AwesomeMath
April 7-24, 2006

Admission Test B Solutions

1. Find all positive integers n for which $3n-4$, $4n-5$, and $5n-3$ are all prime.

   Solution: We make a parity argument to show that the only solution is $n=2$. Notice that if n is even, then $3n-4$ is even. If n is odd, then $5n-3$ is even. Since the only even prime is 2, one of the two numbers must equal 2. If $3n-4=2$, then $n=2$, $4n-5=3$ and $5n-3=7$. These numbers are all prime, so $n=2$ is a solution. If $5n-3=2$, then $n=1$, $4n-5=-1$, and $3n-4=-1$. These numbers are not all prime, so this is not a solution. So the only solution is $\boldsymbol{n=2}$.

2. Find the greatest 9-digit number whose digits' product is 9!.

   Solution: We use the Greedy Algorithm to construct the largest possible integer of this form. Proceeding place by place, starting at the left, we select the largest available digit. Since $9! = 2^7 \cdot 3^4 \cdot 5 \cdot 7$, we can begin with a maximum of two nines followed by two eights, leaving $2 \cdot 5 \cdot 7$ in the product. We then can make one seven, no sixes, one five, no fours or threes, and one two. To create a nine digit number, we add two trailing ones. Thus, our constructed number is $\mathbf{998,875,211}$.
   Solution by Jonathan Schneider.

3. Given three squares with side lengths 2, 3, and 6, cut two of them and reassemble the resulting five pieces into a square of side length 7. (By cut we mean dissect the square into two pieces by a polygonal line.)

   Solution: Many different dissections suffice; all involve cutting the 6×6 square at least once. For example,

```
6666 66 333 2 2
6666 66 333 2 2
66666 6 333
66666 6
666666
666666

2222333
6666333
6666333
6666666
6666666
6666666
6666666
```

4. The numbers 246, 462, and 624 are all divisible by 6. What is the greatest possible common divisor of the three-digit numbers $\underline{abc}$, $\underline{bca}$, and $\underline{cab}$ when a, b, and c are all different?

   Solution: Let k be the greatest common factor of $\underline{abc}$, $\underline{bca}$, and $\underline{cab}$. Notice that a, b, and c must all be single-digit positive numbers. We begin by showing that that k can have no prime factors but 2, 3, and 37. Suppose by way of contradiction that r is a prime factor of all three numbers other than 2, 3, or 37. Notice that $r|100b + 10c + a$ and $r|10(100a + 10b + c)$. So $r|10(100a + 10b + c) - (100b + 10c + a)$. So $r|999a$. Similarly, $r|999b$ and $r|999c$. Since r is a prime number other than 2, 3, or 37, it cannot divide 999, so it must divide a, b, and c. But no prime divides three different one-digit numbers other than 2 and 3. Notice also that since 4 does not divide 999 and 4 does not divide three different single-digit positive numbers, 4 cannot divide k. so the highest power of 2 that can divide k is $2^1 = 2$.

   If 37 divides k, then 2 does not because we verify by inspection that among the three digit multiples of 74, no three are of the form $\underline{abc}$, $\underline{bca}$, and $\underline{cab}$. And since a, b, and c are distinct, $111 = 3 \cdot 37$ cannot divide k. So if 37 divides k, then neither 2 nor 3 does, and no higher power of 37 can divide k, so therefore $k = 37$. Similarly, we verify that 81 does not divide k because no triplet of three-digit multiples of 81 are in the desired form. Therefore the highest power of 3 that can divide k is $3^3 = 27$.

   Finally, we notice that 54 divides the numbers 486, 648, and 864. 54 is divisible by the highest powers of 2 and 3 that can divide k, and is greater than 37. So the maximum greatest common factor of the three numbers is **54**.

5. The harmonic mean of two positive integers is 2006. Find the greatest possible value of their arithmetic mean.

   Solution: Let the two integers be x and y. Assume without loss of generality that $x \geq y$. We are given

$$\begin{aligned}
\frac{2}{\frac{1}{x} + \frac{1}{y}} &= 2006 \\
\frac{xy}{x+y} &= 1003 \\
xy - 1003(x+y) &= 0 \\
(x-1003)(y-1003) &= 1003^2 = 17^2 \cdot 59^2
\end{aligned}$$

   By letting $x - 1003$ and $y - 1003$ be each pair of factors of 1003^2, we see that the sum $x + y$ is maximized (and therefore their arithmetic mean, $\frac{x+y}{2}$ is maximized) when $x - 1003 = 1003^2$ and $y - 1003 = 1$. So $\frac{x+y}{2} = \mathbf{504008}$.

6. If (a, b, c) is a triple of integers satisfying the system of equations

$$ab - 3c = \frac{abc}{9} + 2$$

$$bc - 3a = \frac{abc}{9} + 3$$

$$ca - 3b = \frac{abc}{9} + 6$$

compute $2a + 3b + 6c$.

Solution: Adding up the three equations yields

$$\frac{abc}{3} - (ab + bc + ca) + 3(a + b + c) + 11 = 0$$

which can be rewritten as

$$(a - 3)(b - 3)(c - 3) = -60$$

But 60 has too many divisors to analyze this new equation by cases, so we continue as follows: subtract the second equation from the first, the third from the second, and the first from the third to obtain

$$(b + 3)(a - c) = -1$$

$$(c + 3)(b - a) = -3$$

$$(a + 3)(c - b) = 4$$

It follows that $|b + 3| = 1$ and $|c + 3| \in \{1, 3\}$, so $b \in \{-4, -2\}$ and $c \in \{-6, -4, -2, 0\}$. Because $b - 3$ and $c - 3$ divide -60, $b = -2$ and $c \in \{-2, 0\}$. But $b \neq c$, or else $(c - b) = 0$. So $c = 0$, and it follows that $a = -1$. So $2a + 3b + 6c = \mathbf{-8}$.

7. Is there an equiangular hexagon whose side lengths are (in some order) 2006, 2007, 2008, 2009, 2010, and 2011?

Solution: We construct such a hexagon. Begin with an equilateral triangle of side length 6027. Remove from the corners of this triangle three noncongruent smaller equilateral triangles: one each of side length 2009, 2010, and 2011. The remaining hexagon is equiangular, and has side lengths (in order) 2006, 2011, 2007, 2009, 2008, and 2010.

8. Let S be a subset of $\{1, 2, 3, \ldots, 15\}$ such that the product of any three distinct elements of S is not a square. Determine the maximum number of elements in S.

Solution: Call the 15-element set U. Consider these four disjoint subsets of U: {1, 4, 9}, {6, 8, 12}, {2, 7, 14}, {3, 5, 15}. In each of these four sets, the product of the elements is a perfect square. So none of these sets can be a subset of S. Since there are four of them, S must not contain at least one element of each of them. So the maximum number of elements in S is less than or equal to 11.

Suppose that S has 11 elements. Then S must contain 10, since the only four members of U not in S are each in one of the four disjoint sets listed above. Therefore $\{5, 8\}$ cannot be a subset of S, nor can $\{6, 15\}$ (since $5 \cdot 8 \cdot 10 = 20^2$ and $6 \cdot 15 \cdot 10 = 30^2$). Since only one member of $\{1, 4, 9\}$ is not in S, at least two of them are. Therefore $\{3, 12\}$ cannot be a subset of S (since $3 \cdot 12 \cdot 1 = 6^2$, $3 \cdot 12 \cdot 4 = 12^2$, and $3 \cdot 12 \cdot 9 = 18^2$.) So one member each of $\{5, 8\}$, $\{6, 15\}$, and $\{3, 12\}$ must not be in S. But all of these numbers are in two of the four listed disjoint sets. So we must remove three members of these two sets, and therefore we must remove at least two members of one of the sets. But we have to remove at least one member of each of the sets, so we must remove at least 5 members of the union of the four sets. So S cannot have 11 elements.

But S can have 10 elements; for example, $S = \{4, 5, 6, 7, 9, 10, 11, 12, 13, 14\}$. So the maximum number of elements of S is **10**.
Solution by Jonathan Schneider.

9. A company reports annually. It has been noted that the company recorded a profit over every period of p consecutive years and a loss over every period of q consecutive years. Find (in terms of p and q) the maximum possible length of time the company has been in business.

Solution: The problem is a generalization of IMO 1977, #2. The answer is $\boldsymbol{p+q-\gcd(p,q)-1}$, where $\gcd(p, q)$ denotes the greatest common divisor of p and q.

For a detailed argument, we refer the student to Mathematical Puzzles: A Connoisseur's Collection by Peter Winkler (©2004, A. K. Peters, Ltd.), pp. 17 - 19. In addition, visit
http://www.artofproblemsolving.com/Forum/resources-1-16-1977-367411.html
where longtime deputy leader of the Romanian IMO Team Bogdan Enescu writes that the Olympiad problem from which this question is derived is in his "Top 10 of all IMO problems."

10. The result of the addition

```
      AWESOME
         MATH
+      SUMMER
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is a seven-digit number whose digits are all equal. Different letters stand for different decimal digits. There are only two possible numbers that `AWESOME` can represent. What are they?

Solution: The four solutions are

$$9752185 + 8964 + 238850 = 9999999$$

$$9752185 + 8960 + 238854 = 9999999$$

$$2071357 + 5286 + 145579 = 22222222$$

$$2071357 + 5289 + 145576 = 22222222$$

So the two possible values for `AWESOME` are 9752185 and 2071357.