# Course Descriptions

| Subject | Beginning (level 1) courses | Intermediate (level 2) courses | Advanced (level 3) courses |
|---|---|---|---|
| Algebra | Algebra 1.5 | Algebra 2.5 | Algebra 3.5 |
| Combinatorics | Math Counts with Proofs | Counting Strategies | Combinatorial Argument |
| Geometry | Elements of Geometry | Computational Geometry | Geometry Proofs |
| Number Theory | Number Sense | Modular Arithmetic | Number Theory |
| | These courses are computationally oriented with a touch on proofs. They are suited for most USA math competitions (MathCounts National level, AMC10, AMC12, ARML, and the entry level of AIME). | These courses are about half computational problems and half proofs. They are well suited for the hard end of AIME and the entry level of Math Olympiad contests. | These courses are proof oriented. They are well suited for students who can easily pass AIME and are seriously preparing for Math Olympiad contests. |

## Algebra courses

- *Algebra 1.5*

  Develops essential skills such as factoring, grouping, recognizing roots, telescoping sums/products, and rationalizing. Solving (systems of) equations/inequalities (linear, absolute value, quadratic, rational, radical) is the main theme of the course. Discriminants, Viete's relations, and symmetric polynomials also play a central role. **This is the entry level algebra course. It covers all AMC levels and easy end of AIME and ARML. This course is a good fit for students with MathCounts state level experience, AMC10/12 scores approaching AIME qualifying cuts, or an AIME score between 1 and 3.**

- *Algebra 2.5*

  Studies special systems of equations, discriminants, Viete's relations, symmetric polynomials, functional properties. Introduces (weighted) AM–GM–HM and Cauchy–Schwarz inequalities. **This is the intermediate level algebra course. It covers the hard end of AMC12, and the medium to hard end of ARML and AIME. A student with an AIME score between 4 and 7 should be a good fit for this course.**

- *Algebra 2.5*

  Discusses functional equations, classical inequalities such as AM-GM-HM, Cauchy-Schwarz, Power-mean, and Jensens inequalities, as well as Muirhead's and Schur's inequalities, and inequalities related to symmetric polynomials. **This is the advanced level algebra course. It covers the hard end of AIME and all levels of USAMO. A student with a strong algebra background and an AIME score of 8 or above should consider this course.**

## Combinatorics courses

- *Math Counts with Proofs*

  Studies the addition and multiplication principles, permutations and combinations, and probability. Teaches how to deal with over-counting and many useful properties of integer divisors. It also introduces mathematical proofs using pigeonhole principle, well-ordering, etc. **This is the entry level combinatorics course. It covers MathCounts, all the AMC levels, and the easy end of AIME and ARML. This course is a good fit for students with MathCounts state level experience, AMC10/12 scores approaching AIME qualifying cuts, or AIME scores between 1 and 3.**

- *Counting Strategies*

  Discusses counting strategies such as the addition and multiplication principles, permutations and combinations, properties of the binomial coe!cients, bijections, recursions, and the inclusion- exclusion principle. **This is the intermediate level combinatorics course. It covers the hard end of AMC12, the medium to hard end of AIME and ARML, as well as the beginning USAMO level. A student with an AIME score between 4 and 7 should be a good fit for this course.**

- *Combinatorial Arguments*

  Introduces methods of mathematical proofs, including induction, proofs by contradiction, the Pigeonhole Principle, the well-ordering principle, colorings, assigning weights, bijections/mappings, recursion, calculating in two ways, and combinatorial constructions. Topics may include graph theory and combinatorial geometry. A focal point of the course is combinatorial number theory. **This is the advanced level combinatorics course. It covers the hard end of AIME and the medium to hard end of USAMO. A student who is familiar with mathematics proofs and has an AIME score of 8 or above should consider this course.**

## Geometry courses

- *Elements of Geometry*

  Deals with computational geometry in two dimensions using Euclidean methods, including manipulation of angles and lengths, as well as the basic properties of polygons, circles, and the relations between figures. Analytic geometry is also a focal point. **This is the entry level geometry course. It covers MathCounts, all AMC levels, and the easy end of AIME and ARML. This course is a good fit for students with MathCounts state level experience, AMC10/12 scores approaching AIME qualifying cuts, or AIME scores between 1 and 3.**

- *Computational Geometry*

  Studies non-synthetic techniques in solving geometry problems: coordinate geometry, vectors (2- and 3-dimensional), planes, spheres, trigonometry, and complex numbers. Features many important geometric themes: The Law of Sines and the Law of Cosines, Ptolemys theorem, Cevas theorem, Menelauss theorem, Stewarts theorem, Herons and Brahmaguptas formulas, Brocard points, dot product and the vector form of the Law of Cosines, the Cauchy-Schwarz inequality, 3-dimensional coordinate systems, as well as linear representation and traveling on the earth (sphere). **This is the intermediate level geometry course. It covers the hard end of AMC12, the medium to hard end of AIME and ARML. A student with an AIME score between 4 and 7 should consider this course.**

- *Geometric Proofs*

  Focuses on classical topics such as concurrency, collinearity, cyclic quadrilaterals, special centers/points of triangles, and geometric constructions. Introduces important transformations  translation, reflections, and spiral similarities, with a touch on projective and inversive geometry. **This is the advanced level geometry course. It covers the hard end of AIME and the medium to hard end of USAMO. A student with a strong background in geometry and an AIME score of 8 or above should consider this course.**

## Number Theory courses

- *Number Sense*

  Studies divisibility, factoring, numerical systems, divisors and arithmetic functions of divisors. Setting-up and solving linear Diophantine equations is also a focal point of the course. **This is the entry level number theory course. It covers MathCounts, all AMC levels, and the easy end of AIME and ARML. This course is a good fit for students with MathCounts state level experience, AMC10/12 scores approaching AIME qualifying cuts, or AIME scores between 1 and 3.**

- *Modular Arithmetic*

  Develops essential skills in number theory: divisibility, the division algorithm, prime numbers, the Fundamental Theorem of Arithmetic, GCD, LCM, Bezouts identity, the Euclidean algorithm, modular arithmetic, and divisibility criteria in the decimal system. Studies numerical functions such as the number of divisors or the sum of divisors of integers. **This is the intermediate level number theory course. It covers the hard end of AMC12 and the medium to hard end of AIME and ARML. A student qualified for AIME with a score between 4 and 7 should be a good fit for this course.**

- *Number Theory*

  Focuses on in-depth discussions of Diophantine equations, residue classes, quadratic reciprocity, Fermats little theorem, Eulers theorem, primitive roots, and Eulers totient function, etc. **This is the advanced level number theory course. It covers the hard end of AIME and the medium to hard end of USAMO. A student with a strong background in number theory and an AIME score of 8 or above should consider this course.**

# Sample Problems

## Algebra

**Algebra 1.5**

1. If $a+b=1$ and $a^2+b^2=2$, evaluate $a^4+b^4$.
2. Simplify
$$\frac{(1+ax)^2-(a+x)^2}{(1+bx)^2-(b+x)^2} \div \frac{(1+ay)^2-(a+y)^2}{(1+by)^2-(b+y)^2}.$$
3. Let $a,b,c$ be distinct real numbers such that
$$a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}.$$
Prove that $|abc|=1$.
4. Evaluate
$$\sum_{n=0}^{\infty}\frac{n}{n^4+n^2+1}.$$

**Algebra 2.5**

1. Let $a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_n$ be real numbers. Prove that
$$(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2)\geq(a_1b_1+a_2b_2+\cdots+a_nb_n)^2.$$
2. Find all positive integers $a,b,c$ such that the equations
$$x^2-ax+b=0,\quad x^2-bx+c=0,\quad x^2-cx+a=0$$
have integer roots.
3. Let $f(x)=ax^2+bx+c$ be a quadratic function with integer coe!cients with the property that for every positive integer $n$ there is an integer $c_n$ such that $n$ divides $f(c_n)$. Prove that $f$ has rational zeros.
4. Let $a,b$ integer numbers. Solve the equation
$$(ax-b)^2+(bx-a)^2=x$$
when it is known that it has an integer root.

**Algebra 3.5**

1. Find all polynomials with complex coe!cients such that $P(x^2)=P^2(x)$ is identically true.
2. Let $a,b,c\geq 0$. Prove that
$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}+\frac{1}{a+b+c+1}\geq 1.$$
3. Let $a,b,c,d,e,f$ be positive integers such that $S=a+b+c+d+e+f$ is a divisor of $ab+bc+ca-(de+ef+fd)$ and $abc+def$. Show that $S$ is a composite number.
4. Find all real polynomials with real coeficients $P(x)$ which satisfy the equality
$$P(a-b)+P(b-c)+P(c-a)=2P(a+b+c)$$
for all triples $a,b,c$ of real numbers such that $ab+bc+ca=0$.

# Combinatorics

### Math Counts with Proofs

1. How many even integers between 4000 and 7000 have four different digits?

2. How many ordered triples $(x, y, z)$ of non-negative integers have the property that $x + y + z = 8$?

3. There are three men and eleven women taking a dance class. In how many different ways can each man be paired with a woman partner and then have the eight remaining women be paired into four pairs of two?

4. We want to paint some identically-sized cubes so that each face of each cube is painted a solid color and each cube is painted with six different colors. If we have seven different colors to choose from, how many distinguishable cubes can we produce?

### Counting Strategies

1. How many positive integers less than 5000 are multiples or 3, 5 or 7, but not multiples of 35?

2. Let $m$ be a positive integer, and let $n = 2^m$ Prove that the numbers
$$\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n-1}$$
are all even. How many of these numbers are divisible by 4?

3. A number of $n$ tennis players take part in a tournament in which each of them plays exactly one game with each of the others. If $x_i$ and $y_i$ denote the number of wins and losses, respectively, of the $i$th player, prove that
$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2.$$

4. Let $S$ be a set with 9 elements and let $A_1, A_2, \dots, A_{13}$ be distinct subsets of $S$, each having 3 elements. Prove that among these subsets there exist two, $A_i$ and $A_j$, such that $|A_i \cap A_j| = 2$.

### Combinatorial Arguments

1. The numbers $1, 2, \dots, 49$ are placed in a $7 \times 7$ table. We then add the numbers in each row and each column. Among these 14 sums we have a even numbers and b odd numbers. Is it possible that $a = b$?

2. The numbers $a_1, a_2, \dots, a_{108}$ are written on a circle such that the sum of any 20 consecutive numbers equals 1000. If $a_1 = 1, a_{19} = 19$, and $a_{50} = 50$, find $a_{100}$.

3. An even number, $2n$, of knights arrive at King Arthurs court, each one of them having at most $n - 1$ enemies. Prove that Merlin the wizard can assign places for them at a round table in such a way that every knight is sitting only next to friends.

4. On an $8 \times 8$ chessboard whose squares are colored black and white in an arbitrary way we are allowed to simultaneously switch the colors of all squares in any $3 \times 3$ and $4 \times 4$ region. Can we transform any coloring of the board into one where all the squares are black?

# Geometry

### Elements of Geometry

1. Let $ABCD$ be a parallelogram, and let $M$ and $N$ be the midpoints of sides $BC$ and $CD$, respectively. Suppose $AM = 2, AN = 1$, and $\angle MAN = 60°$. Computer $AB$.

2. How large an equilateral triangle can one fit inside a square with side length 2?

3. Charlyn walks completely around the boundary of a square whose sides are each 5 km long. From any point on her path, she can see exactly 1 km horizontally in all directions. What is the area of the region consisting of all points Charlyn can see during her walk?

4. Points $A$ and $B$ lie on a circle centered at $O$, and $\angle AOB = 60°$ A second circle is internally tangent to the first and tangent to both $OA$ and $OB$. What is the ratio of the area of the smaller circle to that of the larger circle?

### Computational Geometry

1. In quadrilateral $ABCD$, $BC = 8$, $CD = 12$, $AD = 10$, and $\angle A = \angle B = 60°$. Given that $AB = p + \sqrt{q}$, where $p$ and $q$ are positive integers, find $p + q$.

2. (a) Let $G$ be the centroid of triangle $ABC$. Prove that for any point $M$,
$$MA^2 + MB^2 + MC^2 = 3MG^2 + AG^2 + BG^2 + CG^2.$$
   (b) Let $I$ be the incenter of triangle $ABC$. Prove that for any point $X$,
$$a \cdot AX^2 + b \cdot BX^2 + c \cdot CX^2 = (a + b + c) \cdot IX^2 + a \cdot IA^2 + b \cdot IB^2 + c \cdot IC^2.$$

3. (a) Prove that in any triangle
$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{1}{8}.$$
   (b) Prove that in any triangle
$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} \leq \frac{9}{4}.$$

4. Let $M, N, P, Q, R, S$ be the midpoints of the sides $AB, BC, CD, DE, EF, FA$ of a hexagon. Prove that
$$RN^2 = MQ^2 + PS^2$$
   if and only if $MQ \perp PS$.

### Geometric Proofs

1. Prove that for any point $M$ inside parallelogram $ABCD$, the following relation holds:
$$MA \cdot MC + MB \cdot MD \geq AB \cdot BC.$$

2. Let $AD$ be an altitude in $\triangle ABC$. Point $P$ is on segment $AD$. Let $E$ bet the intersection of $BP$ and $AC$. Let $F$ be the intersection of $CP$ and $AB$. Prove that $\angle ADE = \angle ADF$.

3. Let $w_1$ be a circle smaller than and internally tangent to the circle $w_2$ at $T$. A tangent to $w_1$ (at $T'$) intersects $w_2$ at $A$ and $B$. If $A, T'$, and $B$ are fixed, what is the locus of $T$?

4. Let $ABCD$ be a quadrilateral, and let $E$ and $F$ be points on sides $AD$ and $BC$, respectively, such that $AE : ED = BF : FC$. Ray $FE$ meets rays $BA$ and $CD$ at $S$ and $T$, respectively. Prove that the circumcircles of triangles $SAE, SBF, TCF$, and $TDE$ pass through a common point.

# Number Theory

### Number Sense

1. Show that the number 101010 cannot be a difference of two squares of integers.
2. Let a unit step be the diagonal of a unit square. Starting from the origin, go one step to $(1,1)$. The turn $90^\circ$ counterclockwise (to the left) and go two steps to $(-1,3)$. Then turn $90^\circ$ counterclockwise (to the left) and go three steps to $(-4,0)$. At each step you continue to turn $90^\circ$ counterclockwise and increase the length of the movement by one at each step. What is the final position after 100 moves?
3. Find all positive integers a and b such that $a^2 + b^2 = \operatorname{lcm}(a,b) + 7\gcd(a,b)$.
4. Compute the sum of the greatest odd divisor of each of the numbers $2006, 2007, \ldots, 4012$.

### Modular Arithmetic

1. Show that $\frac{1}{9}(10^n + 3 \cdot 4^n + 5)$ is an integer for all $n \geq 1$.
2. Show that if $a^5 \pm 2b^5$ is divisible by 11, then both $a$ and $b$ are divisible by 11.
3. If $\{a_1, a_2, \ldots, a_{p-1}\}$ and $\{b_1, b_2, \ldots, b_{p-1}\}$ are complete sets of nonzero residue classes modulo some odd prime $p$, show that $\{a_1b_1, a_2b_2, \ldots, a_{p-1}b_{p-1}\}$ is not a set of complete residue classes modulo $p$.
4. Given that $a + b\sqrt[3]{2} + c\sqrt[3]{4} = 0$, where $a, b, c$ are integers, show that $a = b = c = 0$.

### Number Theory

1. Find all solutions to $2^k = 9^m + 7^n$.
2. Let $p$ be a prime, and let $k$ be a nonnegative integer. Calculate
$$\sum_{n=1}^{p-1} n^k \pmod{p}.$$
3. Prove that the equation $x^2 + 7xy - y^2 = 401$ has no integer solutions.
4. Determine all positive integers $n$ for which there is an integer $m$ such that $2^n - 1$ is a divisor of $m^2 + 9$.