AwesomeMath Admission Test Cover Sheet

Your name (please print) _____ _____
Last *First*

Admission Test A/B/C (circle one)

Contact Information _____ (phone number)

(please print) _____ (email address)

Number of pages (not including this cover sheet) _____

AwesomeMath
February 3-6, 2006

Admission Test A

- Solve as many of the problems as you can. If you can solve five or more of the ten questions, we strongly encourage you to apply, but don't be discouraged if you can't.
- You should include all significant steps in your reasoning and computation. We are interested in your ability to present your work, so unsupported answers will receive much less credit than well-reasoned progress towards a solution without a correct answer.
- In this document, you will find a cover sheet and an answer sheet. Print out each one, and make several copies of the blank answer sheet. Fill out the top of each answer sheet as you go, and then fill out the cover sheet when you're finished. **Start each problem on a new answer sheet.**
- **All the work you present must be your own**.
- **Don't be intimidated!** Some of the problems involve complex mathematical ideas, but all can be solved using only elementary techniques, admittedly combined in clever ways.
- **Be patient and persistent!** Learning comes more from struggling with problems than from solving them. Problem-solving becomes easier with experience. Success is not a function of cleverness alone.
- **You must postmark or submit your solutions by e-mail by Monday, Feb. 6, 2006**. Make sure that the cover sheet is the first page of your submission, and that it is completely filled out.

  Solutions are to be mailed to the following address:

  Dr. Titu Andreescu
  School of Natural Sciences and Mathematics
  The University of Texas at Dallas
  2601 N. Floyd Road, FN 33
  Richardson, TX 75083

  If you e-mail your solutions, please send them to

  titu.andreescu@utdallas.edu

  E-mailed solutions may be written and scanned or typed in TeX. They should be sent as an attachment in either .doc or .pdf format. If you write and scan your solutions, insert the scans into a .doc or .pdf file, and send just the one file.

Please go the next page for the problems.

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Admission Test A

1. Place ten 1's and six 0's in a 4×4 array such that each row has an even number of 1's and each column has an odd number of 1's.

2. Let n be a positive integer, and let $S_n = 1 + 2 \cdots + n$. What are all the possible units digits (in the decimal representation) of S_n? Prove your result.

3. Given a 6×6 square, show that it is possible to dissect it into 8 incongruent rectangles with integer side lengths. Prove that if the square is dissected into 9 rectangles with integer side lengths, then some two of the rectangles must be congruent to each other.

4. Given distinct real numbers x, y, and z such that
$$\frac{z}{x+y} < \frac{x}{y+z} < \frac{y}{z+x},$$
write x, y, and z in increasing order from left to right. Justify your result *algebraically.*

5. In the coordinate-plane, consider square $ABCD$ with $A = (12, 19)$ and $C = (3, 22)$. Find the coordinates of B and D.

6. Let T be a subset of $\{1, 2, \ldots, 2003\}$. An element a of T is called *isolated* if neither $a - 1$ nor $a + 1$ is in T. Determine the number of five-element subsets of T that contain no isolated elements.

7. A square is *inscribed* in a triangle if all of the vertices of the square lie on the sides of the triangle. Given a right triangle, there are two obvious ways to inscribe a square in it. The first is to place a corner of the square at the vertex with the right angle. The second way is to place a side of a square on the hypotenuse of the right triangle. Which way will result in a bigger square, or will they be equal?

8. Given nonzero real numbers a, b, and c such that the quadratic equations (in x) $ax^2+bx+c = 0$, $bx^2+cx+a = 0$, $cx^2+ax+b = 0$ share a common root, find all possible values of $\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}$.

9. Equilateral triangle ABC is inscribed in circle ω. Point P lies on minor arc $\widehat{BC}$. Segments AP and BC meet at D. Given that $BP = 21$ and $CP = 28$, compute $\frac{BD}{DC}$ and PD.

10. A heap of 2006 balls consists of 1003 10-gram balls and 1003 9.9-gram balls. We wish to pick out two heaps of balls with equal numbers of balls in them but different total weights. What is the minimal number of weighings needed to do this? (The balance scale reports the weight of the objects in the left pan, minus the weight of the objects in the right pan.)

Suggestions for Writing Proofs
(By Tiankai Liu)

- All proofs should be written neatly and coherently in paragraphs of standard American English. Mathematical symbols like $\equiv$ and $\leq$ should be used only in equations, not as verbs or prepositions in a sentence. Do not write things like "all of the $\triangle$'s angles are $\leq 90°$"—this should be "$\angle A, \angle B, \angle C \leq 90°$" or "all of the triangle's angles are at most $90°$." Avoid the symbols $\wedge \vee \because \therefore \forall \exists$; instead, write out "and," "or," "because," "therefore," "for all," "there exists." Similarly, do not use $\Longrightarrow$ or $\Rightarrow\!\!\Leftarrow$ except as part of a sequence of equations.

- Write a statement using words rather than symbols unless this would be unnecessarily awkward. Do not invent more notation than is necessary to explain your solution.

- Write true statements. Do not write something that is only partially true, and then say how to fix it later. If you assume something in one of your statements, say clearly what you are assuming. Define all terms you make up. If you use figures, graphs, tables, etc., explain thoroughly what they represent.

- Use the following formats for common proof patterns:

  - *Proof by induction:* (1) state the claim, (2) check the base case, (3) prove the induction step, and (4) conclude with the words "induction is complete."
  - *Proof by contradiction:* (1) state the claim, (2) state that you are assuming the opposite, (3) derive a contradiction (and say why it is a contradiction), and (4) conclude that the claim follows.
  - *Proof by case analysis:* (1) state the claim, (2) state the various cases, (3) say why they exhaust all possibilities, (4) analyze the cases one by one, and (5) conclude that the claim follows.

- Draw accurate diagrams with compass and straightedge for geometry problems. This is for your own good as well as the graders'.

AwesomeMath Admission Test Answer Sheet

Your name (please print) _____

Admission Test A/B/C (circle one) Problem Number ___ Page ___ of ___

Write neatly! Write all work inside the box. Do NOT write on the back of the page.