Problems for Mathematical Reflections 5

Juniors

J25. Let k be a real number different from 1. Solve the system of equations

$$\begin{cases} (x+y+z)(kx+y+z) = k^3+2k^2 \\ (x+y+z)(x+ky+z) = 4k^2+8k \\ (x+y+z)(x+y+kz) = 4k+8. \end{cases}$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

J26. A line divides an equilateral triangle into two parts with the same perimeter and having areas S_1 and S_2, respectively. Prove that

$$\frac{7}{9} \leq \frac{S_1}{S_2} \leq \frac{9}{7}$$

Proposed by Bogdan Enescu, "B.P. Hasdeu" National College, Romania

J27. Consider points M, N inside the triangle ABC such that $\angle BAM = \angle CAN, \angle MCA = \angle NCB, \angle MBC = \angle CBN$. M and N are izogonal points. Suppose $BMNC$ is a cyclic quadrilateral. Denote T the circumcenter of $BMNC$, prove that $MN \perp AT$.

Proposed by Ivan Borsenco, University of Texas at Dallas

J28. Let p be a prime such that $p \equiv 1 (\bmod 3)$ and let $q = \lfloor \frac{2p}{3} \rfloor$. If

$$\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(q-1)q} = \frac{m}{n}$$

for some integers m and n, prove that $p | m$.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

J29. Find all rational solutions of the equation

$$\left\{x^2\right\} + \{x\} = 0.99$$

Proposed by Bogdan Enescu, "B.P. Hasdeu" National College, Romania

J30. Let a, b, c be three nonnegative real numbers. Prove the inequality

$$\frac{a^3+abc}{b+c} + \frac{b^3+abc}{a+c} + \frac{c^3+abc}{a+b} \geq a^2+b^2+c^2.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

Seniors

S25. Prove that in any acute-angled triangle ABC,

$$\cos^3 A + \cos^3 B + \cos^3 C + \cos A \cos B \cos C \geq \frac{1}{2}$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

S26. Consider a triangle ABC and let I_a be the center of the circle that touches the side BC at A' and the extensions of sides AB and AC at C' and B', respectively. Denote by X the second intersections of the line $A'B'$ with the circle with center B and radius BA' and by K the midpoint of CX. Prove that K lies on the midline of the triangle ABC corresponding to AC.

Proposed by Liubomir Chiriac, Princeton University

S27. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\sqrt[3]{\frac{a^2+bc}{b^2+c^2}} + \sqrt[3]{\frac{b^2+ca}{c^2+a^2}} + \sqrt[3]{\frac{c^2+ab}{a^2+b^2}} \geq \frac{9\sqrt[3]{abc}}{a+b+c}$$

Proposed by Pham Huu Duc, Australia

S28. Let M be a point in the plane of triangle ABC. Find the minimum of

$$MA^3 + MB^3 + MC^3 - \frac{3}{2}R \cdot MH^2,$$

where H is the orthocenter and R is the circumradius of the triangle ABC.

Proposed by Hung Quang Tran, Hanoi, Vietnam

S29. Prove that for any real numbers a, b, c the following inequality holds

$$3(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ac + a^2) \geq a^3b^3 + b^3c^3 + c^3a^3.$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

S30. Let $p > 5$ be a prime number and let

$$S(m) = \sum_{i=0}^{\frac{p-3}{2}} \frac{m^{2i}}{2i+1}.$$

Prove that the numerator of $S(1)$ is divisible by p if and only if the numerator of $S(3)$ is divisible by p.

Proposed by Iurie Boreico, Moldova

Undergraduate

U25. Calculate the following sum $\displaystyle\sum_{k=0}^{\infty} \frac{2k+1}{(4k+1)(4k+3)(4k+5)}$.

Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania

U26. Let $f : [a, b] \to \mathbb{R}$ ($0 < a < b$) be a continuous function on $[a, b]$ and differentiable on (a, b). Prove that there is a $c \in (a, b)$ such that

$$\frac{2}{a-c} < \ f'(c) < \frac{2}{b-c}$$

Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania

U27. Let k be a positive integer. Evaluate

$$\int_0^1 \left\{\frac{k}{x}\right\}^2 dx$$

where $\{a\}$ is the *fractional part* of a.

Proposed by Ovidiu Furdui, Western Michigan University

U28. Let f be the function defined by

$$f(x) = \sum_{n \geq 1} |\sin n| \cdot \frac{x^n}{1 - x^n}.$$

Find in a closed form a function g such that $\displaystyle\lim_{x \to 1^-} \frac{f(x)}{g(x)} = 1$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

U29. Let A be a square matrix of order n, for which there is a positive integer k such that $kA^{k+1} = (k+1)A^k$. Prove that $A - I_n$ is invertible and find its inverse.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

U30. Let n be a positive integer. What is the largest cardinal of a subgroup G of $GL_n(\mathbb{Z})$ such that for any matrix $A \in G$, all elements of $A - I_n$ are even?

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Olympiad

O25. For any triangle ABC, prove that
$$\cos\frac{A}{2}\cot\frac{A}{2}+\cos\frac{B}{2}\cot\frac{B}{2}+\cos\frac{C}{2}\cot\frac{C}{2}\geq\frac{\sqrt{3}}{2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)$$

Proposed by Darij Grinberg, Germany

O26. Consider a triangle ABC and let O be its circumcenter. Denote by D the foot of the altitude from A and by E the intersection of AO and BC. Suppose tangents to the circumcircle of triangle ABC at B and C intersect at T and that AT intersects this circumcircle at F. Prove that the circumcircles of triangles DEF and ABC are tangent.

Proposed by Ivan Borsenco, University of Texas at Dallas

O27. Let a, b, c be positive numbers such that $abc = 4$ and $a, b, c > 1$. Prove that

$$(a-1)(b-1)(c-1)(\frac{a+b+c}{3}-1)\leq(\sqrt[3]{4}-1)^4$$

Proposed by Marian Tetiva, Birlad, Romania

O28. Let ϕ be Euler's totient function. Find all natural numbers n such that the equation $\phi(\ldots(\phi(x))) = n$ (ϕ iterated k times) has solutions for any natural k.

Proposed by Iurie Boreico, Moldova

O29. Let $P(x)$ be a polynomial with real coefficients of degree n with n distinct real zeros $x_1 < x_2 < ... < x_n$. Suppose $Q(x)$ is a polynomial with real coefficients of degree $n-1$ such that it has only one zero on each interval (x_i, x_{i+1}) for $i = 1, 2, ..., n-1$. Prove that the polynomial $Q(x)P'(x) - Q'(x)P(x)$ has no real zero.

Proposed by Khoa Lu Nguyen, Massachusetts Institute of Technology

O30. Prove that equation

$$\frac{1}{x_1^2}+\frac{1}{x_2^2}+...+\frac{1}{x_n^2}=\frac{n+1}{x_{n+1}^2}$$

has a solution in positive integers if and only of $n \geq 3$.

Proposed by Oleg Mushkarov, Bulgarian Academy of Sciences, Sofia