Junior problems

J73. Let

$$a_n = \begin{cases} n^2 - n, & \text{if 4 divides } n^2 - n \\ n - n^2, & \text{otherwise.} \end{cases}$$

Evaluate $a_1 + a_2 + \ldots + a_{2008}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J74. A triangle has altitudes h_a, h_b, h_c and inradius r. Prove that

$$\frac{3}{5} \le \frac{h_a - 2r}{h_a + 2r} + \frac{h_b - 2r}{h_b + 2r} + \frac{h_c - 2r}{h_c + 2r} < \frac{3}{2}.$$

Proposed by Oleh Faynshteyn, Leipzig, Germany

J75. Jimmy has a box with n not necessarily equal matches. He is able to construct with them a cyclic n-gon. Jimmy then constructs other cyclic n-gons with these matches. Prove that all of them have the same area.

Proposed by Ivan Borsenco, University of Texas at Dallas

J76. Let $a, b, c \ge 1$ be real numbers such that $a + b + c = 2abc$. Prove that

$$\sqrt[3]{(a+b+c)^2} \ge \sqrt[3]{ab-1} + \sqrt[3]{bc-1} + \sqrt[3]{ca-1}.$$

Proposed by Bruno de Lima Holanda, Fortaleza, Brazil

J77. Prove that in each triangle

$$\frac{1}{r}\left(\frac{b^2}{r_b} + \frac{c^2}{r_c}\right) - \frac{a^2}{r_b r_c} = 4\left(\frac{R}{r_a} + 1\right).$$

Proposed by Dorin Andrica, Babes-Bolyai University and Khoa Lu Nguyen, MIT

J78. Let p and q be odd primes. Prove that for any odd integer $d > 0$ there is an integer r such that the numerator of the rational number

$$\sum_{n=1}^{p-1} \frac{[n \equiv r (\mathrm{mod}\ q)]}{n^d}$$

is divisible by p, where $[Q]$ is equal to 1 or 0 as the proposition Q is true or false.

Proposed by Robert Tauraso, Roma, Italy

Senior problems

S73. The zeros of the polynomial $P(x) = x^3 + x^2 + ax + b$ are all real and negative. Prove that $4a - 9b \leq 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S74. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$(a^a + b^a + c^a)\left(a^b + b^b + c^b\right)(a^c + b^c + c^c) \geq \left(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}\right)^3.$$

Proposed by Jose Luis-Diaz Barrero, Spain

S75. Let ABC be a right triangle with $\angle A = 90°$. Let D be an arbitrary point on BC and let E be its reflection in the side AB. Denote by F and G the intersections of AB with lines DE and CE, respectively. Let H be the projection of G onto BC and let I be the intersection of HF and CE. Prove that G is the incenter of triangle AHI.

Proposed by Son Hong Ta, Ha Noi University, Vietnam

S76. Let x, y, and z be complex numbers such that

$$(y + z)(x - y)(x - z) = (z + x)(y - z)(y - x) = (x + y)(z - x)(z - y) = 1.$$

Determine all possible values of $(y + z)(z + x)(x + y)$.

Proposed by Alex Anderson, New Trier High School, Winnetka, USA

S77. Let ABC be a triangle and let X be the projection of A onto BC. The circle with center A and radius AX intersects line AB at P and R and line AC at Q and S such that $P \in AB$ and $Q \in AC$. Let $U = AB \cap XS$ and $V = AC \cap XR$. Prove that lines BC, PQ, UV are concurrent.

Proposed by F. J. Garcia Capitan, Spain and J. B. Romero Marquez, Spain

S78. Let $ABCD$ be a quadrilateral inscribed into a circle $C(O, R)$ and let (O_{ab}), (O_{bc}), (O_{cd}), (O_{ad}) be the symmetric circles to $C(O)$ with respect to AB, BC, CD, DA, respectively. The pairs of circles $(O_{ab}), (O_{ad})$; $(O_{ab}), (O_{bc})$; $(O_{bc}), (O_{cd})$; $(O_{cd}), (O_{ad})$ intersect again at A', B', C', D'. Prove that A', B', C', D' lie on a circle of radius R.

Proposed by Mihai Miculita, Oradea, Romania

Undergraduate problems

U73. Prove that there is no polynomial $P \in \mathbb{R}[X]$ of degree $n \geq 1$ such that $P(x) \in \mathbb{Q}$ for all $x \in \mathbb{R} \backslash \mathbb{Q}$.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

U74. Prove that there is no differentiable function $f : (0, 1) \to \mathbb{R}$ for which $\sup\limits_{x \in E} |f'(x)| = M \in \mathbb{R}$, where E is a dense subset of the domain, and $|f|$ is nowhere differentiable on $(0, 1)$.

Proposed by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

U75. Let P be a complex polynomial of degree $n > 2$ and let A and B be 2×2 complex matrices such that $AB \neq BA$ and $P(AB) = P(BA)$. Prove that $P(AB) = cI_2$ for some complex number c.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania

U76. Let $f : [0, 1] \to \mathbb{R}$ be an integrable function such that $\displaystyle\int_0^1 xf(x)dx = 0$. Prove that $\displaystyle\int_0^1 f^2(x)dx \geq 4 \left(\int_0^1 f(x)dx \right)^2$.

Proposed by Cezar Lupu, University of Bucharest, Romania and Tudorel Lupu, Constanza, Romania

U77. Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^2. Prove that if the function $\sqrt{f(x)}$ is differentiable, then its derivative is a continuous function.

Suggested by Gabriel Dospinescu, Ecole Normale Superieure, France

U78. Let $n = \prod_{i=1}^{k} p_i$, where $p_1, p_2, \ldots, p_k$ are distinct odd primes. Prove that there is a $A \in M_n(\mathbb{Z})$ with $A^m = I_n$ if and only if the symmetric group S_{n+k} has an element of order m.

Proposed by Jean-Charles Mathieux, Dakar University, Senegal

Olympiad problems

O73. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+a+b+c \geq \frac{2(a+b+c)^3}{3(ab+bc+ca)}.$$

Proposed by Pham Huu Duc, Ballajura, Australia

O74. Consider a non-isosceles acute triangle ABC such that $AB^2+AC^2=2BC^2$. Let H and O be the orthocenter and the circumcenter of triangle ABC, respectively. Let M be the midpoint of BC and let D be the intersection of MH with the circumcircle of triangle ABC such that H lies between M and D. Prove that AD, BC, and the Euler line of triangle ABC are concurrent.

Proposed by Daniel Campos Salas, Costa Rica

O75. Let a, b, c, d be positive real numbers such that $a^2+b^2+c^2+d^2=1$. Prove that

$$\sqrt{1-a}+\sqrt{1-b}+\sqrt{1-c}+\sqrt{1-d} \geq \sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}.$$

Proposed by Vasile Cartoaje, Ploiesti, Romania

O76. A triple of different subsets S_i, S_j, S_k of a set with n elements is called a "triangle". Define its perimeter by

$$\left|(S_i \cap S_j) \cup (S_j \cap S_k) \cup (S_k \cap S_i)\right|.$$

Prove that the number of triangles with perimeter n is $\dfrac{1}{3}(2^{n-1}-1)(2^n-1)$.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

O77. Consider the polinomials $f, g \in \mathbb{R}[X]$. Prove that there is a nonzero polynomial $P \in \mathbb{R}[X, Y]$ such that $P(f, g) = 0$.

Proposed by Iurie Boreico, Harvard University, USA

O78. Let ABC be a triangle and let M, N, P be the midpoints of sides BC, CA, AB, respectively. Denote by X, Y, Z the midpoints of the altitudes emerging from vertices A, B, C, respectively. Prove that the radical center of the circles AMX, BNY, CPZ is the center of the nine-point circle of triangle ABC.

Proposed by Cosmin Pohoata, Bucharest, Romania