Vectors Conquering Hexagons

Iurie Boreico, Liubomir Chiriac

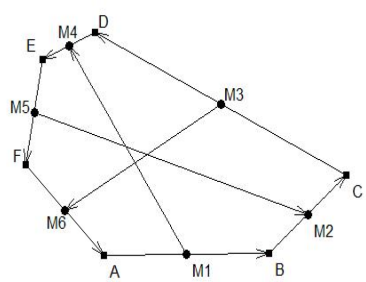
Abstract. Many geometry problems involving hexagons often prove to be extremely challenging. This is due to the fact that finding a synthetic solution is sometimes very complicated. In this article we present some efficient methods of solving such problems by using vectors.

We start with some basic techniques widely used in many problems involving hexagons. The first two examples illustrate some standard vector manipulations.

Problem 1. Let $ABCDEF$ be a convex hexagon and let $M_1, M_2, M_3, M_4, M_5, M_6$ be the midpoints of AB, BC, CD, DE, EF, FA, respectively. Prove that $M_1M_4 \perp M_3M_6$ if and only if

$$M_5M_2^2 = M_1M_4^2 + M_3M_6^2.$$

Solution. Let $\overrightarrow{AB} = a, \overrightarrow{BC} = b, \ldots, \overrightarrow{FA} = f$. Then $\overrightarrow{M_1M_4} = \frac{a}{2} + b + c + \frac{d}{2}$ and $\overrightarrow{M_1M_4} = -\frac{d}{2} - e - f - \frac{a}{2}$. Therefore $\overrightarrow{M_1M_4} = \frac{b+c-e-f}{2}$. Analogously, $\overrightarrow{M_5M_2} = \frac{a+f-c-d}{2}$ and $\overrightarrow{M_3M_6} = \frac{d+e-a-b}{2}$.



Note that

$$\overrightarrow{M_1M_4} + \overrightarrow{M_3M_6} + \overrightarrow{M_5M_2} = 0,$$

thus these vectors can form a triangle (call it T). It is enough to observe that each of the conditions $M_5M_2^2 = M_1M_4^2 + M_3M_6^2$ and $M_1M_4 \perp M_3M_6$ is equivalent to the fact that T is a right triangle and the conclusion follows.

Problem 2. Let $A_1A_2A_3A_4A_5A_6$ be a convex hexagon. Denote by B_i the midpoints of the sides A_iA_{i+1} (indices are taken modulo 6). Prove that triangles $B_1B_3B_5$ and $B_2B_4B_6$ have the same centroid.

Solution. Denote by G_1 the centroid of $\triangle B_1B_3B_5$ and by G_2 the centroid of $\triangle B_2B_4B_6$. From the quadrilaterals $G_2G_1B_1B_2$, $G_2G_1B_3B_4$, and $G_2G_1B_5B_6$ we get (1):

$$3\overrightarrow{G_2G_1} = \overrightarrow{G_2B_2} + \overrightarrow{B_2B_1} - \overrightarrow{G_1B_1} + \overrightarrow{G_2B_4} + \overrightarrow{B_4B_3} - \overrightarrow{G_1B_3} + \overrightarrow{G_2B_6} + \overrightarrow{B_6B_5} - \overrightarrow{G_1B_5}.$$

In order to simplify a bit this expression we need an easy (though famous) result.

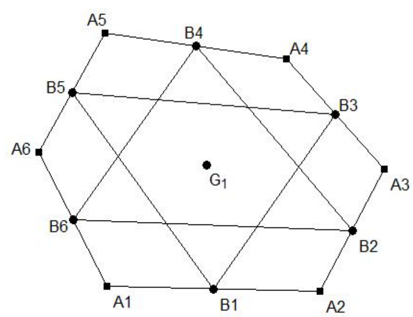
Lemma. If G is the centroid of the triangle ABC then for any point X

$$\overrightarrow{XA} + \overrightarrow{XB} + \overrightarrow{XC} = 3\overrightarrow{XG}.$$

Proof. Let M be the midpoint of BC. Then from $\triangle XBC$ we get $\overrightarrow{XB} + \overrightarrow{XC} = 2\overrightarrow{XM}$. Moreover, from $\triangle XAM$ $\overrightarrow{XA} = \overrightarrow{XM} + \overrightarrow{MA} = \overrightarrow{XM} + 3\overrightarrow{MG}$. Summing up the last two relations we get

$$\overrightarrow{XA} + \overrightarrow{XB} + \overrightarrow{XC} = 3(\overrightarrow{XM} + \overrightarrow{MG}) = 3\overrightarrow{XG}.$$

Note that for $X = G$ it yields $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = 0$. The lemma is proved.



Back to our problem, according to the lemma we get

$$\overrightarrow{G_1B_1} + \overrightarrow{G_1B_3} + \overrightarrow{G_1B_5} = 0 = \overrightarrow{G_2B_2} + \overrightarrow{G_2B_4} + \overrightarrow{G_2B_6}$$

and (1) becomes

$$3\overrightarrow{G_2G_1} = \overrightarrow{B_2B_1} + \overrightarrow{B_4B_3} + \overrightarrow{B_6B_5} = \frac{1}{2}(\overrightarrow{A_3A_1} + \overrightarrow{A_5A_3} + \overrightarrow{A_1A_5}) = 0.$$

This clearly implies $G_1 = G_2$ and we are done.

After solving these questions it is time to proceed to something more challenging. In this sense, we present the following useful idea.

Theorem. Let ABC be a triangle. Consider three vectors u, v, w that sum up to 0 (thus form a triangle). Then either of the following statements implies that the triangle formed by u, v, w is similar to triangle ABC:

a) $\frac{|u|}{AB} = \frac{|v|}{BC} = \frac{|w|}{CA}$;

b) $u || \overrightarrow{AB}, v || \overrightarrow{BC}, w || \overrightarrow{CA}$;

Conversely, a) and b) imply that the vectors u, v, w form a triangle.

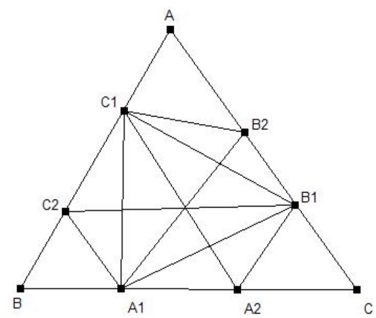
Proof. Both statements follow from the similarity criteria of two triangles. Note that a) follows from the side similarity criterion and b) from the angles similarity criterion (i.e. two triangles with parallel sides are similar). The converse follows from the construction of a triangle T formed by the vectors u, v and $w' = -u - v$. We have $\frac{|u|}{AB} = \frac{|v|}{BC}$ and the angle between u and v equals the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Hence T is similar to ABC which implies $\frac{|w'|}{CA} = \frac{|u|}{AB} = \frac{|w|}{CA}$ and $w || \overrightarrow{CA} || w'$, so w, w' have the same length and the same direction, thus $w = w' = -u - v$, implying $u + v + w = 0$.

While so simple, the theorem is quite useful. Part a) gets an angle relationship provided a length relationship, and part b) gives a length relationship (derived from similarity) provided an angle relationship (from parallelism). The result is a straightforward consequence of the basic triangle similarity properties, however it is more valuable than that. The usefulness of this theorem stems from its statement in vector language, and thus is more general in its application, because the fact that three vectors form a triangle does not imply that their corresponding segments do. Here is an illustration:

Problem 3. Let ABC be an equilateral triangle. The points A_1, A_2, B_1, B_2, C_1 and C_2 are chosen on the sides BC, BC, CA, CA, AB, AB respectively such that they form a convex hexagon with all of its sides equal. Prove that A_1B_2, B_1C_2, and C_1A_2 concur.

(IMO, 2005)

Solution. We cannot see any triangle similarity here (it is hidden beneath a construction that we do not see). However, its consequence may be applied. The vectors $\overrightarrow{A_1A_2}, \overrightarrow{B_1B_2}, \overrightarrow{C_1C_2}$ are equal in length and parallel to the sides of triangle ABC (which are also equal in length), so by the converse of the theorem we deduce that their sum is zero. Clearly, $\overrightarrow{A_1A_2} + \overrightarrow{A_2B_1} + \overrightarrow{B_1B_2} + \overrightarrow{B_2C_1} + \overrightarrow{C_1C_2} + \overrightarrow{C_2A_1} = 0$, hence $\overrightarrow{A_2B_1} + \overrightarrow{B_2C_1} + \overrightarrow{C_2A_1} = 0$.



But these vectors have the same length! Thus we apply the theorem again to see that they form an equilateral triangle, so they are rotations of each other (by 120° and 240°), as we know $\overrightarrow{A_1A_2}, \overrightarrow{B_1B_2}, \overrightarrow{C_1C_2}$ are. Hence triangles $A_1A_2B_1, B_1B_2C_1$, $C_1C_2A_1$ are also rotations of each other so they are congruent. Thus $A_1B_1 = B_1C_1 = C_1A_1$ and so $A_1B_1C_1$ is equilateral. As $C_1B_2 = B_2B_1$, we deduce that A_1, B_2 lie on the perpendicular bisector of B_1C_1, so A_1B_2 is the perpendicular bisector of B_1C_1. Analogously, B_1C_2 and C_1B_2 are also perpendicular bisectors in triangle $A_1B_1C_1$, so the three lines concur.

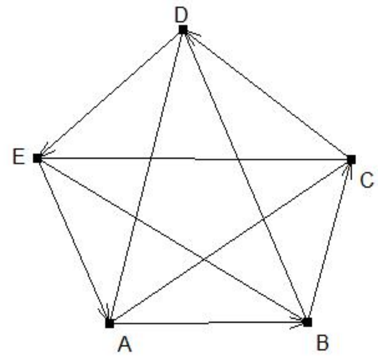
A well-known concept about vectors is their scalar product. It is widely used in geometry. Here we will present another product, less known but still useful.

The vectorial product of two vectors u, v (which we will denote here by $u \times v$) is the product of their lengths and the sine of the clockwise angle between them (thus the angle between u and v and the angle between v and u are not equal, they actually sum up to 360°). Clearly, $u \times v$ is twice the area of the triangle formed by the vectors u and v, possibly with a minus sign if one needs and angle greater than π to rotate u clockwise until it becomes parallel to v. In analytic geometry, if $u = (x, y)$ and $v = (x_1, y_1)$, then $u \times v = xy_1 - yx_1$.

The vectorial product is, like the scalar product, distributive with respect to addition. However, it is not commutative: in fact $a \times b + b \times a = 0$. Is is clear that two vectors u and v are parallel (or opposite) if and only if $u \times v = 0$, in particular $u \times u = 0$. Next we show a simple application of this concept.

Problem 4. Let $ABCDE$ be a pentagon. Suppose that four of its sides are parallel to the diagonals opposite to them. Prove that the fifth side is also parallel to the diagonal opposite to it.

Solution. Assume that $AB||CE, BC||DA, CD||EB$, and $DE||AC$. We must prove $EA||BD$. Let $a = \overrightarrow{AB}, b = \overrightarrow{BC}, c = \overrightarrow{CD}, d = \overrightarrow{DE}, e = \overrightarrow{EA}$.



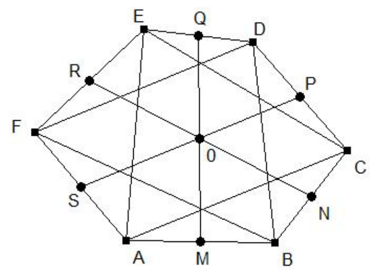
In terms of vectorial product, the problem becomes: $a \times (c + d) = b \times (d + e) = c \times (e + a) = d \times (a + b) = 0$ implies $e \times (b + c) = 0$. However, this directly follows from the identity $a \times (c+d) + b \times (d+e) + c \times (e+a) + d \times (a+b) + e \times (b+c) = 0$. We can check it by opening the brackets and grouping the terms into pairs of the form $a \times c + c \times a$, which are zero by definition.

Remark. Another proof of the identity is the fact that $a \times (c + d) = 2[ABC] - 2[ABE]$ and by summing up with the other analogous relations, all the areas will cancel each other. Indeed, as $a+b+c+d+e = 0$, $a \times (c+d) = -a \times (a+b+e) = (b + e) \times a = b \times a + e \times a = 2[ABC] - 2[ABE]$ (the sign is reversed if a, b, c, d, e go counterclockwise). This fact leads to a proof without using vectorial products, as $AB||DE$ is equivalent to $[ABC] = [ABE]$. However, our original proof works well even without the assumption that the pentagon is convex, which would pose significant difficulties to the synthetic proof. The next problem is another (harder) application of the vectorial product.

Problem 5. Let $ABCDEF$ be a convex hexagon. The midpoints of its sides are M, N, P, Q, R, S, respectively (M is the midpoint of AB, N of BC and so on). Prove that MQ, NR, PS are concurrent if and only if triangles ACE and BDF have equal areas.

Solution. Let us take a point O and set $\overrightarrow{OA} = a, \overrightarrow{OB} = b, \overrightarrow{OC} = c, \overrightarrow{OD} = d, \overrightarrow{OE} = e, \overrightarrow{OF} = f$. Then $\overrightarrow{OM} = \frac{a+b}{2}$ and so on.

The condition on the areas of ACE and BDF is easy to formulate: it rewrites as $(a - e) \times (c - e) = (b - f) \times (d - f)$ and after opening the brackets we write it as $a \times c + c \times e + e \times a = b \times d + d \times f + f \times b$.



Now it is harder to characterize the condition for concurrency of MQ, NR, PS. To ease it, we can recall the fact that we may choose O arbitrarily. Particularly, it would be comfortable to choose O as the intersection point of MQ and NR, implying that $\overrightarrow{OM} \times \overrightarrow{OQ} = \overrightarrow{ON} \times \overrightarrow{OP} = 0$, or $(a+b) \times (d+e) = (b+c) \times (e+f) = 0$. The condition now transforms into the collinearity of O, P, S, which can be rewritten as $(c+d) \times (f+a) = 0$.

The first condition is cyclic, the second would be if we could relax it (that two expressions equal zero imply the third equals zero). To make the second condition cyclic, let us sum up these expressions.

We have $(a+b) \times (d+e) + (b+c) \times (e+f) + (c+d) \times (f+a)$, which we can not bring to the first condition as it contains for example the factor $b \times e$ which is not present there. To eliminate the factor consider this expression: $(a+b) \times (d+e) - (b+c) \times (e+f) + (c+d) \times (f+a)$. Now, if we open the brackets, we get $(a \times d + d \times a) + (b \times e - b \times e) + (c \times f - c \times f) + a \times e - c \times + c \times a + b \times d - b \times f + d \times f = (b \times d + d \times f + f \times b) - (a \times c + c \times e + e \times a)$.

We have $(a+b) \times (d+e) = (b+c) \times (e+f) = 0$, $(c+d) \times (f+a) = 0$ is equivalent to $(a+b) \times (d+e) - (b+c) \times (e+f) + (c+d) \times (f+a) = 0$. But as we see from the brackets, this is equivalent to $a \times c + c \times e + e \times a = b \times d + d \times f + f \times b$.

We have developed some strong skills in dealing with vectors (and hexagons). Now we can boldly attack one of the hardest geometry problems proposed at the International Mathematical Olympiad in the last decade.

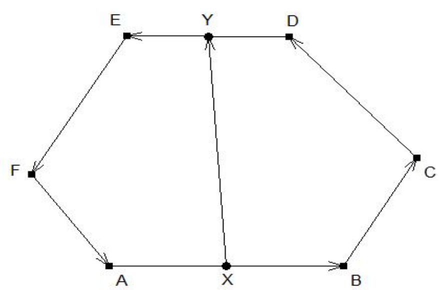
Problem 6. In the convex hexagon $ABCDEF$ it is known that the distance between the midpoints of any two opposite sides is equal to $\frac{\sqrt{3}}{2}$ times the sum of their length. Prove that all the angles of the hexagon are equal.

(IMO, 2003)

Solution. Consider $\overrightarrow{AB} = a, \overrightarrow{BC} = b, \ldots, \overrightarrow{FA} = f$. If X is the midpoint of AB and Y is the midpoint of DE, then

$$\frac{a}{2} + b + c + \frac{d}{2} = \overrightarrow{XY} = -\frac{a}{2} - f - e - \frac{d}{2} \Longrightarrow \overrightarrow{XY} = \frac{b - e + c - f}{2}.$$

Consequently, if $a - d = x, c - f = y, e - b = z$, then $\overrightarrow{XY} = \frac{y-z}{2}$.



By using this notation, the given property becomes

$$|\frac{y-z}{2}| = \frac{\sqrt{3}}{2}(|a| + |d|) \geq \frac{\sqrt{3}}{2}|a - d| \Longrightarrow |x|\sqrt{3} \leq |y - z|. \quad (1)$$

In conclusion $3|x|^2 \leq |y|^2 - 2yz + |z|^2$ and by writing similar expressions for the other two pairs of opposite sides we get $3|y|^2 \leq |x|^2 - 2xz + |z|^2$ and $3|z|^2 \leq |y|^2 - 2yx + |x|^2$. It is clear that these inequalities imply $|x + y + z|^2 \leq 0$, hence $x + y + z = 0$ and we have equality in (1) and in all of its analogous relations. In particular, we obtained that the opposite sides of the given hexagon are parallel.

Furthermore, consider a triangle UVW such that $\overrightarrow{UV} = x, \overrightarrow{VW} = y$ and $\overrightarrow{WU} = z$. We know from (1) that the length of the median $WM = \frac{1}{2}|y - z|$ is $\frac{\sqrt{3}}{2}UV$ (and of course, similar relations hold for the other medians). Choose a point W_1 on the same side of UV as W such that triangle UVW_1 is equilateral. Now, if $\angle UWV < 60^\circ$, then the vertex W lies outside the circumcircle of $\triangle UVW_1$, which is internally tangent to the circle centered at M with radius $\frac{\sqrt{3}}{2}UV$. A similar contradiction is obtained in the case when $\angle UWV > 60^\circ$, thus $\triangle UVW$ should necessarily be equilateral. Finally, this assertion implies the fact that the hexagon is equiangular and the problem is solved.

Based on the solution to this problem we can easily solve another difficult question:

Problem 7. Consider a convex hexagon $ABCDEF$ with $AD = BC + EF$, $BE = CD + FA$, and $CF = AB + DE$. Prove that

$$\frac{AB}{DE} = \frac{EF}{BC} = \frac{CD}{FA}.$$

(Kazakhstan, 2006)

Solution. As in the previous problem, let $\overrightarrow{AB} = a, \overrightarrow{BC} = b, \ldots, \overrightarrow{FA} = f$. It is easy to get that $\overrightarrow{AD} = \frac{1}{2}(a+b+c-d-e-f)$. Thus setting $a-d = x, c-f = y, e-b = z$, the given conditions become

(1) $\frac{|x-z+y|}{2} = |b| + |e| = |b| + |-e| \geq |b - e| = |z|$, with equality when $BC||EF$;

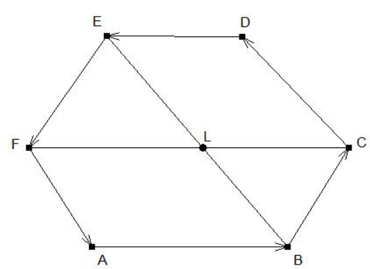
(2) $\frac{|y-x+z|}{2} \geq |x|$;

(3) $\frac{|z-y+x|}{2} \geq |y|$.

At this point some new notations are clearly needed. Let $x - z + y = k, y - x + z = l$ and $z - y + x = m$. Then (1), (2), (3) become:

$$|k| \geq |l + m|, |l| \geq |k + m|, |m| \geq |l + k|.$$

By squaring these relations and summing up we obtain $0 \geq |x + y + z|^2$ implying $x + y + z = 0$. Therefore all previous inequalities are actually equalities. In conclusion, $ABCDEF$ has all of its opposite sides parallel.



In addition, since $k + l + m = x + y + z$, it follows that $k + l + m = 0$ (i.e. $\overrightarrow{AD}, \overrightarrow{CF}, \overrightarrow{EB}$ can form a triangle). Hence,

$$\overrightarrow{FC} = \overrightarrow{AD} + \overrightarrow{EB} = \frac{k+m}{2} = x = a - d \Longleftrightarrow AB||DE||CF.$$

Analogously, it can be inferred that $BC||AD||EF$ and $AF||BE||CD$. Finally, if $BE \cap CF = \{L\}$, since $ABLF$ and $CDEL$ are parallelograms,

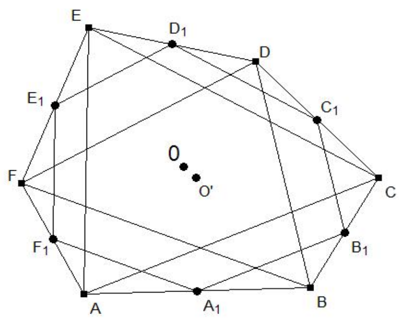
$$\frac{AB}{DE} = \frac{FL}{LC} = \frac{EF}{BC} = \frac{EL}{LB} = \frac{CD}{AF}.$$

Finally, we are ready for another "tough nut" recently proposed for a Team Selection Test in Vietnam.

Problem 8. Let $ABCDEF$ be a convex hexagon with perimeter P, and let the midpoints of its sides form the hexagon $A_1B_1C_1D_1E_1F_1$ with perimeter P'. If $A_1B_1C_1D_1E_1F_1$ has all angles equal, then prove that $P' \leq \frac{\sqrt{3}}{2}P$.

(Vietnam TST, 2004)

Solution. We can assume that A_1 is the midpoint of AB, B_1 is the midpoint of BC, and so on. Then A_1B_1 is parallel to AC and $\overrightarrow{A_1B_1} = \frac{1}{2}\overrightarrow{AC}$. Analogously, $\overrightarrow{C_1D_1} = \frac{1}{2}\overrightarrow{CE}$ and $\overrightarrow{E_1F_1} = \frac{1}{2}\overrightarrow{EA}$. As the angles of $A_1B_1C_1D_1E_1F_1$ are congruent, they all equal $120°$. Hence the supports of the vectors $\overrightarrow{A_1B_1}, \overrightarrow{C_1D_1}, \overrightarrow{E_1F_1}$ form an equilateral triangle. However, $\overrightarrow{A_1B_1} + \overrightarrow{C_1D_1} + \overrightarrow{E_1F_1} = \frac{1}{2}(\overrightarrow{AC} + \overrightarrow{CE} + \overrightarrow{EA}) = 0$, so we conclude that these three vectors form an equilateral triangle and so ACE is equilateral. Similarly, we get that BDF is also equilateral, with sides parallel to the sides of ACE, but inversely oriented.



Let O be the circumcenter of triangle ACE and O' be the circumcenter of triangle BDF. Set $\overrightarrow{OO'} = x$. We have $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OO'} + \overrightarrow{O'B} = u + x$, where $u = \overrightarrow{AO} + \overrightarrow{O'B}$. Analogously, $\overrightarrow{CD} = v + x, \overrightarrow{EF} = w + x$, where $v = \overrightarrow{CO} + \overrightarrow{O'D}, w = \overrightarrow{EO} + \overrightarrow{O'F}$. It is clear that $\overrightarrow{CO}, \overrightarrow{O'D}$ are rotations of $\overrightarrow{AO}, \overrightarrow{O'B}$ by $120°$, so v is rotation of u by $120°$. The same thing is true for w. Now we prove that $|u + x| + |v + x| + |w + x| \geq |u| + |v| + |w|$. Indeed, if we construct points XYZ such that $\overrightarrow{OX} = u, \overrightarrow{OY} = v, \overrightarrow{OZ} = w$, then O will be the center of the equilateral triangle XYZ (since u, v, w are rotates of each other by $120°$). The inequality will then become $O'X + O'Y + O'Z \geq OX + OY + OZ$, true since O is the Fermat-Toricelli point of triangle XYZ. Now, if the sidelength of ACE is a and the sidelength of BDF is b, then $AO = \frac{a}{\sqrt{3}}, BO' = \frac{b}{\sqrt{3}}$ and the angle between $\overrightarrow{AO}$ and $\overrightarrow{BO'}$ is $60°$. Thus by the Law of Cosines, $|u| = |\overrightarrow{AO} + \overrightarrow{O'B}| = \frac{1}{\sqrt{3}}\sqrt{a^2 - ab + b^2} \geq \frac{1}{2\sqrt{3}}(a+b)$. Therefore $AB + CD + EF \geq \frac{3}{2\sqrt{3}}(a+b) = \frac{\sqrt{3}}{2}(a+b)$. Analogously, $BC + DE + FA \geq \frac{\sqrt{3}}{2}(a+b)$, so $P \geq \sqrt{3}(a+b)$. It remains to see that $P' = \frac{3}{2}(a+b)$ thus $P \geq \frac{2}{\sqrt{3}}P'$, as desired.

Bibliography

[1] Complex Numbers from A to ... Z, Titu Andreescu and Dorin Andrica, Birkhauser, 2005

[2] A report on the 46th International Mathematical Olympiad in Mexico, 2005.

[3] 44th International Mathematical Olympiad, *Short-listed Problems.* Tokyo, Japan, July 2003.

[4] *The Vietnamese Mathematical Olympiad in 2004.* Hanoi, July 2004.

[5] www.mathlinks.ro

Authors:

Iurie Boreico, Harvard University, USA, iurie.boreico@gmail.com

Liubomir Chiriac, Princeton University, USA, LiubomirC@gmail.com