A History of a Solved Conjecture

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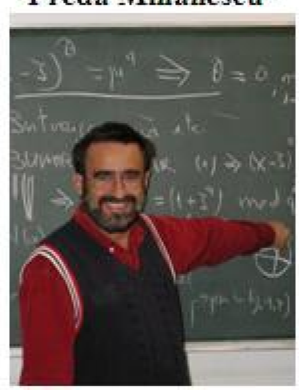
Abstract

The present article describes the landmarks in the history of Catalan's problem and gives an overview of Mihailescu's brilliant solution.

"In the summer of 1999, I heard of Catalan's conjecture and obtained the first partial results, which had some impact. Two years later, in the fall of 2001, I finished its proof." (*Preda Mihailescu*, December 16, 2009)

Eugène Charles Catalan[1] **Vs.** **Preda Mihăilescu**[2]



[1] was born in 1814, in Bruges (now Belgium, then part of the First French Empire), the only child of a French jeweller by the name of Joseph Catalan. In 1825, he traveled to Paris and learned mathematics at Ecole Polytechnique where he met Joseph Liouville (1833). In 1834 Catalan was expelled from university and went to Chalons-sur-Marne, where he got a job after graduating. Catalan came back to Ecole Polytechnique, and, with the help of Liouville, obtained his degree in mathematics in 1841. He went on to Charlemagne College to teach descriptive geometry. Though he was politically active and strongly left-wing, leading him to participate in the 1848 Revolution, Catalan had an animated career and also sat in the France's Chamber of Deputies. The University of Liege appointed him chair of analysis in 1865. In 1879, still in Belgium, he became a journal editor where he published as a foot note Paul-Jean Busschop's theory after refusing it in 1873 - letting Busschop know that it was too empirical. In 1883, he worked for the Belgian Academy of Sciences in the field of number theory. He died in Liege in 1894.

[2] (born May 23, 1955) is a Romanian mathematician, best known for his proof of Catalan's conjecture. Born in Bucharest, he is the brother of Vintila Mihailescu. After leaving Romania in 1973, he settled in Switzerland. Mihailescu studied mathematics and informatics in Switzerland, receiving his Ph.D. from ETH Zurich in 1997. His thesis, Cyclotomy of rings and primality testing, was written under the direction of Erwin Engeler and Hendrik Lenstra. For several years, Mihailescu did research at the University of Paderborn, Germany. Since 2005 he is a professor at the Georg-August University of Gottingen.

THEOREM.

The equation

$$x^u - y^v = 1 \tag{1}$$

has in natural numbers unique solution $(x, y; u, v) = (3, 2; 2, 3)$. This was proved in the special case $x = 3, y = 2$, by Gersonides,[3] in 1343. Approximately 100 years before Catalan, *Leonhard Euler* (1707-1783) solved the Diophantine equation

$$x^3 - y^2 = \pm 1. \tag{2}$$

Equation (2) was solved in recent years by methods of algebraic number theory. In 1844, August Leopold Crelle (1780-1855), editor of Crelle's Journal, the prestigious magazine (published in German, English, and French) and known today as *Journal fur die reine und angewandte Mathematik (Journal for Pure and Applied Mathematics), receives a letter from Catalan: "I beg you, sir, to announce in your journal the following theorem that I believe is true, although I have not yet succeeded in completely proving it; perhaps others will be more successful. Two consecutive whole numbers, other than 8 and 9, cannot be consecutive powers; that is, the equation $x^m - y^n = 1$ in which the unknowns are positive integers only admits a single solution...."* Editor's note published in [1] became a famous conjecture. Slowly mathematicians began to obtain interesting results for some special cases. First it was observed that the equation is equivalent to

$$x^p - y^q = 1 \ (x > 0, y > 0),$$

where p and q are prime numbers. The case $q = 2$ was resolved by *V.A. Lebesgue* in 1850 ([7]). After a little more than 100 years, in 1964, *Chao Ko*(1910-2002), solved the case $p = 2$ ([2]). The equation became

$$x^p - y^q = 1 \quad (xy \neq 0, \text{ where } p \text{ and } q \text{ are odd prime numbers}) \tag{4}$$

J.W.S Cassels (1922-) wrote equation (4) in the form

$$(x - 1)\frac{x^p - 1}{x - 1} = y^q \tag{5}$$

and noted that the greatest common divisor of the two factors in the left-hand side must be 1 (*case I*) or p (*case II*). In 1960 ([3]), it was proved that equation (4) has no solutions in *case I*

$$\text{because the system } x - 1 = a^q, \frac{x^p - 1}{x - 1} = b^q, y = ab \text{ has not solution.}$$

[3] Rabbi Levi Ben Gershon(1288-1344) - Jewish mathematician and astronomer

A study of the number of solutions (x, y) for p and q fixed was done by *Alan Baker*(1939-)[4], who gave logarithmic approximations and Robert Tijdeman (1943 -) who showed in [11] that equation (4) has at most a finite number of solutions (for $p < 7 \cdot 10^{11}$ and $q < 7 \cdot 10^{16}$, where $p < q$). Both tried to find lower bounds for p and q. In [6] it was showed that $p, q > 10^6$, then $p, q > 3 \cdot 10^8$ and in [8] it was proved that $\max(p, q) \approx 8 \cdot 10^{16}$ respectively $\min(p, q) > 10^7$ (conditions of *Wieferich*[5]). In [9], Preda Mihailescu proved that the congruences of *Wieferich*[6] in fact hold without any class number condition. Important for the final demonstration, which used results of *Galois* group and cyclotomic fields ([10]) was the estimation $|x| > q^p$ ([4] and [5]). A sketch of Mihailescu's proof can be found on the internet at [12].

Remark. In 2010, Preda Mihailescu expected to complete the review process of his proof of *Leopoldt* [7] ([15],[16]) conjecture. The proof (not yet validated) is available at [17] (also see the explanation at that at [18].)

And he will not stop here:

" ... I work at a theory in mathematics that will catch very well; it will be useful and welcome. It is under examination and professionals take time to verify it. If things are as I see them, there will be a deep simplification that the mathematicians will definitely enjoy. I will be able to share more in one year."

Preda Mihailescu, December 16, 2009

REFERENCES[8]

1. E. Catalan, Note extraite d'une lettre adressee a l'editeur, J. Reine Angew. Math. 27 (1844), 192.
2. Chao Ko, On the Diophantine equation Sci. Sinica (Notes) 14(1964), 457-460.
3. J.W.S. Cassels, On the equation, II, Proc. Cambridge Philos. Soc. 56 (1960), 97-103.
4. S. Hyyro, Uber das Catalansche Problem, Ann. Univ. Turku, Ser. A I no. 79 (1964), 8 pp.

[4] received the Fields Medal in 1970 at the age of 31

[5] $p^{q-1} \equiv 1 \pmod{q^2}$; $q^{p-1} \equiv 1 \pmod{p^2}$

[6] *Arthur Josef Alwin Wieferich* (1884-1954) - German mathematician

[7] *Heinrich-Wolfgang Leopoldt* (1927-) - German mathematician

[8] full details can be studied and a bibliography of [13], respectively [14]

5. S. Hyyro, Uberdie Gleichung und das Catalansche Problem, Ann. Acad. Sci. Fenn., Ser. A no. 355(1964), 50 pp.

6. K. Inkeri, On Catalan's conjecture, J. Number Theory 34 (1990), 142-152.

7. V.A. Lebesgue, Sur l'impossibilite en nombres entiers de l'equation , Nouv. Ann. Math. 9 (1850), 178-181.

8. M. Mignotte, Catalan's equation just before 2000, Number Theory (Turku, 1999), de Gruyter, Berlin, 2001, pp. 247-254

9. P. Mihailescu, A class number free criterion for Catalan's conjecture, J. Number Theory 99 (2003), 225-231.

10. P. Mihailescu, Primary cyclotomic units and a proof of Catalan's conjecture, prprint (September 2, 2002), submitted.

11. R. Tijdeman, On the equation of Catalan, Acta Arith. 29 (1976), 197-209.

12. `http://www.dpmms.cam.ac.uk/Seminars/Kuwait/abstracts/L30.pdf`

13. T. Metsankyla, Catalan's Conjecture: another old Diophantine problem solved, Bulletin of the American Mathematical Society 41 (1): 43-57. `http://www.ams.org/bull/2004-41-01/S0273-0979-03-00993-5/S0273-0979-03-00993-5.pdf`.

14. Y. F. Bilu, Catalan's conjecture (after Mihailescu), Sem. Bourbaki, 55eme annee, nr. 909 (2002/03), 24 pp.

15. `http://londonnumbertheory.wordpress.com/2009/11/08/leopoldts-conjecture/`

16. `http://arxiv.org/PS_cache/arxiv/pdf/0905/0905.1274v3.pdf`

17. `http://londonnumbertheory.files.wordpress.com/2009/11/mihailescu.pdf`

18. `http://londonnumbertheory.files.wordpress.com/2009/12/mihailescu2.pdf`

19. N. Stanciu, Retrospectiva unei conjecturi rezolvate, GMB, No. 2 / 2010. pag. 57-60.

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