Problems for Mathematical Reflections 4

Juniors

J19. Let a, b be real numbers such that $3(a+b) \geq 2|ab+1|$. Prove that

$$9(a^3+b^3) \geq |a^3b^3+1|$$

Dr. Titu Andreescu, University of Texas at Dallas

J20. Prove that:

a) There are infinitely many quadruples (a, b, c, d) of pairwise distinct positive integers such that $ab+cd=(a+b)(c+d)$.

b) For any such quadruple, $max(a, b, c, d) \geq \dfrac{4\sqrt{3}}{\sqrt{3}+1}(a+b+c+d)$.

Ivan Borsenco, University of Texas at Dallas

J21. A $(2m+1)\times(2n+1)$ grid is colored with two colors. A $1\times\ 1$ square is called row-dominant if there are at least $n+1$ squares of its color in its row. Define column-dominant squares in the same way. Prove that there are at least $m+n+1$ both column-dominant and row-dominant squares.

Iurie Boreico, Moldova

J22. There are n 1's written on a board. At each step we can select two of the numbers on the board and replace them by $\sqrt[3]{\frac{a^2b^2}{a+b}}$. We keep applying this operation until there is only one number left. Prove that this number is not less than $\dfrac{1}{\sqrt[3]{n}}$.

Liubomir Chiriac, Princeton University

J23. Let $ABCDEF$ be a hexagon with parallel opposite sides, and let $FC\cap AB = X_1$, $FC\cap ED = X_2$, $AD\cap EF = Y_1$, $AD\cap BC = Y_2$, $BE\cap CD = Z_1$, $BE\cap AF = Z_2$. Prove that if X_1, Y_1, Z_1 are collinear then X_2, Y_2, Z_2 are also collinear and in this case the lines $X_1Y_1Z_1$ and $X_2Y_2Z_2$ are parallel.

Santiago Cuellar

J24. Consider a triangle ABC and a point P in its interior. Denote by d_a, d_b, d_c the distances from P to the triangle's sides. Prove that

$$2S\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{R}\right) \geq d_a+d_b+d_c$$

where S and R are the triangle's area and circumradius, respectively.

Ivan Borsenco, University of Texas at Dallas

Seniors

S19. Let ABC be a scalene triangle. A point P is called nice if AD, BE, CF are concurrent, where D, E, F are the projections of P onto BC, CA, AB, respectively. Find the number of nice points that lie on the line OI.

Iurie Boreico, Moldova and Ivan Borsenco, University of Texas at Dallas

S20. Let ABC be an acute triangle and let P be a point in its interior. Prove that:

$$(AP + BP + CP)^2 \geq \sqrt{3}(PA \cdot BC + PB \cdot CA + PC \cdot AB).$$

Khoa Lu Nguyen, M.I.T

S21. Let p be a prime number and let $a_1, a_2, ...a_n$ be distinct positive integers between not exceeding $p - 1$. Suppose that

$$p | a_1^k + a_2^k + ... + a_n^k$$

for $k = 1, 2, ..., p - 2$. Find $\{a_1, a_2 ..., a_n\}$.

Pascual Restrepo Mesa,Universidad de los Andes, Colombia

S22. Let n and k be positive integers. Eve gives Adam k apples. However, she can first give him bitter apples, at most n. The procedure goes as follows: Eve gives Adam an apple at a time and Adam can either eat it (and find out whether it's sweet or not), or throw it away. Adam knows that the bitter apples come first, and the sweet last. Find, in terms of n, the least value of k for which Adam can be sure he eats more sweet apples than bitter.

Iurie Boreico, Moldova

S23. Let a, b, c, d be positive real numbers. Prove that

$$3(a^2 - ab + b^2)(c^2 - cd + d^2) \geq 2(a^2c^2 - abcd + b^2d^2).$$

Dr. Titu Andreescu, University of Texas at Dallas

S24. Let ABC be an acute-angled triangle inscribed in a circle $\mathcal{C}$. Consider all equilateral triangles DEF with vertices on $\mathcal{C}$. The Simpson lines of D, E, F with respect to the triangle ABC form a triangle T. Find the greatest possible area of this triangle.

Iurie Boreico, Moldova and Ivan Borsenco, University of Texas at Dallas

Undergraduate

U19. Let f_0 be a real-valued function, continuous on the interval $[0,1]$ and for each integer $n \geq 0$ let $f_{n+1}(x) = \int_0^x f_n(t)dt$. Suppose that there is a positive integer k with the property that $f_k(1) = \frac{1}{(k+1)!}$. Prove that there exists x_0 such that $f_0(x_0) = x_0$.

Dr. Titu Andreescu, University of Texas at Dallas

U20. Prove that there is no entire function f such that $f(f(x)) = e^x$ for all real numbers x, but there is an infinitely many times differentiable function with this property.

Gabriel Dospinescu, Ecole Normale Superieure, Paris

U21. Evaluate

$$\int_0^1 \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\} dx$$

where $\{a\}$ is the fractional part of a.

Ovidiu Furdui, Western Michigan University

U22. Let $||.||$ be a norm on $\mathbb{C}^n$ and define $|||A||| = \sup_{||x|| \leq 1} ||Ax||$ for any complex matrix A in $M_n(\mathbb{C})$. Let $a < 2$ and let G be a subgroup of $GL_n(\mathbb{C})$ such that $|||A - I_n||| \leq$ for all $A \in G$. Prove that G is finite.

Gabriel Dospinescu and Alexandre Thiery, Ecole Normale Superieure, Paris

U23. Evaluate the sum

$$\sum_{k=0}^{n-1} \frac{1}{1 + 8sin^2(\frac{k\pi}{n})}$$

Dorin Andrica and Mihai Piticari

U24. Find all linear maps $f : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ such that $f(I_n) = I_n$ and $f(A^k) = f^k(A)$ for some integer $k > 1$ and all $A \in M_n(\mathbb{C})$.

Gabriel Dospinescu, Ecole Normale Superieure, Paris

Olympiad

O19. Let a, b, c be positive real numbers. Prove that:

a) $(a^3+b^3+c^3)^2 \geq (a^4+b^4+c^4)(ab+bc+ac)$

b) $9(a^4+b^4+c^4)^2 \geq (a^5+b^5+c^5)(a+b+c)^3$.

Ivan Borsenco, University of Texas at Dallas

O20. The incircle of triangle ABC touches AC at E and BC at D. The excircle corresponding to A touches the side BC at A_1 and the extensions of AB, AC at C_1 and B_1, respectively. Let $DE \cap A_1B_1 = L$. Prove that L lies on the circumcircle of triangle A_1BC_1.

Liubomir Chiriac, Princeton University

O21. Let p be a prime number. Find the least degree of a polynomial f with integer coefficients such that $f(0), f(1)...f(p-1)$ are perfect $(p-1)$-th powers.

Pascual Restrepo Mesa, Universidad de los Andes, Colombia

O22. Consider a triangle ABC and points P, Q in its plane. Let A_1, B_1, C_1 and A_2, B_2, C_2 be cevians in this triangle. Denote by U, V, W the second intersections of circles $(AA_1A_2), (BB_1B_2), (CC_1C_2)$ with circle (ABC), respectively. Let X be the point of intersection of AU with BC. Similarly define Y and Z. Prove that X, Y, Z are collinear.

Khoa Lu Nguyen, M.I.T and Ivan Borsenco, University of Texas at Dallas

O23. Let ABC be a triangle and let A_1, B_1, C_1 be the points where the angle bisectors of A, B and C meet the circumcircle of triangle ABC, respectively. Let M_a be the midpoint of the segment connecting the intersections of segments A_1B_1 and A_1C_1 with segment BC. Define M_b and M_c analogously. Prove that AM_a, BM_b, and CM_c are concurrent if and only if ABC is isosceles.

Dr. Zuming Feng, Phillips Exeter Academy, New Hampshire

O24. Find all positive integers a, b, c such that

$$2^n a + b | c^n + 1$$

for every positive integer n.

Gabriel Dospinescu, Ecole Normale Superieure, Paris