Junior problems

J169. If $x, y, z > 0$ and $x + y + z = 1$, find the maximum of

$$E(x, y, z) = \frac{xy}{x+y} + \frac{yz}{y+z} + \frac{zx}{z+x}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Ercole Suppa, Teramo, Italy

By the AM–GM inequality,

$$\begin{aligned} E(x, y, z) &\leq \frac{\frac{(x+y)^2}{4}}{x+y} + \frac{\frac{(y+z)^2}{4}}{y+z} + \frac{\frac{(z+x)^2}{4}}{z+x} = \\ &= \frac{x+y}{4} + \frac{y+z}{4} + \frac{z+x}{4} = \\ &= \frac{x+y+z}{2} = \frac{1}{2} \end{aligned}$$

Therefore the maximum of $E(x, y, z)$ is $\frac{1}{2}$, occurring for example if

$$x = y = z = \frac{1}{3}$$

□

Also solved by Arkady Alt, San Jose, California, USA; Michel Bataille, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Sayan Mukherjee; Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy.

J170. In the interior of a regular pentagon $ABCDE$ consider the point M such that triangle MDE is equilateral. Find the angles of triangle AMB.

Proposed by Catalin Barbu, Vasile Alecsandri College, Bacau, Romania

Solution by Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy

We have that

$$\angle MEA = \angle AED - \angle MED = 108^\circ - 60^\circ = 48^\circ.$$

Morever, $AE = EM$ implies that

$$\angle EAM = \angle EMA = \frac{1}{2}(180^\circ - 48^\circ) = 66^\circ.$$

By symmetry

$$\angle ABM = \angle MBC = \frac{1}{2}\angle ABC = 54^\circ.$$

Finally

$$\angle MAB = \angle EAB - \angle DCM = 108^\circ - 66^\circ = 42^\circ, \quad \angle AMB = 180^\circ - \angle MAB - \angle ABM = 84^\circ.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Sayan Mukherjee; Raul A. Simon, Chile; Ercole Suppa, Teramo, Italy; Vicente Vicario Garcia, Huelva, Spain.

J171. If different letters represent different digits, could the addition

$$\begin{array}{r} AXXXU \\ BXXV \\ CXXY \\ +DEXXZ \\ --------- \\ XXXXX \end{array}$$

be correct?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Francisco Javier Garcia Capitan, Spain

We assume that leading digits A, B, C, D and X cannot be 0.

Now, A and D are distinct numbers, so $A+D \geqslant 3$. In addition, the sum in the fourth column (we refer the columns from right to left) has carrying, thus we have in fact $X > 3$.

The sum in the first column has carry as well, because if not, we have $4X \equiv X \mod 10$ in the second column, and this is impossible. Call R the carrying of the sum in the first column. We'll have $R + 4X = 10R + X$ adding the second column, thus $X = 3R$. Since $X > 3$, it can be $X = 6, R = 2$ or $X = 9, R = 3$, but the latter is impossible, since $U + V + X + Y <= 6+7+8+9 = 30$ and $X = 9, R = 3$ implies $U + V + Y + Z = 39$.

If we must have $X = 6, R = 2$, which implies $U + V + Y + Z = 26$ and some of

$$(1) \begin{cases} B+C+E=8 \\ A+D=5 \end{cases}, \quad (2) \begin{cases} B+C+E=18 \\ A+D=4 \end{cases}.$$

If we take $E = 0$ and the remaining digits such that $\{B, C\} = \{1, 7\}$, $\{A, D\} = \{2, 3\}$ and $\{U, V, Y, Z\} = \{4, 5, 8, 9\}$ then (1) holds and we have a solution of the problem.

$$\begin{array}{ccccccc} & & 2 & 6 & 6 & 6 & 4 \\ & & & 1 & 6 & 6 & 5 \\ & & & 7 & 6 & 6 & 8 \\ + & & 3 & 0 & 6 & 6 & 9 \\ - & - & - & - & - & - & - \\ & & 6 & 6 & 6 & 6 & 6 \end{array}$$

Note that the conditions (2) are impossible since they imply that

$$\begin{aligned} &(U+V+Y+Z)+X+(B+C+E)+(A+D) \\ =&26+6+18+4=54 \neq 45, \end{aligned}$$

which is the sums of the ten digits.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Andrea Ligori, Università di Roma "Tor Vergata", Roma, Italy.

J172. Let P be a point situated in the interior of an equilateral triangle ABC and let A', B', C' be the intersectionsof lines AP, BP, CP with sides BC, CA, AB, respectively. Find P such that

$$A'B^2 + B'C^2 + C'A^2 = AB'^2 + BC'^2 + CA'^2.$$

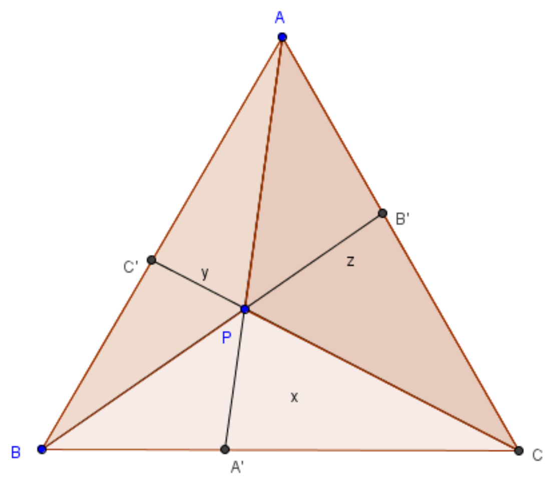
Proposed by Catalin Barbu, Vasile Alecsandri College, Bacau, Romania

Solution by Sayan Mukherjee

Let us denote the areas of $\triangle BPC, \triangle APB, \triangle APC$ as x, y, z respectively. we have,

$$\frac{BA'}{A'C} = \frac{\triangle AA'B}{\triangle AA'C} = \frac{\triangle BPA'}{\triangle CPA'} = \frac{\triangle AA'B - \triangle BPA'}{\triangle AA'C - \triangle CPA'} = \frac{\triangle APB}{\triangle APC} = \frac{y}{z}.$$

Let $AB = BC = CA = a$, then $BA' = \dfrac{ay}{y+z}$; and $A'C = \dfrac{az}{y+z}$. Similarly we obtain four other relations.



Substituting these values in our first relation we obtain,

$$\frac{y^2}{(y+z)^2} + \frac{z^2}{(z+x)^2} + \frac{x^2}{(x+y)^2} = \frac{z^2}{(y+z)^2} + \frac{x^2}{(z+x)^2} + \frac{y^2}{(x+y)^2};$$

Which, on transposition, is equivalent to

$$\frac{y^2 - z^2}{(y+z)^2} + \frac{z^2 - x^2}{(z+x)^2} + \frac{x^2 - y^2}{(x+y)^2} = 0.$$

This relation can be simplified to $\sum_{cyc} \dfrac{x-y}{x+y} = 0$, which rewrites into

$$\frac{(x-y)(z-x)(z-y)}{(x+y)(y+z)(z+x)} = 0.$$

Hence the first relation actually holds true if and only if $x = y$ or $y = z$ or $z = x$. So, if P is such that at least two of the triangles $\triangle APB, \triangle BPC, \triangle CPA$ have the same area, then the first relation holds good. So, P must lie on any of the three medians of $\triangle ABC$.

Also solved by Michel Bataille, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Francisco Javier Garcia Capitan, Spain; Raul A. Simon, Chile; Ercole Suppa, Teramo, Italy;Vicente Vicario Garcia, Huelva, Spain.

J173. Let a and b be rational numbers such that

$$|a| \leq \frac{47}{|a^2 - 3ab^2|} \quad \text{and} \quad |b| \leq \frac{52}{|b^2 - 3a^2|}.$$

Prove that $a^2 + b^2 \leq 17$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Michel Bataille, France

Let $X = a^3 - 3ab^2$ and $Y = 3a^2b - b^3$. From the hypothesis, we have $|X| \leq 47$ and $|Y| \leq 52$. Observing that $X + iY = (a+ib)^3$ (easily checked), we deduce that

$$a^2 + b^2 = |a+ib|^2 = \left|(a+ib)^3\right|^{\frac{2}{3}} = \left((X^2+Y^2)^{\frac{1}{2}}\right)^{\frac{2}{3}} = (X^2+Y^2)^{\frac{1}{3}} \leq (47^2+52^2)^{\frac{1}{3}}.$$

Since $47^2 + 52^2 = 4913 = 17^3$, it follows that $a^2 + b^2 \leq 17$.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note that $a = b = 0$ clearly satisfies $a^2 + b^2 < 17$, therefore $a^2 - 3b^2$ and $b^2 - 3a^2$ are nonzero rationals, since otherwise 3 would be the square of a rational, absurd. Define $x = a^2$, $y = b^2$, hence $x(x-3y)^2 \leq 47^2$ and $y(3x-y)^2 \leq 52^2$. Adding these two inequalities results in $(x+y)^3 \leq 47^2 + 52^2 = 17^3$, or $x + y = a^2 + b^2 \leq 17$. Note that equality is reached whenever equality holds simultaneously on both inequalities given as condition in the problem statement, for example when $a = \pm 1$ and $b = \pm 4$.

J174. The incircle of triangle ABC touches sides BC, CA, AB at D, E, F, respectively. Let K be a point on side BC and let M be the point on the line segment AK such that $AM = AE = AF$. Denote by L and N the incenters of triangles ABK and ACK, respectively. Prove that K is the foot of the altitude from A if and only if $DLMN$ is a square.

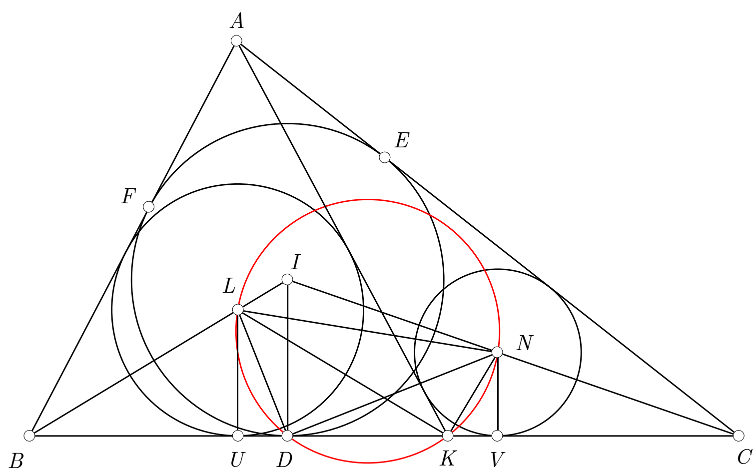
Proposed by Bogdan Enescu, B.P.Hasdeu College, Buzau, Romania

Solution by Ercole Suppa, Teramo, Italy

We begin by proving two lemmas.

Lemma 1. The points D, K lie on the circle with diameter LN.

Proof. Suppose wlog that $c < b$. Let I, U, V be the incenter of ABC and the points where the circles (L), (N) touch the side BC; let r, r_1, r_2 be the inradii of the circles (I), (L), (N), as shown in figure.



Denote $a = BC$, $b = CA$, $c = AB$, $m = BK$, $n = KC$, $x = AK$.

Since L and N are the incenters of $\triangle ABK$ and $\triangle ACK$ we have

$$\angle LKN = \angle LKA + \angle AKN = \frac{1}{2}(\angle BKA + \angle AKC) = 90^\circ$$

In order to prove that $\angle LDN = 90^\circ$ we will show that

$$LD^2 + DN^2 = LN^2 \tag{1}$$

The Pythagora's theorem yields $LD^2 = r_1^2 + UD^2$, $ND^2 = r_2^2 + DV^2$ and

$$LN^2 = UV^2 + (r_1 - r_2)^2 = UD^2 + DV^2 - 2UD \cdot DV + r_1^2 + r_2^2 - 2r_1r_2$$

Therefore to establish (3) it is enough to show that $UD \cdot DV = r_1 r_2$.

We clearly have

$$UD = BD - BU = \frac{a+c-b}{2} - \frac{m+c-x}{2} = \frac{a+x-b-m}{2} \tag{2}$$

$$DV = DC - CV = \frac{a+b-c}{2} - \frac{m+b-x}{2} = \frac{a+x-c-n}{2} \tag{3}$$

From (2) and (3), plugging $n = a - m$ into the expression, we obtain

$$UD \cdot DV = \frac{(x+a-b-m)(x-c+m)}{4} \tag{4}$$

From the similar triangles $\triangle BUL \sim \triangle BDI$, $\triangle CVN \sim \triangle CDI$, it follows that

$$LU : ID = BU : BD \quad \Rightarrow \quad r_1 = r \cdot \frac{c+m-x}{a+c-b} \tag{5}$$

$$NV : ID = CV : CD \quad \Rightarrow \quad r_2 = r \cdot \frac{b+n-x}{a+b-c} \tag{6}$$

From (5),(6) using the well known formula

$$r^2 = \frac{(b+c-a)(a+c-b)(a+b-c)}{4(a+b+c)}$$

and plugging $n = a - m$ into the expression, we obtain

$$r_1 r_2 = \frac{(b+c-a)(a+b-m-x)(c+m-x)}{4(a+b+c)} \tag{7}$$

Using (4) and (7) we have

$$UD \cdot DV - r_1 r_2 = \frac{ax^2 - ac^2 + a^2m - b^2m + c^2m - am^2}{2(a+b+c)} \tag{8}$$

From the Stewart's theorem we get

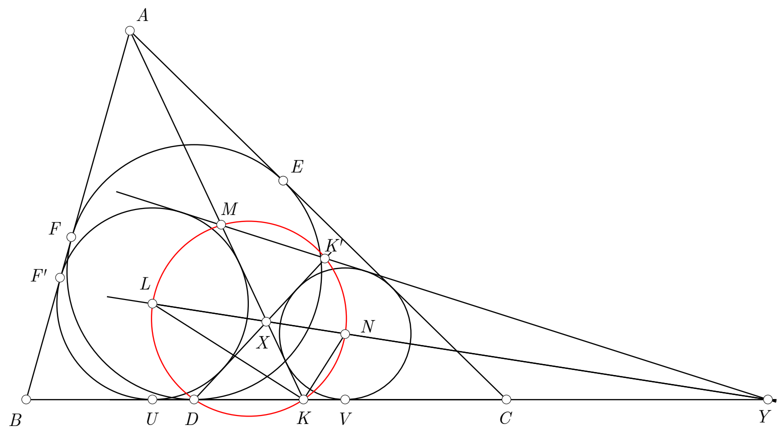
$$x^2 = \frac{mb^2 + (a-m)c^2 - am(a-m)}{a}$$

Finally, plugging x^2 into (8), after a boring calculation, we have

$$UD \cdot DV - r_1 r_2 = \frac{\left(c^2 - am\right)(m+n-a)}{2(a+b+c)} = 0$$

Thus $LD^2 + DN^2 = LN^2$ and the lemma is proved. ■

Lemma 2. If M is the second intersection point of AK with the circle γ circumscribed to $DKNL$, then $DM \perp LN$ and $AM = AE = AF$.



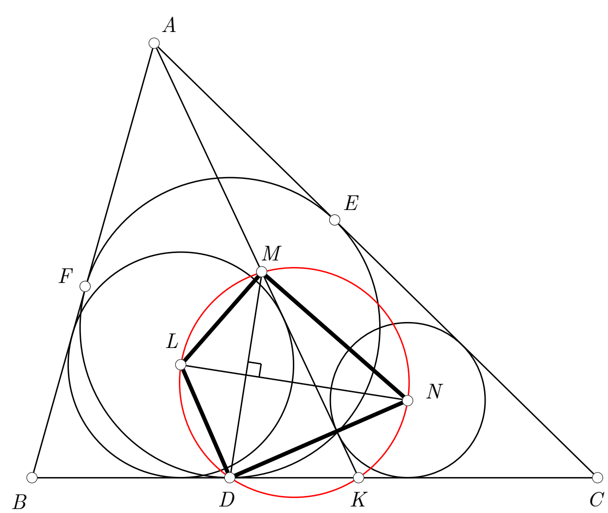
Proof. Let the incircle of triangle ABK touches side AB at F'. According to LEMMA 1 the center of γ is the midpoint of LN, so the point M lies on the external tangent to the circles (L), (N). Therefore, because of simmetry of the figure, we have $DM \perp LN$ and

$$\begin{aligned} AM &= AF' - UD = AF' - (BD - BU) \\ &= \frac{c+x-m}{2} - \frac{a+c-b}{2} + \frac{c+m-x}{2} \\ &= \frac{b+c-a}{2} = AF \end{aligned}$$

The lemma is proved. ■

Considering now the original problem, from LEMMA 1 and LEMMA 2 it follows that

- $DLMN$ is cyclic;
- $\angle LDN = \angle LMN = 90°$;
- $DM \perp LN$.



Therefore $DLMN$ is a square if and only if MD is a diameter of the circumcircle of $DLMN$, i.e. $\angle MKD = 90^\circ$ (i.e. $AK \perp BC$).

Also solved by Michel Bataille, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Sayan Mukherjee.

Senior problems

S169. Let $k > 1$ be an odd integer such that $a^k + b^k = c^k + d^k$ for some positive integers a, b, c, d. Prove that $\frac{a^k+b^k}{a+b}$ is not a prime.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note first that we need to assume that $\{a, b\} \neq \{c, d\}$, since otherwise we may take $k = 3$, $a = c = 2$, $b = d = 1$, and $\frac{a^k+b^k}{a+b} = 3$ is prime. Moreover, as long as $\{a, b\} = \{c, d\}$, the condition $a^k + b^k = c^k + d^k$ in the problem is irrelevant, so I will assume that $\{a, b\} \neq \{c, d\}$. Note also that, if $a = b > 1$, then $\frac{a^k+b^k}{a+b} = a^{k-1}$ cannot be a prime because $k \geq 3$, whereas $a = b = 1$ is only possible if $c = d = 1$. We may then assume wlog that $a \geq b + 1 \geq 2$, hence for any odd $k \geq 3$,

$$a^k + b^k - (a + b)^2 \geq a^3 + b^3 - a^2 - b^2 - 2ab = (a + b)(a^2 - ab + b^2 - a - b) =$$

$$= (a + b)(a - b - 1)^2 + b(a + b)(a - 2) + (a + b)(a - b - 1) \geq 0,$$

with equality iff $k = 3$, $a = 2$ and $b = 1$, possible only if $\{c, d\} = \{1, 2\}$.

We may therefore assume that $\{a, b\} \neq \{c, d\}$, where moreover wlog $a > b$ and $c \geq d$, and by the previous result, $\frac{a^k+b^k}{a+b} > a + b$, and similarly $\frac{c^k+d^k}{c+d} > c + d$. Note that, if $a = c$, then $b^k = d^k$, or $b = d$, in contradiction with our assumption, and similarly if $b = d$, hence either $a > c \geq d > b$, or $c > a > b > d$.

In the first case, note that

$$(a - c)\left(a^{k-1} + ca^{k-2} + \cdots + c^{k-1}\right) = (d - b)\left(b^{k-1} + db^{k-2} + \cdots + d^{k-1}\right),$$

and since $a^i c^j > b^i d^j$ because $a > b$ and $c \geq d$, we have $a - c < d - b$, or $a + b < c + d$. Therefore, $\frac{a^k+b^k}{a+b} > \frac{c^k+d^k}{c+d}$, and if $\frac{a^k+b^k}{a+b}$ is prime, it cannot divide $\frac{c^k+d^k}{c+d}$, hence it divides $c + d$, or $a^k + b^k \leq (a + b)(c + d) < (c + d)^2 < c^k + d^k$, absurd.

In the second case, we obtain similarly that $a + b > c + d$, and if $\frac{a^k+b^k}{a+b}$ divides $c + d$, we have $a^k + b^k \leq (a + b)(c + d) < (a + b)^2$, contradiction, hence $\frac{a^k+b^k}{a+b}$ divides $\frac{c^k+d^k}{c+d}$, or $c + d$ divides $a + b$, and since $a + b > c + d$, then $a + b \geq 2(c + d) > 2c > a + b$, absurd.

The conclusion follows.

S170. Consider $n (n \geq 6)$ circles of radius $r < 1$ that are pairwise tangent and all tangent to a circle of radius 1. Find r.

Proposed by Catalin Barbu, Vasile Alecsandri College, Bacau, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We will assume that the problem statement means that n circles or radius r are all tangent to a given circle of radius 1, and each circle of radius r is tangent to another two circles of radius r, since it is well known that at most four circles can be tangent to each other, and at most three of them would have the same radius, since either one of the four circles contains the other three inside them, or one of them is in the gap formed by the other three.

With this assumption, clearly all centers of the circles with radii r are at a distance $1 + r$ from the center of the circle with radius 1, and at a distance $2r$ from their closest neighbours of radius r, and by cyclic symmetry, these n centers are at the vertices of a regular n-gon, with sidelength $2r$ and circumradius $1 + r$, hence considering the isosceles triangle with equal sides of length $1 + r$ and sidelength $2r$ for the other side, where the different angle is $\frac{360^\circ}{n}$, we have $r = (1 + r) \sin \frac{180^\circ}{n}$, or equivalently

$$r = \frac{\sin \frac{180^\circ}{n}}{1 - \sin \frac{180^\circ}{n}}.$$

Also solved by Raul A. Simon, Chile.

S171. Prove that if the polynomial $P \in \mathbf{R}[X]$ has n distinct real zeros, then for any $\alpha \in \mathbf{R}$ the polynomial $Q(X) = \alpha X P(X) + P'(X)$ has at least $n-1$ distinct real zeros.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy

Let be $a_1 < a_2 < \cdots < a_n$ be the real roots of P and

$$P(x) = Q(x) \prod_{j=1}^{n} (x - a_j)$$

where $Q \in \mathbb{R}[x]$ has no real roots. Therefore

$$R(a_i) = \alpha a_i P(a_i) + P'(a_i) = P'(a_i) = Q(a_i) \prod_{j \neq i} (a_i - a_j)$$

and it's clear that in the last product has $n - i - 1$ negative factors. Since $Q(x)$ has constant sign, it follows that $R(a_i)R(a_{i+1}) < 0$ for $i = 1, \ldots, n-1$, and by Intermediate Value Theorem there exists at least a zero of R in (a_i, a_{i+1}). This means that R has at least $n-1$ distinct real zeros.

Also solved by Arkady Alt, San Jose, California, USA; Michel Bataille, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Vicente Vicario Garcia, Huelva, Spain.

S172. Let a, b, c be positive real numbers. Prove that

$$\sum_{cyc} \frac{a^2b^2(b-c)}{a+b} \geq 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

Clearing the denominators, the inequality is equivalent to

$$\sum_{cyc}(a^4c^2b + c^4b^3) \geq \sum_{cyc} 2a^3b^2c$$

but this follows from the AM–GM $(a^4c^2b + c^4b^3)/2 \geq a^3b^2c$

Also solved by Arkady Alt, San Jose, California, USA; Michel Bataille, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Sayan Mukherjee; Ercole Suppa, Teramo, Italy.

S173. Let

$$f_n(x,y,z)=\frac{(x-y)z^{n+2}+(y-z)x^{n+2}+(z-x)y^{n+2}}{(x-y)(y-z)(x-z)}.$$

Prove that $f_n(x,y,z)$ can be written as a sum of monomials of degree n and find $f_n(1,1,1)$ for all positive integers n.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Michel Bataille, France

It is easily verified that

$$\frac{1}{(x-y)(x-z)}+\frac{1}{(y-x)(y-z)}+\frac{1}{(z-y)(z-x)}=0.$$

This allows one to transform $f_n(x,y,z)$ as follows:

$$\begin{aligned}
f_n(x,y,z) &= \frac{x^{n+2}}{(x-y)(x-z)}+\frac{y^{n+2}}{(y-x)(y-z)}+\frac{z^{n+2}}{(z-y)(z-x)}\\
&= \frac{-x^{n+2}}{(y-x)(y-z)}+\frac{-x^{n+2}}{(z-y)(z-x)}+\frac{y^{n+2}}{(y-x)(y-z)}+\frac{z^{n+2}}{(z-y)(z-x)}\\
&= \frac{y^{n+2}-x^{n+2}}{(y-x)(y-z)}+\frac{z^{n+2}-x^{n+2}}{(z-y)(z-x)}\\
&= \frac{y^{n+1}+y^nx+\cdots+x^ny+x^{n+1}}{(y-z)}+\frac{z^{n+1}+z^nx+\cdots+x^nz+x^{n+1}}{(z-y)}\\
&= \frac{(y^{n+1}-z^{n+1})+x(y^n-z^n)+\cdots+x^n(y-z)}{y-z},
\end{aligned}$$

and finally

$$f_n(x,y,z)=(y^n+y^{n-1}z+\cdots+yz^{n-1}+z^n)+x(y^{n-1}+y^{n-2}z+\cdots+yz^{n-2}+z^{n-1})+\cdots+x^{n-1}(y+z)+x^n.$$

Thus, $f_n(x,y,z)$ is the sum of all monomials $x^ay^bz^c$ over all triples (a,b,c) of nonnegative integers satisfying $a+b+c=n$. This answers the first part of the question.
As for the second part, $f_n(1,1,1)$ is just the total number of monomials in the above sum, that is, $(n+1)+n+\cdots+2+1=\frac{(n+1)(n+2)}{2}$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Moubinool Omarjee , Paris France; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

S174. Prove that for each positive integer k the equation

$$x_1^3 + x_2^3 + \cdots + x_k^3 + x_{k+1}^2 = x_{k+2}^4$$

has infinitely many solutions in positive integers with $x_1 < x_2 < \cdots < x_{k+1}$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

First solution by Arkady Alt, San Jose, California, USA

Since

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2 (k+1)^2}{4} = (1 + 2 + \cdots + k)^2$$

then by substitution

$$x_1 = x, x_2 = 2x, \ldots, x_k = kx, x_{k+1} = (1 + 2 + \cdots + k)\, y^2, x_{k+2} = (1 + 2 + \cdots + k)\, y$$

in original equation, we obtain

$$\left(1^3 + 2^3 + \cdots + k^3\right) x^3 + (1 + 2 + \cdots + k)^2 y^4 = (1 + 2 + \cdots + k)^4 y^4 \iff x^3 = ay^4$$

where $a = (1 + 2 + \cdots + k)^2 - 1 = \dfrac{(k-1)(k+2)\left(k^2 + k + 2\right)}{4}$. Since $x = a^7 n^4,\ y = a^5 n^3$ for any positive integer n , then $x^3 = ay^4$. Hence

$$x_1 = a^7 n^4, x_2 = 2a^7 n^4, \ldots, x_k = ka^7 n^4, x_{k+1} = (1 + 2 + \cdots + k)\, a^{10} n^6, x_{k+2} = (1 + 2 + \cdots + k)\, a^5 n^3$$

for any positive integer n is a solution to the original equation and obviously, $x_1 < x_2 < \cdots < x_{k+1}$.

Second solution by the authors

For any positive integer n we have the well-known identity:

$$1^3 + 2^3 + \cdots + n^3 + (n+1)^3 + \cdots + (n+k)^3 = \left(\frac{(n+k)(n+k+1)}{2}\right)^2,$$

that is

$$\left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 + \cdots + (n+k)^3 = \left(\frac{(n+k)(n+k+1)}{2}\right)^2.$$

Consider the positive integers n such that the triangular number $t_{n+k} = \frac{(n+k)(n+k+1)}{2}$ is a perfect squares. There are infinitely many such integers since the relation $t_{n+k} = u^2$ is equivalent to the Pell's equation $(2n + 2k + 1)^2 - 2u^2 = 1$. The fundamental solution to this Pell equation is $(3, 2)$, hence all these integers are given by the sequence (n_s), where

$$2n_s + 2k + 1 + u_s\sqrt{2} = (3 + 2\sqrt{2})^s,$$

for s big enough such that $n_s \geq 1$.

We can take

$$x_1 = n_s + 1, \cdots, x_k = n_s + k, x_{k+1} = \frac{n_s(n_s+1)}{2}, x_{k+2} = u_s.$$

It is clear that for s big enough we have $n_s \geq 1$ and $\frac{n(n+1)}{2} > n + k$, hence we get an infinite family of solutions.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

Undergraduate problems

U169. Sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ are defined by $x_1 = 2, y_1 = 1$, and $x_{n+1} = x_n^2 + 1$, $y_{n+1} = x_n y_n$ for all n. Prove that for all $n \geq 1$

$$\frac{x_n}{y_n} < \frac{651}{250}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Andrea Ligori, Università di Roma "Tor Vergata", Roma, Italy

The inequality holds for $1 \leq n \leq 4$ because $x_1/y_1 = 2$, $x_2/y_2 = 5/2$, $x_3/y_3 = 13/5$, and $x_4/y_4 = 677/260$,. Moreover, for $n \geq 5$

$$\frac{x_n}{y_n} = \frac{x_{n-1}^2+1}{x_{n-1}y_{n-1}} = \frac{x_{n-1}}{y_{n-1}} + \frac{1}{y_n} = \frac{x_4}{y_4} + \sum_{k=5}^{n} \frac{1}{y_k} < \frac{677}{260} + \sum_{k=5}^{\infty} \frac{1}{y_k}.$$

Hence it suffices to show that

$$\sum_{k=4}^{\infty} \frac{1}{y_k} \leq \frac{651}{250} - \frac{677}{260} = \frac{1}{6500}.$$

We have that for $n \geq 1$

$$x_n \geq x_{n-1}^2 \geq x_{n-2}^4 \geq x_{n-3}^8 \geq \cdots \geq 2^{2^{n-1}}$$

and,

$$y_n = x_{n-1}y_{n-1} \geq 2^{2^{n-2}} y_{n-1} \geq 2^{2^{n-2}+2^{n-3}} y_{n-2} \geq \cdots \geq 2^{2^{n-2}+2^{n-3}+\cdots+1} = 2^{2^{n-1}-1}.$$

This means that $y_n \geq 2^{2^{n-1}-1} \geq 2^{3n} = 8^n$ for $n \geq 5$, and

$$\sum_{k=5}^{\infty} \frac{1}{y_k} \leq \sum_{k=5}^{\infty} \frac{1}{8^k} = \frac{1}{7 \cdot 8^4} = \frac{1}{28672} < \frac{1}{6500}.$$

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

U170. Sequences $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}$ are defined as follows: $x_1 = \alpha$, $y_1 = \beta$, with $|\alpha| \neq |\beta| \neq 0$, and

$$x_{n+1} = \max(x_n - y_n, x_n + y_n),$$
$$y_{n+1} = \min(x_n - y_n, x_n + y_n),$$

for all $n \geq 1$. Prove that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty.$$

Proposed by Bogdan Enescu, B.P.Hasdeu College, Buzau, Romania

Solution by Michel Bataille, France

It is easily seen that (y_n) has no limit (finite or not) if $\alpha = 0, \beta = 1$. We will show the following results: $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$ if $\alpha \neq 0$ and $|\alpha| \neq |\beta|$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ if $\alpha = \beta = 0$, and otherwise, $\lim_{n\to\infty} x_n = \infty$ and (y_n) has no limit either finite or infinite.
For all real numbers a, b, we have $\max(a,b) = \frac{1}{2}(a+b+|a-b|)$ and $\min(a,b) = \frac{1}{2}(a+b-|a-b|)$. It follows that for all $n \geq 1$,

$$x_{n+1} = x_n + |y_n|,\ y_{n+1} = x_n - |y_n| \qquad (1).$$

First, suppose $|\alpha| < |\beta|$ that is, $-|\beta| < \alpha < |\beta|$. Then, using (1), $x_2 = \alpha + |\beta|,\ y_2 = \alpha - |\beta| < 0,\ x_3 = 2|\beta|,\ y_3 = 2\alpha$ so that $x_4 = 2(|\beta| + |\alpha|),\ y_4 = 2(|\beta| - |\alpha|),\ x_5 = 4|\beta|,\ y_5 = 4|\alpha|$. An easy induction yields $x_{2n} = 2^{n-1}(|\beta| + |\alpha|),\ y_{2n} = 2^{n-1}(|\beta| - |\alpha|),\ x_{2n+1} = 2^n|\beta|,\ y_{2n+1} = 2^n|\alpha|$ for all integer $n \geq 2$. Thus, $\lim_{n\to\infty} x_n = \infty$ (since $\lim_{n\to\infty} 2^n = \infty$ and $|\beta| > 0$) and $\lim_{n\to\infty} y_n = \infty$ if $\alpha \neq 0$ while (y_n) has no limit (finite or not) if $\alpha = 0$ (since then $\lim_{n\to\infty} y_{2n} = \infty$ and $\lim_{n\to\infty} y_{2n+1} = 0$).
If $\alpha > |\beta|$, with the help of (1) again, we obtain

$$x_{2n-1} = 2^{n-1}\alpha,\ y_{2n-1} = 2^{n-1}|\beta|,\ x_{2n} = 2^{n-1}(|\beta| + \alpha),\ y_{2n} = 2^{n-1}(\alpha - |\beta|)$$

for all $n \geq 2$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$ follows.
If $\alpha < -|\beta|$, similarly,

$$x_{2n+1} = 2^n|\alpha|,\ y_{2n+1} = 2^n|\beta|,\ x_{2n+2} = 2^n(|\alpha| + |\beta|),\ y_{2n+2} = 2^n(|\alpha| - |\beta|)$$

for all $n \geq 2$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$ again.
Lastly, in the same way it is readily seen that if α and β are equal or opposite nonzero real numbers, then $\lim_{n\to\infty} x_n = \infty$ while (y_n) has no limit (finite or not). Also, we clearly have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ if $\alpha = \beta = 0$ and the proof is complete.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Andrea Ligori, Università di Roma "Tor Vergata", Roma, Italy.

U171. Let A be a matrix of order n such that $A^{10} = O_n$. Prove that

$$\frac{1}{4}A^4 + \frac{1}{2}A^3 + \frac{1}{2}A^2 + A + I_n$$

is invertible.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, California, USA

Let $f(A) = -\frac{1}{4}A^3 - \frac{1}{2}A^3 - \frac{1}{2}A - I_n$ and $B = Af(A)$. Since for any two polynomial $P(x)$ and $Q(x)$ and any matrix A holds $P(A)Q(A) = Q(A)P(A)$ then $B^{10} = (Af(A))^{10} = A^{10}f(A)^{10} = 0_n$.Hence, $\left(\frac{1}{4}A^4 + \frac{1}{2}A^3 + \frac{1}{2}A^2 + A + I_n\right)\left(I_n + B + B^2 + \cdots + B^9\right) = (I_n - B)\left(I_n + B + B^2 + \cdots + B^9\right) = I_n - B^{10} = I_n$.Thus, $I_n + B + B^2 + \cdots + B^9$ is inverse matrix for $\frac{1}{4}A^4 + \frac{1}{2}A^3 + \frac{1}{2}A^2 + A + I_n$.

Also solved by Michel Bataille, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Moubinool Omarjee , Paris France; Andrea Ligori, Università di Roma "Tor Vergata", Roma, Italy.

U172. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing invertible function such that for all $x \in \mathbb{R}$, $f(x) + f^{-1}(x) = e^x - 1$ for all $x \in R$. Prove that f has at most one fixed point.

Proposed by Samin Riasat, University of Dhaka, Bangladesh

Solution by Emanuele Natale, Università di Roma "Tor Vergata", Roma, Italy

If x_0 is a fixed point then $f(x_0) = f^{-1}(x_0) = x_0$ and

$$f(x_0) + f^{-1}(x_0) = 2x_0 = e^{x_0} - 1$$

which has two solutions: 0 and a some $c > 0$. So, if f has more than one fixed point then it has just these two fixed points: 0 and c. Since f is a strictly increasing invertible function then f^{-1} is strictly increasing too.

Take $x < 0$ then $f(x) < f(0) = 0$, $f^{-1}(x) < f^{-1}(0) = 0$. Moreover $f(x) < x$ otherwise $f(x) > x$ and $x > f^{-1}(x)$. In both cases

$$e^x - 1 = f(x) + f^{-1}(x) < x + 0 = x$$

which is a contradiction because for $x < 0$ we have that $x < e^x - 1$.

Therefore c is the only possible fixed point.

Also solved by Michel Bataille, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

U173. Let θ be a real number. Prove that

$$\sum_{k=0}^{n-1} \frac{\sin\left(\frac{2k\pi}{n} - \theta\right)}{3 + 2\cos\left(\frac{2k\pi}{n} - \theta\right)} = \frac{(-1)^n n \sin(n\theta)}{5F_n^2 + 4(-1)^n \sin^2\left(\frac{n\theta}{2}\right)},$$

where F_n denotes the n^{th} Fibonacci number.

Proposed by Javier Buitrago, Universidad National de Colombia, Colombia

Solution by G. C. Greubel, Newport News, USA

Let

$$P_n = \frac{\sin\left(\frac{2k\pi}{n} - \theta\right)}{3 + 2\cos\left(\frac{2k\pi}{n} - \theta\right)}. \tag{1}$$

It is readily clear that

$$P_n = \frac{1}{2}\frac{d}{d\theta}\left[\ln\left(3 + 2\cos\left(\frac{2k\pi}{n} - \theta\right)\right)\right]. \tag{2}$$

From this we have

$$\begin{aligned} \sum_{k=0}^{n-1} P_n &= \frac{1}{2}\frac{d}{d\theta}\sum_{k=0}^{n-1}\left[\ln\left(3 + 2\cos\left(\frac{2k\pi}{n} - \theta\right)\right)\right] \\ &= \frac{1}{2}\frac{d}{d\theta}\ln\left\{S_n\right\}, \end{aligned} \tag{3}$$

where

$$S_n = \prod_{k=0}^{n-1}\left[3 + 2\cos\left(\frac{2k\pi}{n} - \theta\right)\right]. \tag{4}$$

Taking $n = 1$ we have

$$\begin{aligned} S_1 &= \prod_{k=0}^{0}\left[3 + 2\cos\left(2k\pi - \theta\right)\right] \\ &= 3 + 2\cos\theta \\ &= 5 - 4\sin^2\left(\frac{\theta}{2}\right). \end{aligned} \tag{5}$$

In the case when $n = 2$ we have

$$\begin{aligned} S_2 &= \prod_{k=0}^{1}\left[3 + 2\cos\left(\frac{2k\pi}{2} - \theta\right)\right] \\ &= (3 + 2\cos\theta)(3 + 2\cos(\pi - \theta) \\ &= 9 - 4\cos^2\theta \\ &= 5 + 4\sin^2\theta. \end{aligned} \tag{6}$$

The two cases of n given here have the general form

$$S_n = 5F_n^2 + 4(-1)^n \sin^2\left(\frac{n\theta}{2}\right). \tag{7}$$

Further values of n lead to the same result. Now using (7) in (4) leads to

$$\begin{aligned}
\sum_{k=0}^{n-1} P_n &= \frac{1}{2}\frac{d}{d\theta}\ln\left\{5F_n^2 + 4(-1)^n \sin^2\left(\frac{n\theta}{2}\right)\right\} \\
&= \frac{1}{2S_n}\cdot\frac{d}{d\theta}\left[5F_n^2 + 4(-1)^n \sin^2\left(\frac{n\theta}{2}\right)\right] \\
&= \frac{1}{2S_n}\left[2(-1)^n n\sin(n\theta)\right] \\
&= \frac{(-1)^n n\sin(n\theta)}{S_n}. \qquad (8)
\end{aligned}$$

By combining the components of this result leads to the expression

$$\sum_{k=0}^{n-1} \frac{\sin\left(\frac{2k\pi}{n}-\theta\right)}{3+2\cos\left(\frac{2k\pi}{n}-\theta\right)} = \frac{(-1)^n\cdot n\cdot\sin(n\theta)}{5F_n^2 + 4(-1)^n\sin^2\left(\frac{n\theta}{2}\right)}. \tag{9}$$

Also solved by Michel Bataille, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

U174. Let p be a prime. A linear recurrence of degree n in $\mathbb{F}_p$ is a sequence $\{a_k\}_{k\geq 0}$ in $\mathbb{F}_p$ satisfying a relation of the form

$$a_{i+n} = c_{n-1}a_{i+n-1} + \cdots + c_1 a_{i+1} + c_0 a_i \text{ for all } i \geq 0,$$

where $c_0, c_1, \ldots, c_{n-1} \in \mathbb{F}_p$ and $c_0 \neq 0$.

(a) What is the maximal possible period of a linear recurrence of degree n in $\mathbb{F}_p$?

(b) How many distinct linear recurrences of degree n have this maximal period?

Proposed by Holden Lee, Massachusetts Institute of Technology

Solution by Holden Lee, Massachusetts Institute of Technology

(a) $p^n - 1$. There are only p^n possibilities for n consecutive terms of the sequence. If $(0, \ldots, 0)$ appears then nothing else can appear. (Pf. of existence below)

(b) Suppose a recurrence has period $p^n - 1$. Note that by Vandermonde's determinant the functions $f_\alpha(t) = \alpha^t$ where $\alpha \in \mathbb{F}_{p^n}^\times$ are a basis for the space of functions $\mathbb{Z}/(p^n - 1)\mathbb{Z} \to \mathbb{F}_{p^n}$. Then the general term can be written as

$$a_k = b_1\alpha_1^k + \cdots + b_m\alpha_m^k.$$

for some m, distinct $\alpha_i \in \mathbb{F}_{p^n}^\times$, and $b_1, \ldots, b_m \in \mathbb{F}_{p^n}^\times$. By plugging into the recurrence, we get that all the α_i must be roots of the characteristic polynomial. If α_i has degree d_i then $\alpha_i^{p^{d_i}-1} = 1$. Thus in order for the period to be $p^n - 1$ the characteristic polynomial must be irreducible (so the roots have degree n). The roots have the same order; the common order is the least N so that the characteristic polynomial divides $x^N - 1$. Again using the fact the f_α are linearly independent we can show that the period has to be this common order and not less. Hence the characteristic polynomial has to divide the cyclotomic polynomial $\phi_{p^n-1}(x)$. Each factor of it in $\mathbb{F}_p$ has degree n, so there are $\varphi(p^n - 1)/n$ valid polynomials. For each of these we can choose $a_0, \ldots, a_{n-1}$ in $p^n - 1$ different ways. (Note a nonzero sequence cannot satisfy two recurrences corresponding to two different irreducible characteristic polynomials.) Answer: $\frac{(p^n-1)\varphi(p^n-1)}{n}$.

Olympiad problems

O169. Let a, b, c, d be real numbers such that

$$(a^2+1)(b^2+1)(c^2+1)(d^2+1) = 16.$$

Prove that

$$-3 \leq ab+bc+cd+da+ac+bd-abcd \leq 5.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA, and Gabriel Dospinescu, Ecole Normale Superieure, France

Solution by Michel Bataille, France

Consider the complex number $Z = (1+ia)(1+ib)(1+ic)(1+id)$. An easy calculation yields:

$Re(Z) = 1 - (ab+bc+cd+da+ac+bd) + abcd$ and $|Z|^2 = (a^2+1)(b^2+1)(c^2+1)(d^2+1)$.

Now, the hypothesis gives $|Z| = 4$ so that the inequality $|Re(Z)| \leq |Z|$ writes as

$$|(ab+bc+cd+da+ac+bd-abcd)-1| \leq 4$$

that is,

$$-3 \leq ab+bc+cd+da+ac+bd-abcd \leq 5,$$

as required.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Sayan Mukherjee; Pedro H. O. Pantoja, Natal-RN, Brazil.

O170. Let a and b be positive integers such that a does not divide b and b does not divide a. Prove that there is an integer x such that $1 < x \leq a$ and both a and b divide $x^{\phi(b)+1} - x$, where ϕ is Euler's totient function.

Proposed by Vahgan Aslanyan, Yerevan, Armenia

Solution by Sayan Mukherjee

We have, $x\left(x^{\varphi(b)} - 1\right)$ is divisible by x and when $\gcd(b, x) = 1$; applying Euler's theorem this is also divisible by b.

So, letting $x = \dfrac{a}{\gcd(a,b)}$ we get, using $\gcd\left(\dfrac{a}{\gcd(a,b)}, b\right) = 1$, that

$$\frac{a}{\gcd(a,b)}\left[\left(\frac{a}{\gcd(a,b)}\right)^{\varphi(b)} - 1\right] \equiv 0 \bmod \left(\frac{a}{\gcd(a,b)} \cdot b\right) \equiv 0 \mod \operatorname{lcm}(a,b).$$

Therefore the required x is $\dfrac{a}{\gcd(a,b)}$ since this also satisfies $1 < x \leq a$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Tigran Hakobyan, Armenia.

O171. Prove that in any convex quadrilateral $ABCD$,

$$\sin\left(\frac{A}{3}+60^\circ\right)+\sin\left(\frac{B}{3}+60^\circ\right)+\sin\left(\frac{C}{3}+60^\circ\right)+\sin\left(\frac{D}{3}+60^\circ\right)$$
$$\geq \frac{1}{3}(8+\sin A+\sin B+\sin C+\sin D).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

We employ the theory of Schür–concave functions, see A.W.Marshall, I.Olkin, Inequalities: Theory of Majorization and Its Applications, Academic Press, 1979.

Let's recall few notions about the theory as exposed in chapters 3 and 4 of the cited book.

For any $x=(x_1,x_2,\ldots,x_n)\in\mathbf{R}^n$ let $x_{[1]}\geq x_{[2]}\geq\ldots\geq x_{[n]}$ denote the components of x in the decreasing order. We write

$$x\prec y \quad \text{if} \quad \sum_{i=1}^k x_{[i]}\leq\sum_{i=1}^k y_{[i]} \quad k=1,2,\ldots,n-1, \quad \sum_{i=1}^n x_{[i]}=\sum_{i=1}^n y_{[i]}$$

When $x\prec y$ we say that x is majorized by y or that y majorizes x. A typical example is

$$\Big(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\Big)\prec\Big(\frac{1}{n-1},\frac{1}{n-1},\ldots,\frac{1}{n},0\Big)\prec(\frac{1}{2},\frac{1}{2},0,\ldots,0)\prec(1,0,\ldots,0)$$

Two other examples are

$$\Big(\frac{2s}{4},\frac{2s}{4},\frac{2s}{4},\frac{2s}{4}\Big)\prec\Big(a_1,a_2,a_3,a_4\Big)$$

where $(a_1,\ldots,a_3)$ are the sides of a quadrilateral and

$$\Big(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\Big)\prec(A,B,C,D)\prec(\pi,\pi,0,0)$$

where A,B,C,D, are the angles of a quadrilateral.

Definition A function $\varphi\colon\mathcal{D}\subseteq\mathbf{R}^n\to\mathbf{R}$ is said to be "Schur–convex" (concave) if for any $x\in\mathcal{D}$, $y\in\mathcal{D}$

$$x\prec y \implies \varphi(x)\leq\varphi(y) \quad (\varphi(x)\geq\varphi(y))$$

Now let $\mathcal{D}$ be a subset of $\mathbf{R}^n$ such that: i) $x\in\mathcal{D}\Longrightarrow x'\in\mathcal{D}$ if the components of x' are permutated respect to the components of x. ii) $\mathcal{D}$ is convex and has non–empty interior. We have the theorem (the proof is at page 57 of [1]; see also the remark [**A.4.a**]).

Theorem *Let $\mathcal{D}$ be the set $\{x \in \mathbf{R}^4 :\ 0 \le x_1 \le a, 0 \le x_2 \le a, 0 \le x_3 \le a, 0 \le x_4 \le a\}$ and let $\varphi\colon \mathcal{D} \to \mathbf{R}$ be a continuously differentiable function. Necessary and sufficient condition for φ to be Schur–convex (concave) are*

1) φ is symmetric

2) $(x_1 - x_2)(\varphi_{x_1} - \varphi_{x_2}) \ge 0 \quad (\le 0)$ for any $x \in \mathcal{D}$.

To apply the theorem to our inequality we define $\mathcal{D} = \{x \in \mathbf{R}^4,\ 0 \le x_i \le \pi\}$ and $\varphi\colon \mathcal{D} \to \mathbf{R}$,

$$\varphi(A,B,C,D) = \sin\left(\frac{A+\pi}{3}\right) + \sin\left(\frac{B+\pi}{3}\right) + \sin\left(\frac{C+\pi}{3}\right) + \sin\left(\frac{D+\pi}{3}\right) +$$

$$-\frac{1}{3}(8 + \sin A + \sin B + \sin C + \sin D)$$

and we show that $(A-B)(\varphi_A - \varphi_B) \ge 0$ if $A \ge B$ namely the function φ is Schür–convex. This implies that

$$\varphi(A,B,C,D) \ge \varphi(\pi/2, \pi/2, \pi/2, \pi/2) = 0$$

and the result is achieved.

We have to prove that $\varphi_A - \varphi_B \ge 0$ if $A \ge B$.

$$\varphi_A - \varphi_B = \cos\frac{A+\pi}{3} - \cos\frac{B+\pi}{3} - (\cos A - \cos B)$$

and it's nonnegative since the function $\cos x$ is strictly decreasing between $x = 0$ and $x = \pi$ and $A - B > \frac{A+\pi}{3} - \frac{B+\pi}{3} = \frac{A-B}{3}$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Sayan Mukherjee.

O172. Prove that if a 7×7 square board is covered by 38 dominoes such that each domino covers exactly two squares of the board, then it is possible to remove one domino after which the remaining 37 cover the board.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Consider a graph with 49 vertices, representing each vertex one of the squares in the board, where two vertices are joined by and edge iff their corresponding squares are covered by the same domino. Assume that the board can be covered by 38 dominoes so that no matter which domino is removed, at least one square becomes uncovered. Therefore, no edge in the graph may join two vertices, such that both of them are ends of other edges. Consider two vertices V_1 and V_2, joined by an edge. Clearly, one of them (wlog V_2) cannot have any other edges. If V_1 has other vertices joined to it through edges, these must also have no other edges connecting them to further vertices in the graph, or all points that may be reached from V_1 through edges of the graph form a tree with root V_1 and leaves $V_2, V_3, \ldots$, no other vertices involved. Clearly, no vertex can be joined to more than 4 other vertices, otherwise by Dirichlet's principle, two dominoes would cover the same two squares, and one of them could be removed leaving all 49 squares covered. Therefore, the graph may be decomposed in disjoint subgraphs, each one of them with k vertices, such that $k-1$ of them are joined by an edge to the remaining vertex, no other vertices or edges are present in each subgraph, ie, denoting n_k the number of subgraphs with k vertices (where clearly $k = 2, 3, 4, 5$) we have $2n_2 + 3n_3 + 4n_4 + 5n_5 = 49$ and $n_2 + 2n_3 + 3n_4 + 4n_5 = 38$. Assume that $n_5 = 4 - d$ where $d \geq 0$, hence $n_2 + n_3 + n_4 = 7 + d$, whereas $n_2 + 2n_3 + 3n_4 = 22 + 4d > 3n_2 + 3n_3 + 3n_4$, absurd, hence $n_5 \geq 5$. Now, on the board, a graph with $k = 5$ would represent a cross-shaped pentamino with one square joined to its four neighbours, hence no more than 2 squares on each side of the board, and none of its corners, can be covered by a domino involved in each one of these n_5 subgraphs. It follows that the n_5 subgraphs cover no more than 33 squares, ie $n_5 \leq 6$.

Assume that $n_5 = 5$, then $2n_2 + 3n_3 + 4n_4 = 24$ and $n_2 + 2n_3 + 3n_4 = 18$. Note now that

$$0 = 3 \cdot 24 - 4 \cdot 18 = 2n_2 + n_3,$$

and since $n_2, n_3 \geq 0$, we have $n_2 = n_3 = 0$, $n_4 = 6$. Note however that each corned must be covered by one of the n_4 subgraphs because it cannot be covered by one of the n_5 subgraphs, and each n_4 subgraph is a T-shaped tetramino, hence each one of the four corners of the board must be covered by one of the two squares forming the horizontal bar of the T, leaving one "prisoner" square, limiting on one side with the side of the board, and on two sides with sides of this T-shaped tetramino. This square must therefore be covered, next to each corner square, by another one of the n_4 subgraphs, hence $n_4 \geq 8$, contradiction. Therefore $n_5 = 6$.

Assume finally that $n_5 = 6$, then $2n_2 + 3n_3 + 4n_4 = 19$ and $n_2 + 2n_3 + 3n_4 = 14$, or

$$1 = 3 \cdot 19 - 4 \cdot 14 = 2n_2 + n_3,$$

or since $n_2, n_3 \geq 0$, it follows that $n_2 = 0$, $n_3 = 1$, and $n_4 = 4$. As in the case of $n_5 = 5$, it follows that one of the four corners of the board must be covered by the n_3 subgraph, while the other three must be covered by n_4 subgraphs, yielding $n_4 \geq 6$ (two of the n_4 subgraphs must be placed on or close to each corner), contradiction again. The conclusion follows.

It is possible however to cover the board with $n_5 = 3$ pentaminoes, $n_4 = 7$ tetraminoes and $n_3 = 2$ trominoes (leaving $n_2 = 0$), or the board can be covered by 37 dominoes, such that no domino can be removed without leaving uncovered squares.

Also solved by G.R.A.20 Math Problems Group, Roma, Italy.

O173. Find all triples (x, y, z) of integers such that

$$\frac{x^3+y^3+z^3}{3} - xyz = 2010 \max\{\sqrt[3]{x-y}, \sqrt[3]{y-z}, \sqrt[3]{z-x}\}.$$

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Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote $u = \sqrt[3]{x-y}$, $v = \sqrt[3]{y-z}$ and $w = \sqrt[3]{z-x}$, hence clearly $u^3+v^3+w^3 = 0$, which is known to have integral solutions only if at least one of u, v, w is zero, while

$$\frac{x^3+y^3+z^3}{3} - xyz = \frac{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}{3} =$$

$$= \frac{(x+y+z)(u^6+v^6+w^6)}{6},$$

$$(x+y+z)(u^6+v^6+w^6) = 12060u,$$

where we have assume wlog by cyclic symmetry in the variables that $u = \max\{u, v, w\}$. Note also that $u \geq 0$, with equality iff $u = v = w = 0$, since if $u \leq 0$, then $0 = u^3+v^3+w^3 \leq 0$, equality must hold, hence $u = v = w = 0$. We then have two possible cases:

- If $u = 0$, then $u = v = w = 0$, or $x = y = z$. Note that any $x = y = z$ results in both sides of the proposed equation being zero, hence $(x, y, z) = (r, r, r)$ is a solution for any integer r.
- If $u > 0$, and since one of v, w is necessarily zero, while $v^3 + w^3 = -u^3$, then the other one necessarily equals $-u$, ie (u, v, w) is some permutation of $(u, 0, -u)$, yielding

  $$(x+y+z)u^5 = 6030 = 2 \cdot 3^2 \cdot 5 \cdot 67,$$

  or since the RHS is not divisible by any fifth power, $u = 1$, and $x + y + z = 6030$, where $x = y + 1$. Now, since one of v, w is zero, either $z = y$ or $z = x$, yielding respectively $3y+1 = 6030$ or $3y+2 = 6030$, impossible since 6030 is a multiple of 3, but $3y+1, 3y+1$ are not.

It follows that (x, y, z) is an integral solution of the proposed equation if $x = y = z = r$ for some integer r.

O174. The point O is considered inside of the convex quadrilateral $ABCD$ of area S. Suppose that K, L, M, N are interior points of the sides AB, BC, CD, DA, respectively. If $OKBL$ and $OMDN$ are parallelograms of areas S_1 and S_2, respectively, prove that

(a) $\sqrt{S_1} + \sqrt{S_2} < 1.25\sqrt{S}$;

(b) $\sqrt{S_1} + \sqrt{S_2} < C_0\sqrt{S}$, where $C_0 = \max_{0<\alpha<\frac{\pi}{4}} \frac{\sin(2\alpha+\frac{\pi}{4})}{\cos\alpha}$.

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Solution by the authors

Without loss of generality we can suppose that points O and D are not on different sides of the line AC. Let $S_{ABC} = a, S_{ACD} = b$, and $S_{OAC} = x$. We have $S_{OKB} = S_{OBL} = S_{KLB} = \frac{S_1}{2}$ and

$$\frac{S_{OKB}}{S_{OAB}} \cdot \frac{S_{OBL}}{S_{OBC}} = \frac{KB}{AB} \cdot \frac{BL}{BC} = \frac{S_{KBL}}{S_{ABC}},$$

frpm which we find $S_1 = \frac{2S_{OAB} \cdot S_{OBC}}{a}$. Similarly, we obtain $S_2 = \frac{2S_{OAD} \cdot S_{OCD}}{b}$; hence

$$\sqrt{S_1} + \sqrt{S_2} \leq \frac{S_{OAB} + S_{OBC}}{\sqrt{2a}} + \frac{S_{OAD} + S_{OCD}}{\sqrt{2b}} = \frac{a+x}{\sqrt{2a}} + \frac{b-x}{\sqrt{2b}} = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}} - \frac{\sqrt{a}-\sqrt{b}}{\sqrt{2ab}}x.$$

If $a \geq b$, then $\sqrt{S_1} + \sqrt{S_2} \leq \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}} \leq \sqrt{a+b} = \sqrt{S}$. If $a < b$, then the point O can not be outside the parallelogram $ABCE$, thus $x \leq a$, so that

$$\sqrt{S_1} + \sqrt{S_2} \leq \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}} - \frac{\sqrt{a}-\sqrt{b}}{\sqrt{2ab}}a = \frac{b+\sqrt{2ab}-a}{\sqrt{2n}}.$$

Let $\frac{a}{b} = \tan^2\alpha$, where $\alpha \left[0, \frac{\pi}{4}\right]$. Then $\frac{b+\sqrt{2ab}-a}{\sqrt{2b}} : \sqrt{a+b} = \frac{\sin\left(2\alpha+\frac{\pi}{4}\right)}{\cos\alpha} \leq C_0$. Consequently $\sqrt{S_1}+\sqrt{S_2} \leq \frac{b+\sqrt{2ab}-a}{\sqrt{2b}} \leq C_0\sqrt{S}$. When $\alpha = \frac{\pi}{4}$, $\frac{\sin(2\alpha+\pi/4)}{\cos\alpha} = 1$, that is $C_0 \geq 1$. Thus in all cases $\sqrt{S_1} + \sqrt{S_2} \leq C_0\sqrt{S}$. If the quadrilateral satisfies the following conditions: $AB = BC, AD = CD$, $\frac{S_{ABC}}{\tan\alpha_0}$ where $C_0 = \frac{\sin\left(2\alpha_0+\frac{\pi}{4}\right)}{\cos\alpha_0}$ and $ABCO$ is a parallelogram, then $\sqrt{S_1} + \sqrt{S_2} = C_0\sqrt{S}$.

a) To prove the inequality it is sufficient to prove that when $0 \leq \alpha \leq \frac{\pi}{4}, \sin\left(2\alpha + \frac{\pi}{4}\right) < 1.25\cos\alpha$. Indeed, let $\phi \in \left[0, \frac{\pi}{4}\right]$ and $\cos\phi = \frac{4}{5}$, then, if $0 \leq \alpha < \phi$, then $\sin(2\alpha+\phi) \leq 1 = \frac{5}{4}\cos\phi$. And if $\phi \leq \alpha \leq \frac{\pi}{4}$, then $\tan\phi = \frac{3}{4} > \sqrt{2}-1 = \tan\frac{\pi}{8}$; hence $\phi > \frac{\pi}{8}$ and

$$\sin\left(2\alpha + \frac{\pi}{4}\right) \leq \sin\left(2\phi + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \cdot \frac{31}{25} < \frac{\sqrt{2}}{2} \cdot \frac{5}{4} \leq 1.25\cos\alpha.$$

Remark. Using derivatives it is possible to prove that

$$\tan\alpha_0 = \sqrt[3]{\sqrt{2}+1} - \sqrt[3]{\sqrt{2}-1} = 0.59\ldots \quad \text{while} \quad C_0 = 1.11\ldots.$$