A note on the breaking point of a simple inequality

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Introduction

Every student who is starting to become acquainted with inequality problems comes across Mongolian inequality

$$\left(\frac{x+y+z}{3}\right)^3 \geq \frac{(x+y)(y+z)(z+x)}{8}$$

Another inequality which bears some similarity to it

$$\left(\frac{x+y+z}{3}\right)\left(\frac{xy+yz+zx}{3}\right) \leq \frac{(x+y)(y+z)(z+x)}{8}.$$

is also popular in problem solving communities. We can see that the direction is reversed in the second inequality. So it occurred to the authors to find the limit point where the inequality is still true. This is the problem we will solve

Problem

Find the minimum value for a such that the inequality

$$\left(\frac{x+y+z}{3}\right)^a \left(\frac{xy+yz+zx}{3}\right)^{\frac{3-a}{2}} \geq \frac{(x+y)(y+z)(z+x)}{8} (1)$$

holds for all positive x, y, z.

Solution

Let's try to maximize the RHS keeping the LHS fixed.

Set $m = x+y+z$, $n = xy+yz+zx$, $p = xyz$.

We rewrite the LHS as $(\frac{m}{3})^a(\frac{n}{3})^{\frac{3-a}{2}}$ and the RHS as $\frac{mn-p}{8}$. So keeping m, n fixed and decreasing p will keep the LHS fixed but will increase the RHS. Now how small can we make p? We must ensure that the equation $x^3-mx^2+nx-p=0$ has three positive real solutions, or that the line $y=p$ intersects the graph of $y = x^3-mn^2+nx$ in three points to the right of the y- axis. Therefore the extremal case is either when the line intersects the graph on y- axis (so that moving further would produce a point of intersection to the left of y- axis - a negative solution) which means one of the variables is zero, or when the line touches the graph (because moving further would yield fewer points of intersections) which is equivalent to two variables being equal. Therefore it suffices to look at these cases one by one.

First case: Assume that $z = 0$. Also, because the inequality is symmetric and homogeneous we can assume $y = 1$. The inequality becomes now

$$\left(\frac{x+1}{3}\right)^a \left(\frac{x}{3}\right)^{\frac{3-a}{2}} \geq \frac{(x+1)x}{8} (2)$$

Squaring, we get $64(x+1)^{2a}x^{3-a} \geq 3^{3+a}(x+1)^2x^2$, or $64(x+1)^{2(a-1)} \geq 3^{3+a}x^{a-1}$, which is equivalent to

$$\left(\frac{(x+1)^2}{3x}\right)^{a-1} \geq \frac{81}{64}$$

As $\dfrac{(x+1)^2}{3x}$ can take values as large as we want, we conclude that $a - 1 > 0$.

Then the minimal value of $\dfrac{(x+1)^2}{3x}$ is $\dfrac{4}{3}$ which occurs for $x = 1$ so we must compute a for $x = 1$, which gives us

$$\left(\frac{4}{3}\right)^{a-1} \geq \frac{81}{64}$$

$$a \geq 1 + \frac{4\ln 3 - 6\ln 2}{2\ln 2 - \ln 3} = \frac{3\ln 3 - 4\ln 2}{2\ln 2 - \ln 3} = 1.81884\ldots$$

This is the minimal value for a in this case.

Second case: Assume that $y = z$. Again, since the inequality is homogeneous, we may suppose $y = z = 1$. Then the inequality can be rewritten as

$$\left(\frac{x+2}{3}\right)^a \left(\frac{2x+1}{3}\right)^{\frac{3-a}{2}} \geq \frac{(x+1)^2}{4} (3)$$

Squaring this is yields $16(x+2)^{2a}(2x+1)^{3-a} \geq (x+1)^4 3^{3+a}$, or

$$\left(\frac{(x+2)^2}{3(2x+1)}\right)^a \geq \frac{27(x+1)^4}{16(2x+1)^3} (4)$$

That is, $a \geq \dfrac{\ln E}{\ln F}$ (5) , where $E = \dfrac{27(x+1)^4}{16(2x+1)^3}$, $F = \dfrac{(x+2)^2}{3(2x+1)}$

The last step is possible because $\ln F \geq 0$ since $F \geq 1$, which is equivalent to $(x-1)^2 = (x+2)^2 - 3(2x+1) \geq 0$.

So we need to compute the maximum of the value $f(x) = \dfrac{\ln E}{\ln F}$ for all x except $x = 1$. (For $x = 1$ we can check that any value of a actually gives equality.) Now we shall prove that $f(x)$ is decreasing on $[0, +\infty)$, hence has its maximal value at $x = 0$, which is exactly the value we computed in the previous case, as we have the same triple $(1, 1, 0)$.

First of all let's prove that $E \geq F$. This is equivalent to

$$81(x+1)^4 \geq 16(x+2)^2(2x+1)^2$$

and follows from AM-GM for the numbers $x+2$ and $2x+1$

$$\frac{3(x+1)}{2} \geq 2\sqrt{(x+1)(2x+1)}$$

Now we prove that $f'(x) \geq 0$, or, as $f(x) \geq 0$ (since $E \geq F$), it suffices to prove that $\dfrac{f'(x)}{f(x)} \geq 0$. As $f = \dfrac{\ln E}{\ln F}$, we deduce $\dfrac{f'}{f} = \dfrac{(\ln E)'}{\ln E} - \dfrac{(\ln F)'}{\ln F}$.

Further $\dfrac{(\ln E)'}{\ln E} = \dfrac{E'}{E \ln E}$, and $\dfrac{(\ln F)'}{\ln F} = \dfrac{F'}{F \ln F}$. So we need to prove that

$$\frac{E'}{E} \ln F - \frac{F'}{F} \ln E \geq 0$$

As $\dfrac{E'}{E} = \dfrac{2(x-1)}{(x+1)(2x+1)}$ (6) , $\dfrac{F'}{F} = \dfrac{2(x-1)}{(x+2)(2x+1)}$ (7) and using the fact

that $E \geq F \geq 1$ or $\ln E \geq \ln F \geq 0$ it suffices to show that

$$g(x) = (x-1)((x+2)\ln F - (x+1)\ln E) \geq 0 (8)$$

$$g'(x) = ((x+2)\ln F - (x+1)\ln E)' = \ln F - \ln E + (x+2)\frac{F'}{F} - (x+1)\frac{E'}{E}$$

As $(x+2)\dfrac{F'}{F} = (x+1)\dfrac{E'}{E}$ because of (6) and (7) we deduce that $g'(x) = \ln F - \ln E \leq 0$, so this function is decreasing. Now it's clear that for

$$x = 1 : (x+2)\ln F - (x+1)\ln E = 0$$

$$x < 1 : (x+2)\ln F - (x+1)\ln E \ \geq 0$$

$$x > 1 : (x+2)\ln F - (x+1)\ln E \leq 0$$

so in any case $g(x) = (x-1)((x+2)\ln F - (x+1)\ln E) \leq 0$ thus $f'(x) \leq 0$ and the maximal value is at 0. We have already proved in the first case that

$$a \geq \frac{3\ln 3 - 4\ln 2}{2\ln 2 - \ln 3} = 1.81884\ldots$$

and our proof is complete.