Junior problems

J133. A sequence $(a_n)_{n\geq 2}$ of real numbers greater than 1 satisfies the relation

$$a_n = \sqrt{1 + \frac{(n+1)!}{2\left(a_2 - \frac{1}{a_2}\right)\cdots\left(a_{n-1} - \frac{1}{a_{n-1}}\right)}}$$

for all $n > 2$. Prove that if $a_k = k$ for some $k \geq 2$, then $a_n = n$ for all $n \geq 2$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Aravind Srinivas L, Chennai, India

Consider for some $n \in \mathbb{N}$ such that $n > 2$, we have by the definition of the sequence that : $a_n = \sqrt{1 + \frac{(n+1)!}{2\left(a_2 - \frac{1}{a_2}\right)\cdots\left(a_{n-1} - \frac{1}{a_{n-1}}\right)}}$. Squaring both sides and subtracting by 1 on both sides, we get:

$$a_n^2 - 1 = \frac{(n+1)!}{2(a_2 - \frac{1}{a_2})\cdots(a_{n-1} - \frac{1}{a_{n-1}})} \tag{1}$$

Now, consider a_{n+1}. Writing it according to the definition of the sequence, we have :

$$a_{n+1} = \sqrt{1 + \frac{(n+1)!(n+2)}{2\left(a_2 - \frac{1}{a_2}\right)\cdots\left(a_{n-1} - \frac{1}{a_{n-1}}\right)\left(a_n - \frac{1}{a_n}\right)}}.$$

Using (1) here, we get

$$a_{n+1}^2 = 1 + a_n(n+2) \qquad (2).$$

Now if $a_n = n$ for some $n \geq 2$, we can say by (2) that $a_{n+t} = n + t$ for $t \in \mathbb{N}$ Here, we take the sign of the sequence term (square root) depending upon the sign of the square root of the previous terms of the sequence as from (1) we know that $a_{n+t} > a_n$. This is because it is given that each of $a_n > 1$ in the first line of the problem statement and we use it here in this part, to say that sice it is given so, we can take only the positive sign of the value of the sequence for each term. In (2), if we replace n by $n-1$, we get $a_n = 1 + a_{n-1}(a_n + 1)$ which gives us easily that $a_{n-1} = \frac{n^2-1}{n+1} = n - 1$. And in this way, we can argue for all $n \geq 3$ and also thus get from $a_3 = 3$ that $a_2 = 2$. Thus, we have that $a_n = n$ for all $n \geq 2$, $n \in \mathbb{N}$. We have thus exhausted all the given conditions in the problem statement.

Second solution by Arkady Alt , San Jose ,California, USA

Let $b_n := \left(a_2 - \frac{1}{a_2}\right) \cdots \left(a_n - \frac{1}{a_n}\right), n \geq 2$, then for $n > 2$

$$a_n = \sqrt{1 + \frac{(n+1)!}{2b_{n-1}}} \iff a_n^2 - 1 = \frac{(n+1)!}{2b_{n-1}}$$

and, since

$$\frac{b_n}{b_{n-1}} = a_n - \frac{1}{a_n} = \frac{a_n^2 - 1}{a_n},$$

we have $\frac{b_n}{b_{n-1}} = \frac{(n+1)!}{2b_{n-1}a_n} \iff b_n = \frac{(n+1)!}{2a_n}$. Letting $b_{n-1} = \frac{n!}{2a_{n-1}}$ in $a_n^2 - 1 = \frac{(n+1)!}{2b_{n-1}}$ for $n > 2$ give us $a_n^2 - 1 = (n+1)\,a_{n-1}$. Thus, $a_{n+1}^2 = 1 + (n+2)\,a_n, n \geq 2$. Let $a_k = k$ for some $k \geq 2$. Then $a_n = n$ for any $n \geq k$. Indeed, since $a_k = k$ and in supposition $a_n = n, n \geq k$ we obtain $a_{n+1}^2 = 1 + (n+2)\,a_k = 1 + (n+2)\,n = (n+1)^2$ then by induction $a_n = n$ for any $n \geq k$ If $k > 2$ then for any $2 < n \leq k$ from supposition $a_n = n$ follows $a_{n-1} = \frac{a_n^2 - 1}{n+1} = \frac{n^2 - 1}{n+1} = n - 1$. Thus, by induction $a_n = n$ for any $2 \leq n \leq k$.

Also solved by G. C. Greubel, Newport News, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain

J134. How many positive integers n less than 2009 are divisible by $\lfloor \sqrt[3]{n} \rfloor$.

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania

First solution by the author

Let $a = \lfloor \sqrt[3]{n} \rfloor$. Then $a \leq \sqrt[3]{n} < a+1$, i.e. $a^3 \leq n < (a+1)^3$. Because $\lfloor \sqrt[3]{n} \rfloor$ divides n, it follows that $n = ab$ for some positive integer b. We get $a^3 \leq ab < (a+1)^3$, hence

$$a^2 \leq b < \frac{(a+1)^3}{a} = a^2 + 3a + 3 + \frac{1}{a} \leq a^2 + 3a + 4.$$

All positive integers n with the property that $\lfloor \sqrt[3]{n} \rfloor$ divides n are given by

$$a^3, a^3 + a, a^3 + 2a, \ldots, a^3 + (3a+3)a, \ a \in \mathbb{Z}_+^*.$$

Note that the largest a with $a^3 < 2009$ is $a = 12$ and we have $12^3 + 12k \leq 2009$ is equivalent to $12k \leq 281$, i.e. $k \leq 23 + \frac{5}{12}$, hence $k = 23$. The desired number is given by

$$\sum_{a=1}^{11} (3a+4) + 24 = 3 \cdot \frac{11 \cdot 12}{2} + 44 + 24 = 198 + 44 + 24 = 266.$$

Second solution by G. C. Greubel, Newport News, VA

The solution here first seeks the integer values of n such that $\sqrt[3]{n}$ is an integer. These values of n are given by, where $1 \leq n \leq 2008$,

$$\{1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728\}.$$

It is seen that these values are cubes. A set of values of n can be created such that the floor function is a constant. This is seen here with the values of $\lfloor \sqrt[3]{n} \rfloor$:

n	$\lfloor \sqrt[3]{n} \rfloor$
$1-7$	1
$8-26$	2
$27-63$	3
$64-124$	4
$125-215$	5
$216-342$	6
$343-511$	7
$512-728$	8
$729-999$	9
$1000-1330$	10
$1331-1727$	11
$1728-2008$	12

. (1)

Writing out the number of terms in each group such that the value of

$$\phi_n = \frac{n}{\lfloor \sqrt[3]{n} \rfloor}$$

are integers provide the desired result sought. When this is done it is seen that for $1 \leq n \leq 7$ provides 7 integer values. When $8 \leq n \leq 26$ there are 10 integer values. Each of the above grouped values can be determined in the same way. It becomes evident that the number of terms in each group that are integers is equal to the number of terms that are multiples of the respective $\lfloor \sqrt[3]{n} \rfloor$ values. To show this consider the range $8 \leq n \leq 26$. The values of ϕ_n that yield integers are ϕ_8, ϕ_{10}, ϕ_{12}, ϕ_{14}, ϕ_{16}, ϕ_{18}, ϕ_{20}, ϕ_{22}, ϕ_{24}, ϕ_{26}. These are the ten values that yield integer values.

A similar procedure is used for each of the other cases. The last set of values for n in the above chart is $1728 \leq n \leq 2008$. It is not a complete range as compared to the other ranges of n in the table. The number of values in this set that produce integer values is 24. Adding a column in the above table of integer values of ϕ_n is given by

n	$\lfloor \sqrt[3]{n} \rfloor$	$\phi_n = int.$
$1-7$	1	7
$8-26$	2	10
$27-63$	3	13
$64-124$	4	16
$125-215$	5	19
$216-342$	6	22
$343-511$	7	25
$512-728$	8	28
$729-999$	9	31
$1000-1330$	10	34
$1331-1727$	11	37
$1728-2008$	12	24

. (2)

Summing the last column in this table yields the total number of integers divisible by $\lfloor \sqrt[3]{n} \rfloor$. The number is 266.

Third solution by Carlo Pagano, Universita di Roma "Tor Vergata", Roma, Italy

Let $a = \lfloor \sqrt[3]{N} \rfloor$ where N is a positive integer. The number of positive integers

n less or equal than N which are divisible by $\lfloor \sqrt[3]{n} \rfloor$ is given by

$$\begin{aligned}
\sum_{i=1}^{a-1} \left\lceil \frac{(i+1)^3 - i^3}{i} \right\rceil + \left\lceil \frac{N+1-a^3}{a} \right\rceil &= \sum_{i=1}^{a-1} \left\lceil 3i + 3 + \frac{1}{i} \right\rceil + \left\lceil \frac{N+1-a^3}{a} \right\rceil \\
&= \sum_{i=1}^{a-1} \left(3i + 3 + \left\lceil \frac{1}{i} \right\rceil \right) + \left\lceil \frac{N+1}{a} \right\rceil - a^2 \\
&= \frac{3a(a-1)}{2} + 4(a-1) - a^2 + \left\lceil \frac{N+1}{a} \right\rceil \\
&= \frac{a^2 + 5a - 8}{2} + \left\lceil \frac{N+1}{a} \right\rceil .
\end{aligned}$$

By letting $N = 2008$, we have that $a = 12$ and the answer to the above question is

$$\frac{144 + 5 \cdot 12 - 8}{2} + \left\lceil \frac{2009}{12} \right\rceil = 266.$$

Also solved by John T. Robinson, Yorktown Heights, NY, USA; Ercole Suppa, Teramo, Italy; Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain.

J135. Find all n for which the number of diagonals of a convex $n-$gon is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by John T. Robinson, Yorktown Heights, NY, USA

Since the number of diagonals of a convex n-gon is $n(n-3)/2$, we seek to find positive integer solutions to the Diophantine equation

$$\frac{n(n-3)}{2} = k^2$$

$$n^2 - 3n = 2k^2$$

$$4n^2 - 12n = 8k^2$$

$$4n^2 - 12n + 9 = 8k^2 + 9$$

$$(2n-3)^2 = 2(4k^2) + 9$$

$$x^2 - 2y^2 = 9\ ,$$

where $x = 2n - 3$ and $y = 2k$. We see that

$$x^2 = 2y^2 \mod 9\ .$$

However since a square is congruent to 0, 1, 4, or 7 modulo 9, and twice a square is therefore congruent to 0, 2, 8, or 5 modulo 9, it follows that we must have both $x^2 = 0 \mod 9$ and $2y^2 = 0 \mod 9$. Furthermore if $2y^2 = 0 \mod 9$, multiplying by the inverse of 2 modulo 9 (that is by 5) we have $y^2 = 0 \mod 9$. Therefore $3|x$ and $3|y$; letting $x = 3X$ and $y = 3Y$ we now have

$$X^2 - 2Y^2 = 1\ .$$

This is just an instance of Pell's equation, and the positive integer solutions for X are known to be p_{2k}, where p_k/q_k is the kth convergent in the continued fraction expansion of $\sqrt{2}$ (recurrence relations and closed form expressions for p_k, q_k, p_{2k}, etc., can be found in the literature). It follows that for all n for which the number of diagonals of a convex n-gon is a perfect square, n is of the form

$$n = \frac{3p_{2k} + 3}{2} = \frac{3(p_{2k} + 1)}{2}\ .$$

For example, the first five such n are as follows (using $p_2 = 3$, $p_4 = 17$, $p_6 = 99$, $p_8 = 577$, $p_{10} = 3363$): 6, 27, 150, 867, 5046.

Second solution by Ercole Suppa, Teramo, Italy

The number of diagonals of a convex n-gon is given by

$$\binom{n}{2} - n = \frac{n(n-1)}{2} - n = \frac{n^3 - n}{2}$$

so it is enough to solve the following diophantine equation

$$n^2 - 3n = 2m^2, \qquad n, m \in \mathbb{N} \tag{1}$$

From (1) follows that

$$n^2 - 3n \equiv 2m^2 \pmod 3 \qquad \Rightarrow \qquad n \equiv 0, m \equiv 0 \pmod 3$$

Thus, setting $n = 3a$ and $m = 3y$ ($a, y \in \mathbb{N}$), the equation (1) is equivalent to

$$\begin{aligned} 9a^2 - 9a &= 18y^2 && \Leftrightarrow \\ a^2 - a &= 2y^2 && \Leftrightarrow \\ 4a^2 - 4a + 1 &= 8y^2 + 1 && \Leftrightarrow \\ (2a-1)^2 - 8y^2 &= 1 && \Leftrightarrow \\ x^2 - 8y^2 &= 1 && \end{aligned} \tag{2}$$

where we have put $x = 2a - 1$.

The equation (2) is a Pell's equation $x^2 - Dy^2 = 1$ with fundamental solution $x_1 = 3$, $y_1 = 1$, so all positive solutions are of the form x_k, y_k, where

$$x_k + y_k\sqrt{8} = \left(x_1 + y_1\sqrt{D}\right)^k$$

The solutions x_k, y_k can be computed from the formulas

$$\begin{cases} x_k = \frac{1}{2}\left[\left(x_1 + y_1\sqrt{D}\right)^k + \left(x_1 - y_1\sqrt{D}\right)^k\right] \\ y_k = \frac{1}{2\sqrt{D}}\left[\left(x_1 + y_1\sqrt{D}\right)^k + \left(x_1 - y_1\sqrt{D}\right)^k\right] \end{cases}$$

or from the recursive formulas

$$\begin{cases} x_{k+1} = x_1 x_k + D y_1 y_k \\ y_{k+1} = x_1 y_k + x_k y_1 \end{cases} \iff \begin{cases} x_{k+1} = 3x_k + 8y_k \\ y_{k+1} = 3y_k + x_k \end{cases} \tag{3}$$

From (3), by a simple calculation, we get

$$x_{k+2} = 6 \cdot x_{k+1} - x_k \qquad , \qquad x_1 = 3, x_2 = 17 \tag{4}$$

Since $n = 3a = \frac{3}{2}(x+1)$ the recurrence (4) yields

$$\frac{3}{2}(x_{k+2}+1) = 6 \cdot \frac{3}{2}(x_{k+1}+1) - \frac{3}{2}(x_k+1) + 6 \qquad \Longleftrightarrow$$

$$\boxed{n_{k+2} = 6n_{k+1} - n_k + 6 \qquad , \qquad n_1 = 6, n_2 = 27} \tag{5}$$

By means of Mathematica we have listed the first 10 solutions given by (5):

$$6, 27, 150, 867, 5046, 29403, 171366, 998787, 5821350, 33929307$$

The proof is finished.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

If the sides of the n-gon are not counted as diagonals, then there are $\binom{n}{2} - n = \frac{n(n-3)}{2}$ diagonals. Note that, if n is not a multiple of 3, n and $n-3$ are relatively prime since their difference 3 is divisible by their greatest common divider. In this case, and since $n(n-3) = 2t^2$ for some positive integer t when the number of diagonals is a perfect square t^2, either $n = 2u^2$ and $n - 3 = v^2$, or $n = u^2$ and $n - 3 = 2v^2$ for relatively prime integers u, v. Clearly $v^2 - 2u^2$ and $u^2 - 2v^2$ must respectively be multiples of 3 and since all squares are congruent 0 or 1 modulus 3, this means that u, v are both multiples of 3, and n and $n - 3$ are both multiples of 9, contradiction. Hence $n = 3m$ is a multiple of 3, and $9m(m-1) = 2t^2$. Now $m, m-1$ are always relatively prime, hence $t = 3uv$ with u, v relatively prime, and either $m = 2u^2$ and $m - 1 = v^2$, or $m = u^2$ and $m-1 = 2v^2$. In the first case, $v^2-2u^2 = -1$, and in the second case $u^2-2v^2 = 1$. These are Pell equations, which solved by standard means yield respective non-negative solutions $v_k = \frac{(\sqrt{2}+1)^{2k+1}-(\sqrt{2}-1)^{2k+1}}{2}$, $u_k = \frac{(\sqrt{2}+1)^{2k+1}+(\sqrt{2}-1)^{2k+1}}{2\sqrt{2}}$, and $u_k = \frac{(\sqrt{2}+1)^{2k}+(\sqrt{2}-1)^{2k}}{2}$, $v_k = \frac{(\sqrt{2}+1)^{2k}-(\sqrt{2}-1)^{2k}}{2\sqrt{2}}$, for all non-negative integers k. Note then that $n_k = 6u_k^2$ for the first solution, and $n_k = 3u_k^2$ for the second solution. Both cases may be written as one:

$$n_k = 3\left(\frac{(\sqrt{2}+1)^k + (\sqrt{2}-1)^k}{2}\right)^2,$$

for all positive integer k. Note that $k = 0$ yields $n = 3$, and a triangle has 0^2 diagonals. The first few values, other than 3, are $n = 6, 27, 150, 867, 5046, \ldots$.

If the sides of the n-gon are counted as diagonals, then there are $\binom{n}{2} = \frac{n(n-1)}{2}$ diagonals. Note that this is the same equation that m satisfies in the case where $n = 3m$ is a solution when the sides of the n-gon are not counted as diagonals, ie, if sides are counted as diagonals, the number of diagonals is a perfect square for

$$n_k = \left(\frac{(\sqrt{2}+1)^k + (\sqrt{2}-1)^k}{2} \right)^2,$$

where k is any positive integer, yielding values $n = 9, 50, 289, 1682, 9801, \ldots$.

Fourth solution by Carlo Pagano, Universita di Roma "Tor Vergata", Roma, Italy

We have to solve the diophantine equation

$$\frac{n(n-3)}{2} = m^2$$

that is

$$n^2 - 3n - 2m^2 = 0$$

for $n \geq 3$ and $m \geq 0$. Note that $n^2 + m^2 \equiv 0 \pmod 3$ which implies that both n and m are divisible by 3. So, by letting $n = 3N$ and $3M$, the equation becomes

$$N^2 - N - 2M^2 = 0$$

that is

$$(2N-1)^2 - 2(2M)^2 = 1.$$

Since the non negative solutions of the Pell's equation $X^2 - 2Y^2 = 1$ are

$$X_k + Y_k\sqrt{2} = (3 + 2\sqrt{2})^k \quad \text{for } k \geq 0,$$

we have that (note that X_k is odd)

$$n_k = 3N_k = \frac{3(X_k+1)}{2} \quad \text{for } k \geq 0$$

that is $3, 6, 27, 150, 867, 5046 \ldots$.

J136. Let a, b, c be the sides, m_a, m_b, m_c the medians, h_a, h_b, h_c the altitudes, and l_a, l_b, l_c the angle bisectors of a triangle ABC. Prove that the diameter of the circumcircle of triangle ABC is equal to

$$\frac{l_a^2}{h_a}\sqrt{\frac{m_a^2-h_a^2}{l_a^2-h_a^2}}.$$

Proposed by Panagiote Ligouras, "Leonardo da Vinci" High School, Bari, Italy

First solution by Arkady Alt, San Jose, California, USA

We should assume that $AB \neq AC$ because otherwise $l_a = h_a$ and in this case expression is undefined. Let AA_1 be angle bisector, then

$$\angle A_1AC = 90^\circ - \angle AA_1C = 90^\circ - B - \frac{A}{2} = \frac{C-B}{2}$$

and $h_a = l_a \cos\dfrac{C-B}{2}$. Hence, $\dfrac{l_a^2}{h_a^2} - 1 = \dfrac{1}{\cos^2\dfrac{C-B}{2}} - 1 = \tan^2\dfrac{C-B}{2}$. Since,

$$m_a^2 - h_a^2 = \frac{2\left(b^2+c^2\right)-a^2}{4} - \frac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{4a^2} = \frac{\left(b^2-c^2\right)^2}{4a^2}$$

then

$$\frac{l_a^2}{h_a}\sqrt{\frac{m_a^2-h_a^2}{l_a^2-h_a^2}} = \frac{l_a^2}{h_a^1}\sqrt{\frac{\frac{m_a^2-h_a^2}{h_a^2}}{\frac{l_a^2}{h_a^2}-1}} = \frac{1}{\cos^2\dfrac{C-B}{2}} \cdot \frac{\left|b^2-c^2\right|}{2a} \cdot \left|\cot\frac{C-B}{2}\right| = \frac{\left|b^2-c^2\right|}{a\left|\sin\left(B-C\right)\right|}$$

$$= \frac{4R^2\left|\sin^2 B - \sin^2 C\right|}{2R\sin A\left|\sin\left(B-C\right)\right|} = \frac{R\left|2\sin^2 B - 2\sin^2 C\right|}{\sin A\left|\sin\left(B-C\right)\right|} = \frac{R\left|\cos 2C - \cos 2B\right|}{\sin A\left|\sin\left(B-C\right)\right|}$$

$$= \frac{2R\left|\sin\left(B+C\right)\sin\left(B-C\right)\right|}{\sin A\left|\sin\left(B-C\right)\right|} = 2R.$$

Second solution by Aravind Srinivas L, Chennai, India

We know use the following relations: $l_a^2 = bc[1-(\frac{a}{b+c})^2]$ $h_a = \frac{bc}{2R}$ $m_a^2 = \frac{2b^2+2c^2-a^2}{4}$. We have to prove that

$$\frac{l_a^2}{h_a}\sqrt{\frac{m_a^2-h_a^2}{l_a^2-h_a^2}} = 2R$$

(R-circumradius of triangle ABC) Plugging the values of l_a^2, h_a, m_a^2 in the given to be proved equality, we have to prove after squaring both sides of the equality that

$$\frac{(b+c+a)^2(b+c-a)^2}{(b+c)^2} \cdot \frac{R^2(2b^2+2c^2-a^2)-b^2c^2((b^2+c^2)^2)}{4R^2a(b+c+a)(b+c-a)-b^2c^2((b+c)^2)} = 1$$

which after expanding using the relation $R = \frac{abc}{4\triangle}$ simplies to the Hero's Forumla for Area of a Triangle, that is
$4\triangle = \sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}$. Here, $\triangle$ denotes area of triangle.

Also solved by Ercole Suppa, Teramo, Italy; Daniel Lasaosa, Universidad Publica de Navarra, Spain.

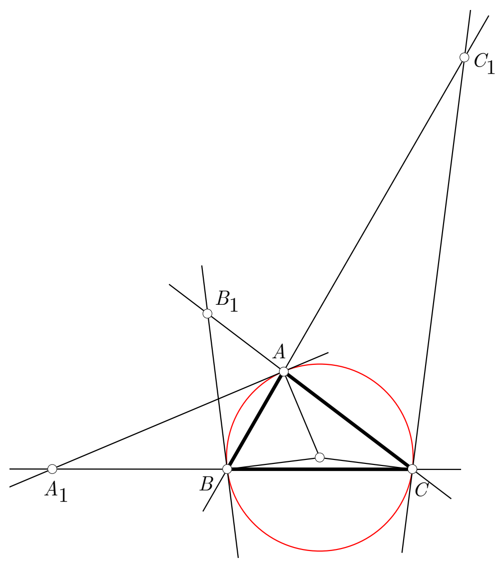
J137. Let ABC be a triangle and let tangents to the circumcircle at A, B, C intersect BC, AC, AB at points A_1, B_1, C_1, respectively. Prove that

$$\frac{1}{AA_1}+\frac{1}{BB_1}+\frac{1}{CC_1}=2\max\left(\frac{1}{AA_1},\frac{1}{BB_1},\frac{1}{CC_1}\right).$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Ercole Suppa, Teramo, Italy

Let us denote $a = BC$, $b = AC$, $c = AB$, $\alpha = \angle BAC$, $\beta = \angle ABC$, $\gamma = \angle ACB$ and suppose without loss of generality that $a \geq b \geq c$ (so $\alpha \geq \beta \geq \gamma$).



Since $\angle A_1AB = \gamma$ we have $\angle A_1AC = \alpha + \gamma$. Therefore the sinus theorem yields

$$\frac{AA_1}{\sin\gamma}=\frac{AC}{\sin\left(180^\circ-\alpha-2\gamma\right)} \quad\Rightarrow\quad AA_1=\frac{b\sin\gamma}{\sin\left(\beta-\gamma\right)} \quad\Rightarrow$$

$$\frac{1}{AA_1}=\frac{\sin\beta\cos\gamma-\cos\beta\sin\gamma}{b\sin\gamma}=\frac{1}{2R}\cot\gamma-\frac{\cos\beta}{b}=\frac{1}{2R}\left(\cot\gamma-\cot\beta\right) \qquad (1)$$

Similarly we obtain

$$\frac{1}{BB_1}=\frac{1}{2R}\left(\cot\gamma-\cot\alpha\right) \qquad (2)$$

and

$$\frac{1}{CC_1}=\frac{1}{2R}\left(\cot\beta-\cot\alpha\right) \qquad (3)$$

Adding (1), (2), (3) we get

$$\frac{1}{AA_1} + \frac{1}{BB_1} + \frac{1}{CC_1} = \frac{1}{R}\left(\cot\gamma - \cot\alpha\right) \tag{4}$$

In order to complete the proof we must show that

$$\max\left(\frac{1}{AA_1}, \frac{1}{BB_1}, \frac{1}{CC_1}\right) = \frac{1}{BB_1} = \frac{1}{2R}\left(\cot\gamma - \cot\alpha\right) \tag{5}$$

Now, if $\triangle ABC$ is an acute-angled triangle, $\cot\gamma \geq \cot\beta \geq \cot\alpha > 0$ and this implies that

$$\cot\gamma - \cot\alpha \geq \cot\gamma - \cot\beta \qquad \Longleftrightarrow \qquad \frac{1}{BB_1} \geq \frac{1}{AA_1} \tag{6}$$

$$\cot\gamma - \cot\alpha \geq \cot\beta - \cot\alpha \qquad \Longleftrightarrow \qquad \frac{1}{BB_1} \geq \frac{1}{CC_1} \tag{7}$$

whereas if $\triangle ABC$ is an obtuse-angled or a right triangle we have $\cot\gamma \geq \cot\beta$, $\cot\alpha \leq 0$ and the relations (6) and (7) are verified anyway. The proof is complete. □

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Assume that $\angle B > \angle C$. Since AA_1 is tangent to the circumcircle of ABC, the semiinscribed angle $\angle BAA_1 = \angle BCA = C$, hence since $\angle ABA_1 = \pi - B$, we conclude $\angle AA_1B = B - C$, and by the Sine Law, $AA_1 = \frac{AB\sin B}{\sin(B-C)}$, or

$$\frac{2R}{AA_1} = \frac{\sin(B-C)}{\sin B \sin C} = \frac{1}{\tan C} - \frac{1}{\tan B}.$$

Restoring generality, $\frac{2R}{AA_1} = \left|\frac{1}{\tan B} - \frac{1}{\tan C}\right|$, and similarly for its cyclic permutations. Assume now wlog that $A \geq B \geq C$. Clearly,

$$\max\left(\frac{1}{AA_1}, \frac{1}{BB_1}, \frac{1}{CC_1}\right) = \frac{1}{BB_1} = \frac{1}{\tan C} - \frac{1}{\tan B},$$

while

$$\frac{1}{AA_1}+\frac{1}{BB_1}+\frac{1}{CC_1} = \left(\frac{1}{\tan C} - \frac{1}{\tan B}\right)+\left(\frac{1}{\tan C} - \frac{1}{\tan A}\right)+\left(\frac{1}{\tan B} - \frac{1}{\tan A}\right) =$$

$$= \frac{2}{\tan C} - \frac{2}{\tan A}.$$

The conclusion follows. Note that if wlog $AB = AC$, the perpendicular bisector of BC, which passes through A, is a diameter of the circumcircle of ABC, hence perpendicular to the tangent to the circumcircle at A, or A_1 is "at infinity" (the tangent to the circumcircle at A is parallel to BC), hence we may say $\frac{1}{AA_1} = 0$. Although A_1 is not defined, we may write $\frac{1}{AA_1} = 0$. Note that if ABC is isosceles, wlog at A, then by symmetry $\frac{1}{BB_1} = \frac{1}{CC_1} > 0$, hence the conclusion also holds in this case. Finally, if ABC is equilateral, $\frac{1}{AA_1} = \frac{1}{BB_1} = \frac{1}{CC_1} = 0$, and the conclusion also holds.

J138. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{b^2+c^2}+\frac{b^3}{a^2+c^2}+\frac{c^3}{a^2+b^2} \geq \frac{a+b+c}{2}.$$

Proposed by Mircea Becheanu, University of Bucharest, Romania

First solution by Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy

By Cauchy–Schwarz in the form $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c} \geq \frac{(x+y+z)^2}{a+b+c}$ we have

$$\frac{a^3}{b^2+c^2}+\frac{b^3}{c^2+a^2}+\frac{c^3}{a^2+b^2} \geq \frac{(a^2+b^2+c^2)^2}{a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)} \geq \frac{a+b+c}{2}$$

Clearing the denominators yields

$$\sum_{\text{sym}} a^4 + a^2b^2 \geq \sum_{\text{sym}} abc^2 + ac^3$$

Now the result follows by the Muirhead's theorem since $[4,0,0] \succ [3,1,0]$ and $[2,2,0] \succ [2,1,1]$. The underlying AGM's are $(a^4+a^4+a^4+b^4)/4 \geq \sqrt[4]{a^{12}b^4} = a^3b$, $(a^2b^2+a^2c^2)/2 \geq \sqrt{a^4b^2c^2} = a^2bc$.

Second solution by Aravind Srinivas L, Chennai, India

After multiplying both sides of the inequality by $(a^2+b^2)(b^2+c^2)(c^2+a^2)$, we get to prove that : $2\sum_{cyc} a^3(c^2+a^2)(a^2+b^2) \geq (\sum_{cyc} a)(a^2+b^2)(b^2+c^2)(c^2+a^2)$.
After expanding and collecting and adding the like terms out of the expansion, we get that the inequality is equivalent to: $\sum_{sym} a^7 + 2\sum_{sym} a^5b^2 + \sum_{sym} a^3b^2c^2 \geq \sum_{sym} a^5b^2 + \sum_{sym} a^3b^4 + \sum_{sym} ab^2c^4 + \sum_{sym} a^3b^2c^2$. This is equivalent to : $\sum_{sym} a^7 + \sum_{sym} a^5b^2 \geq \sum_{sym} a^3b^4 + \sum_{sym} ab^2c^4$. Notice that a, b, c are positive real numbers. So, applying the Muirhead's Inequality we have $\sum_{sym} a^7b^0c^0 \geq \sum_{sym} a^3b^4c^0$ and $\sum_{sym} a^5b^2c^0 \geq \sum_{sym} ab^2c^4$. Adding this, we prove the result.

Senior problems

S133. There are 144 lilypads in a row and are colored red, green, blue, red, green, blue and so on. Prove that the number of ways for a frog to reach the last lilypad from the first lilypad in a sequence of left-to-right jumps between lilypads of different color is a multiple of 3.

Proposed by Brian Basham, Massachusetts Institute of Technology, USA

First solution by John T. Robinson, Yorktown Heights, NY, USA

Divide the lilypads into consecutive groups of 3, where group 1 is the last (far right) group, group 2 is the next to last group, etc., with group 48 (144/3) being the first group. Let $R(n)$, $G(n)$, and $B(n)$ be the number of ways of jumping left-to-right between lilypads of different color starting from the red, green, or blue respectively lilypad in group n and ending on the last (blue) lilypad. We have $R(1) = 2$, $G(1) = 1$, $B(1) = 0$, and the following recurrence relations for $n > 1$:

$$B(n) = \sum_{1\le i\le n-1} R(i) + G(i)\ ,$$

$$G(n) = B(n) + \left(\sum_{2\le i\le n-1} (R(i) + B(i)\right) + R(1) + 1\ ,$$

$$R(n) = G(n) + B(n) + \left(\sum_{2\le i\le n-1} (G(i) + B(i)\right) + G(1) + 1\ .$$

These can be shown to be equivalent after some straightforward manipulations to the following:

$$B(n) = R(n-1) + G(n-1) + B(n-1)\ ,$$

$$G(n) = 2R(n-1) + 2G(n-1) + B(n-1)\ ,$$

$$R(n) = 4R(n-1) + 3G(n-1) + 2B(n-1)\ .$$

In matrix form we have

$$\begin{pmatrix} R(n) \\ G(n) \\ B(n) \end{pmatrix} = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} R(n-1) \\ G(n-1) \\ B(n-1) \end{pmatrix}$$

for $n \ge 2$, with $(R(1), G(1), B(1)) = (2,1,0)$. Working modulo 3, the above becomes

$$\begin{pmatrix} R' \\ G' \\ B' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} R \\ G \\ B \end{pmatrix} \mod 3$$

(where 2 above could of course also be represented by -1 if desired). Calculating, we find that (modulo 3) the initial R,G,B vector (1) 2,1,0 is carried by the above matrix to successively (2) 2,0,0; (3) 2,1,2; (4) 0,2,2; (5) 1,0,1; (6) 0,0,2; (7) 1,2,2; (8) 2,2,2; (9) 0,1,0; (10) 0,2,1; (11) 2,2,0; (12) 2,2,1; (13) 1,0,2; and (14) 2,1,0. We therefore have $R(n+13) = R(n) \bmod 3$, with $R(4) = R(6) = R(9) = R(10) = 0$. It follows that $R(48) = R(3 \cdot 13 + 9) = R(9) = 0 \bmod 3$, thus establishing that the number of ways for a frog to reach the last lilypad from the first lilypad is a multiple of 3.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Number the lilypads $R_n, G_n, B_n, R_{n-1}, \ldots, B_2, R_1, G_1, B_1$ from left to right when there are $3n$ of them, and denote by $b(n), g(n)$ respectively the number of ways to reach B_1 from R_n jumping first to a blue lilypad and to a green lilypad, respectively. Let us see how may paths there are when we increase by 3 the number of lilypads. Starting from R_{n+1}, the possible starting sequences jumping first to a green lilypad are: $R_{n+1}G_{n+1}B_{n+1}R_k\ldots$, $R_{n+1}G_{n+1}B_{n+1}G_k\ldots$, $R_{n+1}G_{n+1}R_k\ldots$, $R_{n+1}G_{n+1}B_k\ldots$ and $R_{n+1}G_k\ldots$, where $k \le n$. There are clearly as many paths of the first type and of the third type, as paths going from R_k to B_1, added over all $k \le n$. Substitution of $R_{n+1}G_{n+1}B_{n+1}$, $R_{n+1}G_{n+1}$ and R_{n+1} by R_n in the second, fourth and fifth types of paths established one-to-one correspondences between these paths, and all paths starting in R_n and jumping first to a green, blue and green lilypad, respectively, for a total of $2g(n) + b(n)$ additional paths. Hence

$$g(n+1) = b(n) + 2g(n) + 2\sum_{k=1}^{n}(b(k) + g(k)) = 3b(n) + 5g(n) - b(n-1) - 2g(n-1).$$

Similarly, considering all paths that start from R_{n+1} and jump first to a blue lilypad, we obtain the following types of paths: $R_{n+1}B_{n+1}R_k\ldots$, $R_{n+1}B_{n+1}G_k\ldots$, $R_{n+1}B_k\ldots$, where $k \le n$, for a total of

$$b(n+1) = b(n) + g(n) + \sum_{k=1}^{n}(b(k) + g(k)) = 3b(n) + 2g(n) - b(n-1) - g(n-1).$$

Note that $b(1) = g(1) = 1$, corresponding respectively to paths R_1B_1 and $R_1G_1B_1$, and $b(2) = 4$, $g(2) = 7$ by direct application of the formulas (this may also be verified by counting all possible paths from R_2 to B_1). Now, counting modulus 3, clearly $g(n+1) \equiv 2g(n) - b(n-1) - 2g(n-1) \pmod 3$ and $b(n+1) \equiv 2g(n) - b(n-1) - g(n-1)$, or counting modulus 3, $g(3) = 2$ and $b(3) = 0$, and similarly $g(4) = 1$ and $b(4) = 2$. We may then easily find that, modulus 3, the series $g(1), g(2), \ldots$ is $1, 1, 2, 1, 1, 1, 0, 2, 0, 2, 1, 0, 0, 1, 1, \ldots$ and the series $b(1), b(2), \ldots$ is $1, 1, 0, 2, 0, 2, 1, 0, 0, 1, 1, 2, 1, 1, 1, \ldots$, repeating itself periodically since $b(14) = b(1)$, $b(15) = b(2)$, and $g(14) = g(1)$, $g(15) = g(2)$. Therefore, $b(k+13) = b(k)$ and $g(k+13) = g(k)$ for all $k \geq 1$, and since $144 = 3 \cdot 48$, and $48 = 3 \cdot 13 + 9$, we obtain $g(48) = g(9) = 0$ and $b(48) = b(9) = 0$, or the total number of paths, modulus 3, is $0 + 0 = 0$, hence multiple of 3. The conclusion follows.

Note that the number of paths will be a multiple of 3 iff $n \equiv 4, 6, 9, 10 \pmod{13}$, ie, iff the total number of lilypads is of the form $39k + 12$, $39k + 18$, $39k + 27$ or $39k + 30$ for some non-negative integer k, and $144 = 3 \cdot 39 + 27$.

Also solved by Aravind Srinivas L, Chennai, India.

S134. Find all triples (x, y, z) of integers satisfying the system of equations

$$x + y = 5z$$
$$xy = 5z^2 + 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by G. C. Greubel, Newport News, VA

The equations to solve are

$$x + y = 5z \tag{3}$$
$$xy = 5z^2 + 1. \tag{4}$$

Solving for y in (1) yields

$$y = 5z - x. \tag{5}$$

Using this in (2) provides

$$x(5z - x) = 5z^2 + 1. \tag{6}$$

This is a quadratic equation of the form

$$x^2 - (5z)x + (5z^2 + 1) = 0. \tag{7}$$

Solving for x provides

$$x = \frac{5z}{2} \pm \frac{1}{2}\sqrt{5z^2 - 4}. \tag{8}$$

This also implies that the value of y is

$$y = \frac{5z}{2} \mp \frac{1}{2}\sqrt{5z^2 - 4}. \tag{9}$$

In order to seek solution of integer type requires that the square root terms be integers. This requires that z be an integer and that $5z^2 - 4$ be a square integer. Playing with integer values one can determine rather quickly that the integer form of z, that satisfies the requirement that $5z^2 - 4$ be an integer, is and odd Fibonacci number, F_{2n+1}. When this value is used in $5z^2 - 4$ it is evident from the identity $5F_{2n+1}^2 - 4 = L_{2n+1}^2$ that our squared integer sought is an odd Lucas number. Using these values in the equation for x, namely (6), we have

$$\begin{aligned} x &= \frac{5}{2}F_{2n+1} \pm \frac{1}{2}\sqrt{5F_{2n+1}^2 - 4} \\ &= \frac{5}{2}F_{2n+1} \pm \frac{1}{2}L_{2n+1} \\ &= \frac{1}{2}\left(5F_{2n+1} \pm L_{2n+1}\right). \end{aligned} \tag{10}$$

By using the relation $5F_{2n+1} = L_{2n} + L_{2n+2} = L_{2n+1} + 2L_{2n}$ we then have

$$x = \frac{1}{2}\left(2L_{2n} + L_{2n+1} \pm L_{2n+1}\right). \tag{11}$$

This leads to the two values $x = \{L_{2n+2}, L_{2n}\}$. The values for y are then the reverse of these, namely, $y = \{L_{2n}, L_{2n+2}\}$.

From the values given above the integer solutions are

$$(x, y, z) = \begin{cases} (L_{2n}, L_{2n+2}, F_{2n+1}) \\ (L_{2n+2}, L_{2n}, F_{2n+1}) \end{cases} \tag{12}$$

Second solution by Brian Bradie, Christopher Newport University, Newport News, VA

Multiplying the second equation by 5 and substituting for $5z$ from the first equation yields

$$x^2 - 3xy + y^2 + 5 = 0.$$

By the quadratic formula, we then have

$$x = \frac{3}{2}y \pm \frac{1}{2}\sqrt{5(y^2 - 4)}.$$

In order for x to be an integer, we must have $y^2 - 4 = 5n^2$ for some integer n. It would then follow that

$$x = \frac{3}{2}y \pm \frac{5}{2}n \quad \text{and} \quad z = \frac{1}{2}y \pm \frac{1}{2}n.$$

If we consider the equation $y^2 - 5n^2 = 4$ modulo 4, we find that y and n must be of the same parity; consequently, x and z will always be integers. Now, the positive integer solutions of $y^2 - 5n^2 = 4$ are the ordered pairs (y_k, n_k), where y_k and n_k satisfy the recurrence relations

$$y_0 = 2, \quad y_1 = 3, \quad y_k = 3y_{k-1} - y_{k-2}$$

and

$$n_0 = 0, \quad n_1 = 1, \quad n_k = 3n_{k-1} - n_{k-2}.$$

Thus,

$$y_k = \left(\frac{3 + \sqrt{5}}{2}\right)^k + \left(\frac{3 - \sqrt{5}}{2}\right)^k = L_{2k}$$

and

$$n_k = \frac{1}{\sqrt{5}}\left[\left(\frac{3 + \sqrt{5}}{2}\right)^k + \left(\frac{3 - \sqrt{5}}{2}\right)^k\right] = F_{2k},$$

where L_j and F_j denote the j-th Lucas number and the j-th Fibonacci number, respectively. Using the identity

$$L_{2k} = F_{2k-1} + F_{2k+1},$$

we find

$$\frac{1}{2}(y_k + n_k) = \frac{1}{2}(L_{2k} + F_{2k}) = \frac{1}{2}(F_{2k-1} + F_{2k} + F_{2k+1}) = F_{2k+1}.$$

On the other hand

$$\frac{1}{2}(y_k - n_k) = \frac{1}{2}(y_k + n_k) - n_k = F_{2k+1} - F_{2k}.$$

Moreover,

$$\frac{3}{2}y_k + \frac{5}{2}n_k = 3F_{2k+1} + F_{2k} = 2F2k + 1 + F_{2k+2} = F_{2k+1} + F_{2k+3} = L_{2k+2},$$

and

$$\frac{3}{2}y_k - \frac{5}{2}n_k = 3F_{2k+1} - 4F_{2k}.$$

Thus, for each $k \geq 0$, the triples

$$(L_{2k+2}, L_{2k}, F_{2k+1}) \quad \text{and} \quad (3F_{2k+1} - 4F_{2k}, L_{2k}, F_{2k+1} - F_{2k})$$

are solutions of (1). Because the equations in (1) are invariant under the transformation $(x, y, z) \to (-x, -y, -z)$, the triples

$$(-L_{2k+2}, -L_{2k}, -F_{2k+1}) \quad \text{and} \quad (-3F_{2k+1} + 4F_{2k}, -L_{2k}, -F_{2k+1} + F_{2k})$$

are also solutions of (1).

Also solved by John T. Robinson, Yorktown Heights, NY, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S135. Zeroes are written at every vertex of a regular $n-$gon. Every minute, Bob picks a vertex, adds 2 to the number written at that vertex, and subtracts 1 from the numbers written at the two adjacent vertices. Prove that, no matter how long Bob plays, he will never be able to achieve a configuration in which a 1 is written at one vertex, a -1 is written at another, and a zero is written everywhere else.

Proposed by Timothy Chu, Lynbrook Highschool, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Number the vertices from 1 to n counterclockwise, and denote x_i the number of times that a vertex has been picked before arriving to a given configuration (ie, to a given distribution of values in the vertices of the n-gon), and use cyclic notation such that $x_{n+i} = x_i$. Clearly, the amount assigned to vertex i is $2x_i - x_{i-1} - x_{i+1}$. Note that we may thus assume wlog that $\min\{x_i\} = 0$, since we may subtract $\min\{x_i\}$ from each x_i, resulting in the same configuration since each $2x_i - x_{i-1} - x_{i+1}$ does not change. Assume that in the ending configuration, $x_1 = -1$, and that $x_k = 0$ for some $k \neq 1$. Clearly, since $2x_k - x_{k-1} - x_{k+1}$ is equal to either 0 or 1, x_{k-1} and x_{k+1} are non-positive, ie $x_{k-1} = x_{k+1} = 0$, and 0 is assigned to the k-th vertex. Note that we may travel from the k-th vertex to the vertex that has 1 assigned, clockwise or counterclockwise, without going through the vertex that has -1 assigned. By trivial induction forward or backward, we conclude that $x_i = 0$ for the vertex that has 1 assigned, absurd since the value that has 1 assigned would have a non-positive value. Hence $x_1 = 0$, and $x_2 + x_n = 1$, or wlog by symmetry since we may number the vertices clockwise instead of counterclockwise, $x_2 = 1$ and $x_n = 0$. Note therefore that $x_{n-1} = 0$ or the n-th vertex would have a negative value assigned. Now, traveling backward from the n-th vertex, we must encounter the vertex that has 1 assigned. By trivial backwards induction, and since 0 is assigned to all vertices from the n-th down to the vertex that has 1 assigned, we conclude that $x_i = 0$ for the vertex that has 1 assigned, or its value is non-positive, contradiction. The conclusion follows.

S136. A weightlifter lifts a barbell which has n equal weights on both of its sides. At each step he takes out weights from one of two sides of the barbell. However, if the difference between the weights on the sides is greater than k weights, the barbell will turn and fall. What is the least number of steps required to take out all the weights?

Proposed by Iurie Boreico, Harvard University, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

On the first step, the lifter may take out at most k weights from one side. Before the last step, there can be weights only at one side of the barbell, and there may be at most k of them. Hence there are two steps where the lifter takes out at most k weights. On any other step, the lifter may take out at most $2k$ weights, this being possible only if before the step there are k weights more on one side, and the lifter takes out $2k$ weights from the other side. There are therefore at least $\frac{2n-2k}{2k} = \frac{n}{k} - 1$ such steps, for a total of at least $\lceil \frac{n}{k} \rceil + 1$, where for any real x, $\lceil x \rceil$ denotes the least integer larger than or equal to x. This number of steps may indeed be achieved with the following strategy: the lifter takes out exactly k weights from one side on the first step, then alternates taking out exactly $2k$ weights from each side as many times as it is possible, ie until both sides have less than $2k$ weights. Since after each such step there are clearly exactly k weights more on one side than on the other, there are less than k weights on one of the sides, and exactly two more steps are needed in order to finish the process. If the total number of steps was greater than $\lceil \frac{n}{k} \rceil + 1$, the lifter would have removed, in all but the last two steps, exactly $k + 2k \left(\lceil \frac{n}{k} \rceil - 1 \right) \geq 2n - k$, contradiction since there would then be no more than k weights left to remove, all on one of the sides. The lifter may therefore use exactly $\lceil \frac{n}{k} \rceil + 1$ steps to remove all weights.

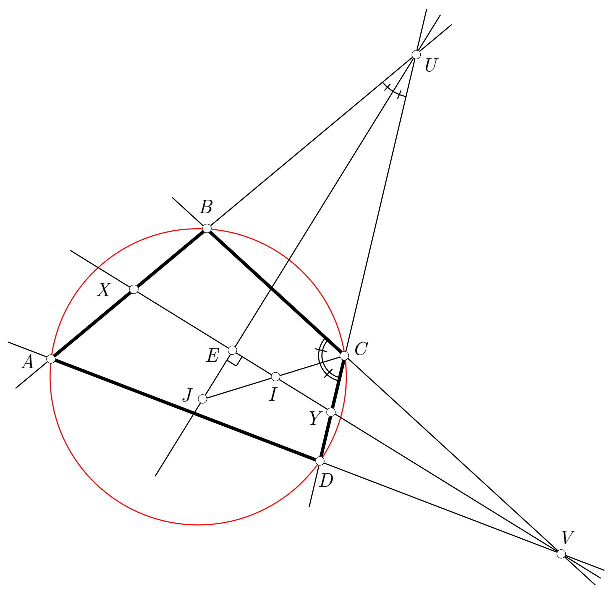
S137. Let $ABCD$ be a cyclic quadrilateral and let $\{U\} = AB \cap CD$ and $\{V\} = BC \cap AD$. The line that passes through V and is perpendicular to the angle bisector of angle $\angle AUD$ intersects UA and UD at X and Y, respectively. Prove that

$$AX \cdot DY = BX \cdot CY.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Ercole Suppa, Teramo, Italy

Let E, J be the intersection points of the line XY with the angle bisectors of angles $\angle AUD$, $\angle BCD$ respectively and let $I = XY \cap CJ$, as shown in figure.



Claim: The line XY is the angle bisector of angle $\angle AVB$.

Proof. Let us denote with A, B, C, D the measures of internal angles of quadrilateral $ABCD$. We have

$$\angle AVB = 180^\circ - (A + B) \qquad , \qquad \angle AUD = 180^\circ - (A + D) \tag{1}$$

and

$$\begin{aligned}\angle CIV = \angle EIJ = 90^\circ - \angle EJI = 90^\circ - \angle UJC = \\ = 90^\circ - (180^\circ - \angle JUC - \angle JCU) = \\ = \angle JUC + \angle JCU - 90^\circ = \\ = \frac{180^\circ - (A+D)}{2} + \left(180^\circ - \frac{C}{2}\right) - 90^\circ = \\ = \frac{C}{2} - \frac{D}{2} + 90^\circ - \frac{C}{2} = 90^\circ - \frac{D}{2}\end{aligned} \tag{2}$$

From (1), (2) it follows that

$$\begin{aligned}\angle BVX = \angle CVI = 180^\circ - \angle CIV - \angle ICV = \\ = 180^\circ - \left(90^\circ - \frac{D}{2}\right) - \left(180^\circ - \frac{C}{2}\right) = \\ = \frac{D}{2} + \frac{C}{2} - 90^\circ = 90^\circ - \frac{A+B}{2} = \frac{1}{2}\angle AVB\end{aligned}$$

and the claim is proven. ■

Coming back to the problem, applying the internal bisector theorem to the triangles $\triangle VCD$ and $\triangle VAB$ we obtain

$$CY : YD = VC : VD \qquad \Rightarrow \qquad VC = \frac{CY}{YD} \cdot VD \tag{3}$$

$$BX : XA = VB : VA \qquad \Rightarrow \qquad VB = \frac{BX}{XA} \cdot VA \tag{4}$$

From (3), (4), taking into account that $VC \cdot VB = VD \cdot VA$, we get

$$\frac{CY}{YD} \cdot VD \cdot \frac{BX}{XA} \cdot VA = VD \cdot VA \qquad \Rightarrow \qquad AX \cdot DY = BX \cdot CY$$

and the proof is complete. □

Second solution by Maxim Ignatiuc, Moldova

First, let's denote UC, CY, YD, BU, XB, AF as a, b, c, d, e and f, respectively.Triangle $\triangle UXY$ is isosceles ,as bisector of $\angle U$ is perpendicular to XY.Considering this fact, $a + b = d + e$. Let k be equal to $a + b$. Consequently, $a = k - b$ and $d = k - e$. As we have a cyclic quadrilateral, we consider the equation:

$$(k - e)(k + f) = (k - b)(k + c)$$

Moving k to the left side, we get:

$$k = (bc - ef)/(c - b + e - f)$$

Now,we apply the Menelaus theorem for triangle $\triangle AUD$ with a secant BV:

$$\frac{AV}{DV} \cdot \frac{b+c}{k-b} \cdot \frac{k-e}{e+f} = 1$$

Then the same operation is done for $\triangle AUD$ with a secant XV :

$$\frac{AV}{DV} \cdot \frac{c}{k} \cdot \frac{k}{f} = 1$$

Equilizing both parts gives us:

$$\frac{f}{c} \cdot \frac{b+c}{k-b} \cdot \frac{k-e}{e+f} = 1$$

Substituting the value of k:

$$\frac{f}{c} \cdot \frac{b+c}{\frac{(b-e)(b+f)}{c-b+e-f}} \cdot \frac{\frac{(b-e)(e+c)}{c-b+e-f}}{e+f}$$

$$f(b+c)(c+e) = c(b+f)(e+f)$$

Factorization yields to:

$$(c-f)(be-cf) = 0$$

$$(YD - AX)(XB \cdot CY - YD \cdot AX) = 0$$

As $YD \neq AX$ we get the desired answer:

$XB \cdot CY = YD \cdot AX$.

Also solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S138. Let a, b, c be positive real numbers such that $\sqrt{a}+\sqrt{b}+\sqrt{c}=3$. Prove that

$$8(a^2+b^2+c^2) \geq 3(a+b)(b+c)(c+a).$$

Proposed by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

First solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

The inequality is equivalent to

$$a+b+c=3, \qquad 8(a+b+c)^2(a^4+b^4+c^4) \geq 27(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

that is

$$16\sum_{\text{sym}} a^5b + 4\sum_{\text{sym}} a^6 + 8\sum_{\text{sym}} a^4bc \geq 19\sum_{\text{sym}} a^4b^2 + 9\sum_{\text{sym}} (abc)^2$$

and the result follows by Muirhead's inequality since $[4,2,0] \prec [6,0,0]$, $[4,2,0] \prec [5,1,0]$, $[2,2,2] \prec [4,1,1,]$. The underlying AGM's are respectively $(a^6+a^6+b^6)/3 \geq a^4b^3$, $(a^5b+a^5b+a^5b+ba^5)/4 \geq a^4b^2$, $(a^4bc+b^4ac+c^4ab)/3 \geq (abc)^2$.

Second solution by Ercole Suppa, Teramo, Italy

By AM-GM inequality we have

$$3(a+b)(b+c)(c+a) \leq 3\left(\frac{2a+2b+2c}{3}\right)^3 = \frac{8}{9}(a+b+c)^3 \tag{1}$$

On the other hand, taking $x_i = a^{1/3}$, $y_i = b^{2/3}$, $p = \frac{3}{2}$, $q = 3$, in the Hölder's inequality

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} {x_i}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} {y_i}^q\right)^{\frac{1}{q}}$$

we get

$$a+b+c = a^{\frac{1}{3}}a^{\frac{2}{3}} + b^{\frac{1}{3}}b^{\frac{2}{3}} + c^{\frac{1}{3}}c^{\frac{2}{3}} \leq \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^{\frac{2}{3}} \left(a^2+b^2+c^2\right)^{\frac{1}{3}} \qquad \Rightarrow$$

$$(a+b+c)^3 \leq \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \left(a^2+b^2+c^2\right) \tag{2}$$

Since $\sqrt{a}+\sqrt{b}+\sqrt{c}=3$ from (2) it follows that

$$(a+b+c)^3 \leq 9\left(a^2+b^2+c^2\right) \tag{3}$$

From (1) and (3) we get the desired inequality. The equality holds if $a=b=c=1$.

Also solved by Arkady Alt, San Jose, California, USA; G. C. Greubel, Newport News, VA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Aravind Srinivas L, Chennai, India.

Undergraduate roblems

U133. Let f be a continuous real valued function defined on $[0,1]$ such that $\int_0^1 f(x)dx = \int_0^1 xf(x)dx$. Prove that there is a real number $c \in (0,1)$ for which $c \cdot f(c) = 2\int_c^0 f(x)dx$.

Proposed by Duong Viet Thong, Nam Dinh University, Vietnam

First solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

Define $H(x) = \int_0^x \left(\int_0^y f(z)dz\right) dy$.

$$H(1) = \int_0^1 \left(\int_0^y f(z)dz\right) dy = \int_0^1 \left(\int_z^1 f(z)dy\right) dz = \int_0^1 (1-z)f(z)dz = 0$$

Since trivially $H(0)=0$, by Rolle's theorem, via the continuity of $f(x)$, we have the existence of $x_0 \in (0,1)$ such that $H'(x_0) = 0$ that is $\int_0^{x_0} f(x)dx = 0$. Now define the function $F(x) = x^2 \int_0^x f(y)dy$ and observe that $F(0) = F(x_0) = 0$. Rolle's theorem again ensures that $F'(c) = 0$ for some $c \in (0, x_0)$ that is

$$2c \cdot \int_0^c f(x)dx + c^2 f(c) = 0$$

and we are done.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let $F(x) = \int_0^x f(y)dy$, which clearly exists and is continuous and differentiable, with first derivative $F'(x) = f(x)$. Integrating by parts, the condition given in the problem statement about $f(x)$ is written, in terms of $F(x)$, as follows:

$$F(1) = \int_0^1 xF'(x)dx = F(1) - \int_0^1 F(x)dx; \qquad \int_0^1 F(x)dx = 0.$$

We need to prove that $c \in (0,1)$ exists such that $cF'(c) = -2F(c)$, or equivalently, such that $0 = c^2F'(c) + 2cF(c) = G'(c)$, where we have defined $G(x) = x^2F(x)$. Note now that $z \in (0,1)$ exists such that $G(z) = 0$; indeed, either $F(x) = 0$ for all $x \in [0,1]$, or $F(x)$ changes sign so that positive and negative contributions to its integral over $[0,1]$ cancel out, the intermediate value theorem guaranteeing in this latter case that $F(z) = 0$ for some $z \in (0,1)$. Clearly, $G(0) = G(z) = 0$, hence by Rolle's theorem $x \in (0,z) \subset (0,1)$ exists such that $G'(c) = 0$. The conclusion follows.

U134. Let $f : [0, \infty) \to \mathbb{R}$ be a function such that $f(x_1) + f(x_2) \geq 2f(x_1 + x_2)$ for all $x_1, x_2 \geq 0$. Prove that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(x_1 + x_2 + \cdots + x_n)$$

for all $x_1, x_2, \ldots, x_n \geq 0$.

Proposed by Mihai Piticari, Romania

Solution by John T. Robinson, Yorktown Heights, NY, USA

We want to show that

$$\sum_{1 \leq i \leq m} f(x_i) \geq mf\left(\sum_{1 \leq i \leq m} f(x_i)\right)$$

for $m \geq 1$. This holds trivially for $m = 1$, and we are given that this holds for $m = 2$; assume now that this holds for $m = n - 1$ (with $n \geq 3$). Then for $1 \leq j \leq n$ we have, where unless otherwise indicated all summation indices range from 1 to n, the following.

$$\sum_i f(x_i) \geq f(x_j) + (n-1)f\left(\sum_{i \neq j} x_i\right)$$

Adding these n inequalities gives

$$n\sum_i f(x_i) \geq \sum_j f(x_j) + (n-1)\sum_j f\left(\sum_{i \neq j} x_i\right)$$

Subtracting out the common sum and then dividing by $(n-1)$ gives

$$\sum_i f(x_i) \geq \sum_j f\left(\sum_{i \neq j} x_i\right) \quad \text{(A)}$$

We also have, again for $1 \leq j \leq n$, the following.

$$f(x_j) + f\left(\sum_{i \neq j} x_i\right) \geq 2f\left(\sum_i x_i\right)$$

Adding these n inequalities gives

$$\sum_j f(x_j) + \sum_j f\left(\sum_{i \neq j} x_i\right) \geq 2nf\left(\sum_i x_i\right) \quad \text{(B)}$$

Adding inequalities (A) and (B) gives

$$2\sum_i f(x_i) + \sum_j f\left(\sum_{i \neq j} x_i\right) \geq 2nf\left(\sum_i x_i\right) + \sum_j f\left(\sum_{i \neq j} x_i\right)$$

Cancelling the common term on both sides and dividing by 2 we have

$$\sum_i f(x_i) \geq nf\left(\sum_i x_i\right)$$

which establishes the result for $m = n$; therefore by induction the result holds for all $m \geq 1$ as was to be proved. Note by the way that only $f(x_1) + f(x_2) \geq 2f(x_1 + x_2)$ for all x_i in the domain of f was used above; that is the result also holds for a wider class of functions.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

U135. Suppose that $f, g : (0, \infty) \to (a, \infty)$ are continuous convex functions such that f is increasing and continuously differentiable. Prove that if

$$f'(x) \geq \frac{f(g(x)) - f(x)}{x}$$

for all $x > 0$, then $g(x) \leq 2x$ for all $x > 0$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

If $g(x_0) \leq x_0 < 2x_0$ for all $x_0 \in \mathbb{R}^+$, the proposed result is clearly true. Assume therefore that $x_0 \in \mathbb{R}^+$ exists such that $g(x_0) > x_0$. Since $f(x)$ is convex, $f'(x)$ is non-decreasing, thus applying the condition from the problem statement to any such x_0, we find

$$x_0 f'(x_0) \geq f(g(x_0)) - f(x_0) = \int_{x_0}^{g(x_0)} f'(x)dx \geq \int_{x_0}^{g(x_0)} f'(x_0)dx = f'(x_0)(g(x_0) - x_0).$$

Since furthermore $f'(x_0) > 0$ because f is increasing, $x_0 \geq g(x_0) - x_0$. The conclusion follows.

U136. Let P be a non-constant polynomial. Prove that there are infinitely many positive integers n such that $(P(n))^n$ is not a power of a prime.

Proposed by Cezar Lupu, University of Bucharest, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We begin the solution with the following

Claim 1: If $P(x)$ is a polynomial of degree $m \geq 1$, and $P(n), P(n+1), \ldots, P(n+m)$ are integers for some integer n, then $m!P(x)$ is a polynomial with integral coefficients.

Proof: The result is clearly true for $m = 1$, since if $P(x) = ax + b$, then $P(n+1) - P(n) = a$ must be an integer if $P(n), P(n+1)$ are integers, hence $b = P(n) - an$ must be an integer too. Assume that the result is true for a given m, and for any polynomial $P(x)$ of degree $m+1$ with coefficient $a \neq 0$ for x^{m+1}, define $Q(x) = P(x+1) - P(x)$. Clearly $Q(x)$ is a polynomial with degree m and coefficient $(m+1)a$ for x^m. Note that if $P(n), P(n+1), \ldots, P(n+m+1)$ are integers, so are $Q(n), Q(n+1), \ldots, Q(n+m)$, or applying the hypothesis of induction, $(m+1)!a$ is an integer. Note therefore that $R(x) = P(x) - a(x-n)(x-n-1)\ldots(x-n-m)$ is a polynomial of m-th degree, such that $P(x) = R(x)$ are integers for $x = n, n+1, \ldots$, or by hypothesis of induction, the coefficients of $m!R(x)$ are integers, hence the coefficients of $(m+1)!P(x) = (m+1)!R(x) - (m+1)!a(x-n)(x-n-1)\ldots(x-n-m)$ are integers too. The conclusion follows.

U137. Suppose that k and n are positive integers with $n > 1$ and that $A_1, A_2, \dots, A_k$, $B_1, B_2, \dots, B_k$ are $n \times n$ matrices with real entries such that for each matrix X with real entries satisfying $X^2 = O_n$, the matrix $A_1XB_1+A_2XB_2+\cdots+A_kXB_k$ is nilpotent. Prove that $A_1B_1 + A_2B_2 + \cdots + A_kB_k$ is of the form aI_n for some real number a.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

Solution by the author

Observe that the condition implies that $Tr(X(B_1A_1 + ... + B_nA_n)) = 0$ for all matrices X with $X^2 = O_n$. We claim that this also holds for all matrices X of zero trace. This is a quite standard result: any matrix of zero trace is conjugate to a matrix whose diagonal is zero. The proof is by induction: Let X be a nonzero matrix of zero trace. There exists a vector e_1 such that e_1 is not collinear to Xe_1. Take a basis of the form e_1, $e_2 = Xe_1$, e_3,...,e_n, and observe that when writing the matrix of the associated linear map in this base, we obtain a 0 on position (1,1) and a matrix X' of zero trace whose lines and columns are the $n-1$ last lines and columns of X. By applying the induction hypothesis to this new matrix, we can easily conclude. Therefore, if X is a matrix of zero trace, there is $P \in GL_n(\mathbb{R})$ such that $P^{-1}XP = \sum_{i\neq j} a_{ij}E_{ij}$, where E_{ij} is the standard elementary matrix whose unique nonzero entry is a 1 on position (i, j). Because $E_{ij}^2 = O_n$, this shows that X lies in the vector space spanned by the matrix of square equal to O_n. Therefore, if $U = B_1A_1 + ... + B_nA_n$, we must have $Tr(XU) = 0$ whenever $Tr(X) = 0$. This implies that the linear forms $f(X) = Tr(UX)$ and $g(X) = Tr(X)$ have the same kernel (if f is zero, we are done, because in this case $U = O_n$) and thus they are proportional. That is, there exists $a \in \mathbb{C}$ such that $Tr(UX) = aTr(X)$ for all X and this means that $U = aI_n$.

U138. Let q be a Fermat prime and let $n \leq q$ be a positive integer. Let p be a prime divisor of $1 + n + \cdots + n^{q-1}$. Define a function λ on real numbers x by $\lambda(x) = x - \frac{x^2}{2} + \cdots - \frac{x^{p-1}}{p-1}$. Prove that p divides the numerator of the fraction

$$\sum_{j=0}^{\log_2 \frac{q-1}{2}} \frac{\lambda(n^{2^j})(n^{pq-p} - 1)}{(n^p - 1)(n^{2^j p} + 1)}$$

when it is written in lowest termns.

Proposed by David B. Rush, Massachusetts Institute of Technology, USA

No solutions has yet been received.

Undergraduate roblems

O133. Let a, b, c and x, y, z be positive real numbers such that $\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}=\sqrt[3]{m}$ and $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{n}$. Prove that

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z} \geq \frac{m}{n}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote $A=\sqrt[3]{a}$, $B=\sqrt[3]{b}$, $C=\sqrt[3]{c}$, $A+B+C=\sqrt[3]{m}$, and define $u=\frac{A}{\sqrt{x}}$, $v=\frac{B}{\sqrt{y}}$, $w=\frac{C}{\sqrt{z}}$. Clearly $\sqrt{n}=\frac{A}{u}+\frac{B}{v}+\frac{C}{w}$, or the result that we need to prove is equivalent to showing that, for all positive reals A, B, C, u, v, w, the following inequality holds:

$$\sqrt{\frac{Au^2+Bv^2+Cw^2}{A+B+C}} \geq \frac{A+B+C}{\frac{A}{u}+\frac{B}{v}+\frac{C}{w}}.$$

This is the inequality between weighted quadratic and harmonic means of u, v, w and respective weights A, B, C, which is well known to be always true, with equality holding iff $u=v=w$, ie, iff

$$\frac{\sqrt[3]{a}}{\sqrt{x}}=\frac{\sqrt[3]{b}}{\sqrt{y}}=\frac{\sqrt[3]{c}}{\sqrt{z}}.$$

Also solved by Magkos Athanasios, Kozani, Greece; Perfetti Paolo, Università degli studi di Tor Vergata Roma, Italy; Arkady Alt, San Jose, California, USA.

O134. Let p be a prime and let n be an integer greater than 4. Prove that if a is an integer that is not divisible by p, then the polynomial $ax^n - px^2 + px + p^2$ is irreducible in $\mathbb{Z}[X]$.

Proposed by Mircea Becheanu, University of Bucharest, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Assume the contrary, hence nonconstant polynomials $B(x) = \sum_{i=0}^{m} b_i x^i$ and $C(x) = \sum_{j=0}^{n-m} c_j x^j$ exist such that $P(x) = ax^n - px^2 + px + p^2 = B(x)C(x)$, where clearly $1 \le m \le n-1$. Since $b_0 c_0 = p^2$ are integers, and wlog $b_0, c_0 > 0$ since we may change the sign of $B(x), C(x)$ simultaneously without altering the problem, then either $b_0 = c_0 = p$, or $b_0 = p^2$ and $c_0 = 1$.

In the first case, clearly $b_0 c_1 + b_1 c_0 = p$, or $b_1 + c_1 = 1$, hence b_1, c_1 cannot be multiples of p simultaneously. Now, $b_0 c_2 + b_1 c_1 + b_2 c_0 = -p$, or $b_1 c_1$ is a multiple of p, hence wlog since we may exchange the roles of $B(x)$ and $C(x)$ without altering the problem, b_1 is a multiple of p but c_1 is not. Now, for all integer j such that $n > j \ge 3$, $b_0 c_j + b_1 c_{j-1} + \cdots + b_j c_0 = 0$, hence $b_1 c_{j-1} + b_2 c_{j-2} + \cdots + b_{j-1} c_1$ is a multiple of p for all $n > j \ge 3$, where b_1 is a multiple of p but c_1 is not. Taking $j = 3$, $b_2 c_1$ must be a multiple of p, hence p divides b_2. By trivial induction, if $b_1, b_2, \ldots, b_k$ are multiples of p, taking $j = k+2$, we obtain that $b_{k+1} c_1$ must be a multiple of p, hence b_{k+1} is also a multiple of p. Since we may choose j as high as $n-1$, $b_0, b_1, \ldots, b_{n-2}$ are clearly multiples of p (with $b_{m+1} = b_{m+2} = \cdots = b_{n-2} = 0$ if $m < n-1$). Since $a = b_m c_{n-m}$ is not a multiple of p, then $m = n-1$, and $C(x) = c_1 x + p$. Call $-c_1 = q$, so $b_{n-1} = -\frac{a}{q}$, and for all $n-1 \ge j \ge 3$, $0 = b_{j-1} c_1 + b_j c_0 = -q b_{j-1} + p b_j$, or after trivial induction, $b_2 = -\frac{p^{n-3} a}{q^{n-2}}$. Now, $b_1 c_1 + b_2 c_0 = -p$, or $b_1 = \frac{(b_2+1)p}{q} = \frac{p^{n-2} a + p q^{n-2}}{q^{n-1}}$. Since p, q are mutually prime (otherwise a would be a multiple of p), then q^{n-1} divides a. Note also that $\frac{p}{q}$ is a root of $ax^n - px^2 + px + p^2$, or $0 = \frac{p^2}{q}\left(p^{n-2}\frac{a}{q^{n-1}} - \frac{p}{q} + 1 + q\right)$, and $\frac{p}{q} = p^{n-2}\frac{a}{q^{n-1}} + q + 1$ is an integer, yielding $q = 1$, or p is a root of the original polynomial. We conclude that $p^4 < |ap^n| = |p^2(p-2)| < p^3$, contradiction.

In the second case, and since c_0 is not a multiple of p, but $b_0 c_j + b_1 c_{j-1} + \cdots + b_j c_0$ is a multiple of p for $j = 0, 1, \ldots, n-1$ (note that the sum is p^2 for $j = 0$, p for $j = 1$, $-p$ for $j = 2$ and 0 for $n-1 \ge j \ge 3$), we conclude after trivial induction that $b_j c_0$, hence b_j, must be a multiple of p for $j = 0, 1, 2, \ldots, n-1$, and since $m \le n-1$, then $a = b_m c_{n-m}$ is a multiple of p, contradiction.

O135. In a convex quadrilateral $ABCD, AC \cap BD = \{E\}, AB \cap CD = \{F\}$, and EF intersects the sides AD and BC at X and Y. Let M and N be the midpoints of AD and BC, respectively. Prove that quadrilateral $BCMX$ is cyclic if and only if quadrilateral $ADNY$ is cyclic.

Proposed by Andrei Ciupan, Tudor Vianu High School, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Applying Menelaus' theorem to triangles CDG, ACD and ACG, we find

$$\frac{CF}{FD} \cdot \frac{DA}{AG} \cdot \frac{GB}{BC} = 1, \qquad \frac{AX}{XD} \cdot \frac{DF}{FC} \cdot \frac{CE}{EA} = 1, \qquad \frac{AE}{EC} \cdot \frac{CB}{BG} \cdot \frac{GD}{DA} = 1.$$

Multiplying these three equalities, we find $\frac{GA}{GD} = \frac{AX}{DX} = \frac{GX-GA}{GD-GX}$, or $GX = \frac{2GA \cdot GD}{GA+GD}$, and since M is the midpoint of AD, $2GM = GA + GD$, yielding $GX \cdot GM = GA \cdot GD$. Note therefore that $BCMX$ is cyclic iff $GX \cdot GM = GB \cdot GC$ iff $GB \cdot GC = GA \cdot GD$ iff $ABCD$ is cyclic. Similarly, we find that $ADNY$ is cyclic iff $ABCD$ is cylcic. The conclusion follows.

O136. For a positive integer n and a prime p, denote by $v_(n)$ the nonnegative integer for which $p^{v_p(n)}$ divides n but $p^{v_p(n)+1}$ does not. Prove that $v_5(n) = v_5(F_n)$, where F_n denotes the n^{th} Fibonacci number.

Proposed by David B. Rush, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Claim: If a series is given by $x_0 = 0$, x_1 not divisible by 5, and $x_{n+2} = ax_{n+1} + bx_n$ for all $n \geq 0$, where $a \equiv b \equiv 1 \pmod 5$, then 5 divides x_n iff 5 divides n, $v_5(x_5) = 1$, and furthermore $x_{5k+10+r} = a'x_{5k+5+r} + b'x_{5k+r}$ for all non-negative integers k, r, where $a' \equiv b' \equiv 1 \pmod 5$.

Proof: Clearly, $(x_n) = (0, x_1, x_1, 2x_1, 3x_1, 5x_1, 8x_1, 13x_1, 21x_1, 34x_1, 55x_1, \dots)$, where $x_0 = 0$, $x_5 = 5x_1$ and $x_{10} = 55x_1$ are the only terms shown divisible by 5 since x_1 is not a multiple of 5. Moreover, 5^2 does not divide $x_5 = 5x_1$ since again 5 does not divide x_1. Now, any series established by the recurrence relation $x_{n+2} = ax_{n+1} + bx_n$ obeys the general form $x_n = A\lambda_+^n + B\lambda_-^n$, where λ_+, λ_- are the two roots of the equation $\lambda^2 = a\lambda + b$, and constants A, B are determined by the initial conditions x_0, x_1. Now, it is well known that $x_{n+2m} = a'x_{n+m} + bx_n$ where $\lambda^2 = a'\lambda + b'$ has roots λ_+^m, λ_-^m for any positive integer m, or taking $m = 5$, and by Cardano-Vieta relations,

$$a' = \lambda_+^5 + \lambda_-^5 = (\lambda_+ + \lambda_-)^5 - 5(\lambda_+\lambda_-)(\lambda_+ + \lambda_-)^3 + 5(\lambda_+\lambda_-)^2(\lambda_+ + \lambda_-) =$$

$$= a^5 + 5ab(a^2 + b) \equiv a^5 \equiv 1 \pmod 5,$$

$$b' = -\lambda_+^5\lambda_-^5 = -(\lambda_+\lambda_-)^5 = b^5 \equiv 1 \pmod 5.$$

Furthermore, since we have already calculated all terms of the sequence (x_n) up to $n = 10$, we may easily compute the remainders of successive terms in sequences $(x_{5n}), (x_{5n+1}), (x_{5n+2}), (x_{5n+3}), (x_{5n+4})$, where $n \geq 0$ is a non-negative integer in each case, since $x_{5k+10+r} = x_{5k+5+r} + x_{5k+r}$ by direct application of the previous result with $n = 5k + r$. These five series turn out easily to be modulus-periodic, with respective remainders $(0, 0, 0, \dots)$, $(x_1, 3x_1, 4x_1, 2x_1, x_1, 3x_1, \dots)$, $(x_1, 3x_1, 4x_1, 2x_1, x_1, 3x_1, \dots)$, $(2x_1, x_1, 3x_1, 4x_1, 2x_1, x_1, \dots)$ and $(3x_1, 4x_1, 2x_1, x_1, 3x_1, 4x_1, \dots)$. Note that since x_1 is not divisible by 5, no term in the sequences $(x_{5n+1}), (x_{5n+2}), (x_{5n+3}), (x_{5n+4})$ is divisible by 5, but all terms in the sequence (x_{5n}) are. The conclusion follows.

Starting from any sequence (x_n) that satisfies the hypotheses of the previous claim (and the Fibonacci series clearly does), note that we may then define a new sequence (x'_n) by $x'_n = \frac{x_{5n}}{5}$. By the claim, this new sequence is made of integers, and satisfies the hypotheses of the claim. Hence, for any positive integer multiple of 5, hence of the form $n = b5^a$ where a is a positive integer and b is a positive integer not divisible by 5, by trivial induction $\frac{x_n}{5^a} = x'_b$ is an integer, and is the b-th term in a sequence made of integers that satisfies the hypotheses of the claim, hence not a multiple of 5 since b is not a multiple of 5, or $v_5(x_n) = v_5(n) = a$ whenever n is a multiple of 5. By direct application of the claim, $v_5(x_n) = v_5(n) = 0$ whenever n is not a multiple of 5. The conclusion follows.

O137. Find the locus of centers of the equilateral triangles inscribed in a given square.

Proposed by Oleg Mushkarov, Bulgarian Academy of Sciences, Sofia

First solution by John T. Robinson, Yorktown Heights, NY, USA

By choosing coordinates appropriately, we can assume that the given square is a unit square with vertices at (0,0), (0,1), (1,1), and (1,0).

Claim: The locus of centers of equilateral triangles inscribed in the given unit square is an interior square, centered in the unit square and with sides parallel to the unit square, with vertices at $(1-\sqrt{3}/3, 1-\sqrt{3}/3)$, $(1-\sqrt{3}/3, \sqrt{3}/3)$, $(\sqrt{3}/3, \sqrt{3}/3)$, and $(\sqrt{3}/3, 1-\sqrt{3}/3)$.

Proof: Consider one of the four maximal area inscribed equilateral triangles with vertices at $(\sqrt{3}-1, 0)$, $(0, \sqrt{3}-1)$, and $(1,1)$. Note that the midpoint of the upper side is at $(1/2, \sqrt{3}/2)$. Although the x-coordinate is necessarily 1/2, it is interesting that the y-coordinate is $\sqrt{3}/2$, since another inscribed equilateral triangle, the one with vertices $(1/2, 0)$, $(0, \sqrt{3}/2)$, and $(1, \sqrt{3}/2)$, has the same midpoint for its upper side. This suggests (but does not prove) that various other inscribed equilateral triangles could be obtained by rotating the line passing through the points $(0, \sqrt{3}-1)$ and $(1,1)$ about the point $(1/2, \sqrt{3}/2)$. The approach taken now is to try this out and then check if it works; if it does work out then the guess is proved to be correct.

In more detail, as the previously mentioned line is rotated clockwise, the vertex $(1,1)$ will remain on the right side of the unit square and drop some distance, say y, and the vertex on the left side of the unit square, $(0, \sqrt{3}-1)$, will go up the same distance; therefore the top two vertices will be at $(1, 1-y)$ and $(0, \sqrt{3}-1+y)$. Note that $0 \leq y \leq 1-(\sqrt{3}-1) = 2-\sqrt{3}$. The vertex on the bottom side of the unit square will be at some point $(x, 0)$. We now have (calling the three triangle side lengths a, b, c) the following.

$$a^2 = (1-x)^2 + (1-y)^2 = x^2 + y^2 + 2 - 2x - 2y$$

$$b^2 = 1 + (2 - 2y - \sqrt{3})^2 = 4y^2 + 8 + 4\sqrt{3}y - 8y - 4\sqrt{3}$$

$$c^2 = x^2 + (\sqrt{3} - 1 + y)^2 = x^2 + y^2 + 4 + 2\sqrt{3}y - 2y - 2\sqrt{3}$$

From $a^2 = c^2$ we have

$$x^2 + y^2 + 2 - 2x - 2y = x^2 + y^2 + 4 + 2\sqrt{3}y - 2y - 2\sqrt{3}$$

$$x = \sqrt{3}(1-y) - 1$$

Finally we check if this really *does* give an equilateral triangle after all by comparing a^2 and b^2 with this expression for x substituted in the expression for a^2 above; we find

$$\begin{aligned} a^2 &= (\sqrt{3}(1-y)-1)^2 + y^2 + 2 - 2(\sqrt{3}(1-y)-1) - 2y \\ &= 3 + 3y^2 + 1 - 2\sqrt{3} - 6y + 2\sqrt{3}y + y^2 + 2 - 2\sqrt{3} + 2\sqrt{3}y + 2 - 2y \\ &= 4y^2 + 8 - 4\sqrt{3} - 8y + 4\sqrt{3}y = b^2 \end{aligned}$$

showing that the inscribed triangle is in fact equilateral. Could there be any other equilateral triangles with an upper side? No, for the following reason: if instead of the above we assume $(1,1)$ goes to $(1, 1-y_1)$ and $(0, \sqrt{3}-1)$ goes to $(0, \sqrt{3}-1+y_2)$, then we have 2 equations (e.g. $a^2 = b^2$, $a^2 = c^2$) in 3 unknowns, which means there must be a one-paramter family of solutions, which is in fact what we have already derived.

Finally, the center of the triangle with vertices at $(1, 1-y)$, $(0, \sqrt{3}-1+y)$, and $(\sqrt{3}(1-y)-1, 0)$ is now found to be

$$\text{center} \;=\; \left(\frac{\sqrt{3}(1-y)}{3}, \frac{\sqrt{3}}{3} \right).$$

As y varies from $2-\sqrt{3}$ to 0, the result is a line segment with endpoints $((3-\sqrt{3})/3, \sqrt{3}/3)$ and $(\sqrt{3}/3, \sqrt{3}/3)$. Rotating the unit square into its three other orientations (ninety agrees apart) establishes the claimed result.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Assume wlog that the sides of the square are lines $x = \pm 1$ and $y = \pm 1$ on a given system of coordinates with origin at the center of the square. If the center and circumradius of the inscribed triangle are respectively (x, y) and R, the vertices of the triangle are, for a given angle α, $(x + R\cos\alpha, y + R\sin\alpha)$, $(x + R\cos(\alpha + 120°), y + R\sin(\alpha + 120°))$ and $(x + R\cos(\alpha - 120°), y + R\sin(\alpha - 120°))$. Note that there are two possible positions for the triangle: either two vertices are on two opposite sides and the third vertex is on one of the other two sides, or one vertex coincides with one vertex of the square, and the other two vertices lie on the two sides of the square that do not share that vertex of the square; this is clearly so because the altitude of the triangle is less than its side, hence a side of the triangle cannot lie on a side of the square. Assuming wlog that the vertices of the triangle lie on lines $y = \pm 1$ and $x = 1$ (we may always rotate $90°$ the square and the triangle around the origin of coordinates without changing the problem), we find conditions

$$y + R\sin(\alpha + 120°) = -y - R\sin(\alpha - 120°) = 1; \qquad x + R\cos\alpha = 1.$$

From the first condition, we conclude that $R\cos\alpha\sin 120^\circ = 1$, hence $x = 1 - \frac{1}{\sin 120^\circ} = -\frac{2-\sqrt{3}}{\sqrt{3}}$, independently on R and α. Note that when one of the vertices of the triangle is at vertex $(-1,1)$ of the square, we find the additional condition

$$2 - \frac{2}{\sqrt{3}} = 1 + x = -R\cos(\alpha + 120^\circ) = \frac{\sqrt{3}}{2} R\sin\alpha + \frac{1}{\sqrt{3}},$$

$$y = \frac{1}{2} R\sin\alpha = \frac{2-\sqrt{3}}{\sqrt{3}} = -x.$$

Since we may again rotate the figure by 90° without altering the problem, we may easily see that the locus of the centers of the triangles is a square, determined by lines $x = \pm\frac{2-\sqrt{3}}{\sqrt{3}}$, $y = \pm\frac{2-\sqrt{3}}{\sqrt{3}}$, such that the locus is at one of the vertices of this square when the triangle touches the original square at one of its vertices. Note therefore that the locus is the result of performing a contraction of the original square, with center at the center of the square, and scaling factor $\frac{2-\sqrt{3}}{\sqrt{3}} = \frac{1}{3+2\sqrt{3}}$.

O138. Consider a regular hexagon with side 1. There are only two ways to tile this hexagon with rhombi with side 1. Each of these two tilings involve three rhombi of different types. Prove that no matter how we tile a regular hexagon of side n with rhombi with side 1, the number of rhombi of each type is the same.

Proposed by Ivan Borsenco, MIT and Iurie Boreico, Harvard University, USA

First solution by John T. Robinson, Yorktown Heights, NY, USA

Suppose a regular hexagon with sides of length n, oriented so that it has vertical left and right sides (and therefore a single vertex at the top and another at the bottom), is triangulated with unit side triangles each having a vertical edge. Call a triangle with a vertex to the left of the vertical edge an L triangle, and a triangle with a vertex to the right of the vertical edge an R triangle. The three types of rhombi are then formed by joining left to right L and R triangles at the vertical edge, R and L triangles at the lower edge, and R and L triangles at the upper edge. Call these three rhombi types (respectively) F (for flat, since the leftmost vertex is level with the rightmost vertex), D (for down, since the right vertical edge is lower than the left vertical edge), and U (for up, since the right vertical edge is higher than the left vertical edge).

Consider now the n R triangles on the left side of the hexagon. Each can only be part of a U or D rhombus. Similarly, to the right of this U or D rhombus, there can only be another U or D rhombus. Continuing, the result is a path starting from the left side of the hexagon and ending somewhere (as far as has been determined at this point) on the right side of the hexagon. This path could have one or more upper corners, as shown in the figure (part a) for the path starting from the R triangle labelled 1 (and similarly for the other paths).

Next note that each upper corner can be eliminated by repeated application of the tiling-preserving operations illustrated in the figure (part b). Also note that we can't have an unending sequence of U,D rhombi pairs below an upper corner, since when we run into one of the lower hexagon sides the only rhombus that will fit with the side is a F rhombus, therefore any such sequence has to end with an F rhombus. By tiling-preserving it is meant that if the hexagon is tiled before the operation, it is still tiled afterwards, and only the rhombi indicated in the figure are changed (essentially the F rhombus at the bottom is moved to the top, and all U,D pairs are changed to D,U pairs).

Considering now the path starting at 1 as shown in the figure (part c), the result is a path that has no upper corners and therefore has exactly one bottom corner, ending somewhere on the right side of the hexagon. Can this path end anywhere other than at $1'$, say at some right side triangle k' for $k > 1$? No, because then there is only one possible path starting at $(k-1)'$ going back to

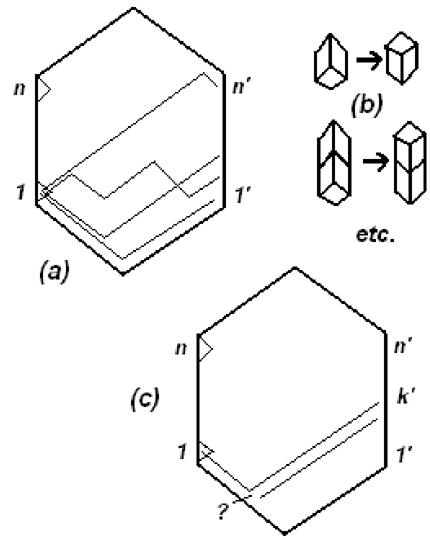


Figure for Problem O138

the left, but when this path runs into the bottom left side of the hexagon we are left with an isolated triangle. The same reasoning applies to the paths starting at 2,3,...,n. That is each path that starts at k $(1 \leq k \leq n)$ must end at k'.

From the preceding we see that there are at least $2n^2$ rhombi of types U and D (that is, $2n$ Us and Ds for each of n paths). Let f, u, and d be the number of rhombi of types F, U, and D. We have

$$f + u + d = 3n^2 \text{ and}$$

$$u + d \geq 2n^2 .$$

Substituting $u + d = 3n^2 - f$ gives

$$3n^2 - f \geq 2n^2 ; \;\; f \leq n^2 .$$

By rotational symmetry we also have $u \leq n^2$ and $d \leq n^2$. Since $f+u+d = 3n^2$, the only solution is

$$f = u = d = n^2$$

as was to be shown.

Notes on the problem: this problem has previously appeared in the literature, and the usual approach seems to be to establish a bijection between such tilings and plane partitions contained in a cube of side n (these can be thought of as irregular staircases consisting of stacks of unit cubes, starting at one corner of a cube and ending at the opposite corner, more or less). Another approach to proving the result was investigated by Dijkstra. The intent here was to give a different (hopefully correct, and possibly original) proof.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Number counterclockwise the sides of the hexagon as $1, 2, 3, 4, 5, 6$. Call rhombi of type A, B, C respectively, those whose sides are parallel to sides $2, 3, 5, 6$, $1, 3, 4, 6$ and $1, 2, 4, 5$. Rotate now the hexagon such that the sides $1, 4$ are horizontal, side 1 being at the bottom and side 4 being at the top. For each one of the segments of length 1 in which side 1 is divided, a rhombus of type B or C must be used in the tiling such that its bottom horizontal side coincides with said segment. Consider now the top horizontal side of each one of these n rhombi. Clearly, this segment must also be the bottom horizontal side of another rhombus of type B or C, and so on, until reaching the n segments of length 1 in which the top horizontal side of the hexagon is reached. There are clearly at least $2n^2$ rhombi of types B or C that must be used in the tiling along this path. Rotating the hexagon such that sides 2 and 3 are horizontal and at the bottom, clearly there must also be at least $2n^2$ rhombi of types C or A that must be used to cover the horizontal segments of length 1 of side 2, and at least $2n^2$ rhombi of types A or B that must be used to cover the horizontal segments of length 1 of side 3. Since each rhombus may be counted at most twice in this way (eg, each rhombi of type A may be counted when covering sides 2 and 3, but not when covering side 1), then the total number of rhombi is at least $\frac{3 \cdot 2n^2}{2} = 3n^2$. Now, this means that the total surface covered by these rhombi is at least $6n^2$ times the area of a unit triangle, and the area of the hexagon is exactly the area of $6n^2$ unit triangles; just consider the hexagon divided in 6 equal triangles of sidelength n, where the sides of these 6 triangles are the sides of the hexagon plus its main diagonals. Therefore, all inequalities must be equalities, ie, each rhombus has been counted exactly twice, or $2n^2 = a + b = b + c = c + a$, yielding trivially $a = b = c = n^2$. The conclusion follows.