{2}\cos\frac{B-C}{2} = 1-\cos A+\cos B+\cos C,$$

hence the left inequality is equivalent to

$$4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le \sum_{\text{cyc}}\sqrt{2\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}},$$

and after squaring both sides, to

$$2(1+\cos A)(1+\cos B)(1+\cos C) \le \frac{3+\cos A+\cos B+\cos C}{2} + 2\sum_{\text{cyc}}\sqrt{\cos^2\frac{A}{2}\sin B\sin C}.$$

Now,

$$4\cos^2\frac{A}{2}\sin B\sin C = (1+\cos A)(\cos A+\cos(B-C)) \ge (\cos A+\cos(B-C))^2,$$

and similarly for its cyclic permutations, or after some algebra, it suffices to show that

$$1+4\cos A\cos B\cos C \le \cos A+\cos B+\cos C.$$

Now, it is relatively well known that $1+\frac{r}{R} = \cos A+\cos B+\cos C$, and $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, or the inequality is trivially true and strict when ABC is obtuse, and when ABC is acute, it suffices to show that

$$\cos A\cos B\cos C \le \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

or equivalently,

$$\tan A \tan B \tan C \geq 8 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

Now, from the well known relation $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ and the AM-GM inequality, we obtain $\tan A \tan B \tan C \geq 3\sqrt{3}$, with equality iff ABC is equilateral, while $\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \leq \frac{3\sqrt{3}}{8}$ is another well known relation, equivalent to the fact that, from all triangles with the same circumradius, the one with largest perimeter is equilateral. The conclusion follows, equality holds in the left equality iff ABC is equilateral.

Undergraduate problems

U187. Let p be a a prime such that $p \equiv 3 (\text{mod } 8)$ or $p \equiv 5 (\text{mod } 8)$, and $p = 2q + 1$ where q is a prime. Evaluate $\omega^2 + \omega^4 + \cdots + \omega^{2^{p-1}}$, with $\omega \neq 1$ a root of order p of unity.

Proposed by Dorin Andrica and Mihai Piticari, Romania

Solution by Vahagn Aslanyan, Yerevan State University, Armenia

$p \equiv \pm 3 (mod\ 8)$ therefore 2 is quadratic non-residue modulo p, i.e. $(\frac{2}{p}) = -1$. Hence by Euler's formula $2^{\frac{p-1}{2}} \equiv -1 (mod\ p)$. So $2^q \equiv -1 (mod\ p)$. Denote the order of 2 modulo p by γ. We have $\gamma | p - 1 = 2q$ with q a prime, therefore $\gamma = 1$ or $\gamma = 2$ or $\gamma = q$ or $\gamma = 2q$. The first case is impossible, in second case $p | 2^\gamma - 1 = 3 \Rightarrow p = 3$ but in this case $q = \frac{p-1}{2} = 1$ is not prime (contradiction). The third case is impossible because $2^q \equiv -1 (mod\ p)$. Thus $\gamma = 2q = p - 1$,i.e. 2 is a primitive root modulo p. Now we may conclude that $2, 2^2, ..., 2^{p-1}$ are pairwise incongruent modulo p, so they are congruent to numbers $1, 2, ..., p - 1$ modulo p in some order. So $\omega^2 + \omega^4 + ... + \omega^{2^{p-1}} = \omega + \omega^2 + ... + \omega^{p-1} = \frac{\omega(\omega^{p-1} - 1)}{\omega - 1} = \frac{\omega^p - \omega}{\omega - 1} = \frac{1 - \omega}{\omega - 1} = -1$, because $\omega \neq 1$ and ω is a root of order p of unity.

Also solved by Omran Kouba, Institute for Applied Sciences and Technology, Syria; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy.

U188. Let G be a finite group in which for every positive integer m the number of solutions in G of the equation $x^m = e$ is at most m. Prove that G is cyclic.

Proposed by Roberto Bosch Cabrera, Florida, USA

First solution by Jose Hernandez Santiago, Oaxaca, Mexico

Let n be the order of G. For every $d \in \mathbb{N}$, let us denote with A_d the subset of G that consists of all elements that have order d. Now, if $f(d) := |A_d|$, we affirm that $f(d) \leq \phi(d)$ for every $d \in \mathbb{N}$, where ϕ is the totient function of Euler. Indeed, if $f(d) = 0$, the purported inequality follows from the possitivity of the totient function. Otherwise, fix a $g \in G$ with order d. It follows that all elements of G of order d belong to the set $\{1, g, g^2, \ldots, g^{d-1}\}$. Since g^i has order d iff $(d, i) = 1$, it is straightforward to conclude that $f(d) = \phi(d)$ in this case ($f(d) \neq 0$, that is).

Besides, the identities $n = |G| = |\bigsqcup_{d|n} A_d| = \sum_{d|n} |A_d| = \sum_{d|n} f(d)$ and $n = \sum_{d|n} \phi(d)$ imply at once that none of the inequalities $f(d) \leq \phi(d)$ can be strict when d is a divisor of n. In particular, this implies that

$$|A_n| = f(n) = \phi(n) > 0$$

and we're done.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let $x \in G$, and let n be the cardinal of G. Clearly, $\{x, x^2, x^3, \ldots, x^n\}$ is a set of n elements in G, hence either e is in that set, or two elements in that set are equal, $x^i = x^j$ for $1 \leq i < j \leq n$ and $x^{j-i} = e$. In either case, a positive integral exponent $m \leq n$ exists such $x^m = e$. For a given element $x \in G$, let k be the minimum such exponent, and for $k = 1, 2, \ldots, n$, let G_k be the set of elements such that $x^k = e$ and $x^j \neq e$ for all $j \in \{1, 2, \ldots, k-1\}$. Clearly, all elements in G are in one of $G_1, G_2, \ldots, G_n$, which are disjoint by definition, hence $G_1 \cup G_2 \cup \cdots \cup G_n$ is a partition of G, where some of the G_k may be empty.

We prove first that, if $G_k \neq \emptyset$ for some $k \geq 2$, then k divides n. Indeed, let $x \in G_k$, or $X = \{e, x, x^2, x^3, \ldots, x^{k-1}\}$ is a set of k distinct elements of G, since otherwise positive integral exponents i and $j \leq k-i-1$ exist such that $x^i = x^{i+j}$, and $x^j = e$, in contradiction with the definition of x. Now, for any $y \in G \backslash X$, we define the set $Y = \{y, yx, yx^2, \ldots, yx^{k-1}\}$. The k elements in this set are clearly distinct, since otherwise positive integral exponents i and $j \leq k-i$ exist such that $yx^i = yx^{i+j}$ and $x^j = e$. Note that, for any $z \in G \backslash (X \cup Y)$, if $zx^i = yx^j$, for wlog $i \geq j$, then $zx^{i-j} = y$, and $z \in Y$, contradiction, hence x, we can assign all elements of G either to X or to several disjoint subsets Y, until all elements are assigned to a set of either form, ie, G is partitioned into X and several sets of the form Y, hence since X, and each Y, have cardinal k, we conclude that k divides n.

We prove next that, if $G_k \neq \emptyset$ for some prime $k \geq 2$, then the cardinal of G_k is exactly $\varphi(k)$, where φ denotes Euler's totient function. Indeed, let $x \in G_k$, or as we already know $X = \{e, x, x^2, x^3, \ldots, x^{k-1}\}$ is a set of k distinct elements of G. Clearly all these elements satisfy $(x^j)^k = (x^k)^j = e^j = e$, or no other element in G may satisfy $x^k = e$, and $G_k \subset X$. Let now $r \in \{1, 2, \ldots, k-1\}$ such that r is prime with k. If $x^{jr} = e$, let s be the remainder modulus k of jr, or $x^{jr} = x^s = e$, and $s = 0$, ie k divides jr. But r and k are coprime, hence k divides j, and $x^r \in G_k$, hence G_k has at least $\varphi(k)$ elements (one for each $r \in \{1, 2, \ldots, k-1\}$ which is coprime with k). On the other hand, if $j \in \{1, 2, \ldots, k-1\}$ has a common divider $d > 1$ with k, then $(x^j)^{\frac{k}{d}} = (x^k)^{\frac{j}{d}} = e^{\frac{j}{d}} = e$, and $x^j \notin G_k$. We conclude that there are exactly $\varphi(k)$ elements in X that are also in G_k.

Now, for each integer n, if D is the set of its divisors, it is well known that

$$\sum_{d \in D} \varphi(d) = n,$$

or since the cardinal $|G_k|$ of G_k is 0 if either $k \notin D$ or $k \in D$ but $G_k = \emptyset$, but the cardinal of G_k is $\varphi(k)$ if $k \in D$ and $G_k \neq \emptyset$, then

$$n = |G| = \sum_{k=1}^{n} |G_k| = \sum_{d \in D} |G_d| \leq \sum_{d \in D} \varphi(d) = n,$$

with equality iff $G_k \neq \emptyset$ for all $k \in D$. Since equality must hold, then $G_d \neq \emptyset$ for all $d \in D$, in particular $G_n \neq \emptyset$, and for any $x \in G_n$, $\{x, x^2, \ldots, x^{n-1}, x^n = e\}$ contains n distinct elements of G, hence x generates G, and G is cyclic. The conclusion follows.

Also solved by Henry Ricardo, NY, USA; Sunil Ghosh, University of Cambridge, UK.

U189. Let $a_1, \dots, a_n, b_1, \dots, b_n$ be distinct complex numbers such that $a_k + b_l \neq 0$ for all $k, l = 1, 2, \dots, n$. Solve the system of equations

$$\frac{x_1}{a_k + b_1} + \frac{x_2}{a_k + b_2} + \dots + \frac{x_n}{a_k + b_n} = \frac{1}{a_k}, \ k = 1, 2, \dots, n.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We subtract, from each one of the equations for $k = 1, 2, \dots, n-1$, the equation for $k = n$ multiplied by $\frac{a_n + b_n}{a_k + b_n}$, and multiply the result by $\frac{a_k + b_n}{a_n - a_k}$ (which is clearly a well defined nonzero complex number since $a_n \neq a_k$ and $a_k + b_n \neq 0$), yielding

$$\frac{a_k + b_n}{a_n - a_k} \sum_{i=1}^{n-1} \left(\frac{x_i}{a_k + b_i} - \frac{(a_n + b_n)x_i}{(a_n + b_i)(a_k + b_n)} \right) = \sum_{i=1}^{n-1} \frac{(b_n - b_i)x_i}{(a_n + b_i)(a_k + b_i)} =$$

$$= \frac{a_k + b_n}{a_n - a_k} \left(\frac{1}{a_k} - \frac{a_n + b_n}{a_n(a_k + b_n)} \right) = \frac{b_n}{a_k a_n}.$$

Define now, for $i = 1, 2, \dots, n-1$, $z_i = \frac{a_n(b_n - b_i)}{b_n(a_n + b_i)} x_i$, or each one of the previous $n-1$ equations rewrites as

$$\sum_{i=1}^{n-1} \frac{z_i}{a_k + b_i} = \frac{1}{a_k},$$

ie, we fall back upon the same system, but with one less variable and one less equation. Now, for each solution $(z_1, z_2, \dots, z_{n-1})$ of the reduced system (ie, the system with $n-1$ variables and equations), substitution of the corresponding values of $x_1, x_2, \dots, x_{n-1}$ for each one of these solutions, in any of the n equations in the original system, yields a unique value of x_n, and clearly two distinct solutions of the reduced system must have at least one z_i with different value, hence a corresponding x_i with distinct value in the solution for the original system, or each solution for the reduced system generates exactly one distinct solution for the original system. Reciprocally, any solution from the original system yields, after elimination of the variable x_n, a solution for the reduced system, which by the previous argument, is unique. It follows that the system with n equations has exactly as many solutions as the system with $n-1$ equations, and the values of x_i $(i = 1, 2, \dots, n-1)$ that solve the original system can be found by multiplying the corresponding values of z_i that solve the reduced system, each one of them multiplied by $\frac{b_n(a_n + b_i)}{a_n(b_n - b_i)}$.

By trivial backwards induction, we find that the system with n equations has exactly as many solutions as the system with one equation, $\frac{x_1}{a_1 + b_1} = \frac{1}{a_1}$, which clearly has a unique solution $x_1 = \frac{a_1 + b_1}{a_1}$. Clearly, for the system of two equations, the value of x_1 will be

$$\frac{b_2(a_2 + b_1)}{a_2(b_2 - b_1)} \cdot \frac{a_1 + b_1}{a_1} = \frac{b_2(a_1 + b_1)(a_2 + b_1)}{a_1 a_2 (b_2 - b_1)},$$

and again by trivial induction, the value of x_1 in the unique solution for the system with n equations will be

$$x_1 = \frac{b_2 b_3 \dots b_n (a_1 + b_1)(a_2 + b_1) \dots (a_n + b_1)}{a_1 a_2 \dots a_n (b_2 - b_1)(b_3 - b_1) \dots (b_n - b_1)}.$$

By cyclical symmetry between variables, we deduce that

$$x_i = \frac{1}{b_i} \frac{a_1 a_2 \dots a_n}{b_1 b_2 \dots b_n} \frac{(a_1 + b_i)(a_2 + b_i) \dots (a_n + b_i)}{(b_{i+1} - b_i)(b_{i+2} - b_i) \dots (b_{i+n-1} - b_i)},$$

where cyclical notation has been used, ie $b_k = b_{n+k}$ for all positive integer k. It follows that these values for each one of the x_i, $i = 1, 2, \dots, n$, form the unique solution for the proposed system.

Second solution by Omran Kouba, Institute for Applied Sciences and Technology, Syria

We will start by fixing some notation. The vector space of complex polynomials of degree smaller than n will be denoted by $\mathbb{C}_n[X]$, the polynomial $(X+b_1)(X+b_2)\cdots(X+b_n)$ will denoted by $B(X)$, and finally, we will write $\Delta_n^{(k)}$ for the set $\{1,2,\ldots,n\}\setminus\{k\}$.

We consider the linear isomorphism $\Phi : \mathbb{C}^n \longrightarrow \mathbb{C}_n[X]$ defined as follows : for a given vector $\mathbf{v} = (x_1,\ldots,x_n) \in \mathbb{C}^n$ we consider the rational function $F_{\mathbf{v}}(X)$ defined by

$$F_{\mathbf{v}}(X) = \frac{x_1}{X+b_1} + \frac{x_2}{X+b_2} + \cdots + \frac{x_n}{X+b_n}$$

and we define $Q_{\mathbf{v}} = \Phi(\mathbf{v})$ by $Q_{\mathbf{v}}(X) = B(X)F_{\mathbf{v}}(X)$. Conversely, given a polynomial $Q \in \mathbb{C}_n[X]$, then we see immediately that $\mathbf{v}_Q = \Phi^{-1}(Q)$ is the vector $\mathbf{v}_Q = (x_1,\ldots,x_n)$ defined by

$$x_j = \operatorname{Res}\left(\frac{Q(X)}{B(X)}, -b_j\right) = \frac{Q(-b_j)}{B'(-b_j)} = \frac{Q(-b_j)}{\prod_{i\in\Delta_n^{(j)}}(b_i-b_j)}, \quad \text{for } 1 \le j \le n.$$

Now $\mathbf{v} = (x_1,\ldots,x_n)$ is a solution to the considered system if and only if $a_k F_{\mathbf{v}}(a_k) = 1$ for $1 \le k \le n$. This is equivalent to the fact that $(a_i)_{1\le i\le n}$ are zeros of the polynomial $XQ_{\mathbf{v}}(X) - B(X)$, whose degree is smaller or equal to n. So $(x_1,\ldots,x_n)$ is a solution to the considered system if and only if there exists a constant λ such that

$$XQ_{\mathbf{v}}(X) - B(X) = \lambda \prod_{k=1}^{n}(a_k - X),$$

Substituting $X=0$ allows us to see that λ is uniquely determined : $\lambda = -\prod_{k=1}^n(b_k/a_k)$. We arrive to the conclusion that $\mathbf{v} = (x_1,\ldots,x_n)$ is a solution to the considered system if and only if

$$\Phi(\mathbf{v}) = Q_{\mathbf{v}}(X) = \frac{1}{X}\left(B(X) - (b_1\ldots b_n)\prod_{k=1}^{n}\left(1 - \frac{X}{a_k}\right)\right),$$

or

$$\mathbf{v} = \Phi^{-1}\left(\frac{1}{X}\left(B(X) - (b_1\ldots b_n)\prod_{k=1}^{n}(1 - X/a_k)\right)\right),$$

That is, for $1 \le j \le n$, we have

$$x_j = \left(\prod_{i=1}^{n}(1 + b_j/a_i)\right)\left(\prod_{i\in\Delta_n^{(j)}}\frac{b_i}{b_i-b_j}\right) = \left(1 + \frac{b_j}{a_j}\right)\prod_{i\in\Delta_n^{(j)}}\frac{b_i(a_i+b_j)}{a_i(b_i-b_j)}.$$

which is the desired solution.

U190. Evaluate

$$\lim_{n\to\infty}\left(n\frac{\sqrt[n+1]{(n+1)!)}}{\sqrt[n]{n!}}-(n-1)\frac{\sqrt[n]{n!)}}{\sqrt[n-1]{(n-1)!}}\right).$$

Proposed by Arkady Alt, San Jose, California, USA

Solution by the author

We will use inequality $\left(\frac{n}{e}\right)^n\sqrt{an}<n!<\left(\frac{n}{e}\right)^n\sqrt{an}\cdot e^{\frac{1}{12n}}$.

Let $L_n:=\frac{n}{e}\sqrt[2n]{an}, R_n:=L_n\cdot e^{\frac{1}{12n^2}}, a_n=n\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}-(n-1)\frac{\sqrt[n]{n!}}{\sqrt[n-1]{(n-1)!}}$.

Since $L_n<\sqrt[n]{n!}<R_n$ then $n\frac{L_{n+1}}{R_n}-(n-1)\frac{R_n}{L_{n-1}}<a_n<n\frac{R_{n+1}}{L_n}-(n-1)\frac{L_n}{R_{n-1}}$.

1. We will prove that $\lim_{n\to\infty}n\left(\frac{L_{n+1}}{R_n}-\frac{L_{n+1}}{L_n}\right)=0$.

Noting that $\lim_{n\to\infty}\frac{L_{n+1}}{L_n}=\lim_{n\to\infty}\frac{(n+1)\sqrt[2n+2]{a(n+1)}}{n\sqrt[2n]{an}}=1$ we obtain

$\lim_{n\to\infty}n\left(\frac{L_{n+1}}{R_n}-\frac{L_{n+1}}{L_n}\right)=\lim_{n\to\infty}\frac{nL_{n+1}}{R_n}\left(1-e^{\frac{1}{12n^2}}\right)=$

$\lim_{n\to\infty}\frac{nL_{n+1}}{12n^2L_n}\cdot\lim_{n\to\infty}12n^2\left(1-e^{\frac{1}{12n^2}}\right)\cdot\lim_{n\to\infty}e^{-\frac{1}{12n^2}}=-\frac{1}{12}\lim_{n\to\infty}\frac{1}{n}\cdot\frac{L_{n+1}}{L_n}=0.$

Similarly we obtain $\lim_{n\to\infty}n\left(\frac{R_{n+1}}{L_n}-\frac{L_{n+1}}{L_n}\right)=0$

2. Using **1.** we obtain $\lim_{n\to\infty}\left(n\frac{L_{n+1}}{R_n}-(n-1)\frac{R_n}{L_{n-1}}\right)=$

$\lim_{n\to\infty}\left(n\frac{L_{n+1}}{L_n}-(n-1)\frac{L_n}{L_{n-1}}+n\left(\frac{L_{n+1}}{R_n}-\frac{L_{n+1}}{L_n}\right)-(n-1)\left(\frac{R_n}{L_{n-1}}-\frac{L_n}{L_{n-1}}\right)\right)=$

$\lim_{n\to\infty}\left(n\frac{L_{n+1}}{L_n}-(n-1)\frac{L_n}{L_{n-1}}\right)$ and $\lim_{n\to\infty}\left(n\frac{R_{n+1}}{L_n}-(n-1)\frac{L_n}{R_{n-1}}\right)=$

$\lim_{n\to\infty}\left(n\frac{L_{n+1}}{L_n}-(n-1)\frac{L_n}{L_{n-1}}\right)$.Let $\alpha_n:=\ln\frac{L_{n+1}}{L_n}=\ln\left(\frac{(n+1)\sqrt[2n+2]{a(n+1)}}{n\sqrt[2n]{an}}\right)=$

$\ln\frac{(n+1)\sqrt[2n+2]{a(n+1)}}{n\sqrt[2n]{an}}=\ln\left(1+\frac{1}{n}\right)-\frac{\ln a}{n(n+1)}+\frac{\ln(n+1)}{2(n+1)}-\frac{\ln n}{2n}$ then

$n\frac{L_{n+1}}{L_n}=ne^{\alpha_n}$, $(n-1)\frac{L_n}{L_{n-1}}=(n-1)e^{\alpha_{n-1}}$ and we obtain

$\lim_{n\to\infty}\left(n\frac{L_{n+1}}{L_n}-(n-1)\frac{L_n}{L_{n-1}}\right)=\lim_{n\to\infty}\left(ne^{\alpha_n}-(n-1)e^{\alpha_n-1}\right)=$

$\lim_{n\to\infty}\left(n\left(e^{\alpha_n}-1\right)-(n-1)\left(e^{\alpha_{n-1}}-1\right)\right)+1.$

Note that $\lim_{n\to\infty}n\alpha_n=\lim_{n\to\infty}n\left(\ln\left(1+\frac{1}{n}\right)-\frac{\ln a}{n(n+1)}+\frac{\ln(n+1)}{2(n+1)}-\frac{\ln n}{2n}\right)=$

$=\lim_{n\to\infty}n\ln\left(1+\frac{1}{n}\right)-\lim_{n\to\infty}\frac{\ln a}{n+1}+\frac{1}{2}\lim_{n\to\infty}\left(\frac{n\ln(n+1)}{(n+1)}-\ln n\right)=$

$$1+\frac{1}{2}\lim_{n\to\infty}\left(\frac{n\ln(n+1)}{(n+1)}-\ln n\right)=1+\frac{1}{2}\lim_{n\to\infty}\left(\frac{\ln\left(1+\frac{1}{n}\right)^{n}-\ln n}{n+1}\right)=1$$

Since $\lim\limits_{n\to\infty}\frac{e^{\alpha_n}-1}{\alpha_n}=1$ then $\lim\limits_{n\to\infty}n\left(e^{\alpha_n}-1\right)=\lim\limits_{n\to\infty}n\alpha_n\lim\limits_{n\to\infty}\frac{e^{\alpha_n}-1}{\alpha_n}=1$ and,
therefore, $\lim\limits_{n\to\infty}\left(n\left(e^{\alpha_n}-1\right)-(n-1)\left(e^{\alpha_{n-1}}-1\right)\right)=1-1=0.$

Thus, $\lim\limits_{n\to\infty}a_n=1.$

Also solved by Omran Kouba, Institute for Applied Sciences and Technology, Syria; Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Moubinool Omarhee, Paris, France; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

U191. For a positive integer n define $a_n = \prod_{k=1}^{n}(1+\frac{1}{2^k})$. Prove that

$$2-\frac{1}{2^n} \le a_n < 3-\frac{1}{2^{n-1}}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

First solution by Angel Plaza, University of Las Palmas de Gran Canaria, Spain

$1+\sum_{k=1}^{n}\frac{1}{2^k} \le \prod_{k=1}^{n}\left(1+\frac{1}{2^k}\right)$, with equallity only for $n=1$. Since $\sum_{k=1}^{n}\frac{1}{2^k} = \frac{1/2-1/2^{n+1}}{1/2} = 1-\frac{1}{2^n}$ the LHS inequality is obtained.

For the RHS inequality, taking logarithms we have

$$\ln a_n = \ln\prod_{k=1}^{n}\left(1+\frac{1}{2^k}\right) \;=\; \sum_{k=1}^{n}\ln\left(1+\frac{1}{2^k}\right) < \sum_{k=1}^{n}\frac{1}{2^k} = 1-\frac{1}{2^n}.$$

Therefore $a_n < e^{1-\frac{1}{2^n}}$. Note that this upper bound for a_n is sharper than the proposed in the problem.

Second solution by Arkady Alt, San Jose, USA

1. Right hand link of double inequality (by Math. Induction).

Let $b_n := 2-\frac{1}{2^n}, n\in\mathbb{N}$.We will prove that $\frac{b_{n+1}}{b_n} \le \frac{a_{n+1}}{a_n}, n\in\mathbb{N}$.

$$\frac{b_{n+1}}{b_n} \le \frac{a_{n+1}}{a_n} \iff \frac{2-\frac{1}{2^{n+1}}}{2-\frac{1}{2^n}} \le 1+\frac{1}{2^{n+1}} \iff \frac{\frac{1}{2^{n+1}}}{2-\frac{1}{2^n}} \le \frac{1}{2^{n+1}} \iff 1 \le 2-\frac{1}{2^n} \iff 1 \le 2^n.$$

Note that $2-\frac{1}{2^1} = a_1$.Since $\frac{b_{n+1}}{b_n} \le \frac{a_{n+1}}{a_n}, n\in\mathbb{N}$ then from supposition $a_n \le b_n$, $n\in\mathbb{N}$

follows $b_{n+1} = b_n\cdot\frac{b_{n+1}}{b_n} \le a_n\cdot\frac{a_{n+1}}{a_n} = a_{n+1}$.

2. Left hand link of double inequality.

Proof 1.

For $n=1,2$ inequality $a_n < 3-\frac{1}{2^{n-1}}$ holds (by direct calculation).

Let $n\ge 3$. Since $1+\frac{1}{2^k} < e^{\frac{1}{2^k}}$ then $\prod_{k=1}^{n}\left(1+\frac{1}{2^k}\right) < e^{\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}} = e^{1-\frac{1}{2^n}} < e$.

We have $e < 2.8 < 3-\frac{1}{2^{n-1}}$ for $n\ge 3$.

Proof 2.

By AM-GM inequality we have

$$\prod_{k=1}^{n}\left(1+\frac{1}{2^k}\right) \le \left(\frac{\sum_{k=1}^{n}\left(1+\frac{1}{2^k}\right)}{n}\right)^n = \left(1+\frac{\sum_{k=1}^{n}\frac{1}{2^k}}{n}\right)^n < \left(1+\frac{1}{n}\right)^n < e < 2.8 < 3-\frac{1}{2^{n-1}}$$

for $n\ge 3$.

Also solved by Omran Kouba, Institute for Applied Sciences and Technology, Syria; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Andrea Ligori Università di Roma "Tor Vergata", Italy; Lorenzo Pascali Università di Roma "La Sapienza", Italy.

U192. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with finite lateral limits at any point in $\mathbb{R}$. Prove that

(a) f is integrable on any interval $[a, b]$;

(b) If $F(x) = \int_0^x f(t)dt$ is differentiable at any point in $\mathbb{R}$, then f has finite limit at any point in $\mathbb{R}$.

Proposed by Sorin Radulescu and Mihai Piticari, Romania

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let D be the set of points such that $z \in D$ iff

$$\lim_{x \to z_-} f(x), \qquad \lim_{x \to z_+} f(x), \qquad f(z),$$

are not all equal, where $\lim_{x \to z_-}$ denotes the lateral limit from the left, and $\lim_{x \to z_+}$ denotes the lateral limit from the right. Assume that D has an accumulation point y. Therefore, and since we may invert the function around y without altering the problem by using a composition $g(x) = f(2y - x)$, wlog there exists a strictly increasing sequence $(z_n)_{n \geq 0}$ such that $z_n \in D$ for all $n \geq 0$, and $\lim_{n \to \infty} z_n = y$. Now, since finite lateral limits exist for y, in particular for every $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $y > x > y - \delta$, then

$$\left| f(x) - \lim_{x \to y_-} f(x) \right| < \frac{\epsilon}{2}.$$

In particular, for any $0 < \delta' < \frac{\delta}{4}$, and any $y - \delta' > x_0 > y + \delta'$, we have that, for all $x_0 + \delta' > x > x_0 - \delta'$, then $|f(x) - f(x_0)| < \epsilon$, or $f(x_0)$ is continuous at x_0, hence $x_0 \notin D$. Now, this means that there is an open interval $(y - \delta', y + \delta')$ around y such that, any $x_0 \neq y$ in this interval, is not in D, in contradiction with y being an accumulation point in D. Contradiction, hence D has no accumulation points. Note also that $x \in D$ iff f is not continuous in x, or equivalently, f is continuous in x iff $x \notin D$.

Any finite interval $[a, b]$ contains a finite number of points in D, since otherwise an accumulation point of D would exist in $[a, b]$. Clearly, for every other point $x \in [a, b]$, f is continuous at x, or $[a, b]$ may be partitioned in a finite number n of intervals (x_{i-1}, x_i) for $i = 1, \ldots, n$, and a finite number $n + 1$ of points $\{x_0, x_1, \ldots, x_n\}$, such that $x_0 = a$, $x_n = b$, and $x_i \in D$ for all $i = 1, 2, \ldots, n - 1$, but $x \notin D$ for all $x \in [a, b]$ such that $x \notin \{x_0, x_1, \ldots, x_n\}$. Since f is continuous and finite in each one of these open intervals, then

$$I_i = \int_{x_{i-1}}^{x_i} f(x)dx$$

exists and is a finite real for each $i = 1, 2, \ldots, n$, and since $f(x_i)$ is a finite real for each $i = 0, 1, \ldots, n-1$, the contribution to the integral from each isolated point x_i is 0, yielding

$$\int_a^b f(x)dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)dx \sum_{i=1}^{n} I_i,$$

clearly real and finite since it is the sum of a finite number of finite reals. The conclusion to part (a) follows.

Assume that $F(x)$ is differentiable at a given $z \in \mathbb{R}$. Note that, for all $x \in \mathbb{R}$ and $\Delta > 0$,

$$\frac{F(x + \Delta) - F(x)}{\Delta} = \frac{1}{\Delta} \int_x^{x+\Delta} f(x)dx.$$

Since $\lim_{x \to z_+} f(x) = L_+$ exists, then for every $\epsilon > 0$, a $\delta > 0$ exists such that for each $z + \delta > x > z$, then $|f(x) - L_+| < \epsilon$, or for any $0 < \Delta < \delta$,

$$\left| \frac{F(x + \Delta) - F(x)}{\Delta} - L_+ \right| \leq \frac{1}{\Delta} \int_x^{x+\Delta} |f(x) - L_+| \, dx < \frac{1}{\Delta} \int_x^{x+\Delta} \epsilon dx = \epsilon,$$

or equivalently,

$$\lim_{\Delta\to 0}\frac{F(x+\Delta)-F(x)}{\Delta}=L_+,$$

and similarly, denoting $L_-=\lim_{x\to z_-}f(x)$, we have

$$\lim_{\Delta\to 0}\frac{F(x)-F(x-\Delta)}{\Delta}=L_-.$$

Since $F(x)$ is differentiable at z, then $L_+=L_-$ is a finite real, or f has a finite limit at z. The conclusion to part (b) follows. Note that, by the previous argument, we can also conclude that $\frac{dF(x)}{dx}$ takes, at each real x, the value of the limit of f at x, independently of whether f is continuous at x or not.

Note: Any finite-valued and piecewise continuous function, with isolated discontinuity points, will clearly satisfy part (a), for example a square wave such that $f(x)=1$ if $[x]$ is even and $f(x)=-1$ if $[x]$ is odd, where $[x]$ denotes the integral part of x, its indefinite integral being a triangular wave, plus an additive integration constant. A finite-valued function f that has isolated discontinuities, but is everywhere else continuous, satisfies part (b); for example a function whose value is a finite real constant a at every non-integral real, but whose value is a different finite real constant $b\neq a$ at every integer, clearly has finite limit a at every real, its indefinite integral being of the form $ax+C$, where C is an integration constant.

Second solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

(a). We prove that the set of points where the limits right and left are unequal is actually a countable set yielding the integrability of f. We need a definition and a lemma

Definition *Let $B\subset\mathbf{R}$ be an uncountable set. A point $b\in B$ is said "point of condensation" if in any open interval (a,c), $a<b<c$, there are uncountable many points of B.* The Lemma we need is

Lemma 1 (proof at the end). *Any point of an uncountable set $B\subset\mathbf{R}$ is a condensation point except at most a countable subset of B.*

We prove the

Proposition *The set of points where there exists the right and left limits but they are different is at most countable*

Define

$$\begin{aligned}
&l_-(p)=\lim_{x\to p^-}f(x),\qquad l_+(p)=\lim_{x\to p^+}f(x)\\
&X^{(N)}=\big\{x:\forall\,\delta>0\ \exists\ x_1\in(x-\delta,\delta),\ x_2\in(x,x+\delta)\\
&\qquad\qquad\qquad\qquad\wedge\ 1/(N-1)>|f(x_1)-f(x_2)|\geq 1/N\big\}\\
&Y^{(N)}=\big\{x:\forall\,\delta>0\ \exists\ x_1\in(x-\delta,\delta),\ x_2\in(x,x+\delta)\\
&\qquad\qquad\qquad\qquad\wedge\ N>|f(x_1)-f(x_2)|\geq N-1\big\}
\end{aligned}$$

By hypotheses $\mathbf{R}=\Big[\bigcup_{N=2}^{\infty}X^{(N)}\Big]\cup\Big[\bigcup_{N=2}^{\infty}Y^{(N)}\Big]$ thus we suppose that at least one of the sets $X^{(N)}$'s or $Y^{(N)}$'s, say $X^{(N_0)}$, is uncountable. Let $p\in X^{(N_0)}$ a condensation point and let's suppose that in any left–neighborhood of p there are uncountably many points of $X^{(N_0)}$ (at left and/or right of p there must be uncountably many points of $X^{(N_0)}$ by definition of condensation point). Let $\{x_n\}_{n\geq 1} \in X^{(N_0)}$ be a sequence converging toward right to p that is $x_n \nearrow p$. The definition of $X^{(N_0)}$ yields the existence of two sequences $\{y_n\}_{n\geq 1}$ and $\{z_n\}_{n\geq 1}$ such that $y_n < x_n < z_n$, $z_n - y_n \to 0$ and $|f(y_n) - f(z_n)| \geq 1/N_0$. Now we have

$$\lim_{n\to\infty} f(x_n) = l_-(p), \qquad \lim_{n\to\infty} f(y_n) = l_-(p) \qquad \lim_{n\to\infty} f(z_n) = l_-(p)$$

$$|f(x_n) - l_-(p)| = |f(x_n) - f(y_n) + f(y_n) - l_-(p) + f(z_n) - f(z_n)|$$

and $|f(y_n) - f(p)| \to 0$, $|f(z_n) - f(p)| \to 0$ but $|f(y_n) - f(z_n)| \geq 1/N_0$. The contradiction drops if we suppose that each $X^{(N)}$ and Y is countable implying that the point of discontinuity of f are a countable set and this in turn implies the Riemann–integrability of the function.

Proof of the Lemma 1. Let B_c be the set of condensation points of B and $B^{(i)}$ the set of the isolated points of B. Notice that $B^{(i)} \subset B$ but in general $B_c \not\subset B$. It is well known that $B^{(i)}$ is countable. B' denotes the set of the accumulation points of B. We have

$$B = (B' \cap B) \cup B^{(i)}, \qquad B' \cap B = (B_c \cap B) \cup ((B' \cap B)\backslash B_c)$$

Since $B^{(i)}$ is countable, it suffices to prove that $((B' \cap B)\backslash B_c)$ is countable. Let $p \in ((B' \cap B)\backslash B_c)$ and $U \ni p$ an interval $(p - r, p + r)$ with r a rational number. Since $p \notin B_c$, r can be chosen such that in U there are countable points of B. The set of rational numbers is countable and then the set $((B' \cap B)\backslash B_c)$ is also countable. Since B is uncountable, $B_c \cap B$ is uncountable concluding the proof.

(b). We need the lemma

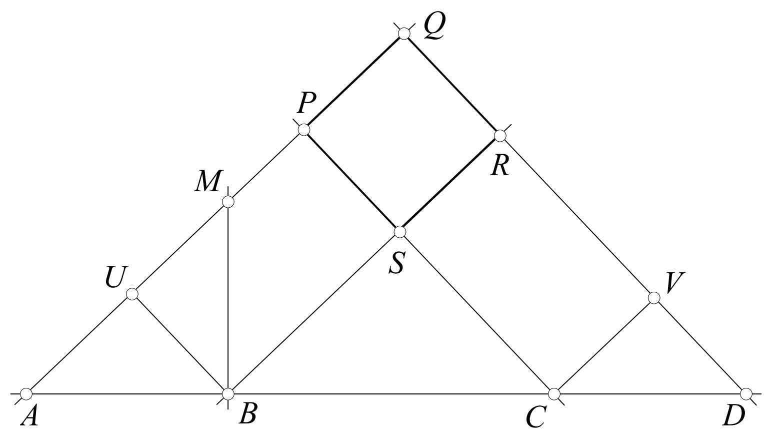
Lemma 2 *Let $f\colon \mathbf{R} \to \mathbf{R}$ be a differentiable function. If $\lim_{x\to x_0^\pm} f'(x) = l^\pm$ ($\pm\infty$ allowed), then $\lim_{h\to 0^\pm} h^{-1}(f(x_0 + h) - f(x_0)) = l^\pm$*

Let's suppose $F(x) = \int_0^x f(t)dt$ differentiable. Via the proof of (a) we know that the discontinuity points of f are at most countable and then the continuity points are dense in $\mathbb{R}$. Let $Cont(f)$ and $Disc(f)$ denote respectively the discontinuity and the continuity points of f and let $p \in Disc(f)$. There exists a sequence $\{x_n\}$, $x_n \in Cont(f)$ such that $x_n \searrow p$ and a sequence $\{x_n'\}$ such that $x_n' \nearrow p$. By the Fundamental theorem of the Calculus we have $F'(x_n) = f(x_n)$ and $F'(x_n') = f(x_n')$. Since $f(x_n) \to l_+(p)$, by the Lemma if follows $F_+'(p) = l_+(p)$ while $F_-'(p) = l_-(p)$ (respectively the limits right and left of the derivatives of $F(x)$) and then $l_+(p) = l_-(p)$. Since a Riemann–integrable function is bounded, $l_+(p) = l_-(p)$ is finite.

O187. Points A, B, C, D are situated in this order on a line. Through A, B and C, D construct parallel lines a, b and c, d such that their points of intersection are vertices of a square and find the sidelength of this square in terms of u, v, w where $u = AB$, $v = BC$, $w = CD$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Francisco Javier Garcia Capitan, Priego de Cordoba, Spain



Let M be the intersection of line a and the perpendicular to line AB at B. Let U, V be the orthogonal projections of B, C on lines a, d, respectively. From similar triangles $UBW \sim UAB \sim VCD$ we have

$$\frac{BM}{UB} = \frac{AB}{UA} = \frac{CD}{VC}.$$

But, since $PQRS$ is a square, we have $UB = VC$, therefore we also have $BM = CD$, that gives a very easy construction: we simply erect BM equal to CD, a is the line AM, and b is the parallel to a through B. On the other hand, the lines c and d are the perpendiculars to a through C, D respectively.

Now, the side-length x of the square equals to BU, the $B-$height of triangle ABM with $AB = u$ and $BM = w$. Therefore we have

$$x = \frac{BA \cdot BM}{AM} = \frac{uw}{\sqrt{u^2 + w^2}}.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Samer SerajOntario, Canada.

O188. Let $a_1, a_2, \ldots, a_n$ be nonzero real numbers, not necessarily distinct. What is the maximum number of subsets A of $\{1, 2, \ldots, n\}$ such that $\sum\limits_{i\in A} a_i = 0$ if (a) $n = 2010$ (b) $n = 2011$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

Solution by by the author

The answer is $\binom{n}{[n/2]}$, for any n. Call a subset B of $S = \{1, 2, ..., n\}$ good if $\sum_{i\in B} a_i = 0$. If $i \in S$ and B is good, call i good with respect to B if either $a_i > 0$ and $i \in B$ or $a_i < 0$ and $i \notin B$. Finally, for a good set B let A_B be the set of those $i \in S$ that are good with respect to B. We claim that this is a Sperner family, i.e. no two A_B's are comparable with respect to inclusion.

Suppose that B_1 and B_2 are two good sets such that $A_{B_1} \subset A_{B_2}$. Then

$$\sum_{i\in B_1, a_i>0} a_i \leq \sum_{i\in B_2, a_i>0} a_i,$$

as any $i \in B_1$ for which $a_i > 0$ is in B_2. A similar argument gives

$$\sum_{i\in B_1, a_i<0} a_i \leq \sum_{i\in B_2, a_i<0} a_i.$$

Using the fact that B_1, B_2 are good, it is easy to deduce from here that all these inequalities are equalities (using the fact that

$$\sum_{i\in B_2, a_i>0} a_i = -\sum_{i\in B_2, a_i<0} a_i)$$

and so $B_1 = B_2$.

Using Sperner's theorem, we deduce that there are at most $\binom{n}{[n/2]}$ good subsets. This is achieved by taking the first $[n/2]$ a_i's equal to 1 and the remaining a_i's equal to -1.

O189. Find the locus of the orthocenter of triangle ABC, where A, B, C are distinct points on a given sphere.

Proposed by Jesik Min, Korean Minjok Leadership Academy

Solution by Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy

It is known that the orthocenter H, the centroid G and the circumcenter O' are collinear (*Euler line*) and $O'H = 3O'G$. Hence, since the centroid of a proper triangle is strictly inside the triangle we have that $|O'G| < r$ and, by Pythagoras Theorem,

$$\begin{aligned}|OH| &= \sqrt{|OO'|^2 + |O'H|^2} = \sqrt{|OO'|^2 + |3O'G|^2} \\ &< \sqrt{|OO'|^2 + 9r^2} \leq 3\sqrt{|OO'|^2 + r^2} = 3R.\end{aligned}$$

On the other hand, any point P inside the sphere of center O and radius $3R$ is the orthocenter of some triangle with distinct vertices on the given sphere. In fact, it is easy to take A, B, C on a great circle so that OG is equal to $OP/3$. In this case $O = O'$ and $OH = 3OG = OP$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Samer SerajOntario, Canada.

O190. Let ABC be a triangle with sidelengths a, b, c and medians m_a, m_b, m_c. Prove that

$$m_a + m_b + m_c \leq \frac{1}{2}\sqrt{7(a^2+b^2+c^2) + 2(ab+bc+ca)}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Arkady Alt, San Jose, California, USA

Since

$$4m_b m_c = \sqrt{\left(2\left(a^2+c^2\right)-b^2\right)\left(2\left(a^2+b^2\right)-c^2\right)} = \sqrt{4a^4 + 2a^2\left(b^2+c^2\right) + 5b^2c^2 - 2b^4 - 2c^4} =$$
$$\sqrt{4a^4 + 2a^2\left(b^2+c^2\right) + b^2c^2 - 2\left(b^2-c^2\right)^2} = \sqrt{\left(2a^2+bc\right)^2 - 2\left(\left(b^2-c^2\right)^2 - a^2(b-c)^2\right)} \leq$$
$$\sqrt{\left(2a^2+bc\right)^2 - 2(b-c)^2(a+b+c)(b+c-a)} \leq \sqrt{\left(2a^2+bc\right)^2} = 2a^2+bc$$ and
$$4\left(m_a^2+m_b^2+m_c^2\right) = 3\sum_{cyc} a^2$$ then $$4\left(m_a+m_b+m_c\right)^2 = 3\sum_{cyc} a^2 + 8\sum_{cyc} m_b m_c \leq$$
$$3\sum_{cyc} a^2 + 2\sum_{cyc}\left(2a^2+bc\right) = 7\sum_{cyc} a^2 + 2\sum_{cyc} bc.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Samer SerajOntario, Canada; Scott H. Brown, Auburn University, USA.

O191. Let I be the incenter of a triangle ABC and let IA_1, IB_1, IC_1 be symmedians of triangles BIC, CIA, AIB, respectively. Prove that AA_1, BB_1, CC_1 are concurrent at some point P. Prove that P lies on the line $G\Gamma$, where G is centroid and Γ is Gergonne point of triangle ABC.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

It is well known that the centroid G is at a distance $\frac{h_a}{3}$ from side BC (easily provable by Thales theorem, and since the centroid divides the median in two segments in proportion $2:1$, the longer one between vertex and centroid and the shorter one between centroid and midpoint of the opposite side), where $h_a = \frac{2S}{a}$ is the altitude from A, and S the area of ABC. Therefore, the (nonexact) trilinear coordinates of G are $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$.

It is also well known that the point T where the incircle touches side BC is at a distance $\frac{c+a-b}{2}$ from B, and at a distance $\frac{a+b-c}{2}$ from C, hence at a ratio of distances to AB and AC equal to $\frac{c+a-b}{2}\sin B : \frac{a+b-c}{2}\sin C = (c+a-b)b : (a+b-c)c$. It follows that, since line passing through vertex A and point P, with respective (nonexact) trilinear coordinates $A \equiv (1,0,0)$ and $P \equiv (\alpha_P, \beta_P, \gamma_P)$, has equation $\gamma_P \beta = \beta_P \gamma$, then line AT has trilinear coordinates

$$(c+a-b)b\beta = (a+b-c)c\gamma,$$

or the trilinear coordinates of Γ satisfy simultaneously this equation and its cyclic permutations, for (nonexact) trilinear coordinates $\left(\frac{1}{a(b+c-a)}, \frac{1}{b(c+a-b)}, \frac{1}{c(a+b-c)}\right)$.

Now, the symmedian AD in triangle ABC divides side BC in two parts which are in proportion $BD : CD = AB^2 : AC^2$, or applying this result to triangle IBC, we have that

$$\frac{BA_1}{CA_1} = \frac{IB^2}{IC^2} = \frac{\sin^2 \frac{C}{2}}{\sin^2 \frac{B}{2}},$$

since $r = IB \sin \frac{B}{2} = IC \sin \frac{C}{2}$. Now,

$$2\sin^2 \frac{C}{2} = 1 - \cos C = \frac{2ab - a^2 - b^2 + c^2}{2ab} = \frac{(b+c-a)(c+a-b)}{2ab},$$

and similarly for $\sin^2 \frac{B}{2}$, or

$$\frac{BA_1}{CA_1} = \frac{c(c+a-b)}{b(a+b-c)},$$

and similarly for the result of cyclic permutation of A, B, C. Hence $\frac{BA_1}{CA_1}\frac{CB_1}{AB_1}\frac{AC_1}{BC_1} = 1$, or by the reciprocal of Menelaus' theorem, AA_1, BB_1, CC_1 are concurrent. Moreover, since the ratio of distance from A_1 to AB, AC is $(c+a-b) : (a+b-c)$, then line AA_1 has equation $(c+a-b)\beta = (a+b-c)\gamma$ in trilinear coordinates. Therefore, the point where AA_1, BB_1, CC_1 concur has (nonexact) trilinear coordinates $\left(\frac{1}{b+c-a}, \frac{1}{c+a-b}, \frac{1}{a+b-c}\right)$, and it will be collinear with G, Γ, iff

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{b+c-a} & \frac{1}{c+a-b} & \frac{1}{a+b-c} \\ \frac{1}{a(b+c-a)} & \frac{1}{b(c+a-b)} & \frac{1}{c(a+b-c)} \end{vmatrix} = 0,$$

which is trivially true since

$$\frac{1}{a+b+c}\frac{1}{a} + \frac{2}{a+b+c}\frac{1}{b+c-a} = \frac{1}{a(b+c-a)},$$

or the third row in the determinant is the linear combination of the first and second rows, with respective coefficients $\frac{1}{a+b+c}, \frac{2}{a+b+c}$. The conclusion follows.

O192. Let p be a prime such that $p \equiv 2 (\text{mod } 3)$. Prove that there are no integers a, b, c of the same parity such that

$$\left(\frac{a}{2} + \frac{b}{2} i\sqrt{3}\right)^p = c + i\sqrt{3}.$$

Proposed by Dorin Andrica and Mihai Piticari, Romania

Solution by Prasanna Ramakrishnan

First, we acknowledge the special case, $p = 2$. Then

$$\left(\frac{a}{2} + \frac{b}{2} i\sqrt{3}\right)^2 = c + i\sqrt{3} \Rightarrow \frac{a^2 + 3b^2}{2} + \frac{ab}{2}(i\sqrt{3}) = c + i\sqrt{3}.$$

It follows that $ab = 2$. Since a and b are integers, this means that one of a and b is ± 1 and the other is ± 2. But this implies that there are of different parity which is impossible. Hence $p = 2$ yields no solution. Then, using the binomial expansion we get that

$$\begin{aligned}\left(\frac{a}{2} + \frac{b}{2} i\sqrt{3}\right)^p &= \frac{\sum_{k=0}^{p} \binom{p}{k} a^{p-k} b^k (i\sqrt{3})^k}{2^p} \\ &= \frac{\sum_{\text{even } k}^{p} \binom{p}{k} a^{p-k} b^k \cdot 3^{\frac{k}{2}}}{2^p} + \frac{\sum_{\text{odd } k}^{p} \binom{p}{k} a^{p-k} b^k \cdot 3^{\frac{k-1}{2}}}{2^p}(i\sqrt{3}) = c + i\sqrt{3}\end{aligned}$$

and so we notice that

$$c = \frac{\sum_{\text{even } k}^{p} \binom{p}{k} a^{p-k} b^k \cdot 3^{\frac{k}{2}}}{2^p} \tag{1}$$

and

$$1 = \frac{\sum_{\text{odd } k}^{p} \binom{p}{k} a^{p-k} b^k \cdot 3^{\frac{k-1}{2}}}{2^p}. \tag{2}$$

From (2),

$$2^p = \frac{\sum_{\text{odd } k}^{p} \binom{p}{k} a^{p-k} b^k \cdot 3^{\frac{k-1}{2}}}{2^p}.$$

Since 2^p and a^{p-k} are positive, b^k must also be positive. Since k is odd, b is positive. After expanding the summation, $b^p < 2^p$ and so, $b < 2$ which implies that $b = 1$. Similarly $a^{p-1} < 2^p$ which implies that $a = \pm 1$ (since $p \geq 5$ and a, b are of the same parity). So then we check $a = b = 1$ and $a = -1$ and $b = -1$, niether of which yields solutions and we are done..