Junior problems

J145. Find all nine-digit numbers $aaaabbbbb$ that can be written as a sum of fifth powers of two positive integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J146. Let $A_1A_2A_3A_4A_5$ be a convex pentagon and let $X \in A_1A_2$, $Y \in A_2A_3$, $Z \in A_3A_4$, $U \in A_4A_5$, $V \in A_5A_1$ be points such that A_1Z, A_2U, A_3V, A_4X, A_5Y intersect at P. Prove that

$$\frac{A_1X}{A_2X} \cdot \frac{A_2Y}{A_3Y} \cdot \frac{A_3Z}{A_4Z} \cdot \frac{A_4U}{A_5U} \cdot \frac{A_5V}{A_1V} = 1.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J147. Let $a_0 = a_1 = 1$ and

$$a_{n+1} = 1 + \frac{a_1^2}{a_0} + \cdots + \frac{a_n^2}{a_{n-1}}$$

for $n \geq 1$. Find a_n in closed form.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J148. Find all n such that for each $\alpha_1, \ldots, \alpha_n \in (0, \pi)$ with $\alpha_1 + \cdots + \alpha_n = \pi$ the following equality holds

$$\sum_{i=1}^{n} \tan \alpha_i = \frac{\sum_{i=1}^{n} \cot \alpha_i}{\prod_{i=1}^{n} \cot \alpha_i}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J149. Let $ABCD$ be a quadrilateral with $\angle A \geq 60^\circ$. Prove that

$$AC^2 < 2(BC^2 + CD^2).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J150. Let n be an integer greater than 2. Find all real numbers x such that $\{x\} \leq \{nx\}$, where $\{a\}$ denotes the fractional part of a.

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania and Mihai Piticari, "Dragos-Voda" National College, Romania

Senior problems

S145. Let k be a nonzero real number. Find all functions $f : R \longrightarrow \mathbb{R}$ such that

$$f(xy) + f(yz) + f(zx) - k\left[f(x)f(yz) + f(y)f(zx) + f(z)f(xy)\right] \geq \frac{3}{4k},$$

for all $x, y, z \in R$.

Proposed by Marin Bancos, North University of Baia Mare, Romania

S146. Let m_a, m_b, m_c be the medians, k_a, k_b, k_c the symmedians, r the inradius, and R the circumradius of a triangle ABC. Prove that

$$\frac{3R}{2r} \geq \frac{m_a}{k_a} + \frac{m_b}{k_b} + \frac{m_c}{k_c} \geq 3.$$

Proposed by Pangiote Ligouras, Bari, Italy

S147. Let $x_1, \ldots, x_n, a, b > 0$. Prove that the following inequality holds

$$\frac{x_1^3}{(ax_1 + bx_2)(ax_2 + bx_1)} + \cdots + \frac{x_n^3}{(ax_n + bx_1)(ax_1 + bx_n)} \geq \frac{x_1 + \cdots + x_n}{(a+b)^2}.$$

Proposed by Marin Bancos, North University of Baia Mare, Romania

S148. Let n be a positive integer and let a, b, c be real numbers such that $a^2 b \geq c^2$. Find all real numbers $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ for which

$$x_1 y_1 + \cdots + x_n y_n = \frac{a}{2}$$

and

$$x_1^2 + \cdots + x_n^2 + b(y_1^2 + \cdots + y_n^2) = c.$$

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania

S149. Prove that in any acute triangle ABC,

$$\frac{1}{2}\left(1 + \frac{r}{R}\right)^2 - 1 \leq \cos A \cos B \cos C \leq \frac{r}{2R}\left(1 - \frac{r}{R}\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S150. Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in a circle $C(O, R)$ and circumscribed about a circle $\omega(I, r)$. Denote by R_i the radius of the circle tangent to A_iA_{i+1} and tangent to the extensions of the sides $A_{i-1}A_i$ and $A_{i+1}A_{i+2}$. Prove that the sum $R_1 + R_2 + R_3 + R_4$ does not depend on the position of points A_1, A_2, A_3, A_4.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Undergraduate problems

U145. Consider the determinant

$$D_n = \begin{vmatrix} 1 & 2 & \cdots & n \\ 1 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^n & \cdots & n^n \end{vmatrix}.$$

Find $\lim\limits_{n\to\infty} (D_n)^{\frac{1}{n^2 \ln n}}$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U146. Let n be a positive integer. For all $i, j = 1, ..., n$ define $S_n(i,j) = \sum_{k=1}^{n} k^{i+j}$. Evaluate the determinant $\Delta = |S_n(i,j)|$.

Proposed by Dorin Andrica,"Babes-Bolyai" University, Romania

U147. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and let $c \in \mathbb{R}$ such that

$$\int\limits_a^b f(x)\, dx \neq (b-a)\, f(c),$$

for all $a, b \in \mathbb{R}$. Prove that

$$f'(c) = 0.$$

Proposed by Bogdan Enescu, "B. P. Hasdeu" National College, Romania

U148. Let $f : [0,1] \Rightarrow \mathbb{R}$ be a continuous non-decreasing function. Prove that

$$\frac{1}{2}\int_0^1 f(x)dx \leq \int_0^1 xf(x)dx \leq \int_{\frac{1}{2}}^1 f(x)dx.$$

Proposed by Duong Viet Thong, Hanoi University of Science, Vietnam

U149. Find all real numbers a for which there are functions $f, g : [0,1] \to \mathbb{R}$ such that for all

$$(f(x) - f(y))(g(x) - g(y)) \geq |x - y|^a$$

for all $x, y \in [0,1]$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

U150. Let (a_n) and (b_n) be sequences of positive transcendental numbers such that for all positive integers p the series $\sum_n (a_n^p + b_n^p)$ converges. Suppose that for all positive integers p there is a positive integer q such that $\sum_n a_n^p = \sum_n b_n^q$. Prove that there is an integer r and a permutation σ of the set of positive integers such that

$$a_n = b_{\sigma(n)}^r.$$

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

Olympiad problems

O145. Find all positive integers n for which

$$\left(1^4+\frac{1}{4}\right)\left(2^4+\frac{1}{4}\right)\cdots\left(n^4+\frac{1}{4}\right)$$

is the square of a rational number.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O146. Find all pairs (m,n) of positive integers such that $\varphi(\varphi(n^m))=n$, where φ is Euler's totient function.

Proposed by Marco Antonio Avila Ponce de Leon, Mexico

O147. Let H be the orthocenter of an acute triangle ABC, and let A', B', C' be the midpoints of sides BC, CA, AB. Denote by A_1 and A_2 the intersections of circle $C(A', A'H)$ with side BC. In the same way we define points B_1, B_2 and C_1, C_2, respectively. Prove that points $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclic.

Proposed by Catalin Barbu, Bacau, Romania

O148. Let ABC be a triangle and let A_1, A_2 be the intersections of the trisectors of angle A with the circumcircle of ABC. Define analogously points B_1, B_2, C_1, C_2. Let A_3 be the intersection of lines B_1B_2 and C_1C_2. Define analogously B_3 and C_3. Prove that the incenters and circumcenters of triangles ABC and $A_3B_3C_3$ are collinear.

Proposed by Daniel Campos Salas, Costa Rica

O149. A circle is divided into n equal sectors. We color the sectors in $n-1$ colors using each of the colors at least once. How many such colorings are there?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O150. Let n be a positive integer, $\varepsilon_0, ..., \varepsilon_{n-1}$ the n^{th} roots of unity, and a, b complex numbers. Evaluate the product

$$\prod_{k=0}^{n-1}(a+b\varepsilon_k^2).$$

Proposed by Dorin Andrica, "Babes-Bolyai" University, Romania