The SOS-Schur method

Bach Ngoc Thanh Cong, Nguyen Vu Tuan, Nguyen Trung Kien

The SOS-Schur method connects two well-known results, the sum of squares method and Schur inequality. The idea of the SOS-Schur method is to reduce a three variable inequality to an inequality of the following type

$$f(a,b,c) = M(a-b)^2 + N(a-c)(b-c) \geq 0.$$

Let us consider a classical example:

Example 1. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+c}{a+b} + \frac{b+c}{b+a} + \frac{c+a}{c+b}.$$

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Solution. Without loss of generality, assume that $c = \min(a,b,c)$. Note that for $x, y, z > 0$ we have

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} - 3 = \frac{1}{xy}(x-y)^2 + \frac{1}{xz}(x-z)(y-z).$$

Therefore the given inequality can be rewritten as

$$\left(\frac{1}{ab} - \frac{1}{(a+c)(b+c)}\right)(a-b)^2 + \left(\frac{1}{ac} - \frac{1}{(a+b)(a+c)}\right)(a-c)(b-c) \geq 0.$$

The last inequality is clearly true, because $c = \min(a,b,c)$, and we are done. $\square$

The above example is proved by a simple technique: rewriting the desired inequality as sum of two nonnegative numbers. We transform the inequality into the standard SOS-Schur form:

$$f(a,b,c) = M(a-b)^2 + N(a-c)(b-c) \geq 0.$$

After that we try to prove that, if $c = \min(a,b,c)$ or $c = \max(a,b,c)$, then M and N are nonnegative.

The following identities are useful in solving inequalities on three variables using the SOS-Schur method.

$$\begin{aligned}
a^2+b^2+c^2-ab-bc-ca &= (a-b)^2 + (a-c)(b-c)\\
a^3+b^3+c^3-3abc &= (a+b+c)(a-b)^2 + (a+b+c)(a-c)(b-c)\\
(a+b)(b+c)(c+a)-8abc &= 2c(a-b)^2 + (a+b)(a-c)(b-c)\\
ab^2+bc^2+ca^2-3abc &= c(a-b)^2 + b(a-c)(b-c)
\end{aligned}$$

$$a^2b^2+b^2c^2+c^2a^2-abc(a+b+c)=c^2(a-b)^2+ab(a-c)(b-c)$$
$$a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2=(a+b)^2(a-b)^2+(a+c)(b+c)(a-c)(b-c)$$
$$a^4+b^4+c^4-a^3b-b^3c-c^3a=(a^2+b^2+ab)(a-b)^2+(b^2+bc+c^2)(a-c)(b-c).$$

The next step is to establish the relationship between the SOS-Schur and SOS representation for symmetric polynomials. To find this common representation, we rely on the standard sum of squares representation. Suppose that we have the following symmetric expression:

$$f(a,b,c)=S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2.$$

How to transform it to the SOS-Schur form? The answer is simple: observe that

$$(a-b)^2+(b-c)^2+(c-a)^2=2(a-b)^2+2(a-c)(b-c).$$

Hence

$$\begin{aligned} f(a,b,c) &= (S_a-S_c)(b-c)^2+(S_b-S_c)(a-c)^2+S_c\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \\ &= 2S_c(a-b)^2+\left(\frac{(b-c)(S_a-S_c)}{a-c}+\frac{(a-c)(S_b-S_c)}{b-c}+2S_c\right)(a-c)(b-c) \end{aligned}$$

Note that $f(a,b,c)$ is a symmetric polynomial, while S_a, S_b, and S_c are semi-symmetric. Thus

$$f(a,b,c)=M(a-b)^2+N(a-c)(b-c),$$

where $M=2S_c$ and $N=\dfrac{(b-c)(S_a-S_c)}{a-c}+\dfrac{(a-c)(S_b-S_c)}{b-c}+2S_c$.

Alternatively, from a SOS representation,

$$\begin{aligned} f(a,b,c) &= S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 \\ &= (S_a+S_b)(a-c)(b-c)+(S_b+S_c)(b-a)(c-a)+(S_c+S_a)(c-b)(a-b) \\ &= \left(\frac{c(S_a-S_b)}{(a-b)}+\frac{aS_b-bS_a}{(a-b)}+S_c\right)(a-b)^2+(S_a+S_b)(a-c)(b-c). \end{aligned}$$

Thus

$$f(a,b,c)=M(a-b)^2+N(a-c)(b-c),$$

where $M=\dfrac{c(S_a-S_b)}{(a-b)}+\dfrac{aS_b-bS_a}{(a-b)}+S_c$ and $N=S_a+S_b$.

It follows that the main difficulty is actually to transform a cyclic expression into a standard SOS-Schur form. All above gives us a theoretical sense of this method. The following appications illustrate the special advantage of the SOS-Schur method.

Example 2. Let a, b, c be positive real numbers such that $a \geq b \geq c$. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Solution. We have

$$\begin{aligned}
f(a,b,c) &= a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \\
&= \big[a^2b(a-b) - ab^2(a-b)\big] + \big[b^2c(b-c) - ab^2(b-c)\big] + \\
&\quad + \big[c^2a(c-a) - ab^2(c-a)\big] \\
&= ab(a-b)^2 + (ab+ac-b^2)(a-c)(b-c).
\end{aligned}$$

Clearly the last expression is positive, and we are done. $\square$

Example 3. Let a, b, c be positive real numbers. Prove that

$$\frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b} + \frac{3(ab+bc+ca)}{(a+b+c)^2} \geq 4.$$

Solution. Without loss of generality, assume that $c = \min(a, b, c)$. We have

$$\begin{aligned}
\frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b} - 3 &= \frac{1}{(a+c)(b+c)}(a-b)^2 + \frac{1}{(a+b)(b+c)}(a-c)(b-c) \\
\frac{3(ab+bc+ca)}{(a+b+c)^2} - 1 &= -\frac{1}{(a+b+c)^2}(a-b)^2 - \frac{1}{(a+b+c)^2}(a-c)(b-c)
\end{aligned}$$

Hence the given inequality can be rewritten as

$$f(a,b,c) = M(a-b)^2 + N(a-c)(b-c),$$

where $M = \dfrac{1}{(a+c)(b+c)} - \dfrac{1}{(a+b+c)^2}$ and $N = \dfrac{1}{(a+b)(b+c)} - \dfrac{1}{(a+b+c)^2}$.

Clearly $M, N \geq 0$, and the inequality is proved. $\square$