On some geometric inequalities

Tran Quang Hung

Abstract

In this article we use a purely algebraic inequality to prove a variety of geometric inequalities.

1 Introduction

In the recently published article: *An unexpectedly useful inequality* by Pham Huu Duc [1], the following inequality was proved

$$(b+c)x+(c+a)y+(a+b)z \geq 2\sqrt{(xy+yz+zx)(ab+bc+ca)}\ \forall a,b,c,x,y,z \geq 0.$$

The inequality was presented along with its algebraic applications. This inequality not only has many applications in algebra but also it has many applications in geometry. We start with a nice proof of this result that appeared in [2]:

Proposition 1. For all real numbers a,b,c,x,y,z such that $ab+bc+ca \geq 0$ and $xy+yz+zx \geq 0$ the following inequality holds

$$(b+c)x+(c+a)y+(a+b)z \geq 2\sqrt{(xy+yz+zx)(ab+bc+ca)}.$$

Proof. Using Cauchy-Schwarz inequality we get

$$\begin{aligned}
&(b+c)x+(c+a)y+(a+b)z = (a+b+c)(x+y+z)-(ax+by+cz)\\
&= \sqrt{[a^2+b^2+c^2+2(ab+bc+ca)][x^2+y^2+z^2+2(xy+yz+zx)]}-(ax+by+cz)\\
&\geq 2\sqrt{(xy+yz+zx)(ab+bc+ca)}+\sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)}-(ax+by+cz)\\
&\geq 2\sqrt{(xy+yz+zx)(ab+bc+ca)}.
\end{aligned}$$

The next inequality can be proved as a corollary:

Corollary 1. For all real positive numbers a,b,c,x,y,z the following inequality is true

$$\frac{x}{y+z}\,a+\frac{y}{z+x}\,b+\frac{z}{x+y}\,c \geq \sqrt{3(ab+bc+ca)}.$$

Proof. Let us replace in Proposition 1 (x,y,z) with $\left(\frac{x}{y+z},\frac{y}{z+x},\frac{z}{x+y}\right)$. Note that

$$\frac{xy}{(z+x)(z+y)}+\frac{yz}{(x+y)(x+z)}+\frac{zx}{(y+z)(y+x)} \geq \frac{3}{4},$$

and the conclusion follows.

Proposition 2. Let P be a point in the plane of triangle ABC, then

$$\frac{PA \cdot PC}{bc} + \frac{PC \cdot PA}{ca} + \frac{PA \cdot PB}{ab} \geq 1.$$

where a, b, c are the sides of the triangle.

Proof. There are many ways to prove this inequality; we use complex numbers. Let the complex coordinates of A, B, C and P be $A(a), B(b), C(c)$ and $P(p)$, respectively. Using identity

$$(b-c)(p-b)(p-c)+(c-a)(p-c)(p-a)+(a-b)(p-a)(p-b) = (a-b)(b-c)(c-a),$$

we have

$$\begin{aligned}
&BC \cdot PB \cdot PC + CA \cdot PC \cdot PA + AB \cdot PA \cdot PB \\
&\qquad = |(b-c)(p-b)(p-c)| + |(c-a)(p-c)(p-a)| + |(a-b)(p-a)(p-b)| \\
&\qquad \geq |(b-c)(p-b)(p-c) + (c-a)(p-c)(p-a) + (a-b)(p-a)(p-b)| \\
&\qquad = |(a-b)(b-c)(c-a)| = AB \cdot BC \cdot CA.
\end{aligned}$$

Dividing both sides by $AB \cdot BC \cdot CA$ we get

$$\frac{PB \cdot PC}{bc} + \frac{PC \cdot PA}{ca} + \frac{PA \cdot PB}{ab} \geq 1.$$

Note that the equality holds if and only if $P = H$, where H is the orthocenter of triangle ABC.

Let us combine the ideas of the first two propositions in the following statement:

Proposition 3. Let P be a point in the plane of triangle ABC, and let x, y, z be real numbers such that $xy + yz + zx \geq 0$. Then

$$(y+z)\frac{PA}{a} + (z+x)\frac{PB}{b} + (x+y)\frac{PC}{c} \geq 2\sqrt{xy+yz+zx}.$$

Proof. We apply Proposition 1 for $\left(\frac{PA}{a}, \frac{PB}{b}, \frac{PC}{c}\right)$ and (x, y, z) to get

$$\begin{aligned}
&(y+z)\frac{PA}{a} + (z+x)\frac{PB}{b} + (x+y)\frac{PC}{c} \\
&\quad \geq 2\sqrt{(xy+yz+zx)\left(\frac{PA \cdot PC}{bc} + \frac{PC \cdot PA}{ca} + \frac{PA \cdot PB}{ab}\right)} \\
&\quad \geq 2\sqrt{xy+yz+zx},
\end{aligned}$$

as desired.

We continue with a few classical problems that can be solved with the help of the established results.

2 Applications

Problem 1. Consider triangle ABC and a point P in its plane. Prove that

$$\frac{PA}{a} + \frac{PB}{b} + \frac{PC}{c} \geq \sqrt{3}.$$

Solution. Plugging $x = y = z = 1$ in the Proposition 3, we get

$$2\left(\frac{PA}{a} + \frac{PB}{b} + \frac{PC}{c}\right) \geq 2\sqrt{3},$$

and we are done.

Problem 2. Consider triangle ABC and a point P in its plane. Prove that

$$a \cdot PA + b \cdot PB + c \cdot PC \geq 4K_{ABC},$$

where K_{ABC} is the area of triangle ABC.

Solution. Let a, b, c be the triangle's sides. Denote $x = \frac{b^2+c^2-a^2}{2}, y = \frac{a^2+c^2-b^2}{2}, z = \frac{a^2+b^2-c^2}{2}$. Then

$$xy + yz + zx = \frac{2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)}{4} = 4K_{ABC}^2 \geq 0.$$

Hence using Proposition 3 for these (x, y, z) we get

$$a^2 \cdot \frac{PA}{a} + b^2 \cdot \frac{PB}{b} + c^2 \cdot \frac{PC}{c} = a \cdot PA + b \cdot PB + c \cdot PC \geq 4K_{ABC}.$$

Problem 3. Let P be a point in the plane of triangle ABC. Prove that

$$PA + PB + PC \geq 6r,$$

where r is the inradius of the incircle of triangle ABC.

Solution. Let $x = s - a, y = s - b, z = s - c$, where a, b, c are the triangle's sides and s is the semiperimeter. Then using Proposition 3 we get

$$PA + PB + PC \geq 2\sqrt{(s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a)}.$$

Recall that $(s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a) = r(4R + r)$, where R and r are the circumradius and inradius, respectively. Thus we get a much stronger inequality

$$PA + PB + PC \geq 2\sqrt{r(4R + r)}.$$

3 References

[1] Pham Huu Duc, An unexpectedly useful inequality, Mathematical Reflections 2008, Issue 1.

[2] Manlio, Blackmouse, Canhang, Mathlinks Forum 2007, http://www.mathlinks.ro/viewtopic.php?t=187355.

[3] Dragoslav S. Mitrinovic, J. Pecaric,V. Volenec, Recent Advances in Geometric Inequalities.

[4] Bottema, Oene; Djordjevic, R.Z.; Janic, R.; Mitrinovic, D.S.; and Vasic, P.M., Geometric Inequalities.

Tran Quang Hung
Ha Noi National University, Vietnam
email: hung100486@yahoo.com