On vector properties of an equilateral triangle

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The equilateral triangle is one of the basic geometry figures. Every mathematician appreciates its harmony and beauty. In this article we prove a vector based relation that helps us investigate the nature of the equilateral triangle through different problems.

Theorem 1. Let O be the center of an equilateral triangle ABC and let M be an arbitrary point in its plane. Denote by M_a, M_b and M_c the projections of M on the lines BC, CA, and AB, respectively. The following relation holds

$$\overrightarrow{MM_a} + \overrightarrow{MM_b} + \overrightarrow{MM_c} = \frac{3}{2}\overrightarrow{MO}.$$

Proof: Let AA_1, BB_1, and CC_1 be the altitudes of triangle ABC. Denote by S_a, S_b and S_c the algebraic areas of triangles MBC, MCA, and MAB. First of all observe that

$$\frac{S_a}{S_{ABC}} = \frac{\overrightarrow{MM_a}}{\overrightarrow{AA_1}}.$$

Therefore

$$\overrightarrow{MM_a} = \frac{\overrightarrow{MM_a}}{\overrightarrow{AA_1}} \cdot \overrightarrow{AA_1} = \frac{3}{2} \cdot \frac{S_a}{S_{ABC}} \cdot \overrightarrow{AO} = \frac{3}{2} \cdot \frac{S_a}{S_{ABC}} \cdot (\overrightarrow{AM} + \overrightarrow{MO}).$$

Analogously,

$$\overrightarrow{MM_b} = \frac{3}{2} \cdot \frac{S_b}{S_{ABC}} \cdot (\overrightarrow{BM} + \overrightarrow{MO}), \qquad \overrightarrow{MM_c} = \frac{3}{2} \cdot \frac{S_c}{S_{ABC}} \cdot (\overrightarrow{CM} + \overrightarrow{MO}).$$

Thus

$$\overrightarrow{MM_a} + \overrightarrow{MM_b} + \overrightarrow{MM_c} = \frac{3(S_a\overrightarrow{AM} + S_b\overrightarrow{BM} + S_c\overrightarrow{CM})}{2S_{ABC}} + \frac{3(S_a + S_b + S_c)}{2S_{ABC}} \cdot \overrightarrow{MO}.$$

Observe that $S_a + S_b + S_c = S_{ABC}$ and recall the fundamental formula for algebraic areas in triangle ABC

$$S_a \cdot \overrightarrow{AM} + S_b \cdot \overrightarrow{BM} + S_c \cdot \overrightarrow{CM} = \overrightarrow{0}.$$

It follows that

$$\overrightarrow{MM_a} + \overrightarrow{MM_b} + \overrightarrow{MM_c} = \frac{3}{2}\overrightarrow{MO},$$

and the proof is complete.

Further, we want to bring to the reader's attention three problems that can be solved using the result in Theorem 1.

Problem 1. Consider an equilateral triangle ABC and let M and N be two arbitrary points in the plane of ABC. Denote by M_a and N_a the projections of M and N on the line BC. Analogously we define M_b, N_b and M_c, N_c. Prove that

$$\overrightarrow{M_aN_a} + \overrightarrow{M_bN_b} + \overrightarrow{M_cN_c} = \frac{3}{2}\overrightarrow{MN}.$$

Solution: From Theorem 1, we have

$$\overrightarrow{MM_a} + \overrightarrow{MM_b} + \overrightarrow{MM_c} = \frac{3}{2}\overrightarrow{MO}, \quad \overrightarrow{NN_a} + \overrightarrow{NN_b} + \overrightarrow{NN_c} = \frac{3}{2}\overrightarrow{NO}.$$

It follows that

$$\sum \overrightarrow{M_aN_a} = \sum(\overrightarrow{M_aM} + \overrightarrow{MN} + \overrightarrow{NN_a}) = \frac{3}{2}\left(-\overrightarrow{MO} + 2\overrightarrow{MN} + \overrightarrow{NO}\right) = \frac{3}{2}\overrightarrow{MN},$$

and the problem is solved.

Problem 2. Consider an equilateral triangle ABC and let M and N be two arbitrary points in the plane of ABC. Denote by M_a and N_a the projections of M and N on the line BC. Analogously we define M_b, N_b and M_c, N_c. Prove that

$$M_aN_a^2 + M_bN_b^2 + M_cN_c^2 = \frac{3}{2}MN^2.$$

Solution: Because M_a and N_a are projections of M and N onto line BC, we get

$$\overrightarrow{M_aN_a} \cdot \overrightarrow{MN} = \overrightarrow{M_aN_a} \cdot (\overrightarrow{MM_a} + \overrightarrow{M_aN_a} + \overrightarrow{NN_a}) = \overrightarrow{M_aN_a} \cdot \overrightarrow{M_aN_a} = M_aN_a^2.$$

Thus

$$M_aN_a^2 + M_bN_b^2 + M_cN_c^2 = (\overrightarrow{M_aN_a} + \overrightarrow{M_bN_b} + \overrightarrow{M_cN_c}) \cdot \overrightarrow{MN}.$$

Using the result in problem 1:

$$\overrightarrow{M_aN_a} + \overrightarrow{M_bN_b} + \overrightarrow{M_cN_c} = \frac{3}{2}\overrightarrow{MN},$$

we obtain

$$M_aN_a^2 + M_bN_b^2 + M_cN_c^2 = \frac{3}{2}\overrightarrow{MN} \cdot \overrightarrow{MN} = \frac{3}{2}MN^2.$$

Problem 3. Let O be the center of equilateral triangle ABC with circumradius R and let M be an arbitrary point in the plane of ABC. Denote by M_a, M_b and M_c the projections of M on the lines BC, CA and AB, respectively. Prove that

$$MM_a^2 + MM_b^2 + MM_c^2 = \frac{3}{2}MO^2 + \frac{3}{4}R^2.$$

Solution: Denote by A_1, B_1, C_1 the midpoints of BC, CA, AB. We have

$$MM_a^2 + MM_b^2 + MM_c^2 = \sum(\overrightarrow{MM_a})^2 = \sum(\overrightarrow{MO} + \overrightarrow{OA_1} + \overrightarrow{A_1M_a})^2 =$$

$$= \sum(MO^2 + OA_1^2 + A_1M_a^2 + 2\overrightarrow{MO}\cdot\overrightarrow{OA_1} + 2\overrightarrow{MO}\cdot\overrightarrow{A_1M_a} + 2\overrightarrow{OA_1}\cdot\overrightarrow{A_1M_a}) =$$

$$= 3MO^2 + \frac{3}{4}R^2 + \sum A_1M_a^2 + 2\overrightarrow{MO}\cdot\sum\overrightarrow{OA_1} + 2\overrightarrow{MO}\cdot\sum\overrightarrow{A_1M_a} + 2\sum\overrightarrow{OA_1}\cdot\overrightarrow{A_1M_a}.$$

It is not difficult to see that $\sum\overrightarrow{OA_1} = 0$ and $\sum\overrightarrow{OA_1}\cdot\overrightarrow{A_1M_a} = 0$.

Observe that M_a and A_1 are projections of M and O onto the line BC, therefore $\overrightarrow{A_1M_a}\cdot\overrightarrow{MO} = -A_1M_a^2$. Also, applying the result in problem 2 for points M and O we obtain

$$A_1M_a^2 + B_1M_b^2 + C_1M_c^2 = \frac{3}{2}MO^2.$$

It follows that

$$MM_a^2 + MM_b^2 + MM_c^2 = 3MO^2 + \frac{3}{4}R^2 + \sum A_1M_a^2 - 2\sum A_1M_a^2 = \frac{3}{2}MO^2 + \frac{3}{4}R^2.$$

From this problem we have two nice corollaries.

Corollary 1. For every point M in the plane of an equilateral triangle the following inequality holds

$$MM_a^2 + MM_b^2 + MM_c^2 \geq 3R^2.$$

Corollary 2. The geometrical locus of points M that satisfy $MM_a^2 + MM_b^2 + MM_c^2 = k$ is a circle centered at O with radius $\frac{2}{3}\sqrt{k - \frac{3}{4}R^2}$.

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