Nice numbers

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Sums of reciprocals of positive integers have been used to express other rational numbers since the ancient times. In particular, because the ancient Egyptians used them in such a way, the reciprocals of positive integers are referred to as Egyptian fractions. In 1202, Pisano showed that every rational number can be represented as a sum of distinct Egyptian fractions:

$$r = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}, \quad \text{with} \quad 1 \le a_1 < a_2 < \cdots < a_k.$$

In particular, every positive integer can be represented in this form. In what follows we consider representations of the unity,

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1, \quad 1 \le a_1 \le a_2 \le \cdots \le a_k,$$

where the terms in the representations are not necessarily distinct. For example,

$$\frac{1}{1} = 1,$$
$$\frac{1}{2} + \frac{1}{2} = 1,$$
$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1.$$

It is not difficult to see that in such representations, the sum of the denominators a_1, a_2,..., a_k cannot be equal to numbers such as, for instance, 2, 3, or 5 (why?). It is also apparent that $a_1 + a_2 + \cdots + a_k$ can be equal to certain numbers such as 1, 4, or 11.

An integer n will be called *nice* if it can be written as

$$n = a_1 + a_2 + \cdots + a_k,$$

where $a_1, a_2, \ldots, a_k$ are positive integers (not necessarily distinct) such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1.$$

In what follows we will try to find out which numbers are nice.

Certainly, you do not need a hint to figure out a way to solve the following exercise.

Exercise 1. Show that every perfect square is nice.

If $a_1, a_2, \ldots, a_k$ are summands in a representation of a nice number, then

$$\frac{1}{2a_1} + \frac{1}{2a_2} + \cdots + \frac{1}{2a_k} = \frac{1}{2}.$$

Based on this observation, solve the following two exercises.

Exercise 2. Show that if n is a nice number, then so is $2n + 2$.

Exercise 3. Show that 10, 20, 22, 24, 34 are nice numbers.

Observe that $\frac{1}{4}+\frac{1}{4} = \frac{1}{3}+\frac{1}{6} = \frac{1}{2}$ and then solve the following two exercises.

Exercise 4. Show that 17 and 18 are nice numbers.

Exercise 5. Show that if n is a nice number, then so are $2n+8$ and $2n+9$.

Exercise 6. Use the previous exercise to show that 26, 27, 28, 29, 30, and 31 are nice numbers.

Based on an observation similar to the one preceding Exercise 2 and the fact that $\frac{2}{3} = \frac{1}{3} + \frac{1}{3} = \frac{1}{2} + \frac{1}{6}$ solve the following exercises.

Exercise 7. Show that if n is a nice number, then so are $3n+6$ and $3n+8$.

Exercise 8. Show that 33, 35, and 39 are nice.

You have certainly noticed by now that it is very useful to represent $\frac{1}{k}$ as $\frac{1}{x} + \frac{1}{y}$, for some positive integers x and y, especially in the case of a small k.

Exercise 9. For $k = 2, 3, 4$, find all possible ways to represent $\frac{1}{k}$ as a sum of two Egyptian fractions

$$\frac{1}{k} = \frac{1}{x} + \frac{1}{y},$$

where $x \leq y$.

Hint: the above relation leads to $xy = k(x + y)$, which is equivalent to

$$(x - k)(y - k) = k^2.$$

You will obtain the following representations

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{3} + \frac{1}{6},$$

$$\frac{1}{3} = \frac{1}{6} + \frac{1}{6} = \frac{1}{4} + \frac{1}{12},$$

$$\frac{1}{4} = \frac{1}{8} + \frac{1}{8} = \frac{1}{6} + \frac{1}{12} = \frac{1}{5} + \frac{1}{20}.$$

Exercise 10. Show that all possible representations of $\frac{1}{6}$ as a sum of two Egyptian fractions are

$$\frac{1}{6} = \frac{1}{12} + \frac{1}{12} = \frac{1}{10} + \frac{1}{15} = \frac{1}{9} + \frac{1}{18} = \frac{1}{8} + \frac{1}{24} = \frac{1}{7} + \frac{1}{42}.$$

Exercise 11. Use the representations from Exercise 10 and the ones obtained before to show that the numbers 32, 37, 47, 51, and 55 are nice.

Exercise 12. Verify that numbers $24, 25, \ldots, 55$ are all nice (use Exercise 2 and Exercise 5 for the numbers not already mentioned in the previous exercises).

Consider the statement

$$S(n): \text{ the integers } n, n+1, n+2, \ldots, 2n+7 \text{ are all nice.}$$

Exercise 12 shows that the statement $S(24)$ is true.

Exercise 13. Using the above remark and Exercise 5, prove by induction that all numbers $n \geq 24$ are nice.

You are now left with the task of trying to find out which numbers $n \leq 23$ can be represented as $n = a_1 + a_2 + \cdots + a_k$, where $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1$. Recall the inequality between the arithmetic mean (A-M) and the harmonic mean (H-M) of k positive real numbers:

$$\frac{a_1 + a_2 + \cdots + a_k}{k} \geq \frac{k}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}}. \tag{1}$$

Taking into account that $a_1 + a_2 + \cdots + a_k \leq 23$ and $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1$ yields $k^2 \leq 23$, i.e., $k \leq 4$.

The case $k = 1$ is trivial. For $k = 2$, $\frac{1}{a_1} + \frac{1}{a_2} = 1$ yields $a_1 = a_2 = 2$. So far you only obtained the nice numbers 1 and 4.

For $k = 3$, the equation $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = 1$ with $a_1 \leq a_2 \leq a_3$ gives either $a_1 = a_2 = a_3 = 3$ or $a_1 = 2$. In the latter case the problem reduces to the equation $\frac{1}{a_2} + \frac{1}{a_3} = \frac{1}{2}$, which was solved in Exercise 9. Thus $a_1 = a_2 = a_3 = 3$, or $a_1 = 2$, $a_2 = 4$, $a_3 = 4$, or $a_1 = 2$, $a_2 = 3$ $a_3 = 6$, and the only nice numbers obtained for $k = 3$ are 9, 10, and 11.

Exercise 14. Show that the only nice numbers $n \leq 23$ that can be obtained for $k = 4$ are 16, 17, 18, 20, and 22.

In conclusion, you proved that the nice numbers are 1, 4, 9, 10, 11, 16, 17, 18, 20, 22, and all numbers greater than or equal to 24.

It can be shown that, for a fixed k, there are only finitely many nice numbers n that can be represented as $n = a_1 + a_2 + \cdots + a_k$, where $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1$. The smallest such n is k^2, which follows from the inequality (1). It is far more difficult task to find the largest such n.