## Junior problems

J97. Let $a, b, c, d$ be integers such that $a+b+c+d=0$. Prove that $a^5+b^5+c^5+d^5$ is divisible by 30.

*Proposed by Johan Gunardi, Jakarta, Indonesia*

*First solution by Magkos Athanasios, Kozani, Greece*

Because $x^5 \equiv x \pmod 5$, $x^3 \equiv x \pmod 3$, $x^2 \equiv x \pmod 2$ then $x^5 \equiv x^3 \equiv x \pmod 3$ and $x^5 \equiv x \pmod 2$ for any integer $x$. Therefore, $a^5+b^5+c^5+d^5 \equiv a+b+c+d \equiv 0 \pmod k$ where $k=2,3,5$. Thus, $a^5+b^5+c^5+d^5 \vdots 30$. For any natural $m>1$ product of $m$ consequtive integers allways divisible by $m$, then $x^2-x=(x-1)x$ divisible by $2, x^3-x=(x-1)x(x+1)$ divisible by 3 and $x^5-x=x\left(x^2-1\right)\left(x^2+1\right)=x\left(x^2-1\right)\left(x^2-4+5\right)=(x-2)(x-1)x(x+1)(x+2)+5x\left(x^2-1\right)$ divisible by 5.

*Second solution by John T. Robinson, Yorktown Heights, NY, USA*

We will prove that $x^5 \equiv x \pmod{30}$. First note that $x^5-x=(x-1)\cdot x\cdot(x+1)\cdot(x^2+1)$. Therefore we want to show that $(x-1)\cdot x\cdot(x+1)\cdot(x^2+1)$ is always divisible by $30=2\cdot 3\cdot 5$. In the following assume $x-1>0$. Then there will always be factors of 2 and 3 among $x-1, x$, and $x+1$, and in some cases a factor of 5 as well. The only triples $x-1, x, x+1$ for which there is not a factor of 5 are those of the form

$$5\cdot n+1,\ 5\cdot n+2,\ 5\cdot n+3,$$

or

$$5\cdot n+2,\ 5\cdot n+3,\ 5\cdot n+4.$$

In the first case, $x=5\cdot n+2$, so $x^2+1=25\cdot n^2+20\cdot n+5$, which gives a factor of 5. In the second case, $x=5\cdot n+3$, so $x^2+1=25\cdot n^2+30\cdot n+10$, which also gives a factor of 5. It follows that $x^5-x$ is always divisible by 30 and hence the conclusion.

*Third solution by Nguyen Manh Dung, Hanoi University of Science, Vietnam*

We will prove a lemma first:
**Lemma.** Let $x$ be an integer, then $30|x^5-x$
**Proof.** We have $x^5-x=(x-1)x(x+1)\left(x^2+1\right)$.
Because $6|\,(x-1)x(x+1)$ so $6|x^5-x$.

- If $x\equiv 0(mod\ 5)$ then $5|x$.

- If $x \equiv 1 (mod\ 5)$ then $5 | x+1$.
- If $x \equiv 2 (mod\ 5)$ then $5 | x^2+1$.
- If $x \equiv 3 (mod\ 5)$ then $5 | x^2+1$.
- If $x \equiv 4 (mod\ 5)$ then $5 | x+1$.

So for all integer $x$, $5 | x^5 - x$. But $\gcd(5,6) = 1$ , it follows that $30 | x^5 - 1$. Coming back to the problem, from the Lemma, we have

$$\begin{aligned} a^5+b^5+c^5+d^5 &= \left(a^5+b^5+c^5+d^5\right) - (a+b+c+d) \\ &= \left(a^5-a\right)+\left(b^5-b\right)+\left(c^5-c\right)+\left(d^5-d\right) \vdots\ 30 \end{aligned}$$

So $a^5+b^5+c^5+d^5$ is divisible by 30.

*Fourth solution by Miguel Amengual Covas, Mallorca, Spain*

Since $a+b+c+d=0$, we have $d=-a-b-c$ and

$$\begin{aligned} a^5+b^5+c^5+d^5 &= a^5+b^5+c^5-(a+b+c)^5 \\ &= -5(a+b)(b+c)(c+a)(a^2+b^2+c^2+ab+bc+ca) \end{aligned}$$

(see Example 3 on page 437 of H. S. Hall and S. R. Knight, Higher Algebra, Fourth edition, London, (1899).) Hence $a^5+b^5+c^5+d^5$ is divisible by 5. By the Pigeonhole principle, at least two of the numbers $a,b,c$ have the same parity. Hence at least one of the sums $a+b, b+c, c+a$ is even, making $a^5+b^5+c^5+d^5$ divisible by 2. Finally, we consider $a,b,c$ modulo 3. The possibilities are $(0,0,0),(0,0,1),(0,0,2),(0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1),(0,2,2),$ $(1,1,1),(1,1,2),(1,2,2)$ and $(2,2,2)$. A simple check proves that $a^5+b^5+c^5+d^5$ is divisible by 3. Therefore $a^5+b^5+c^5+d^5$ is divisible by the least common multiple of 2, 3 and 5, i.e., $a^5+b^5+c^5+d^5$ is divisible by 30.

*Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Michel Bataille, France; Oles Dobosevych, Ukraine; Ivanov Andrei, Moldova.*

J98. Find all primes $p$ and $q$ such that 24 does not divide $q+1$ and $p^2q+1$ is a perfect square.

*Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

*First solution by Bojan Joveski, Yahya Kemal College, Skopje, Macedonia*

*i)* If one of $p$ or $q$ is even

Since $x^2 = p^2q+1$, x is odd, ie. $x^2 \equiv 1 \pmod 8$

$$\begin{aligned} p^2q+1 &\equiv 1 \pmod 8 \\ p^2q &\equiv 0 \pmod 8 \end{aligned}$$

Because both $p$ and $q$ are primes, and 2 is the only even prime, *p=q=2*

*ii)* If both $p, q > 2$ in that case $x > 2$

$$\begin{aligned} p^2q+1 &= x^2 \\ p^2q &= (x-1)(x+1) \end{aligned}$$

Since x is even, x-1 and x+1 will be both odd, and they are relatively prime because $\gcd(x-1, x+1) = \gcd(x-1, 2) = 1$

So,

$$x-1 = q, x+1 = p^2 \tag{1}$$

or

$$x-1 = p^2, x+1 = q \tag{2}$$

Since $x > 2$ and the $gcd(x-1, x+1) = 1$ the other cases are not possible.

(1) $\Rightarrow q+1 = p^2-1$

*i)* If *p=3*, we have solution *q=7*

*ii)* If $p \neq 3$, then $p^2 \equiv 1 \pmod 3$ and $p^2 \equiv 1 \pmod 8$ which contradicts with the condition that 24 doesn't divide *q+1*

(2) $\Rightarrow p^2+2 = q$ For $p = 3$, we have solution $q = 11$ If $p \neq 3$, then $p^2+2 \equiv 0 \pmod 3$ which cannot be prime.

So, all the solutions are: $(p, q) \in \{(2,2), (3,7), (3,11)\}$

*Second solution by Ivanov Andrei, Moldova*

Let $p^2q+1=r^2$. Then

$$(r-1)(r+1)=p^2q$$

Because $p$ and $q$ are primes we can have the folloing posibillities:

1) $\begin{array}{c} r-1=1 \\ r+1=p^2q \end{array}$

We have $r=2$ and $p^2q=3$, which does not have any solutions.

2) $\begin{array}{c} r-1=p^2q \\ r+1=1 \end{array}$

Then $r=0$ and $p^2q=-1$, which does not have any solutions.

3) $\begin{array}{c} r-1=p \\ r+1=pq \end{array}$

After substracting this 2 equations we obtain $pq=p+2 \Leftrightarrow p(q-1)=2$. The single solution is $(p,q)=(2,2)$ which gives us $p^2q+1=9$.

4) $\begin{array}{c} r-1=pq \\ r+1=p \end{array}$

Beause $pq>p$ this system does not have solutions.

5) $\begin{array}{c} r-1=q \\ r+1=p^2 \end{array}$

By substracting we obtain $p^2-1=q+1$. If $p=2$ then $q=2$ (we already have this solution) and for $p=3$ we get $q=7$, so $(p,q)=(3,7)$ and $p^2q+1=64$. But for $p>3$ we have $p^2-1$ is divisible by 24 (because $p^2-1=(p-1)(p+1)$ and one of these numbers is divisible by 3, one by 4 and the other by 2), so $q+1$ is divisible by 24, contradiction.

6) $\begin{array}{c} r-1=p^2 \\ r+1=q \end{array}$

Then $q=p^2+2$. If $p=3$ then $q=11$ and we have the solution $(p,q)=(3,11)$ and $p^2q+1=100$. If $p$ is not divisible by 3 then $p^2\equiv 1 \pmod 3$ and $p^2+2\equiv 0 \pmod 3$, so $q$ is divisible by 3 and $q=3$, but then $p=1$, which is not a prime.

*Third solution by Oles Dobosevych, Ukraine*

Let $p^2q+1=a^2$. Or $p^2q=(a+1)(a-1)$.

As we have, that $a+1>a-1$ and $p,q$ are primes we would have the following cases:

a) $a+1=p^2q$, $a-1=1$;

b) $a+1=pq$, $a-1=p$;

c) $a+1=p^2$, $a-1=q$;

d) $a+1=q$, $a-1=p^2$.

a) From the second equation we would have that $a=2$. But on the other hand, as we have that $p, q$ are primes we can say that $p, q \geq 2$. So $3=p^2q \geq 8$, contradiction.

b) If we subtract from first equation the second we would have that $2=pq-p$. The righthand side of our equation is divisible on $p$. Thus the lefthand side must be divisible too so $p=2$ and $q=2$. This pair satisfies our conditions as $24 \nmid 3$ and $2^2 \cdot 2+1=9$ is perfect square.

c) If we subtract from the first equation the second we would have that $p^2-1=q+1$. First let us prove that if $p \neq 2$ and $p \neq 3$ we would have that $q+1 \vdots 24$. As, we have that $p$ is prime (not equal to two or three) we may write that $p^2-1 \equiv 0 \mod 8$ and $p^2-1 \equiv 0 \mod 3$. Because 2 and 3 are relative prime we have that $p^2-1 \equiv 0 \mod 24$ or $q+1 \vdots 24$ which is a contradiction. So in this case $p$ can be equal to two or three. In this way we find two solutions $(2;2)$ and $(3;7)$. A simple check shows that they are indeed correct solutions.

d) If we subtract from the first equation the second we would have that $p^2+2=q$. If $p \neq 3$, then $p^2 \equiv 1 \mod 3$ and thus $q \vdots 3$. This means that $q=3$, $p=1$ but we would get that $p$ which is not prime. If $p=3$, $q=11$ it is easy to check that this pair is right.

**Answer:**$(2;2)$**,** $(3;7)$**,** $(3;11)$**.**

*Fourth solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain*

Let us assume that $p=2$. Then, $4q+1$ is an odd perfect square, or it may be written as $(2n+1)^2=4n(n+1)+1$, and since $n(n+1)=q$ must be a prime, then $q=2$. Note that $q+1=3$ is not divisible by 24 and $p^2q+1=9$ is a perfect square.

Let us now assume that $q=2$. Then, $2p^2+1$ is an odd perfect square, or it may be written as $(2n+1)^2=4n(n+1)+1$, and $p^2=2n(n+1)$. Clearly $p$ must divide 2, or $p=2$, and we fall back on the previous case.

Any other solution must have $p, q$ odd, hence $p^2q+1$ is an even perfect square, or it may be written as $4m^2$, and $p^2q=4m^2-1=(2m+1)(2m-1)$. Clearly, $2m+1$ and $2m-1$ are relatively prime, hence either one of them is 1 (absurd since then $p^2q=3$), or one equals $p^2$ and the other one equals $q$.

Assume first that $2m+1=q=p^2+2$, and since $q=3$ yields $p=1$ absurd, then $p^2+2$ is not a multiple of 3. But if $p$ is not a multiple of 3, $p^2+2=(p-1)(p+1)+3$ must be a multiple of 3 since either $p-1$ or $p+1$ is a multiple of 3. Hence $p=3$, $q=11$, yielding $q+1=12$ not divisible by 24, and $p^2q+1=100$ a perfect square.

Assume finally that $q = 2m - 1 = p^2 - 2$. Since $p$ is an odd integer, then $q+1 = p^2 - 1 = (2m+1)^2 - 1 = 4m(m+1)$, and since either $m$ or $m+1$ is even, then 8 divides $q+1$. Furthermore, if 3 does not divide $p$, then $p^2 \equiv 1 \pmod 3$, and 24 divides $q+1$, absurd. So $p = 3$, $q = 7$, yielding $q+1 = 8$ not divisible by 24 and $p^2 q + 1 = 64$, a perfect square.

The only solutions are then $p = q = 2$, or $p = 3$, $q = 7$, or $p = 3$, $q = 11$.

J99. In a triangle $ABC$, let $\phi_a, \phi_b, \phi_c$ be the angles between medians and altitudes emerging from the same vertex. Prove that one of the numbers $\tan\phi_a, \tan\phi_b, \tan\phi_c$ is the sum of the other two.

*Proposed by Oleh Faynshteyn, Leipzig, Germany*

*First solution by Arkady Alt, San Jose, California, USA*

If we define $\phi_a, \phi_b, \phi_c$ as oriented angles between medians and altitudes ( let it be counterclockwise orientation) then statement of problems becomes Prove that $\tan\phi_a + \tan\phi_b + \tan\phi_c = 0$. Since $a = b\cos C + c\cos B$ and $\tan\phi_a = \dfrac{\frac{a}{2} - c\cos B}{c\sin B}$ then, applying the Sine Theorem we obtain

$$\begin{aligned}\tan\phi_a &= \frac{b\cos C - c\cos B}{c\sin B}\\ &= \frac{2R\sin B\cos C - 2R\sin C\cos B}{2R\sin C\sin B}\\ &= \frac{\sin B\cos C - \sin C\cos B}{\sin C\sin B}\\ &= \cot C - \cot B\end{aligned}$$

therefore

$$\sum_{cyc}\tan\phi_a = \sum_{cyc}(\cot C - \cot B) = 0.$$

*Second solution by Magkos Athanasios, Kozani, Greece*

Let $AM$ be the median and $AD$ the altitude emerging from vertex $A$. It is obvious that

$$\tan\phi_a = \frac{MD}{AD}.$$

Recalling that $|b^2 - c^2| = 2aMD$, we obtain

$$\tan\phi_a = \frac{|b^2 - c^2|}{2ah_a} = \frac{|b^2 - c^2|}{4F},$$

where $F$ is the area of triangle $ABC$. Similarly, we have

$$\tan\phi_b = \frac{|c^2 - a^2|}{4F}, \quad \tan\phi_c = \frac{|a^2 - b^2|}{4F}.$$

Without loss of generality assume that $a \ge b \ge c$. We find then that $\tan\phi_b = \tan\phi_a + \tan\phi_c$.

*Third solution by Ivanov Andrei, Moldova*

Suppose that $\angle A \le \angle B \le \angle C$. Let $M$ be midpoint of $AC$ and $H$ be the altitude from $B$. Then $A, M, H, C$ lie on $AC$ in this order. Then $AH = BH \cot \angle A$ and $HC = BH \cot \angle C$, so

$$\begin{aligned} MH = MC - HC = \frac{1}{2}AC - HC = \frac{1}{2}(AH - BH) \\ = \frac{1}{2}BH(\cot \angle A - \cot \angle C) \end{aligned}$$

Then $\tan \phi_b = \frac{MH}{BH} = \frac{1}{2}(\cot \angle A - \cot \angle C)$. By analogy $\tan \phi_a = \frac{1}{2}(\cot \angle B - \cot \angle C)$ and $\tan \phi_c = \frac{1}{2}(\cot \angle B - \cot \angle A)$. Note that

$$\begin{aligned} \tan \phi_c + \tan \phi_b &= \frac{1}{2}(\cot \angle A - \cot \angle C + \cot \angle B - \cot \angle A) \\ &= \frac{1}{2}(\cot \angle B - \cot \angle C) = \tan \phi_a. \end{aligned}$$

*Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Michel Batailll, France; Oles Dobosevych, Ukraine.*

J100. Consider the set of points from the plane such that the distance between any two points is a real number from the interval $[a, b]$. Prove that the number of these points is finite.

*Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

*First solution by John T. Robinson, Yorktown Heights, NY, USA*

Note that $a$ must be positive (otherwise there could be an infinite number of points), that is we assume $0 < a \leq b$. Pick any point: all other points must be at a distance $b$ or less from this point, that is all points are in a disc $D$ of radius $b$. If the set of points are collinear, then $1 + 2\frac{b}{a}$ (which is finite) is an upper bound on the number of points. If the points are not all collinear, then there exists a triangulation of the set of points into a set of non-overlapping triangles (for example, by constructing the Voronoi tessellation determined by the points, and then using that to generate a Delaunay triangulation, although any method of triangulation will work for the current purpose). For every triangle, each side is of length at least a, therefore every triangle has a minimum area $T$. An upper bound on the number of triangles is found by dividing the area of D by $T$, which is finite. Since the number of triangles is finite, the number of points is therefore finite.

*Second solution by Oles Dobosevych, Ukraine*

First notice, that if $a = 0$ then there exists such a set of points, that the number of points in it is infinite. For example let $A_1$, $A_2$ be points such that $A_1A_2 = b$ and $A_n$ be the midpoint of $A_1A_{n-1}$. Now let us prove that when $a > 0$ then the number of points is finite. First of all notice that if we draw the circle with center in one of the points and radius $2b$ all points must be inside this circle. Also notice that in the circle with radius $\frac{a}{4}$ there cannot be two different points of the set. It means that in the square with side $\frac{\sqrt{2}a}{4}$ there cannot be two different points because the square is inscribed in the circle with radius $\frac{a}{4}$. Now let us find such a positive integer $k$ that we can put a circle with radius $2b$ into a square with side $\frac{\sqrt{2}ka}{4}$. Such a $k$ exists; for example it can be $k = \left[\frac{4b}{\frac{\sqrt{2}a}{4}}\right] + 1$. Notice that all points must be inside the square with side $\frac{\sqrt{2}ka}{4}$. If we divide square with side $\frac{\sqrt{2}ka}{4}$ into $k^2$ squares with side $\frac{\sqrt{2}a}{4}$, we will get that all points must lie inside large square and any to point cannot lie in the same small square. From pigeonhole principle we get that it's can't be more then $k^2$ points.

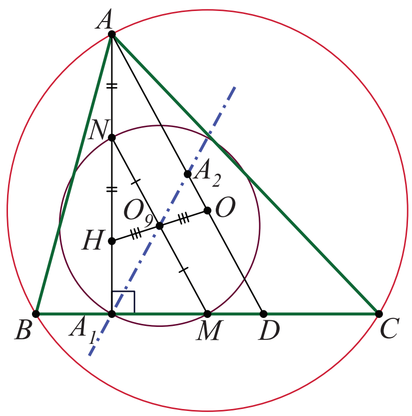
*Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain*

Call $P$ any set of such points, and take any element $p \in P$. For any $\delta > 0$, draw the circle $\Gamma$ with center $p$ and radius $b + \delta$. Clearly, all points of $P$ are contained inside $\Gamma$, since otherwise their distance to $p$ would be larger than $b$. Call $N = \lceil \frac{4b+4\delta}{a} \rceil$, ie, the smallest integer not smaller than $\frac{4b+4\delta}{a}$, then take any square $\Sigma$ circumscribed to $\Gamma$, and divide it into $N^2$ smaller squares by drawing lines parallel to the sides of $\Sigma$. The sidelength of each smaller square is then $\frac{2b+2\delta}{N} \le \frac{a}{2}$, hence the maximum distance between two points on or inside each square is at most $\frac{a}{\sqrt{2}}$, clearly smaller than $a$. Each smaller square contains in its interior or on its boundary at most one point that belongs to $P$, hence the number of points in $P$ is at most $N^2$. The conclusion follows.

J101. Consider triangle $ABC$ with circumcenter $O$ and orthocenter $H$. Let $A_1$ be the projection of $A$ onto $BC$ and let $D$ be the intersection of $AO$ with $BC$. Denote by $A_2$ the midpoint of $AD$. Similarly, we define $B_1, B_2$ and $C_1, C_2$. Prove that $A_1A_2, B_1B_2, C_1C_2$ are concurrent.

*Proposed by Andrea Munaro, Italy and Ivan Borsenco, MIT, USA*

*First solution by Ivanov Andrei, Moldova*




We will prove that lines $A_1A_2$, $B_1B_2$, $C_1C_2$ are concurrent at $O_9$ (midpoint of $OH$). Let $A_2'$ be the intersection point of the lines $A_1O_9$ with $AD$. We need to prove that $A_2'$ is midpoint of $AD$. Let $M$ and $N$ be midpoints of the segments $AH$ and $BC$. Then $A_1, M, N$ lie on the nine point circle of the triangle $ABC$. Then $MN$ is diameter of this circle because $\angle MA_1N = 90°$. So lines $MN$ and $OH$ intersect at $O_9$. From this $O_9N$ is midline in triangle $OHA \Rightarrow MN \parallel AD$. So

$$\frac{O_9N}{AA_2'} = \frac{A_1A_2'}{AO_9} = \frac{O_9M}{A_2'D} \Leftrightarrow \frac{AA_2'}{A_2'D} = \frac{O_9N}{O_9M} = 1.$$

*Second solution by Oles Dobosevych, Ukraine*

Let us solve this problem using directed angles. Let $\measuredangle CAB = \alpha$, $\measuredangle ABC = \beta$, and $\measuredangle BCA = \gamma$. Because $BB_1$, $CC_1$ and $AA_1$ are altitudes then $\measuredangle B_1A_1C = \measuredangle CAB = \alpha$ and $\measuredangle CB_1A_1 = \measuredangle C_1B_1A = \measuredangle ABC = \beta$. Also as $\measuredangle ABB_1 + \measuredangle BB_1A + \measuredangle B_1AB = 0$, and $\measuredangle B_1AB = \alpha$, $\measuredangle BB_1A = \frac{\pi}{2}$ we get $\measuredangle ABB_1 = \frac{\pi}{2} - \alpha$. On the other hand, $\measuredangle BOA = 2\measuredangle BCA = 2\gamma$ and since $BO = AO$ it means that triangle $BOA$ is isosceles, and $\measuredangle ABO = \measuredangle OAB$. Because $\measuredangle ABO + \measuredangle BOA + \measuredangle OAB = 0$ we get that $2\measuredangle ABO + 2\gamma = 0$. This implies that

$\measuredangle ABO = \frac{\pi}{2} - \gamma$ (because if $\measuredangle ABO = \pi - \gamma$ we will get that $\measuredangle BAO = \pi - \gamma$, so $\angle ABO + \angle BAO + \angle OAB = 2\pi$, which is false). Also

$$\measuredangle ABO = \measuredangle ABB_1 + \measuredangle B_1BB_2$$

$$\frac{\pi}{2} - \gamma = \frac{\pi}{2} - \alpha + \measuredangle B_1BB_2$$

$$\measuredangle B_1BB_2 = \alpha - \gamma$$

$$\measuredangle B_1BD_2 + \measuredangle BD_2B_1 + \measuredangle D_2B_1B = 0$$

$$\alpha - \gamma + \measuredangle BD_2B_1 + \frac{\pi}{2} = 0$$

$$\measuredangle BD_2B_1 = \frac{\pi}{2} - \alpha + \gamma.$$

$B_2$ is the midpoint of $BD_2$ $B_1B_2D_2$ is isosceles. So $\measuredangle BD_2B_1 = \measuredangle D_2B_1B_2 = \frac{\pi}{2} - \alpha + \gamma$. Since $\measuredangle CB_1B_2 = \measuredangle CB_1A_1 + \measuredangle A_1B_1B_2$ and

$$\alpha + \beta + \gamma = 0$$

$$\frac{\pi}{2} - \alpha + \gamma = \beta + \measuredangle A_1B_1B_2$$

$$\measuredangle A_1B_1B_2 = \frac{\pi}{2} + 2\gamma.$$

Furthermore $0 = \measuredangle CB_1A_1 + \measuredangle A_1B_1B_2 + \measuredangle B_2B_1C_1 + \measuredangle C_1B_1A$.

$$0 = \beta + \frac{\pi}{2} + 2\gamma + \measuredangle B_2B_1C_1 + \beta$$

$$\measuredangle B_2B_1C_1 = \frac{\pi}{2} + 2\alpha.$$

In the same way we can calculate $\measuredangle C_1A_1A_2 = \frac{\pi}{2} + 2\beta$, $\measuredangle A_2A_1B_1 = \frac{\pi}{2} + 2\gamma$, $\measuredangle B_1C_1C_2 = \frac{\pi}{2}+2\alpha$, $\measuredangle C_2C_1A_1 = \frac{\pi}{2}+2\beta$. So $\angle A_1B_1B_2 = \angle A_2A_1B_1$, $\angle B_1C_1C_2 = \angle B_2B_1C_1$, $\angle C_1A_1A_2 = \angle C_2C_1A_1$. So we can write that

$$\frac{\sin\angle A_1B_1B_2 \sin\angle B_1C_1C_2 \sin\angle C_1A_1A_2}{\sin\angle B_2B_1C_1 \sin\angle C_2C_1A_1 \sin\angle A_2A_1B_1} = 1$$

From the trigonometric from of Ceva's theorem we conclude that $A_1A_2$, $B_1B_2$, $C_1C_2$ are concurrent.

*Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Mihai Miculita, Oradea, Romania.*

J102. Evaluate

$$\binom{2008}{3} - 2\binom{2008}{4} + 3\binom{2008}{5} - 4\binom{2008}{6} + \cdots - 2004\binom{2008}{2006} + 2005\binom{2008}{2007}.$$

*Proposed by Zuming Feng, Phillips Exeter Academy, USA*

*First solution by John T. Robinson, Yorktown Heights, NY, USA*

Let $f(x) = \frac{(1+x)^{2008}}{x^2}$. Then expanding this out, taking the derivative, and evaluating the resulting expression at $x = -1$, we find (since $f'(-1) = 0$) that the sum given in the problem statement is equal to $2008 + 2006 - 2 = 4012$.

*Second solution by Manh Dung Nguyen, Vietnam*

The expression equals to

$$\begin{aligned}
E &= \sum_{n=1}^{2005} (-1)^n \, n \binom{2008}{n+2} \\
&= \sum_{n=1}^{2005} \left[ (-1)^{n+1} (n+2) \binom{2008}{n+2} - 2 (-1)^{n+1} \binom{2008}{n+2} \right] \\
&= \sum_{n=1}^{2005} \left[ 2008 (-1)^{n+1} \binom{2007}{n+1} + 2 (-1)^{n+2} \binom{2008}{n+2} \right] \\
&= 2008 \sum_{n=2}^{2006} (-1)^n \binom{2007}{n} + 2 \sum_{n=3}^{2007} (-1)^n \binom{2008}{n}
\end{aligned}$$

On the other hand,

$$\sum_{n=0}^{2007} (-1)^n \binom{2007}{n} = (1-1)^{2007} = 0$$

It follows that

$$\sum_{n=2}^{2006} (-1)^n \binom{2007}{n} + \binom{2007}{0} - \binom{2007}{1} - \binom{2007}{2007}$$

or

$$\sum_{n=2}^{2006} (-1)^n \binom{2007}{n} = 2007$$

Similarly,

$$\sum_{n=3}^{2007} (-1)^n \binom{2008}{n} = -2013022$$

Hence

$$E = 2007 \times 2008 - 2 \times 2013022 = 4012$$

*Third solution by Oles Dobosevych, Ukraine*

Let

$$A = \binom{2008}{3} - 2\binom{2008}{4} + 3\binom{2008}{5} + \ldots + 2005\binom{2008}{2007}$$

From the Binomial Theorem we can write that

$$-\binom{2008}{0} + \binom{2008}{1} - \binom{2008}{2} + \ldots - \binom{2008}{2008} = (-1+1)^{2008} = 0. \qquad (1)$$

Using (1) we get that

$$\begin{aligned} A &= A + 2\left(-\binom{2008}{0} + \binom{2008}{1} - \binom{2008}{2} + \ldots - \binom{2008}{2008}\right) \\ &= \left(-2\binom{2008}{0} + 2\binom{2008}{1} - 2\binom{2008}{2008}\right) \\ &+ \left(-2\binom{2008}{2} + 3\binom{2008}{3} - \ldots + 2007\binom{2008}{2007}\right). \quad (2) \end{aligned}$$

Since $k\binom{n}{k} = n\binom{n-1}{k-1}$, we can rewrite (2), as

$$A = -2 + 2008 \cdot 2 - 2 + 2008\left(-\binom{2007}{1} + \binom{2007}{2} + \ldots + \binom{2007}{2006}\right) \qquad (3)$$

But as we have that $\binom{n}{k} = \binom{n}{n-k}$ (3) can be rewrite as

$$A = -4 + 4016 + 2008 \cdot 0 = 4012$$

*Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Michel Bataille, France.*

## Senior problems

S97. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n \geq \left(\sqrt[n]{x_1x_2\cdots x_n}\right)^{n-1}\sqrt{\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}}.$$

*Proposed by Arkady Alt, San Jose, California, USA*

*First solution by Manh Dung Nguyen, Hanoi University of Science, Vietnam*

Without loss of generality we may assume that $x_1+x_2+\cdots+x_n = n$.
The inequality is equivalent to

$$(x_1x_2\cdots x_n)^{\frac{2(n-1)}{n}}\left(x_1^2+x_2^2+\cdots+x_n^2\right) \leq n$$

For n = 2, the inequality reduces to $(x_1-x_2)^4 \geq 0$.
For $n \geq 3$ , assume that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ and apply the **EV- Theorem:** For $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n, x_1+x_2+\cdots+x_n = n$ and $x_1^2+x_2^2+\cdots+x_n^2 =$ constant The product $x_1x_2\cdots x_n$ is maximal when $0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n$ Consequently, it suffices to show the inequality for $x_1 = x_2 = \cdots = x_{n-1} = x$ and $x_n = y$ where $0 \leq x \leq 1 \leq y$ and $(n-1)x+y = n$. Under the circumstance, the inequality reduces to

$$x^{\frac{2(n-1)^2}{n}}y^{\frac{2(n-1)}{n}}\left[(n-1)x^2+y^2\right] \leq n.$$

For $x = 0$, the inequality is trivial. For $x > 0$ , it is equivalent to $f(x) \leq 0$ where

$$f(x) = \frac{2(n-1)^2}{n}\ln x + \frac{2(n-1)}{n}\ln y + \ln\left[(n-1)x^2+y^2\right] - \ln n.$$

With $y = n-(n-1)x$. We have $y' = -(n-1)$ and

$$\begin{aligned}\frac{nf'(x)}{2(n-1)^2} &= \frac{1}{x} - \frac{1}{y} + \frac{n(x-y)}{(n-1)\left[(n-1)x^2+y^2\right]} \\ &= \frac{(y-x)\left[\left[(n-1)x-y\right]^2+(n-2)y(x+y)\right]}{(n-1)xy\left[(n-1)x^2+y^2\right]} \geq 0.\end{aligned}$$

Therefore, the function $f(x)$ is strictly increasing on $(0,1]$ and hence $f(x) \leq f(1) = 0$. Equality occurs if and only if $x_1 = x_2 = \cdots = x_n$

*Second solution by Michel Bataille, France*

By homogeneity, we may suppose $x_1x_2\cdots x_n = 1$ and then prove

$$(x_1 + x_2 + \cdots + x_n)^{2n} \geq n^{2n-1}S$$

where $S = x_1^2 + x_2^2 + \cdots + x_n^2$. Using AM-GM,

$$\begin{aligned}(x_1 + x_2 + \cdots + x_n)^2 &= S + 2\sum_{1\leq i<j\leq n} x_ix_j \\ &\geq S + 2\cdot\frac{n(n-1)}{2}\left((x_1x_2\cdots x_n)^{n-1}\right)^{2/(n(n-1))} \\ &= S + n(n-1),\end{aligned}$$

hence it suffices to show that

$$(S + n(n-1))^n \geq n^{2n-1}S.$$

Now, by AM-GM again,

$$S + n(n-1) = S + n + n + \cdots + n \geq n\left(Sn^{n-1}\right)^{1/n}$$

and so

$$(S + n(n-1))^n \geq n^nSn^{n-1} = n^{2n-1}S,$$

as desired.

*Also solved by Oles Dobosevych, Ukraine; Daniel Lasaosa, Universidad Publica de Navarra, Spain.*

S98. Let $n$ be a positve integer. Prove that $\prod_{d|n}\frac{\phi(d)}{d}\le\left(\frac{\phi(n)}{n}\right)^{\frac{\tau(n)}{2}}$, where $\tau(n)$ is the number of divisors of $n$ and $\phi(n)$ is Euler's totient function.

*Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

*First solution by Arkady Alt, San Jose, California, USA*

Let $P(n):=\prod_{d|n}\dfrac{\varphi(d)}{d}$ and $F(n):=P(n)^{\frac{1}{\tau(n)}}$.Then, since $\varphi(n)$ is a multiplicative function, for any two relatively prime positive integers $n$ and $m$ holds

$$\begin{aligned}
P(nm)&=\prod_{d|nm}\frac{\varphi(d)}{d}=\prod_{s|n}\prod_{t|m}\frac{\varphi(st)}{st}\\
&=\prod_{s|n}\prod_{t|m}\left(\frac{\varphi(s)}{s}\cdot\frac{\varphi(t)}{t}\right)\\
&=\prod_{s|n}\left(\left(\frac{\varphi(s)}{s}\right)^{\tau(m)}\cdot P(m)\right)\\
&=P(m)^{\tau(n)}\cdot\left(\prod_{s|n}\left(\frac{\varphi(s)}{s}\right)\right)^{\tau(m)}\\
&=P(m)^{\tau(n)}\cdot P(n)^{\tau(m)}
\end{aligned}$$

and, therefore, due to multiplicativity of $\tau(n)$ we obtain

$$\begin{aligned}
P(nm)^{\frac{1}{\tau(nm)}}&=\left(P(n)^{\tau(m)}\cdot P(m)^{\tau(n)}\right)^{\frac{1}{\tau(n)\tau(m)}}\\
&=P(n)^{\frac{1}{\tau(n)}}\cdot P(m)^{\frac{1}{\tau(m)}}\\
&\iff F(nm)=F(n)F(m).
\end{aligned}$$

Since

$$\prod_{d|n}\frac{\varphi(d)}{d}\le\left(\frac{\varphi(n)}{n}\right)^{\frac{\tau(n)}{2}}$$

$$\iff P(n)^{\frac{1}{\tau(n)}}\le\sqrt{\frac{\varphi(n)}{n}}\iff F(n)\le\sqrt{\frac{\varphi(n)}{n}}$$

then, using multiplicativity of $F(n)$ and $\varphi(n)$, suffices to prove latter inequality for any $n=q^k$, where $q$ is prime and $k$ is natural number.

Since $\dfrac{\varphi\left(q^i\right)}{q^i} = \dfrac{q^i - q^{i-1}}{q^i} = 1 - \dfrac{1}{q}$, for $i = 1, 2, ..., k$, $\tau\left(q^k\right) = k + 1$, $P\left(q^k\right) = \prod_{d|q^k, d>1}\left(1 - \dfrac{1}{q}\right) = \prod_{i=1}^{k}\left(1 - \dfrac{1}{q}\right) = \left(1 - \dfrac{1}{q}\right)^k$ then

$$
\begin{aligned}
F\left(q^k\right) &\leq \sqrt{\frac{\varphi\left(q^k\right)}{q^k}} \\
&\iff \left(1 - \frac{1}{q}\right)^{\frac{k}{k+1}} \leq \left(1 - \frac{1}{q}\right)^{\frac{1}{2}} \\
&\iff \left(1 - \frac{1}{q}\right)^{\frac{k-1}{2(k+1)}} \leq 1 \iff \left(1 - \frac{1}{q}\right)^{k-1} \leq 1.
\end{aligned}
$$

*Second solution by Samin Riasat, Notre Dame College, Dhaka, Bangladesh*

To verify this we only need to note that the number of prime divisors of $ab$ is always less than or equal to the number of primes dividing $a$ plus the number of primes dividing $b$, with equality iff $a$ and $b$ have no common prime divisor. Therefore the inequality follows, since $\phi$ contains each prime divisor exactly once.

Now using the inequality above we deduce that

$$
\begin{aligned}
\prod_{d|n} \frac{\phi(d)}{d} &= \sqrt{\prod_{d|n} \frac{\phi(d)}{d} \cdot \frac{\phi(n/d)}{n/d}} \\
&\geq \sqrt{\prod_{d|n} \frac{\phi(n)}{n}} \\
&= \sqrt{\left(\frac{\phi(n)}{n}\right)^{\tau(n)}}
\end{aligned}
$$

which was what we wanted. Equality holds iff $n$ is square-free i.e. of the form $p_1 p_2 \cdots p_k$, where the $p_i$ are distinct prime numbers.

*Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Michel Bataille, France.*

S99. Let $ABC$ be an acute triangle. Prove that

$$\frac{1-\cos A}{1+\cos A}+\frac{1-\cos B}{1+\cos B}+\frac{1-\cos C}{1+\cos C}\leq\left(\frac{1}{\cos A}-1\right)\left(\frac{1}{\cos B}-1\right)\left(\frac{1}{\cos C}-1\right).$$

*Proposed by Daniel Campos Salas, Costa Rica*

No solutions has yet been received.

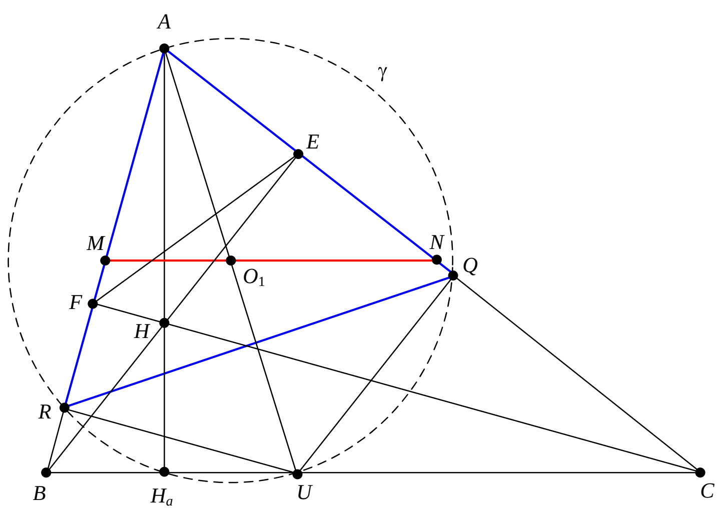
S100. Let $ABC$ be an acute triangle with altitudes $BE$ and $CF$. Points $Q$ and $R$ lie on segments $CE$ and $BF$, respectively, such that $\frac{CQ}{QE} = \frac{FR}{RB}$. Determine the locus of the circumcenter of triangle $AQR$ when $Q$ and $R$ vary.

*Proposed by Alex Anderson, Washington University in St. Louis, USA*

*First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain*

Denote by $P$ the point in segment $BC$ such that $\frac{BP}{PC} = \frac{BR}{RF} = \frac{EQ}{QC}$. Clearly, by Thales' theorem, $PR \parallel CF \perp AB$, and $PQ \parallel BE \perp AC$, or $AP$ is a diameter of the circumcircle of $AQR$. As $R$ varies continuously from $B$ to $F$, and hence $Q$ varies continuously from $E$ to $C$, then $P$ varies continuously from $B$ to $C$, and the locus of the circumcenter of $AQR$ is clearly the segment joining the midpoints of $AB$ and $AC$.

*Second solution by Ercole Suppa, Teramo, Italy*




If $Q$ is a point of $CE$, the point $R$ can be costructed in the following way:

- through point $Q$ draw a line parallel to $BE$ to intersect $BC$ at point $U$;
- through point $U$ draw a line parallel to $CF$ to intersect $AB$ at point $R$;

From Thales' theorem we have:

$$\frac{CQ}{QE} = \frac{CU}{UB} \qquad , \qquad \frac{CU}{UB} = \frac{FR}{RB}$$

so the point $R$ satisfies the relation:

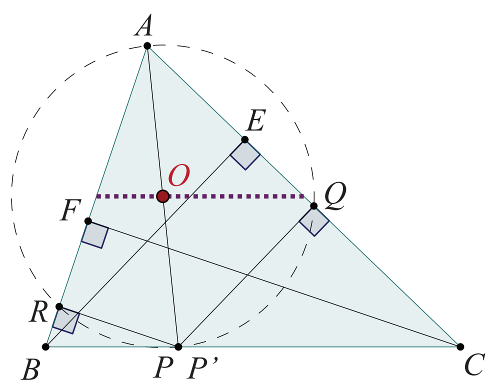
$$\frac{CQ}{QE} = \frac{FR}{RB}$$

The circle $\gamma$ with diameter $AU$ contains $H_a$, $Q$, $R$ because

$$\angle AQU = \angle ARU = \angle AH_aU = 90^\circ$$

Thus the circumcenter of $\triangle AQU$ is the mid-point $O_1$ of $AU$. This implies that the required locus is the set of mid-points of the cevians $AU$, where $U$ is a variable point of $BC$. In other words the locus is the segment joining the mid-points $M$, $N$ of the sides $AB$, $AC$.

*Third solution by Ivanov Andrei, Moldova*




Let $P, P' \in BC$, such that $PR \perp AB$ and $P'Q \perp AC$. Then $P'Q \parallel BE$ and $PR \parallel CF$, so

$$\frac{BP}{PC} = \frac{BR}{RF} \quad \text{and} \quad \frac{BP'}{P'C} = \frac{EQ}{QC}$$

But therefore from the given relation

$$\frac{BP}{PC} = \frac{BP'}{P'C} \iff \frac{BC}{PC} = \frac{BC}{P'C}$$

So $P'C = PC$ and $P \equiv P'$. Quadrilateral $ARPQ$ is cyclic (because $m(\angle PRA) = m(\angle PQA) = 90^\circ$), so the circumcenter of the triangle $ARQ$ is midpoint of the segment $AP$. When $Q$ and $R$ vary, $P$ moves on $BC$ and ints midpoint always lies on the midline of the triangle $ABC$.

S101. Let $a, b, c$ be distinct real numbers. Prove that

$$\left(\frac{a}{a-b}+1\right)^2+\left(\frac{b}{b-c}+1\right)^2+\left(\frac{c}{c-a}+1\right)^2 \geq 5.$$

*Proposed by Roberto Bosch Cabrera, University of Havana, Cuba*

*First solution by Pham Huu Duc, Ballajura, Australia*

Observe that

$$\begin{aligned}
ab(a-b)+bc(b-c)+ca(c-a) &= ab(a-b)+bc[b-a+a-c]+ca(c-a)\\
&= (a-b)(ab-bc)+(c-a)(ca-bc)\\
&= b(a-b)(a-c)+c(c-a)(a-b)\\
&= (a-b)(c-a)[c-b]\\
&= -(a-b)(b-c)(c-a)
\end{aligned}$$

Using this, we obtain

$$\sum_{cyc}\frac{a}{a-b}\left(1-\frac{b}{b-c}\right) = -\sum_{cyc}\frac{ca}{(a-b)(b-c)} = -\sum_{cyc}\frac{ab(a-b)}{(a-b)(b-c)(c-a)} = 1$$

that is,

$$\sum_{cyc}\frac{a}{a-b} = 1+\sum_{cyc}\frac{a}{a-b}\cdot\frac{b}{b-c}$$

Hence

$$\begin{aligned}
\sum_{cyc}\left(\frac{a}{a-b}+1\right)^2 &= 3+\sum_{cyc}\left(\frac{a}{a-b}\right)^2+2\sum_{cyc}\frac{a}{a-b}\\
&= 3+\sum_{cyc}\left(\frac{a}{a-b}\right)^2+2\left[1+\sum_{cyc}\frac{a}{a-b}\cdot\frac{b}{b-c}\right]\\
&= 5+\left(\sum_{cyc}\frac{a}{a-b}\right)^2
\end{aligned}$$

from which the desired result follows.

*Second solution by Manh Dung Nguyen, Hanoi University of Science, Vietnam*

By expanding, the inequality becomes

$$\sum_{cyclic}\left(\frac{a}{a-b}\right)^2+2\sum_{cyclic}\frac{a}{a-b} \geq 2.$$

On the other hand,

$$\sum_{cyclic} \tfrac{a}{a-b} = \tfrac{a(b-c)(c-a)+b(c-a)(a-b)+c(a-b)(b-c)}{(a-b)(b-c)(c-a)}$$
$$= \tfrac{ab(c-a)+bc(a-b)+ca(b-c)}{(a-b)(b-c)(c-a)} + 1 = \sum_{cyclic} \left(\tfrac{a}{a-b}\right)\left(\tfrac{b}{b-c}\right) + 1.$$

It follows that

$$\sum_{cyclic} \left(\frac{a}{a-b}\right)^2 + 2\sum_{cyclic} \frac{a}{a-b} = 2 + \left(\frac{a}{a-b} + \frac{b}{b-c} + \frac{c}{c-a}\right)^2 \geq 2$$

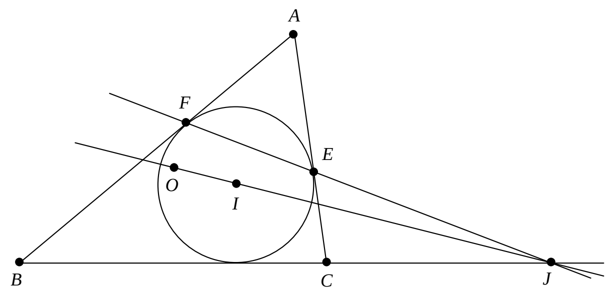
as desired.
Equality occurs if and only if $\frac{a}{a-b} + \frac{b}{b-c} + \frac{c}{c-a} = 0$.

*Also solved by Dimitar Trenevski, Skopje, Macedonia; Magkos Athanasios, Kozani, Greece; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Ercole Suppa, Teramo, Italy; Oles Dobosevych, Ukraine.*

S102. Consider triangle $ABC$ with circumcenter $O$ and incenter $I$. Let $E$ and $F$ be the points of tangency of the incircle with $AC$ and $AB$, respectively. Prove that $EF$, $BC$, $OI$ are concurrent if and only if $r_a^2 = r_b r_c$, where $r_a, r_b, r_c$ are the radii of the excircles.

*Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

*First solution by Ercole Suppa, Teramo, Italy*




We will use homegeneous barycentric coordinates with respect $\triangle ABC$. Denote as usual by $a$, $b$, $c$,$s$,$\Delta$ the sides $BC$, $CA$, $AB$, the semiperimeter and the area of the triangle respectively. We have:

$$E(s-c:0:s-a) \quad , \quad F(s-b:s-a:0) \quad , \quad I(a:b:c)$$
$$O\left(a^2\left(-a^2+b^2+c^2\right), b^2\left(a^2-b^2+c^2\right), c^2\left(a^2+b^2-c^2\right)\right)$$

Hence the equations of the lines $EF$, $IO$ are:

$$\begin{aligned} EF \quad &: \quad (s-a)x-(s-b)y-(s-c)z=0 \\ IO \quad &: \quad bc(b-c)(s-a)x+ac(c-a)(s-b)y+ab(a-b)(s-c)z=0 \end{aligned}$$

The equation of the line $BC$ is $x=0$. Therefore the lines $EF$, $IO$, $BC$ are concurrent if and only if:

$$\begin{vmatrix} s-a & b-s & c-s \\ bc(b-c)(s-a) & ac(c-a)(s-b) & ab(a-b)(s-c) \\ 1 & 0 & 0 \end{vmatrix} = 0 \qquad \Longleftrightarrow$$
$$a(a+b-c)(a-b+c)\left(b^2+c^2-ab-ac\right)=0$$

On the other hand we have:

$$\begin{aligned} r_a^2 - r_b r_c &= \frac{\Delta^2}{(s-a)^2} - \frac{\Delta^2}{(s-b)(s-c)} = \\ &= \frac{s(s-b)(s-c)}{s-a} - s(s-a) = \\ &= \frac{(a+b+c)\left(b^2+c^2-ab-ac\right)}{2(a-b-c)} \end{aligned}$$

and thus we are done.

*Second solution by Michel Batailll, France*

We show that $r_a^2 = r_b r_c$ and the concurrency of $EF, BC, OI$ are both equivalent to the condition $a(b+c) = b^2 + c^2$.

If $s$ denotes as usual the semiperimeter of $\Delta ABC$, it is known that $r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}}$. Similar results hold for $r_b$ and $r_c$ and so

$$r_a^2 = r_b r_c \Leftrightarrow (s-a)^2 = (s-b)(s-c) \Leftrightarrow (b+c-a)^2 = (c+a-b)(a+b-c) \Leftrightarrow a(b+c) = b^2+c^2.$$

Now, let us use areal (barycentric) coordinates relatively to $ABC$. We have $E(s-c, 0, s-a)$, $F(s-b, s-a, 0)$ from which we deduce

$$b(s-b)\mathbf{E} - c(s-c)\mathbf{F} = (s-a)(s-b)\mathbf{C} - (s-a)(s-c)\mathbf{B}.$$

It follows that the point of intersection of the lines $BC$ and $EF$ is $U(0, -(s-c), s-b)$ and this point is on the line $OI$ if and only if

$$\begin{vmatrix} 0 & a & a\cos A \\ -(s-c) & b & b\cos B \\ s-b & c & c\cos C \end{vmatrix} = 0$$

(since $I(a,b,c)$ and $O(a\cos A, b\cos B, c\cos C)$.) This condition easily rewrites as

$$b(s-b)(\cos B - \cos A) + c(s-c)(\cos C - \cos A) = 0 \qquad (*).$$

Now, using the law of cosines,

$$\cos B - \cos A = \frac{c^2+a^2-b^2}{2ca} - \frac{b^2+c^2-a^2}{2bc} = \frac{2(a-b)s(s-c)}{abc}$$

(readily checked) and similarly $\cos C - \cos A = \frac{2(a-c)s(s-b)}{abc}$.

Carrying into $(*)$, we obtain $(a-b)b + (a-c)c = 0$ that is, $a(b+c) = b^2 + c^2$.

This completes the proof.

*Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Mihai Miculita, Oradea, Romania.*

## Undergraduate problems

U97. Prove that

$$f(x) = \begin{cases} 1, \ x \geq 0 \\ \operatorname{arccot} \dfrac{1}{x}, \ x < 0, \end{cases}$$

does not have antiderivatives.

*Proposed by Dinu Ovidiu Gabriel, Valcea, Romania*

*Solution by Michel Bataill, France*

For $x < 0$, $f(x) = \operatorname{arccot} \frac{1}{x} \in (\frac{\pi}{2}, \pi)$. Thus, $f(x) > \frac{\pi}{2}$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$. It follows that $f$ does not satisfy the Darboux property and so $f$ cannot be the derivative of some function $F$. This follows from the fact that a derivative $F'$ satisfies the Darboux property: if $a < b$ and $m$ is between $F'(a)$ and $F'(b)$, say $F'(a) < m < F'(b)$, then $m = F'(c)$ for some $c \in (a, b)$. For completeness we give a short proof of this well-known property:
Consider $G(x) = F(x) - mx$. Then, $G'(a) < 0$ and $G'(b) > 0$ so that we certainly have $G(x_1) < G(a)$ and $G(x_2) < G(b)$ for some $x_1, x_2 \in (a, b)$. Now, $G$ is continuous on $[a, b]$, hence attains its minimum at, say $c$, on $[a, b]$. Because of $x_1, x_2$, we must have $c \in (a, b)$ and since $G'(c) = 0$, we have $F'(c) = m$.

U98. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative such that

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx.$$

Prove that there exists $\xi \in (0,1)$ such that $f(\xi) = f'(\xi)\int_0^{\xi} f(x)dx$.

*Proposed by Cezar Lupu, Univeristy of Bucharest, Romania*

*First solution by Arin Chaudhuri*

Define

$$h(x) = \int_0^x f(t)dt.$$

From the continuity of $f$ it follows that $h$ is continuously differentiable and $h'(x) = f(x)$ and by the formula for integration by parts we have

$$\begin{aligned}
\int_0^1 h(t)dt &= [th(t)]_0^1 - \int_0^1 th'(t)dt \\
&= h(1) - \int_0^1 tf(t)dt \\
&= \int_0^1 f(t)dt - \int_0^1 tf(t)dt \\
&= 0.
\end{aligned}$$

Clearly $h(0) = 0$. We claim there exists an $u \in (0,1]$ such that $h(u) = 0$. We prove it by contradiction. Assume no such $u$ exists then clearly $h$ cannot change sign on $(0,1]$, for then the continuity of $h$ along with the intermediate value theorem would imply the existence of such an $u$. Hence we must have either $h(t) > 0$ for all $t \in (0,1]$ or $h(t) < 0$ for all $t \in (0,1]$. Now, if $h(t) > 0$ for all $t \in (0,1]$ then by compactness of $[1/2,1]$ and continuity of $h$ there exists a $v \in [1/2,1]$ with $h(v) = \inf_{t\in[1/2,1]} h(t)$ and by our assumption we have $h(v) > 0$. Since $h(t) \geq 0$ for all $t \in [0,1]$ we have,

$$\int_0^1 h(t)dt = \int_0^{\frac{1}{2}} h(t)dt + \int_{\frac{1}{2}}^1 h(t)dt \geq \int_{\frac{1}{2}}^1 h(t)dt \geq \frac{1}{2}\inf_{t\in[1/2,1]} h(t) > 0.$$

However the above contradicts the fact the integral of $h$ is 0. We are led to a similar contradiction if we assume $h(t) < 0$ for all $t \in (0,1]$. Hence such an $u$ exists.

Now define

$$m(t) = h(t)\exp(-f(t)).$$

Clearly, $m(0) = m(u) = 0$, by rolle's theorem there exists a $\xi \in (0,u) \subseteq (0,1)$ with $m'(\xi) = 0$, i.e.,

$$(h'(\xi) - f'(\xi)h(\xi))\exp(-f(\xi)) = 0.$$

Clearly, $\exp(-f(\xi)) > 0$ hence

$$h'(\xi) - f'(\xi)h(\xi) = 0$$

or equivalently

$$f(\xi) = f'(\xi)\int_0^{\xi} f(x)dx.$$

*Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain*

Define $F : [0,1] \to \mathbb{R}$ as $F(x) = \int_0^x f(y)dy$. By definition, $F(0) = 0$, while clearly $F(1) = \int_0^1 f(x)dx = \int_0^1 xf(x)dx$, yielding

$$F(1) = \int_0^1 \frac{d\,(xF(x))}{dx}dx = \int_0^1 F(x)dx + \int_0^1 xf(x)dx = \int_0^1 F(x)dx + F(1),$$

and $\int_0^1 F(x)dx = 0$. Consider function $\int_0^x F(y)dy$, defined in $[0,1]$. Clearly, it is zero at $x = 0$ and $x = 1$, hence by Rolle's theorem there exists $u \in (0,1)$ such that its derivative is $F(u) = 0$. Define now $g(x) = F(x)e^{-f(x)}$. Clearly, $g(0) = g(u) = 0$ since $F(0) = F(u) = 0$ while $e^{-f(0)}$ and $e^{-f(u)}$ are finite real numbers, or again by Rolle's theorem there exists a $\xi \in (0,u) \subset (0,1)$ such that

$$0 = g'(\xi) = \left(f(\xi) - F(\xi)f'(\xi)\right)e^{-f(\xi)}.$$

Since the second factor in the RHS cannot be zero, then $f(\xi) = f'(\xi)F(\xi)$. The conclusion follows.

*Third solution by Michel Bataille, France*

Let $F(x) = \int_0^x f(t)\,dt$. Integrating by parts, we have

$$\int_0^1 xf(x)\,dx = [xF(x)]_0^1 - \int_0^1 F(x)\,dx = \int_0^1 f(x)\,dx - \int_0^1 F(x)\,dx$$

and the hypothesis yields $\int_0^1 F(x)\,dx = 0$. This implies $G(1) = G(0) (= 0)$ if $G(x)$ denotes $\int_0^x F(t)\,dt$. Since $G$ is differentiable on $[0,1]$ with $G'(x) = F(x)$, Rolle's theorem provides $a \in (0,1)$ such that $G'(a) = 0$ that is, $F(a) = 0$.
This said, let $H(x) = F(x)e^{-f(x)}$. The function $H$ is differentiable on $[0,1]$ and

$$H'(x) = F'(x)e^{-f(x)} - F(x)f'(x)e^{-f(x)} = (f(x) - F(x)f'(x))e^{-f(x)}.$$

Furthermore, $H(0) = H(a) (= 0)$ (since $F(0) = F(a) = 0$) and Rolle's theorem again gives $\xi \in (0,a)$ (hence $\xi \in (0,1)$) such that $H'(\xi) = 0$. Such a $\xi$ satisfies

$$f(\xi) = f'(\xi)F(\xi) = f'(\xi)\int_0^{\xi} f(x)\,dx.$$

U99. Let $a$ and $b$ be positive real numbers such that $a+b=a^4+b^4$. Prove that

$$a^a b^b \le 1 \le a^{a^3} b^{b^3}.$$

*Proposed by Vasile Cartoaje, University of Ploiesti, Romania*

*First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain*

If $x=1$ equality is clear, both sides of the inequality being zero.

If $x>1$, $\ln x = \int_1^x \frac{dz}{z} < \int_1^x dz = x-1$, since $z>1$ in the open integration interval.

If $x<1$, $\ln x = -\int_x^1 \frac{dz}{z} > -\int_x^1 dz = x-1$, since again $z>1$ in the open integration interval.

Taking $x=\frac{1}{a}$ easily produces $a^3\ln a \ge a^3-a^2$, while taking $x=a$ results in $a\ln a \le a^2-a$, and similarly for $b$. Since the problem is equivalent to showing that $a\ln a + b\ln b \le 0 \le a^3\ln a + b^3\ln b$, it suffices to prove that, given positive reals $a,b$ such that $a+b=a^4+b^4$, then $a^3+b^3\ge a^2+b^2$ and $a^2+b^2\le a+b$. The problem will be finished by proving these last two inequalities.

Define first $f(x)=a^x+b^x$. Clearly, $f'(x)=a^x\ln a+b^x\ln b$ and consequently $f''(x)=a^x\ln^2 a+b^x\ln^2 b\ge 0$, or $f$ is convex, strictly unless $a=b=1$, and since $f(1)=f(4)$, then $f(2)\le f(1)$, yielding $a^2+b^2\le a+b$, with equality iff $a=b=1$.

Note finally that, since $8(a+b)=8(a^4+b^4)\ge(a+b)^4$, where the inequality between arithmetic and quartic means has been used, then $ab\le\frac{(a+b)^2}{4}\le 1$ because of the AM-GM inequality, with equality iff $a=b=1$, and

$$(a+b+1)(a^3+b^3-a^2-b^2)=(a+b-a^2-b^2)(1-ab)\ge 0,$$

with equality iff $a=b=1$. The conclusion follows, and both proposed inequalities turn into equalities iff $a=b=1$.

*Second solution by Arkady Alt, San Jose, California, USA*

Due to the symmetry of constrain we can assume that $a\ge b$. Since

$a+b=a^4+b^4$ can be equivalently rewritten in the form

$a^3\left(1+\left(\frac{b}{a}\right)^4\right)=1+\frac{b}{a}$ or in the form $b^3\left(1+\left(\frac{a}{b}\right)^4\right)=1+\frac{a}{b}$ then,

denoting $t:=\frac{a}{b}$, we obtain following parametrization for $a$ and $b$ :

$a=\sqrt[3]{\dfrac{t^4+t^3}{t^4+1}}$ , $b=\sqrt[3]{\dfrac{t+1}{t^4+1}}, t\geq 1.$

With this parametrization we have $a=bt$ and then:

1. $a^ab^b\leq 1 \iff \left(a^3\right)^a\left(b^3\right)^b\leq 1 \iff \left(a^3\right)^{bt}\left(b^3\right)^b\leq 1 \iff \left(a^3\right)^t b^3\leq 1 \iff$

$$\left(\frac{t^4+t^3}{t^4+1}\right)^t\frac{t+1}{t^4+1}\leq 1 \iff t^{3t}\leq\left(\frac{t^4+1}{t+1}\right)^{t+1} \iff t^3\leq\left(\frac{t^4+1}{t+1}\right)^{\frac{t+1}{t}},$$

where latter inequality is right, because applying Bernoulli's Inequality

(1) $\qquad (1+x)^\alpha\geq 1+\alpha x,\ x>-1, \alpha\ \geq 1$

we obtain $\left(\dfrac{t^4+1}{t+1}\right)^{\frac{t+1}{t}}=\left(1+\dfrac{t^4-t}{t+1}\right)^{\frac{t+1}{t}}\geq 1+\dfrac{t^4-t}{t+1}\cdot\dfrac{t+1}{t}=t^3;$

2. $1\leq a^{a^3}b^{b^3} \iff 1\leq\left(a^3\right)^{a^3}\left(b^3\right)^{b^3} \iff 1\leq\left(a^3\right)^{t^3b^3}\left(b^3\right)^{b^3} \iff 1\leq\left(a^3\right)^{t^3}b^3 \iff$

$$1\leq\left(\frac{t^4+t^3}{t^4+1}\right)^{t^3}\frac{t+1}{t^4+1} \iff \left(\frac{t^4+t^3}{t^4+1}\right)^{t^3}\geq\frac{t^4+1}{t+1}=1+\frac{t\left(t^3-1\right)}{t+1}.$$

Consider two cases.

If $t\leq\sqrt[3]{2}$ then by Bernoulli's Inequality (1) we have

$\left(\dfrac{t^4+t^3}{t^4+1}\right)^{t^3}=\left(1+\dfrac{t^3-1}{t^4+1}\right)^{t^3}\geq 1+\dfrac{t^3\left(t^3-1\right)}{t^4+1}$ and

$\dfrac{t^3\left(t^3-1\right)}{t^4+1}\geq\dfrac{t^4-t}{t+1} \iff \dfrac{t^2}{t^4+1}\geq\dfrac{1}{t+1} \iff t^3+t^2\geq t^4+1 \iff$

$t^3(1-t)+\left(t^2-1\right)\geq 0 \iff (t-1)\left(t+1-t^3\right)\geq 0$ because

$t+1-t^3\geq t+1-2=t-1\geq 0;$

If $t>\sqrt[3]{2}$ then $t^3>2$ and applying inequality

(2) $(1+x)^\alpha\geq 1+\alpha x+\dfrac{\alpha(\alpha-1)}{2}x^2,\ x>-1, \alpha\ \geq 2$ (can by obtained by application (1) to $h'(x)$where $h(x):=(1+x)^\alpha-1-\alpha x-\dfrac{\alpha(\alpha-1)}{2}x^2$)

we get $\left(\dfrac{t^4+t^3}{t^4+1}\right)^{t^3}=\left(1+\dfrac{t^3-1}{t^4+1}\right)^{t^3}\geq 1+t^3\cdot\dfrac{t^3-1}{t^4+1}+$

$\dfrac{t^3\left(t^3-1\right)}{2}\cdot\left(\dfrac{t^3-1}{t^4+1}\right)^2$. Thus, suffices to prove that

$\dfrac{t^3\left(t^3-1\right)}{t^4+1}+\dfrac{t^3\left(t^3-1\right)^3}{2\left(t^4+1\right)^2}\geq\dfrac{t\left(t^3-1\right)}{t+1} \iff$

$$(3) \qquad \frac{t^2}{t^4+1}+\frac{t^2\left(t^3-1\right)^2}{2\left(t^4+1\right)^2}-\frac{1}{t+1}\geq 0.$$

Multiplying left hand side of inequality (3) by $2\left(t^4+1\right)^2(t+1)$ we obtain $2t^2\left(t^4+1\right)(t+1)+t^2\left(t^3-1\right)^2(t+1)-2\left(t^4+1\right)^2=t^9-t^8+2t^7-2t^5-4t^4+3t^3+3t^2-2=(t-1)\left(t^8+2t^6+2t^5-4t^3-t^2+2t+2\right)=$
$(t-1)\left(\left(t^8-t^2\right)+2\left(t^6-t^3\right)+2\left(t^5-t^3\right)+2t+2\right)\geq 0.$

U100. Let $f\colon [0,1] \to \mathbf{R}$ be an integrable function such that

- $|f(x)| \leq 1$ and $\int_0^1 xf(x)dx = 0$,
- $F(x) \doteq \int_0^x f(y)dy \geq 0$.

Prove that $\displaystyle\int_0^1 f^2(x)dx + 5\int_0^1 F^2(x)dx \geq 6\int_0^1 f(x)F(x)dx.$

*Proposed by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy*

*First solution by Michel Bataille, France*

We show that $\int_0^1 f(x)F(x)\,dx \geq 0$ and prove the better inequality

$$\int_0^1 f^2(x)\,dx + 5\int_0^1 F^2(x)\,dx \geq 10\int_0^1 f(x)F(x)\,dx.$$

Since $f$ is integrable and bounded on $[0,1]$, the functions $g$ and $h$ defined by

$$g(x,y) = f(y), \qquad h(x,y) = f(x)f(y)$$

are integrable on the triangle defined by the conditions $0 \leq y \leq x \leq 1$. Therefore, by Fubini's formula,

$$\int_0^1 \left(\int_0^x f(y)dy\right)dx = \int_0^1 \left(\int_y^1 f(y)dx\right)dy$$

and

$$\int_0^1 \left(\int_0^x f(x)f(y)dy\right)dx = \int_0^1 \left(\int_y^1 f(x)f(y)dx\right)dy$$

that is,

$$\int_0^1 F(x)dx = \int_0^1 (1-y)f(y)dy \quad \text{and} \quad \int_0^1 f(x)F(x)\,dx = F^2(1) - \int_0^1 f(y)F(y)\,dy.$$

Using the hypothesis $\int_0^1 xf(x)\,dx = 0$, it follows that

$$\int_0^1 F(x)dx = \int_0^1 f(x)dx = F(1) \quad \text{and} \quad \int_0^1 f(x)F(x)\,dx = \frac{1}{2}F^2(1) \geq 0.$$

Moreover, since $A^2 + B^2 \geq 2AB$ for real $A, B$, we obtain

$$\int_0^1 f^2(x)\,dx + \int_0^1 F^2(x)\,dx = \int_0^1 (f^2(x) + F^2(x))\,dx \geq 2\int_0^1 f(x)F(x)\,dx$$

and, from the Cauchy-Schwarz inequality

$$4\int_0^1 F^2(x)\,dx = \int_0^1 2^2\,dx \int_0^1 F^2(x)\,dx \geq \left(\int_0^1 2F(x)\,dx\right)^2 = 4F^2(1)$$
$$= 8\int_0^1 f(x)F(x)\,dx.$$

By addition, this yields

$$\int_0^1 f^2(x)\,dx + 5\int_0^1 F^2(x)\,dx \geq 10\int_0^1 f(x)F(x)\,dx.$$

*Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain*

We will prove the more general and stronger result

$$\alpha\int_0^1 f^2(x)dx + \beta\int_0^1 F^2(x)dx \geq 2(\alpha+\beta)\int_0^1 f(x)F(x)dx,$$

with equality iff either $\alpha = \beta = 0$ or $f(x) = F(x) = 0$. Note first that direct integration yields

$$\int_0^1 f(x)dx = F(1) = \int_0^1 \frac{d(xF(x))}{dx}dx = \int_0^1 xf(x)dx + \int_0^1 F(x)dx,$$

or $\int_0^1 f(x)dx = \int_0^1 F(x)dx = F(1)$. Moreover,

$$F^2(1) = \int_0^1 \frac{dF^2(x)}{dx}dx = 2\int_0^1 f(x)F(x)dx.$$

It then suffices to show that

$$\left(\int_0^1 1^2dx\right)\left(\int_0^1 f^2(x)dx\right) = \int_0^1 f^2(x)dx \geq F^2(1) = \left(\int_0^1 f(x)dx\right)^2,$$

$$\left(\int_0^1 1^2dx\right)\left(\int_0^1 F^2(x)dx\right) = \int_0^1 F^2(x)dx \geq F^2(1) = \left(\int_0^1 F(x)dx\right)^2,$$

which are clearly guaranteed by Cauchy-Schwarz inequality, with equality iff $f(x)$ and $F(x)$ are constant, respectively. The conclusion follows. Both inequalities are equalities iff $f(x) = F(x) = 0$. Otherwise, equality in the proposed inequality is reached iff $\alpha = \beta = 0$.

U101. Consider a sequence of positive real numbers $a_1, a_2, \ldots$ such that for each term in the sequence we have $Aa_n^k \leq a_{n+1} \leq Ba_n^k$, where $A, B, k \in \mathbb{R}^+$. Prove that for all terms $e^{\alpha+\gamma k^n} \leq a_n \leq e^{\beta+\gamma k^n}$, for some $\alpha, \beta, \gamma \in \mathbb{R}^+$.

*Proposed by Zoran Sunic, Texas A&M University, USA*

*First solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy*

We have $e^{\ln A + k \ln a_1} \leq a_2 \leq e^{\ln B + k \ln a_1}$ to be compared with $e^{\alpha+\gamma k^2} \leq a_2 \leq e^{\beta+\gamma k^2}$ so we rewrite $e^{\ln A + (k^2 \ln a_1)/k} \leq a_2 \leq e^{\ln B + (k^2 \ln a_1)/k}$. Now if $0 < A < 1$ and $0 < a_1 < 1$ we cannot have $e^{\beta+\gamma k^2} < e^{\ln A + (k^2 \ln a_1)/k}$ so I think the statement of the problem should be modified. Doing $n$ steps we get

$$e^{\ln A + k^{n-1} \ln a_1 + \ln A \sum_{j=1}^{n-2} k^j} \leq a_n \leq e^{\ln B + k^{n-1} \ln a_1 + \ln B \sum_{j=1}^{n-2} k^j}$$

or

$$e^{\ln A + (k^n \ln a_1)/k + k^n \ln A \sum_{j=1}^{n-2} k^{j-n}} \leq a_n \leq e^{\ln B + (k^n \ln a_1)/k + k^n \ln B \sum_{j=1}^{n-2} k^{j-k}}$$

If $a_1 > 1$ we can take $\gamma = (\ln a_1)/k$. If $A \geq 1$, we can have

$$\alpha = \ln A + \ln A \sum_{j=1}^{n-2} k^j, \qquad \beta = \max\Big\{\ln B + \ln B \sum_{j=1}^{n-2} k^j, 1\Big\}$$

hence a possible modification of the statement could be $a_1 > 1$, $A > 1$

*Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain*

Numbers $\alpha, \beta, \gamma$ may not always be positive reals, since if $a_1, B$ are less than unity, then $a_n < 1$ for all $n$, or $\alpha + \gamma k^n$ must be negative for all $n$. Note also that, when $k = 1$ and $A \neq B$, we may choose $a_n = A^n$ if $A < 1$, or $a_n = B^n$ if $B > 1$, both diverging, and we may not find respectively a lower bound of the form $e^{\alpha+\gamma}$ or an upper bound of the form $e^{\beta+\gamma}$. The case $A = B$ when $k = 1$ trivially produces $a_n = A^{n-1} a_1$, bound only by positive reals when $A = B = 1$, in which case it suffices to take $\alpha + \gamma \leq \ln a_1 \leq \beta + \gamma$ to satisfy the condition of the problem.

We shall show however that it is possible to find reals $\alpha, \beta, \gamma$ such that the condition is met whenever $k \neq 1$. Note that $A^{\frac{k^{n-1}-1}{k-1}} a_1^{k^{n-1}} \leq a_n \leq B^{\frac{k^{n-1}-1}{k-1}} a_1^{k^{n-1}}$ for all $n$, easily provable by induction. Note also that $A \leq B$, and if $A = B$, the sequence is uniquely defined by $\ln a_n = k^{n-1} \ln a_1 + \frac{k^{n-1}-1}{k-1} \ln A$, allowing the choice $\gamma = \frac{\ln a_1}{k} + \frac{\ln A}{k(k-1)}$ and $\alpha = \beta = -\frac{\ln A}{k-1}$.

Assume first that $k < 1$ and $A < B$. Clearly the sequence $\{a_n\}$ is bounded, since

$$\max\{1, B^{\frac{1}{1-k}}\}\max\{1, a_1\} \geq B^{\frac{1-k^{n-1}}{1-k}} a_1^{k^{n-1}} \geq a_n \geq$$
$$\geq A^{\frac{1-k^{n-1}}{1-k}} a_1^{k^{n-1}} \geq \min\{1, A^{\frac{1}{1-k}}\}\min\{1, a_1\},$$

and we may choose $\alpha = \ln\inf\{a_n\}$, $\beta = \ln\sup\{a_n\}$, $\gamma = 0$.

Assume finally that $k > 1$ and $A < B$, and take $\alpha = -\frac{\ln B}{k-1}$, $\beta = -\frac{\ln A}{k-1}$. Defining $c_1, C_1$ as follows, it must clearly be true that

$$c_1 = \frac{\ln a_1 - \beta}{k} = \frac{\ln a_1}{k} + \frac{\ln A}{k(k-1)} \leq \gamma \leq \frac{\ln a_1}{k} + \frac{\ln B}{k(k-1)} = \frac{\ln a_1 - \alpha}{k} = C_1.$$

We may clearly choose such a $\gamma$ since $\frac{\ln A}{k(k-1)} < \frac{\ln B}{k(k-1)}$. Define now $c_n, C_n$ as follows:

$$C_n = \frac{\ln a_n - \alpha}{k^n}; \qquad c_n = \frac{\ln a_n - \beta}{k^n}.$$

Clearly $C_n > c_n$, while furthermore

$$C_{n+1} \leq \frac{\ln B + k\ln a_n + \frac{\ln B}{k-1}}{k^{n+1}} = \frac{\ln a_n + \frac{\ln B}{k-1}}{k^n} = C_n,$$

and similarly $c_{n+1} \geq c_n$, or $(c_n, C_n)$ is a sequence of nonempty nested intervals that define at least one real number that we may call $\gamma$. The conclusion follows.

U102. Points on the real axis are colored red and blue. We know there exists a function $f\colon \mathbb{R} \to \mathbb{R}^+$ such that if $x, y$ have distinct color then $\min\{f(x), f(y)\} \leq |x-y|$. Prove that every open interval contains a monochromatic open interval.

*Proposed by Iurie Boreico, Harvard University, USA*

*First solution by Roberto Bosch Cabrera, Cuba*

We suppose the contrary: there exist an open interval $(a, b)$ such that no contains any monochromatic open interval. Let $r_0 \in (a, b)$ red (this point exist because in other case the interval $(a, b)$ is blue; contradiction). In the neighborhood $V_{r_0} = (r_0 - \epsilon, r_0 + \epsilon)$ with $\epsilon > 0$ exists infinitely many blue points, because in other case we will have $b_1, ..., b_n$ the only blue points, so we consider $b = \min\{|r_0 - b_1|, ..., |r_0 - b_n|\}$ and set $W_{r_0} = (r_0 - \frac{b}{2}, r_0 + \frac{b}{2})$; the latter is an monochromatic (red) open interval, contained in $(a, b)$, contradiction. Now we consider $V_{r_0} = (r_0 - \epsilon_0, r_0 + \epsilon_0)$ with $0 < \epsilon_0 < f(r_0)$, let $b_1$ a fixed blue point in this interval; so we have $\min\{f(r_0), f(b_1)\} \leq |r_0 - b_1| < \epsilon_0$ and hence $\min\{f(r_0), f(b_1)\} = f(b_1) < \epsilon_0$. Now we consider $W_{b_1} = (b_1 - \epsilon_1, b_1 + \epsilon_1)$ such that $0 < \epsilon_1 < f(b_1)$ and $r_0 \in W_{b_1}$. By analogy we obtain that $f(r_0) < \epsilon_1$. Hence $f(r_0) < \epsilon_1 < f(b_1) < \epsilon_0 < f(r_0)$ contradiction! We are done.

*Second solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy*

We proceed by contradiction supposing that there exists an open interval, say $U$, such that in every open subinterval there are two point of different colors. This means that the points of different colors are dense everywhere in $U$. Let $r \in U$ a red point. By density there exists a sequence $\{b_k\}$ of blue points such that $b_k \to r$. The fact that $\min\{f(r), f(b_k)\} \leq |r - b_k|$ implies that $\lim_{b_k \to r} \min\{f(r), f(b_k)\} = 0$ and this in turn implies that definitively $f(b_k) \leq f(r)$. Otherwise if frequently it is $f(b_k) > f(r)$, we couldn't have $\lim_{b_k \to r} \min\{f(r), f(b_k)\} = 0$ by $f(r) > 0$. The same thing occurs at the blue points and the conclusion is that the function is never zero ($f \in \mathbf{R}^+$) but its limit is zero at any point. This is proven to be impossible. For we define $A_n = \{x \in [a, b] \subset U : \frac{1}{n+1} < f(x) \leq \frac{1}{n}\}$ for any integer $n \geq 1$. At least one of the $A_n's$ has infinite points possessing at least an accumulation point, say $p$. Of course the limit of $f$ for $x \to p$, which exists, cannot be zero contrary to what affirmed about the function $f$.

## Olympiad problems

O97. Find all odd primes $p$ such that both of the numbers

$$1 + p + p^2 + \cdots + p^{p-2} + p^{p-1} \quad \text{and} \quad 1 - p + p^2 + \cdots - p^{p-2} + p^{p-1}$$

are primes.

*Proposed by Xiaoshen Mou, Shanghai, China*

No solutions has yet been received.

O98. Let $a, b, c$ be positive real numbers such that $abc = 1$. Prove that

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \leq \sqrt[3]{3(3 + a + b + c + ab + bc + ca)}.$$

*Proposed by Cezar Lupu, University of Bucharest, Romania*

*First solution by Magkos Athanasios, Kozani, Greece*

Set $\sqrt[3]{a} = x,\ \sqrt[3]{b} = y,\ \sqrt[3]{c} = z$. We have then $x, y, z > 0$ and $xyz = 1$. What we have to prove assumes the form

$$3(3 + x^3 + y^3 + z^3 + (xy)^3 + (yz)^3 + (zx)^3) \geq (x + y + z)^3.$$

Now, recall Schur's inequality

$$A^3 + B^3 + C^3 + 5ABC \geq (A + B)(B + C)(C + A).$$

Setting $A = xy, B = yz, C = zx$, we find $(xy)^3 + (yz)^3 + (zx)^3) + 5 \geq (x + y)(y + z)(z + x)$, since $xyz = 1$. Then it is enough to prove that

$$3(3+x^3+y^3+z^3+(x+y)(y+z)(z+x)-5) \geq x^3+y^3+z^3+3(x+y)(y+z)(z+x),$$

or equivalently that $x^3 + y^3 + z^3 \geq 3$, which is true by the AM-GM inequality.

*Second solution by Nguyen Manh Dung, Hanoi University of Science, Vietnam*

Setting $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$. The inequality becomes

$$\sqrt[3]{\frac{x}{y}} + \sqrt[3]{\frac{y}{z}} + \sqrt[3]{\frac{z}{x}} \leq \sqrt[3]{3\left(3 + \frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right)}$$

By the Holder inequality, we have:

$$\begin{aligned}\sqrt[3]{3\left(3 + \frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right)} &= \sqrt[3]{3\,(x + y + z)\left(\frac{1}{y} + \frac{1}{z} + \frac{1}{z}\right)} \\ &\geq \left(\sqrt[3]{\frac{x}{y}} + \sqrt[3]{\frac{y}{z}} + \sqrt[3]{\frac{z}{x}}\right).\end{aligned}$$

Hence we are done. Equality holds if and only if $x = y = z$ or $a = b = c = 1$.

*Third solution by Roberto Bosch Cabrera, Cuba*

Let $\sqrt[3]{a} = x$, $\sqrt[3]{b} = y$, $\sqrt[3]{c} = z$, so the inequality can be rewrite as

$$(x + y + z)^3 \leq 3(3 + x^3 + y^3 + z^3 + x^3y^3 + y^3z^3 + z^3x^3)$$

with $x, y, z$ positive real numbers such that $xyz = 1$.

Now set $x = \frac{p}{q}, y = \frac{q}{r}, z = \frac{r}{p}$, we need to prove that

$$\left(p \cdot \frac{1}{q} + q \cdot \frac{1}{r} + r \cdot \frac{1}{p}\right)^3$$

$$\leq 3\left(3 + p^3 \cdot \frac{1}{q^3} + q^3 \cdot \frac{1}{r^3} + r^3 \cdot \frac{1}{p^3} + p^3 \cdot \frac{1}{r^3} + q^3 \cdot \frac{1}{p^3} + r^3 \cdot \frac{1}{q^3}\right)$$

but

$$3\left(3 + p^3 \cdot \frac{1}{q^3} + q^3 \cdot \frac{1}{r^3} + r^3 \cdot \frac{1}{p^3} + p^3 \cdot \frac{1}{r^3} + q^3 \cdot \frac{1}{p^3} + r^3 \cdot \frac{1}{q^3}\right)$$

$$= 3\left(p^3 + q^3 + r^3\right)\left(\frac{1}{q^3} + \frac{1}{r^3} + \frac{1}{p^3}\right).$$

Now from Hőlder's inequality we have that

$$(a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3)^3 \leq (a_1^3 + a_2^3 + a_3^3)(b_1^3 + b_2^3 + b_3^3)(c_1^3 + c_2^3 + c_3^3).$$

Setting

$$\begin{aligned} (a_1, a_2, a_3) &= (1, 1, 1) \\ (b_1, b_2, b_3) &= (p, q, r) \\ (c_1, c_2, c_3) &= \left(\frac{1}{q}, \frac{1}{r}, \frac{1}{p}\right) \end{aligned}$$

we obtain our inequality.

*Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Michel Bataille, France; Dimitar Trenevski, Skopje, Macedonia; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Oles Dobosevych, Ukraine.*

O99. Let $AB$ be a chord that is not a diameter of circle $\omega$. Let $T$ be a mobile point on $AB$. Construct circles $\omega_1$ and $\omega_2$ that are externally tangent to each other at $T$ and internally tangent to $\omega$ at $T_1$ and $T_2$, respectively. Let $X_1 \in AT_1 \cap TT_2$ and $X_2 \in AT_2 \cap TT_1$. Prove that $X_1X_2$ passes through a fixed point.

*Proposed by Alex Anderson, Washington University in St. Louis, USA*

*First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain*

Call $P$ any point on line $AB$, and define $Y_1 \in PT_1 \cap TT_2$, $Y_2 \in PT_2 \cap TT_1$. We shall prove the more general result that $Y_1Y_2$ passes through a fixed point independently on the choice of $P$. Clearly, the proposed problem represents the particular case $Y_1 = X_1$ and $Y_2 = X_2$ when $P = A$.

Call $O, O_1, O_2$ the respective centers of $\omega, \omega_1, \omega_2$, $D$ and $E$ the respective second points where $TT_1$ and $TT_2$ intersect $\omega$, and $M$ the midpoint of $AB$. Clearly, $T_1O_1T$ and $T_1OD$ are isosceles at $O_1$ and $O$, respectively, while lines $T_1O_1$ and $T_1O$ coincide, and lines $T_1T$ and $T_1D$ coincide. Clearly, $OD \parallel O_1T \perp AB$, and similarly $OE \perp AB$, or $DE$ is the diameter of $\omega$ which is the perpendicular bisector of $AB$. Call now $F \in ET_1 \cap DT_2$. Since $DT_1 \perp EF$ and $ET_2 \perp DT_1$ because $DE$ is a diameter of $\omega$, then $T = DT_1 \cap ET_2$ is the orthocenter of triangle $DEF$, and the perpendicular to $DE$ through $T$, ie $AB$, is the altitude from $F$ onto $DE$. Furthermore, $M$ is the foot of this altitude, and $O$ is the midpoint of $DE$, or $OMT_1T_2$ is cyclic, its circumcircle being the nine-point circle of $DEF$. Call now $N = T_1T_2 \cap DE$. Clearly, the power of $N$ with respect to the nine-point circle of $DEF$ is $NM \cdot NO = NT_1 \cdot NT_2$, and its power with respect to $\omega$ is $ND \cdot NE = NT_1 \cdot NT_2$, or calling $\rho$ the radius of $\omega$, we find that $(ON - OM)ON = NM \cdot NO = ND \cdot NE = (ON + \rho)(ON - \rho)$, or $OM \cdot ON = \rho^2$, and $N$ is the inverse of $M$ with respect to $\omega$, independently on the choice of $T$. We will now show that $Y_1Y_2$ passes through $N$, regardless of the choice of $T$ and $P$.

Define trilinear coordinates $(\alpha, \beta, \gamma)$ of any given point in the plane such that these coordinates are respectively proportional to the directed distances from the point to sides $EF, FD, DE$, for any positive proportionality constant. Clearly, the trilinear coordinates of $D, E, F, T_1, T_2, M$ are respectively $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(0, \cos F, \cos E)$, $(\cos F, 0, \cos D)$ and $(\cos E, \cos D, 0)$. Moreover, any point $P$ on line $FM = AB$ has coordinates $(\cos E, \cos D, \kappa)$ for some real $\kappa$. Now, points on lines $DE$ and $T_1T_2$ respectively satisfy equations $\gamma = 0$ and

$$0 = \begin{vmatrix} \alpha & \beta & \gamma \\ 0 & \cos F & \cos E \\ \cos F & 0 & \cos D \end{vmatrix} = \cos F \left(\alpha \cos D + \beta \cos E - \gamma \cos F\right),$$

or $N \equiv (\cos E, -\cos D, 0)$. Similarly, points on lines $PT_1$ and $TT_2 = ET_2$ respectively satisfy equations $\gamma \cos F = \beta \cos E + \alpha \left(\frac{\kappa \cos F}{\cos E} - \cos D\right)$ and $\gamma \cos F = \alpha \cos D$, for

$$Y_1 \equiv \left(1, \frac{2\cos D}{\cos E} - \frac{\kappa \cos F}{\cos^2 E}, \frac{\cos D}{\cos F}\right).$$

Similarly,

$$Y_2 \equiv \left(\frac{2\cos E}{\cos D} - \frac{\kappa \cos F}{\cos^2 D}, 1, \frac{\cos E}{\cos F}\right).$$

Now, since

$$\begin{vmatrix} \cos E & -\cos D & 0 \\ 1 & \frac{2\cos D}{\cos E} - \frac{\kappa \cos F}{\cos^2 E} & \frac{\cos D}{\cos F} \\ \frac{2\cos E}{\cos D} - \frac{\kappa \cos F}{\cos^2 D} & 1 & \frac{\cos E}{\cos F} \end{vmatrix} =$$

$$= \cos E \left(\frac{\cos D}{\cos F} - \frac{\kappa}{\cos E}\right) - \cos D \left(\frac{\kappa}{\cos D} - \frac{\cos E}{\cos F}\right) = 0,$$

then points $Y_1, Y_2, N$ are collinear. Since $N$ is a fixed point that depends only on $M$, and does not move when $T$ and $P$ move in $AB$, the conclusion follows.

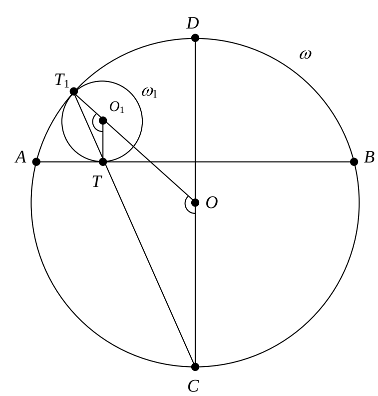
*Second solution by Ercole Suppa, Teramo, Italy*

In the following proof we'll use two lemmae:

LEMMA 1. Let $AB$ be a chord that is not a diameter of circle $\omega$, let $T$ be a point on $AB$, let $\omega_1$ be a circle internally tangent to $\omega$ at $T_1$ and tangent to $AB$ to the point $T$, let $C$, $D$ be the intersection points of $\omega$ with the perpendicular bisector of $AB$, with $C$ and $T_1$ lying on opposite sides of $AB$. The points $C$, $T$, $T_1$ are collinear.

*Proof.*

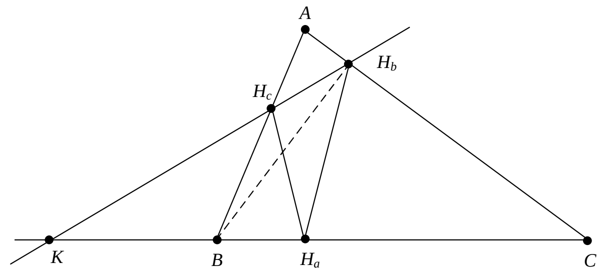
Let $O_1$ be the center of $\omega_1$. We have $TO_1 \| CD$ because $TO_1$ and $CD$ are both perpendicular to $AB$. The isosceles triangles $\Delta T_1O_1T$ and $\Delta T_1OC$ are similars because $\angle T_1O_1T = \angle T_1OC$. Thus $\angle O_1TT_1 = \angle OCT_1$ and this implies that $C$, $T$, $T_1$ are collinear. ■



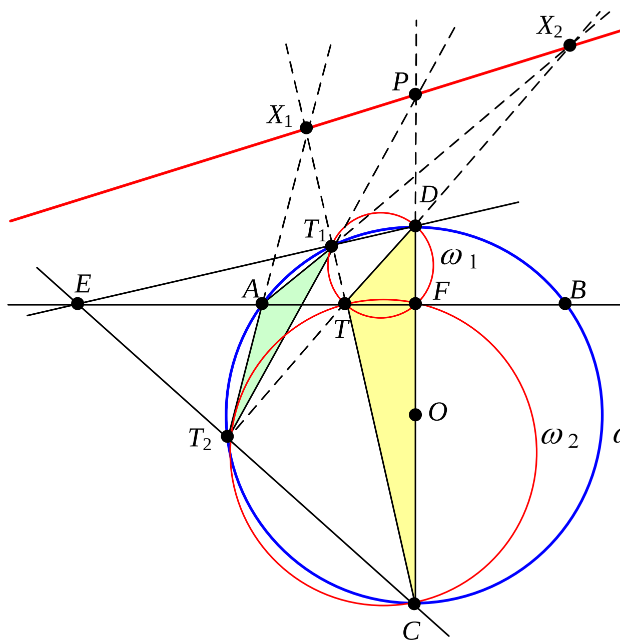
Lemma 2. Let $H_aH_bH_c$ be the orthic triangle of $\triangle ABC$ and let $K$ be the intersection point of $H_bH_c$ with the line $BC$. The point $K$ is the harmonic conjugate of $H_a$ with respect to $B$ and $C$.

*Proof.*

We suppose, without loss of generality, that $c < b$. Since in $\triangle KH_bH_a$ the lines $BH_b$, $AC$ are the internal and external angle bisectors, the points $K$ and $H_a$ are harmonic conjugates with respect to $B$ and $C$. ■




Now we can proove that all lines $X_1X_2$ pass through a fixed point.




Let $C$, $D$ be the intersection points of $\omega$ with the perpendicular bisector of $AB$ (with $C$ and $T_1$ on the opposite sides of $AB$) and let $F$ be the middle point of $AB$. We have:

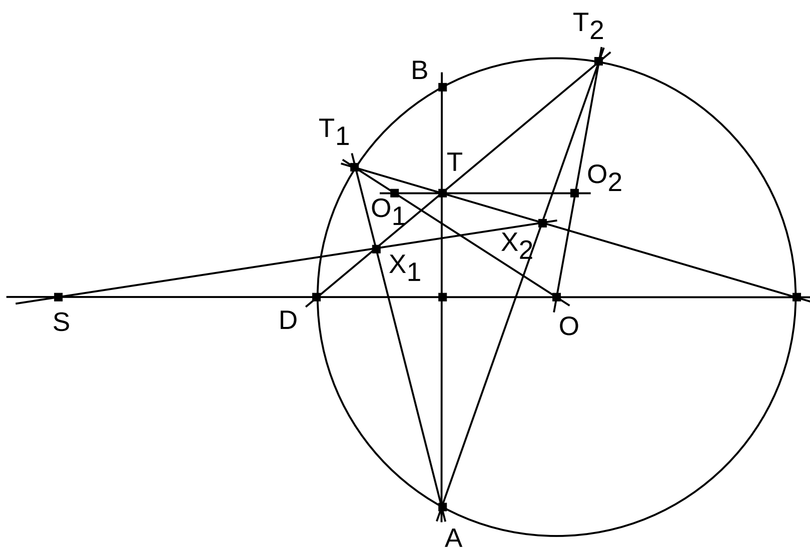
- by Lemma 1 the point $C$, $T$, $T_1$ are collinear; similarly the point $D$, $T$, $T_2$ are collinear;

- $T$ is the orthocenter of triangle $\triangle CDE$;
- the quadrilateral $TFDT_1$ is cyclic because $CT_1 \perp T_1D$ and $EF \perp FD$; similarly the quadrilateral $CFTT_2$ is cyclic; denote with $\omega_1$, $\omega_2$ the circumcircles of $TFDT_1$ and $CFTT_2$;
- the lines $CT_1$, $BA$, $DT_2$ concur in the point $E$, radical center of three circles $(ABC)$, $(TCT_1)$, $(TDT_2)$;
- thus the triangles $AT_1T_2$ , $CTD$ are perspective. Hence based on the Desarques theorem, we conclude that the points $X_1 = AT_2 \cap TT_1$, $P = T_1T_2 \cap CD$, $X_2 = AT_1 \cap TT_2$ are collinear;
- by LEMMA 2 the point $P$ is harmonic conjugate of $F$ with respect to the points $C$ and $D$, i.e. $P$ is the pole of the line $AB$ wrt the circle $\omega$.

Then $P$ is a fixed point independent from the choice of the point $T$ and the proof is completed.

*Third solution by Francisco Javier Garcia Capitan, Spain*

The solution follows at once from the construction of circles $\omega_1$, $\omega_2$ and Pascal theorem.




Let $DE$ be the diameter of circle $\omega$ perpendicular to the given chord $AB$. We have that $T_1$ and $T_2$ are homothetic centers of circles $w$ and $w_1$, $w$ and $w_2$. So, if $O_1T$ and $OE$ are parallel radii in same direction, we have that $OO_1$ and $ET$ meet at $T_1$, and in the same way, $OO_2$ and $DT$ meet at $T_2$. Now, by applying the Pascal theorem to the hexagon $AAT_1EDT_2$ we have that the intersection points $S = AA \cap ED$, $X_1 = AT_1 \cap DT_2$, and $X_2 = ET_1 \cap AT_2$ are collinear. But

$S$, the intersection of diameter $DE$ and the tangent at $A$ is always the same point, the pole of the line $AB$ with respect to the circle $\omega$, so the problem is solved.

O100. Let $p$ be a prime. Prove that $p(x) = x^p + (p-1)!$ is irreducible in $\mathbb{Z}[X]$.

*Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

*First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain*

We shall prove that $x^p + (p-1)!$ is irreducible in $\mathbb{Q}[X]$. For $p = 2$, $x^2 + 1 = (x+i)(x-i)$ is clearly irreducible in $\mathbb{Q}[X]$. For any odd prime $p$, exchanging $x$ by $-x$, we find that $x^p + (p-1)!$ is irreducible in $\mathbb{Q}[X]$ iff $x^p - (p-1)!$ is irreducible in $\mathbb{Q}[X]$ (or for that purpose in $\mathbb{R}[X]$). The roots of $x^p - (p-1)!$ are clearly $\sqrt[p]{(p-1)!}u_k$, where $k = 0, 1, \dots, p-1$ and $u_0, u_1, \dots, u_{p-1}$ are the $p$-th roots of unity. Each root of unity is of the form $\cos\left(\frac{2k\pi}{p}\right) + i\sin\left(\frac{2k\pi}{p}\right)$, where in any factorization of $x^p - (p-1)!$ in $\mathbb{R}[X]$, complex conjugate roots appear in the same factor. The product of the binomials corresponding to two complex conjugate roots of unity is clearly $x^2 - 2\cos\left(\frac{2k\pi}{p}\right) + 1$, ie, any factorization of $x^p - (p-1)!$ in two polynomials from $\mathbb{R}[X]$ must have independent terms of the form $\pm\left(\sqrt[p]{(p-1)!}\right)^q$ and $\mp\left(\sqrt[p]{(p-1)!}\right)^{p-q}$, give or take a real constant that would also multiply the highest-degree term. If we prove that $\left(\sqrt[p]{(p-1)!}\right)^q$ is irrational, we will then have actually proved that $x^p + (p-1)!$ is irreducible in $\mathbb{Q}[X]$, but since $p, q$ are coprime, this condition is equivalent to $\left(\sqrt[p]{(p-1)!}\right)$ being irrational, which is clearly true, since a $p$-th root of an integer is rational iff it is an integer, ie, iff the integer itself is a perfect $p$-th power. Call $m$ any prime lower than $p$; if $(p-1)!$ is a perfect $p$-th power, the multiplicity of $m$ in $(p-1)!$ is at least $p$, or

$$p \leq \left\lfloor \frac{p-1}{m} \right\rfloor + \left\lfloor \frac{p-1}{m^2} \right\rfloor + \cdots \leq (p-1)\sum_{k=1}^{\infty} \frac{1}{m^k} = \frac{p-1}{m-1} \leq p-1,$$

clearly absurd. It follows that $x^p + (p-1)!$ is irreducible in $\mathbb{Q}[X]$.

*Second solution by Johan Gunardi, Indonesia*

Let $q$ be the greatest prime less than $p$. Note that the coefficients of $x^{p-1}, x^{p-2}, \dots, x^0$ are divisible by $q$. If $q^2$ does not divide $(p-1)!$, then we are done by Eisenstein's Criterion. Consider the factors of $(p-1)! = 1 \cdot 2 \cdot \ldots \cdot q \cdot \ldots \cdot (p-1)$. Clearly, $1 \cdot 2 \cdot \ldots \cdot (q-1)$ is not divisible by $q$. It remains to prove that $(q+1)(q+2)\dots(p-1)$ is not divisible by $q$. By Bertrand's Postulate, there is at least one prime between $q$ and $2q$, $p$ is one such prime. Hence, there is no multiple of $q$ among $(q+1), (q+2), \dots, (p-1)$. Therefore $q^2$ does not divide $(p-1)!$, and our proof is complete.

*Also solved by John T. Robinson, Yorktown Heights, NY, USA.*

O101. Let $a_0, a_1, \ldots, a_6$ be real numbers greater than $-1$. Prove that

$$\frac{a_0^2+1}{\sqrt{a_1^5+a_1^4+1}}+\frac{a_1^2+1}{\sqrt{a_2^5+a_2^4+1}}+\cdots+\frac{a_6^2+1}{\sqrt{a_0^5+a_0^4+1}} \geq 5$$

whenever

$$\frac{a_0^3+1}{\sqrt{a_1^5+a_1^4+1}}+\frac{a_1^3+1}{\sqrt{a_2^5+a_2^4+1}}+\cdots+\frac{a_6^3+1}{\sqrt{a_0^5+a_0^4+1}} \leq 9.$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*First solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy*

It suffices to prove that

$$\frac{a_0^3+a_0^2+2}{\sqrt{a_1^5+a_1^4+1}}+\frac{a_1^3+a_1^2+2}{\sqrt{a_2^5+a_2^4+1}}+\ldots+\frac{a_n^3+a_n^2+2}{\sqrt{a_0^5+a_0^4+1}} \geq 14$$

We observe that $(a_{n+1} \doteq a_0)$

$$\sum_{k=0}^{6} \frac{a_k^3+a_k^2+2}{\sqrt{a_{k+1}^5+a_{k+1}^4+1}}=\sum_{k=0}^{6} \frac{a_k^3+a_k^2+2}{\sqrt{a_{k+1}^2+a_{k+1}+1}\sqrt{a_{k+1}^3-a_{k+1}+1}}$$

which we rewrite as

$$\sum_{k=0}^{6} \frac{a_k^3-a_k+1}{\sqrt{a_{k+1}^2+a_{k+1}+1}\sqrt{a_{k+1}^3-a_{k+1}+1}}+\sum_{k=0}^{6} \frac{a_k^2+a_k+1}{\sqrt{a_{k+1}^2+a_{k+1}+1}\sqrt{a_{k+1}^3-a_{k+1}+1}}$$

namely

$$\sum_{k=0}^{6} \frac{\sqrt{a_k^3-a_k+1}}{\sqrt{a_{k+1}^2+a_{k+1}+1}}+\sum_{k=0}^{6} \frac{\sqrt{a_k^2+a_k+1}}{\sqrt{a_{k+1}^3-a_{k+1}+1}} \geq 2\cdot 7=14$$

the last inequality allowed by the AGM and we are done.

*Second solution by Samin Riasat, Notre Dame College, Dhaka, Bangladesh*

We have the identity

$$a^5+a^4+1=(a^2+a+1)(a^3-a+1)$$

Assume by contradiction that

$$\frac{a_0^3+1}{a_1^5+a_1^4+1}+\frac{a_1^3+1}{a_2^5+a_2^4+1}+\cdots+\frac{a_6^3+1}{a_0^5+a_0^4+1} \leq 9$$

then

$$\frac{a_0^2+1}{a_1^5+a_1^4+1}+\frac{a_1^2+1}{a_2^5+a_2^4+1}+\cdots+\frac{a_6^2+1}{a_0^5+a_0^4+1}\leq 5.$$

Adding these two inequalities we get

$$\frac{a_0^3+a_0^2+2}{a_1^5+a_1^4+1}+\frac{a_1^3+a_1^2+2}{a_2^5+a_2^4+1}+\cdots+\frac{a_6^3+a_6^2+2}{a_0^5+a_0^4+1}\leq 14. \qquad (1)$$

Denote $b_i = a_i^2 + a_i + 1$ and $c_i = a_i^3 - a_i + 1$, where $i = 0, 1, 2, \ldots, 6$. Then $a_i > -1$ implies $b_i$ and $c_i$ are positive, and the last inequality can be written as

$$\sum_{i=0}^{6}\frac{b_i+c_i}{\sqrt{b_{i+1}c_{i+1}}}\leq 14$$

But since the sequences $\{b_i + c_i\}_{i=0}^{6}$ and $\left\{\frac{1}{\sqrt{b_ic_i}}\right\}_{i=0}^{6}$ are oppositely sorted, we get from Rearrangement inequality,

$$\sum_{i=0}^{6}\frac{b_i+c_i}{\sqrt{b_{i+1}c_{i+1}}}\geq\sum_{i=0}^{6}\frac{b_i+c_i}{\sqrt{b_ic_i}}\geq 7\cdot 2=14$$

where the last inequality follows from AM-GM. This contradicts (1) and therefore our assumption was false. Hence the conclusion follows.

*Third solution by Arkady Alt, San Jose, California, USA*

Let $A_k := \dfrac{a_0^k+1}{\sqrt{a_1^5+a_1^4+1}}+\dfrac{a_1^k+1}{\sqrt{a_2^5+a_2^4+1}}+\ldots+\dfrac{a_6^k+1}{\sqrt{a_0^5+a_0^4+1}}, k=2,3.$

We will prove that

$A_2 + A_3 = \dfrac{a_0^3+a_0^2+2}{\sqrt{a_1^5+a_1^4+1}}+\dfrac{a_1^3+a_1^2+2}{\sqrt{a_2^5+a_2^4+1}}+\ldots+\dfrac{a_6^3+a_6^2+2}{\sqrt{a_0^5+a_0^4+1}}\geq 14.$

Since $x > -1$ then $x^3+x^2+2 = x^2(x+1)+2 > 0,\ x^5+x^4+1 = x^4(x+1)+1 >$
$0$

and, moreover, for any $x > -1$ holds inequality

(1) $$\frac{x^3+x^2+2}{\sqrt{x^5+x^4+1}}\geq 2.$$

Indeed, $\dfrac{x^3+x^2+2}{\sqrt{x^5+x^4+1}}\geq 2 \iff \left(x^3+x^2+2\right)^2\geq 4\left(x^5+x^4+1\right)$ and $\left(x^3+x^2+2\right)^2 -$
$4\left(x^5+x^4+1\right) = \left(x^2(x+1)+2\right)^2 - 4x^4(x+1) - 4 = x^4(x+1)^2 + 4x^2(x+1) -$
$4x^4(x+1) =$
$x^2(x+1)\left(x^2(x+1)+4-4x^2\right) = x^2(x+1)^2(x-2)^2\geq 0.$

Let $x_i := a_i^3 + a_i^2 + 2,\ y_i := \dfrac{1}{\sqrt{a_i^5 + a_i^4 + 1}}, i = 0, 1, ..., 6.$

Since $sign\left((x_i - x_j)(y_j - y_i)\right) = sign\left((x_i - x_j)\left(y_j^2 - y_i^2\right)\right) = sign\left((a_i - a_j)^2\right)$ then, applying Rearrangement Inequality and inequality (1) obtain

$A_2 + A_3 = x_0 y_1 + x_1 y_2 + ... + x_6 y_0 \geq \sum_{i=0}^{6} x_i y_i = \sum_{i=0}^{6} \dfrac{a_i^3 + a_i^2 + 2}{\sqrt{a_i^5 + a_i^4 + 1}} \geq 7 \cdot 2 =$ 14.

Since $A_2 + A_3 \geq 14$ and $A_3 \leq 9$ then $A_2 \geq 14 - A_3 \geq 14 - 9 = 5$.

*Fourth solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain*

Clearly $(x^3 - x + 1)(x^2 + x + 1) = x^5 + x^4 + 1$, or using the AM-GM inequality, $x^3 + x^2 + 2 \geq 2\sqrt{x^5 + x^4 + 1}$ for all non-negative reals $x$. This is true in particular for all the $a_i$, hence

$$\left(\frac{a_0^2 + 1}{\sqrt{a_1^5 + a_1^4 + 1}} + \cdots + \frac{a_6^2 + 1}{\sqrt{a_0^5 + a_0^4 + 1}}\right) + \left(\frac{a_0^3 + 1}{\sqrt{a_1^5 + a_1^4 + 1}} + \cdots + \frac{a_6^3 + 1}{\sqrt{a_0^5 + a_0^4 + 1}}\right)$$

$$\geq 2\left(\frac{\sqrt{a_0^5 + a_0^4 + 1}}{\sqrt{a_1^5 + a_1^4 + 1}} + \cdots + \frac{\sqrt{a_6^5 + a_6^4 + 1}}{\sqrt{a_0^5 + a_0^4 + 1}}\right) \geq 14,$$

since the bracket in the middle term is the sum of 7 elements of product 1, and the AM-GM inequality has been used again. Therefore, the sum of both elements is not smaller than 14, and if one of them does not exceed 9, then the other one is at least 5. The result follows.

Note that we may use the AM-GM inequality since, for all $x \geq -1$, it holds $x^3 - x + 1 > 0$ and $x^2 + x + 1 > 0$. In fact, the proposed result would be true, not only for $a_i \geq -1$, but for all $a_i$ larger than the negative real root of $x^3 - x + 1$. Note also that 5 and 9 may be exchanged by any pair of non-negative real numbers such that their sum is 14.

O102. A hive is placed in the Cartesian plane and its cells are regular hexagons with two unit sides parallel to $y$ axis. A bee lives in a cell centered at the origin. It wants to visit another bee whose cell contains the point of coordinates $(2008, 2008)$. The bee can move from a cell to any of the six neighboring cells in one second. What is the minimum number of seconds needed for the bee to reach the other bee? Find how many different routes of optimal time exist.

*Proposed by Iurie Boreico, Harvard University and Ivan Borsenco, MIT, USA*

*First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain*

Clearly, the vectors from the center of one cell of the hive to the centers of its 6 neighbors are of the form $(\pm\sqrt{3}, 0)$ or $\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{3}{2}\right)$, and all cells have a center of the form $\left(m\frac{\sqrt{3}}{2}, n\frac{3}{2}\right)$, where $m$ and $n$ are integers of the same parity. In particular, point $(2008, 2008)$ will be inside a cell with center $\left(m\frac{\sqrt{3}}{2}, \frac{4017}{2}\right)$. Since $3\cdot 2319^2 = 16133283 = 4\cdot 2008^2 + 5027$ and $3\cdot 2317^2 = 4036800 = 2008^2 - 22789$, then the center of the cell that contains point $(2008, 2008)$ is $\left(2319\frac{\sqrt{3}}{2}, 1339\frac{3}{2}\right)$. Assume that the bee makes respectively $a, b, c, d, e, f$ moves of the forms $(\sqrt{3}, 0)$, $(-\sqrt{3}, 0)$, $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$, $\left(\frac{\sqrt{3}}{2}, -\frac{3}{2}\right)$, $\left(-\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$, $\left(-\frac{\sqrt{3}}{2}, -\frac{3}{2}\right)$, then it ends up in cell with center $\left((2a-2b+c+d-e-f)\frac{\sqrt{3}}{2}, (c-d+e-f)\frac{3}{2}\right)$, and optimal routes occur whenever $a+b+c+d+e+f$ is minimum, under constraints $2a-2b+c+d-e-f = 2319$ and $c-d+e-f = 1339$, where $a, b, c, d, e, f$ are non-negative integers. Clearly $b = d = e = f = 0$, yielding $a = 490$, $c = 1339$, for a minimum time of 1829 seconds. The total number of possible paths is the number of ways in which we may order the 1829 moves that the bee makes, ie, the number of ways in which we may choose the 490 moves of type $a$ among all the 1829 moves, ie, $\binom{1829}{490} = \binom{1829}{1339}$.

*Second solution by John T. Robinson, Yorktown Heights, NY, USA*

Label the cells using pairs of integer coordinates as follows: cell $(i, j)$ is the cell reached from the cell centered at the origin by moving $i$ cells to the right (or $-i$ cells to the left for $i$ negative) and $j$ cells up at a 60 degree angle from the $x-$axis (or $-j$ cells down in the opposite direction for $j$ negative). It is then easy to compute that the center of cell $(i, j)$ in $x-y$ coordinates is $\left(\left(\frac{\sqrt{3}}{2}\right)\cdot(2\cdot i+j), \left(\frac{3}{2}\right)\cdot j\right)$. Keeping in mind that the radius of the incircle of a hexagon with unit sides is $\frac{\sqrt{3}}{2} = 0.866...$, we see that since the cell with cell coordinates $(490, 1339)$ has its center in $x-y$ coordinates at $(2008.313..., 2008.5)$, this is the target cell. Finally, starting at the cell at the origin, only moves $R$

to the right, which moves from cell $(i, j)$ to cell $(i+1, j)$, and moves $U$ up and to the right at a 60 degree angle from the $x-$axis, which moves from cell $(i, j)$ to cell $(i, j+1)$, result in a new position which is closer to the target cell (note that this remains true for $i < 490$ and $j < 1339$). Therefore a total of at least 490 $R$ moves and 1339 $U$ moves are required, which takes 1829 seconds. Also, since the R and U moves can be made in any order, there are a total of

$$\binom{1829}{490} = \binom{1829}{1339} = \frac{1829!}{490! \cdot 1339!}$$

optimal time paths (this is about $9 \cdot 10^4 59$ paths).