Junior problems

J181. Let a, b, c, d be positive real numbers. Prove that

$$\left(\frac{a+b}{2}\right)^3+\left(\frac{c+d}{2}\right)^3 \leq \left(\frac{a^2+d^2}{a+d}\right)^3+\left(\frac{b^2+c^2}{b+c}\right)^3$$

Proposed by Pedro H. O. Pantoja, Natal-RN, Brazil

Solution by Roberto Bosch Cabrera, Florida, USA

Using the well known inequality $\left(\frac{x+y}{2}\right)^3 \leq \frac{x^3+y^3}{2}$, we have that

$$\left(\frac{a+b}{2}\right)^3+\left(\frac{c+d}{2}\right)^3 \leq \frac{a^3+b^3}{2}+\frac{c^3+d^3}{2},$$

with equality if and only if $a=b$ and $c=d$. Now notice that if we let $\frac{a}{d}=x$,

$$\frac{a^3+d^3}{2} \leq \left(\frac{a^2+d^2}{a+d}\right)^3 \Leftrightarrow x^3+1 \leq 2\left(\frac{x^2+1}{x+1}\right)^3.$$

But,

$$\begin{aligned} x^3+1 &\leq 2\left(\frac{x^2+1}{x+1}\right)^3 \\ \Leftrightarrow x^6-3x^5+3x^4-2x^3+3x^2-3x+1 &\geq 0 \\ \Leftrightarrow (x-1)^4(x^2+x+1) &\geq 0 \end{aligned}$$

so we obtain that

$$\frac{a^3+d^3}{2} \leq \left(\frac{a^2+d^2}{a+d}\right)^3$$

with equality if and only if $a=d$. Analogously

$$\frac{b^3+c^3}{2} \leq \left(\frac{b^2+c^2}{b+c}\right)^3$$

with equality if and only if $b=c$. Hence the original inequality is true with equality if and only if $a=b=c=d$.

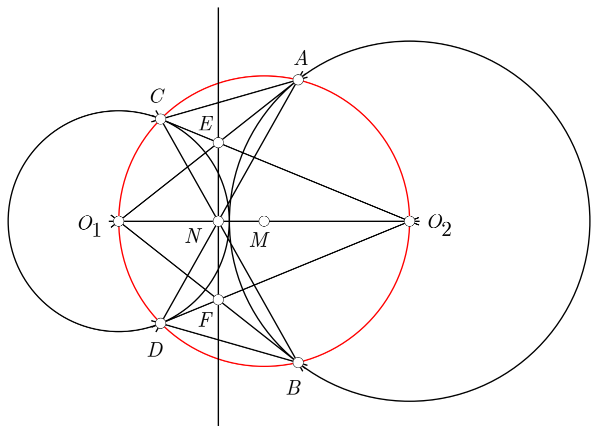
Also solved by Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

J182. Circles $C_1(O_1, r)$ and $C_2(O_2, R)$ are externally tangent. Tangent lines from O_1 to C_2 intersect C_2 at A and B, while tangent lines from O_2 to C_1 intersect C_1 at C and D. Let $O_1A \cap O_2C = \{E\}$ and $O_1B \cap O_2D = \{F\}$. Prove that $EF \cap O_1O_2 = AD \cap BC$.

Proposed by Roberto Bosch Cabrera, Florida, USA

First solution by Ercole Suppa, Teramo, Italy

Let M be the midpoint of O_1O_2 and observe that A, B, C, D lie on the circunference with center M and radius MO_1, as shown in figure.



Let $N = AD \cap BC$ and note that N lies on O_1O_2 because AD is the reflection of BC across the line O_1O_2. Since $O_1C = O_1D$ we have $\angle CAO_1 = \angle O_1AD$, so O_1A is the internal bisector of $\angle CAN$. By the same argument, since $O_2A = O_2B$, we have $\angle O_2CA = \angle BCO_2$, so O_2C is the internal bisector of $\angle ACN$.

Therefore E is the incenter of $\triangle ACN$, so $\angle CNE = \angle ENA$. Similarly F is the incenter of $\triangle BND$, so $\angle DNF = \angle FNB$. Consequently we have

$$\angle CNE = \frac{1}{2}\angle CNA = \frac{1}{2}\angle DNB = \angle FNB$$

and this implies E, N, F are collinear. Thus $EF \cap O_1O_2 = AD \cap BC = \{N\}$ and the claim is proved.

Second solution by Gabriel Alexander Chicas Reyes, El Salvador

We will show that the result holds in the more general case of any two circles such that neither of them is contained in the other.

Note that $\angle O_1AO_2 = \angle O_1BO_2 = \pi/2$ since O_1A is tangent to C_2. In the same way $\angle O_1CO_2 = \angle O_1DO_2 = \pi/2$, so that the points A, C, O_1, D, B, O_2 all lie on the circle of diameter O_1O_2. Now applying Pascal's theorem to the cyclic hexagon ADO_2CBO_1 it follows that $P := AD \cap BC$, $AO_1 \cap CO_2 = E$ and $BO_1 \cap DO_2 = F$ are collinear. In other words, AD, BC and EF concur.

Now it remains to observe that by symmetry O_1O_2 is the perpendicular bisector of AB and CD, and thus P lies on O_1O_2. Since we just proved that EF goes through P, it follows that AD, BC, EF and O_1O_2 concur, as we wanted.

Also solved by Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Robert Bosch Cabrera, USA.

J183. Let x, y, z be real numbers. Prove that

$$(x^2+y^2+z^2)^2+xyz(x+y+z)\geq\frac{2}{3}(xy+yz+zx)^2+(x^2y^2+y^2z^2+z^2x^2).$$

Proposed by Neculai Stanciu, George Emil Palade, Buzau, Romania

First solution by Arkady Alt, San Jose, California, USA

The original inequality will follow from the following sharper inequality

$$\left(x^2+y^2+z^2\right)^2+xyz\,(x+y+z)-\left(x^2y^2+y^2z^2+z^2x^2\right)\geq(xy+yz+zx)^2\,. \tag{1}$$

Indeed, for any real u, v, w we have

$$\begin{aligned} u^2+v^2+w^2\geq uv+vw+wu &\iff (u+v+w)^2\geq 3\,(uv+vw+wu)\\ &\iff (u-v)^2+(v-w)^2+(w-u)^2\geq 0. \end{aligned}$$

Then

$$\left(x^2+y^2+z^2\right)^2\geq 3\left(x^2y^2+y^2z^2+z^2x^2\right)$$

and

$$x^2y^2+y^2z^2+z^2x^2\geq xyz\,(x+y+z)\,.$$

Therefore,

$$\begin{aligned} \left(x^2+y^2+z^2\right)^2+xyz\,(x+y+z)-\left(x^2y^2+y^2z^2+z^2x^2\right) &\geq 2\left(x^2y^2+y^2z^2+z^2x^2\right)+xyz\,(x+y+z)\\ &\geq x^2y^2+y^2z^2+z^2x^2+2xyz\,(x+y+z)\\ &=(xy+yz+zx)^2\geq\frac{2}{3}\,(xy+yz+zx)^2\,. \end{aligned}$$

Remark. Equality in (1) occurs if and only if $x=y=z$ and in original inequality equality occurs if and only if $x=y=z=0$.

Second solution by G. C. Greubel, Newport News, VA

By expanding the terms $(x^2+y^2+z^2)^2$ and $(xy+yz+zx)^2$ and equating the terms on both sides leads to

$$x^4+y^4+z^4+\frac{1}{3}(x^2y^2+y^2z^2+z^2x^2)-\frac{1}{3}xyz(x+y+z)\geq 0.$$

By the AM-GM inequality we have

$$x^2y^2+y^2z^2+z^2x^2\geq xyz(x+y+z).$$

Hence it is enough to prove that

$$x^4+y^4+z^4\geq 0$$

which is obvious. Equality occurs if and onyl if $x=y=z=0$.

Also solved by Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Robert Bosch Cabrera, USA; Gabriel Alexander Chicas Reyes, El Salvador; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

J184. Find all quadruples (x, y, z, w) of integers satisfying the system of equations

$$x + y + z + w = xy + yz + zx + w^2 - w = xyz - w^3 = -1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Daniel Campos Salas, Costa Rica

Note that

$$(x + y)(y + z)(z + x) = (x + y + z)(xy + yz + zx) - xyz = 2.$$

This implies that $(x+y, y+z, z+x)$ equals some permutation of $(1, 1, 2)$, $(-1, -1, 2)$ or $(1, -1, -2)$. It follows that (x, y, z) equals some permutation of $(0, 1, 1)$, $(1, 1, -2)$ or $(0, 1, -2)$.

The first case implies that $w = -1 - (x + y + z) = -3$ and $w^3 = 1 + xyz = 1$, which is a contradiction. The second case implies that $w = -1 - (x + y + z) = -1$, $w^2 - w = -1 - (xy + yz + zx) = 2$ and $w^3 = xyz+1 = -1$, from where it follows that $w = -1$. The third case implies that $w = -1-(x+y+z) = -2$ and $w^3 = xyz + 1 = 1$, which is a contradiction. It follows that all posible quadruples (x, y, z, w) are $(1, 1, -2, -1)$ with all posible permutations among x, y, z

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note first that

$$(x+w+1)(y+w+1)(z+w+1) = w^3 - 1 + (w+1)(-w^2 + w - 1) + (w+1)^2(-w-1) + (w+1)^3 = -2,$$

or $x + w + 1$ divides -2, hence x may take one of the following values:

- $x = -w - 3$; then $y + z = -1 - w - x = 2$, and $yz = -1 + w - w^2 + 2(w + 3) = -w^2 + 3w + 5$, leading to $w^3 - 1 = xyz = (w + 3)(w^2 - 3w - 5) = w^3 - 14w - 15$, or $w = -1$, for $x = -2$ and $yz = 1$. Note that $(y - z)^2 = (y + z)^2 - 4yz = 0$, or $y = z = 1$ because $y + z = 2$.
- $x = -w - 2$; then $y + z = -1 - w - x = 1$, and $yz = -1 + w - w^2 + (w + 2) = 1 + 2w - w^2$, leading to $w^3 - 1 = xyz = (w + 2)(w^2 - 2w - 1) = w^3 - 5w - 2$; no solution in integers exists in this case because no integer w satisfies $5w = -1$.
- $x = -w$; then $y + z = -1 - w - x = -1$, and $yz = -1 + w - w^2 - w = -w^2 - 1$, leading to $w^3 - 1 = xyz = w^3 + w$, or $w = -1$, for $x = 1$ and $yz = -2$. Note that $(y - z)^2 = (y + z)^2 - 4yz = 1 + 8 = 3^2$ for $y - z = \pm 3$, with solutions $y = 1$ and $z = -2$, or $y = -2$ and $z = 1$.
- $x = 1 - w$; then $y + z = -1 - w - x = -2$, and $yz = -1 + w - w^2 + 2(1 - w) = 1 - w - w^2$, leading to $w^3 - 1 = xyz = (w - 1)(w^2 + w - 1) = w^3 - 2w + 1$, or $w = 1$, for $x = 0$ and $yz = -1$. Note that $(y - z)^2 = (y + z)^2 - 4yz = 8$ is not a perfect square, hence no solution in integers exist in this case.

Restoring generality, all solutions are $w = -1$, and (x, y, z) a permutation of $(1, 1, -2)$.

J185. Let $H(x,y)=\frac{2xy}{x+y}$ be the harmonic mean of the positive real numbers x and y. For $n\geq 2$, find the greatest constant C such that for any positive real numbers $a_1,\ldots,a_n,b_1,\ldots,b_n$ the following inequality holds

$$\frac{C}{H(a_1+\cdots+a_n,b_1+\cdots+b_n)}\leq\frac{1}{H(a_1,b_1)}+\cdots+\frac{1}{H(a_n,b_n)}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Roberto Bosch Cabrera, Florida, USA

Let

$$S=\frac{H(a_1+\cdots+a_n,b_1+\cdots+b_n)}{H(a_1,b_1)}+\cdots+\frac{H(a_1+\cdots+a_n,b_1+\cdots+b_n)}{H(a_n,b_n)}$$

Note that

$$\begin{aligned}
S &= \frac{(a_1+\cdots+a_n)(b_1+\cdots+b_n)}{(a_1+\cdots+a_n)+(b_1+\cdots+b_n)}\cdot\left[\frac{a_1+b_1}{a_1b_1}+\cdots+\frac{a_n+b_n}{a_nb_n}\right]\\
&= \frac{(a_1+\cdots+a_n)(b_1+\cdots+b_n)}{(a_1+\cdots+a_n)+(b_1+\cdots+b_n)}\cdot\left[\frac{1}{a_1}+\cdots+\frac{1}{a_n}+\frac{1}{b_1}+\cdots+\frac{1}{b_n}\right]\\
&= \frac{\frac{1}{a_1}+\cdots+\frac{1}{a_n}+\frac{1}{b_1}+\cdots+\frac{1}{b_n}}{\frac{1}{a_1+\cdots+a_n}+\frac{1}{b_1+\cdots+b_n}}\geq n^2
\end{aligned}$$

The last inequality is true because

$$\begin{aligned}
\frac{1}{a_1}+\cdots+\frac{1}{a_n} &\geq n^2\cdot\frac{1}{a_1+\cdots+a_n}\\
\frac{1}{b_1}+\cdots+\frac{1}{b_n} &\geq n^2\cdot\frac{1}{b_1+\cdots+b_n}
\end{aligned}$$

by the AM-HM inequality. Equality occurs if and only if $a_1=\cdots=a_n$ and $b_1=\cdots=b_n$ so $C=n^2$.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Pedro H. O. Pantoja, Natal-RN, Brazil.

J186. Let ABC be a right triangle with $AC = 3$ and $BC = 4$ and let the median AA_1 and the angle bisector BB_1 intersect at O. A line through O intersects hypotenuse AB at M and AC at N. Prove that

$$\frac{MB}{MA} \cdot \frac{NC}{NA} \leq \frac{4}{9}.$$

Proposed by Valcho Milchev, Kardzhali, Bulgaria

First solution by Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy

We prove the inequality by using analytic geometry. Let $A = (3, 0)$, $B = (0, 4)$ and $C = (0, 0)$. Moreover, since $\cos(\widehat{B}) = 4/5$, it follows that

$$A_1 = (0, 2) \quad \text{and} \quad B_1 = \left(BC\tan(\widehat{B}/2), 0\right) = \left(4\sqrt{\frac{1-\cos(\widehat{B})}{1+\cos(\widehat{B})}}, 0\right) = \left(\frac{4}{3}, 0\right).$$

Hence the equations of the median AA_1 and the angle bisector BB_1 are respectively

$$2x + 3y = 6 \quad \text{and} \quad 3x + y = 4.$$

The intersection is given by

$$O = \left(\frac{6}{7}, \frac{10}{7}\right).$$

Let $a(7x - 6) + b(7y - 10) = 0$ be a line l through O then

$$NA = \frac{5(3a - 2b)}{7a} \quad \text{and} \quad MA = \frac{25(3a - 2b)}{7(3a - 4b)}.$$

Therefore

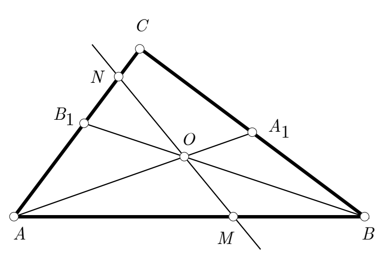
$$\frac{5}{MA} + \frac{3}{NA} = \frac{14}{5},$$

and by AGM inequality

$$\frac{MB}{MA} \cdot \frac{NC}{NA} = \left(\frac{5}{MA} - 1\right) \cdot \left(\frac{3}{NA} - 1\right) \leq \frac{1}{4}\left(\frac{5}{MA} + \frac{3}{NA} - 2\right)^2 = \frac{4}{25}.$$

So the maximum of the LHS is attained when the line l is vertical: $x = 6/7$.

Second solution by Ercole Suppa, Teramo, Italy



We use coordinate geometry. The point C is the intersection of the circle with center A and radius 3 and the circle with center B and radius 4. Assuming wlog $A(0, 0)$, $B(5, 0)$ the point C is obtained by solving the following system

$$\begin{cases} x^2 + y^2 = 9 \\ (x-5)^2 + y^2 = 16 \end{cases}$$

which, after some algebra, gives us $C\left(\frac{9}{5}, \frac{12}{5}\right)$. Thus, since A_1 is the midpoint of BC, we get $A_1\left(\frac{17}{5}, \frac{6}{5}\right)$.

Since BB_1 is the bisector of $\angle ABC$ we have

$$\frac{AO}{OA_1} = \frac{AB}{BA_1} = \frac{5}{2} \qquad \Rightarrow \qquad O\left(\frac{17}{7}, \frac{6}{7}\right)$$

Thus a line through O has equation

$$\ell_m : \qquad 7mx - 7y + 6 - 17m = 0$$

A simple calculation show that ℓ_m intersects AC iff $-\frac{27}{11} \leq m \leq \frac{6}{17}$, whereas ℓ_m intersects AB iff $m \leq -\frac{1}{3}$ or $m \geq \frac{6}{17}$. Therefore ℓ_m intersects both AB, AC iff

$$-\frac{27}{11} \leq m \leq -\frac{1}{3}$$

After some more algebra, we find:

$$M\left(\frac{17m-6}{7m}, 0\right), \qquad N\left(\frac{51m-18}{21m-28}, \frac{68m-24}{21m-28}\right)$$

Therefore, by using the euclidean distance formula, we obtain:

$$\begin{aligned} &4 \cdot MA^2 \cdot NA^2 - 9 \cdot MB^2 \cdot NC^2 = \\ =&\frac{4\left(851m^2 - 2676m - 306\right)\left(2039m^2 + 636m + 666\right)}{2401m^2(3m-4)^2} \end{aligned} \tag{1}$$

Now observe that $2039m^2 + 636m + 666 > 0$ for every $m \in \mathbb{R}$ and

$$851m^2 - 2676m - 306 \geq 0$$

if and only if

$$m \in \left(-\infty, \frac{1338 - 105\sqrt{186}}{851}\right] \cup \left[\frac{1338 + 105\sqrt{186}}{851}, +\infty\right)$$

From (1) and the previous results, taking into account $-\frac{1}{3} < \frac{1338-105\sqrt{186}}{851}$, it follows that

$$4 \cdot MA^2 \cdot NA^2 - 9 \cdot MB^2 \cdot NC^2 \geq 0 \qquad \Leftrightarrow \qquad \frac{MB}{MA} \cdot \frac{NC}{NA} \leq \frac{4}{9}$$

so the desired inequality is established.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

Senior problems

S181. Let a and b be positive real numbers such that

$$|a-2b| \le \frac{1}{\sqrt{a}} \quad \text{and} \quad |2a-b| \le \frac{1}{\sqrt{b}}.$$

Prove that $a+b \le 2$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Anthony Erb Lugo, San Juan, Puerto Rico

We start by squaring both inequalities,

$$a^2 - 4ab + 4b^2 \le \frac{1}{a} \text{ and } 4a^2 - 4ab + b^2 \le \frac{1}{b}$$

Next, we eliminate the fractions in each inequality,

$$a^3 - 4a^2b + 4b^2a \le 1 \text{ and } 4a^2b - 4ab^2 + b^3 \le 1$$

Now add the two inequalities together,

$$a^3 + b^3 \le 2$$

By multiplying by 4 and applying Hölder's Inequality, we have that,

$$8 \ge (1+1)(1+1)(a^3+b^3) \ge (a+b)^3$$

And finally,

$$2 \ge a+b$$

Which is what we wanted to prove, so we're done.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Pública de Navarra, Spain; G. C. Greubel, Newport News, USA; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Prithwijit De, Mumbai, India; Roberto Bosch Cabrera, Havana, Cuba.

S182. Let a, b, c be real numbers such that $a > b > c$. Prove that for each real number x the following inequality holds

$$\sum_{\text{cyc}} (x-a)^4(b-c) \geq \frac{1}{6}(a-b)(b-c)(a-c)[(a-b)^2+(b-c)^2+(c-a)^2].$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Daniel Campos Salas, Costa Rica

Note that the coefficients of x^4 and x^3 vanish. Therefore the left-hand side equals

$$\sum_{cyc} 6x^2a^2(b-c) - 4xa^3(b-c) + a^4(b-c).$$

Note that each of the coefficients of the polynomial on the left-hand side vanish when $a = b$, $b = c$ or $c = a$. Therefore $(a-b)(b-c)(a-c)$ divides each of them. It is easy to show that

$$\sum_{cyc} a^2(b-c) = (a-b)(b-c)(a-c),$$

$$\sum_{cyc} a^3(b-c) = (a-b)(b-c)(a-c)(a+b+c),$$

$$\sum_{cyc} a^4(b-c) = (a-b)(b-c)(a-c)(a^2+b^2+c^2+ab+bc+ca).$$

It results that the inequality is equivalent to

$$6x^2 - 4x(a+b+c) + (a^2+b^2+c^2+ab+bc+ca) \geq \frac{1}{3}(a^2+b^2+c^2-ab-bc-ca),$$

or equivalently,

$$6\left(x - \frac{a+b+c}{3}\right)^2 = 6x^2 - 4x(a+b+c) + \frac{2}{3}(a+b+c)^2 \geq 0,$$

and we're done.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

S183. Let $a_0 \in (0,1)$ and $a_n = a_{n-1} - \frac{a_{n-1}^2}{2}$ for $n \geq 1$. Prove that for all $n \geq 1$,

$$\frac{n}{2} < \frac{1}{a_n} - \frac{1}{a_0} < \frac{n+1+\sqrt{n}}{2}.$$

Proposed by Arkady Alt, San Jose, California, USA

Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

LHS. Note that $a_n \in (0,1)$ for any n. Indeed, by math induction, because the function $h(x) = x - \dfrac{x^2}{2}$ is increasing on $[0,1]$ and $a_0 \in (0,1)$ then the assumption $a_n \in (0,1)$ yields

$$a_{n+1} = h(a_n) \in (h(0), h(1)) = \left(0, \frac{1}{2}\right) \subset (0,1).$$

Furthermore,

$$\frac{1}{a_n} = \frac{1}{a_{n-1} - \frac{a_{n-1}^2}{2}} = \frac{1}{a_{n-1}} + \frac{1}{2}\frac{1}{1 - \frac{a_{n-1}}{2}}$$

$$\frac{1}{a_n} - \frac{1}{a_0} = \sum_{k=1}^{n}\left(\frac{1}{a_k} - \frac{1}{a_{k-1}}\right) = \sum_{k=1}^{n}\frac{1}{2 - a_{k-1}} > \sum_{k=1}^{n}\frac{1}{2} = \frac{n}{2}$$

RHS. We proceed by induction. For $n = 0$ clearly holds. Let's suppose it true for $1 \leq n \leq r$. For $n = r+1$ we have

$$\frac{1}{a_{n+1}} - \frac{1}{a_0} = \frac{1}{a_{n+1}} - \frac{1}{a_n} + \frac{1}{a_n} - \frac{1}{a_0} = \frac{1}{2 - a_n} + \frac{1}{a_n} - \frac{1}{a_0}$$

By using the induction hypotheses for $n \geq 1$

$$\frac{1}{2 - a_n} + \frac{1}{a_n} - \frac{1}{a_0} \leq \frac{1}{2 - a_n} + \frac{n+1+\sqrt{n}}{2} \leq \frac{n+2+\sqrt{n+1}}{2}$$

namely

$$a_n \leq 2\frac{\sqrt{n+1} - \sqrt{n}}{1 + \sqrt{n+1} - \sqrt{n}} = 2\frac{1}{\sqrt{n+1} + \sqrt{n}}\frac{1}{1 + \sqrt{n+1} - \sqrt{n}}$$

which is implied by

$$a_n \leq \frac{2}{2\sqrt{n+1}}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}\sqrt{n}} \tag{1}$$

so we need to show (1) for $n \geq 1$. By the LHS. we have

$$\frac{n}{2} \leq \frac{1}{a_n} - \frac{1}{a_0} \Rightarrow a_n \leq \frac{2a_0}{2 + na_0} \leq \frac{2}{2+n}$$

since the function $x/(1+cx)$ increases for $0 < x < 1$ if $c > 0$. Thus

$$a_n \leq \frac{2}{2+n} \leq \frac{1}{\sqrt{2}\sqrt{n}} \iff (n-2)^2 \geq 0$$

and we are done.

Also solved by Ajat Adriansyah, Indonesia; Albert Stadler, Switzerland; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy.

S184. Let $H_n = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, $n \geq 2$. Prove that

$$e^{H_n} > \sqrt[n]{n!} \geq 2^{H_n}.$$

Proposed by Tigran Hakobyan, Vanadzor, Armenia

First solution by Arkady Alt, San Jose, California, USA

LHS. By the AM-GM inequality we have

$$\sqrt[n]{n!} < \frac{1+2+\cdots+n}{n} = \frac{n+1}{2}.$$

Since

$$\left(1+\frac{1}{n}\right)^n < e \iff 1+\frac{1}{n} < e^{\frac{1}{n}} \iff \frac{n+1}{n} < e^{\frac{1}{n}}$$

for any positive integer n, then

$$\prod_{k=2}^{n} e^{\frac{1}{k}} > \prod_{k=2}^{n} \frac{k+1}{k} \iff e^{H_n} > \frac{n+1}{2} \implies e^{H_n} > \frac{n+1}{2} > \sqrt[n]{n!}.$$

RHS. Note that

$$n+1 > 2^{(n+1)H_{n+1}-nH_n}, n \geq 2.$$

Indeed, since

$$(n+1)\, H_{n+1} - nH_n = (n+1)\, H_n + 1 - nH_n = 1 + H_n$$

then

$$n+1 > 2^{(n+1)H_{n+1}-nH_n} \iff n+1 > 2^{1+H_n} \iff 2^{H_n} < \frac{n+1}{2}, n \geq 2. \tag{1}$$

We will prove inequality (1) by math induction.

1. Base case

If $n = 2$ then $2^{H_2} = \sqrt{2}$ and $\sqrt{2} < \dfrac{3}{2} \iff 8 < 9$.

2. Step case

Note that $\dfrac{n+2}{n+1} > 2^{\frac{1}{n+1}}$. Indeed, by the AM-GM Inequality

$$\sqrt[n+1]{2} = \sqrt[n+1]{2 \cdot 1 \cdot 1 \cdots 1} < \frac{2+n\cdot 1}{n+1} = \frac{n+2}{n+1}.$$

Since $2^{\frac{1}{n+1}} < \dfrac{n+2}{n+1}$ and by supposition of math induction $2^{H_n} < \dfrac{n+1}{2}$, $n \geq 2$. Then

$$2^{H_n} \cdot 2^{\frac{1}{n+1}} < \frac{n+1}{2} \cdot \frac{n+2}{n+1} \iff 2^{H_{n+1}} < \frac{n+2}{2}.$$

And again by math induction, since for $n = 2$ we have $2! = 2^{2H_2}$ and from supposition $n! \geq 2^{nH_n}$ it follows that

$$\begin{aligned}(n+1)! &= (n+1)\, n! \geq (n+1)\, 2^{nH_n} \\ &> 2^{(n+1)H_{n+1}-nH_n} \cdot 2^{nH_n} = 2^{(n+1)H_{n+1}}.\end{aligned}$$

Then $n! \geq 2^{nH_n} \iff \sqrt[n]{n!} \geq 2^{H_n}$ for any $n \geq 2$.

Also solved by Albert Stadler, Switzerland; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Pública de Navarra, Spain; G. C. Greubel, Newport News, USA; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy.

S185. Let A_1, A_2, A_3 be non-collinear points on parabola $x^2 = 4py$, $p > 0$, and let $B_1 = l_2 \cap l_3, B_2 = l_3 \cap l_1, B_3 = l_1 \cap l_2$ where l_1, l_2, l_3 are tangents to the parabola at points A_1, A_2, A_3, respectively. Prove that $\dfrac{[A_1A_2A_3]}{[B_1B_2B_3]}$ is a constant and find its value.

Proposed by Arkady Alt, San Jose, California, USA

First solution by Evangelos Mouroukos, Agrinio, Greece

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) denote the coordinates of the points A_1, A_2 and A_3 respectively. The equations of the lines ℓ_1, ℓ_2 and ℓ_3 are

$$\ell_1 : xx_1 = 2p(y + y_1),$$

$$\ell_2 : xx_2 = 2p(y + y_2),$$

$$\ell_3 : xx_3 = 2p(y + y_3).$$

Solving the system of ℓ_2 and ℓ_3, we find that the point B_1 has coordinates

$$\left(\frac{2p(y_3 - y_2)}{x_3 - x_2}, \frac{x_2y_3 - x_3y_2}{x_3 - x_2}\right).$$

Since $y_3 - y_2 = \dfrac{x_3^2 - x_2^2}{4p} = \dfrac{(x_3 - x_2)(x_3 + x_2)}{4p}$ and $x_2y_3 - x_3y_2 = \dfrac{x_2x_3^2 - x_3x_2^2}{4p}$, we find that

$$B_1\left(\frac{x_2 + x_3}{2}, \frac{x_2x_3}{4p}\right)$$

and similarly

$$B_2\left(\frac{x_3 + x_1}{2}, \frac{x_3x_1}{4p}\right),$$

$$B_3\left(\frac{x_1 + x_2}{2}, \frac{x_1x_2}{4p}\right).$$

We compute the area of triangle $A_1A_2A_3$:

$$[A_1A_2A_3] = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}\right| = \frac{1}{2}\left|\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}\right| =$$

$$= \frac{1}{2}\left|\begin{vmatrix} x_2 - x_1 & \dfrac{(x_2 - x_1)(x_2 + x_1)}{4p} \\ x_3 - x_1 & \dfrac{(x_3 - x_1)(x_3 + x_1)}{4p} \end{vmatrix}\right| = \frac{1}{8p}\left|(x_2 - x_1)(x_3 - x_1)\begin{vmatrix} 1 & x_2 + x_1 \\ 1 & x_3 + x_1 \end{vmatrix}\right| =$$

$$= \frac{1}{8p}\left|(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)\right|.$$

A similar calculation yields

$$[B_1B_2B_3] = \frac{1}{2}\left|\begin{vmatrix} \dfrac{x_2 + x_3}{2} & \dfrac{x_2x_3}{4p} & 1 \\ \dfrac{x_3 + x_1}{2} & \dfrac{x_3x_1}{4p} & 1 \\ \dfrac{x_1 + x_2}{2} & \dfrac{x_1x_2}{4p} & 1 \end{vmatrix}\right| = \frac{1}{16p}\left|\begin{vmatrix} x_2 + x_3 & x_2x_3 & 1 \\ x_3 + x_1 & x_3x_1 & 1 \\ x_1 + x_2 & x_1x_2 & 1 \end{vmatrix}\right| =$$

$$= \frac{1}{16p}\left|\begin{vmatrix} x_2+x_3 & x_2x_3 & 1 \\ x_2-x_1 & x_3(x_2-x_1) & 0 \\ x_1-x_3 & x_2(x_1-x_3) & 0 \end{vmatrix}\right| = \frac{1}{16p}\left|(x_2-x_1)(x_1-x_3)\begin{vmatrix} x_2+x_3 & x_2x_3 & 1 \\ 1 & x_3 & 0 \\ 1 & x_2 & 0 \end{vmatrix}\right| =$$

$$= \frac{1}{16p}\left|\,(x_2-x_1)\,(x_3-x_1)\,(x_3-x_2)\,\right|.$$

We conclude that $\boxed{\dfrac{[A_1A_2A_3]}{[B_1B_2B_3]} = 2}$.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let x_i be the value of x for A_i $(i = 1, 2, 3)$. The slope of $x^2 = 4py$ at $x = x_i$ is clearly $\frac{x_i}{2p}$, or l_i has equation $y = \frac{x_i(2x-x_i)}{4p}$. It follows that B_1 has coordinates satisfying $y = \frac{x_2(2x-x_2)}{4p} = \frac{x_3(2x-x_3)}{4p}$, or $2x(x_2 - x_3) = (x_2 + x_3)(x_2 - x_3)$. Since $x_2 \neq x_3$ (otherwise $A_2 = A_3$), it follows that $x = \frac{x_2+x_3}{2}$ and $y = \frac{x_2x_3}{4p}$ for B_1, and similarly by cyclic permutation for B_2, B_3.

Using the vector product, and since $\overrightarrow{A_1A_i} = \left(x_i - x_1, (x_i - x_1)\frac{x_i+x_1}{4p}\right)$ for $i = 2, 3$, it follows that

$$[A_1A_2A_3] = \frac{|x_1-x_2||x_2-x_3||x_3-x_1|}{8p}.$$

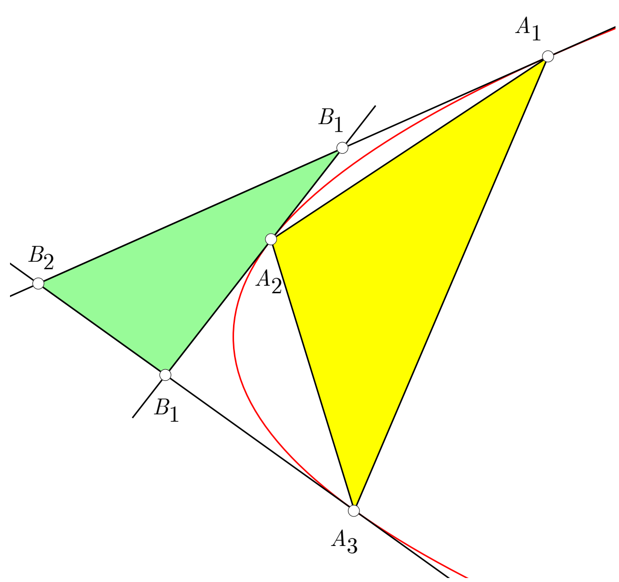
Similarly, and since $\overrightarrow{B_1B_i} = \left(\frac{x_1-x_i}{2}, \frac{x_j(x_1-x_i)}{4p}\right)$, where $\{i, j\} = \{2, 3\}$, it also follows that

$$[B_1B_2B_3] = \frac{|x_1-x_2||x_2-x_3||x_3-x_1|}{16p} = \frac{[A_1A_2A_3]}{2}.$$

The conclusion follows, the proposed ratio is always equal to 2.

Third solution by Ercole Suppa, Teramo, Italy

Let $A_1 = \left(u, \frac{1}{4p}u^2\right)$, $A_2 = \left(v, \frac{1}{4p}v^2\right)$, $A_3 = \left(w, \frac{1}{4p}w^2\right)$.



Observe that the equation of the tangent to the parabola $x^2 = 4py$ at its point $T\left(t, \frac{1}{4p}t^2\right)$ is

$$2t(x-t) - 4p\left(y - \frac{1}{4p}t^2\right) = 0 \qquad \Leftrightarrow \qquad 2tx - 4py - t^2 = 0$$

Therefore the equations of the lines ℓ_1, ℓ_2, ℓ_3 are:

$$\begin{aligned}
\ell_1 &: 2ux - 4py - u^2 = 0 \\
\ell_2 &: 2vx - 4py - v^2 = 0 \\
\ell_1 &: 2wx - 4py - w^2 = 0
\end{aligned}$$

After some algebra we get

$$B_1 = \left(\frac{v+w}{2}, \frac{vw}{4p}\right), \qquad B_2 = \left(\frac{u+w}{2}, \frac{uw}{4p}\right), \qquad B_3 = \left(\frac{u+v}{2}, \frac{uv}{4p}\right)$$

$$[A_1A_2A_3] = \frac{1}{2}\left|\det\begin{pmatrix} u & \frac{1}{4p}u^2 & 1 \\ v & \frac{1}{4p}v^2 & 1 \\ w & \frac{1}{4p}w^2 & 1 \end{pmatrix}\right| = \frac{1}{8}\left|\frac{(u-v)(u-w)(v-w)}{p}\right| \tag{*}$$

$$[B_1B_2B_3] = \frac{1}{2}\left|\det\begin{pmatrix} \frac{v+w}{2} & \frac{vw}{4p} & 1 \\ \frac{u+w}{2} & \frac{uw}{4p} & 1 \\ \frac{u+v}{2} & \frac{uv}{4p} & 1 \end{pmatrix}\right| = \frac{1}{16}\left|\frac{(u-v)(u-w)(v-w)}{p}\right| \tag{**}$$

Finally, by using (*),(**) we obtain $\frac{[A_1A_2A_3]}{[B_1B_2B_3]} = 2$, establishing the result.

Also solved by Albert Stadler, Switzerland; Roberto Bosch Cabrera, Havana, Cuba; G.R.A.20 Problem Solving Group, Roma, Italy; Daniel Campos Salas, Costa Rica.

S186. We wish to assign probabilities p_k, $k = 0, 1, 2, 3$, to random variables X_1, X_2, and X_3 taking values in the set $\{0, 1, 2, 3\}$ (some of them possibly with probability 0), such that the X_i, $i = 1, 2, 3$, will be identically distributed with $P(X_i = k) = p_k$, $k = 0, 1, 2, 3$, and $X_1 + X_2 + X_3 = 3$. Prove that this is possible if and only if $p_2 + p_3 \le 1/3$, $p_1 = 1 - 2p_2 - 3p_3$, and $p_0 = p_2 + 2p_3$.

Proposed by Shai Covo, Kiryat-Ono, Israel

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

The three random variables cannot clearly be independent, since otherwise their sum would not necessarily be 3. We need therefore to build first a joint probability distribution, calculate therefrom the individual, marginal probability distributions, then apply the condition that p_k is the same for the three variables. Note that, for the joint probability distribution, the entire universe of possible outcomes is $(0, 0, 3), (0, 3, 0), (3, 0, 0), (0, 1, 2), (1, 2, 0), (2, 0, 1), (0, 2, 1), (2, 1, 0), (1, 0, 2), (1, 1, 1)$, containing 10 events (clearly no other joint event has the three random variables adding up to 3). We will assign probabilities a_1 to a_{10} to these events. Calculating the marginal probabilities for each one of the variables to take each one of the four values $\{0, 1, 2, 3\}$, and imposing the conditions given in the problem statement, we find

$$p_0 = a_1 + a_2 + a_4 + a_7 = a_1 + a_3 + a_6 + a_9 = a_2 + a_3 + a_5 + a_8,$$

$$p_1 = a_5 + a_9 + a_{10} = a_4 + a_8 + a_{10} = a_6 + a_7 + a_{10},$$

$$p_2 = a_6 + a_8 = a_5 + a_7 = a_4 + a_9, \qquad p_3 = a_1 = a_2 = a_3.$$

Now,

$$p_0 - 2p_3 - p_2 = a_7 - a_9 = a_9 - a_8 = a_8 - a_7.$$

Now, the three last terms are equal and add up to 0, hence $a_7 = a_8 = a_9$, and $p_0 = p_2 + 2p_3$; since furthermore $p_0 + p_1 + p_2 + p_3 = 1$, elimination of p_0 yields $p_1 = 1 - 2p_2 - 3p_3$. Similarly, we find $a_4 = a_5 = a_6$. Denote therefore $u = a_1 = a_2 = a_3$, $v = a_4 = a_5 = a_6$, $w = a_7 = a_8 = a_9$ and $t = a_{10}$, hence

$$p_0 = 2u + v + w, \qquad p_1 = v + w + t, \qquad p_2 = v + w, \qquad p_3 = u,$$

and

$$p_2 + p_3 = u + v + w \le \frac{3u + 3v + 3w + t}{3} = \frac{p_0 + p_1 + p_2 + p_3}{3} = \frac{1}{3},$$

with equality iff $t = 0$, ie iff the outcome $(1, 1, 1)$ can never happen.

Note that we could have proposed the problem with the condition that the expectation value of $X_1 + X_2 + X_3$ is 3, rather than the value itself, while keeping the probabilities and the outcomes of each variable independent of each other, and two out of the three results would still be true; in this case, note that the expectation value of each X_i is $p_1 + 2p_2 + 3p_3 = 1$, while $p_0 + p_1 + p_2 + p_3 = 1$, or $p_0 = 1 - p_2 - p_3 - (1 - 2p_2 - 3p_3) = p_2 + 2p_3$. However, we may take $p_0 = \frac{7}{12}$, $p_1 = 0$, $p_2 = \frac{1}{4}$, $p_3 = \frac{1}{6}$, and the other two results would be true, but $p_2 + p_3 = \frac{5}{12} > \frac{4}{12} = \frac{1}{3}$.

Undergraduate problems

U181. Consider sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and $(B_n)_{n\geq 2}$, where $a_0 = b_0 = 1$, $a_{n+1} = a_n + b_n$, $b_{n+1} = (n^2+n+1)a_n + b_n$, $n \geq 1$. Evaluate $\lim_{n\to\infty} B_n$, where

$$B_n = \frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}}.$$

Proposed by Neculai Stanciu, George Emil Palade, Buzau, Romania

Solution by Arkady Alt, San Jose, California, USA

Proposition 1.

$$a_n = 2n!, n \geq 1.$$

Proof. By replacing (b_n, b_{n+1}) with $(a_{n+1} - a_n, a_{n+2} - a_{n+1})$ in

$$b_{n+1} = \left(n^2+n+1\right)a_n + b_n$$

we obtain

$$a_{n+2} - a_{n+1} = \left(n^2+n+1\right)a_n + a_{n+1} - a_n \iff a_{n+2} = 2a_{n+1} + \left(n^2+n\right)a_n, n \geq 0.$$

We also have

$$a_1 = a_0 + b_0 = 2, b_1 = \left(0^2+0+1\right)a_0 + b_0 = 2, a_2 = a_1 + b_1 = 4.$$

Since $a_1 = 2\cdot 1!, a_2 = 2\cdot 2!$ and from supposition $a_n = 2n!, a_{n+1} = 2\,(n+1)!$ it follows that

$$\begin{aligned}a_{n+2} &= 2\cdot 2\,(n+1)! + n\,(n+1)\cdot 2n! = 2n!\,(2n+2+n\,(n+1)) \\ &= 2n!\left(n^2+3n+2\right) = 2n!\,(n+1)\,(n+2) \\ &= 2\,(n+2)!.\end{aligned}$$

Then by math induction we have $a_n = 2n!, n \geq 1.$■ Thus,

$$B_n = \frac{(n+1)^2}{\sqrt[n+1]{2\,(n+1)!}} - \frac{n^2}{\sqrt[n]{2n!}}.$$

Proposition 2.

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e.$$

Proof. We have

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}.$$

Since

$$\left(1+\frac{1}{k}\right)^k < e < \left(1+\frac{1}{k}\right)^{k+1} \iff \frac{(k+1)^k}{k^k} < e < \frac{(k+1)^{k+1}}{k^{k+1}}, k\in\mathbb{N}$$

then

$$\prod_{k=1}^{n}\frac{(k+1)^k}{k^k}<e^n<\prod_{k=1}^{n}\frac{(k+1)^{k+1}}{k^{k+1}} \iff \prod_{k=1}^{n}\frac{(k+1)^k}{k^{k-1}}\cdot\prod_{k=1}^{n}\frac{1}{k}<e^n<\prod_{k=1}^{n}\frac{(k+1)^{k+1}}{k^k}\cdot\prod_{k=1}^{n}\frac{1}{k}$$

$$\iff \frac{(n+1)^n}{n!}<e^n<\frac{(n+1)^{n+1}}{n!} \iff \left(\frac{n+1}{e}\right)^n<n!<\frac{(n+1)^{n+1}}{e^n}$$

$$\iff \frac{n+1}{e}<\sqrt[n]{n!}<\frac{(n+1)\sqrt[n]{n+1}}{e}$$

$$\iff \frac{ne}{(n+1)\sqrt[n]{n+1}}<\frac{n}{\sqrt[n]{n!}}<\frac{ne}{n+1}.$$

Hence,

$$\lim_{n\to\infty}\frac{ne}{n+1}=\lim_{n\to\infty}\frac{ne}{(n+1)\sqrt[n]{n+1}}=e \implies \lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=e.\blacksquare$$

Proposition 3.

$$\lim_{n\to\infty}\left(\frac{n^2}{\sqrt[n]{n!}}-\frac{n^2}{\sqrt[n]{2n!}}\right)=e\ln 2.$$

Proof. Since $\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=e$ and $\lim_{n\to\infty}n\left(\sqrt[n]{2}-1\right)=\ln 2\lim_{n\to\infty}\frac{n}{\ln 2}\left(e^{\frac{\ln 2}{n}}-1\right)=\ln 2$ then

$$\lim_{n\to\infty}\left(\frac{n^2}{\sqrt[n]{n!}}-\frac{n^2}{\sqrt[n]{2n!}}\right)=\lim_{n\to\infty}\frac{n^2}{\sqrt[n]{2n!}}\left(\sqrt[n]{2}-1\right)=\lim_{n\to\infty}\frac{1}{\sqrt[n]{2}}\cdot\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\cdot\lim_{n\to\infty}n\left(\sqrt[n]{2}-1\right)=e\ln 2.\blacksquare$$

Let $\delta_n=\frac{n^2}{\sqrt[n]{2n!}}-\frac{n^2}{\sqrt[n]{n!}}+e\ln 2$ then

$$\lim_{n\to\infty}\delta_n=0$$

and

$$\frac{n^2}{\sqrt[n]{2n!}}=\frac{n^2}{\sqrt[n]{n!}}-e\ln 2+\delta_n.$$

Let $\overline{B}_n=\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}$, then

$$B_n=\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-e\ln 2+\delta_{n+1}-\left(\frac{n^2}{\sqrt[n]{n!}}-e\ln 2+\delta_n\right)=\overline{B}_n+\delta_{n+1}-\delta_n$$

and, therefore,

$$\lim_{n\to\infty}\left(B_n-\overline{B}_n\right)=0.$$

To prove following proposition we need inequality for $n!$ which is more accurate then the one above - which help us find

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}.$$

For our purpose it the following is sufficient.

Lemma.(The proof is in Appendix). *There is positive constant a such that for any $n\in\mathbb{N}$ the following inequality holds*

$$\left(\frac{n}{e}\right)^n\sqrt{an}<n!<\left(\frac{n}{e}\right)^n\sqrt{an}\cdot e^{\frac{1}{12n}}. \tag{1}$$

Proposition 4.

$$\lim_{n\to\infty}(n+1)\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}-\frac{n}{\sqrt[n]{n!}}\right)=0.$$

Proof. Since

$$(1) \iff \frac{n}{e}\sqrt[2n]{an} < \sqrt[n]{n!} < \frac{n}{e}\sqrt[2n]{an}\cdot e^{\frac{1}{12n^2}} \iff \frac{e^{1-\frac{1}{12n^2}}}{\sqrt[2n]{an}} < \frac{n}{\sqrt[n]{n!}} < \frac{e}{\sqrt[2n]{an}}$$

we have

$$\begin{aligned}\frac{n+1}{\sqrt[n+1]{(n+1)!}} - \frac{n}{\sqrt[n]{n!}} &< \frac{e}{\sqrt[2(n+1)]{a\,(n+1)}} - \frac{e^{1-\frac{1}{12n^2}}}{\sqrt[2n]{an}} = \frac{e}{\sqrt[2n]{an}e^{\frac{1}{12n^2}}}\left(\frac{\sqrt[2n]{an}e^{\frac{1}{12n^2}}}{\sqrt[2(n+1)]{a\,(n+1)}} - 1\right)\\ &= \frac{e}{\sqrt[2n]{an}e^{\frac{1}{12n^2}}}\left(e^{\alpha_n} - 1\right),\end{aligned}$$

where

$$\alpha_n = \ln\frac{\sqrt[2n]{an}e^{\frac{1}{12n^2}}}{\sqrt[2(n+1)]{a\,(n+1)}} = \frac{1}{2}\left(\frac{\ln an}{n} - \frac{\ln a\,(n+1)}{n+1} + \frac{1}{6n^2}\right).$$

Also note that $\frac{n+1}{\sqrt[n+1]{(n+1)!}} > \frac{n}{\sqrt[n]{n!}}, n \in \mathbb{N}$. Indeed,

$$\begin{aligned}\frac{n+1}{\sqrt[n+1]{(n+1)!}} > \frac{n}{\sqrt[n]{n!}} &\iff \frac{(n+1)^{n(n+1)}}{((n+1)!)^n} > \frac{n^{n(n+1)}}{(n!)^{n+1}}\\ &\iff \frac{(n+1)^{n^2}}{(n!)^n} > \frac{n^{n(n+1)}}{(n!)^{n+1}}\\ &\iff n! > \frac{n^{n(n+1)}}{(n+1)^{n^2}}\\ &\iff n! > \left(\frac{n}{\left(1+\frac{1}{n}\right)^n}\right)^n \impliedby n! > \left(\frac{n}{e}\right)^n.\end{aligned}$$

Thus,

$$0 < (n+1)\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}} - \frac{n}{\sqrt[n]{n!}}\right) < \frac{(n+1)\,e}{\sqrt[2n]{an}e^{\frac{1}{12n^2}}}\left(e^{\alpha_n}-1\right).$$

Since $\lim\limits_{n\to\infty}\frac{e}{\sqrt[2n]{an}e^{\frac{1}{12n^2}}} = e$ and $\lim\limits_{n\to\infty}\alpha_n = 0$ yields $\lim\limits_{n\to\infty}\frac{e^{\alpha_n}-1}{\alpha_n} = 1$ then it suffices to prove that

$$\lim_{n\to\infty}(n+1)\,\alpha_n = 0.$$

We have

$$\begin{aligned}\lim_{n\to\infty}(n+1)\,\alpha_n &= \frac{1}{2}\lim_{n\to\infty}(n+1)\left(\frac{\ln an}{n} - \frac{\ln a\,(n+1)}{n+1} + \frac{1}{6n^2}\right)\\ &= \frac{1}{2}\lim_{n\to\infty}(n+1)\left(\frac{\ln an}{n} - \frac{\ln an}{n+1} + \frac{\ln an}{n+1} - \frac{\ln a\,(n+1)}{n+1} + \frac{1}{6n^2}\right)\\ &= \frac{1}{2}\lim_{n\to\infty}(n+1)\left(\frac{\ln an}{n\,(n+1)} - \frac{\ln\left(1+\frac{1}{n}\right)}{n+1} + \frac{1}{6n^2}\right)\\ &= \frac{1}{2}\left(\lim_{n\to\infty}\frac{\ln an}{n} - \lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right) + \lim_{n\to\infty}\frac{n+1}{6n^2}\right) = 0.\blacksquare\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\overline{B}_n = \lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2+n}{\sqrt[n]{n!}} + \frac{n}{\sqrt[n]{n!}}\right) = \lim_{n\to\infty}(n+1)\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}} - \frac{n}{\sqrt[n]{n!}}\right) + \lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} = e$$

and, therefore,

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty} \left(\left(B_n - \overline{B}_n\right) + \overline{B}_n\right) = \lim_{n\to\infty} \left(\left(B_n - \overline{B}_n\right)\right) + \lim_{n\to\infty} \overline{B}_n = e.$$

Appendix. Proof of the Lemma.

1. First we will prove inequality

$$e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}},$$

for any $n \in \mathbb{N}$.

Note that sequence $\left(\left(1+\frac{2}{n-1}\right)^n\right)_{n\geq 2}$ is decreasing. Indeed,

$$\begin{aligned}\left(1+\frac{2}{n-1}\right)^n > \left(1+\frac{2}{n}\right)^{n+1} &\iff \left(\frac{n+1}{n-1}\right)^n > \left(\frac{n+2}{n}\right)^{n+1}\\ &\iff \left(\frac{n\,(n+1)}{(n-1)\,(n+2)}\right)^n > 1+\frac{2}{n}\\ &\iff \left(1+\frac{2}{(n-1)\,(n+2)}\right)^n > 1+\frac{2}{n}.\end{aligned}$$

Applying the inequality $(1+a)^n \geq 1+na+\frac{n(n-1)}{2}a^2, a>0, n\in\mathbb{N}$ to $a=\frac{2}{(n-1)(n+2)}$ yields

$$\begin{aligned}\left(1+\frac{2}{(n-1)\,(n+2)}\right)^n &\geq 1+\frac{2n}{(n-1)\,(n+2)}+\frac{n\,(n-1)}{2}\cdot\frac{4}{(n-1)^2\,(n+2)^2}\\ &= 1+\frac{2n}{(n-1)\,(n+2)}+\frac{2n}{(n-1)\,(n+2)^2}\end{aligned}$$

and

$$\begin{aligned}1+\frac{2n}{(n-1)\,(n+2)}+\frac{2n}{(n-1)\,(n+2)^2} > 1+\frac{2}{n} &\iff (n+3)\,n^2 > (n-1)\,(n+2)^2\\ &\iff n^3+3n^2 > n^3+3n^2-4\\ &\iff 4>0.\blacksquare\end{aligned}$$

Since $\left(1+\frac{2}{n-1}\right)^n > \left(1+\frac{2}{n}\right)^{n+1} > \left(1+\frac{2}{n+1}\right)^{n+2}$ then

$$\left(1+\frac{2}{n-1}\right)^n > \left(1+\frac{2}{n+1}\right)^{n+2}. \tag{2}$$

By replacing n with $2n+1$ in **(2)** yields

$$\left(1+\frac{1}{n}\right)^{2n+1} > \left(1+\frac{1}{(n+1)}\right)^{2n+3} \iff \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} > \left(1+\frac{1}{n+1}\right)^{n+1+\frac{1}{2}}$$

and, since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} = e$ then

$$e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}},\ n\in\mathbb{N}.$$

2. Consider Taylor representation for the function $\ln\frac{1+x}{1-x}$ on $x\in(0,1)$:

$$\begin{aligned}\ln\frac{1+x}{1-x}&=\ln(1+x)-\ln(1-x)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k}-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(-x)^k}{k}\\&=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k}+\sum_{k=1}^{\infty}\frac{x^k}{k}\\&=\sum_{k=1}^{\infty}\frac{\left((-1)^{k-1}+1\right)x^k}{k}\\&=\sum_{k=1}^{\infty}\frac{2x^{2k-1}}{2k-1}.\end{aligned}$$

Since $\frac{n+1}{n}=\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}$ then by replacing x with $\frac{1}{2n+1}$ in $\ln\frac{1+x}{1-x}=\sum\limits_{k=1}^{\infty}\frac{2x^{2k-1}}{2k-1}$ we obtain

$$\begin{aligned}\ln\frac{n+1}{n}&=\sum_{k=1}^{\infty}\frac{2}{2k-1}\cdot\frac{1}{(2n+1)^{2k-1}}=\frac{2}{2n+1}\left(1+\sum_{k=2}^{\infty}\frac{1}{(2k-1)(2n+1)^{2(k-1)}}\right)\\&<\frac{2}{2n+1}\left(1+\sum_{k=1}^{\infty}\frac{1}{3(2n+1)^{2k}}\right)=\frac{2}{2n+1}\left(1+\frac{1}{3}\cdot\frac{\frac{1}{(2n+1)^2}}{1-\frac{1}{(2n+1)^2}}\right)\\&=\frac{1}{n+\frac{1}{2}}\left(1+\frac{1}{12n(n+1)}\right)\implies\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}<e^{1+\frac{1}{12n(n+1)}}.\end{aligned}$$

Thus we have the double inequality

$$e<\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}<e^{1+\frac{1}{12n(n+1)}}.$$

Since

$$\begin{aligned}e<\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}&\iff e<\frac{(n+1)^{n+\frac{1}{2}}}{n^{n+\frac{1}{2}}}\iff e(n+1)<\frac{(n+1)^{n+1+\frac{1}{2}}}{n^{n+\frac{1}{2}}}\\&\iff\frac{e^{n+1}(n+1)!}{e^n n!}<\frac{(n+1)^{n+1+\frac{1}{2}}}{n^{n+\frac{1}{2}}}\\&\iff\frac{n^{n+\frac{1}{2}}}{e^n n!}<\frac{(n+1)^{(n+1)+\frac{1}{2}}}{e^{n+1}(n+1)!},n\ge1\end{aligned}$$

then the sequence $(a_n)_{n\ge1}$ where $a_n=\frac{n^{n+\frac{1}{2}}}{e^n n!}$, is increasing. Since

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}<e^{1+\frac{1}{12n(n+1)}}&\iff\frac{(n+1)^{n+1+\frac{1}{2}}}{n^{n+\frac{1}{2}}}<\frac{(n+1)!}{n!}\cdot e^{n+1-n+\frac{1}{12n}-\frac{1}{12(n+1)}}\\&\iff\frac{(n+1)^{n+1+\frac{1}{2}}}{(n+1)!e^{n+1-\frac{1}{12(n+1)}}}<\frac{n^{n+\frac{1}{2}}}{n!e^{n-\frac{1}{12n}}}\end{aligned}$$

then the sequence $(b_n)_{n\ge1}$, where $b_n=\frac{n^{n+\frac{1}{2}}e^{\frac{1}{12n}}}{n!e^n}=a_ne^{\frac{1}{12n}}$ is decreasing. Since $e^{-1}=a_1\le a_n<b_n\le b_1=e^{-\frac{11}{12}}$ and $b_n=a_ne^{\frac{1}{12n}}$ then both sequences converge to the same limit. Let $\frac{1}{a}=\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n$

then $e^{\frac{11}{12}} < a < e$ and

$$a_n < \frac{1}{a} < b_n \iff \frac{1}{b_n} < a < \frac{1}{a_n} \iff \frac{n!e^n}{n^{n+\frac{1}{2}}e^{\frac{1}{12n}}} < a < \frac{e^n n!}{n^{n+\frac{1}{2}}}$$
$$\iff \left(\frac{n}{e}\right)^n \sqrt{an} < n! < \left(\frac{n}{e}\right)^n \sqrt{an} \cdot e^{\frac{1}{12n}}.$$

Remark. Using Vallis' formula we can obtain $a = 2\pi$ but here we don't need that.

Also solved by Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Daniel Lasaosa, Universidad Pública de Navarra, Spain; G. C. Greubel, Newport News, USA; Roberto Bosch Cabrera, Havana, Cuba; Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy.

U182. Find all continuous functions f on $[0,1]$ such that $f(x) = c$ if $x \in \left[0, \frac{1}{2}\right]$ and $f(x) = f(2x-1)$ if $x \in \left(\frac{1}{2}, 1\right]$, where c is a given constant.

Proposed by Arkady Alt, San Jose, California, USA

Solution by Vahagn Aslanyan, Yerevan, Armenia

We will show that for every positive integer n

$$f(x) = c, x \in \left[0, 1 - \frac{1}{2^n}\right].$$

We will induct on n. For $n = 1$ it is the condition of the problem and therefore is true. Now suppose it is true for n. Let us prove it for $n+1$. If $x \in [\frac{1}{2}, 1 - \frac{1}{2^{n+1}}]$, then $2x - 1 \in [0, 1 - \frac{1}{2^n}] \Rightarrow$ (by induction hypothesis) $f(2x-1) = c \Rightarrow f(x) = c$. Thus $f(x) = c$ if $x \in [0,1)$. Because f is continuous we have

$$f(1) = f\left(\lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right)\right) = \lim_{n\to\infty} f\left(\left(1 - \frac{1}{2^n}\right)\right) = \lim_{n\to\infty} c = c.$$

Then $f(x) = c$ for $x \in [0,1]$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Roberto Bosch Cabrera, USA; Emanuele Natale, Università di Roma "Tor Vergata", Roma, Italy; Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy.

U183. Let m and n be positive integers. Prove that

$$\sum_{k=0}^{n} \frac{1}{k+m+1}\binom{n}{k} \leq \frac{(m+2n)^{m+n+1} - n^{m+n+1}}{(m+n+1)(m+n)^{m+n+1}}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Daniel Campos Salas, Costa Rica

Applying the weighted AM-GM we have that

$$\left(x + \frac{n}{m+n}\right)^{m+n} = \left(\frac{mx + n(1+x)}{m+n}\right)^{m+n} \geq x^m(1+x)^n,$$

for $x \geq 0$. Note

$$\int_0^1 x^m(1+x)^n dx = \int_0^1 \left(\sum_{k=0}^{n} \binom{n}{k} x^{m+k}\right) dx = \sum_{k=0}^{n} \frac{1}{k+m+1}\binom{n}{k},$$

and that

$$\int_0^1 \left(x + \frac{n}{m+n}\right)^{m+n} dx = \frac{(m+2n)^{m+n+1} - n^{m+n+1}}{(m+n+1)(m+n)^{m+n+1}}.$$

Therefore the result follows from integrating both sides of the inequality, and we're done.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

U184. Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions such that $\int\limits_a^b f(x)dx = 0$. Prove that there is some $c \in (a, b)$ satisfying

$$f'(c) \int\limits_c^b g(x)dx + g'(c) \int\limits_c^b f(x)dx = 2f(c)g(c).$$

Proposed by Duong Viet Thong, National Economics University, Vietnam

First solution by Ajat Adriansyah, Indonesia

Define

$$S(t) = \left(\int_a^t f(x)\,dx\right)\left(\int_b^t g(x)\,dx\right)$$

By condition $\int_a^b f(x)\,dx = 0$ we have $S(a) = S(b) = 0$, thus by Rolle's Theorem there exists $\xi \in (a, b)$ such that $S'(\xi) = 0$. Notice that

$$S'(t) = f(t)\int_b^t g(x)\,dx + g(t)\int_a^t f(x)\,dx$$

and also by condition $\int_a^b f(x)\,dx = 0$ we have $S'(b) = 0 = S'(\xi)$, therefore by Rolle's Theorem we can find $c \in (\xi, b) \subset (a, b)$ such that $S''(c) = 0$, that is

$$\begin{aligned} S''(c) &= -f'(c)\int_c^b g(x)\,dx + g(c)f(c) + g'(c)\int_a^c f(x)\,dx + g(c)f(c) \\ &= 2f(t)g(t) - f'(c)\int_c^b g(x)\,dx + g'(c)\left(\int_a^b f(x)\,dx - \int_c^b f(x)\,dx\right) \\ &= 2f(t)g(t) - f'(c)\int_c^b g(x)\,dx - g'(c)\int_c^b f(x)\,dx \end{aligned}$$

Therefore $S''(c) = 0$ implies the

$$2f(c)g(c) = f'(c)\int_c^b g(x)\,dx + g'(c)\int_c^b f(x)\,dx$$

Second solution by Evangelos Mouroukos, Agrinio, Greece;

Consider the functions $F, G, H : [a, b] \Rightarrow \mathbb{R}$ defined by

$$F(x) = \int_x^b f(t)dt,$$

$$G(x) = \int_x^b g(t)dt,$$

$$H(x) = F(x)G(x)$$

for $x \in [a, b]$. We have that $F(a) = F(b) = 0$, hence also $H(a) = H(b) = 0$. An application of Rolle's Theorem to H yields the existence of a number $x_0 \in (a, b)$ such that $H'(x_0) = 0$. Since

$$H'(x) = F'(x)G(x) + F(x)G'(x) = -f(x)G(x) - F(x)g(x),$$

it follows that $H'(b)=0$. Applying Rolle's Theorem to H' on the interval $[x_0,b]$, we obtain the existence of a number $c\in(a,b)$ such that $H''(c)=0$. Since

$$H''(x) = -f'(x)G(x) + f(x)g(x) - g'(x)F(x) + f(x)g(x) =$$

$$= -f'(x)\int_x^b g(t)dt - g'(x)\int_x^b f(t)dt + 2f(x)g(x),$$

substituting $x=c$ we find that

$$f'(c)\int_c^b g(x)dx + g'(c)\int_c^b f(x)dx = 2f(c)g(c)$$

as desired.

Third solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

Consider the functions

$$f(x)\int_b^x g(y)dy + g(x)\int_b^x f(y)dy \doteq H(x), \qquad F(x) = \int_b^x f(y)dy$$

Integrating by parts we get

$$\int_b^a H(x)dx = \left[F(x)\int_b^x g(y)dy\right]\Big|_b^a - \int_b^a F(y)g(y)dy + \int_b^a g(x)F(x)dx = 0$$

since both $F(a)$ and $F(b)$ are zero. It follows that $H(\xi)=0$ for some $a<\xi<b$. Since trivially $H(b)=0$, Rolle's theorem yields that for a point $c\in(\xi,b)$, $H'(c)=0$ holds namely

$$H'(c) = f'(c)\int_b^c g(x)dx + g'(c)\int_b^c f(x)dx + 2f(c)g(c) = 0$$

and we are done.

Also solved by Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

U185. Determine if there is a non-constant complex analytic function satisfying the conditions:

(i) $f(f(z)) = f(z)$ for all complex numbers z

(ii) there is a complex number z_0, such that $f(z_0) \neq z_0$.

Proposed by Harun Immanuel, Airlangga University, Indonesia

First solution by Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy

The answer is no. Suppose that f is an analytic non-costant function on $\mathbb{C}$ then $f(\mathbb{C})$ is dense in $\mathbb{C}$. Indeed, take $a \in \mathbb{C}$, we want to show that a is an accumulation point of $f(\mathbb{C})$. If $a \in f(\mathbb{C})$ it's clear. If not then $g(z) := \frac{1}{(a-f(z))}$ is an analytic function on $\mathbb{C}$, and if a is not an accumulation point of $f(\mathbb{C})$ then g is bounded. Hence, by Liouville theorem, g is constant and therefore f is costant too.

Now, by (i), $f(f(z)) = f(z)$ for all $z \in \mathbb{C}$ which means that f acts as the identity on $f(\mathbb{C})$. Hence, by the previous remark, $f(z) = z$ holds in a dense subset of $\mathbb{C}$ and, by continuity, we have that $f(z) = z$ for all $z \in \mathbb{C}$ which si in contradiction with (ii).

Second solution by Emanuele Natale, Università di Roma "Tor Vergata", Roma, Italy

Since f is complex analytic and because of property (ii), it cannot be constant in any open set and $\frac{df(z)}{dz} = 0$ at most for numerable isolated z; moreover, we can take its derivative

$$\frac{df(z)}{dz} = \frac{df(f(z))}{dz} = \frac{\partial f(f(z))}{\partial f(z)} \frac{df(z)}{dz}$$

from which we see that $\frac{\partial f(z)}{\partial z} = 1$ except for at most numerable isolated z;

but then, by the continuity of $\frac{df(z)}{dz}$, $\frac{\partial f(z)}{\partial z} = 1$ must be true for every complex number z and then it is not possible that there is a complex number z_0 such that $f(z_0) \neq z_0$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

U186. Let $(A, +, \cdot)$ be a finite ring of characteristic ≥ 3 such that $1 + x \in U(A) \bigcup \{0\}$ for each $x \in U(A)$. Prove that A is a field.

Proposed by Sorin Radulescu, Aurel Vlaicu College, Bucharest and Mihai Piticari, Dragos Voda College, Campulung Moldovenesc, Romania

Solution by by the authors

Solution by the authors. We need the following auxiliary results.

Lemma 1. *If $(A, +, \cdot)$ is finite ring such that $U(A) \cup \{0\}$ is a field, then it has no nilpotent elements.*

Proof. Let $a \neq 0$ be a nilpotent element. Then $1 + a$ is invertible, hence $1 + a \in U(A)$. It follows $a = (1 + a) - 1 \in U(A)$, that is the element a is invertible, not possible. $\square$

Lemma 2. *Let $(A, +, \cdot)$ be a finite ring. Then $(A, +, \cdot)$ is a field if and only if A has no nilpotent elements and the equation $x^2 = x$ has only solutions 0 and 1.*

Proof. $(A, +, \cdot)$ is a field, then clearly it has no nilpotent elements and the equation $x^2 = x$ has only solutions 0 and 1. Conversely, because A is finite, it follows that for any $a \neq 0$ there are positive integers $p > q$ such that $a^p = a^q$. Then, we can write $a^q = a^{2p-q} = a^{3p-2q} = \cdots = a^{kp-(k-1)q} = \cdots$. Let $s = kp - (k-1)q$ such that $s > 2q$. We have $a^q = a^s$ and by multiplication by a^{s-2q} it follows $a^{2(s-q)} = a^{s-q}$. We get $a^{2n} = a^n$, where $n = s - q$.

Now, assume that $a \in A, a \neq 0$. Then $a^{2n} = a^n$ for some positive integer n. That is $(a^n)^2 = a^n$ and since the equation $x^2 = x$ has only solutions 0 and 1, it follows $a^n = 0$ or $a^n = 1$. The first situation is not possible since A has no nilpotent elements, hance we have $a^n = 1$, that is a is invertible. Therefore $(A, +, \cdot)$ is a field. $\square$

If $x, y \in U(A) \cup \{0\}$ then it is clear that $xy \in U(A) \cup \{0\}$. Let $x, y \in U(A), x \neq y$. From $xy^{-1} \in U(A)$ it follows $1 - xy^{-1} \in U(A)$, hence $(1 - xy^{-1})y \in U(A)$, that is $y - x \in U(A)$. This means that $U(A) \cup \{0\}$ is stable with respect the summation, hence $U(A) \cup \{0\}$ is a field. According to Lemma 1 it follows that A has no nilpotent elements. Let $a \in A$ be a solution to the equation $x^2 = x$. Then $a^2 = a$ and $(2a-1)^2 = 4a^2 - 4a + 1 = 1$, hence $2a - 1$ is invertible, i.e.$2a - 1 \in U(A)$. We get $2a = (2a-1) + 1 \in U(A)$. If $2a = 0$, then $a = 0$. If $2a = c$, where $c \in U(A)$, then $a = 2^{-1}c$, that is a is invertible. From the relation $a^2 = a$ it follows $a = 1$, and we are done via the result in Lemma 2.

Remark. The result in the problem is not true if the characteristic of A is 2. Indeed, for the ring $A = (\mathbb{Z}_2)^n$, we have $U(A) \cup \{0\} = \{0, 1\} \cong \mathbb{Z}_2$ but A is not a field.

Olympiad problems

O181. Let a, b, c be the sidelengths of a triangle. Prove that

$$\sqrt{\frac{abc}{-a+b+c}}+\sqrt{\frac{abc}{a-b+c}}+\sqrt{\frac{abc}{a+b-c}} \geq a+b+c.$$

Proposed by Titu Andreescu, Unversity of Texas at Dallas, USA and Gabriel Dospinescu, Ecole Normale Superieure, France

First solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

We change variables

$$a=\frac{1}{2}(x+y-z),\ b=\frac{1}{2}(y+z-x),\ c=\frac{1}{2}(z+x-y),$$

or

$$y=a+b,\ z=b+c,\ x=a+c$$

This sets the inequality as

$$\sqrt{(x+y)(y+z)(z+x)}\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{y}}\right) \geq 2\sqrt{2}(x+y+z)$$

Squaring we get

$$(x+y)(y+z)(z+x)\sum_{\text{cyc}}\left(\frac{1}{x}+\frac{2}{\sqrt{xy}}\right) \geq 8(x+y+z)^2$$

By AGM $\sqrt{xy} \leq (x+y)/2$ whence

$$(x+y)(y+z)(z+x)\sum_{\text{cyc}}\left(\frac{1}{x}+\frac{4}{x+y}\right) \geq 8(x+y+z)^2$$

By clearing the denominators we get

$$\sum_{\text{sym}} x^3y^2 \geq \sum_{\text{sym}} x^3yz$$

which follows also by the AGM since

$$(x^3y^2+x^3z^2)/2 \geq x^3yz, \quad \text{and cyclic}$$

and we are done.

Second solution by Daniel Campos Salas, Costa Rica

Applying Hlder's inequality it follows that

$$\left(\sum_{cyc}\sqrt{\frac{abc}{-a+b+c}}\right)^2\left(\sum_{cyc}\frac{a^2(-a+b+c)}{bc}\right) \geq \left(\sum_{cyc} a\right)^3.$$

It is enough to prove that

$$\sum_{cyc} a \geq \sum_{cyc} \frac{a^2(-a+b+c)}{bc}.$$

It's easy to verify that the last inequality is equivalent to Schur's inequality

$$\sum_{cyc} a^2(a-b)(a-c) \geq 0,$$

and we're done.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Andrea Fanchini, Cantu, Italy.

O182. On side BC of triangle ABC consider m points, on CA n points, and on AB s points. Join the points from the sides AB and AC with the points on side BC. Determine the maximum number of the points of intersection situated in the interior of triangle ABC.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote respectively by M, N, S the set of points on sides BC, CA, AB. Since the problem statement says nothing about the locations of points on these sides, we will assume that no three lines intersect at a point inside the triangle; otherwise, for each one of the lines PQ (with $P \in M$ and $Q \in N \cup S$) that concurs at a given point with two or more other lines, we can move ever so slightly P on BC until it passes through no point where two other lines concur, thus increasing the number of intersection points (line PQ will now meet each one of the lines with which it concurred at different intersection points, instead of at one single intersection point).

Note now that we can establish a bijection between the pairs of pairs of points, one pair of points taken from M, another taken from $N \cup S$, and the number of intersection points in the interior of ABC. Indeed, exactly two lines pass through any intersection point inside the triangle. Each one of these two lines passes through one point on BC, and another point either on CA or AB. Note also that the two points on BC cannot coincide, neither can the points on CA, AB, otherwise either the lines would be the same, or their intersection point would be on the border, not in the interior, of ABC. We may thus associate exactly one pair of pairs of points to each intersection point.

Reciprocally, consider any pair of points $P, Q \in M$, and any pair of points $X, Y \in N \cup S$. Note that PQX is a triangle contained entirely inside ABC, except for segment PQ and vertex X, who are on its border. Therefore, Y is outside PQX, since it is a point on the border of ABC, not on BC, and distinct from X. Similarly, X is outside PQY, P is outside QXY, and Q is outside PXY, or P, Q, X, Y are the vertices of a convex quadrilateral. Note that $PQ \cap XY$ is clearly on line BC, while out of PX, QY and PY, QX, two will be sides of the quadrilateral and will thus meet outside ABC, and two will be their diagonals and will meet in the interior of ABC; no other intersection points of any two lines defined by these pairs of points may be in the interior of ABC. We may thus associate exactly one intersection point in the interior of ABC to each such pair of pairs of points.

Since the number of pairs of pairs of points is $\binom{m}{2}\binom{n+s}{2}$, this is also the maximum number of intersection points in the interior of ABC.

Second solution by Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy

First, we notice that the maximum number of points of intersection can be obtained as soon as the intersections occur between no more than two segments. So we just count all the possible intersections between the segments. To achieve this we first count the intersections between the sides BC and CA, then between BC and AB and finally between AB and CA. Between BC and CA there are $\binom{n}{2}\binom{m}{2}$ points, between BC and AB there are $\binom{s}{2}\binom{m}{2}$ points, and between AB and CA there are $sn\binom{m}{2}$ points. Hence the maximum number of the points of intersection we are looking for is given by the sum of these three numbers:

$$\binom{m}{2}\left(\binom{n}{2}+\binom{s}{2}+sn\right).$$

O183. Evaluate

$$\sum_{k=1}^{2010} \tan^4\left(\frac{k\pi}{2011}\right).$$

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

First solution by G.R.A.20 Math Problems Group, Roma, Italy

We will show that if n is an odd positive integer then

$$S_n := \sum_{k=1}^{n-1} \tan^4\left(\frac{k\pi}{n}\right) = 2\binom{n}{2}^2 - 4\binom{n}{4}$$

and for $n = 2011$ it gives 5451632830730. It is easy to verify that

$$\prod_{k=0}^{n-1}(x - t_k) = \operatorname{Re}\left((x+i)^n\right) = \sum_{k=0}^{(n-1)/2} \binom{n}{2k} x^{n-2k}(-1)^k.$$

where $t_k = \tan\left(\frac{k\pi}{n}\right)$. Hence, letting

$$\begin{aligned}
\prod_{k=0}^{n-1}\left(x^4 - t_k^4\right) &= -\prod_{k=0}^{n-1}(x - t_k)(-x - t_k)(ix - t_k)(-ix - t_k) \\
&= \left(\sum_{k=0}^{(n-1)/2} \binom{n}{2k} x^{n-2k}(-1)^k\right)^2 \left(\sum_{k=0}^{(n-1)/2} \binom{n}{2k} x^{n-2k}\right)^2 \\
&= x^{4n} - \left(2\binom{n}{2}^2 - 4\binom{n}{4}\right) x^{4n-4} + R(x),
\end{aligned}$$

where $R(x)$ is the sum of all the terms of lesser powers. On the other hand

$$\prod_{k=0}^{n-1}\left(x^4 - t_k^4\right) = x^{4n} - S_n x^{4n-4} + R(x)$$

and the desired result follows.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

From the de Moivre formula, it follows that 2011α is an integer multiple of π iff

$$0 = \sin\alpha \sum_{k=0}^{1005} \binom{2011}{2k+1}(-1)^k x^{1005-k}(1-x)^k = \sin\alpha P(x) = \sin\alpha \sum_{j=0}^{1005} c_j x^j,$$

where we have defined $x = \cos^2\alpha$. Note also that, if α itself is not an integer multiple of π, then $\sin\alpha \neq 0$, or the 1005 real roots of $P(x) = 0$ are the 1005 distinct values of $\cos^2\alpha$ such that 2011α is an integral multiple of $\frac{\pi}{2011}$, ie the 1005 distinct values that $\cos^2\alpha$ may take for $\alpha = \frac{k\pi}{2011}$ for $k = 1, 2, \ldots, 1005$, which are the same than the 1005 distinct values of $\cos^2\alpha$ for $\alpha = \frac{k\pi}{2011}$ for $k = 2010, 2009, \ldots, 1006$, since clearly $\cos\left(\frac{k\pi}{2011}\right) = -\cos\left(\frac{(2011-k)\pi}{2011}\right)$. Denote these 1005 distinct values of $\cos^2\alpha$ by $x_1, x_2, \ldots, x_{1005}$, and note that, since the 0-th, first and second degree coefficients of $(1-x)^k$ are respectively $1, -k, \binom{k}{2}$, we have

$$c_0 = \binom{2011}{2011}(-1)^{1005} = -1,$$

$$c_1 = \binom{2011}{2011}(-1)^{1005}(-1005) + \binom{2011}{2009}(-1)^{1004} = \frac{2011^2-1}{2},$$

$$c_2 = -\binom{2011}{2011}\binom{1005}{2} + \binom{2011}{2009}(-1004) - \binom{2011}{2007} = -\frac{2011^4 - 10\cdot 2010^2 + 9}{24}.$$

Note next that

$$\tan^4\alpha = \frac{(1-\cos^2\alpha)^2}{\cos^4\alpha} = 1 - \frac{2}{\cos^2\alpha} + \frac{1}{\cos^4\alpha},$$

or

$$\sum_{k=1}^{2010}\tan^4\left(\frac{k\pi}{2010}\right) = 2010 - 4\sum_{k=1}^{1005}\frac{1}{x_k} + 2\sum_{k=1}^{1005}\frac{1}{x_k^2}.$$

Using Cardano-Vietta relations,

$$\sum_{k=1}^{1005}\frac{1}{x_k} = -\frac{c_1}{c_0} = \frac{2011^2-1}{2},$$

$$\sum_{k=1}^{1005}\frac{1}{x_k^2} = \left(\sum_{k=1}^{1005}\frac{1}{x_k}\right)^2 - 2\frac{c_2}{c_0} = \frac{(2011^2-1)^2}{4} - \frac{2011^4 - 10\cdot 2010^2 + 9}{12} =$$

$$= \frac{2011^4 + 2\cdot 2011^2 - 3}{6}.$$

Plugging these values,

$$\sum_{k=1}^{2010}\tan^4\left(\frac{k\pi}{2010}\right) = 2011\frac{2011^3 - 4\cdot 2011 + 3}{3}.$$

Also solved by Albert Stadler, Switzerland; Arkady Alt, San Jose, California, USA; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Andrea Fanchini, Cantu, Italy.

O184. Points A, B, C, D lie on a line in this order. Using a straight edge and a compas construct parallel lines a and b through A and B, and parallel lines c and d through C and D, such that their points of intersection are vertices of a rhombus.

Proposed by Mihai Miculita, Oradea, Romanaia

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote $u = AB$, $v = CD$, and construct any triangle XYZ two of whose sidelengths are $YZ = u$, $ZX = v$. Draw lines a, b forming an angle $\angle XYZ$ with AB, and lines c, d forming an angle $\angle ZXY$ with AB. Denote $P = b \cap c$, $Q = b \cap d$, $R = a \cap d$ and $S = a \cap c$. Clearly CBP and DAR are similar to XYZ. Consider now the line ℓ parallel to AB through P, and denote $A' = \ell \cap a$, $D' = \ell \cap d$. Clearly $PA'S$ and $D'PQ$ are again similar to XYZ. Using the Sine Law,

$$PQ = \frac{PD' \sin \angle QD'P}{\sin \angle PQD'} = \frac{v \sin \angle ZXY}{\sin \angle YZX} = \frac{uv}{XY},$$

$$PS = \frac{PA' \sin \angle SA'P}{\sin \angle PSA'} = \frac{u \sin \angle XYZ}{\sin \angle YZX} = \frac{uv}{XY},$$

where we have used that $PA' = AB = u$ and $PD' = CD = v$ because $ABPA'$ and $DCPD'$ are parallelograms. Note therefore that $PQRS$ is a parallelogram with two adjacent sides $PQ = PS$ of equal length, hence a rhombus. The conclusion follows.

Also solved by Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy

O185. Find the least integer $n \geq 2011$ for which the equation

$$x^4 + y^4 + z^4 + w^4 - 4xyzw = n$$

is solvable in positive integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Define $x^2 - y^2 = a$, $z^2 - w^2 = b$ and $xy - zw = c$, or $a^2 + b^2 + 2c^2 = x^4 + y^4 + z^4 + w^4 - 4xyzw = n$. Every square of an odd number $2m + 1$ is $4m(m + 1) + 1$, or since exactly one of $m, m + 1$ is even, every odd perfect square leaves a remainder of 1 when divided by 8, hence $2c^2$ leaves a remainder of 0 or 2 when divided by 8 because every even perfect square is a multiple of 4. By the same reason, $a = x^2 - y^2$ cannot leave a remainder of 2 when divided by 4, or a^2 and b^2 leave remainders of 0 or 1 when divided by 8. It follows that the possible remainders of n when divided by 8 are 0 (when a, b, c are all even), 1 (when one of a, b is odd, and the other two and c are even), 2 (when either a, b are odd and c is even, or a, b are even and c is odd), 3 (when c and one of a, b are odd, and the other one is even), or 4 (when a, b, c are all even). Therefore, the lowest integers $n \geq 2011$ such that the proposed equation has a solution could be $2011, 2012, 2016, 2017$. Note that $19^4 + 3 \cdot 18^4 - 4 \cdot 19 \cdot 18^3 = 2017$, or there is a solution for $n = 2017$. We will next prove that no solution exists for $n = 2011, 2012, 2016$.

By the previous arguments, since $n = 2016$ is a multiple of 8, there may be solutions only when a, b, c are all even. This means that x, y have the same parity, and so do z, w. Since xy, zw must also have the same parity, either x, y, z, w are all even, or are all odd. In the first case, define integers $x' = \frac{x}{2}$, $y' = \frac{y}{2}$, $z' = \frac{z}{2}$ and $w' = \frac{w}{2}$, hence $x'^4 + y'^4 + z'^4 + w'^4 - 4x'y'z'w' = \frac{2016}{16} = 126 \equiv 6 \pmod 8$, and no solution exists in this case. When x, y, z, w are all odd, $a = x^2 - y^2$ and $b = z^2 - w^2$ are multiples of 8. By sheer trial (we need "only" to try $a \leq b \in \{0, 8, 16, 24, 32, 40\}$), we find that no such multiples of 8 exist such that $2016 - a^2 - b^2$ is twice a perfect square, or no solutions exist in this case either.

Again by the previous arguments, since $n = 2012 \equiv 4 \pmod 8$, we would need a, b odd, or x, y have opposite parity, and xy is even, and similarly zw is also even, hence c is even, contradiction. No solution exists either with $n = 2012$.

Finally, $n = 2011 \equiv 3 \pmod 8$, or wlog b, c are odd and a is even. Since $b = z^2 - w^2$ is odd, z, w have opposite parity, hence zw is even, and xy is odd, hence x, y are both odd, and $a = x^2 - y^2$ is a multiple of 8. Again by sheer trial, we take $a = 0, 8, 16, 24, 32, 40$ and try to express $2016 - a^2$ as the sum of an odd perfect square b^2 and twice another odd perfect square $2c^2$, finding only the following solutions:

- $a = 0$, $b = 43$, $c = 9$. Since $b = 43 = (z + w)(z - w)$ is prime, it follows that $zw = 21 \cdot 22 = 462$, and $a = 0$ results in $x = y$, or $x^2 = 462 \pm 9$, false since $453, 471$ are not perfect squares.
- $a = 8$, $b = 37$, $c = 17$. Similarly $zw = 19 \cdot 18 = 342$, $xy = 3 \cdot 1 = 3$, and $c \neq |xy - zw|$.
- $a = 8$, $b = 43$, $c = 7$. Then, $zw = 462$, $xy = 3$, and again $c \neq |xy - zw|$.
- $a = 40$, $b = 13$, $c = 11$. Then $zw = 42$, $xy = 99$ or $xy = 21$, and again $c \neq |xy - zw|$ in either case.
- $a = 40$, $b = 19$, $c = 5$. Then $zw = 90$, $xy = 99$ or $xy = 21$, and $c \neq |xy - zw|$.

No solutions exist either when $n = 2011$.

We conclude that the minimum such n is $n = 2017$.

Also solved by Albert Stadler, Switzerland.

O186. Let n be a positive integer. Prove that each odd common divisor of

$$\binom{2n}{n}, \binom{2n-1}{n}, \dots, \binom{n+1}{n}$$

is a divisor of $2^n - 1$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

First solution by Daniel Campos Salas, Costa Rica

Using the identity $\binom{m+1}{k} - \binom{m}{k} = \binom{m}{k-1}$, we get that any odd common divisor of the numbers is also a common divisor of

$$\binom{2n-1}{n-1}, \binom{2n-2}{n-1}, \dots, \binom{n+1}{n-1}, \binom{n+1}{n}.$$

Repeating this argument till we get $\binom{n+1}{k}$ for some k, we get that any odd common divisor of the original set of numbers is also a common divisor of

$$\binom{n+1}{1}, \binom{n+1}{2}, \dots, \binom{n+1}{n},$$

and of their sum, which equals $2^{n+1} - 2 = 2(2^n - 1)$. Therefore, we conclude that any odd common divisor of the original set also divides $2^n - 1$, and we're done.

Second solution by Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy

By Vandermonde's identity,

$$\binom{n+i+1}{n} = \sum_{j=0}^{i} \binom{n+1}{n-j}\binom{i}{j}.$$

By using this formula for $i = 0, \dots, n$, we find that

$$a := \gcd\left(\binom{n+1}{n}, \binom{n+2}{n}, \dots, \binom{2n}{n}\right) = \gcd\left(\binom{n+1}{n}, \binom{n+1}{n-1}, \dots, \binom{n+1}{1}\right).$$

Therefore, if d is divisor of a then d divides the sum

$$\sum_{j=1}^{n} \binom{n+1}{j} = 2^{n+1} - 2 = 2(2^n - 1).$$

It follows that if d is odd then d divides $2^n - 1$.

Third solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Clearly $\binom{n+1}{n} = n + 1$, or any odd common divisor of $\binom{2n}{n}, \binom{2n-1}{n}, \dots, \binom{n+1}{n}$ is also a divisor of $n + 1$. Assume that $n + 1$ is not an odd prime or the power of an odd prime, and let p be any odd prime that divides $n + 1$ with multiplicity u, hence p^u divides $n + 1$, but p^{u+1} does not. Now, consider

$$\binom{n+p^u}{n} = \frac{(n+p^u)(n+p^u-1)\dots(n+1)}{p^u(p^u-1)(p^u-2)\dots 1}.$$

Note that for any positive integer k, $n+k+1-k = n+1$ is divisible by p^u, but $n+1+p^u$ is the smallest integer larger than $n+1$ that is divisible by p^u, hence for all $k = 1, 2, \dots, p^u - 1$, $n+k+1$ is not divisible by p^u. It follows that p divides $n + k + 1$ and k with the same multiplicity for all $k = 1, 2, \dots, p^u - 1$.

Moreover, p^u divides $n+1$ but p^{u+1} does not, or $\binom{n+p^u}{n}$ is not a multiple of p, where $p^u \le n-2$, because p^u and $n+1$ share a common divisor p not smaller than 3 and $p^u \le n+1-p$ because p^u is a proper divisor of $n+1$. Thus $n+1 < n+p^u < 2n$, hence p does not divide the common divisor of $\binom{2n}{n}, \binom{2n-1}{n}, \dots, \binom{n+1}{n}$. We conclude that, if $n+1$ is not a prime or the power of a prime, then no odd prime divisor p of $n+1$ divides all of $\binom{2n}{n}, \binom{2n-1}{n}, \dots, \binom{n+1}{n}$, ie, the only odd common divisors of $\binom{2n}{n}, \binom{2n-1}{n}, \dots, \binom{n+1}{n}$ are ± 1, which clearly divide 2^n-1 for any n.

Assume now that $n+1 = p^u$ is an odd prime or the power of an odd prime p for some positive integer multiplicity $u \ge 1$. Then, consider

$$\binom{n+p^{u-1}}{n} = \frac{(n+p^{u-1})(n+p^{u-1}-1)\dots(n+1)}{p^{u-1}(p^{u-1}-1)(p^{u-1}-2)\dots 1}.$$

Note that for any positive integer k, $n+k+1-k = n+1$ is divisible by p^u, hence by p^{u-1}, but $n+1+p^{u-1}$ is the smallest integer larger than $n+1$ which is divisible by p^{u-1}. It follows that, for $k = 1, 2, \dots, p^{u-1}-1$, p divides k and $n+k+1$ with the same multiplicity. Moreover, p^u divides $n+1$ but p^{u+1} does not, hence p, but not p^2 divides $\binom{n+p^{u-1}}{n}$, and the only odd common divisors of $\binom{2n}{n}, \binom{2n-1}{n}, \dots, \binom{n+1}{n}$ are ± 1, and maybe $\pm p$. It suffices therefore to show that, when $n+1$ is a power of an odd prime p, p divides 2^n-1. Since $n+1 = p^u$, using Fermat's little theorem we find

$$2^n = 2^{p^u-1} = \left(2^{p-1}\right)^{(p^{u-1}+p^{u-2}+\dots+1)} \equiv 1^{(p^{u-1}+p^{u-2}+\dots+1)} \equiv 1 \pmod p.$$

The conclusion follows. Note that the previous argument is also valid when $n+1 = p$ is an odd prime itself, since taking $u = 1$ results in $\binom{n+1}{n} = n+1 = p$, thus again the only odd common divisors of $\binom{2n}{n}, \binom{2n-1}{n}, \dots, \binom{n+1}{n}$ are ± 1 and maybe $\pm p$, and $2^n = 2^{p-1} \equiv 1 \pmod p$.