Junior problems

J79. Find all integers that can be represented as $a^3+b^3+c^3-3abc$ for some positive integers a, b, and c.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J80. Characterize triangles with sidelengths in arithmetical progression and lengths of medians also in arithmetical progression.

Proposed by Daniel Lasaosa, Universidad Publica de Navarra, Spain

J81. Let a, b, c be positive real numbers such that

$$\frac{1}{a^2+b^2+1}+\frac{1}{b^2+c^2+1}+\frac{1}{c^2+a^2+1}\geq 1.$$

Prove that $ab+bc+ca\leq 3$.

Proposed by Alex Anderson, New Trier High School, Winnetka, USA

J82. Let $ABCD$ be a quadrilateral whose diagonals are perpendicular. Denote by $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ the centers of the nine-point circles of triangles ABC, BCD, CDA, DAB, respectively. Prove that the diagonals of $\Omega_1\Omega_2\Omega_3\Omega_4$ intersect at the centroid of $ABCD$.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

J83. Find all positive integers n such that a divides n for all odd positive integers a not exceeding $\sqrt{n}$.

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

J84. Al and Bo play the following game: there are 22 cards labeled 1 through 22. Al chooses one of them and places it on a table. Bo then places one of the remaining cards at the right of the one placed by Al such that the sum of the two numbers on the cards is a perfect square. Al then places one of the remaining cards such that the sum of the numbers on the last two cards played is a perfect square, and so on. The game ends when all the cards were played or no more card can be placed on the table. The winner is the one who played the last card. Does Al have a winning strategy?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Senior problems

S79. Let $a_n = \sqrt[4]{2} + \sqrt[n]{4}$, $n = 2, 3, 4, \ldots$. Prove that $\frac{1}{a_5} + \frac{1}{a_6} + \frac{1}{a_{12}} + \frac{1}{a_{20}} = \sqrt[4]{8}$.

Proposed by Titu Andreescu, University of Texas at Dallas

S80. Let ABC be a triangle and let M_a, M_b, M_c be the midpoints of sides BC, CA, AB, respectively. Let the feet of the perpendiculars from vertices M_b, M_c in triangle AM_bM_C be C_2 and B_1; the feet of the perpendiculars from vertices M_a, M_b in triangle CM_aM_b be B_2 and A_1; the feet of the perpendiculars from vertices M_c, M_a in triangle BM_aM_c be A_2 and C_1. Prove that the perpendicular bisectors of B_1C_2, C_1A_2, and A_1B_2 are concurrent.

Proposed by Vinoth Nandakumar, Sydney University, Australia

S81. Consider the polynomial

$$P(x) = \sum_{k=0}^{n} \frac{1}{n+k+1} x^k,$$

with $n \geq 1$. Prove that the equation $P(x^2) = (P(x))^2$ has no real roots.

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

S82. Let a and b be positive real numbers with $a \geq 1$. Further, let s_1, s_2, s_3 be nonnegative real numbers for which there is a real number x such that

$$s_1 \geq x^2, \;\; as_2 + s_3 \geq 1 - bx.$$

What is the least possible value of $s_1 + s_2 + s_3$ in terms of a and b (the minimum is taken over all possible values of x)?

Proposed by Zoran Sunic, Texas A&M University, USA

S83. Find all complex numbers x, y, z of modulus 1, satisfying

$$\frac{y^2+z^2}{x} + \frac{x^2+z^2}{y} + \frac{x^2+y^2}{z} = 2(x+y+z).$$

Proposed by Cosmin Pohoata, Bucharest, Romania

S84. Let ABC be an acute triangle and let ω and Ω be its incircle and circumcircle, respectively. Circle ω_A is tangent internally to Ω at A and tangent externally to ω. Circle Ω_A is tangent internally to Ω at A and tangent internally to ω. Denote by P_A and Q_A the centers of ω_A and Ω_A, respectively. Define the points P_B, Q_B, P_C, Q_C analogously. Prove that

$$\frac{P_AQ_A}{BC} + \frac{P_BQ_B}{CA} + \frac{P_CQ_C}{AB} \geq \frac{\sqrt{3}}{2}.$$

Proposed by Cezar Lupu, Univeristy of Bucharest, Romania

Undergraduate problems

U79. Let $a_1 = 1$ and $a_n = a_{n-1} + \ln n$. Prove that the sequence $\sum_{i=1}^{n} \frac{1}{a_i}$ is divergent.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

U80. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function at the origin satisfying $f(0) = 0$ and $f'(0) = 1$. Evaluate

$$\lim_{x\to 0} \frac{1}{x} \sum_{n=1}^{\infty} (-1)^n f\left(\frac{x}{n}\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U81. The sequence $(x_n)_{n\geq 1}$ is defined by

$$x_1 < 0, \quad x_{n+1} = e^{x_n} - 1, \quad n \geq 1.$$

Prove that $\lim_{n\to\infty} n x_n = -2$.

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

U82. Evaluate

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{k}{n}\right)^{\frac{n}{k^3}}.$$

Proposed by Cezar Lupu, Univeristy of Bucharest, Romania

U83. Find all functions $f : [0, 2] \to (0, 1]$ that are differentiable at the origin and satisfy $f(2x) = 2f^2(x) - 1$, for all $x \in [0, 1]$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U84. Let f be a three times differentiable function on an interval I, and let $a, b, c \in I$. Prove that there exists $\xi \in I$ such that

$$f\left(\frac{a+2b}{3}\right) + f\left(\frac{b+2c}{3}\right) + f\left(\frac{c+2a}{3}\right) - f\left(\frac{2a+b}{3}\right) - f\left(\frac{2b+c}{3}\right) - f\left(\frac{2c+a}{3}\right) =$$

$$= \frac{1}{27}(a-b)(b-c)(c-a)f'''(\xi).$$

Proposed by Vasile Cirtoaje, University of Ploiesti, Romania

Olympiad problems

O79. Let $a_1, a_2, \ldots, a_n$ be integer numbers, not all zero, such that $a_1+a_2+\ldots+a_n = 0$. Prove that

$$|a_1 + 2a_2 + \ldots + 2^{k-1}a_k| > \frac{2^k}{3},$$

for some $k \in \{1, 2, \ldots n\}$.

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College, Romania

O80. Let n be an integer greater than 1. Find the least number of rooks such that no matter how they are placed on an $n \times n$ chessboard there are two rooks that do not attack each other, but at the same time they are under attack by third rook.

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

O81. Let $a, b, c, x, y, z \geq 0$. Prove that

$$(a^2 + x^2)(b^2 + y^2)(c^2 + z^2) \geq (ayz + bzx + cxy - xyz)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas

O82. Let $ABCD$ be a cyclic quadrilateral inscribed in the circle $C(O, R)$ and let E be the intersection of its diagonals. Suppose P is the point inside $ABCD$ such that triangle ABP is directly similar to triangle CDP. Prove that $OP \perp PE$.

Proposed by Alex Anderson, New Trier High School, Winnetka, USA

O83. Let $P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$, $a_n \neq 0$, be a polynomial with complex coefficients such that there is an m with

$$\left|\frac{a_m}{a_n}\right| > \binom{n}{m}.$$

Prove that the polynomial P has at least a zero with the absolute value less than 1.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O84. Let $ABCD$ be a cyclic quadrilateral and let P be the intersection of its diagonals. Consider the angle bisectors of the angles $\angle APB$, $\angle BPC$, $\angle CPD$, $\angle DPA$. They intersect the sides AB, BC, CD, DA at $P_{ab}, P_{bc}, P_{cd}, P_{da}$, respectively and the extensions of the same sides at $Q_{ab}, Q_{bc}, Q_{cd}, Q_{da}$, respectively. Prove that the midpoints of $P_{ab}Q_{ab}, P_{bc}Q_{bc}, P_{cd}Q_{cd}, P_{da}Q_{da}$ are collinear.

Proposed by Mihai Miculita, Oradea, Romania