A PAIR OF INEQUALITIES FOR THE SUMS OF THE MEDIANS AND SYMMEDIANS OF A TRIANGLE

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In this note $A,\ B,\ C$ denote the vertices of a triangle, a, b, c the sides and (m_a, m_b, m_c), (w_a, w_b, w_c) and (s_a, s_b, s_c) the medians, angle-bisectors, and symmedians, respectively.

It is well-known (see [1].8.20) that

$$w_a + w_b + w_c \leq m_a + m_b + m_c \tag{1}$$

Our aim is to establish an inverse inequality to (1). We define

$$\Omega = \frac{1}{2}(|a-b| + |b-c| + |c-a|) = \max(a,b,c) - \min(a,b,c).$$

Theorem. In every triangle

$$s_a + s_b + s_c \leq m_a + m_b + m_c \leq s_a + s_b + s_c + 2\Omega. \tag{2}$$

$$m_a + m_b + m_c \leq w_a + w_b + w_c + \Omega. \tag{3}$$

Equalities holds if and only if the triangle is equilateral.

Proof. Taking into account that $\dfrac{m_a}{s_a} = \dfrac{b^2+c^2}{2bc}$ one finds that $m_a \geq s_a$, with equality if and only if $b = c$. Therefore

$$s_a + s_b + s_c \leq m_a + m_b + m_c \tag{4}$$

with equality for $a = b = c$.

Let $m_a = AA_1,\ w_a = AA_2,\ s_a = AA_3$. According to the theorem of Steiner

$$\frac{BA_3}{A_3C} = \frac{c^2}{b^2};\ \text{hence, } BA_3 = \frac{ac^2}{b^2+c^2}.$$

In the same manner

$$A_3C = \frac{ab^2}{b^2+c^2}.$$

Therefore, in the triangle AA_1A_3 we find that

$$0 \leq m_a - s_a \leq |A_1A_3| = \frac{a|b^2-c^2|}{2(b^2+c^2)}.$$

Since $a(b+c) < (b+c)^2 \leq 2(b^2+c^2)$, we obtain

$$\frac{a|b^2-c^2|}{2(b^2+c^2)} \leq |b-c|,$$

with equality if and only if $b = c$. This implies

$$m_a - s_a \leq |b - c|$$

hence

$$m_a + m_b + m_c - (s_a + s_b + s_c) \leq |a - b| + |b - c| + |c - a|. \tag{5}$$

From (4) and (5), we conclude (2). Observe that the equality occurs if and only if the triangle is equilateral.

In order to prove (3), let us consider the triangle AA_1A_2. We have

$$A_1A_2 = \frac{a|b - c|}{2(b + c)}$$

which together with $a < b + c$ implies

$$m_a - w_a \leq |A_1A_2| \leq \frac{|b - c|}{2}$$

with equality only when $b = c$. In conclusion,

$$m_a + m_b + m_c - (w_a + w_b + w_c) \leq \frac{1}{2}(|a - b| + |b - c| + |c - a|)$$

and the theorem is proved.

Using same technique one can prove that the following inequality:

$$g_a + g_b + g_c + \Omega \geq n_a + n_b + n_b$$

where $g_a,\ g_b,\ g_c$ and $n_a,\ n_b,\ n_c$ denote the Gergonne cevians and the Nagel cevians respectively.

References

[1] BOTTEMA O., DJORDJEVIC R. Z., JANICR R., MITRINOVIC D. S., VASIC P. M.: Geometric Inequalities, Wolters - Noordhoff Publishing House, Groningen, 1969