An original method of proving inequalities

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In this paper we present an original method for proving inequalities.

Problem (Bulgarian TST 2003). *Let a, b, and c be positive real numbers whose sum is 3. Prove that*

$$\frac{a}{b^2+1}+\frac{b}{c^2+1}+\frac{c}{a^2+1} \geq \frac{3}{2}$$

All contestants who solved the problem found the following computational solution.

Solution 1. Clearing denominators, the inequality becomes $2(a^3c^2+b^3a^2+c^3b^2+a^3+b^3+c^3+ac^2+ba^2+cb^2+a+b+c) \geq 3(a^2b^2c^2+a^2b^2+b^2c^2+c^2a^2+a^2+b^2+1)$. Substituting 3 for $a+b+c$, the inequality can be broken into

$$\begin{aligned} \frac{3}{2}(a^3c^2+ac^2) &\geq 3a^2c^2 \text{ (by AM-GM) and the 2 permutations} \\ a^3+a^3+1 &\geq 3a^2 \text{ (by AM-GM) and the 2 permutations} \end{aligned}$$

$$\frac{1}{2}(a^3c^2+ac^2+b^3a^2+ba^2+c^3b^2+cb^2) \geq \frac{1}{2}\cdot 6a^{\frac{4}{3}}b^{\frac{4}{3}}c^{\frac{4}{3}} \geq 3a^2b^2c^2,$$

the last inequality being true because $abc \leq 1$, which follows from $a+b+c=3$ and the AM-GM inequality.

Solution 2. The inequality $\frac{a}{b^2+1}+\frac{b}{c^2+1}+\frac{c}{a^2+1} \geq \frac{3}{2}$ is equivalent to $a-\frac{a}{b^2+1}+b-\frac{b}{c^2+1}+c-\frac{c}{a^2+1} \leq \frac{3}{2}$, so $\frac{ab^2}{b^2+1}+\frac{bc^2}{c^2+1}+\frac{ca^2}{a^2+1} \leq \frac{3}{2}$. Because $a^2+1 \geq 2a$ (and the two permutations), it follows that the left hand side is less than or equal to $\frac{1}{2}(ab+bc+ca) \leq \frac{3}{2}$, since $3(ab+bc+ca) \leq (a+b+c)^2 = 9$.

From the second solution we find the following problem:

Problem. *Let n be an integer greater than 3 and let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n=2$. Find the minimum of the expression $\frac{a_1}{a_2^2+1}+\frac{a_2}{a_3^2+1}+\cdots+\frac{a_n}{a_1^2+1}$.*

Note that the increased number of variables thwarts any attempt to resolve the problem in the manner of the first solution.

Solution. Because $a_1 + a_2 + \cdots + a_n = 2$, the problem is equivalent to finding the maximum of the expression $a_1 - \frac{a_1}{a_2^2+1} + \cdots + a_n - \frac{a_n}{a_1^2+1}$, i.e. of the expression $\frac{a_1a_2^2}{a_2^2+1} + \cdots + \frac{a_na_1^2}{a_1^2+1}$. Because $a_1^2 + 1 \geq 2a_1, \cdots, a_n^2 + 1 \geq 2a_n$, the expression does not exceed $\frac{a_1a_2^2}{2a_1} + \cdots + \frac{a_na_1^2}{2a_1} = \frac{1}{2}(a_1a_2 + \cdots + a_na_1)$.

For the final step, the following result is useful:

Lemma. *If $n \geq 4$, then for all $a_1, a_2, \cdots, a_n \geq 0$,*

$$4(a_1a_2 + \cdots + a_{n-1}a_n + a_na_1) \leq (a_1 + a_2 + \cdots + a_n)^2.$$

Proof. Let $f(a_1, a_2, \cdots, a_n) = 4(a_1a_2 + \cdots + a_na_1) - (a_1 + \cdots + a_n)^2$. We prove by induction on n that $f(a_1, a_2, \cdots, a_n) \leq 0$.

For $n = 4$ the inequality is $4(a_1 + a_3)(a_2 + a_4) \leq (a_1 + a_2 + a_3 + a_4)^2$, which is a direct consequence of the AM-GM inequality. For the inductive step, let $a_{n-1} = \min\{a_1, a_2, \cdots, a_n\}$. Then

$$\begin{aligned}
&f(a_1, a_2, \cdots, a_n) - f(a_1, \cdots, a_{n-2}, a_{n-1} + a_n) \\
&= 4(a_{n-1}a_n + a_1a_n - a_{n-2}(a_{n-1} + a_n) - (a_{n-1} + a_n)a_1) \\
&= -4(a_{n-2}a_{n-1} + (a_{n-2} - a_{n-1})a_n + a_1a_{n-1}) \\
&\leq 0
\end{aligned}$$

Hence, $f(a_1, a_2, \cdots, a_n) \leq f(a_1, a_2, \cdots, a_{n-2}, a_{n-1} + a_n)$. By the inductive hypothesis, this expression is at most 0, and the conclusion follows. □

Coming back to the problem, we have

$$\begin{aligned}
\frac{1}{2}(a_1a_2 + \cdots + a_{n-1}a_n + a_na_1) &= \frac{4(a_1a_2 + \cdots + a_{n-1}a_n + a_na_1)}{8} \\
&\leq \frac{(a_1 + a_2 + \cdots + a_n)^2}{8} = \frac{2^2}{8} = \frac{1}{2}.
\end{aligned}$$

Hence $\frac{a_1a_2^2}{a_2^2+1} + \cdots + \frac{a_na_1^2}{a_1^2+1} \leq \frac{1}{2}$, so $\frac{a_1}{a_2^2+1} + \frac{a_2}{a_3^2+1} + \cdots + \frac{a_{n-1}}{a_n^2+1} + \frac{a_n}{a_1^2+1} \geq \frac{3}{2}$. Equality holds when, for example, $a_1 = a_2 = 1, a_3 = \cdots = a_n = 0$, so the minimum is indeed $\frac{3}{2}$.