## Solutions for Mathematical Reflections 6(2006)

### Juniors

J31. Find the least perimeter of a right-angled triangle whose sides and altitude are integers.

Proposed by Ivan Borsenco, University of Texas at Dallas

*Solution by Jos Alejandro Samper Casas, Colegio Helvetia de Bogota, Colombia*

**Solution.** The answer for the least possible perimeter is 60. It holds for a right-angled triangle $(15, 20, 25)$, whose altitude is 12.

Let $x, y, z$ be a Pythagorean triple with $z$ the hypotenuse and let $h$ be the altitude of the triangle. Let $d = gcd(x, y, z)$ be the greatest common divisor of $x, y, z$. We have that $x = d \cdot a$ , $y = d \cdot b$ and $z = d \cdot c$ for $a, b, c$ a primitive Pythagorean triple. Now calculating the area of the triangle in two ways we obtain that $h = \frac{xy}{z} = \frac{abd}{c}$. Using the fact that $gcd(ab, c) = 1$, we get $c|d$ which tells us that $c \leq d$, since both are positive integers. Because the perimeter is equal to $d(a + b + c) \geq c(a + b + c)$, we can minimize it be taking $c = d$. Then the altitude of a right-angled triangle having sides $ca, cb, c^2$ (with $c^2$ the hypotenuse) is $ab$ and is an integer.

We know that $a, b, c$ is a primitive Pythagorean triple if and only if there exist $m, n \in \mathbb{Z}^+$ such that $gcd(m, n) = 1$, $m \not\equiv n \pmod 2$ $m > n > 0$ that satisfy $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$. Replacing in (1), we notice that all we need to find is the minimum value of

$$p = (m^2 + n^2)(2m^2 + 2mn).$$

Clearly $m > n > 0$, therefore $m \geq 2$ and $n \geq 1$. Thus

$$p \geq (2^2 + 1)(2 \cdot 2^2 + 2 \cdot 2 \cdot 1) = 60.$$

Now the triangle with sides $(15, 20, 25)$ satisfies all the conditions of the original problem and its perimeter is 60. The problem is solved.

*Also solved by Ashay Burungale, India*

J32. Let $a$ and $b$ be real numbers such that

$$9a^2 + 8ab + 7b^2 \le 6.$$

Prove that $7a + 5b + 12ab \le 9$.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

*First solution by Ashay Burungale, India*

**Solution.** We start from the initial inequality

$$9a^2 + 8ab + 7b^2 \le 6.$$

By making simple transformations the inequality is equivalent to

$$2(a-b)^2 + 7(a - \frac{1}{2})^2 + 5(b - \frac{1}{2})^2 + 7a + 5b + 12ab \le 9.$$

Clearly from this we can derive that

$$7a + 5b + 12ab \le 9.$$

*Second solution by Daniel Campos Salas, Costa Rica*

**Solution.** Let $a = x + \dfrac{1}{2}$ and $b = y + \dfrac{1}{2}$. The given condition turns to

$$9x + 9x^2 + 4x + 4y + 8xy + 7y + 7y^2 = 13x + 9x^2 + 11y + 7y^2 + 8xy \le 0.$$

The result follows from the following chain of inequalities:

$$\begin{aligned}
7a + 5b + 12ab &= 7\left(x + \frac{1}{2}\right) + 5\left(y + \frac{1}{2}\right) + 12\left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) \\
&= 9 + 13x + 11y + 12xy \\
&\le 9 + 13x + 11y + 8xy + 4x^2 + y^2 \\
&\le 9 + (13x + 9x^2 + 11y + 7y^2 + 8xy) \\
&\le 9,
\end{aligned}$$

and we are done.

J33. Consider the sequence: 31, 331, 3331,... whose $n$th term has $n$ $3s$ followed by a 1. Prove that this sequence contains infinitely many composite numbers.

Proposed by Wing Sit, University of Texas at Dallas

*First solution by Jose Alejandro Samper Casas, Colegio Helvetia de Bogota, Colombia*

**Solution.** Let $a_n$ denote the n-th term of the sequence. We will prove that there exist infinitely many $j$ so that $31|a_j$.

We first note that

$$a_n = 1 + \sum_{k=1}^{n} 3 \cdot 10^k = \frac{10^{n+1} - 7}{3}.$$

Now $a_1 = 31 = \frac{10^2-7}{3}$. Fermat's Little Theorem tells us that :

$$31|10^{30} - 1 \implies 31|10^{30k} - 1 \implies 31|10^{30k+2} - 10^2$$
$$\implies 31|10^{30k+2} - 10^2 + (10^2 - 7) = 10^{30k+2} - 7$$

Because $gcd(31, 3) = 1$ we have that $31|\frac{10^{30k+2}-7}{3} = a_{30k+1}$ and we are done.

*Second solution by Aleksandar Ilic, Serbia*

**Solution.** Let's compute $n$-th term:

$$a_n = \underbrace{33\ldots33}_{n}1 = 1 + 3 \cdot 10 \cdot \underbrace{11\ldots11}_{n} = 1 + 30 \cdot \frac{10^n - 1}{9} = \frac{10^{n+1} - 7}{3}.$$

It's easy to see that all numbers are relatively prime with 2, 3 and 5, and that array is strictly increasing. Let $p$ be any prime that divides some member $a_n$ of the sequence (we can take $p = 31$).

$$p \mid a_n \quad \leftrightarrow \quad 10^{n+1} \equiv 7 (\text{mod } p).$$

From Fermat's Little Theorem we get $10^{p-1} \equiv 1 \mod p$. Now we can consider members with indexes of form $n + k \cdot (p - 1)$, and the problem is solved - because they are composite.

$$p \mid a_{n+k\cdot(p-1)} \quad \leftrightarrow \quad 10^{n+1+k(p-1)} \equiv 10^{n+1} \cdot \left(10^{p-1}\right)^k \equiv 7 (\text{mod } p).$$

*Also solved by Ashay Burungale, India*

J34. Let $ABC$ be a triangle and let $I$ be its incenter. Prove that at least one of $IA, IB, IC$ is greater than or equal to the diameter of the incircle of $ABC$.

Proposed by Magkos Athanasios, Kozani, Greece

*First solution by Andrea Munaro, Italy*

**Solution.** Let $r$ be the inradius of triangle $ABC$. Assume contradiction, that $IA < 2r$, $IB < 2r$, $IC < 2r$. Adding them we get

$$IA + IB + IC < 6r.$$

Denoting by $A, B, C$ the angles of a triangle and using the fact that $I$ is the intersection point of the angle-bisectors we get

$$IA = \frac{r}{\sin\frac{A}{2}},\ IB = \frac{r}{\sin\frac{B}{2}},\ IC = \frac{r}{\sin\frac{C}{2}}.$$

Thus

$$r\left(\frac{1}{\sin\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}} + \frac{1}{\sin\frac{C}{2}}\right) < 6r \text{ or } \frac{1}{\sin\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}} + \frac{1}{\sin\frac{C}{2}} < 6$$

Consider the function $f(x) = \frac{1}{\sin x}$. $f(x)$ is convex because

$$f''(x) = \frac{1+\cos^2 x}{\sin^3 x} > 0 \text{ in } (0, \pi).$$

Using Jensen's inequality

$$\frac{1}{\sin\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}} + \frac{1}{\sin\frac{C}{2}} \geq 3\left(\frac{1}{\sin\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)}\right) = \frac{3}{\sin\frac{\pi}{6}} = 6.$$

Contradiction. Hence at least one of $IA, IB, IC$ is greater than or equal to $2r$, and we are done.

*Second solution by Courtis G. Chryssostomos, Larissa, Greece*

Assume contradiction, that $AI < 2r$, $BI < 2r$, $CI < 2r$.
Using the fact that $AI \sin \frac{A}{2} = r$, $BI \sin \frac{B}{2} = r$, $CI \sin \frac{C}{2} = r$ we get

$$\frac{1}{2} < \sin \frac{A}{2}, \ \frac{1}{2} < \sin \frac{B}{2}, \ \frac{1}{2} < \sin \frac{C}{2}$$

or

$$\frac{3}{2} < \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2}.$$

Take the function $f(x) = \sin \frac{x}{2}$, $x \in (0, \pi)$. Calculating first and second derivative we get $f'(x) = \frac{1}{2} \cos \frac{x}{2}$, $f"(x) = -\frac{1}{4} \sin \frac{x}{2} < 0$. The function is concave down, thus from Jensen Inequality we have

$$f(A) + f(B) + f(C) \leq 3 \cdot f\left(\frac{A+B+C}{3}\right), \ \Rightarrow$$

$$\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \leq 3 \cdot \sin \left(\frac{A+B+C}{3}\right) = \frac{3}{2}.$$

Contradiction, and the conclusion follows.

*Third solution by Ashay Burungale, India*

**Solution.** Recall Erdos-Mordell inequality: for any point $P$ inside the triangle we have

$$AP + BP + CP \geq 2(d_a + d_b + d_c),$$

where $d_a, d_b, d_c$ are distances from the point $P$ to the triangle's sides. Using this inequality for $P = I$ we have

$$AI + BI + CI \geq 6r.$$

Thus there exist one from $IA, IB, IC$ that is greater than or equal to the diameter of the incircle.

*Fourth solution by Aleksandar Ilic, Serbia*

**Solution.** Let projection incenter $I$ on side $AB$ be point $C'$. From sine theorem on triangle $\triangle AIC'$ we have that $AI\sin\frac{\alpha}{2} = r$. Analogously, we get $BI\sin\frac{\beta}{2} = r$ and $CI\sin\frac{\gamma}{2} = r$. We will prove inequality

$$AI \cdot BI \cdot CI \geq (2r)^3,$$

from which at least one of $AI, BI, CI$ is greater that $2r$. Inequality is equivalent to

$$1 \geq 8\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.$$

Function $\sin x$ is concave (non-convex) on interval $[0,\pi]$, so we can use Jensen's inequality and Power Mean:

$$\frac{\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}}{3} \leq \sin\frac{\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2}}{3} = \sin\frac{\pi}{6} = \frac{1}{2}.$$

$$\frac{\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}}{3} \geq \sqrt[3]{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}.$$

Combining above inequalities, the problem is solved.

*Also solved by Daniel Campos Salas, Costa Rica*

J35. Prove that among any four positive integers greater than or equal to 1 there are two, say $a$ and $b$, such that

$$\frac{\sqrt{(a^2-1)(b^2-1)}+1}{ab} \geq \frac{\sqrt{3}}{2}$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

*First solution by Daniel Campos Salas, Costa Rica*

**Solution.** Let $(a, b, c, d)$ be these numbers. Since these are greater than or equal to 1 there exist $\alpha, \beta, \theta, \delta$, in $(0, 90°]$, such that

$$\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right) = (\sin\alpha, \sin\beta, \sin\theta, \sin\delta).$$

Then,

$$N_{a,b} = \frac{\sqrt{(a^2-1)(b^2-1)}+1}{ab} = \sqrt{\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{b^2}\right)} + \frac{1}{ab} =$$
$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta).$$

By the Pigeonhole Principle we have that at least two elements of the set $[\alpha, \beta, \theta, \delta]$ lie on one of the intervals $(0°, 30°], (30°, 60°], (60°, 90°]$, which implies that there are two of them such that their difference is least than or equal to $30°$. Since the function $\cos x$ is strictly decreasing on $[0°, 90°]$, and there exist two elements, say $\alpha$ and $\beta$ such that $30° \geq \alpha - \beta \geq 0°$, we conclude that $N_{a,b} = \cos(\alpha-\beta) \geq \cos 30° = \dfrac{\sqrt{3}}{2}$.

*Second solution by Aleksandar Ilic, Serbia*

**Solution.** Let these four numbers be $a$, $b$, $c$ and $d$. From the condition in the problem we can find numbers $x, y, z, t \in [0, \frac{\pi}{2}]$, such that

$$\frac{1}{a} = \cos x, \frac{1}{b} = \cos y, \frac{1}{c} = \cos z, \text{ and } \frac{1}{d} = \cos t.$$

$$\frac{\sqrt{(a^2-1)(b^2-1)}+1}{ab} = \sqrt{1-\cos^2 x}\cdot\sqrt{1-\cos^2 y} + \cos x\cos y = \cos(x-y).$$

By the Pigeonhole Principle, among four numbers $x, y, z, t$, there are two with absolute value less or equal than $\frac{\pi}{6}$. Let $|x-y| \leq \dfrac{\pi}{6}$ then $\cos(x-y) \geq \cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$, and we are done.

*Third solution by David E. Narvaez, Universidad Tecnologica de Panama*

**Solution.** Let the four numbers be $n_1$, $n_2$, $n_3$ and $n_4$. Since $\sec\theta$ reaches every real number greater than or equal to 1 for $0 \le \theta$, there exist four positive numbers $\theta_1$, $\theta_2$, $\theta_3$ and $\theta_4$ such that

$$n_1 = \sec\theta_1,\ n_2 = \sec\theta_2,\ n_3 = \sec\theta_3,\ n_4 = \sec\theta_4$$

and $0 \le \theta_1, \theta_2, \theta_3, \theta_4 < \frac{\pi}{2}$. Then, since the interval $\left[0, \frac{\pi}{2}\right)$ can be divided in 3 subintervals of length $\frac{\pi}{6}$, by the pigeonhole principle, there exist two numbers, say $\theta_a$ and $\theta_b$, that are in the same interval. Thus, $|\theta_a - \theta_b| \le \frac{\pi}{6}$, and for those two numbers

$$\begin{aligned}
\frac{\sqrt{(a^2-1)(b^2-1)}+1}{ab} &= \frac{\sqrt{(\sec^2\theta_a - 1)(\sec^2\theta_b - 1)}+1}{\sec\theta_a\sec\theta_b} \\
&= \frac{\tan\theta_a\tan\theta_b + 1}{\sec\theta_a\sec\theta_b} \\
&= \sin\theta_a\sin\theta_b + \cos\theta_a\cos\theta_b \\
&= \cos(\theta_a - \theta_b) \\
\frac{\sqrt{(a^2-1)(b^2-1)}+1}{ab} &\ge \cos\frac{\pi}{6}
\end{aligned}$$

since $\cos\theta$ is decreasing in the interval $\left[0, \frac{\pi}{2}\right]$.

J36. Let $a, b, c, d$ be integers such that $gcd(a, b, c, d) = 1$ and $ad - bc \neq 0$. Prove that the greatest possible value of $gcd(ax + by, cx + dy)$ over all pairs $(x, y)$ of relatively prime is $|ad - bc|$.

Proposed by Iurie Boreico, Moldova

*Solution by Iurie Boreico, Moldova*

**Solution.** We have

$$gcd(ax + by, cx + dy)|(ax + by)c - (cx + dy)a = (bc - ad)y.$$

Analogously, $gcd(ax + by, cx + dy)|(bc - ad)x$ and as $x, y$ are coprime we deduce $gcd(ax + by, cx + dy)|\ |ad - bc|$.

Now we prove that $gcd(ax + by)$ can be $|ad - bc|$ using the following lemma:

*Lemma.* If $\gcd(a, b, c, d) = 1$ then there is an $l$ such that

$$gcd(a + bl, c + dl) = 1.$$

*Proof.* Indeed, $gcd(a + bl, c + dl)|(a + bl)d - (c + dl)b = ad - bc$. Now if $p|ad - bc$ then $p$ does not divide one of $a, b, c, d$. If $p$ does not divide one of $a, c$ then taking $l$ divisible by $p$ ensures that $p$ does not divide $gcd(a + bl, c + dl)$ otherwise $p$ does not divide one of $b, d$ thus taking $l$ not divisible by $p$ ensures that $p$ does not divide $gcd(a + bl, c + dl)$. So for some $l$ $gcd(a + bl, c + dl)$ is not divisible by $p$. Considering all primes $p$ dividing $ad - bc$ and using the Chinese Remainder Theorem we deduce the lemma.

Applying the lemma to $(d, -b, -c, a)$ we find an $l$ such that

$$gcd(d - bl, -c + al) = 1.$$

Then $a(d - bl) + b(-c + al) = ad - bc$ and $c(d - bl) + d(-c + al) = l(ad - bc)$, thus taking $x = d - bl, y = -c + al$ we deduce the desired result.

## Seniors

S31. Prove that for all positive real numbers $a, b, c$ the following inequality

$$\frac{1}{a+b+c}\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \geq \frac{1}{ab+bc+ca}+\frac{1}{2(a^2+b^2+c^2)}.$$

Proposed by Pham Huu Duc, Australia

*Solution by David E. Narvaez, Universidad Tecnologica de Panama*

**Solution.** Since the inequality to prove is homogeneous, we can assume without loss of generality that $a+b+c=1$, in which case we have to prove that

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c} \geq \frac{1}{ab+bc+ac}+\frac{1}{2\left(a^2+b^2+c^2\right)}$$

or, in an equivalent form,

$$\frac{1+(ab+bc+ca)}{(ab+bc+ca)-abc} \geq \frac{2\left(a^2+b^2+c^2\right)+(ab+bc+ca)}{2\left(a^2+b^2+c^2\right)(ab+bc+ca)}$$

Let $p=a^2+b^2+c^2$. Then, because $1=(a+b+c)^2=p+2(ab+bc+ca)$, we have $ab+bc+ca=\frac{1-p}{2}$. Then we are left to prove that

$$\frac{1+\frac{1-p}{2}}{\frac{1-p}{2}-abc} \geq \frac{2p+\frac{1-p}{2}}{2p\frac{1-p}{2}}$$

which, by simple algebra, reduces to

$$-\frac{2p^3-5p^2+4p-1}{2(3p+1)} \leq abc$$

Recalling Newton's formula for symmetric polynomials, we have

$$a^3+b^3+c^3-\left(a^2+b^2+c^2\right)(a+b+c)+(a+b+c)(ab+bc+ca)-3abc=0$$

so

$$abc=\frac{a^3+b^3+c^3+(1-3p)}{6}$$

Thus, our last inequality is equivalent to

$$-\frac{2p^3-5p^2+4p-1}{2(3p+1)} \le \frac{2(a^3+b^3+c^3)+(1-3p)}{6}$$

and to

$$-\frac{3p^3-3p^2+6p-2}{3p+1} \le a^3+b^3+c^3$$

as we can easily check.

From Cauchy-Schwarz inequality, we have that

$$\begin{aligned}\left(a^{\frac{1}{2}}\cdot a^{\frac{3}{2}}+b^{\frac{1}{2}}\cdot c^{\frac{3}{2}}+a^{\frac{1}{2}}\cdot c^{\frac{3}{2}}\right)^2 &\le (a+b+c)\left(a^3+b^3+c^3\right)\\ p^2 &\le a^3+b^3+c^3\end{aligned}$$

Hence, it is sufficient to show that

$$-\frac{3p^3-3p^2+6p-2}{3p+1} \le p^2$$

which is equivalent to

$$3p^3-p^2+3p-1 \ge 0$$

and this is true since the polynomial $3x^3-x^2+3x-1=(x^2+1)(3x-1)$ has only one real root at $\frac{1}{3}$, and positive principal coefficient, and from the Quadratic Mean - Arithmetic Mean inequality we have that

$$\begin{aligned}\sqrt{\frac{a^2+b^2+c^2}{3}} &\ge \frac{a+b+c}{3}\\ p &\ge \frac{1}{3}.\end{aligned}$$

The problem is solved.

S32. Let $ABC$ be a triangle and let $P, Q, R$ be three points lying inside $ABC$. Suppose quadrilaterals $ABPQ$, $ACPR$, $BCQR$ are concyclic. Prove that if the radical center of these circles is the incenter $I$ of triangle $ABC$, then the Euler line of the triangle $PQR$ coincides with $OI$, where $O$ is the circumcenter of triangle $ABC$.

Proposed by Ivan Borsenco, University of Texas at Dallas

*First solution by David E. Narvaez, Universidad Tecnologica de Panama*

**Solution.** Let $k$ be the incircle of triangle $ABC$, and let $D$, $E$, $F$ be the points where $k$ touches sides $BC$, $AC$ and $AB$, respectively. Since the radical axis of circles passing through $ABPQ$, $ACPR$ and $BCQR$ are $AP$, $BQ$ and $CR$, if the radical center is $I$, then $P$, $Q$ and $R$ lie on the angle bisectors $AI$, $BI$ and $CI$, respectively.

We will first show that triangles $DEF$ and $PQR$ are homothetic: Since quadrilaterals $ABPQ$ and $BCQR$ are cyclic,

$$m\angle PQI = m\angle BAP = \frac{1}{2}m\angle A,$$

$$m\angle QPI = m\angle ABQ = \frac{1}{2}m\angle B,$$

$$m\angle RQI = m\angle RCB = \frac{1}{2}m\angle C.$$

Thus we have

$$m\angle PQR + m\angle QPI = m\angle PQI + m\angle RQI + m\angle QPI =$$
$$= \frac{1}{2}(m\angle A + m\angle B + m\angle C) = m\angle PQR + m\angle QPI = \frac{\pi}{2}.$$

It follows that $PI$ is perpendicular to $QR$, and since $PI = AI$ and $AI$ is perpendicular to $EF$, then $QR$ is parallel to $EF$. In a similar way we can show that $RP$ is parallel to $FD$ and $PQ$ is parallel to $DE$, which proves our claim.

From our previous observations we can also conclude that $I$ is the orthocenter of triangle $DEF$. But $I$ is the circumcenter of triangle $DEF$, so $I$ lies on the Euler line of both triangles, and since these lines have to be parallel (from the homothety), then this lines coincide.

Then it is sufficient to show that the Euler line of triangle $DEF$ coincides with $OI$, and this is true since $O$ is the circumcenter of the tangential triangle of $DEF$, which is $ABC$, and this point is known to lie on the Euler line of the triangle.

*Second solution by Aleksandar Ilic, Serbia*

**Solution.** From the fact that $I$ is a radical center, we have that points $P$, $Q$ and $R$ are on segments $AI$, $BI$ and $CI$, respectively. Quadrilateral $ABPQ$ is cyclic, so $\measuredangle IPQ = \frac{\beta}{2}$ and $\measuredangle IQP = \frac{\alpha}{2}$. Analogously, $\measuredangle IQR = \frac{\gamma}{2}$ and $\measuredangle IRQ = \frac{\beta}{2}$. Now, we can calculate angles in triangle $\triangle PQR$, so $\measuredangle RPQ = 90^o - \frac{\alpha}{2}$, $\measuredangle PQR = 90^o - \frac{\beta}{2}$ and $\measuredangle QRP = 90^o - \frac{\gamma}{2}$. From equality $\measuredangle IPQ + \measuredangle PQR = 90^o$, we get that $PI \perp RQ$ and point $I$ is orthocenter of triangle $\triangle PQR$.

Let $S$ be circumcenter of triangle $\triangle PQR$, and circumcenters around $ABPQ$, $ACPR$ and $BCQR$ are $C'$, $B'$ and $A'$, respectively. Points $A'$ and $B'$ are lying on perpendicular bisector of $CR$, so $A'B' \parallel PQ$. The same way we get $B'C' \parallel RQ$ and $A'C' \parallel PR$.

From construction we also get that $\measuredangle OB'C' = \measuredangle OC'B' = \frac{\alpha}{2}$ (or $\measuredangle OB'C'$ and $\measuredangle IAC$ are angles with perpendicular legs). This means that point $O$ is circumcenter for triangle $\triangle A'B'C'$. Triangle $\triangle A'B'C'$ is homothetic to triangle $\triangle PQR$, because they have equal opposite angles. Center of homothety is intersection point of $A'P$, $B'Q$ and $C'R$. Point $S$ is orthocenter for triangle $\triangle A'B'C'$ and circumcenter for triangle $\triangle PQR$. So, center of homothety lies on the intersection of $SI$ and $OS$. Because coefficient of homothety is negative, points $S$, $I$ and $O$ are collinear.

S33. Let $a, b, c$ be nonnegative real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} + \frac{4(ab+bc+ca)}{(a+b)(b+c)(a+c)} \geq ab+bc+ca.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

*Solution by Andrei Frimu, Moldova*

**Solution.** In order to simplify the look of the inequality, we make the substitutions $x = \frac{1}{a} = bc$, $y = \frac{1}{b} = ac$, $z = \frac{1}{c} = ab$.

Using $xyz = 1$, we easily get $\dfrac{1}{a^3(b+c)} = \dfrac{x^2}{y+z}$, and analogously other relations, also

$$\frac{4(ab+bc+ca)}{(a+b)(b+c)(c+a)} = \frac{4(x+y+z)}{(x+y)(y+z)(z+x)}$$

The inequality to prove becomes:

$$\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} + \frac{4(x+y+z)}{(x+y)(y+z)(z+x)} \geq x+y+z.$$

Using the fact $(x+y)(y+z)(z+x) = 2+x^2y+y^2x+x^2z+z^2x+y^2z+z^2y$, the inequality becomes

$$\begin{gathered} x^4+y^4+z^4+xyz(x+y+z)+\sum(x^3y+xy^3)+4(x+y+z) \geq \\ 2(x+y+z)+2xyz(x+y+z)+2\sum x^2y^2+\sum(x^3y+xy^3) \end{gathered}$$

or

$$x^4+y^4+z^4+xyz(x+y+z) \geq 2(x^2y^2+y^2z^2+z^2x^2)$$

However, from Schur inequality applied in the following form:

$$x^2(x-y)(x-z)+y^2(y-x)(y-z)+y^2(y-x)(y-z)+z^2(z-x)(z-y) \geq 0,$$

we get

$$x^4+y^4+z^4+xyz(x+y+z) \geq (xy^3+x^3y)+(x^3z+xz^3)+(yz^3+y^3z) \geq$$

$$\geq 2(x^2y^2+x^2z^2+y^2z^2),$$

and the problem is solved.

*Second solution by Ho Phu Thai, Da Nang, Vietnam*

**Solution.** Substitute $a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z}$, we easily change the original inequality to the homogenous form:

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} + \frac{4xyz(x+y+z)}{(x+y)(y+z)(z+x)} \geq x+y+z.$$

Clearing all denominators, we have:

$$x^4 + y^4 + z^4 + xyz(x+y+z) \geq 2(x^2y^2 + y^2z^2 + z^2x^2)$$

By Muirhead's theorem, $2(x^2y^2 + y^2z^2 + z^2x^2) \leq x^3y + xy^3 + y^3z + yz^3 + z^3x + zx^3$, it suffices to show:

$$x^2(x-y)(x-z) + y^2(y-z)(y-x) + z^2(z-x)(z-y) \geq 0,$$

which is exactly the 4-th degree Schur's inequality. Equality occurs if and only if $a = b = c$ or $a \to 0, b = c$.

*Also solved by Daniel Campos Salas, Costa Rica*

S34. Let $ABC$ be an equilateral triangle and let $P$ be a point on its circumcircle. Find all positive integers $n$ such that

$$PA^n + PB^n + PC^n$$

does not depend upon $P$.

Proposed by Oleg Mushkarov, Bulgarian Academy of Sciences, Sofia

*First solution by Hung Quang Tran, Ha Noi National University, Vietnam*

**Solution.** Let $M$ be the midpoint of the small are $BC$ and denote as $a$ the side of an equilateral triangle. We have

$$f(A) = AB^n + AC^n = 2a^n,$$

$$f(M) = MA^n + MB^n + MC^n = \left(\frac{a}{\cos\frac{\pi}{6}}\right)^n + 2\left(\frac{a}{\tan\frac{\pi}{6}}\right)^n =$$

$$= \left(\left(\frac{2}{\sqrt{3}}\right)^n + 2\left(\frac{1}{\sqrt{3}}\right)^n\right)a^n.$$

If $f(P)$ does not depend upon the position of $P$ on the circumference, then $f(M) = f(A)$. It follows that

$$2a^n = \left(\left(\frac{2}{\sqrt{3}}\right)^n + 2\left(\frac{1}{\sqrt{3}}\right)^n\right)a^n, \text{ or } 2^n + 2 = 2\cdot 3^{\frac{n}{2}}.$$

If $n = 1, 3$ the condition doesn't satisfy the property and for $n = 2$ it satisfies. Suppose $n \geq 4$, then consider a function

$$f(x) = 2^x + 2 - 2\cdot 3^{\frac{x}{2}}, \text{ for }, x \geq 4.$$

We have $f'(x) = 2^x \ln 2 - 3^{\frac{x}{2}} \ln 3$ and it is not difficult to see that $f'(4) > 0$ and $f'(x) > 0$, for $x \geq 4$. Therefore $f(x)$ is a strictly increasing function and should have at most one solution. We claim that $x = 4$ is that unique solution and clearly it satisfies the property.

It is remained to find if $n = 2$ and $n = 4$ satisfy the condition of the problem.

If $n = 2$, by Leibnitz Theorem we immediately get that

$$PA^2 + PB^2 + PC^2 = 2a^2.$$

If $n = 4$, (we may assume that $P$ belongs to the small arc of $BC$) then using Ptolemy Theorem and the Law of Cosines we get

$$PA = PB + PC, \text{ and } a^2 = PB^2 + PC^2 + PB \cdot PC.$$

Thus

$$PA^4 + PB^4 + PC^4 = (PB + PC)^4 + PB^4 + PC^4 =$$
$$= 2(PB^4 + 3PB^3PC + 2PB^2PC^2 + 3PBPC^3 + PC^4) =$$
$$= 2(PB^2 + PB \cdot PC + PC^2)^2 = 2a^4.$$

*Second solution by Aleksandar Ilic, Serbia*

**Solution.** Assume that circumradius is $R = 1$. Then the side of equilateral triangle has length $a = \sqrt{3}$. Consider two special positions for point $P$: when $P \equiv A$ and when $P \equiv A'$, where $A'$ is midpoint for smaller arc $BC$. In these cases we have:

$$P \equiv A : \quad PA^n + PB^n + PC^n = 0^n + AB^n + AC^n = 2 \cdot 3^{\frac{n}{2}}$$

$$P \equiv A' : \quad PA^n + PB^n + PC^n = A'A^n + A'B^n + A'C^n = 2^n + 2$$

From condition in the problem we have to find all $n$, such that $3^{\frac{n}{2}} = 2^{n-1} + 1$. This can be true only for even $n$. By checking, we get that equality holds for $n = 2$ and $n = 4$. In case $n \geq 6$, we have $2^{n-1} + 1 > (\sqrt{3})^n$ by trivial induction.

$$(\sqrt{3})^{n+1} = (\sqrt{3})^n \cdot \sqrt{3} < 2^{n-1}\sqrt{3} + \sqrt{3} < 2^n + 1.$$

Now we'll prove that statement of problem is true in case $n = 2$ or $n = 4$. Assume without loss of generality that point $P$ is on smaller arc $BC$. First apply Ptolemy's theorem on quadrilateral $ABPC$:

$$AB \cdot PC + AC \cdot PB = BC \cdot PA \quad \leftrightarrow \quad PB + PC = PA$$

From the Law of Cosines on triangle $\triangle BCM$ we get

$$PB^2 + PC^2 - 2PB \cdot PC \cdot \cos 120^o = PB^2 + PC^2 + PB \cdot PC = a^2.$$

Let's calculate expression for $n = 2$:

$$PA^2 + PB^2 + PC^2 = (PB + PC)^2 + PB^2 + PC^2 =$$
$$= 2(PB^2 + PC^2 + PB \cdot PC) = 2a^2 = \text{const}.$$

In the case $n = 4$ we get:

$$PA^4 + PB^4 + PC^4 = (PB + PC)^4 + PB^4 + PC^4 =$$

$$= 2(PB^4 + PC^4 + 3PB^2 \cdot PC^2 + 2PB^3 \cdot PC + 2PC^3 \cdot PB) =$$
$$= 2(PB^2 + PC^2 + PB \cdot PC)^2 = 2a^4 = \text{const.}$$

*Also solved by Courtis G. Chryssostomos, Larissa, Greece; David E. Narvaez, Universidad Tecnologica de Panama*

S35. Let $ABC$ be a triangle with the largest angle at $A$. On the line $AB$ consider the point $D$ such that $A$ lies between $B$ and $D$ and $AD = \dfrac{AB^3}{AC^2}$. Prove that $CD \le \sqrt{3} \cdot \dfrac{BC^3}{AC^2}$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

*First solution by Ashay Burungale, India*

**Solution.** Denote $AB = c$, $AC = b$, $BC = a$. If angle $A$ is the greatest, then $a \ge \max(b, c)$. From the Law of Cosines we get

$$CD^2 = AC^2 + AD^2 - 2AC \cdot AD \cos \angle CAD =$$

$$= b^2 + \frac{c^6}{b^4} + 2b \cdot \frac{c^3}{b^2} \cdot \left( \frac{b^2 + c^2 - a^2}{2bc} \right) = \frac{b^6 + c^6 + b^2c^2(b^2 + c^2 - a^2)}{b^4}.$$

Thus it is enough to prove that

$$\frac{b^6 + c^6 + b^2c^2(b^2 + c^2 - a^2)}{b^4} \le \frac{3a^6}{b^4},$$

$$b^6 + c^6 + b^2c^2(b^2 + c^2 - a^2) \le 3a^6,$$

or

$$b^6 + c^6 + b^2c^2(b^2 + c^2) \le 3a^6 + a^2b^2c^2.$$

But note that if $a \ge \max(b, c)$, then

$$b^6 + c^6 + b^2c^2(b^2 + c^2) \le b^6 + c^6 + 2a^2b^2c^2 \le 3a^6 + a^2b^2c^2,$$

and we are done.

*Second solution by Daniel Campos Salas, Costa Rica*

**Solution.** From Stewart's theorem we have

$$AD \cdot BC^2 + AB \cdot CD^2 = BD(AB \cdot AD + AC^2), \text{ or}$$

$$AB \left( \frac{AB^2 \cdot BC^2}{AC^2} + CD^2 \right) = (AB + AD)(AB \cdot AD + AC^2) =$$

$$AB \left( 1 + \frac{AB^2}{AC^2} \right) \left( \frac{AB^4 + AC^4}{AC^2} \right).$$

This implies that

$$\begin{aligned} CD^2 &= \frac{(AB^2+AC^2)(AB^4+AC^4)-(AB\cdot BC\cdot AC)^2}{AC^4} \\ &= \frac{AB^6+AC^6+AB^4\cdot AC^2+AB^2\cdot AC^2(AC^2-BC^2)}{AC^4} \\ &\leq \frac{AB^6+AC^6+AB^4\cdot AC^2}{AC^4} \\ &\leq \frac{3BC^6}{AC^4}, \end{aligned}$$

that completes the proof.

*Also solved by Aleksandar Ilic, Serbia*

S36. Let $P$ be a point in the plane of a triangle $ABC$, not lying on the lines $AB, BC$, or $CA$. Denote by $A_b, A_c$ the intersections of the parallels through $A$ to the lines $PB$, $PC$ with the line $BC$. Define analogously $B_a, B_c, C_a, C_b$. Prove thath $A_b, A_c, B_a, B_c, C_a, C_b$ lie on the same conic.

Proposed by Mihai Miculita, Oradea, Romania

*First solution by Ricardo Barosso Campos, Universidad de Sevilla, Spain*

**Solution.** First of all we remind a famous Carnot Theorem:

Let $ABC$ be a triangle and points $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ such that $X_1, X_2 \in BC,\ Y_1, Y_2 \in AC,\ Z_1, Z_2 \in AB$. If the relation

$$\frac{AZ_1}{Z_1B} \cdot \frac{AZ_2}{Z_2B} \cdot \frac{BX_1}{X_1C} \cdot \frac{BX_2}{X_2C} \cdot \frac{CY_1}{Y_1A} \cdot \frac{CY_2}{Y_2A} = 1$$

holds, then $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ lie on a conic.

We want to prove that $A_b, A_c, B_a, B_c, C_a, C_b$ satisfy this condition.

Let $AP \cap BC = A_p$ $, BP \cap AC = B_p$ $, CP \cap AB = C_p$.

Denote $AB = c, BC = a, CA = b$ and

$$BA_p = m, A_pC = n, CB_p = q, B_pA = r, AC_p = s, C_pB = t.$$

Using the similarity of triangles $BC_pC$ and $BAA_C$ we get

$$\frac{BA_c}{BC} = \frac{BA}{BC_P}, \Rightarrow BA_C = \frac{ca}{t}.$$

Analogously,

$$BC_a = \frac{ca}{m},\ CA_B = \frac{ba}{q},\ CB_a = \frac{ab}{n},\ AC_b = \frac{cb}{r},\ AB_c = \frac{bc}{s}.$$

On the other hand we have

$$BA_b = CA_b - a = \frac{ba}{q} - a = \frac{a(b-q)}{q} = \frac{ar}{q}.$$

Similarly,

$$BC_b = \frac{cq}{r},\ CA_c = \frac{as}{t},\ CB_c = \frac{bt}{s},\ AB_a = \frac{bm}{n},\ AC_a = \frac{cn}{m}.$$

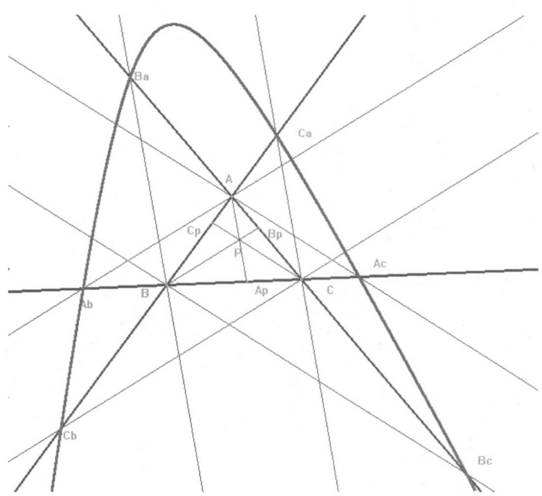
Finally we calculate the product

$$\frac{AC_a}{BC_a} \cdot \frac{AC_b}{BC_b} \cdot \frac{BA_c}{CA_c} \cdot \frac{BA_b}{CA_b} \cdot \frac{CB_a}{AB_a} \cdot \frac{CB_c}{AB_c} =$$

$$= \left(\frac{cn/m}{ca/m} \cdot \frac{cb/r}{cq/r}\right) \cdot \left(\frac{ca/t}{as/t} \cdot \frac{ar/q}{ba/q}\right) \cdot \left(\frac{ab/n}{bm/n} \cdot \frac{bt/s}{bc/s}\right) =$$

$$= \left(\frac{n}{a} \cdot \frac{b}{q}\right) \cdot \left(\frac{c}{s} \cdot \frac{r}{b}\right) \cdot \left(\frac{a}{m} \cdot \frac{t}{c}\right) = \frac{n \cdot r \cdot t}{q \cdot s \cdot m} = 1.$$

Last equality follows from Ceva Theorem, and we are done.




(1) Richter-Gebert, J. :Meditations on Ceva's Theorem (Technical University Munich ,Zentrum Mathematik) (Fields Institute Communications Volume 00, 0000)

*Second solution by Mihai Miculita, Oradea, Romania*

**Solution.** Denote $\{P_a\} = AP \cap BC$, $\{P_b\} = BP \cap AC$, $\{P_c\} = PC \cap AB$, and with

$$\{Q_a\} = B_aC_a \cap BC, \{Q_b\} = A_bC_b \cap AC, \{Q_c\} = A_cB_c \cap AB.$$

If $(l, m, n)$ are barycentric coordinates of point $P$ with respect to triangle $ABC$, then points $P_a, P_b, P_c$ have coordinates:

$$P(0, m, n),\ P_b(l, 0.n),\ P_c(l, m, 0).$$

Observe the following facts

$$AA_b || PB,\ \Rightarrow\ \frac{A_bB}{A_bC} = \frac{AP_b}{AC}\ ;\ \frac{P_bA}{P_bC} = \frac{n}{l},\ \Rightarrow\ \frac{AP_b}{AC} = \frac{n}{l+n}.$$

Therefore

$$\frac{A_bB}{A_bC} = \frac{n}{l+n},\ \Rightarrow\ A_b(0, l+n, -n).$$

Analogously, we get

$$A_c(0, -m, l+m), B_a(m+n, 0, -n), B_c(-l, 0, l+m), C_a(m+n, -m, 0), C_b(-l, l+n, 0).$$

The equation of the line $B_aC_a$ is

$$\begin{vmatrix} x & y & z \\ m+n & 0 & -n \\ m+n & -m & 0 \end{vmatrix} = 0, \iff mn \cdot x + n(m+n) \cdot y + m(m+n) \cdot z = 0.$$

Thus $Q_a$ is the solution of the system

$$x = 0\ ;\ mn \cdot x + n(m+n) \cdot y + m(m+n) \cdot z = 0.$$

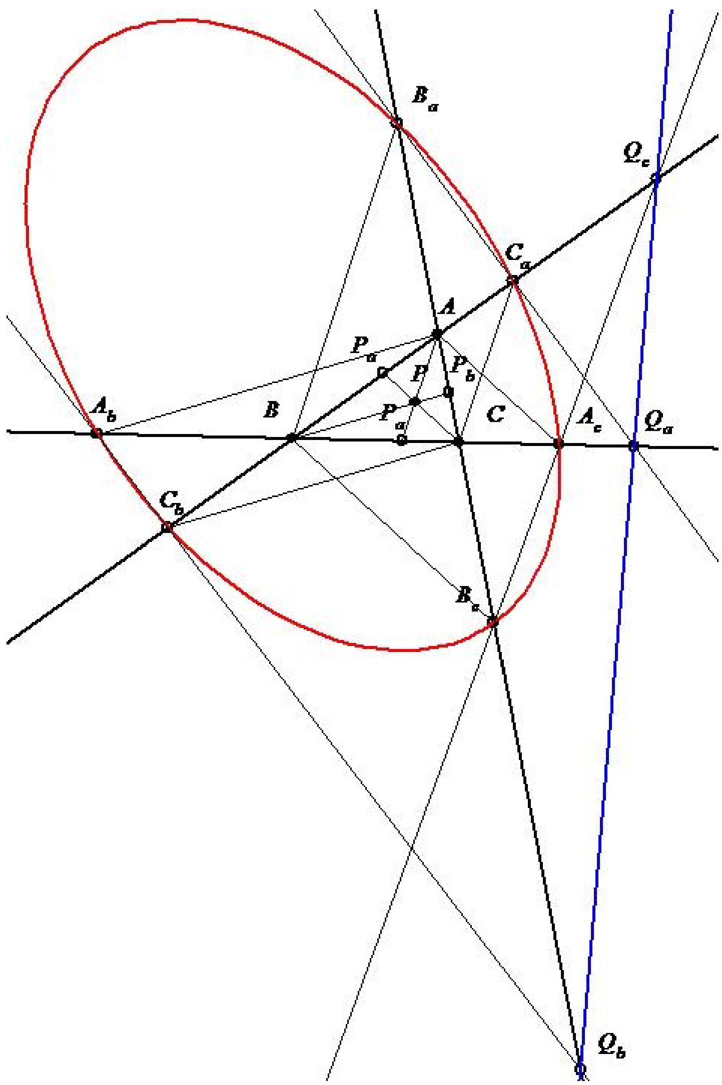
It follows that

$$\frac{y}{m} - \frac{z}{-n},\ \Rightarrow\ Q_a = (0, m, n).$$

Similarly we get the coordinates of the points $Q_b(-l, 0, n)$ and $Q_c(l, -m, 0)$. Because the determinant

$$\begin{vmatrix} 0 & m & -n \\ -l & 0 & n \\ l & -m & 0 \end{vmatrix} = 0,$$

we deduce that points $Q_a, Q_b, Q_c$ are collinear. Using the reciprocal of the Pascal Theorem, we get that points $A_b, A_c, B_a, B_c, C_a, C_b$ lie on the same conic.


$B_a$
$Q_c$
$C_a$
$A$
$P_a$
$P$
$P_b$
$A_b$
$B$
$P_c$
$C$
$A_c$
$Q_a$
$C_b$
$B_c$
$Q_b$

## Undergraduate

U31. Find the minimum of the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{(x^2 - x + 1)^3}{x^6 - x^3 + 1}.$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

*Solution by Daniel Campos Salas, Costa Rica*

**Solution.** Note that $f(0) = 1$. Rewrite our function as

$$f(x) = \frac{(x^2 - x + 1)^3}{x^6 - x^3 + 1} = \frac{(x + \frac{1}{x} - 1)^3}{x^3 + \frac{1}{x^3} - 1} = \frac{(x + \frac{1}{x} - 1)^3}{(x + \frac{1}{x})^3 - 3(x + \frac{1}{x}) - 1}.$$

Denote $y = x + \frac{1}{x}$, then we need to find the minimum of a function

$$g(y) = \frac{(y - 1)^3}{y^3 - 3y - 1}, \text{ where } y \in (-\infty, -2] \cup [+2, +\infty).$$

We have

$$g(y) = \frac{(y - 1)^3}{y^3 - 3y - 1} = 1 + \frac{-3y^2 + 6y}{y^3 - 3y - 1}.$$

Setting the first derivative equal to zero yields the equation

$$3(y - 1)(y^3 - 3y^2 + 2) = 0.$$

The roots of the equation are $1, 1 - \sqrt{3}, 1 + \sqrt{3}$, however only $y = 1 + \sqrt{3}$ belongs to the domain of the function.

Finally we need to check the values at all critical points.

$$\lim_{x \to \pm\infty} g(y) = 1, g(2) = 1, g(-2) = 9, g(1 + \sqrt{3}) = \frac{\sqrt{3}}{2 + \sqrt{3}}.$$

Thus the minimal value of the function is obtained when $y = 1 + \sqrt{3}$ and is $\frac{\sqrt{3}}{2 + \sqrt{3}}$, that is when $x = (1 + \sqrt{3} \pm \sqrt[4]{12})/2$.

U32. Let $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \cdots, b_n$ be sequences of complex numbers. Prove that

$$\mathrm{Re}\left(\sum_{k=0}^{n} a_k b_k\right) \leq \frac{1}{3n+2}\left(\sum_{k=0}^{n} |a_k|^2 + \frac{9n^2+6n+2}{2}\sum_{k=0}^{n} |b_k|^2\right)$$

Proposed by José Luis Díaz-Barrero, Barcelona, Spain

*First solution by Aleksandar Ilic, Serbia*

**Solution.** Let $a_k = x_k + \imath y_k$ and $b_k = X_k - \imath Y_k$, for $0 \leq k \leq n$. We can assume that $x_k, y_k, X_k, Y_k \geq 0$, because we can increase left hand-side using absolute values. Now we have to prove inequality:

$$\sum_{k=0}^{n} (x_k X_k + y_k Y_k) \leq \frac{1}{3n+2}\left(\sum_{k=0}^{n}\left(x_k^2 + y_k^2\right) + \frac{9n^2+6n+2}{2}\sum_{k=0}^{n}\left(X_k^2 + Y_k^2\right)\right).$$

Because of symmetry we only need to prove:

$$2(3n+2)x_k X_k \leq 2x_k^2 + (9n^2+6n+2)X_k^2$$

When consider this as quadratic inequality for variable $x_k$, the discriminant is negative, and thus parabola is above $y = 0$.

$$\Delta = (2(3n+2)X_k)^2 - 4 \cdot 2 \cdot (9n^2+6n+2)X_k^2 =$$

$$= 4X_k^2\left(9n^2+12n+4-18n^2-12n-4\right) = -36n^2X_k^2 < 0.$$

Hence, the problem is solved.

*Second solution by Daniel Campos Salas, Costa Rica*

**Solution.** Let us prove more, namely,

$$\mathrm{Re}\{a_k b_k\} \leq \frac{1}{3n+2}\left(|a_k|^2 + \frac{9n^2+6n+2}{2}|b_k|^2\right).$$

Let $a_k = x + iy$ and $b_k = w + iz$. The inequality is equivalent to

$$xw - yz \leq \frac{1}{3n+2}\left((x^2+y^2) + \frac{9n^2+6n+2}{2}(w^2+z^2)\right).$$

This can be separated into

$$0 \leq \frac{1}{3n+2}\left(x^2 + \frac{9n^2+6n+2}{2}w^2\right) - xw, \tag{1}$$

and

$$0 \leq \frac{1}{3n+2}\left(y^2 + \frac{9n^2+6n+2}{2}z^2\right) + yz. \tag{2}$$

We have that (1) equals

$$\frac{(x-w)^2 + (x-(3n+1)w)^2}{2(3n+2)},$$

and (2) equals

$$\frac{(y+z)^2 + (y+(3n+1)z)^2}{2(3n+2)}.$$

These inequalities prove our claim and complete the proof.

U33. Let $n$ be a positive integer. Evaluate

$$\sum_{r=1}^{\infty} \frac{((n-1)!+1)^r (2\pi i)^r}{r! \cdot n^r} \cdot \prod_{u=0}^{n-1} \prod_{v=0}^{n-1} (n - uv)$$

Proposed by Paul Stanford, University of Texas at Dallas

*Solution by Aleksandar Ilic, Serbia*

**Solution.** We will prove that

$$\sum_{r=1}^{\infty} \frac{((n-1)!+1)^r (2\pi \imath)^r}{r! \cdot n^r} \cdot \prod_{u=0}^{n-1} \prod_{v=0}^{n-1} (n - uv) = 0.$$

If $n$ is composite number, it can be written in form $n = a \cdot b$, where $0 < a, b < n$. Thus product $\prod_{u=0}^{n-1} \prod_{v=0}^{n-1} (n - uv)$ is zero, and we proved equality in this case.

If $n = p$ is prime number, then we use power series for $e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \ldots$ and calculate sum

$$\sum_{r=1} \frac{((p-1)!+1)^r (2\pi \imath)^r}{r! \cdot p^r} = \sum_{r=1} \left( \frac{((p-1)!+1)(2\pi \imath)}{p} \right)^r \frac{1}{r!} = e^{2\pi \imath \cdot \frac{(p-1)!+1}{p}} - 1.$$

From Wilson's theorem we get that $p \mid (p-1)! + 1$, and number $\frac{(p-1)!+1}{p} = k$ is integer. Finally, $e^{2k\pi \imath} = \cos 2k\pi + \imath \sin 2k\pi = 1$. So, our sum equals 0.

*Also solved by Zhao Bin, HUST, China*

U34. Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $f(1) = 0$. Prove that there is a $c \in (0,1)$ such that

$$f(c) = \int_0^c f(x)dx$$

Proposed by Cezar Lupu, University of Bucharest, Romania

*First solution by Aleksandar Ilic, Serbia*

**Solution.** Let $F(x) = \int_0^x f(t)dt$ and $F'(x) = f(x)$ and $F(0) = f(1) = 0$. Consider continuous function $g(x) = F(x) - f(x)$. Without loss of generality, assume that $g(x) > 0$ for $x \in (0,1)$. For $x = 0$ we have $g(0) = -f(0) \geq 0$, and for $x = 1$ we have $g(1) = F(1) \geq 0$.

Let $h(x) = e^{-x}F(x)$. We can compute first derivative for $h(x)$:

$$h'(x) = -e^{-x}F(x) + e^{-x}f(x) = -e^{-x}(F(x) - f(x)) = -e^{-x}g(x) \leq 0$$

So, function $h$ is decreasing function and $h(0) = 0$ and $h(1) = \frac{F(1)}{e} \geq 0$. This gives $h(x) = 0$, and contradiction! Thus there is $c \in (0,1)$ such that $g(c) = 0$ or equivalently $f(c) = F(c)$.

*Second solution by by Zhao Bin, HUST, China*

**Solution.** Introduce

$$g(x) = f(x) - \int_0^x f(t)dt,\ x \in [0,1]$$

Let $f(\epsilon_1) = \max f(x) : x \in [0,1]$, $f(\epsilon_2) = \min f(x) : x \in [0,1]$. Clearly $f(\epsilon_1) \geq 0, f(\epsilon_2) \leq 0$ by $f(1) = 0$. If $f(\epsilon_1) > 0, f(\epsilon_2) < 0$, we have

$$g(\epsilon_1) > 0,\ g(\epsilon_2) < 0.$$

Then by the continuity of $g(x)$ we get that there exist $c$ between $\epsilon_1, \epsilon_2$ such that

$$f(c) = \int_0^c f(x)dx,$$

which proves the statement. If $f(\epsilon_1) = 0, f(\epsilon_2) < 0$, then we have $g(1) = -\int_0^1 f(t)dt > 0$, $g(\epsilon_2) < 0$, thus there exist $c \in (\epsilon_2, 1)$ such that

$$f(c) = \int_0^c f(x)dx.$$

Analogously, if $f(\epsilon_1) > 0, f(\epsilon_2) = 00$ we are done. Also if $f(\epsilon_1) = 0, f(\epsilon_2) = 0$, then $f(x) = 0$, for $x \in [0,1]$ and the problem is solved.

U35. Find all linear maps $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that $f(XY) = f(X)f(Y)$ for all nilpotent matrices $X$ and $Y$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

*Solution by Jean-Charles Mathieux, Dakar University, Sénégal*

**Solution.** We consider $\mathbb{K}$ a zero characteristic field and do not limit the answer to $\mathbb{C}$.

Denote by $f$ a linear map $f : \mathcal{M}_n(\mathbb{K}) \to \mathcal{M}_n(\mathbb{K})$ such that $f(XY) = f(X)f(Y)$ for all nilpotent matrices $X$ and $Y$ . We show that either $f = 0$ or there exists $P \in \mathrm{GL}_n(\mathbb{K})$ such that $f(M) = PMP^{-1}$ for all $M \in \mathcal{M}_n(\mathbb{K})$.

The case $f = 0$ is obvious so let's assume that $f \neq 0$.

Denote by $E_{i,j}$ the matrix of $\mathcal{M}_n(\mathbb{K})$ with all coefficients equal to 0 except on row $i$ and column $j$, where it equals 1 and $F_{i,j} = f(E_{i,j})$.

We proceed in two steps.

*Lemma 1.* $\{F_{i,j}\}_{i,j\in\{1,\dots,n\}}$ is a basis of $\mathcal{M}_n(\mathbb{K})$ such that $F_{i,j}F_{k,l} = \delta_{j,k}F_{i,l}$.

*Lemma 2.* If $\{M_{i,j}\}_{i,j\in\{1,\dots,n\}}$ is a basis of $\mathcal{M}_n(\mathbb{K})$ such that $M_{i,j}M_{k,l} = \delta_{j,k}M_{i,l}$, then there exists $P \in \mathrm{GL}_n(\mathbb{K})$ such that $M_{i,j} = PE_{i,j}P^{-1}$ for all $i, j \in \{1, \dots, n\}$.

*Proof of Lemma 1.* If $\{F_{i,j}\}_{i,j\in\{1,\dots,n\}}$ was not a basis of $\mathcal{M}_n(\mathbb{K})$, we could find $\{\alpha_{i,j}\}_{i,j\in\{1,\dots,n\}} \in \mathbb{K}^{n^2}$ with at least one $\alpha_{l,p} \neq 0$ and $\sum_{i,j\in\{1,\dots,n\}} \alpha_{i,j}F_{i,j} = 0$

For all $i, j, k, l \in \{1, \dots, n\}$ we have $E_{i,j}E_{k,l} = \delta_{j,k}E_{i,l}$ so $F_{i,j}F_{k,l} = \delta_{j,k}F_{i,l}$. This last relation is obvious if $i \neq j$ and $k \neq l$, because $E_{i,j}$ and $E_{k,l}$ are nilpotent. But also $f(E_{i,i}E_{k,l}) = f(E_{i,j_0}E_{j_0,i}E_{k,l}) = f(E_{i,j_0}\delta_{i,k}E_{j_0,l}) = \delta_{i,k}F_{i,l}$ (where we choose $j_0 \neq i$ and $j_0 \neq l$).

There is at least one $F_{k,q} \neq 0$ because $f \neq 0$, $F_{k,l} \times \sum_{i,j\in\{1,\dots,n\}} \alpha_{i,j}F_{i,j} \times F_{p,q} = \alpha_{l,p}F_{k,q} = 0$ which is a contradiction, so $\{F_{i,j}\}_{i,j\in\{1,\dots,n\}}$ is a basis of $\mathcal{M}_n(\mathbb{K})$.

*Proof of Lemma 2.* $M_{i,k}M_{k,j} = M_{i,j}$, so $\mathrm{Im}M_{i,k} = \mathrm{Im}M_{i,j}$, let's denote this space by $\mathcal{V}_i$.

There exist $\{\alpha_{i,j}\}_{i,j\in\{1,\dots,n\}} \in \mathbb{K}^{n^2}$ such that $I_n = \sum_{i,j\in\{1,\dots,n\}} \alpha_{i,j}M_{i,j}$. If $X \in \mathbb{K}^n$, $X = \sum_{i=1}^n \sum_{j=1}^n \alpha_{i,j}M_{i,j}X$, and $\sum_{j=1}^n \alpha_{i,j}M_{i,j}X \in \mathcal{V}_i$, so $\mathbb{K}^n = \mathcal{V}_1 + \mathcal{V}_2 + \cdots + \mathcal{V}_n$.

If $X_i \in \mathcal{V}_i$, and $X_1 + X_2 + \cdots + X_n = 0$ then $M_{i,i}(X_1 + X_2 + \cdots X_n) = X_i = 0$, so $\mathbb{K}^n = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_n$.

If $k \neq j$, $\mathrm{Im} M_{i,k} \subset \mathrm{Ker} M_{i,j}$, so $\bigoplus_{k=1, k\neq j}^{n} \mathrm{Im} M_{i,k} = \mathrm{Ker} M_{i,j}$ (because they have same dimension and the previous inclusion).

Let $Y_1 \in \mathcal{V}_1$, such that $Y_1 \neq 0$. For $i \neq 1, Y_i = M_{i,1} Y_1$. We have $M_{i,j} Y_j = Y_1$. So $\{Y_1, \ldots, Y_n\}$ is a basis of $\mathbb{K}^n$.

We define $P \in \mathrm{GL}_n(\mathbb{K})$ the matrix whose columns are the $Y_i$. We have $PE_{i,j} = M_{i,j} P$, which concludes the proof of lemma 2.

Since $f$ is linear we have proven that $f(M) = PMP^{-1}$ for all $M \in \mathcal{M}_n(\mathbb{K})$.

U36. Let $n$ be an even number greater than 2. Prove that if the symmetric group $\mathfrak{S}_n$ contains an element of order $m$, then $\mathrm{GL}_{n-2}(\mathbb{Z})$ contains an element of order $m$.

Proposed by Jean-Charles Mathieux, Dakar University, Sénégal

*No solutions proposed*

## Olympiad

O31. Let $n$ is a positive integer. Prove that

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}=\sum_{k=0}^{n}2^k\binom{n}{k}^2$$

Proposed by Jean-Charles Mathieux, Dakar University, Sénégal

*First solution by Zhao Bin, HUST, China*

**Solution.** We start from the right side of the equality. First of all, because

$$2^k=\sum_{i=0}^{k}\binom{k}{i}$$

We have

$$\sum_{k=0}^{n}2^k\binom{n}{k}^2=\sum_{k=0}^{n}\sum_{i=0}^{k}\binom{k}{i}\binom{n}{k}^2=\sum_{i=0}^{n}\sum_{k=i}^{n}\binom{k}{i}\binom{n}{k}^2.$$

Then using well-known identities

$$\binom{k}{i}\binom{n}{k}=\binom{n}{i}\binom{n-i}{n-k}$$

and

$$\sum_{k=i}^{n}\binom{n-i}{n-k}\binom{n}{k}=\binom{2n-i}{n}.$$

The first inequality goes directly from the binomial theorem, while the second can be obtained from the combinational way or comparing the the coefficient of $x^n$ in the identity:

$$(1+x)^{n-i}(1+x)^n=(1+x)^{2n-i}.$$

Thus we have

$$\sum_{k=0}^{n}2^k\binom{n}{k}^2=\sum_{i=0}^{n}\sum_{k=i}^{n}\binom{k}{i}\binom{n}{k}^2=\sum_{i=0}^{n}\binom{n}{i}\sum_{k=i}^{n}\binom{n-i}{n-k}\binom{n}{k}=$$

$$=\sum_{i=0}^{n}\binom{n}{i}\binom{2n-i}{n}.$$

Then by substituting $k = n - i$, we easily get

$$\sum_{i=0}^{n} \binom{n}{i} \binom{2n-i}{n} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}.$$

Thus we have proved the desired identity.

*Second solution by José Hernández Santiago, UTM, Oaxaca, México.*

**Solution.** Let $m \in \mathbb{N}$ with $n \geq m$. The desired identity is a special case of the more general result (see Wilf, Herbert S. *Generatingfunctionology.* p. 127),

$$\sum_{k=0}^{m} \binom{m}{k} \binom{n+k}{m} \quad = \quad \sum_{k=0}^{m} 2^k \binom{m}{k} \binom{n}{k}. \tag{3}$$

Setting $m = n$ in (1) we get,

$$\begin{aligned} \sum_{k=0}^{n} 2^k \binom{n}{k}^2 \quad &= \quad \sum_{k=0}^{n} 2^k \binom{n}{k} \binom{n}{k} \\ &= \quad \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{n} \\ &= \quad \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{n+k-n} \\ &= \quad \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}. \end{aligned}$$

Next two solutions were kindly presented by Dr. Scott H. Brown, Auburn University Montgomery.

*Third solution (Benjamin and Bataille)*

**Solution.** Let $[n]$ denote the set $\{1, 2, 3, \ldots, n\}$ and $D$ denote the set of ordered pairs $(A, B)$ where $A$ is a subset of $n$ and $B$ is an n-subset of $[2n]$ that is disjoint from A. We can select elements for $D$ in two ways:

(a) For $0 \le k \le n$, let $M$ be a $k$-subset of $[n]$. Let $A = M^c$, the complement of M, which is an $(n-k)$ subset of $[n]$, let $B$ be an $n$-subset of $\{n+1,\ldots,2n\} \cup M$. As a result we have

$$|D| = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}.$$

(b) For $0 \le k \le n$, choose a $k$-subset $B_1$ from $\{n+1,\ldots,2n\}$ and a $k$-subset $B_2$ of $[n]$. Now form $BB_1 \cup B_2^c$ and choose $A$ from among the $2^k$ subsets of $B_2$. Thus we have

$$|D| = \sum_{k=0}^{n} 2^k \binom{n}{k}^2.$$

*Fourth solution (R.A. Sulanke)*

**Solution.** The proof is based on lattice paths.

(a) Consider the set of paths from $(0,0)$ to $(2n,0)$ using the diagonal step up $(1,1)$ and diagonal step down $(1,-1)$ and the step $(2,0)$. For the "tilted" version consider the path $(0,0)$ to $(0,0)$ using the steps $(0,1),(1,0)$, and $(1,1)$. This path model leads to the following combinatorial proof that for $n \ge 0$,

$$d_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$$

Here $d_n$ counts the unrestricted lattice paths form $(0,0)$ to $(2n,0)$ using the steps $(1,1),(1,-1)$ and $(2,0)$.

(b) Alternatively we determine $d_n$ to be the weighted sum of the paths from $(0,0)$ to $(2n,0)$ where within each path the right-hand turn, or peaks, have weight of 2. As a result, this path model leads to the combinatorial proof that for $n \ge 0$.

$$d_n = \sum_{k=0}^{n} 2^k \binom{n}{k}^2.$$

The problem is solved.

**Remark.** These combinatorial identities are related to the well-known central Delannoy numbers. For more information see:
C. Banderier and S. Schwer, "Why Delannoy Numbers".

*Also solved by Ashay Burungale, India; Aleksandar Ilic, Serbia; David E. Narvaez, Universidad Tecnologica de Panama*

O32. 18. Let $a, b, c > 0$. Prove that

$$\sqrt{\frac{a^2}{4a^2+ab+4b^2}}+\sqrt{\frac{b^2}{4b^2+bc+4c^2}}+\sqrt{\frac{c^2}{4c^2+ca+4a^2}} \leq 1$$

Proposed by Bin Zhao, HUST, China

*First solution by Ho Phu Thai, Da Nang, Vietnam*

**Solution.** By Cauchy-Schwarz inequality

$$\sqrt{\frac{a^2}{4a^2+ab+4b^2}}+\sqrt{\frac{b^2}{4b^2+bc+4c^2}}+\sqrt{\frac{c^2}{4c^2+ca+4a^2}} \leq$$

$$\sqrt{(8a^2+8b^2+8c^2+ab+bc+ca)\sum_{cyc}\frac{a^2}{(4a^2+ab+4b^2)(4a^2+ac+4c^2)}}.$$

We will prove

$$\sum_{cyc}\frac{a^2}{(4a^2+ab+4b^2)(4a^2+ac+4c^2)} \leq \frac{1}{8a^2+8b^2+8c^2+ab+bc+ca}.$$

This is equivalent to

$$64\sum_{sym}a^4b^2+4\sum_{sym}a^4bc+4\sum_{sym}a^3b^3+3\sum_{sym}a^3b^2c \geq 66a^2b^2c^2,$$

which is clearly true from AM-GM inequality.

*Second solution by Vasile Cartoaje, University of Ploiesti, Romania*

**Solution.** By the Cauchy-Schwarz inequality we have

$$\sqrt{\frac{a^2}{4a^2+ab+4b^2}}+\sqrt{\frac{b^2}{4b^2+bc+4c^2}}+\sqrt{\frac{c^2}{4c^2+ca+4a^2}} \leq$$

$$\sum\frac{a^2}{(4a^2+ab+4b^2)(4c^2+ca+4a^2)}\cdot\sum(4c^2+ca+a^2) =$$

$$= \frac{\sum a^2(4b^2+bc+4c^2)\cdot\sum(4c^2+ca+4a^2)}{(4a^2+ab+4b^2)(4b^2+bc+4c^2)(4c^2+ca+4a^2)} = \frac{A}{B},$$

where

$$A = (8\sum b^2c^2 + abc\sum a)(8\sum a^2 + \sum bc),$$
$$B = (4a^2 + ab + 4b^2)(4b^2 + bc + 4c^2)(4c^2 + ca + 4c^2).$$

It suffices to prove that $A \geq B$. We have

$$A = 64(\sum a^2)(\sum b^2c^2) + 8(\sum bc)(\sum b^2c^2) +$$
$$+8abc(\sum a^2)(\sum a) + abc(\sum bc)(\sum a) =$$
$$= 64\sum b^2c^2(b^2+c^2)+195a^2b^2c^2+17abc\sum bc(b+c)+8abc\sum a^3+8\sum b^3c^3.$$

and

$$B = 64\prod(b^2+c^2)+16\sum bc(a^2+b^2)(c^2+a^2)+4abc\sum a(b^2+c^2)+a^2b^2c^2 =$$
$$= 64\sum b^2c^2(b^2+c^2)+129a^2b^2c^2+20abc\sum bc(b+c)+16abc\sum a^3+16\sum b^3c^3.$$

Therefore

$$A - B = 3abc\sum bc(b+c) + 8abc\sum a^3 + 8\sum b^3c^3 - 66a^2b^2c^2 =$$
$$= 3abc[\sum bc(b+c)-6abc]+8abc(\sum a^3-3abc)+8(\sum b^3c^3-3a^2b^2c^2) \geq 0,$$

because

$$\sum bc(b+c) - 6abc = \sum a(b-c)^2 \geq 0,$$

and by the AM-GM inequality,

$$\sum a^3 \geq 3abc, \ \ \sum b^3c^3 \geq 3a^2b^2c^2.$$

The equality occurs if and only if $a = b = c$.

O33. 23. Let $ABC$ be a triangle with cicrumcenter $O$ and incenter $I$. Consider a point $M$ lying on the small arc $BC$. Prove that

$$MA + 2OI \geq MB + MC \geq MA - 2OI$$

Proposed by Hung Quang Tran, Ha Noi University, Vietnam

*Solution by Hung Quang Tran, Ha Noi University, Vietnam*

**Solution.** Using the Law of Cosines in the triangle $AOM$ we get $2AO \cdot OM \cdot \cos AMO = MA^2 + AO^2 - MO^2 = MA^2, \Rightarrow 2\overline{MO} \cdot \overline{MA} = MA^2$, Analogously

$$2\overline{MO} \cdot \overline{MA} = MB^2,\ 2\overline{MO} \cdot \overline{MC} = MC^2$$

From here we have

$$MB + MC - MA = 2\overline{MO} \cdot \left(\frac{\overline{MA}}{MA} + \frac{\overline{MB}}{MB} + \frac{\overline{MC}}{MC}\right)$$

From the Cauchy- Schwarz Inequality we get

$$-MO \cdot \left|\frac{\overline{MA}}{MA} + \frac{\overline{MB}}{MB} + \frac{\overline{MC}}{MC}\right| \leq 2\overline{MO} \cdot \left(\frac{\overline{MA}}{MA} + \frac{\overline{MB}}{MB} + \frac{\overline{MC}}{MC}\right) \leq$$

$$\leq MO \cdot \left|\frac{\overline{MA}}{MA} + \frac{\overline{MB}}{MB} + \frac{\overline{MC}}{MC}\right|$$

We have $MO = R$, we will calculate $\left|\frac{\overline{MA}}{MA} + \frac{\overline{MB}}{MB} + \frac{\overline{MC}}{MC}\right|$.

$$\left|\frac{\overline{MA}}{MA} + \frac{\overline{MB}}{MB} + \frac{\overline{MC}}{MC}\right|^2 = 3 + 2\left(\frac{\overline{MB} \cdot \overline{MC}}{MB \cdot MC} + \frac{\overline{MB} \cdot \overline{MA}}{MB \cdot MA} + \frac{\overline{MA} \cdot \overline{MA}}{MC \cdot MA}\right) =$$

$$= 3 + 2(\cos(\overline{MB}, \overline{MC}) - \cos(\overline{MB}, \overline{MA}) - \cos(\overline{MC}, \overline{MA})) =$$

$$3 - 2(\cos A + \cos B + \cos C) = 3 - 2 \cdot \frac{R + r}{R} = \frac{OI^2}{R^2}.$$

Thus we have

$$MO \cdot \left|\frac{\overline{MA}}{MA} + \frac{\overline{MB}}{MB} + \frac{\overline{MC}}{MC}\right| = OI.$$

It follows that

$$MA + 2OI \geq MB + MC \geq MA - 2OI,$$

and the problem is solved.

O34. Suppose that $f \in \mathbb{Z}[X]$ is a nonconstant monic polynomial such that for infinitely many integers $a$, the polynomial $f(X^2 + aX)$ is reducible in $\mathbb{Q}[X]$. Does it follow that $f$ is also reducible in $\mathbb{Q}[X]$?

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

*First solution by Ashay Burungale, India*

**Solution.** Suppose that $f(x)$ is an irreducible polynomial of degree $d$ and that $f(x^2+ax)$ is a reducible polynomial. If $u$ is a zero of $f(x^2+ax)$, then $t = u^2+au$ is a zero of $f(x)$ and $u = \frac{-a+\sqrt{a^2+4t}}{2}$. Because $f(x^2+ax)$ is reducible, $u$ has degree less than $2d$, so has degree $d$. It follows that $\sqrt{a^2+4t}$ lies in $Q(t)$, or $a^2+4t$ is a square in $Q(t)$. Thus the norm of $a^2+4t$ is a square in $\mathbb{Q}$. But the norm of $a^2+4t = 4^d f(\frac{-a^2}{4}) \in \mathbb{Z}$. Hence we have $4^d(\frac{-a^2}{4}) = b^2$, for some integer $b$. This diophantine equation of degree $2d \geq 4$ has only finitely many integer solutions, and it follows that $f(x)$ is reducible.

*Second solution by Iurie Boreico, Moldova*

**Solution.** Consider the following two lemmas.

*Lemma 1.* For any monic polynomial $f \in Q[X]$ of even degree there exists a polynomial $g \in Q[X]$ such that $\lim_{|x|\to\infty} |\sqrt{f(x)} - g(x)| = 0$.

*Proof.* If $f$ has degree $2n$ the set $g(x) = x^n + a_1 x^{n-1} + \ldots + a_0$ and compute $a_i$ inductively on $i$ such that $f(x) - g^2(x)$ has degree at most $n-1$ (we will get a linear equation in $a_i$ which ensures that $a_i$ are rational). Then as $\sqrt{f(x)} - g(x) = \frac{f(x)-g^2(x)}{\sqrt{f(x)}+g(x)}$, we finish the lemma. The lemma clearly holds not only when the polynomial is monic, but also when its leading coefficient if a perfect square.

*Lemma 2.* If $f \in Z[X]$ and has the leading coefficient a perfect square and $f(m^2)$ is a perfect square for infinitely many integers $m$ then $f(x) = x^k g^2(x)$ for some $k, \in Z, g \in Z[X]$

*Proof.* Assume that $f(0) \neq 0$ otherwise divide by a suitable power of $x$. Applying the lemma 1 for the polynomial $f_1(x) = f(x^2)$ we find $g \in Q(x)$ such that $|\sqrt{f_1(x) - g(x)}|$ tends to zero. Now multiplying by a suitable $n$ we have $|N\sqrt{f_1(x)} - h(x)| < 1$ for all sufficiently big $x$ and

$h \in Z[X]$. If we take $x = m$ such that $f(m^2)$ is a perfect square, we get $N\sqrt{f(m^2)} = h(m)$. This relations holds for infinitely many $m$ thus $f(x^2) = g(x)^2$. As $g$ has no non-zero roots, it's roots must group into pairs of opposite roots (because the roots of $f(x^2)$ do). Thus $g(x) = r(x^2)$ so $f(x) = r^2(x)$, and clearly $r(x) \in Z[X]$ (we can use lemma 1 to prove $r(x) \in Q[X]$ then we deduce $r(x) \in Z[x]$). Now let us solve the problem.

Let $f(x) = P(x) = \prod_{i=1}^{n}(x - w_n)$, then $P(x^2 + ax) = \prod_{i=1}^{n}(y^2 - u_i)$, where $u_i = \frac{a^2}{4} + w_i$ and $y = x + \frac{a}{2}$. Thus $Q(x) = \prod_{i=1}^{n}(x^2 - u_i)$ is irreducible. The $Q(x)$ can be written as $f(x) \cdot g(x)$ where

$$f(x) = u(x^2)h(x), g(x) = \pm v(x^2)h(-x).$$

From all such representations $f(x) \cdot g(x)$, pick up one for which $\deg(f)$ is the least possible. It follows that $gcd(f(x), g(-x))$ is either 1 or $f(x)$, which means that either $h(x) = 1$ or $u(x)|v(x)$. If $u(x)|v(x)$ and $u \neq 1$, then $gcd(f(x), g(x)) \neq 1$. Thus $gcd(f(x), g(x)) = f(x)$, hence $f(x)^2|Q(x)$. This readily implies that $P$ has a double root (if $a$ is sufficiently big), so $(P, P') \neq 1$ and therefore $P$ is irreducible. It follows that we have $h(x) = 1$ or $u(x) = 1$.

a) If $h(x) = 1$, then $f(x) = u(x^2), g(x) = v(x^2), \Rightarrow u(x)v(x) = \pm f(x + \frac{a}{2})$ so $f$ is reducible.

b) If $u(x) = 1$, then $f(x) = h(x),\ g(x) = h(-x)v(x^2), \Rightarrow$ $h(x)h(-x)v(x^2) = Q(x)$. If $h(x)h(-x) = w(x^2)$ we get $w(x)v(x) = \pm f(x + \frac{a}{2})$, thus $f$ is reducible unless $v = 1$. In this case we have $Q(x) = h(x)h(-x)$. Therefore $Q(0) = h(0)^2$ and thus $P(-\frac{a^2}{4})$ is a perfect square. By setting $k(x) = 4^m f(-\frac{x}{4})$ for a suitable $m$, the conditions of lemma 2 are satisfied. Hence $k$ is a perfect square, so must be $P$.

In any case we get that $f = P$ is reducible.

**Remark.** The key assertion that if $f \in Z[X]$ is irreducible and $f(0)$ is not a perfect square then $f(x^2)$ is also irreducible was a problem from Romanian TST 2003.

O35. Let $0 < a < 1$. Find, with proof, the greatest real number $b_0$ such that if $b < b_0$ and $(A_n \subset [0;1])_{n\in\mathbb{N}}$ are finite unions of disjoint segments with total length $a$, then there are two different $i, j \in \mathbb{N}$ such that $A_i \bigcap A_j$ is a union of segments with total length at least $b$. Generalize this result to numbers greater than 2: if $k \in \mathbb{N}$ find the least $b_0$ such that whenever $b < b_0$ and $(A_n \subset [0;1])_{n\in\mathbb{N}}$ are finite unions of disjoint segments with total length $a$, then there are $k$ different $i_1, i_2, \ldots, i_k \in \mathbb{N}$ such that $A_{i_1} \bigcap A_{i_2} \bigcap \ldots \bigcap A_{i_k}$ is a union of segments with total length at least $b$.

Proposed by Iurie Boreico, Moldova

*Solution by Iurie Boreico, Moldova*

**Solution.** We will use the following famous result:

$$\binom{n}{k} > \frac{n^k - k^2 n^{k-1}}{k!}$$

Indeed, $\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!}$. Using the following inequality $xy \geq (x+1)(y-1)$, for $x > y$, we conclude that

$$n(n-1)\ldots(n-k+1) > n^{k-1}(n-1-2-\ldots-(k-1)),$$

from which we deduce the result.

Let us solve the general problem. Pick up some $n$. Suppose $[0,1]$ is divided into intervals of lengths $x_1, x_2, x_3, \ldots, x_m$ by the sets $A_1, A_2, \ldots, A_n$. Let $x_i$ belong to $k_i$ of $A_1, A_2, \ldots, A_n$, and $k_1x_2 + k_2x_2 + \ldots + k_nx_n = na$.

Suppose that each $k$ or the $A_i$ intersect in a set of measure at most $b$. Consider the sum

$$\sum_{1\leq i_1<i_2<\ldots<i_k\neq n} |A_{i_1} \bigcap A_{i_2} \bigcap \ldots \bigcap A_{i_k}|.$$

As we can choose $\binom{n}{k}$ such $k$-uples of subsets, this sum is at most $b\binom{n}{k}$. From the other side this sum is clearly

$$\sum_{i=1}^{m} \binom{k_i}{k} x_i > \frac{1}{k!}\sum_{i=1}^{m} k_i^k x_i - \frac{1}{k!}k^2 \sum_{i=1}^{m} k_i^{k-1} x_i > \frac{1}{k!}\sum_{i=1}^{m} -\frac{k^2}{k!}n^{k-1}.$$

Next, by using Holder's Inequality we conclude that

$$(\sum_{i=1}^{m} k_i^k x_i)^{\frac{1}{k}} (\sum x_i)^{\frac{k-1}{k}} \geq \sum_{i=1}^{m} k_i x_i$$

which translates into

$$\sum_{i=1}^{m} k_i^k x_i \geq (\sum_{i=1}^{m} k_i x_i)^k = n^k a^k.$$

Therefore the sum in contest is at least $\frac{n^k}{k!}(a^k - \frac{k^2}{n})$. But we've prove it is at most $b\binom{n}{k} < b\frac{n^k}{k!}$. Thus if $b < a^k$, we get a contradiction for sufficiently big $n$.

Let us prove that any number greater than $a^k$ fails. Indeed, consider $b > a^k$ and pick up a rational number $\frac{p}{q}$ such that $a < \frac{p}{q} < \sqrt[k]{b}$. Consider $A_i$ be the set of all real numbers from $[0;1]$ which written in base $q$ have the $i$-th digit after zero from the set $\{0, 1, \ldots, p-1\}$. It's obvious that $A_i$ are a union of segments, that $|A_i| = \frac{p}{q} > a$ and $|A_{i_1} \bigcap A_{i_2} \ldots A_{i_k}| = (\frac{p}{q})^k < b$ so this is the required counter-example.

The answer is thus $a^k$.

O36. Let $a_1, a_2, .., a_n$ and $b_1, b_2, ..., b_n$ be real numbers and let $x_{ij}$ be the number of indices $k$ such that $b_k \geq \max(a_i, a_j)$. Suppose that $x_{ij} > 0$ for any $i$ and $j$. Prove that we can find an even permutation $f$ and an odd permutation $g$ such that $\sum\limits_{i=1}^{n} \frac{x_{if(i)}}{x_{ig(i)}} \geq n$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

*Solution by Iurie Boreico, Moldova*

**Solution.** If $n = 2$ then we must prove that $x_{1,1} + x_{2,2} \geq 2x_{1,2}$ which is true because clearly $x_{1,1} \geq x_{1,2}$ and $x_{2,2} \geq x_{1,2}$.

Now assume $n \geq 3$. By AM-GM it suffices to find an even permutation $f$ and an odd permutation $g$ such that

$$\prod_{i=1}^{n} x_{i,f(i)} \geq \prod_{i=1}^{n} x_{j,g(j)}.$$

Set $x_i(\pi) = x_{i,\pi(i)}$ for a permutation $\pi$.

We must find an even permutation $f$ and an odd permutation $g$ such that $\prod_{i=1}^{n} x_i(f) \geq \prod_{i=1}^{n} x_i(g)$. If $X(\pi) = \prod_{i=1}^{n} x_i(\pi)$ we must have $X(f) \geq X(g)$. This will follow directly from the relation $\prod_f X(f) = \prod_g X(g)$ where $f$ runs over all even permutations and $g$ over all odd permutations (recall that the number od even permutation equals the number of odd permutations).

To prove this relation, it suffices to prove that $\prod_f x_i(f) = \prod_g x_i(g)$. This fact follows from the following result:

For any $1 \leq i, j \leq n$, the number of even permutations $f$ such that $f(i) = j$ equals the number of odd permutations $g$ such that $g(i) = j$.

Indeed if we pick up $k, l$ such that $j, k, l$ are pairwise different, then the permutations $\pi$ with $\pi(i) = j$ group into pairs $\pi, \sigma(\pi)$ where $\sigma$ is the transposition exchanging $k$ with $l$. As $\pi, \sigma(\pi)$ have different signs, we deduce there is one even permutation and one odd permutation in each pair, hence the result.