Junior problems

J175. Let $a, b \in (0, \frac{\pi}{2})$ such that $\sin^2 a + \cos 2b \geq \frac{1}{2}\sec a$ and $\sin^2 b + \cos 2a \geq \frac{1}{2}\sec b$. Prove that

$$\cos^6 a + \cos^6 b \geq \frac{1}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Prithwijit De, HBCSE, India

We will use the following well-known trigonometric identities

(a) $\sin^2 x = 1 - \cos^2 x$,

(b) $\cos 2x = 2\cos^2 x - 1$,

(c) $\sec x = \dfrac{1}{\cos x}$.

The inequalities can be written as

$$2\cos^2 b \cos a - \cos^3 a \geq \frac{1}{2} \tag{1}$$

and

$$2\cos^2 a \cos b - \cos^3 b \geq \frac{1}{2}. \tag{2}$$

The signs of the inequalities are preserved because $\cos x$ is positive when $x \in \left(0, \dfrac{\pi}{2}\right)$. Now by squaring both sides of (1) and (2) and adding them we get

$$\cos^6 a + \cos^6 b \geq \frac{1}{2}.$$

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Tigran Hakobyan, Armenia.

J176. Solve in positive real numbers the system of equations

$$\begin{cases} x_1 + x_2 + \cdots + x_n = 1 \\ \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 x_2 \cdots x_n} = n^3 + 1. \end{cases}$$

Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzau, Romania

Solution by Tigran Hakobyan, Armenia

We have

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \geq \frac{n^2}{x_1 + x_2 + \cdots + x_n} = n^2$$

and

$$\frac{1}{x_1 x_2 \cdots x_n} \geq \frac{1}{\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n} = n^n.$$

Thus,

$$n^3 + 1 \geq n^n + n^2$$

which implies that $n \leq 2$. If $n = 1$ we get a contradiction. For $n = 2$ we get

$$\begin{cases} x_1 + x + 2 = 1 \\ \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_1 x_2} = 9 \end{cases}$$

which is $(n, x_1, x_2) \in \left\{ \left(2, \frac{1}{3}, \frac{2}{3}\right), \left(2, \frac{2}{3}, \frac{1}{3}\right) \right\}$.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy.

J177. Let x, y, z be nonnegative real numbers such that $ax + by + cz \leq 3abc$ for some positive real numbers a, b, c. Prove that

$$\sqrt{\frac{x+y}{2}} + \sqrt{\frac{y+z}{2}} + \sqrt{\frac{z+x}{2}} + \sqrt[4]{xyz} \leq \frac{1}{4}(abc + 5a + 5b + 5c).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

From the given condition,

$$3a \geq \frac{ax}{bc} + \frac{y}{c} + \frac{z}{b}$$
$$3b \geq \frac{x}{c} + \frac{by}{ca} + \frac{z}{a}$$
$$3c \geq \frac{x}{b} + \frac{y}{a} + \frac{cz}{ab}.$$

Then

$$3(a+b+c) \geq \frac{x+y}{c} + \frac{y+z}{a} + \frac{z+x}{b} + \left(\frac{ax}{bc} + \frac{by}{ca} + \frac{cz}{ab}\right)$$

hence

$$\begin{aligned} abc + 5(a+b+c) &\geq \left(\frac{x+y}{c} + 2c\right) + \left(\frac{y+z}{a} + 2a\right) + \left(\frac{z+x}{b} + 2b\right) + \left(abc + \frac{ax}{bc} + \frac{by}{ca} + \frac{cz}{ab}\right) \\ &\geq 2\sqrt{2(x+y)} + 2\sqrt{(y+z)} + 2\sqrt{(z+x)} + 4\sqrt[4]{xyz} \end{aligned}$$

and the conclusion follows. The equality holds if and only if $x+y = 2c^2, y+z = 2a^2, z+x = 2b^2$ and $\frac{ax}{bc} = \frac{by}{ca} = \frac{cz}{ab} = abc$. This implies $b^2c^2 = c^2a^2 = 2c^2, c^2a^2 + a^2b^2 = 2a^2, a^2b^2 + b^2c^2 = 2b^2$, that is $b^2 + a^2 = c^2 + b^2 = a^2 + c^2 = 2$, implying $a = b = c = 1$ and $x = y = z = 1$. If $a = b = c = x = y = z = 1$, the equality, as well as the condition of the problem, hold

J178. Find the sequences of integers $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ such that

$$(2+\sqrt{5})^n = a_n + b_n\frac{1+\sqrt{5}}{2}$$

for each $n \geq 0$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

First solution by Arkady Alt, San Jose, California, USA

Let

$$p_n = \frac{\left(2+\sqrt{5}\right)^n + \left(2-\sqrt{5}\right)^n}{2}$$

and

$$q_n = \frac{\left(2+\sqrt{5}\right)^n - \left(2-\sqrt{5}\right)^n}{2\sqrt{5}}$$

for $n = 1, 2, \ldots$. Then $\left(2+\sqrt{5}\right)^n = p_n + q_n\sqrt{5}, n = 0, 1, 2, \ldots$ and both obtained sequences satisfy the same recurrence

$$x_{n+1} = 4x_n + x_{n-1}, n \in \mathbb{N} \tag{1}$$

with initial conditions $p_0 = 1, p_1 = 2, q_0 = 0, q_1 = 1$. It is clear that $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ are sequences of nonnegative integers and since

$$\begin{aligned} a_n + b_n\frac{1+\sqrt{5}}{2} = p_n + q_n\sqrt{5} &\iff a_n + \frac{b_n}{2} + \frac{b_n}{2}\sqrt{5} = p_n + q_n\sqrt{5} \\ &\iff \begin{cases} a_n + \dfrac{b_n}{2} = p_n \\ \dfrac{b_n}{2} = q_n \end{cases} \\ &\iff \begin{cases} a_n = p_n - q_n \\ b_n = 2q_n \end{cases}, n \in \mathbb{N} \cup \{0\} \end{aligned}$$

we have that $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are sequences of integers and can be defined independently by recurrence **(1)** with initial conditions $a_0 = 1, a_1 = 1, b_0 = 0, b_1 = 2$. In explicit form

$$b_n = \frac{\left(2+\sqrt{5}\right)^n - \left(2-\sqrt{5}\right)^n}{\sqrt{5}},\ a_n = \frac{\left(\sqrt{5}-1\right)\left(2+\sqrt{5}\right)^n + \left(\sqrt{5}+1\right)\left(2-\sqrt{5}\right)^n}{2\sqrt{5}}.$$

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Using Newton's binomial formula, exchanging $\sqrt{5}$ by $-\sqrt{5}$ in the first term results in exchanging $\sqrt{5}$ by $-\sqrt{5}$ in the second term, i.e., $(2+\sqrt{5})^n + (2-\sqrt{5})^n = 2a_n + b_n$ and $(2+\sqrt{5})^n - (2-\sqrt{5})^n = \sqrt{5}b_n$, yielding

$$b_n = \frac{(2+\sqrt{5})^n - (2-\sqrt{5})^n}{\sqrt{5}},\ a_n = \frac{(\sqrt{5}-1)(2+\sqrt{5})^n + (\sqrt{5}+1)(2-\sqrt{5})^n}{2\sqrt{5}}.$$

The fact that a_n, b_n are integers for all n may be easily proved considering that they are solutions of the recursive equations $a_n = 4a_{n-1} + a_{n-2}$ and $b_n = 4b_{n-1} + b_{n-2}$, with initial conditions $a_0 = a_1 = 1$ and $b_0 = 0$, $b_1 = 2$.

Also solved by Ivan Dinkov Gerganov, P. R. Slaveikov Secondary School, Bulgaria; Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy; Tigran Hakobyan, Armenia.

J179. Solve in real numbers the system of equations

$$\begin{cases}(x+y)(y^3-z^3)=3(z-x)(z^3+x^3)\\(y+z)(z^3-x^3)=3(x-y)(x^3+y^3)\\(z+x)(x^3-y^3)=3(y-z)(y^3+z^3)\end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy

Without loss of generality assume that $x=0$, then it is clear that $y=z=0$. Because the given system is symmetric we can assume that $x,y,z\neq 0$. Assume that $x=y$, then $x=z$ or $y=-z$. If we assume that $y=-z$ then the first equation becomes $4x^4=6x^4$, contradiction. Now $x\neq y\neq z\neq 0$ and after multiplying all three equations we get

$$\frac{3(x^2-xy+y^2)3(y^2-yz+z^2)3(z^2-zx+x^2)}{(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)}=1$$

which can be written as

$$\left(1-\frac{2xy}{(x-y)^2+3xy}\right)\left(1-\frac{2yz}{(y-z)^2+3yz}\right)\left(1-\frac{2zx}{(z-x)^2+3zx}\right)=\frac{1}{27}.$$

It is not difficult to see that each factor of the LHS is greater than $\frac{1}{3}$ which leads to a contradiction. So the only possible solution is $x=y=z$.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Tigran Hakobyan, Armenia.

J180. Let a, b, c, d be distinct real numbers such that

$$\frac{1}{\sqrt[3]{a-b}} + \frac{1}{\sqrt[3]{b-c}} + \frac{1}{\sqrt[3]{c-d}} + \frac{1}{\sqrt[3]{d-a}} \neq 0.$$

Prove that $\sqrt[3]{a-b} + \sqrt[3]{b-c} + \sqrt[3]{c-d} + \sqrt[3]{d-a} \neq 0.$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Anthony Erb Lugo

Let $\sqrt[3]{a-b}$, $\sqrt[3]{b-c}$, $\sqrt[3]{c-d}$, $\sqrt[3]{d-a}$ by w, x, y, z, respectively and let $S = w + x + y + z$. So that,

$$w^3 + x^3 + y^3 + z^3 = 0(*) \quad \text{and} \quad \frac{1}{w} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \neq 0(**)$$

Since a, b, c and d are distinct, we can conclude that $w, x, y, z \neq 0$, and thus,

$$wxy + wxz + wyz + xyz = wxyz\left(\frac{1}{w} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \neq 0(***)$$

Furthermore, using (*), we have that,

$$\begin{aligned} S^3 &= 6(wxy + wxz + wyz + xyz) + 3(w^2 + x^2 + y^2 + z^2)S \\ S(S^2 - 3(w^2 + x^2 + y^2 + z^2)) &= 6(wxy + wxz + wyz + xyz) \end{aligned}$$

Which implies that $S \neq 0$, otherwise the right hand side would be 0 which is a contradiction of $(***)$, and we're done.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Tigran Hakobyan, Armenia; Daniel Campos Salas, Costa Rica.

Senior problems

S175. Let p be a prime. Find all integers $a_1, \ldots, a_n$ such that $a_1 + \cdots + a_n = p^2 - p$ and all solutions to the equation $px^n + a_1 x^{n-1} + \cdots + a_n = 0$ are nonzero integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let $-r_1, -r_2, \ldots, -r_n$ be the n nonzero integral roots of the equation given in the problem statement, then the equation rewrites as $p(x+r_1)(x+r_2)\ldots(x+r_n) = 0$, or all its coefficients are nonzero integral multiples of p. In particular, a_k is p times the sum of all possible products of k of the r_i's, hence $p + a_1 + a_2 + \cdots + a_n = p(1+r_1)(1+r_2)\ldots(1+r_n) = p^2$, yielding $(1+r_1)(1+r_2)\ldots(1+r_n) = p$, where each one of the $1+r_i$ are integers other than 1, ie, $n-1$ of them must be equal to -1, and the remaining one must be p when n is odd and $-p$ when n is even; in other words, $n-1$ of the r_i's are equal to -2, and the other one is $p-1$ when n is odd and $-p-1$ when n is even. Now, there are $\binom{n-1}{k-1}$ products of k of the r_i such that exactly one of them is not -2, and $\binom{n-1}{k}$ products of k of the r_i such that all of them are -2, ie, when n is odd,

$$\frac{a_k}{p} = \binom{n-1}{k-1}(-2)^{k-1}(p-1) + \binom{n-1}{k}(-2)^k = \frac{(n-1)!}{(n-k)!k!}(-2)^{k-1}(pk+k-2n),$$

and when n is even,

$$\frac{a_k}{p} = \binom{n-1}{k-1}(-2)^{k-1}(-p-1) + \binom{n-1}{k}(-2)^k = -\frac{(n-1)!(2n+pk-k)}{(n-k)!k!}(-2)^{k-1}.$$

Note now that, when n is odd,

$$\sum_{k=1}^{n} \frac{a_k}{p} = \sum_{k-1=0}^{n-1} (p+1)\frac{(n-1)!}{(n-k)!(k-1)!}(-2)^{k-1} + \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}(-2)^k - 1 =$$

$$= (p+1)(1-2)^{n-1} + (1-2)^n - 1 = p-1,$$

while when n is even,

$$\sum_{k=1}^{n} \frac{a_k}{p} = -(p-1)\sum_{k-1=}^{n-1} \frac{(n-1)!}{(n-k)!(k-1)!}(-2)^{k-1} + \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}(-2)^k - 1 =$$

$$= -(p-1)(1-2)^{n-1} + (1-2)^n - 1 = p-1.$$

There can be no other solutions.

Also solved by Anthony Erb Lugo; Tigran Hakobyan, Armenia.

S176. Let ABC be a triangle and let AA_1, BB_1, CC_1 be cevians intersecting at P. Denote by $K_a = K_{AB_1C_1}$, $K_b = K_{BC_1A_1}$, $K_c = K_{CA_1B_1}$. Prove that $K_{A_1B_1C_1}$ is a root of the equation

$$x^3 + (K_a + K_b + K_c)x^2 - 4K_aK_bK_c = 0.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Arkady Alt, San Jose, California, USA

Without loss of generality assume that area of triangle ABC is 1. Let p_a, p_b, p_b be the baricentric coordinates of P, that is $p_a, p_b, p_c > 0, p_a + p_b + p_c = 1$ and

$$p_a\overrightarrow{PA} + p_b\overrightarrow{PB} + p_c\overrightarrow{PC} = 0.$$

Since $\dfrac{AC_1}{BC_1} = \dfrac{p_b}{p_a}$ and $\dfrac{[ACC_1]}{[BCC_1]} = \dfrac{AC_1}{BC_1}$ then

$$[ACC_1] = \frac{p_b}{p_a + p_b}[ABC] = \frac{p_b}{p_a + p_b}.$$

Also, $\dfrac{[AB_1C_1]}{[CB_1C_1]} = \dfrac{AB_1}{CB_1} = \dfrac{p_c}{p_a}$ yields

$$\frac{[AB_1C_1]}{[ACC_1]} = \frac{p_c}{p_c + p_a}.$$

Hence, $K_a = [AB_1C_1] = \dfrac{p_bp_c}{(p_a + p_b)(p_c + p_a)}$. Similarly,

$$K_b = \frac{p_cp_a}{(p_b + p_c)(p_a + p_b)}, \ K_c = \frac{p_ap_b}{(p_c + p_a)(p_b + p_c)}.$$

Let $K = [A_1B_1C_1]$, then

$$\begin{aligned} K &= [ABC] - ([AB_1C_1] + [BC_1A_1] + [CA_1B_1]) \\ &= 1 - (K_a + K_b + K_c) = 1 - \sum_{cyc}\frac{p_bp_c}{(p_a + p_b)(p_c + p_a)} \\ &= \frac{2p_ap_bp_c}{(p_a + p_b)(p_b + p_c)(p_c + p_a)}. \end{aligned}$$

On the other hand, since

$$\frac{K_a}{K} = \frac{p_b + p_c}{2p_a}, \ \frac{K_b}{K} = \frac{p_c + p_a}{2p_b}, \ \frac{K_c}{K} = \frac{p_a + p_b}{2p_c}$$

then

$$\frac{K_aK_bK_c}{K^3} = \frac{(p_a + p_b)(p_b + p_c)(p_c + p_a)}{8p_ap_bp_c} = \frac{1}{4K}.$$

Thus, $K^2 = 4K_aK_bK_c$ and, therefore,

$$K = 1 - (K_a + K_b + K_c) \iff K^3 = K^2 - (K_a + K_b + K_c)\,K^2$$
$$\iff K^3 + (K_a + K_b + K_c)\,K^2 - 4K_aK_bK_c = 0.$$

Second solution by Daniel Campos Salas, Costa Rica

Let x, y, z be the areas of the triangles BPC, CPA, APB, respectively. It's easy to prove that $\frac{AB_1}{B_1C} = \frac{z}{x}$ and $\frac{AC_1}{C_1B} = \frac{y}{x}$. Therefore we have that

$$K_a = \frac{1}{2}AB_1 \cdot AC_1 \sin A = \frac{1}{2} \cdot \frac{zAC}{x+z} \cdot \frac{yAB}{x+y} \sin A = \frac{yz(x+y+z)}{(x+y)(x+z)}.$$

The expressions for K_b and K_c are obtained analogously. It follows that $K_{A_1B_1C_1}$ equals

$$\begin{aligned} & (x+y+z)\left(1 - \frac{yz}{(x+y)(x+z)} - \frac{zx}{(y+z)(y+x)} - \frac{xy}{(z+x)(z+y)}\right) \\ = \; & \frac{2xyz(x+y+z)}{(x+y)(y+z)(z+x)}. \end{aligned}$$

Finally, note that

$$\begin{aligned} & K_{A_1B_1C_1}^3 + K_{A_1B_1C_1}(K_a + K_b + K_c) \\ = \; & K_{A_1B_1C_1}^2(K_{A_1B_1C_1} + K_a + K_b + K_c) \\ = \; & \left(\frac{2xyz(x+y+z)}{(x+y)(y+z)(z+x)})\right)^2 (x+y+z) \\ = \; & \frac{4(xyz)^2(x+y+z)^3}{((x+y)(y+z)(z+x))^2} \\ = \; & 4K_aK_bK_c, \end{aligned}$$

which proves that $K_{A_1B_1C_1}$ is a root of the given polynomial.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S177. Prove that in any acute triangle ABC,

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\geq\frac{5R+2r}{4R}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Arkady Alt, San Jose, California, USA

Since $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, the inequality can be rewritten as

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\geq\frac{5}{4}. \tag{A}$$

Inequality **(A)** is an immediate corollary from more general inequality represented by the following theorem

Theorem. *Let k be any real number such that $k\geq k_*$ where*

$$k_*=\frac{4}{2\sqrt{2-\sqrt{2}}-\sqrt{2}+3}\approx 1.2835.$$

Then for any $\alpha,\beta,\gamma\in\left(0,\frac{\pi}{4}\right]$, such that $\alpha+\beta+\gamma=\frac{\pi}{2}$ the following inequality holds

$$\sin\alpha+\sin\beta+\sin\gamma-k\sin\alpha\sin\beta\sin\gamma\geq\frac{12-k}{8}. \tag{M}$$

Proof. Assuming, due symmetry, that $\alpha\leq\beta\leq\gamma$ and, denoting $\varphi=\alpha+\beta$ we obtain $\gamma=\frac{\pi}{2}-\varphi$, $\beta=\varphi-\alpha$, where the new variables α and φ satisfy the inequalities $0<\alpha\leq\varphi-\alpha\leq\frac{\pi}{2}-\varphi\leq\frac{\pi}{4}$ or equivalently

$$\begin{cases}\dfrac{\pi}{4}\leq\varphi\leq\dfrac{\pi}{3}\\ 2\varphi-\dfrac{\pi}{2}\leq\alpha\leq\dfrac{\varphi}{2}\end{cases}. \tag{1}$$

Since

$$\begin{aligned}\sin\alpha+\sin\beta+\sin\gamma-k\sin\alpha\sin\beta\sin\gamma&=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+\sin\gamma\,(1-k\ \sin\alpha\sin\beta)\\ &=2\sin\frac{\varphi}{2}\cos\left(\frac{\varphi}{2}-\alpha\right)+\cos\varphi\,(1-k\ \sin\alpha\sin(\varphi-\alpha))\end{aligned}$$

then inequality **(M)** can be equivalently rewritten as

$$2\sin\frac{\varphi}{2}\cos\left(\frac{\varphi}{2}-\alpha\right)+\cos\varphi\,(1-k\ \sin\alpha\sin(\varphi-\alpha))\geq\frac{12-k}{8} \tag{2}$$

where variables α and φ are subject to the system **(1).**

Let

$$h(\alpha) = 2\sin\frac{\varphi}{2}\cos\left(\frac{\varphi}{2}-\alpha\right) + \cos\varphi\,(1-k\ \sin\alpha\sin(\varphi-\alpha))$$

for any fixed $\varphi \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]$ and $k_* > \frac{2}{\sqrt{3}} = 1.1547$. We will prove that $h(\alpha)$ is decreasing on $\left[2\varphi - \frac{\pi}{2}, \frac{\varphi}{2}\right]$. Indeed,

$$h'(\alpha) = 2\sin\left(\frac{\varphi}{2}-\alpha\right)\left(\sin\frac{\varphi}{2} - k\cos\varphi\cos\left(\frac{\varphi}{2}-\alpha\right)\right) \le 0$$

on $\left[2\varphi - \frac{\pi}{2}, \frac{\varphi}{2}\right]$ since $\sin\left(\frac{\varphi}{2}-\alpha\right) \ge 0,\ k_* > \frac{2}{\sqrt{3}}$ and

$$\begin{aligned}\sin\frac{\varphi}{2} - k\cos\varphi\cos\left(\frac{\varphi}{2}-\alpha\right) &\le \sin\frac{\varphi}{2} - k\cos\varphi\cos\frac{\varphi}{2} \\ &\le \sin\frac{\pi}{6} - k\cos\frac{\pi}{3}\cos\frac{\pi}{6} = \frac{1}{2} - \frac{k}{2}\cdot\frac{\sqrt{3}}{2} \\ &\le \frac{1}{2} - \frac{\sqrt{3}}{2}\cdot\frac{k_*}{2} < \frac{1}{2} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{3}} \\ &= 0.\end{aligned}$$

Thus, $h(\alpha) \ge h\left(\frac{\varphi}{2}\right) = 2\sin\frac{\varphi}{2} + \cos\varphi\left(1 - k\ \sin^2\frac{\varphi}{2}\right)$ and it remains to prove the inequality

$$2\sin\frac{\varphi}{2} + \cos\varphi\left(1 - k\sin^2\frac{\varphi}{2}\right) \ge \frac{12-k}{8}. \tag{3}$$

Let $t = \sin\frac{\varphi}{2}$ then $\sin\frac{\pi}{8} \le t \le \frac{1}{2}$ and **(3)** is equivalent to

$$\begin{aligned}2t + \left(1-2t^2\right)\left(1-kt^2\right) \ge \frac{12-k}{8} &\iff 16kt^4 - (8k+16)\,t^2 + 16t - (4-k) \ge 0 \\ &\iff (1-2t)^2\left(k\,(2t+1)^2 - 4\right) \ge 0\end{aligned}$$

because $k\,(2t+1)^2 \ge k_*\left(2\sin\frac{\pi}{8}+1\right)^2 = 4$. Since equality in **(2)** occurs if and only if $\alpha = \frac{\varphi}{2}$ and $\varphi = \frac{\pi}{3} \iff \alpha = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$ then in **(M)** equality occurs if and only if $\alpha = \beta = \gamma = \frac{\pi}{6}$. In particular for $k = 2$ and $k = \frac{4}{3}$, replacing (α, β, γ) in **(M)** with $\left(\frac{A}{2}, \frac{B}{2}, \frac{C}{2}\right)$, for any acute triangle ABC we, respectively, obtain inequality **(A)** and inequality

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} - \frac{4}{3}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \ge \frac{4}{3}. \tag{G}$$

(The last one is an inequality due to J. Garfuncel, given in [**RAGI**] without proof in a private communication.)

Remark.

The original inequality immediately follows from **(G).** Indeed,

$$\begin{aligned}\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} &= \sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-\frac{4}{3}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\\ &\quad -\frac{2}{3}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\\ &\geq \frac{4}{3}-\frac{2}{3}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\geq\frac{4}{3}-\frac{2}{3}\cdot\frac{1}{8}\\ &= \frac{5}{4}.\end{aligned}$$

[RAGI]. Mitrinović D.S., Pečarić J. E. , Volenec V. Recent Advances, *Geometric Inequality*, p.269, inequality 5.10.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Assume wlog that $C \geq B \geq A$, and denote $\frac{A+B}{4}=\alpha$, $\frac{B-A}{4}=\delta$. It is well known that $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, or the proposed problem is equivalent to showing that

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\geq\frac{5}{4}.$$

Assume that C is known such that the LHS is minimum, or for that value of C, define

$$f(x,y)=\sin x+\sin y-2\sin x\sin y\sin\frac{C}{2},$$

where $x+y=\frac{A+B}{2}=90^\circ-\frac{C}{2}$ is fixed. Now,

$$f\left(\frac{A}{2},\frac{B}{2}\right)-f(\alpha,\alpha)=f(\alpha-\delta,\alpha+\delta)-f(\alpha,\alpha)=2\sin\alpha(\cos\delta-1)+2\sin^2\delta\sin\frac{C}{2}=$$

$$=4\sin^2\frac{\delta}{2}\left(2\cos^2\frac{\delta}{2}\sin\frac{C}{2}-\sin\alpha\right).$$

Now, if $\delta>0$, and since $\delta=\frac{B-A}{4}<\frac{B}{4}<45^\circ$, we have $\sin^2\frac{\delta}{2}>0$, $2\cos^2\frac{\delta}{2}>2\cos^2(45^\circ)>1$. Moreover, $\frac{C}{2}>\frac{A+B}{4}$, since equality would only hold iff $A=B=C$, which is not true because $\delta>0$. Thus, $f\left(\frac{A}{2},\frac{B}{2}\right)\geq f\left(\frac{A+B}{4},\frac{A+B}{4}\right)$, with equality iff $A=B=90^\circ-\frac{C}{2}$. It therefore suffices to show that, for all $90^\circ>C\geq 60^\circ$, we have

$$2u+(1-2u^2)-2u^2(1-2u^2)\geq\frac{5}{4}.$$

where we have defined $u=\sin\left(45^\circ-\frac{C}{4}\right)$, and therefore $\sin\frac{C}{2}=\cos\left(2\left(45^\circ-\frac{C}{4}\right)\right)=1-2u^2$. After some algebra, this last inequality is equivalent to $(2u-1)^2(4u^2+8u+1)\geq 0$. Since $90^\circ>C\geq 60^\circ$, we have $\frac{45^\circ}{2}<45^\circ-\frac{C}{4}\leq 30^\circ$, or $u>0$. The conclusion follows, equality holds iff $u=\frac{1}{2}$, ie iff $45^\circ-\frac{C}{4}=30^\circ$, or $C=60^\circ$. We conclude that equality holds iff ABC is equilateral.

Third solution by Neculai Stanciu, George Emil Palade, Romania

We have

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\geq\frac{5R+2r}{4R}\iff 2\sum\sin\frac{A}{2}\geq 1+\frac{r}{R}+\frac{3}{2}$$
$$\iff 2\cos\frac{A+B}{2}+2\cos\frac{B+C}{2}+2\cos\frac{C+A}{2}\geq\cos A+\cos B+\cos C+3\cos\frac{A+B+C}{3}$$

which is true from Popoviciu's Inequality for the concave function $\cos x$ on the interval $\left(0,\frac{\pi}{2}\right)$.

Also solved by Daniel Campos Salas, Costa Rica

S178. Prove that there are sequences $(x_k)_{k\geq 1}$ and $(y_k)_{k\geq 1}$ of positive rational numbers such that for all positive integers n and k,

$$(x_k + y_k\sqrt{5})^n = F_{kn-1} + F_{kn}\frac{1+\sqrt{5}}{2},$$

where $(F_m)_{m\geq 1}$ is the Fibonacci sequence.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

Take $x_k = \frac{1}{2}L_k$ and $y_k = \frac{1}{2}F_k$ (see problem J178) where L_n is the n-th Lucas number. Since

$$F_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right) \quad \text{and} \quad L_k = \left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^k$$

it follows that

$$(x_k + y_k\sqrt{5})^n = \left(\frac{1+\sqrt{5}}{2}\right)^{kn} = F_{kn-1} + F_{kn}\left(\frac{1+\sqrt{5}}{2}\right).$$

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Daniel Campos Salas, Costa Rica.

S179. Find all positive integers a and b for which $\frac{(a^2+1)^2}{ab-1}$ is a positive integer.

Proposed by Valcho Milchev, Petko Rachov Slaveikov Secondary School, Bulgaria

Solution by Daniel Campos Salas, Costa Rica

Note $ab-1$ divides $(ab-1)(2a^2+ab+1) = (a^2+ab)^2-(a^2+1)^2$. This implies that $ab-1$ divides $a^2(a+b)^2$, but $ab-1$ and a are coprime, from where we conclude that $ab-1$ divides $(a+b)^2$. Let $k = \frac{(a+b)^2}{ab-1}$. Note that it is not possible for a, b to be equal, since it would imply $\frac{4a^2}{a^2-1} = 4+\frac{4}{a^2-1}$ to be an integer, and 4 has no divisor of the form a^2-1. The equation can also be written in the form

$$a^2-(k-2)ab+b^2+k = 0. \tag{1}$$

Note that for every pair of integers (a, b) satisfying the equation $k = \frac{(a+b)^2}{ab-1}$, the pair $\left(a, \frac{a^2+k}{b}\right)$ and its permutations are also positive integer solutions (it's easy to prove that $\frac{a^2+k}{b}$ is an integer). Iterating this process we find that every solution is part of a family of solutions. Let (a_0, b_0) be a solution for any of these families with the minimal sum. We may assume without loss of generality that $b_0 > a_0$. This implies that $\frac{a_0^2+k}{b_0} \geq b_0$ since $\left(a_0, \frac{a_0^2+k}{b_0}\right)$ and $\left(\frac{a_0^2+k}{b_0}, a_0\right)$ are also solutions to the equation. It follows that

$$k \geq b_0^2-a_0^2 = (b_0-a_0)(a_0+b_0) \geq a_0+b_0.$$

This implies that $\frac{(a_0+b_0)^2}{a_0b_0-1} = k \geq a_0+b_0$, or equivalently, $2 \geq (a_0-1)(b_0-1)$. Note that $a_0 < 3$, since $a_0 < b_0$. If $a_0 = 1$, it follows that $\frac{(b_0+1)^2}{b_0-1}$ is an integer. This implies that $\frac{4}{b_0-1}$, so it follows that b_0 can be equal to $2, 3$ or 5. If $a_0 = 2$, it follows that $\frac{(b_0+2)^2}{2b_0-1}$ is an integer. This implies that $\frac{25}{2b_0-1}$, or equivalently, b_0 can equal 3 or 13, since $b_0 > a_0 = 2$.

For $(a_0, b_0) = (1, 2)$ it follows that $k = 9$. This initial solution generates the family of solutions given by the pairs, (c_n, c_{n+1}), $n \geq 0$, which satisfy, $c_0 = 1, c_1 = 2$, $c_{n+2} = 7c_{n+1}-c_n$, where the recursion holds by (1). The explicit expression for c_n is given by the formula

$$\begin{aligned} c_n &= \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{7+3\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{7-3\sqrt{5}}{2}\right)^n \\ &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{4n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{4n-1} \\ &= F_{4n-1}, \end{aligned}$$

where F_n is the n-th term of the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1}+F_n$ (abusing of the definition we set $F_{-1} = 1$).

For $(a_0, b_0) = (1, 3)$ it follows that $k = 8$. This initial solution generates the family of solutions given by the pairs, (d_n, d_{n+1}), $n \geq 0$, which satisfy, $d_0 = 1, d_1 = 2$, $d_{n+2} = 6d_{n+1} - d_n$, where the recursion holds by (1). The explicit expression for d_n is given by the formula

$$\begin{aligned} d_n &= \frac{1}{2}\left(3+2\sqrt{2}\right)^n + \frac{1}{2}\left(3-2\sqrt{2}\right)^n \\ &= \frac{1}{2}\left((\sqrt{2}+1)^{2n} + (\sqrt{2}-1)^{2n}\right). \end{aligned}$$

For $(a_0, b_0) = (1, 5)$ it follows that $k = 9$. This initial solution generates the family of solutions given by the pairs, (e_n, e_{n+1}), $n \geq 0$, which satisfy, $e_0 = 1, e_1 = 2$, $e_{n+2} = 7e_{n+1} - e_n$, where the recursion holds by (1). The explicit expression for e_n is given by the formula

$$\begin{aligned} e_n &= \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{7+3\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{7-3\sqrt{5}}{2}\right)^n \\ &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{4n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{4n+1} \\ &= F_{4n+1}. \end{aligned}$$

We may interpret this solution as an extension of the first one for negative indexes since $F_{4n+1} = F_{-4n-1}$.

For $(a_0, b_0) = (2, 3)$ it follows that $k = 5$. This initial solution generates the family of solutions given by the pairs, (f_n, f_{n+1}), $n \geq 0$, which satisfy, $f_0 = 1, f_1 = 2$, $f_{n+2} = 3d_{n+1} - f_n$, where the recursion holds by (1). The explicit expression for f_n is given by the formula

$$\begin{aligned} f_n &= \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^{2n} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n} \\ &= L_{2n}, \end{aligned}$$

where L_n is the n-th term of the Lucas sequence defined by $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$.

Therefore, we conclude that all the possible pairs of positive integers are of the form (F_{4n-1}, F_{4n+3}), (L_{2n}, L_{2n+2}) (for all integers n) and (d_n, d_{n+1}) (defined above), and its permutations, and we're done.

Also solved by Tigran Hakobyan, Armenia

S180. Solve in nonzero real numbers the system of equations

$$\begin{cases} x^4 - y^4 = \frac{121x-122y}{4xy} \\ x^4 + 14x^2y^2 + y^4 = \frac{122x+121y}{x^2+y^2}. \end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note first that $x^4 + 14x^2y^2 + y^4 = 4(x^2+y^2)^2 - 3(x^2-y^2)^2 = s^4 - s^2d^2 + d^4$, where we have defined $x + y = s$ and $x - y = d$, while $x^4 - y^4 = sd(x^2+y^2) = \frac{sd(s^2+d^2)}{2}$, $4xy = s^2 - d^2$, $x^2 + y^2 = \frac{s^2+d^2}{2}$. Therefore, the system may be rewritten as

$$\begin{cases} sd(s^2+d^2)(s^2-d^2) = 243d - s, \\ (s^4 - s^2d^2 + d^4)(s^2+d^2) = 243s + d. \end{cases}$$

We can then obtain

$$(243d - s)(243s + d) = sd(s^2+d^2)(s^2-d^2)(243s+d) = (s^4 - s^2d^2 + d^4)(s^2+d^2)(243d - s).$$

Since $s^2+d^2 > 0$ (otherwise $x = y = 0$, in contradiction with the problem statement), it follows that

$$243s^2d(s^2-d^2) + sd^2(s^2-d^2) = 243d(s^4 - s^2d^2 + d^4) - s(s^4 - s^2d^2 + d^4),$$

which after simplification yields $s^5 = 243d^5$, or $s = 3d$. Substitution in both equations yields $d^6 = d$, or since $x \neq y$ (if $x = y \neq 0$ the LHS of the first equation would be zero, but the RHS would not), we find that $d^5 = 1$, ie $s = 3$ and $d = 1$ for $x = 2$, $y = 1$. These values can be clearly shown to satisfy the system by plugging them into the given equations, and no other solutions exist.

Second solution by Arkady Alt, San Jose, California, USA

Since

$$\left(x^4 + 14x^2y^2 + y^4\right)\left(x^2+y^2\right) = 122x + 121y,\ 4xy\left(x^4 - y^4\right) = 121x - 122y$$

and $x, y \neq 0$ then

$$\begin{aligned} &\left(x^4 + 14x^2y^2 + y^4\right)\left(x^2+y^2\right)(x-y) - 4xy\left(x^4-y^4\right)(x+y) \\ &= (122x+121y)(x-y) - (121x-122y)(x+y) \\ &\iff \left(x^2+y^2\right)\left(\left(x^4+14x^2y^2+y^4\right)(x-y) - 4xy\left(x^2-y^2\right)(x+y)\right) \\ &= x^2+y^2 \iff \left(x^4+14x^2y^2+y^4\right)(x-y) - 4xy\left(x^2-y^2\right)(x+y) \\ &= 1 \iff (x-y)^5 = 1 \iff x - y = 1. \end{aligned}$$

Let $t = x + y$ then

$$x^2 - y^2 = t, x^2 + y^2 = \frac{t^2+1}{2},\ 4xy = t^2 - 1,\ y = \frac{t-1}{2},\ 121x - 122y = 121 - \frac{t-1}{2}$$

and the equation $x^4 - y^4 = \dfrac{121x - 122y}{4xy}$ becomes

$$\frac{t\left(t^4-1\right)}{2} = 121 - \frac{t-1}{2} \iff t\left(t^4-1\right) + t - 1 = 242 \iff t^5 = 243 \iff t = 3.$$

Hence,

$$\begin{cases} x - y = 1 \\ x + y = 3 \end{cases} \iff x = 2, y = 1.$$

Undergraduate problems

U175. What is the maximum number of points of intersection that can appear after drawing in a plane l lines, c circles, and e ellipses?

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Andrea Ligori and Emanuele Natale, Università di Roma "Tor Vergata", Roma, Italy

We note that

- the intersection of two circles yields 2 points, so the contribution is $2\binom{c}{2}$;
- the intersection of a line with a circle or an ellipse yields 2 points, so the contribution is $2l(c+e)$;
- the intersection of an ellipse with a circle or another ellipse yields 4 points, so the contribution is $4\binom{e}{2}+4ec$;
- the intersection of two incindent lines yields 1 point so the contribution is $\binom{l}{2}$.

Therefore the final formula is

$$2\binom{c}{2}+2l(c+e)+4\binom{e}{2}+4ec+\binom{l}{2}.$$

It is easy to find a configuration of l lines, c circles, and e ellipses with such a number of intersection points.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

U176. In the space, consider the set of points (a, b, c) where $a, b, c \in \{0, 1, 2\}$. Find the maximum number of non-collinear points contained in the set.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

It is possible to have 16 non-collinear points, taking for example the following ones:

$$(0,0,0),(0,1,0),(1,0,0),(1,2,0),(2,1,0),(2,2,0),(0,0,1),(0,2,1),$$

$$(2,0,1),(2,2,1),(0,1,2),(0,2,2),(1,0,2),(1,2,2),(2,0,2),(2,1,2).$$

Consider a set of non-collinear points of the form (a, b, c) where $a, b, c \in \{0, 1, 2\}$. Note first that for each pair of fixed values (b, c), there may be at most two values of a such that (a, b, c) is in the set, otherwise if points with $a = 0, 1, 2$ are in the set, they would be collinear. The same reasoning applies to all pairs of fixed values (c, a) and (a, b). It follows that, for each fixed value of c, there are at most six pairs of values (a, b) such that (a, b, c) is in the set, and such that no three of them have the same value of a or of b. Assume now that, for a given value of c and two distinct values b_1, b_2 of b, the same two values a_1, a_2 of a exist such that (a, b, c) is in the set. Calling b_3 the third possible value of b, note that neither (a_1, b_3, c) nor (a_2, b_3, c) may be in the set, ie, at most five points with the aforementioned value of c are in the set. We conclude that, if there are six points with the same value of c, then the pairs of values of a such that (a, b, c) is in the set as b takes its three possible values, must be distinct and therefore must be $(0, 1)$, $(1, 2)$, $(0, 2)$, each pair corresponding to each one of the possible values of b. Finally, if $(0, 1, c)$ and $(1, 1, c)$ are in the set, note that $(0, 0, c)$ cannot be in the set, since neither $(0, 2, c)$ nor $(2, 2, c)$ could be on the set, and similarly for $(2, 1, c)$ by symmetry, ie, if there are six points which have the same value of c, they must be either $(0, 0, c)$, $(0, 1, c)$, $(1, 0, c)$, $(1, 2, c)$, $(2, 1, c)$ and $(2, 2, c)$, or $(0, 1, c)$, $(0, 2, c)$, $(1, 0, c)$, $(1, 2, c)$, $(2, 0, c)$ and $(2, 1, c)$.

Assume that it is possible to have 17 non-collinear points. By the previous arguments, exactly two of the planes with constant value of c must contain exactly 6 points, and the remaining plane must contain exactly 5 points. Moreover, both plains containing exactly 6 points must have a different combination out of the two mentioned above, since if (a, b, c) belongs in the set for two distinct values of c, and six distinct pairs of values (a, b), then (a, b, c) cannot be in the set for each one of this six pairs of values and the third value of c. It follows that, the plane containing exactly 5 points, cannot have points at $(0, 1, c)$, $(1, 0, c)$, $(1, 2, c)$ or $(2, 1, c)$, since they would be collinear with the corresponding points of the other two planes. Therefore the plane containing exactly 5 points contains at least points $(0, 0, c)$, $(1, 1, c)$ and $(2, 2, c)$, which are collinear, contradiction. There cannot be more than 16 non-collinear points, but we have produced an example of such 16 non-collinear points, hence the maximum number of non-collinear points is 16.

Also solved by Andrea Ligori, Università di Roma "Tor Vergata", Roma, Italy.

U177. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be integers greater than 1. Prove that there are infinitely many primes p such that p divides $b_i^{\frac{p-1}{a_i}} - 1$ for all $i = 1, 2, \ldots, n$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

Solution by the author

This is a very simple (but rather surprising) application of the following beautiful result.

Theorem 1.1. *(Nagell) Let $f_1, f_2, \ldots, f_n$ be nonconstant polynomials with integer coefficients. Then for infinitely many primes p one can find integers $x_1, x_2, \ldots, x_n$ such that p divides $f_i(x_i)$ for all i.*

For the proof, we refer to T. Andreescu, G. Dospinescu, Problems from the Book. Now, consider the polynomials $f_i(X) = X^{a_i} - b_i$ and add to their collection the polynomials φ_{a_i}, where φ_n is the $n-$th cyclotomic polynomial. Pick a prime $p > b_1 b_2 \cdots b_n + a_1 a_2 \cdots a_n$ as in the theorem and x_i, y_i such that p divides $f_i(x_i)$ and $\varphi_{a_i}(y_i)$. As p does not divide any of the a_is, it is a classical property of cyclotomic polynomials that we must have $p \equiv 1 \pmod{a_i}$. Moreover, p clearly does not divide x_i, since it does not divide b_i. Since $x_i^{a_i} \equiv b_i \pmod{p}$ and $a_i \equiv 1 \pmod{p}$, Fermats little theorem implies that p divides $b_i^{\frac{p-1}{a_i}}$ and we are done.

U178. Let k be a fixed positive integer and let $S_n^{(j)} = \binom{n}{j} + \binom{n}{j+k} + \binom{n}{j+2k} + \cdots,\ j = 0, 1, \ldots, k-1$. Prove that

$$\left(S_n^{(0)} + S_n^{(1)} \cos \frac{2\pi}{k} + \cdots + S_n^{(k-1)} \cos \frac{2(k-1)\pi}{k}\right)^2$$
$$+ \left(S_n^{(1)} \sin \frac{2\pi}{k} + S_n^{(2)} \sin \frac{4\pi}{k} + \cdots + S_n^{(k-1)} \sin \frac{2(k-1)\pi}{k}\right)^2 = \left(2 \cos \frac{\pi}{k}\right)^{2n}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Arkady Alt, San Jose, California, USA

Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ and $D_j = \{j + mk \mid m \in \mathbb{Z}_+ \text{ and } j + mk \leq n\}$. Then $S_n^{(j)} = \sum_p \binom{n}{p}$ and $\bigcup_{j=0}^{k-1} D_j = \{0, 1, 2, \ldots, n\}$. Let

$$a = \sum_{j=0}^{k-1} S_n^{(j)} \cos \frac{2j\pi}{k}$$
$$b = \sum_{j=0}^{k-1} S_n^{(j)} \sin \frac{2j\pi}{k}$$
$$\varepsilon = \cos \frac{2\pi}{k} + i \sin \frac{2\pi}{k} \cdot \sum_{j=0}^{k-1} S_n^{(j)} \left(\cos \frac{2\pi}{k} + i \sin \frac{2\pi}{k}\right)^j.$$

Then $\varepsilon^k = 1$ and

$$\begin{aligned}
a + ib &= \sum_{j=0}^{k-1} S_n^{(j)} \cos \frac{2j\pi}{k} + i \sum_{j=0}^{k-1} S_n^{(j)} \cos \frac{2j\pi}{k} \\
&= \sum_{j=0}^{k-1} S_n^{(j)} \left(\cos \frac{2j\pi}{k} + i \sin \frac{2j\pi}{k}\right) \\
&= \sum_{j=0}^{k-1} S_n^{(j)} \varepsilon^j = \sum_{j=0}^{k-1} \sum_{p \in D_j} \binom{n}{p} \varepsilon^p \\
&= \sum_{p \in \cup D_j} \binom{n}{p} \varepsilon^p = \sum_{p=1}^{n} \binom{n}{p} \varepsilon^p = (1 + \varepsilon)^n \\
&= \left(1 + \cos \frac{2\pi}{k} + i \sin \frac{2\pi}{k}\right)^n = \left(2 \cos \frac{\pi}{k} \left(\cos \frac{\pi}{k} + i \sin \frac{\pi}{k}\right)\right)^n \\
&= \left(2 \cos \frac{\pi}{k}\right)^n \left(\cos \frac{\pi}{k} + i \sin \frac{\pi}{k}\right)^n.
\end{aligned}$$

Hence,

$$\begin{aligned}|a+ib| &= \left|\left(2\cos\frac{\pi}{k}\right)^n\left(\cos\frac{\pi}{k}+i\sin\frac{\pi}{k}\right)^n\right| = \left|\left(2\cos\frac{\pi}{k}\right)^n\right|\left|\left(\cos\frac{\pi}{k}+i\sin\frac{\pi}{k}\right)^n\right| \\ &= \left|\left(2\cos\frac{\pi}{k}\right)\right|^n\left|\left(\cos\frac{\pi}{k}+i\sin\frac{\pi}{k}\right)\right|^n = \left|\left(2\cos\frac{\pi}{k}\right)\right|^n.\end{aligned}$$

Therefore, $a^2 + b^2 = \left(2\cos\frac{\pi}{k}\right)^{2n}$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Daniel Campos Salas, Costa Rica.

U179. Let $f : [0, \infty] \to R$ be a continuous function such that $f(0) = 0$ and $f(2x) \le f(x) + x$ for all $x \ge 0$. Prove that $f(x) < x$ for all $x \in [0, \infty]$.

Proposed by Samin Riasat, University of Dhaka, Bangladesh

Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

Note that

$$f(x) \le f\left(\frac{x}{2}\right) + \frac{x}{2} \le f\left(\frac{x}{4}\right) + \frac{x}{4} + \frac{x}{2} \le f\left(\frac{x}{8}\right) + \frac{x}{8} + \frac{x}{4} + \frac{x}{2} \dots$$

and after n steps we have

$$f(x) \le f\left(\frac{x}{2^n}\right) + \sum_{k=1}^{n} \frac{x}{2^n}$$

The limit $n \to \infty$ yields

$$f(x) \le \lim_{n\to\infty} \left(f\left(\frac{x}{2^n}\right) + \sum_{k=1}^{n} \frac{x}{2^n} \right) = \lim_{n\to\infty} f\left(\frac{x}{2^n}\right) + \lim_{n\to\infty} \sum_{k=1}^{n} \frac{x}{2^n} = 0 + x = x$$

where we have used the continuity of $f(x)$ and $f(0) = 0$ for writing

$$\lim_{n\to\infty} f\left(\frac{x}{2^n}\right) = 0$$

and we are done.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; John Mangual, UCSB; Emanuele Natale, Università di Roma "Tor Vergata", Roma, Italy; Tigran Hakobyan, Armenia; Daniel Campos Salas, Costa Rica.

U180. Let $a_1, \ldots, a_k, b_1, \ldots, b_k, n_1, \ldots, n_k$ be positive real numbers and $a = a_1 + \cdots + a_k, b = b_1 + \cdots + b_k, n = n_1 + \cdots + n_k, k \geq 2$. Prove that

$$\int_0^1 (a_1 + b_1 x)^{n_1} \cdots (a_k + b_k x)^{n_k} dx \leq \frac{(a+b)^{n+1} - a^{n+1}}{(n+1)b}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

By concavity of the logarithm we have

$$\ln \prod_{j=1}^{k} (a_j + b_k x)^{n_j} = n \sum_{j=1}^{k} \frac{n_j}{n} \ln(a_j + b_j x) \leq n \ln\left(\sum_{j=1}^{k} \frac{n_j}{n}(a_j + b_j x)\right)$$

Moreover the monotonicity (increasing) of the logarithm yields

$$n \ln\left(\sum_{j=1}^{k} \frac{n_j}{n}(a_j + b_j x)\right) \leq n \ln\left(\sum_{j=1}^{k} (a_j + b_j x)\right) = \ln(a + bx)^n$$

Now exponentiating and integrating we have

$$\int_0^1 (a_1 + b_1 x)^{n_1} \cdots (a_k + b_k x)^{n_k} dx \leq \int_0^1 (a + bx)^n dx = \frac{(a+b)^{n+1} - a^{n+1}}{(n+1)b}$$

concluding the proof.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Daniel Campos Salas, Costa Rica.

Olympiad problems

O175. Find all pairs (x, y) of positive integers such that $x^3 - y^3 = 2010(x^2 + y^2)$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

Write $x = du, y = dv$ with $d \geq 1$ and u, v relatively prime positive integers. The equation becomes $d(u^3 - v^3) = 2010(u^2 + v^2)$. Thus $u^2 + uv + v^2$ divides $2010(u^2 + v^2)$ and since it is relatively prime to $u^2 + v^2$, we deduce that $u^2 + uv + v^2$ divides $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. We claim that $A = u^2 + uv + v^2$ actually divides 67. It is immediate that A is odd (if not, u, v must be both even). Next, it is easy to see that if 5 divides A, then 5 divides both u, v, a contradiction. Finally, if 3 divides A, we must have $u \equiv v \bmod 3$ and so $u^3 - v^3 = (u - v)A$ is a multiple of 9. Thus $2010(u^2 + v^2)$ is a multiple of 9, which is not the case. Thus A is a divisor of 67 and since $u, v \geq 1$, we deduce that $u^2 + uv + v^2 = 67$. Clearly $u \geq v$, then $67 \geq 3v^2$, thus $v = 4$. Considering each case, we deduce that $v = 2, u = 7$ and so $d = 318$. Hence there is one solution, $x = 7d, y = 2d$ with $d = 318$.

Also solved by Tigran Hakobyan, Armenia.

O176. Let $P(n)$ be the following statement: for all positive real numbers $x_1, x_2, \ldots, x_n$ such that $x_1 + x_2 + \cdots + x_n = n$,

$$\frac{x_2}{\sqrt{x_1+2x_3}} + \frac{x_3}{\sqrt{x_2+2x_4}} + \cdots + \frac{x_1}{\sqrt{x_n+2x_2}} \geq \frac{n}{\sqrt{3}}.$$

Prove that $P(n)$ is true for $n \leq 4$ and false for $n \geq 9$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

First solution by the author

Let $S(x_1, x_2, \ldots, x_n)$ be the left hand side of the inequality. Using Holder's inequality, we obtain

$$S^2(x_2(x_1+2x_3) + \cdots + x_1(x_n+2x_2)) \geq (x_1 + x_2 + \cdots + x_n)^3 = n^3.$$

On the other hand, we have

$$x_2(x_1+2x_3) + \cdots + x_1(x_n+2x_2) = 3(x_1x_2 + x_2x_3 + \cdots + x_nx_1).$$

Using the fact that

$$x_1x_2 + x_2x_3 + \cdots + x_nx_1 \leq n$$

whenever $x_1 + x_2 + \cdots + x_n = n$ and $n \leq 4$. The last fact follows from the fact that

$$ab + bc + ca \leq \frac{(a+b+c)^2}{3}$$

and

$$ab + bc + cd + da = (a+c)(b+d) \leq \frac{(a+b+c+d)^2}{4}.$$

The conclusion follows easily for $n \leq 4$. Chosing x_1, x_2, x_3, x_4 close to $\frac{n}{4}$ and the other variables equal and close to 0, one easily obtains that the expression is smaller than $\frac{n}{\sqrt{3}}$ for $n \geq 9$. The conclusion follows.

Second solution by Arkady Alt, San Jose, California, USA

Let $n \leq 4$.Then, applying consequentially AM-GM and Cauchy inequalities, we obtain

$$\sum_{cyc}^{n} \frac{x_2}{\sqrt{3\,(x_1+2x_3)}} \geq \sum_{cyc}^{n} \frac{2x_2}{3+(x_1+2x_3)} = 2\sum_{cyc}^{n} \frac{x_2^2}{3x_2+(x_1x_2+2x_2x_3)} \geq$$

$$\frac{2\left(\sum_{k=1}^{n} x_k\right)^2}{3\sum_{k=1}^{n} x_k + \sum_{cyc}^{n}(x_1x_2+2x_2x_3)} = \frac{2n^2}{3n+3\sum_{cyc}^{n} x_1x_2}.$$

Thus, $\sum_{cyc}^{n} \frac{x_2}{\sqrt{x_1+2x_3}} \geq \frac{2}{\sqrt{3}} \cdot \frac{n^2}{n+\sum_{cyc}^{n} x_1x_2}$.

For $n = 3,$ since $x_1 + x_2 + x_3 = 3$ we have $\sum_{cyc}^{n} x_1x_2 \leq \frac{(x_1+x_2+x_3)^2}{3} = 3.$

Then $\dfrac{2}{\sqrt{3}} \cdot \dfrac{n^2}{n+\sum_{cyc}^n x_1x_2} = \dfrac{2}{\sqrt{3}} \cdot \dfrac{9}{3+\sum_{cyc}^n x_1x_2} \geq \dfrac{2}{\sqrt{3}} \cdot \dfrac{9}{6} = \sqrt{3} = \dfrac{3}{\sqrt{3}}$.

If $n=4$ then $x_1+x_2+x_3+x_4=4$ and $\sum_{cyc}^n x_1x_2 = (x_1+x_3)(x_2+x_4) \leq \left(\dfrac{(x_1+x_3)+(x_2+x_4)}{2}\right)^2 = 4$.Therefore, $\dfrac{2}{\sqrt{3}} \cdot \dfrac{n^2}{n+\sum_{cyc}^n x_1x_2} =$
$\dfrac{2}{\sqrt{3}} \cdot \dfrac{16}{4+\sum_{cyc}^n x_1x_2} \geq \dfrac{2}{\sqrt{3}} \cdot \dfrac{16}{4+4} = \dfrac{4}{\sqrt{3}}$.

Let $n \geq 9$ and let $x_k = \dfrac{n}{2^k}, k=1,2,...,n$.Then

$$L.H.S. = \sum_{k=1}^{n-2} \frac{x_{k+1}}{\sqrt{x_k+2x_{k+2}}} + \frac{x_n}{\sqrt{x_{n-1}+2x_1}} + \frac{x_1}{\sqrt{x_n+2x_2}} = \sum_{k=1}^{n-2} \frac{\frac{n}{2^{k+1}}}{\sqrt{\frac{n}{2^k}+2\cdot\frac{n}{2^{k+2}}}} +$$

$$\frac{\frac{n}{2^n}}{\sqrt{\frac{n}{2^{n-1}}+2\cdot\frac{n}{2}}} + \frac{\frac{n}{2}}{\sqrt{\frac{n}{2^n}+2\cdot\frac{n}{4}}} = \sum_{k=1}^{n-2} \frac{\sqrt{n}}{\sqrt{2^{k+2}+2^{k+1}}} + \frac{\sqrt{n}}{\sqrt{2^{n+1}+2^{2n}}} + \frac{\sqrt{n}}{\sqrt{\frac{1}{2^{n-2}}+2}}.$$

Since $\sum_{k=1}^{n-2} \dfrac{\sqrt{n}}{\sqrt{2^{k+2}+2^{k+1}}} = \sqrt{n}\sum_{k=1}^{n-2} \dfrac{1}{\sqrt{3\cdot 2^{k+1}}} = \sqrt{\dfrac{n}{3}}\sum_{k=1}^{n-2} \dfrac{1}{2\sqrt{2^{k-1}}} <$
$\dfrac{1}{2}\sqrt{\dfrac{n}{3}} \cdot \dfrac{1}{1-\dfrac{1}{\sqrt{2}}} = \dfrac{1}{2}\sqrt{\dfrac{n}{3}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}-1} = \sqrt{\dfrac{n}{6}}\left(\sqrt{2}+1\right)$, $\dfrac{\sqrt{n}}{\sqrt{2^{n+1}+2^{2n}}} < \dfrac{\sqrt{n}}{2\sqrt{2^n}}$

and $\dfrac{\sqrt{n}}{\sqrt{\dfrac{1}{2^{n-2}}+2}} < \sqrt{\dfrac{n}{2}}$ then $L.H.S. < \sqrt{\dfrac{n}{3}}\left(\dfrac{\sqrt{2}+1}{\sqrt{2}} + \dfrac{\sqrt{3}}{2\sqrt{2^n}} + \dfrac{\sqrt{3}}{\sqrt{2}}\right)$.

Moreover, since $n \geq 9$ we obtain

$1+\dfrac{1}{\sqrt{2}}+\dfrac{\sqrt{3}}{2\sqrt{2^n}}+\dfrac{\sqrt{3}}{\sqrt{2}} < \dfrac{\sqrt{2}+1}{\sqrt{2}}+\dfrac{\sqrt{3}}{2\sqrt{2^9}}+\dfrac{\sqrt{3}}{\sqrt{2}} < 1+\dfrac{1+\sqrt{3}}{\sqrt{2}}+\dfrac{1}{\sqrt{2^9}} = 2.976 < 3$,

and , therefore, $L.H.S. < \sqrt{3n} < \dfrac{n}{\sqrt{3}}$.

So, $P(n)$ is false for $n \geq 9$.

O177. Let P be point situated in the interior of a circle. Two variable perpendicular lines through P intersect the circle at A and B. Find the locus of the midpoint of the segment AB.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

First solution by G.R.A.20 Problem Solving Group, Roma, Italy

We can assume, without loss of generality, that $P = t \in [0,1]$ and the circle $C = \{|z| = 1\}$. Let $A = z = x + iy \in C$ then $B = w = si(z - P) + P \in C$ with some $s > 0$. Hence

$$1 = |w|^2 = (t - sy)^2 + s^2(x - t)^2 \qquad (1)$$

The midpoint of the segment AB is given by $M = (A + B)/2$.

Now we verify that

$$|M - P/2| = \sqrt{2 - |P|^2}/2.$$

In fact, by (1),

$$(2|M - P/2|)^2 = (x - sy)^2 + (s(x - t) + y)^2 = x^2 + y^2 + 1 - t^2 = 2 - t^2.$$

Hence the required locus is a circle with center $P/2$ and radius $\sqrt{2 - |P|^2}/2$.

In the general setting, if the circle C has center at P_0 and radius R then the locus is a circle with center $(P_0 + P)/2$ and radius $\sqrt{2R^2 - |P - P_0|^2}/2$.

Second solution by the author

Let $ABCD$ be a quadrilateral and let M and N be the midpoints of sides AB and CD, respectively. Using the Median Theorem it is easy to prove that the following relation holds :

$$AC^2 + BD^2 + BC^2 + DA^2 = AB^2 + CD^2 + 4MN^2.$$

Let M be the midpoint of the segment AB and let N be the midpoint of the segment OP, where O is the center of the given circle. Applying the relation above in the quadrilateral $ABPO$ we obtain

$$AP^2 + R^2 + BP^2 + R^2 = AB^2 + OP^2 + 4MN^2.$$

It is clear that $AP^2 + BP^2 = AB^2$, hence we get

$$4MN^2 = 2R^2 - OP^2,$$

that is

$$NM = \frac{1}{2}\sqrt{2R^2 - OP^2}.$$

Since the point N is fixed, it follows that the desired locus is the circle of center N and radius $\frac{1}{2}\sqrt{2R^2 - OP^2}$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Daniel Campos Salas, Costa Rica; Tomas Calderon Gomez, Costa Rica.

O178. Let m and n be positive integers. Prove that for each odd positive integer b there are infinitely many primes p such that $p^n \equiv 1 \pmod{b}^m$ implies $b^{m-1} \mid n$.

Proposed by Vahagn Aslanyan, Yerevan, Armenia

Solution by the author

Let $b = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the canonical factorization of b. Because b is odd $p_i > 2$ for $i = 1, 2, \ldots, k$. Let $P = p_1 p_2 \cdots p_k$. Consider the sysmte of conguences $(*) x \equiv p_i + 1 \pmod{p_i^2}, i = 1, 2, \ldots, k$. By the Chinese Remainder Theorem the system $(*)$ has solution. Let that solution be x_0. We have $x_0 \equiv p_i + 1 \pmod{p_i^2}$ for all i. If $x \equiv x_0 \pmod{P^2}$, then x is solution of system $(*)$.

Claim. If $x \equiv x_0 \pmod{P^2}$, then from condition $x^n \equiv 1 \pmod{b^m}$ it follows that $b^{m-1} \mid n$.

Proof. Suppose $p \in \mathcal{P}$ is a prime and a is a positive integer. Let $v_p(a)$ be the degree of p in the canonical factorization of a. For each i, $1 \leq i \leq k, p_i \mid x_0 - 1$. We know that p_i is odd, therefore by a well knwon lemma $v_{p_i}(x^n - 1) = v_{p_i}(x - 1) + v_{p_i}(n)$. But $x \equiv x_0 \equiv p_i + 1 \pmod{p_i^2}$, so $v_{p_i}(x-1) = 1$. Hence $v_{p_i}(n) = v_{p_i}(x^n - 1) - 1 \geq m\alpha_i - 1 \geq (m-1)\alpha_i$ (because $x^n \equiv 1 \pmod{b^m}$). So $p_i^{(m-1)\alpha_i} \mid n$. It is true for all i, and $\gcd(p_i, p_j) = 1$ whenever $i \neq j$, therefore $b^{m-1} = \prod_{i=1}^{k} p_i^{(m-1)\alpha_i} \mid n$. Now from Dirichlet's theorem there are infinitely many primes p such that $p \equiv x_0 \pmod{P^2}$.

Also solved by Tigran Hakobyan, Armenia.

O179. Prove that any convex quadrilateral can be dissected into $n \geq 6$ cyclic quadrilaterals.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Any convex quadrilateral is dissected into two triangles by either of its diagonals; any concave quadrilateral is dissected into two triangles by exactly one of its diagonals; any crossed quadrilateral is already formed by two triangles joined at one vertex, and where two of the sides of each triangle are on the straight line containing two of the sides of the other.

In triangle ABC, let I be the incenter and D, E, F the points where the incircle touches respectively sides BC, CA, AB. Clearly, ABC my be dissected into three cyclic quadrilaterals $AEIF$, $BFID$, $CDIE$.

In triangle ABC, wlog acute at C, consider the circumcenter O, and take a point O' on the perpendicular bisector of AB that is closer to AB than O. The circle with center O' through A, B leaves C outside, hence it must intersect the interior of segments AC, BC at E, D, or $ABDE$ is cyclic.

We may then proceed as follows: write $n = 3 + 3u + v$, where $u \geq 1$ is an integer and $v \in \{0, 1, 2\}$. Dissect (any) quadrilateral $ABCD$ in two triangles, then dissect one of them into three cyclic quadrilaterals. If $v \neq 0$, dissect the other triangle into one cyclic quadrilateral and one triangle, and if $v = 2$, dissect again this latter triangle into one cyclic quadrilateral and one triangle. After this procedure, we have dissected the original quadrilateral into $3+v$ cyclic quadrilaterals ($3, 4, 5$ respectively for $v = 0, 1, 2$) and one triangle. Dissect now this triangle into u triangles (for example dividing one of its sides in u equal parts and joining each point of division with the opposite vertex), and dissect now each one of these u triangles into three cyclic quadrilaterals. We have thus dissected the original quadrilateral into $3 + v + 3u = n$ cyclic quadrilaterals.

Also solved by Daniel Campos Salas, Costa Rica.

O180. Let p be a prime. Prove that each positive integer $n \geq p$, p^2 divides $\binom{n+p}{p}^2 - \binom{n+2p}{2p} - \binom{n+p}{2p}$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

First solution by G.R.A.20 Problem Solving Group, Roma, Italy

Since by Wolstenholme Theorem, for any prime $p > 2$,

$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^2}, \qquad \text{and} \qquad \sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p},$$

it is easy to verify by induction with respect to $n = \lfloor n/p \rfloor \cdot p + [n]_p \geq 0$ that for any prime p

$$\binom{n+ap}{bp} \equiv \binom{\lfloor n/p \rfloor + a}{b} \left(1 + bp \sum_{k=1}^{[n]_p} \frac{1}{k}\right) \pmod{p^2}.$$

Hence by letting $x = \lfloor n/p \rfloor$ and $y = \sum_{k=1}^{[n]_p} \frac{1}{k}$ we obtain

$$\begin{aligned}\binom{n+p}{p}^2 - \binom{n+2p}{2p} - \binom{n+p}{2p} &\equiv (x+1)^2(1+py)^2 - \left(\binom{x+2}{2} + \binom{x+1}{2}\right)(1+2py) \\ &\equiv p^2 y^2 (x+1)^2 \equiv 0 \pmod{p^2}.\end{aligned}$$

Second solution by Campos Salas, Costa Rica

We'll prove it first for $p = 2$. It's easy to verify that the expression equals $4\binom{n+3}{4}$, and it's divisible by 4.

Suppose $p \geq 3$. Note that the greatest power of p that divides $(p!)^2$ and $(2p)!$ is p^2. The following congruences will be taken modulo p^2. It's well-known that $\binom{2p}{p} \equiv 2$. This implies that $\frac{(2p)!}{p^2} \equiv \frac{2(p!)^2}{p^2}$, or equivalently $\frac{p^2}{(p!)^2} \equiv \frac{2p^2}{(2p)!}$. Therefore, the problem is equivalent to prove that p^4 divides

$$\prod_{i=1}^{p}(n+i)\left(\prod_{i=1}^{p}(n+p+i) + \prod_{i=1}^{p}(n-p+i) - 2\prod_{i=1}^{p}(n+i)\right).$$

It would suffice to prove that p^3 divides

$$\prod_{i=1}^{p}(n+p+i) + \prod_{i=1}^{p}(n-p+i) - 2\prod_{i=1}^{p}(n+i).$$

Consider the expression $\left(\prod_{i=1}^{p}(n+x+i) - \prod_{i=1}^{p}(n+i)\right)$ modulo x^3 (this can be done since $p \geq 3$). Note that

$$\prod_{i=1}^{p}(n+x+i)-\prod_{i=1}^{p}(n+i)\equiv\prod_{i=1}^{p}(n+i)\left(x^2s_2-xs_1\right),$$

where s_k represents the sum of all the possible products of k different terms of the set $\{\frac{1}{n+1},...,\frac{1}{n+p}\}$.

Therefore, modulo p^3 we have that

$$\prod_{i=1}^{p}(n+p+i)+\prod_{i=1}^{p}(n-p+i)-2\prod_{i=1}^{p}(n+i)\equiv 2p^2s_2\prod_{i=1}^{p}(n+i).$$

We're left to prove that p divides $s_2\prod_{i=1}^{p}(n+i)$. It's clear that this expression, modulo p, is congruent to $s_2'p!$, where s_2' represents the sum of all the possible products of 2 different terms of the set $\{\frac{1}{1},...,\frac{1}{p}\}$. If a product doesn't have $\frac{1}{p}$ as one of its factors, then is divisible by p when multiplied by $p!$. Then, it is enough to prove that p divides $\frac{1}{1}+\frac{1}{2}+...+\frac{1}{p-1}$ which is true by grouping the k-th term and the $(p-k)$-th term, and we're done.

Third solution by the author

According to the Li-Jen-Shu fromula, for any positive integers n and p we have

$$\sum_{k=0}^{p}\binom{p}{k}^2\binom{n+2p-k}{2p}=\binom{n+p}{p}^2.$$

It follows

$$\sum_{k=1}^{p-1}\binom{p}{k}^2\binom{n+2p-k}{2p}=\binom{n+p}{p}^2-\binom{n+2p}{2p}-\binom{n+p}{2p}.$$

Considering p a prime, we have $\binom{p}{k}\equiv 0 \pmod p$, hence $\binom{p}{k}^2\equiv 0 \pmod{p^2}$. From the above relation, we obtain

$$\sum_{k=1}^{p-1}\binom{p}{k}^2\binom{n+2p-k}{2p}\equiv 0(mod p^2),$$

hence p^2 divides

$$\binom{n+p}{p}^2-\binom{n+2p}{2p}-\binom{n+p}{2p},$$

and we are done.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.