Introductory Level

Introductory Examples

Pigeonhole Principle has practically no prerequisites, but it needs a lot of intuition and creativity, because usually it is well hidden in the statement of a problem. One good way to get acquainted with this method is to solve a good amount of problems. This section of the book does not require outstanding math knowledge from the reader, but it does require ingeniosity. Indeed, all problems can be solved using simple, sometimes tricky observations, without technical complications. Let us go now through some examples of such kind of problems.

The first examples show that Pigeonhole Principle can be applied together with extremal argument.

Example 2.01. Twenty one boys have a total of two hundred dollars in notes. Prove that it is possible to find two boys who have the same amount of money.

Solution. Assume the contrary, all boys have different amount of money. Thus the total amount of money is at least

$$0 + 1 + 2 + \ldots + 20 = \frac{20 \cdot 21}{2} = 210$$

dollars, a contradiction. It follows that we can find two boys who have the same amount of money.

Example 2.02. There are n gentlemen on a banquet. We know that all acquaintances between them are symmetric, if gentleman A knows gentleman B, then gentleman B knows gentleman A. Prove that among them we can find two gentlemen who have the same number of acquaintances.

Solution. Assign to a gentleman a number i if he has i acquaintances. Suppose there are no gentlemen that have the same number. We have n gentlemen and n possible numbers to be assigned: $0, 1, 2, \ldots, n-1$. However, numbers 0 and $n-1$ cannot be assigned at the same time, because it is impossible to have both a gentleman who knows everybody and a gentleman who knows nobody. Thus we have at most $n-1$ different numbers assigned to n gentlemen. By the Pigeonhole Principle there exist two gentlemen with the same assigned numbers. This means that two gentlemen have the same number of acquaintances.

Remark. In the graph language the problem can restated as follows: in every graph there exist two vertices that have the same degree.

Pigeonhole Principle can be met in the geometry type of problems. The main hint is to find a comfortable dissecting of the given figure to apply the principle.

Example 2.03. Prove that among any ten points inside the equilateral triangle of side 1 there are two whose distance apart is at most $\frac{1}{3}$.

Solution. Mark points at intervals of one-third along each side. Join corresponding points to divide the triangle into nine triangles. By the Pigeonhole Principle if we have ten points, at least one of the small triangles must contain two points. No two points in this region are more than $\frac{1}{3}$ distance apart and we are done.

Example 2.04. Fifty-one small insects are placed inside a square of side 1. Prove that at any moment there are at least three insects which can be covered by a single disk of radius $\frac{1}{7}$.

Solution. Subdivide the square into 25 small squares of side $\frac{1}{5}$. By the Pigeonhole Principle there will exist three insects in one of these squares of side $\frac{1}{5}$ with diagonal $\frac{\sqrt{2}}{5}$. A circumcircle of this square has radius $\frac{\sqrt{2}}{10} < \frac{1}{7}$. If we circumscribe a concentric circle with radius $\frac{1}{7}$, it will cover this square completely.

There are a lot of problems that are related with a board that is colored, tiled, or has some numbers or pieces on it. The following two examples give two possible ideas how to tackle them.

Example 2.05. What is maximum number of knights you can place on a chessboard such that none is attacking another?

Solution. Observe that we can have 32 knights on the blackboard that do not attack each other. Place every knight on the white cell, then every knight attacks only black cells and the condition is satisfied. Assume there are at least 33 knights on the chessboard. Divide the chessboard into eight small 2×4 boards. By the Pigeonhole Principle there will exist a 2×4 board that has five knights on it. But we cannot five knights on this small board as we can see there are four pairs of cells that attack each other, a contradiction. It follows that there can be placed at most 32 knights and we are done.

Example 2.06. The squares of an 9×9 checkerboard are filled with numbers $\{1, 2, \ldots, 81\}$. Prove that there exist two squares with a common side, such that the difference between the numbers in them is at least 6.

Solution. In some square there is written 1, and in some square 81. Let us look at the shortest "horizontal-vertical" way from the first square to the second. At each step we move from a square which has a common side with the previous one. It is not difficult to see that this way consists of at most 16 steps. Assume the difference between any two squares on the blackboard does not exceed 5. The difference between 1 and 81 is 80, and since $5 \cdot 16 = 80$, we get that on the path joining them all differences are 5. But we can have only one such path, while on the blackboard these two squares can be connected in more than one way, a contradiction. Thus, there exist two adjacent squares with the difference 6.

Remark. It can be proved that there exist two adjacent squares with the difference 9. For the generalization of this problem see example 4.09.

The following two examples are related with sequences. The first one is about the sequence of numbers being on the circle.

Example 2.07. On a circular arrangement we have zeros and ones, with n terms altogether. Prove that if the number of ones exceeds $n(1-\frac{1}{k})$, there must be string of k consecutive ones.

Solution. Let $a_1, a_2, \ldots, a_n$ be the terms of the sequence on the circle. Suppose r of the terms are ones and there are no k consecutive ones in the circular arrangement of these terms. Then

$$a_1 + a_2 + \ldots + a_k \leq k-1,$$

$$a_2 + a_3 + \ldots + a_{k+1} \leq k-1,$$

$$\ldots$$

$$a_n + a_1 + \ldots + a_{k-1} \leq k-1.$$

Adding these inequalities, we get $kr \leq n(k-1)$, where r is the number of ones. But we have $r > n(1-\frac{1}{k})$. Thus by the Pigeonhole Principle there will exist a string with k consecutive ones.

The second example on sequences brings probably one of the most useful ideas in such type of problems. Remember this idea, because further you will see a lot of its applications.

Example 2.08. Prove that among 70 distinct positive integers not exceeding 200 there exist two whose difference is $4, 5$ or 9.

Solution. Let $a_1, a_2, \ldots, a_{70}$ be the given numbers. Consider 210 numbers

$$a_1, a_2, \ldots, a_{70},\ a_1+4, a_2+4, \ldots, a_{70}+4,\ a_1+9, a_2+9, \ldots, a_{70}+9.$$

All of them do not exceed 209. By the Pigeonhole Principle two of them say $a_i + x$ and $a_j + y$ are equal $(x \neq y)$, where x and y can have the values $0, 4$, or 9. Hence, the difference between a_i and a_j is $4, 5$, or 9, and we are done.

To complete the variety of areas where the Pigeonhole Principle can be met, we give two examples that are related with divisibility.

Example 2.09. Prove that among any three distinct integers we can find two, say a and b, such that the number $a^3b - ab^3$ is a multiple of 10.

Solution. Denote

$$E(a,b) = a^3b - ab^3 = ab(a-b)(a+b)$$

Since if a and b are both odd, then $a+b$ is even, it follows that $E(a,b)$ is always divisible by 2.

Thus we only have to prove that among any three integers we can find two, a and b, with $E(a,b)$ divisible by 5. If one of the numbers is a multiple of 5, the property is true. If $a \equiv b (\mathrm{mod}\ 5)$, then $a-b$ is divisible by 5, and so is $E(a,b)$. If not, consider the pairs $\{1,4\}$ and $\{2,3\}$ of residues classes modulo 5. By the Pigeonhole Principle, the residues of two of the given numbers belong to the same pair. Therefore there exist a and b such that $a+b$ is divisible by 5. Thus $E(a,b)$ is always divisible by 5 and the conclusion follows.

The last example is an old chestnut from Paul Erdös. The short solution does not unveil the difficulty of the problem.

Example 2.10. Prove that among any $n+1$ numbers from the set $\{1,2\ldots,2n\}$ there is one that is divisible by another one.

Solution. Consider $n+1$ numbers $a_1, a_2, \ldots, a_{n+1}$ and write them in the form $a_i = 2^k b_i$, where b_i are odd. Then we have $n+1$ odd numbers $b_1, b_2, \ldots, b_{n+1}$ from the interval $[1, 2n-1]$. But there are only n odd numbers in this interval. By the Pigeonhole Principle there are two indexes k, l such that $b_k = b_l$. Then one of the numbers a_k, a_l is divisible by another one, and we are done.