Some remarks on problem U23

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Abstract

We give three different proofs to identity (2). Some applications are also presented.

1 Introduction

In [3] we proposed the following problem: Evaluate the sum

$$\sum_{k=0}^{n-1} \frac{1}{1+8\sin^2\left(\frac{k\pi}{n}\right)}. \tag{1}$$

In the first issue of the journal "Mathematical Reflections" no solutions were given to this problem. The main purpose of this note is to present three different proofs to the following general result (see also [4]).

Theorem. *For each real number $a \in \mathbb{R} \setminus \{-1, 1\}$ the following equality holds:*

$$\sum_{k=0}^{n-1} \frac{1}{a^2 - 2a\cos\frac{2k\pi}{n} + 1} = \frac{n(a^n+1)}{(a^2-1)(a^n-1)}. \tag{2}$$

Let us remark that the special case $n = 7$ and $a \in (-1, 1)$ makes the object of Problem 49 in the Longlisted Problems of IMO1988 (see [5] page 217).

In the last section we present four applications of our main result.

2 Three proofs to the theorem

Proof 1. (Dorin Andrica and Mihai Piticari) Let $P \in \mathbb{R}[X]$ be a polynomial with real coefficients of degree $n-1$, $P = a_0 + a_1X + \cdots + a_{n-1}X^{n-1}$. If $\alpha \in U_n$ is an n^{th} root of unity, then we have

$$\begin{aligned}
|P(\alpha)|^2 &= P(\alpha)\cdot\overline{P(\alpha)} = P(\alpha)\cdot P(\overline{\alpha}) = P(\alpha)\cdot P\left(\frac{1}{\alpha}\right)\\
&= (a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1})\left(a_0 + \frac{a_1}{\alpha} + \cdots + a_{n-1}\frac{1}{\alpha^{n-1}}\right)\\
&= a_0^2 + a_1^2 + \cdots + a_{n-1}^2 + \sum_{k=1}^{n-1} A_k\alpha^k + \sum_{j=1}^{n-1} B_j\frac{1}{\alpha^j},
\end{aligned}$$

where coefficients A_k, and B_j are different from zero. From relation (see [2, Proposition 3, page 46])

$$\sum_{\alpha \in U_n} \alpha^k = \begin{cases} n, & \text{if } n|k \\ 0, & \text{otherwise} \end{cases}$$

we get the following nice formula

$$\sum_{\alpha \in U_n} |P(\alpha)|^2 = n(a_0^2 + a_1^2 + \cdots + a_{n-1}^2). \tag{3}$$

In order to prove (2) we consider the polynomial

$$P = 1 + aX + \cdots + a^{n-1}X^{n-1} = \frac{a^n X^n - 1}{aX - 1}.$$

Applying formula (3) it follows that

$$\sum_{\alpha \in U_n} |P(\alpha)|^2 = n(1 + a^2 + \cdots + a^{2n-2}) = n\frac{a^{2n} - 1}{a^2 - 1} \tag{4}$$

On the other hand,

$$|P(\alpha)|^2 = \left| \frac{a^n \alpha^n - 1}{a\alpha - 1} \right|^2 = \frac{(a^n - 1)^2}{(a\alpha - 1)(a\overline{\alpha} - 1)} = \frac{(a^n - 1)^2}{a^2 - 2\text{Re }\alpha \cdot a + 1}$$

and formula (2) follows.

Proof 2. (Gabriel Dospinescu, Ecole Normale Superieure, Paris, France) We will use the so called Poisson kernel formula

$$\frac{1 - r^2}{1 - 2r\cos t + r^2} = \sum_{m=-\infty}^{+\infty} r^{|m|} z^m, \quad \text{where} \quad |r| < 1, \tag{5}$$

and $z = \cos t + i \sin t$. Applying it in our case we have

$$\sum_{k=0}^{n-1} \frac{1 - a^2}{1 - 2a\cos\dfrac{2k\pi}{n} + a^2} = \sum_{k=0}^{n-1} \left(\sum_{m=-\infty}^{+\infty} a^{|m|} e^{im\frac{2k\pi}{n}} \right)$$

$$= \sum_{m=-\infty}^{+\infty} a^{|m|} \left(\sum_{k=0}^{n-1} \left(e^{\frac{2\pi i m}{n}} \right)^k \right) = \sum_{m \in \mathbb{Z}} n a^{|m|}$$

$$= n \sum_{j \in \mathbb{Z}} a^{n|j|} = n \frac{1 - a^{2n}}{(1 - a^n)^2} = n \frac{1 + a^n}{1 - a^n},$$

and formula (2) follows.

Proof 3. (Dorin Andrica) Consider the polynomial $P \in \mathbb{C}[X]$ having the factorization $P = \prod_{k=1}^{n}(X^2 + a_k X + b)$. Then

$$\begin{aligned}\frac{P'}{P} &= \sum_{k=1}^{n} \frac{2X + a_k}{X^2 + a_k X + b} = \frac{1}{X} \sum_{k=1}^{n} \frac{2X^2 + a_k X + b - b}{X^2 + a_k X + b} \\ &= X \sum_{k=1}^{n} \frac{1}{X^2 + a_k X + b} + \frac{n}{X} - \frac{b}{X} \sum_{k=1}^{n} \frac{1}{X^2 + a_k X + b} \\ &= \frac{X^2 - b}{X} \sum_{k=1}^{n} \frac{1}{X^2 + a_k X + b} + \frac{n}{X},\end{aligned}$$

and we derive the formula

$$\frac{XP' - nP}{(X^2 - b)P} = \sum_{k=1}^{n} \frac{1}{X^2 + a_k X + b}. \tag{6}$$

For the polynomial

$$P = (X^n - 1)^2 = \prod_{k=0}^{n-1} \left(X^2 - 2X \cos \frac{2k\pi}{n} + 1 \right)$$

we have $P' = 2nX^{n-1}(X^n - 1)$, hence

$$\frac{XP' - nP}{(X^2 - 1)P} = \frac{2nX^n(X^n - 1) - n(X^n - 1)^2}{(X^2 - 1)(X^n - 1)^2} = n \frac{X^n + 1}{(X^2 - 1)(X^n - 1)},$$

and (2) follows from (6).

3 Applications

Application 1. In order to evaluate the sum (1) we take $a = 2$ in identity (2) and get

$$\sum_{k=0}^{n-1} \frac{1}{4 - 4\cos \dfrac{2k\pi}{n} + 1} = \frac{n(2^n + 1)}{3(2^n - 1)},$$

that is

$$\sum_{k=0}^{n-1} \frac{1}{1 + 8\sin^2 \left(\dfrac{k\pi}{n} \right)} = \frac{n(2^n + 1)}{3(2^n - 1)}. \tag{7}$$

Application 2. Let us use the identity (2) to prove the equality

$$\sum_{k=1}^{n-1} \frac{1}{\sin^2\left(\dfrac{k\pi}{n}\right)} = \frac{(n-1)(n+1)}{3}. \tag{8}$$

Indeed, the identity (2) is equivalent to

$$\sum_{k=1}^{n-1} \frac{1}{a^2 - 2a\cos\dfrac{2k\pi}{n} + 1} = \frac{n(a^n+1)}{(a^2-1)(a^n-1)} - \frac{1}{(a-1)^2}.$$

Taking in the right hand side the limit for $a \to 1$ we get

$$\lim_{a\to 1}\left[\frac{n(a^n+1)}{(a^2-1)(a^n-1)} - \frac{1}{(a-1)^2}\right]$$

$$= \frac{1}{2n}\lim_{a\to 1}\frac{(n-1)a^{n+1} - (n+1)a^n + (n+1)a - n + 1}{(a-1)^3}$$

$$= \frac{1}{2n}\lim_{a\to 1}\frac{(n+1)(n-1)a^n - n(n+1)a^{n-1} + (n+1)}{3(a-1)^2}$$

$$= \frac{1}{2n}\lim_{a\to 1}\frac{(n+1)(n-1)na^{n-2}(a-1)}{6(a-1)} = \frac{(n-1)(n+1)}{12}.$$

It follows that

$$\lim_{a\to 1}\sum_{k=1}^{n-1} \frac{1}{a^2 - 2a\cos\dfrac{2k\pi}{n} + 1} = \frac{(n-1)(n+1)}{12},$$

that is

$$\sum_{k=1}^{n-1} \frac{1}{4\sin^2\left(\dfrac{k\pi}{n}\right)} = \frac{(n-1)(n+1)}{12}.$$

Remarks. 1) Using

$$\sin^2\left(\frac{k\pi}{n}\right) = \sin^2\frac{(n-k)\pi}{n}, \quad k = 1, 2, \ldots, n-1,$$

from identity (8) we get:

a) If n is odd, $n = 2m+1$, then

$$\sum_{k=1}^{m} \frac{1}{\sin^2\left(\dfrac{k\pi}{2m+1}\right)} = \frac{2m(2m+2)}{6} = \frac{m(2m+2)}{3}$$

This is equivalent to

$$\sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1} = \frac{m(2m-1)}{3}. \tag{9}$$

b) If n is even, then

$$\sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{k\pi}{n}\right)} = \frac{1}{2}\left(\frac{n^2-1}{3} - 1\right) = \frac{n^2-4}{6}. \tag{10}$$

2) A different method to prove (9) is given in [1, page 147]. Consider the trigonometric equation $\sin(2m+1)x = 0$, with roots

$$\frac{\pi}{2m+1}, \frac{2\pi}{2m+1}, \dots, \frac{m\pi}{2m+1}$$

Expressing $\sin(2m+1)x$ in terms of $\sin x$ and $\cos x$, we obtain

$$\begin{aligned}\sin(2m+1)x &= \binom{2m+1}{1}\cos^{2m} x \sin x - \binom{2m+1}{3}\cos^{2m-2} x \sin^3 x + \dots \\ &= \sin^{2m+1} x \left(\binom{2m+1}{1}\cot^{2m} x - \binom{2m+1}{3}\cot^{2m-2} x + \dots\right)\end{aligned}$$

Set $x = \dfrac{k\pi}{2m+1}$, $k = 1, 2, \dots, m$. Because $\sin^{2m+1} x \neq 0$, we have

$$\binom{2m+1}{1}\cot^{2m} x - \binom{2m+1}{3}\cot^{2m-2} x + \dots = 0.$$

Substituting $y = \cot^2 x$ yields

$$\binom{2m+1}{1} y^m - \binom{2m+1}{3} y^{m-1} + \dots = 0,$$

an algebraic equation with roots

$$\cot^2 \frac{\pi}{2m+1}, \cot^2 \frac{2\pi}{2m+1}, \dots, \cot^2 \frac{m\pi}{2m+1}$$

Using the relations between coefficients and roots, we obtain

$$\sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1} = \frac{\binom{2m+1}{3}}{\binom{2m+1}{1}} = \frac{m(2m-1)}{3}$$

Application 3. We use now the result in our Theorem to prove the following identities:

If n is odd and $n \geq 3$, then

$$\sum_{k=1}^{\frac{n-1}{2}} \frac{1}{\cos^2\left(\dfrac{k\pi}{n}\right)} = \frac{n^2-1}{2} \tag{11}$$

If n is even and $n \geq 4$, then

$$\sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\cos^2\left(\dfrac{k\pi}{n}\right)} = \frac{n^2-4}{6} \tag{12}$$

This is Problem $O71$ and three solutions to it are proposed in "Mathematical Reflections". Here we present a different one directly derived from (2).

In order to prove the identity (11) we note that from (2) we have

$$\begin{aligned}\sum_{k=0}^{n-1} \frac{1}{2+2\cos\dfrac{2k\pi}{n}} &= \lim_{a\to-1} \frac{n(a^n+1)}{(a^2-1)(a^n-1)} \\ &= \frac{n}{4}\lim_{a\to-1}\frac{a^n+1}{a+1} = \frac{n^2}{4}.\end{aligned}$$

That is

$$\sum_{k=0}^{n-1} \frac{1}{\cos^2\left(\dfrac{k\pi}{n}\right)} = n^2,$$

therefore

$$\sum_{k=1}^{n-1} \frac{1}{\cos^2\left(\dfrac{k\pi}{n}\right)} = n^2-1.$$

Using

$$\cos^2\left(\frac{k\pi}{n}\right) = \cos^2\left(\frac{(n-k)\pi}{n}\right), \quad k = 1, \ldots, n-1,$$

the identity (11) follows.

To prove identity (12) by the same method is much more complicated. From (2) we have

$$\sum_{\substack{k=0 \\ k\neq\frac{n}{2}}}^{n-1} \frac{1}{2+2\cos\dfrac{2k\pi}{n}} = \lim_{a\to-1}\left[\frac{n(a^n+1)}{(a^2-1)(a^n-1)} - \frac{1}{(a+1)^2}\right]$$

$$= \lim_{t\to 0}\left\{\frac{n[(t-1)^n+1]}{t(t-2)[(t-1)^n-1]}-\frac{1}{t^2}\right\}$$

$$= -\frac{1}{2}\lim_{t\to 0}\frac{n[(t-1)^n+1]t-(t-2)[(t-1)^n-1]}{t^2[(t-1)^n-1]}$$

$$= -\frac{1}{2}\lim_{t\to 0}\frac{nt\left(2-nt+\dfrac{n(n-1)}{2}t^2+t^3f(t)\right)}{t^2\left(-nt+\dfrac{n(n-1)}{2}t^2+t^3h(t)\right)}$$

$$-\frac{(t-2)\left(-nt+\dfrac{n(n-1)}{2}t^2-\dfrac{n(n-1)(n-2)}{6}+t^3+t^4g(t)\right)}{t^2\left(-nt+\dfrac{n(n-1)}{2}t^2+t^3h(t)\right)}$$

$$= -\frac{1}{2}\cdot\frac{\dfrac{n^2(n-1)}{2}-2\dfrac{n(n-1)(n-2)}{6}-\dfrac{n(n-1)}{2}}{-n}=\frac{n^2-1}{12},$$

since f,g,h are polynomials in t, and so $f(0),g(0),h(0)$ are finite. It follows that

$$\sum_{\substack{k=1\\ k\neq \frac{n}{2}}}^{n-1}\frac{1}{\cos^2\left(\dfrac{k\pi}{n}\right)}=\frac{n^2-4}{3}$$

and then using again

$$\cos^2\left(\frac{k\pi}{n}\right)=\cos^2\left(\frac{(n-k)\pi}{n}\right),\quad k=1,\dots,n-1,$$

we get identity (10). Let us remark that the previous limit was proposed in Problem U72.

Remark. It is possible to derive the relation (12) directly from (8) by using

$$\cos^2\frac{k\pi}{n}=\sin^2\left(\frac{\pi}{2}-\frac{k\pi}{n}\right)=\sin^2\frac{(n-2k)\pi}{2n}=\sin^2\frac{\left(\frac{n}{2}-k\right)\pi}{n},\ k=1,\dots,\frac{n}{2}.$$

Application 4. If $x\in\mathbb{R}$ and $|x|>1$, then

$$\sum_{k=0}^{n-1}\frac{1}{x-\cos\dfrac{2k\pi}{n}}=\frac{2n\left(x+\sqrt{x^2-1}\right)\left[\left(x+\sqrt{x^2-1}\right)^n+1\right]}{\left[\left(x+\sqrt{x^2-1}\right)^2-1\right]\left[\left(x+\sqrt{x^2-1}\right)^n-1\right]}\tag{13}$$

Indeed, we have

$$\sum_{k=0}^{n-1}\frac{1}{a^2-2a\cos\dfrac{2k\pi}{n}+1}=\frac{1}{2a}\sum_{k=0}^{n-1}\frac{1}{\dfrac{1}{2}\left(a+\dfrac{1}{a}\right)-\cos^2\dfrac{2k\pi}{n}}.$$

Let $x = \frac{1}{2}\left(a + \frac{1}{a}\right)$. Then $a = x + \sqrt{x^2 - 1}$, and from identity (2) we get

$$\frac{1}{2a}\sum_{k=0}^{n-1} \frac{1}{x - \cos\dfrac{2k\pi}{n}} = \frac{n(a^n + 1)}{(a^2 - 1)(a^n - 1)},$$

that is the relation (13).

Taking $x = 2$ in (13), $x = 2$, we get

$$\sum_{k=0}^{n-1} \frac{1}{1 + 2\sin^2\left(\dfrac{k\pi}{n}\right)} = \frac{2n\left(2+\sqrt{3}\right)\left[\left(2+\sqrt{3}\right)^n + 1\right]}{\left(3+3\sqrt{3}\right)\left[\left(2+\sqrt{3}\right)^n - 1\right]}. \tag{14}$$

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