Proving inequalities using linear functions

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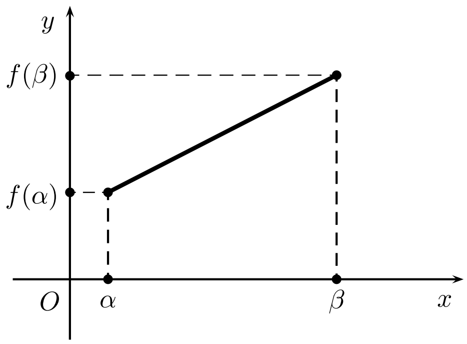
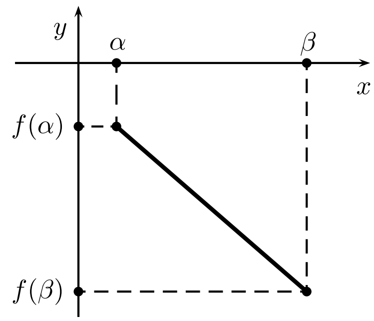
In this note we present a method for proving a class of inequalities based on the simple observation that if a linear real function attaines values of the same sign at the end points of an interval, all of its values are of the same sign on the whole interval. For this purpose, it's crucial to view an expression as a linear function in certain group of the variables.





Theorem 1. *If the function $f(x) = ax + b$ has $f(\alpha) \geq 0$, and $f(\beta) \geq 0$ then $f(x) \geq 0$, $\forall x \in [\alpha, \beta]$.*

This property of linear functions is well illustrated in the figures and it has easily understood geometric interpretation. We will illustrate the idea with two problems.

Problem 1. Let x, y, z be non-negative real numbers such that $x + y + z = 3$, prove that

$$x^2 + y^2 + z^2 + xyz \geq 4.$$

Solution. We rewrite the desired inequality in the form

$$(y + z)^2 - 2yz + x^2 + xyz \geq 4,$$

or

$$yz(x - 2) + 2x^2 - 6x + 5 \geq 0.$$

Set $yz = w$, and view the expression on the left hand as a linear function of w, that is

$$f(w) = (x - 2)w + 2x^2 - 6x + 5.$$

Now we need to find all possible values of w. By AM-GM inequaltiy, $yz \le (y+z)^2/4$. That is

$$w \le (3-x)^2/4,$$

we also have $w \ge 0$ by hypothesis. By theorem 1, it's sufficient to show that $f(0) \ge 0$ and $f(w_0) \ge 0$, where $w_0 = (3-x)^2/4$. It's easy to check that

$$f(0) = 2x^2 - 6x + 5 = 2\Big(x - \frac{3}{2}\Big)^2 + \frac{1}{5} \ge 0,$$
$$f(w_0) = \frac{1}{4}(x-1)^2(x+2) \ge 0.$$

The proof is complete. The equalitiy holds if and only if all the three numbers are equal to 1. □

Actually, in order to determine equality cases, we find all sets of values of the variables so that the values of the linear function at endpoints of interval in question are all zeros. For instance, in problem 1, equation $f(0) = 0$ has no real solution; while we can find out that $f(w_0) = 0$ has root $x = 1$ which leads to $yz \le 1$, $y + z = 2$. Plugging $z = 2 - y$ into the inequality $yz \le 1$, we obtain

$$-y^2 + 2y - 1 \le 0.$$

This yields $y = 1$, then we have $z = 1$.

The next problem has two equality cases determined by solving two equations $f(0) = 0$ and $f(w_0) = 0$. This is also the case when you solve Schur inequality in three variables (the last exercise).

Problem 2. Prove that if x, y, z are non-negative real numbers such that $x + y + z = 1$, then

$$4(x^3 + y^3 + z^3) + 15xyz \ge 1.$$

Determine when equality holds.

Solution. Note that we have the following identity

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b),$$

so that the claimed inequality is equivalently written as

$$(y+z)^3 - 3yz(y+z) + x^3 + \frac{15}{4}xyz \ge \frac{1}{4}.$$

Using the fact that three numbers add up to 1, the above inequality reads

$$(1-x)^3 + yz\Big(\frac{27}{4}x - 3\Big) + x^3 - \frac{1}{4} \ge 0.$$

Put $yz = w$ and consider the left hand member as a linear function of w.

$$f(w) = \Big(\frac{27}{4}x - 3\Big)w + (1-x)^3 + x^3 - \frac{1}{4}.$$

We have $w \leq (1-x)^2/4$, by AM-GM inequality and we also have $w \geq 0$ by hypothesis. By theorem 1, we shall show that $f(0) \geq 0$ and $f(w_0) \geq 0$. Indeed, we have

$$\begin{aligned} f(0) &= (1-x)^3 + x^3 - \frac{1}{4} \\ &= \frac{3}{4}(2x-1)^2; \\ 16f(w_0) &= 16(1-x)^3 + (1-x)^2(27x-12) + 16x^3 - 4 \\ &= 3x(3x-1)^2. \end{aligned}$$

The claimed inequality follows. The proof is complete. Equality occurs if $(x,y,z) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, or any permution of the triple $(x,y,z) = (0, \frac{1}{2}, \frac{1}{2})$. $\square$

Exercise 1 (Mihai Piticari, Dan Popescu, Old & New Inequalities)**.** Prove that

$$5(a^2+b^2+c^2) \leq 6(a^3+b^3+c^3) + 1,$$

where a, b, c are positive real numbers such that $a+b+c=1$.

Exercise 2 (Sefket Arslanagic, CRUX MATHS)**.** Prove the inequaltiy

$$\frac{1}{1-xy} + \frac{1}{1-yz} + \frac{1}{1-zx} \leq \frac{27}{8},$$

where x, y, z are positive real numbers such that $x+y+z=1$.

Exercise 3 (BMO 1979)**.** Let x, y, z be positive real numbers such that $x+y+z=1$, prove that

$$7(xy+yz+zx) \leq 2 + 9xyz.$$

Exercise 4 (USAMO 1979)**.** Prove that if $x, y, z > 0$ and $x+y+z=1$, then $x^3+y^3+z^3+6xyz \geq 1/4$.

Exercise 5 (IMO 1984)**.** Prove that if $x, y, z > 0$ and $x+y+z=1$, then $xy+yz+zx-2xyz \leq 7/27$.

Exercise 6 (I. Schur)**.** Prove that if $a, b, c \geq 0$, then

$$a^3+b^3+c^3+3abc \geq a^2(b+c) + b^2(c+a) + c^2(a+b).$$

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