On some elementary inequalities

Titu Andreescu
University of Texas at Dallas
Dallas, TX, USA

Gabriel Dospinescu
"Louis le Grand" College
Paris, France

A remarkable inequality was proposed by G. Zbaganu as the sixth USAMO 2000 problem: for any nonnegative real numbers $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$, the following inequality holds:

$$\sum_{1\le i,j\le n} \min(a_i a_j, b_i b_j) \le \sum_{1\le i,j\le n} \min(a_i b_j, a_j b_i) \tag{1}$$

Ravi Boppana recently found a beautiful solution to this very difficult question, based on the following lemma.

Lemma 1. *For any positive real numbers $a_1, a_2, ..., a_n$ and any real numbers $x_1, x_2, ..., x_n$, the inequality*

$$\sum_{1\le i,j\le n} x_i x_j \cdot \min(a_i, a_j) \ge 0 \tag{2}$$

holds.

The aim of this paper is to present some applications and refinements of (2), which will lead to an even stronger form of (1). We will also discuss some related problems from various mathematical contests.

Two proofs of the lemma

We present a very simple proof of Lemma 1, as well as a proof involving integrals. The second proof will allow us to prove some other inequalities involving the function.

The elementary solution starts with the simple observation that, without loss of generality, we may assume $a_1 \le a_2 \le \cdots \le a_n$. Then the inequality becomes

$$\sum_{i=1}^{n} a_i \cdot x_i^2 + 2 \cdot \sum_{i=1}^{n-1} a_i x_i \cdot \sum_{j=i+1}^{n} x_j \ge 0.$$

Let us denote $b_i = \sum_{j=i}^{n} x_j$. Then $x_i = b_i - b_{i+1}$ and the above inequality reduces to

$$a_1 b_1^2 + (a_2 - a_1) b_2^2 + ... + (a_n - a_{n-1}) b_n^2 \ge 0,$$

which is true, hence Lemma 1 is proved.

Let us see now an even shorter proof of this result. The crucial observation is the simple identity

$$\min(x, y) = \int_0^\infty \lambda_{[0,x]}(t) \cdot \lambda_{[0,y]}(t)\ dt \tag{3}$$

where λ_A is the characteristic function of the set A and x, y are positive real numbers. Of course, proving (3) is just a formality: it suffices to observe that $\lambda_{[0,x]} \cdot \lambda_{[0,y]}$ equals 1 on $[0, min(x, y)]$ and 0 elsewhere. Using (3), we infer that

$$\sum_{1\le i,j\le n} x_i x_j \cdot \min(a_i, a_j) = \sum_{1\le i,j\le n} \int_0^\infty x_i \lambda_{[0,a_i]}(t) \cdot x_j \lambda_{[0,a_j]}(t)\ dt =$$

$$= \int_0^\infty \sum_{i=1}^n (x_i \cdot \lambda_{[0,a_i]}(t))^2 dt \ge 0$$

and Lemma 1 is proved.

The second proof also shows that (2) can be strengthened to

$$\sum_{1\le i,j\le n} x_i x_j \cdot \min(a_i, a_j) \ge \frac{(\sum_{i=1}^n a_i x_i)^2}{\max_{1\le i\le n} a_i}. \tag{4}$$

Indeed, all we need is to observe that

$$\int_0^\infty \sum_{i=1}^n (x_i \cdot \lambda_{[0,a_i]}(t))^2\ dt = \int_0^{\max_{1\le i\le n} a_i} \sum_{i=1}^n (x_i \cdot \lambda_{[0,a_i]}(t))^2\ dt.$$

and to apply the Cauchy-Schwarz inequality

$$\int_0^{\max_{1\le i\le n} a_i} (\sum_{i=1}^n x_i \cdot \lambda_{[0,a_i]}(t))^2\ dt \ge \frac{(\int_0^{max_{1\le i\le n} a_i} (\sum_{i=1}^n x_i \cdot \lambda_{[0,a_i]}(t) dt))^2}{\max_{1\le i\le n} a_i}.$$

Since

$$\int_0^{max_{1\le i\le n}} (\sum_{i=1}^n x_i \cdot \lambda_{[0,a_i]}(t))\ dt = \sum_{i=1}^n a_i x_i,$$

the conclusion follows.

But Lemma 1 and the relation (3) have other applications besides the USAMO inequality! Before presenting the solution to (1), let us focus on some applications of (2) and (3).

Three related problems

We promised some nice applications of (2) and (3). We will discuss three problems, two of which are from mathematical contests. The third one is a difficult inequality due to Don Zagier. We strongly recommend to the readers to find other solutions to these questions, in order to convince themselves that, despite the following apparently simple solutions, these problems are really difficult.

We begin with an immediate consequence of (3). The way the problem is formulated does not even remotely suggest using (3).

Problem 1. Let $a_1, a_2, \ldots, a_n > 0$ and let $x_1, x_2, \ldots, x_n$ be real numbers such that $\sum_{i=1}^n a_i x_i = 0$.

Prove that

$$\sum_{1 \le i < j \le n} x_i x_j \cdot |a_i - a_j| \le 0.$$

Show that the equality holds if and only if there exists a partition $A_1, A_2, \ldots, A_k$ of the set $\{1, 2, ..., n\}$ such that for all i with $1 \le i \le k$ we have $\sum_{j \in A_i} x_j = 0$ and $a_{j_1} = a_{j_2}$ if $j_1, j_2 \in A_i$.

(Gabriel Dospinescu, Mathlinks Contest)

Solution. Let us use the formula

$$\min(a_i, a_j) = \frac{a_i + a_j - |a_i - a_j|}{2}$$

in (3). Since

$$\sum_{1 \le i, j \le n} x_i x_j (a_i + a_j) = 2 \cdot \left(\sum_{i=1}^n x_i \right) \cdot \left(\sum_{i=1}^n a_i x_i \right) = 0,$$

the first part of the problem follows from (3).

For the second question, observe that in order to have equality, we must have

$$\int_0^\infty \left(\sum_{i=1}^n x_i \cdot \lambda_{[0,a_i]}(t) \right)^2 dt = 0$$

and therefore $f = \sum_{i=1}^n x_i \cdot \lambda_{[0,a_i]}$ is zero at every point of continuity. Now, let $b_1, b_2, ..., b_k$ be the distinct numbers that appear among $a_1, a_2, ..., a_n$ and let $A_i = \{j \mid 1 \le j \le n,\ a_j = b_i\}$. Then $A_1, A_2, ..., A_k$ is a partition of the set $\{1, 2, ..., n\}$ and $\sum_{i=1}^k (\sum_{j \in A_i} x_j) \cdot \lambda_{[0,b_i]} = 0$, except for a finite number of points. We now conclude that $\sum_{j \in A_i} x_j = 0$ for all $1 \le i \le k$. ■

We will see in the following problem that Arthur Engel's principle, according to which any inequality reduces in the end to $x^2 \ge 0$, has merits.

Problem 2. Prove that for any real numbers $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ the following inequality holds

$$\sum_{1 \le i < j \le n} (|a_i - a_j| + |b_i - b_j|) \le \sum_{1 \le i, j \le n} |a_i - b_j|.$$

(Poland, 1999)

Solution. Of course, by considering $M - a_i$ and $M - b_j$ where M is the greatest number among all $2n$ numbers, we may assume that $a_i, b_j \ge 0$.

First of all, we note that the inequality is equivalent to

$$\sum_{1 \le i, j \le n} |a_i - a_j| + \sum_{1 \le i, j \le n} |b_i - b_j| \le 2 \cdot \sum_{1 \le i, j \le n} |a_i - b_j|.$$

The same useful formula $\min(a_i, a_j) = \frac{a_i+a_j-|a_i-a_j|}{2}$ shows that it can be rewritten as

$$\sum_{1\le i,j\le n} \min(a_i, a_j) + \sum_{1\le i,j\le n} \min(b_i, b_j) \ge 2 \cdot \sum_{1\le i,j\le n} \min(a_i, b_j). \tag{5}$$

A simple look at (3) shows now that (5) is a sister of the classical inequality $x^2 + y^2 \ge 2xy$. Indeed, (5) can be rewritten in the form

$$\int_0^\infty f^2(x)\ dx + \int_0^\infty g^2(x)\ dx \ge 2 \cdot \int_0^\infty f(x)g(x)\ dx,$$

where $f = \sum_{i=1}^n \lambda_{[0,a_i]}$ and $g = \sum_{i=1}^n \lambda_{[0,b_i]}$. The last inequality is easy to prove, since it is equivalent to $\int_0^\infty (f(x) - g(x))^2\ dx \ge 0$, which is clearly true. ■

Observe that the same method that was used to find a refinement of (2) allows here to prove the stronger inequality

$$\sum_{1\le i,j\le n} |a_i - b_j| \ge \sum_{1\le i<j\le n} (|a_i - a_j| + |b_i - b_j|) + \frac{(\sum_{i=1}^n a_i - \sum_{i=1}^n b_i)^2}{\max_{1\le i\le n} \max(a_i, b_i)}.$$

Of course, Cauchy-Schwarz cannot be absent from a note on inequalities! Here is a difficult problem which, using (3), boils down to this famous inequality.

Problem 3. Prove that for all $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n \ge 0$ the following inequality holds

$$\left(\sum_{1\le i,j\le n} \min(a_i, a_j)\right) \cdot \left(\sum_{1\le i,j\le n} \min(b_i, b_j)\right) \ge \left(\sum_{1\le i,j\le n} \min(a_i, b_j)\right)^2.$$

(Don Zagier)

Solution. It would be interesting to find another proof to this inequality, but it seems that any attempt to prove it without using (3) fails. Anyway, using (3) the problem becomes very simple. Indeed, the inequality reduces to

$$\left(\int_0^\infty f^2(x)\ dx\right) \cdot \left(\int_0^\infty g^2(x)\ dx\right) \ge \left(\int_0^\infty f(x)g(x)\ dx\right)^2,$$

which is nothing else than, yes, the Cauchy-Schwarz inequality. ■

The reader has already noticed that there are virtually hundreds of inequalities that can be created using this technique. For example, take an inequality such as the classical Schur's inequality

$$x^3 + y^3 + z^3 + 3xyz \ge x^2(y + z) + y^2(z + x) + z^2(x + y)$$

and apply it to $x = A \cdot \lambda_{[0,a]}$, $y = B \cdot \lambda_{[0,b]}$, $z = C \cdot \lambda_{[0,c]}$ in order to deduce (after integration) a stronger form

$$\begin{aligned} &aA^3 + bB^3 + cC^3 + 3ABC \cdot \min(a,b,c) \geq \\ &\qquad AB(A+B)\min(a,b) + BC(B+C)\min(b,c) + CA(C+A)\min(c,a). \end{aligned}$$

There are countless examples and we invite the reader to find other "new" inequalities.

Solution to the USAMO problem

Finally, we go back and discuss a solution to the USAMO 2000 inequality. The reader will be surprised that only one step is enough to solve this problem, once (2) is assumed. Yes, but as we will see, this step is terribly well hidden.

As usual, for very difficult inequalities, identities play the most important role. First of all, let us write the inequality in a simpler form. Observe that continuity reasons show that it suffices to regard all variables positive. Now, put $x_i = \frac{a_i}{b_i}$. The inequality becomes

$$\sum_{1 \leq i,j \leq n} a_i a_j (\min(x_i, x_j) - \min(1, x_i x_j)) \geq 0 \tag{6}$$

One could try at this moment to prove that the matrix $((\min(x_i, x_j) - \min(1, x_i x_j))_{1 \leq i,j \leq n}$ has all eigenvalues nonnegative, but this is not easy at all (at least, we could not find a proof of this assertion yet). Ravi Boppana's observation is the following strange identity.

Lemma 2. *For all positive real numbers x, y the following identity holds*

$$\min(x,y) - \min(1,xy) = f(x)f(y) \cdot \min\left(\frac{|x-1|}{\min(x,1)}, \frac{|y-1|}{\min(y,1)}\right)$$

where $f(x) = \mathsf{sgn}(x-1) \cdot \min(x,1)$.

Impressive, isn't it? Proving this identity is (as usual) not difficult; the hard part is to find it. It suffices to assume that $x \geq y$ and to see that in each of the cases $x \geq y \geq 1$, $x \geq 1 \geq y$ and $1 \geq x \geq y$ the identity holds. We will treat only the first case, when the identity reduces to $y - 1 = \mathsf{sgn}(x-1) \cdot \mathsf{sgn}(y-1) \cdot (y-1)$, which is clearly true. The other two cases are similar.

Using this lemma, the inequality (6) becomes

$$\sum_{1 \leq i,j \leq n} (a_i f(x_i)) \cdot (a_j f(x_j)) \cdot \min\left(\frac{|x_i - 1|}{\min(x_i, 1)}, \frac{|x_j - 1|}{\min(x_j, 1)}\right) \geq 0$$

which is nothing else than (2) disguised. The solution of the celebrated USAMO 2000 problem ends here.

Of course, using (4) and Lemma 2, we can easily establish the following stronger form

$$\sum_{1\le i,j\le n}\min(a_ib_j,a_jb_i)\ge\sum_{1\le i,j\le n}\min(a_ia_j,b_ib_j)+\frac{\left(\sum_{i=1}^n(a_i-b_i)\right)^2}{\max_{1\le i\le n}\frac{|a_i-b_i|}{\min(a_i,b_i)}}.$$

An interesting consequence

We will end this note with an unexpected problem, which is a direct consequence of (1), but whose statement does not suggest this fact. Unfortunately, it seems very difficult to find a solution different from the one that reduces the problem to (1).

Problem 4. Let $x_1,x_2,...,x_n$ be positive real numbers satisfying

$$\sum_{1\le i,j\le n}|1-x_ix_j|=\sum_{1\le i,j\le n}|x_i-x_j|.$$

Prove that these numbers add up to n.

(Gabriel Dospinescu)

Solution. Let us take $b_i=1$ in (1). The formula $\min(x,y)=\frac{x+y-|x-y|}{2}$ allows us to rewrite the inequality (1) in the form

$$2n\cdot\sum_{i=1}^n x_i-\sum_{1\le i,j\le n}|x_i-x_j|\ge n^2+\left(\sum_{i=1}^n x_i\right)^2-\sum_{1\le i,j\le n}|1-x_ix_j|.$$

Using the hypothesis, we now conclude that $\left(\sum_{i=1}^n x_i-n\right)^2\le 0$, which shows that $\sum_{i=1}^n x_i=n$. ■

That is how a simple identity comes handy in solving a variety of difficult problems. The readers will certainly discover other problems that can be solved using these ideas and we strongly encourage them to search for different solutions.