A Metric Relation and its Applications

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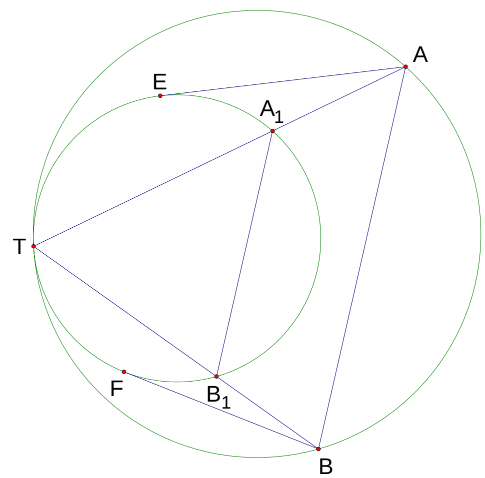
Lemma. Let γ be a circle and let A and B be two arbitrary points on it. A circle ρ touches γ internally at T. Denote by AE and BF the tangent lines to ρ at E and F, respectively. Then $\frac{TA}{TB} = \frac{AE}{BF}$.



Proof. Denote by A_1 and B_1 the second intersections of TA and TB with ρ, respectively. We know that A_1B_1 is parallel to AB. Therefore,

$$\left(\frac{AE}{TA_1}\right)^2 = \frac{AA_1 \cdot AT}{A_1T \cdot A_1T} = \frac{BB_1}{B_1T} \cdot \frac{BT}{B_1T} = \left(\frac{BF}{TB_1}\right)^2.$$

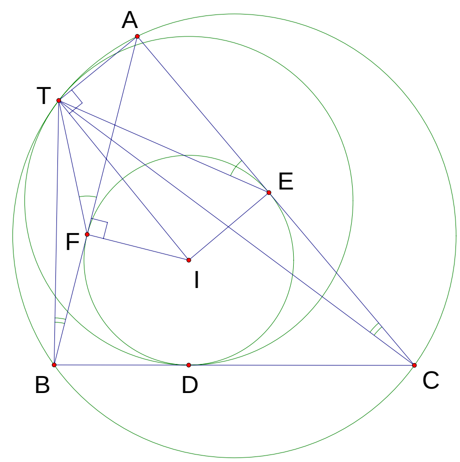
Hence,

$$\frac{AE}{TA_1} = \frac{BF}{TB_1} \implies \frac{AE}{BF} = \frac{TA_1}{TB_1} = \frac{TA}{TB},$$

which completes the proof. □

To illustrate how this lemma works, let us consider some examples. The following problem was proposed by Nguyen Minh Ha, in the Vietnamese Mathematics Magazine, in 2007.

Problem 1. Let Ω be the circumcircle of the triangle ABC and let D be the tangency point of its incircle $\rho(I)$ with the side BC. Let ω be the circle internally tangent to Ω at T, and to BC at D. Prove that $\measuredangle ATI = 90°$.



Solution. Let E and F be the tangency points of $\rho(I)$ with sides CA and AB, respectively. According to the lemma,

$$\frac{TB}{TC} = \frac{BD}{CD} = \frac{BF}{CE}.$$

Therefore triangles TBF and TCE are similar. It follows that $\measuredangle TFA = \measuredangle TEA$, hence the points A, I, E, F, T lie on the same circle. It follows that $\measuredangle ATI = \measuredangle AFI = 90°$ which completes our proof. $\square$

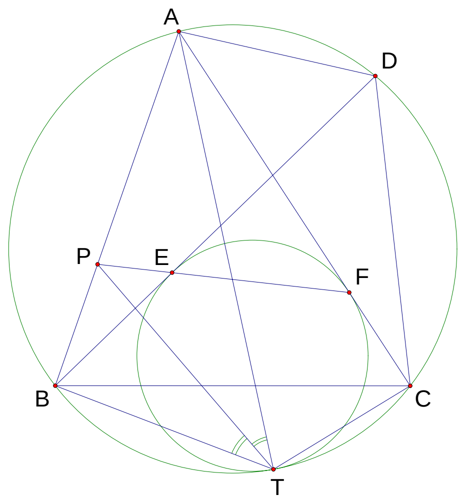
Problem 2. Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let ω be a circle internally tangent to Ω at T, and to DB and AC at E and F, respectively. Let P be the intersection of EF and AB. Prove that TP is the internal angle bisector of the angle $\measuredangle ATB$.

Solution. From our lemma, applied to circles Ω, ω and points A, B, we conclude that $\frac{AT}{BT} = \frac{AF}{BE}$, thus it suffices to prove that

$$\frac{AF}{BE} = \frac{AP}{PB}.$$

Indeed, notice that $\measuredangle PEB = \measuredangle AFP$, and from the Law of Sines, applied to triangles APF, BPE, we have

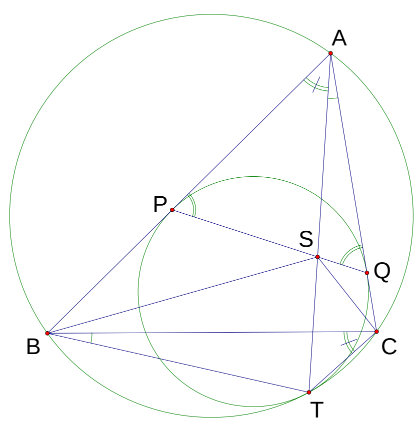
$$\frac{AP}{AF} = \frac{\sin \measuredangle AFP}{\sin \measuredangle APF} = \frac{\sin \measuredangle BEP}{\sin \measuredangle BPE} = \frac{BP}{BE}.$$



Therefore $\frac{AF}{BE} = \frac{AP}{PB}$, which completes our solution. $\square$

The third problem comes from the Moldovan Team Selection Test in 2007, which can be found in [2] and [3].

Problem 3. Let ABC be a triangle and let Ω be its circumcircle. Circles ω is internally tangent to Ω at T, and to sides AB and AC at P and Q, respectively. Let S be the intersection of AT and PQ. Prove that $\measuredangle SBA = \measuredangle SCA$.



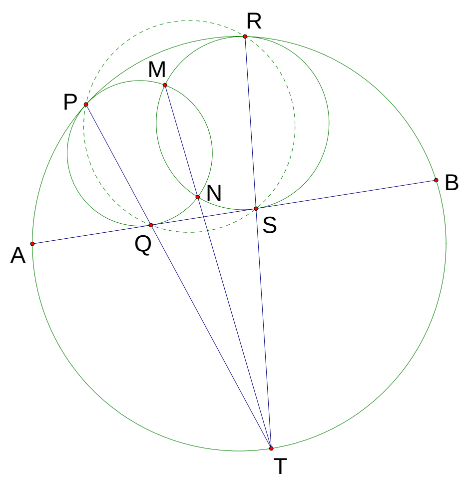
Solution. Using our lemma, we have

$$\frac{BP}{CQ} = \frac{BT}{CT} = \frac{\sin \measuredangle BCT}{\sin \measuredangle CBT} = \frac{\sin \measuredangle BAT}{\sin \measuredangle CAT} = \frac{PS}{QS}.$$

This fact implies that BPS and CQS are similar triangles which in turn implies that $\measuredangle SBA = \measuredangle SCA$. $\square$

Problem 4. Consider a circle (O) and a chord AB. Let circles (O_1), (O_2) be internally tangent to (O) and AB and let M and N their intersection. Prove that MN passes through the midpoint of the arc AB which does not contain M and N.

Solution. Denote by P and Q the tangency points of the circle (O_1) with (O) and AB, respectively. Let R and S be the tangency points of circle (O_2) with (O) and AB, respectively. Let T be the middle point of the arc AB which does not contain M and N.



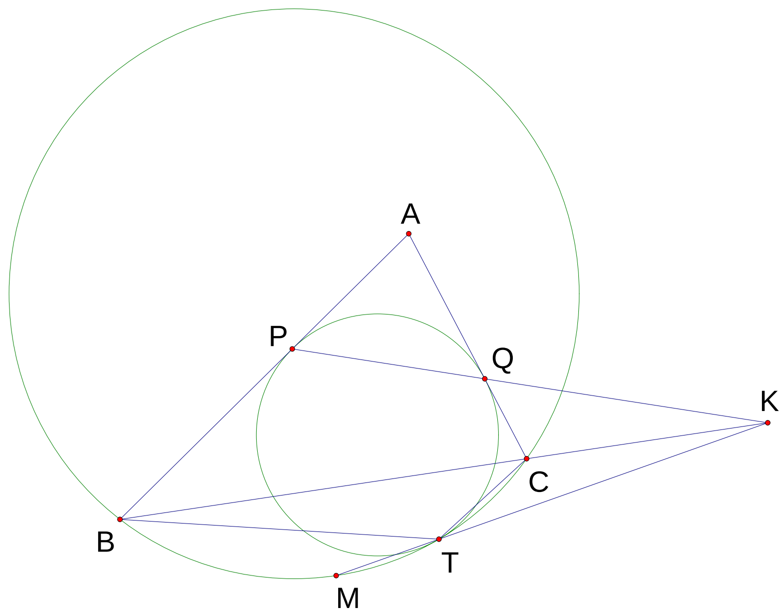
Applying the above lemma to circles (O), (O_1), and points A, B along with their tangent lines AQ, BQ to (O_1) we get $\frac{PA}{PB} = \frac{QA}{AB}$. This means that PQ passes through T. Similarly, RS passes through T. On the other hand, $\measuredangle PQA = \measuredangle QTA + \measuredangle QAT = \measuredangle PRA + \measuredangle ART = \measuredangle PRS$, therefore, points P, Q, R, S lie on a circle which we will denote by (O_3). We have that PQ is the radical axis of (O_1) and (O_3), RS is the radical axis of (O_2) and (O_3), and MN is the radical axis of (O_1) and (O_2). So, MN, PQ, and RS are concurrent at the radical center of the three circles. Hence, we deduce that MN passes through T, which is the midpoint of the arc AB that does not contain M and N. $\square$

We continue with a problem from the MOSP Tests 2007 [4].

Problem 5. Let ABC be a triangle. Circle ω passes through points B and C. Circle ω_1 is tangent internally to ω and also to the sides AB and AC at T, P, and Q, respectively. Let M be midpoint of arc BC (containing T) of ω. Prove that lines PQ, BC, and MT are concurrent.

Solution. Let $K = PQ \cap BC$ and let $K' = MT \cap BC$. Applying Menelaos' Theorem in triangle ABC we obtain

$$\frac{KB}{KB} \cdot \frac{QC}{QA} \cdot \frac{PA}{PB} = 1 \implies \frac{KB}{KC} = \frac{BP}{CQ}.$$



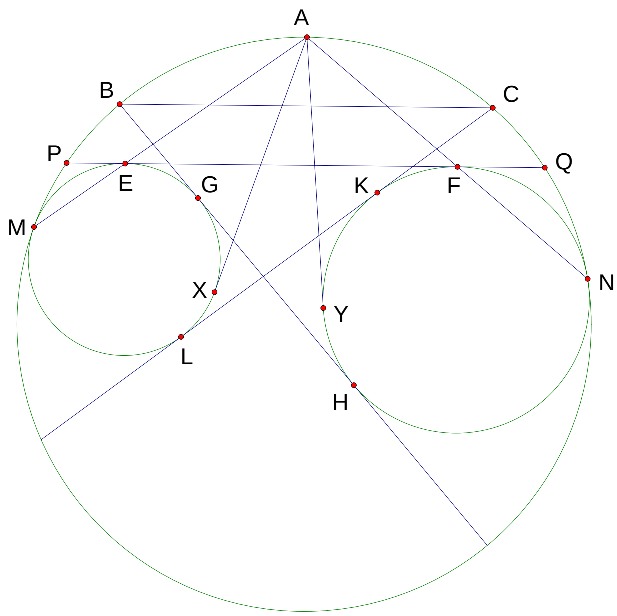
On the other hand, M is the midpoint of arc BC (containing T) of ω so MT is the external bisector of angle $\measuredangle BTC$, therefore $\frac{K'B}{K'C} = \frac{TB}{TC}$. Thus, we are left to prove that $\frac{BP}{CQ} = \frac{TB}{TC}$, which is true according to our lemma and we are done. $\square$

The last problem was given in [5] and is also discussed and proved in [6]. Now, we will present another solution for this nice problem.

Problem 6. Circles (O_1) and (O_2) are internally tangent to a given circle (O) at M and N, respectively. Their internal common tangents intersect (O) at four points. Let B and C be two of them such that B and C lie on the same side with respect to O_1O_2. Prove that BC is parallel to an external common tangent of (O_1) and (O_2).

Solution. Draw the internal common tangents GH, KL of (O_1), (O_2) such that G and L lie on (O_1) and K and H lie on (O_2). Let EF be the external common tangent of (O_1), (O_2) such that E and B lie on the same side with respect to O_1O_2. Denote by P and Q the intersections of EF with (O). We will prove that BC is parallel to PQ. Denote by A be the midpoint of the arc PQ which does not contain M and N. Let AX and AY be the tangents at X and Y to the circles (O_1)

and (O_2). In the solution to Problem 4 we have proved that A, E, and M are collinear; A, F, and N are collinear, and the quadrilateral $MEFN$ is cyclic. Therefore, $AX^2 = AE \cdot AM = AF \cdot AN = AY^2$, i.e. $AX = AY$ (1).



Based on the lemma, $\frac{MA}{AX} = \frac{MB}{BG} = \frac{MC}{CL}$. On the other hand, by the Ptolemy's Theorem, $MA \cdot BC = MB \cdot AC = MC \cdot AB$, therefore

$$AX \cdot BC = BG \cdot AC = CL \cdot AB.$$

Similarly,

$$AY \cdot BC = BH \cdot AC + CK \cdot AB.$$

Thus $AC \cdot (BH - BG) = AB \cdot (CL - CK)$, i.e. $AC \cdot GH = AB \cdot KL$, which implies $AC = AB$. Hence, A is the midpoint of the arc BC of the circle (O). This means that BC is parallel to PQ and our solution is complete. $\square$

References

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[2] Mathlinks, *A circle tangent to the circumcircle and two sides*, http://www.mathlinks.ro/viewtopic.php?t=140464

[3] Mathlinks, *Equal angle*,
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[5] Shay Gueron, *Two Applications of the Generalized Ptolemy Theorem*, American Mathematical Monthly 2002.

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