A minimum problem

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Abstract

Given a triangle, let $R_1,\ R_2,\ R_3$ be the distances from a certain point to the vertices of the triangle. The minimum of $R_1R_2 + R_2R_3 + R_3R_1$ is computed when the considered point runs through triangle's plane.

1 Introduction

Consider a triangle ABC, denote by $a,\ b,\ c$ the lengths of its sides and assume $a \geq b \geq c$. For a point M belonging to triangle's plane denote $MA = R_1$, $MB = R_2$ and $MC = R_3$. The inequality

$$R_1^2 + R_2^2 + R_3^2 \geq \frac{1}{3}(a^2 + b^2 + c^2) \tag{1}$$

is well known and the equality holds if and only if M lies in the center of mass of the triangle.

It is also well known the result of Jacob Steiner (cf. [1]) asserting that, if $A < \dfrac{2\pi}{3}$ then $R_1 + R_2 + R_3$ is minimized if M coincides with Toricelli's point. If $A \geq \dfrac{2\pi}{3}$ then $R_1 + R_2 + R_3 \geq b + c$ and this quantity is minimized when M coincides with A.

2 The minimum of $R_1R_2 + R_2R_3 + R_3R_1$

Making use of the preceding notation we prove the following fact.
Theorem. *If $a \geq b \geq c$ then*

$$S(M) := R_1R_2 + R_2R_3 + R_3R_1 \geq bc. \tag{2}$$

Proof. The desired inequality obviously holds when M lies on one of triangle's vertices. Next assume $R_1, R_2, R_3 > 0$. We take into consideration the two cases which can occur:

Case 1*:* Assume

$$R_1 + R_2 + R_3 > b + c. \tag{3}$$

We have the inequalities $R_2 \geq |c - R_1|$ and $R_3 \geq |b - R_1|$, which imply

$$R_2R_3 \geq |(c - R_1)(b - R_1)| \geq bc - bR_1 - cR_1 + R_1^2,$$

hence $S(M) \geq bc + R_1(R_1 + R_2 + R_3 - b - c) > bc$.

Case 2*:* Assume

$$R_1 + R_2 + R_3 \leq b + c. \tag{4}$$

Denote by $x,\ y,\ z$ the measures of the angles $\widehat{AMB},\ \widehat{BMC},\ \widehat{CMA}$, respectively. If M lies outside of the triangle, then we have either $x = y + z$ or a similar relation. When M does not lie outside of the triangle we have $x + y + z = 2\pi$. It should be said that the cases when one of $x,\ y,\ z$ equals 0 or π are considered as well. In either case we have

$$\begin{aligned}\cos x + \cos y + \cos z + \frac{3}{2} &= \cos(y+z) + \cos y + \cos z + \frac{3}{2}\\ &= 2\cos^2\frac{x+y}{2} - 1 + 2\cos\frac{x+y}{2}\cos\frac{x-y}{2} + \frac{3}{2}\\ &= \frac{1}{2}\left(\left(2\cos\frac{x+y}{2} + \cos\frac{x-y}{2}\right)^2 + \sin^2\frac{x-y}{2}\right)^2\\ &\geq 0.\end{aligned}$$

Thus

$$\cos x + \cos y + \cos x \geq -\frac{3}{2}. \tag{5}$$

Making use of the cosine theorem in each of the triangle $AMB,\ BMC$ and CMA, we can write (5) as

$$\frac{R_1^2 + R_2^2 - c^2}{2R_1R_2} + \frac{R_2^2 + R_3^2 - a^2}{2R_2R_3} + \frac{R_3^2 + R_1^2 - b^2}{2R_3R_1} \geq -\frac{3}{2}.$$

This last inequality is further equivalent to

$$R_1R_2 + R_2R_3 + R_3R_1 \geq \frac{a^2R_1 + b^2R_2 + c^2R_3}{R_1 + R_2 + R_3}. \tag{6}$$

Note that (6) holds even in the case when one of the triangles degenerates. Since $a \geq b$ it then follows $S(M) \geq \dfrac{b^2(R_1 + R_2) + c^2R_3}{R_1 + R_2 + R_3}$. Thus it will suffice to prove that $\dfrac{b^2(R_1 + R_2) + c^2R_3}{R_1 + R_2 + R_3} \geq bc$, that is

$$(b - c)(b(R_1 + R_2) - cR_3) \geq 0. \tag{7}$$

Since $R_1 + R_2 \geq c$, by (4) we get $R_3 \geq b$ hence $b(R_1 + R_2) \geq bc \geq cR_3$. Consequently (7) holds and we are done.

Remark. In order to have equality it is necessary to have equalities in (5), (7) as well as in $a \geq b$, which implies that our triangle is equilateral and M lies on its center. Consequently the equality $S(M) = bc$ holds in the following cases:

a) If $a = b = c$ then M has to lie either on one of the vertices A, B, C or on triangle's center.

b) If $a = b$ then M must coincide either with A or with B.

c) If $a > b$ then M must coincide with A.

We finally note that (6) was proposed by V. Cârtoaje as a problem in a scholar competition that was hold in 1970 in Romania (see [3]). Also, (6) is a special case of a more general inequality to Klamkin [2].

References

[1] R. Courant, H. Robbins, *What is Mathematics?* Oxford University Press, 1941, pp. 373-376.

[2] M. S. Klamkin, *Geometric inequalities via the polar moment of inertia.* Math. Mag. 48(1975), pp. 44-46.

[3] C. Ottescu, L. Panaitopol, *Problems from the mathematical competition.* Editura Didactică şi Pedagogică, Bucharest, 1976 (in Romanian).

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