An unexpectedly useful inequality

Pham Huu Duc

Abstract. A remarkable inequality by T. Andreescu and G. Dospinescu turns out to be useful in solving and obtaining a number of interesting results.

1 Introduction

The following inequality by T. Andreescu and G. Dospinescu appears with the following solution in [**?**] and turns out to be very useful in proving and discovering a number of interesting results:

Theorem 1. Let a, b, c and x, y, z be positive real numbers. Then

$$\frac{x(b+c)}{y+z} + \frac{y(c+a)}{z+x} + \frac{z(a+b)}{x+y} \geq \sqrt{3(ab+bc+ca)}.$$

Proof. The inequality is homogeneous in a, b, and c, so we may assume that $a+b+c=1$. We rewrite the inequality as follows

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{ax}{y+z} + \frac{by}{z+x} + \frac{cz}{x+y} + \sqrt{3(ab+bc+ca)}.$$

We apply the Cauchy-Schwarz Inequality to obtain

$$\frac{ax}{y+z} + \frac{by}{z+x} + \frac{cz}{x+y} \leq \sqrt{\left(\frac{x}{y+z}\right)^2 + \left(\frac{y}{z+x}\right)^2 + \left(\frac{z}{x+y}\right)^2}\sqrt{a^2+b^2+c^2}.$$

Applying the Cauchy-Schwarz Inequality one more time we get

$$\begin{aligned}
&\sqrt{\left(\frac{x}{y+z}\right)^2 + \left(\frac{y}{z+x}\right)^2 + \left(\frac{z}{x+y}\right)^2}\sqrt{a^2+b^2+c^2} + \frac{3}{4}\sqrt{ab+bc+ca} + \\
&\quad + \frac{3}{4}\sqrt{ab+bc+ca} \leq \sqrt{\left(\sum_{\text{cyc}}\left(\frac{x}{y+z}\right)^2 + \frac{3}{4} + \frac{3}{4}\right)\left(\sum_{\text{cyc}} a^2 + \sum_{\text{cyc}} bc + \sum_{\text{cyc}} bc\right)} \\
&\quad = \sqrt{\left(\frac{x}{y+z}\right)^2 + \left(\frac{y}{z+x}\right)^2 + \left(\frac{z}{x+y}\right)^2 + \frac{3}{2}}.
\end{aligned}$$

Thus it suffices to prove the following inequality

$$\left(\frac{x}{y+z}\right)^2 + \left(\frac{y}{z+x}\right)^2 + \left(\frac{z}{x+y}\right)^2 + \frac{3}{2} \leq \left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right)^2,$$

which is equivalent to

$$\frac{3}{4} \leq \frac{yz}{(x+y)(x+z)} + \frac{xz}{(y+z)(y+x)} + \frac{xy}{(z+x)(z+y)}.$$

Clearing denominators this reduces to

$$3(x+y)(y+z)(x+z) \leq 4(x^2y + y^2x + y^2z + z^2y + x^2z + z^2x)$$

or

$$6xyz \leq x^2y + y^2x + y^2z + z^2y + x^2z + z^2x,$$

which is true from the AM-GM inequality, and we are done.

A remarkable feature of this inequality is that no matter how complicated expressions x, y, and z may be, they all vanish in the right-hand side. Using the ideas from the proof above, we can deduce yet another useful and well-known result:

Theorem 2. Let a, b, c and x, y, z be nonnegative real numbers. Then

$$x(b+c) + y(c+a) + z(a+b) \geq 2\sqrt{(xy+yz+zx)(ab+bc+ca)}.$$

The proof of this inequality can be found in [**?**]. The following sections demonstrate the uses of Theorem 1 and Theorem 2 in solving numerous known inequalities. Furthermore, we present strengthenings of some of these results and some new inequalities as well.

2 Applications

1. Let x, y, z be positive real numbers. Prove that

$$xy(x+y-z) + yz(y+z-x) + zx(z+x-y) \geq \sqrt{3(x^3y^3 + y^3z^3 + z^3x^3)}.$$

Solution. Observe that

$$xy(x+y-z)+yz(y+z-x)+zx(z+x-y) = \frac{x(y^3+z^3)}{y+z} + \frac{y(z^3+x^3)}{z+x} + \frac{z(x^3+y^3)}{x+y}.$$

Setting $a = x^3$, $b = y^3$, $c = z^3$ and using Theorem 1 we get

$$\frac{x(y^3+z^3)}{y+z} + \frac{y(z^3+x^3)}{z+x} + \frac{z(x^3+y^3)}{x+y} \geq \sqrt{3(x^3y^3 + y^3z^3 + z^3x^3)},$$

and we are done.

2. Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{a(b^2+c^2)}{a^2+bc} + \frac{b(c^2+a^2)}{b^2+ca} + \frac{c(a^2+b^2)}{c^2+ab} \geq 3.$$

Solution. Let $x = a(b^2+c^2)$, $y = b(c^2+a^2)$, and $z = c(a^2+b^2)$. Then

$$\frac{x(b+c)}{y+z} = \frac{a(b^2+c^2)(b+c)}{b(c^2+a^2)+c(a^2+b^2)} = \frac{a(b^2+c^2)}{a^2+bc}.$$

Using Theorem 1 we have

$$\frac{a(b^2+c^2)}{a^2+bc} + \frac{b(c^2+a^2)}{b^2+ca} + \frac{c(a^2+b^2)}{c^2+ab} \geq \sqrt{3(ab+bc+ca)} = 3.$$

3. Let a, b, c be positive real numbers. Prove that

$$\frac{b^2+c^2}{a(b+c)} + \frac{c^2+a^2}{b(c+a)} + \frac{a^2+b^2}{c(a+b)} \geq \frac{3}{2}\left(\sqrt{\frac{(a+b+c)(a^2+b^2+c^2)}{abc}} - 1\right).$$

Solution. Note that

$$1 + \frac{b^2+c^2}{a(b+c)} = \frac{bc}{ab+ca}\left(\frac{c+a}{b} + \frac{a+b}{c}\right)$$

Applying Theorem 1 for $x = bc, y = ca, z = ab$ we have

$$3 + \sum_{\text{cyc}} \frac{b^2+c^2}{a(b+c)} \geq \sqrt{3}\sqrt{\frac{(a+b)(a+c)}{bc} + \frac{(b+c)(b+a)}{ca} + \frac{(c+a)(c+b)}{ab}}.$$

Furthermore,

$$\sum_{\text{cyc}} \frac{(a+b)(a+c)}{bc} = \sum_{\text{cyc}} \left(1 + \frac{a(a+b+c)}{bc}\right) = 3 + \frac{(a+b+c)(a^2+b^2+c^2)}{abc},$$

hence

$$\begin{aligned} 3+\sum_{\text{cyc}} \frac{b^2+c^2}{a(b+c)} &\geq \sqrt{9+\frac{3(a+b+c)(a^2+b^2+c^2)}{abc}} \\ &\geq \frac{3}{2}\left(1+\sqrt{\frac{(a+b+c)(a^2+b^2+c^2)}{abc}}\right), \end{aligned}$$

where the last inequality follows from the AM-GM inequality. Subtracting 3 from both sides yields the desired result.

4. Let $n \geq 2$ be a real number and let a, b, c be positive real numbers. Prove that

$$\frac{a^n + b^n}{a+b} + \frac{b^n + c^n}{b+c} + \frac{c^n + a^n}{c+a} \geq \sqrt{\frac{3(a^{n-1} + b^{n-1} + c^{n-1})(a^n + b^n + c^n)}{a+b+c}}.$$

Solution. Applying Theorem 2 we get

$$\sum_{\text{cyc}} \frac{b^n + c^n}{b+c} = \sum_{\text{cyc}} \frac{b^n c^n}{b+c}\left(\frac{1}{b^n} + \frac{1}{c^n}\right) \geq$$

$$2\sqrt{\left(\frac{a^{2n}b^n c^n}{(a+b)(a+c)} + \frac{b^{2n}c^n a^n}{(b+c)(b+a)} + \frac{c^{2n}a^n b^n}{(c+a)(c+b)}\right)\left(\frac{1}{b^n c^n} + \frac{1}{c^n a^n} + \frac{1}{a^n b^n}\right)}$$

$$= 2\sqrt{\left(\frac{a^n}{(a+b)(a+c)} + \frac{b^n}{(b+c)(b+a)} + \frac{c^n}{(c+a)(c+b)}\right)(a^n + b^n + c^n)}.$$

It remains to prove that

$$\frac{a^n}{(a+b)(a+c)} + \frac{b^n}{(b+c)(b+a)} + \frac{c^n}{(c+a)(c+b)} \geq \frac{3}{4} \cdot \frac{a^{n-1} + b^{n-1} + c^{n-1}}{a+b+c}.$$

Our inequality reduces to

$$(a^n(b+c) + b^n(a+c) + c^n(a+b)) \geq \frac{3}{4}\left(\frac{a^{n-1} + b^{n-1} + c^{n-1}}{a+b+c}\right)(a+b)(b+c)(c+a)$$

or

$$4\left[(a^{n-1} + b^{n-1} + c^{n-1})(ab + bc + ca) - abc(a^{n-2} + b^{n-2} + c^{n-2})\right](a+b+c)$$

$$\geq 3(a^{n-1} + b^{n-1} + c^{n-1})((a+b+c)(ab+bc+ca) - abc).$$

It suffices to prove that

$$(a^{n-1} + b^{n-1} + c^{n-1})(a+b+c)(ab+bc+ca) \geq$$
$$abc[4(a^{n-2} + b^{n-2} + c^{n-2})(a+b+c) - 3(a^{n-1} + b^{n-1} + c^{n-1})].$$

Using the AM-GM inequality, $(a+b+c)(ab+bc+ca) \geq 9abc$, hence it remains to prove that

$$3(a^{n-1} + b^{n-1} + c^{n-1}) \geq (a^{n-2} + b^{n-2} + c^{n-2})(a+b+c),$$

which is true from Chebyshev's inequality.

3 Conclusion

We have shown the usefulness of Theorem 1 and Theorem 2 in providing elegant solutions to a number of interesting inequalities, very difficult to prove by other means.

We end this article with a list of further inequalities which can be solved by using the ideas presented above.

1. Let a, b, c be positive real numbers. Prove that

$$\frac{ab(a^3+b^3)}{a^2+b^2} + \frac{bc(b^3+c^3)}{b^2+c^2} + \frac{ca(c^3+a^3)}{c^2+a^2} \geq \sqrt{3abc(a^3+b^3+c^3)}.$$

2. Let a, b, c be positive real numbers. Prove that

$$ab\,\frac{a+c}{b+c} + bc\,\frac{b+a}{c+a} + ca\,\frac{c+b}{a+b} \geq \sqrt{3abc(a+b+c)}.$$

3. Let a, b, c be positive real numbers. Then for any real number k, the following inequality holds

$$\frac{a^k+b^k}{a+b} + \frac{b^k+c^k}{b+c} + \frac{c^k+a^k}{c+a} \geq \sqrt{\frac{8(a+b+c)(a^kb^k+b^kc^k+c^ka^k)}{(a+b)(b+c)(c+a)}}.$$

References

[1] Andreescu T., Cîrtoaje V., Dospinescu G., Lascu M., *Old and New Inequalities*, GIL Publishing House, 2004

[2] Vedula N. Murty, problem 3076, *Crux Mathematicorum*, vol. 31, no. 7.

Author: Pham Huu Duc, Ballajura, Australia