Junior problems

J61. Find all pairs (m, n) of positive integers such that

$$m^2 + n^2 = 13(m + n).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J62. Consider a right-angled triangle ABC with $\angle A = 90^\circ$. Let $E \in AC$ and $F \in AB$ such that $\angle AEF = \angle ABC$ and $\angle AFE = \angle ACB$. Denote by E' and F' the projections of E and F onto BC, respectively. Prove that

$$E'E + EF + FF' \leq BC$$

and determine when equality holds.

Proposed by Alex Anderson, New Trier High School, Winnetka, USA

J63. Find the least n such that no matter how we color an $n \times n$ lattice point grid in two colors we can always find a parallelogram with all vertices to be monochromatic.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

J64. Let a, b, c be positive real numbers. Prove that

$$\frac{b+c}{a + \sqrt[3]{4(b^3+c^3)}} + \frac{c+a}{b + \sqrt[3]{4(a^3+c^3)}} + \frac{a+b}{c + \sqrt[3]{4(a^3+b^3)}} \leq 2.$$

Proposed by José Luis Díaz-Barrero, Barcelona, Spain

J65. Prove that the interval $(2^n + 1, 2^{n+1} - 1), n \geq 2$ contains an integer that can be represented as a sum of n prime numbers.

Proposed by Radu Sorici, University of Texas at Dallas, USA

J66. Let $a_0 = a_1 = 1$ and $a_{n+1} = 2a_n - a_{n-1} + 2$ for $n \geq 1$. Prove that $a_{n^2+1} = a_{n+1}a_n$ for all $n \geq 0$.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

Senior problems

S61. Let ABC be a triangle. Prove that

$$\frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\geq 4\sqrt{\frac{R}{r}},$$

where R and r are its circumradius and inradius, respectively.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S62. Let $ABCD$ be a parallelogram and let $X\in AB, Y\in BC, Z\in CD$, and $K\in AD$ such that $XZ\parallel BC\parallel AD$ and $YK\parallel AB\parallel CD$. Let $P=XZ\cap YK$ and $Q=BZ\cap DY$. Prove that A,P,Q are collinear.

Proposed by Juan Bosco Marquez and Francisco Javier Garcia Capitan, Spain

S63. Let a,b,c be positive real numbers such that $ab+bc+ca\geq 3$. Prove that

$$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}}\geq\frac{3}{\sqrt{2}}.$$

Proposed by Pham Huu Duc, Ballajura, Australia

S64. Let ABC be a triangle with centroid G and let g be a line through G. Line g intersects BC at a point X. The parallels to lines BG and CG through A intersect line g at points X_b and X_c, respectively. Prove that

$$\frac{1}{\overrightarrow{GX}}+\frac{1}{\overrightarrow{GX_b}}+\frac{1}{\overrightarrow{GX_c}}=0.$$

Proposed by Darij Grinberg, Germany

S65. Let n be an integer greater than 1 and let X be a set with $n+1$ elements. Let $A_1,A_2,\ldots,A_{2n+1}$ be subsets of X such that the union of any n has at least n elements. Prove that among these $2n+1$ subsets there exist three such that any two of them have a common element.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

S66. Consider a triangle ABC and let D and E be the reflections of vertices B and C into AC and AB, respectively. Let $F=BE\cap CD$ and let H_a be the projection of the altitude from A onto BC. Denote by F_a,F_b,F_c the projections of F onto BC,CA,AB, respectively. Prove that F_a,F_b,F_c,H_a are concyclic.

Proposed by Mihai Miculita, Oradea, Romania

Undergraduate problems

U61. Find the sum of the series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{i!j!}{(i+j+1)!}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U62. Let $x_1, x_2, \ldots, x_n > 0$ such that $x_1 + x_2 + \ldots + x_n = n$ and let $y_k = n - x_k$, $k = 1, 2, \ldots, n$. Prove that

$$x_1^{x_1} \cdot x_2^{x_2} \cdots x_n^{x_n} \geq \left(\frac{y_1}{n-1}\right)^{y_1} \cdot \left(\frac{y_2}{n-1}\right)^{y_2} \cdots \left(\frac{y_n}{n-1}\right)^{y_n}.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

U63. Let f and g be polynomials with complex coefficients and let a be a nonzero complex number. Prove that if

$$(f(x))^3 = (g(x))^2 + a$$

for all $x \in C$, then the polynomials f and g are constant.

Proposed by Magkos Athanasios, Kozani, Greece

U64. Let x be a real number. Define the sequence $(x_n)_{n\geq 1}$ recursively by $x_1 = 1$ and $x_{n+1} = x^n + nx_n$, $n \geq 1$. Prove that

$$\prod_{n=0}^{\infty}\left(1 - \frac{x^n}{x_{n+1}}\right) = e^{-x}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U65. Let A, B, C be 3×3 invertible matrices such that their elements are in the interval $[0, 1]$ and entries in each row sum up to 1. Prove that $AC^{-1}BA^{-1}CB^{-1}$ and $CA^{-1}BC^{-1}AB^{-1}$ have the same trace.

Proposed by Jean-Charles Mathieux, Dakar University, Sénégal

U66. Let $V = \{v_1, v_2, \ldots, v_k, \ldots\}$ be a set of vectors in R^n containing n linearly independent vectors. A finite subset $S \subset V$ is called "crucial" if the set $V \setminus S$ contains no n independent vectors, but every set $V \setminus T$ where $T \subset S$ does. Prove that there are finitely many "crucial" subsets.

Proposed by Iurie Boreico, Harvard University, USA

Olympiad problems

O61. Let a, b, c be positive numbers such that $4abc = a + b + c + 1$. Prove that

$$\frac{b^2+c^2}{a} + \frac{c^2+a^2}{b} + \frac{a^2+b^2}{c} \geq 2(ab+bc+ca).$$

Proposed by Ciupan Andrei, Bucharest, Romania

O62. Consider the Cartesian plane. Let us call a point X rational if both its coordinates are rational numbers. Prove that if a circle passes through three rational points, then it passes through infinitely many of them.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

O63. Let M and N be two point inside the circle $C(O)$ such that O is the midpoint of MN and let S be an arbitrary point on this circle. Let E and F be the second intersections of the lines SM and SN with the circle. Tangents at E and F to $C(O)$ intersect each other at I. Prove that the perpendicular bisector of the segment MN passes through the midpoint of SI.

Proposed by Son Hong Ta, Ha Noi University, Vietnam

O64. Let F_n be the n-th Fibonacci number. Prove that for all $n \geq 4$, $F_n + 1$ is not a prime.

Proposed by Dorin Andrica, Cluj-Napoca, Romania

O65. Let ABC be a triangle and let D, E, and F be the tangency points of its incircle $\gamma(I)$ with BC, CA, and AB, respectively. Let X_1 and X_2 be the intersections of line EF with the circumcircle $\rho(O)$ of triangle ABC. Similarly, define Y_1, Y_2, and Z_1, Z_2. Prove that the radical center of the circles DX_1X_2, EY_1Y_2, and FZ_1Z_2 lies on line OI.

Proposed by Cosmin Pohoata, Romania and Darij Grinberg, Germany

O66. Let m a fixed positive integer. Prove that there is a constant $c(m)$ such that for each integer $n > 0$, there is a prime number $p < c(m)n$ with the following property: the equation $k^{2^m} \equiv n (\bmod p)$ has integer solutions while the equation $k^{2^m} \equiv -n (\bmod p)$ does not have integer solutions.

Proposed by Adrian Zahariuc, Princeton University, USA