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CRACK PATHS IN PLANE SITUATIONS—II. DETAILED FORM OF THE EXPANSION OF THE STRESS INTENSITY FACTORS

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Abstract—In a previous paper, we established the general form of the expansion of the stress intensity factors in powers of the crack extension length, for a crack propagating in a two-dimensional body along an arbitrary kinked and curved path. The aim of the present paper is to calculate precisely the various functions of the geometric and mechanical parameters which appear in this expansion. The functions involved in the case of a straight extension are identified by considering the special case of a crack composed of two straight branches, placed in an infinite body loaded by uniform forces at infinity; the problem is solved with the aid of Muskhelishvili's formalism and conformal mapping. The functions describing the effect of the curvature of the crack extension are determined by studying another special case, identical to the first one except that the crack extension is curved; the method of solution consists of using a perturbative procedure with respect to the curvature parameters to reduce the original problem to a simpler one involving a crack with two straight branches, and solving again the latter problem by conformal mapping. A numerical strategy using these results for the prediction of crack paths over arbitrary long distances is discussed in conclusion.

INTRODUCTION

Let us consider (Fig. 1) an elastic body under plane strain conditions containing a crack with a kinked and curved extension of length s. Let πm $(-1 < m < +1)$ denote the kink angle, C the curvature of the main branch at the angular point 0 and a^*, C^* curvature parameters such that the shape of the extension may be described by

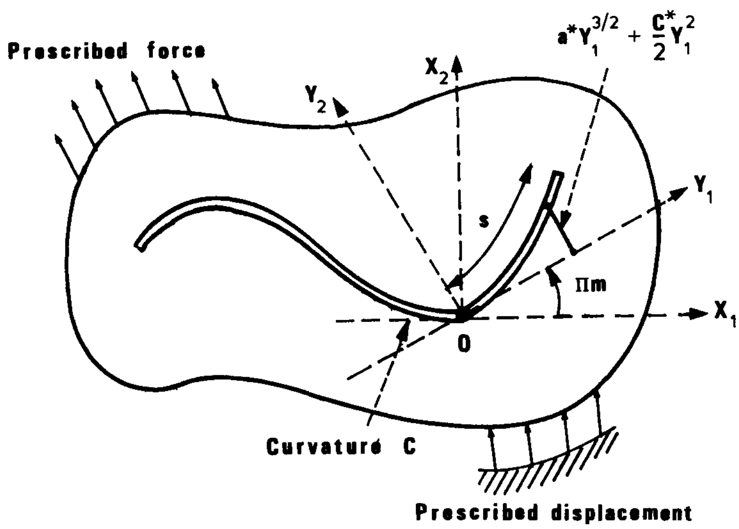

Fig. 1. General problem studied.

$$Y_2 = a^* Y_1^{3/2} + \frac{C^*}{2} Y_1^2 + 0(Y_1^{5/2}), \tag{1}$$

where $0Y_1Y_2$ denotes an orthonormal coordinate system with origin at the angular point and first axis directed along the tangent to the extension at that point.† It was shown in Part I (Leblond, 1989)‡ that the expansion of the stress intensity factors (SIFs) $k_p(s)$ ($p = 1, 2$) at the extended crack tip in powers of s is of the general form

$$k_p(s) = k_p^* + k_p^{(1/2)} \sqrt{s} + k_p^{(1)} s + 0(s^{3/2}), \tag{2}$$

where k_p^*, $k_p^{(1/2)}$, $k_p^{(1)}$ are given (using the Einstein summation convention) by

$$k_p^* = F_{pq}(m) k_q ; \tag{3}$$

$$k_p^{(1/2)} = G_p(m) T + a^* H_{pq}(m) k_q ; \tag{4}$$

$$k_p^{(1)} = Z_p + I_{pq}(m) b_q + C J_{pq}(m) k_q + a^* K_p(m) T + a^{*2} L_{pq}(m) k_q + C^* M_{pq}(m) k_q. \tag{5}$$

In these equations the k_qs, T and the b_qs are the SIFs, non-singular stress and coefficients of the $\sqrt{r}$ terms in the stress expansion at the original crack tip 0; the F_{pq}s, G_ps, H_{pq}s, I_{pq}s, J_{pq}s, K_ps, L_{pq}s and M_{pq}s are functions of the kink angle, which were termed *universal* in Leblond (1989) because they apply to any situation, whatever the geometry and the loading under study; and Z_p is an extra, *non-universal* term in the sense that it depends on the whole geometry of the body considered and cannot be expressed in any simple and general way. This term is nevertheless independent of the curvature parameters a^*, C^* of the crack extension.

Since $G_p(m)T$ does not depend on a^*, eqn (4) can be rewritten in the form

$$k_p^{(1/2)} = [k_p^{(1/2)}]_{a^*=0}^{\pi m} + a^* H_{pq}(m) k_q, \tag{6}$$

where the first term on the right-hand side is the value of $k_p^{(1/2)}$ for a *straight* ($a^* = 0$) extension in the direction πm. Similarly, since Z_p, $I_{pq}(m)b_q$, $CJ_{pq}(m)k_q$, $a^*K_p(m)T$ and $a^{*2}L_{pq}(m)k_q$ are independent of C^*, these terms can be grouped in eqn (5) in order to express $k_p^{(1)}$ as

$$k_p^{(1)} = [k_p^{(1)}]_{C^*=0}^{\pi m, a^*} + C^* M_{pq}(m) k_q, \tag{7}$$

where the notation $[k_p^{(1)}]_{C^*=0}^{\pi m, a^*}$ refers to an extension having a zero C^*.

A strategy for numerical determination of crack paths based on these equations was briefly sketched at the end of Part I and will be further discussed in the conclusion. The use of this procedure requires of course the detailed knowledge of the functions involved. The subject of the present paper is the determination of the latter.

The methods expounded here allow for the evaluation of *all* the functions which appear in eqns (3–5). However the strategy just mentioned is based on eqns (2), (3), (6) and (7) which involve only the F_{pq}s, H_{pq}s and M_{pq}s. The determination of these essential functions will be presented in detail. That of the G_ps and K_ps will also be thoroughly explained because, as will be seen, it provides a check on the correctness of the equations and an estimate of the accuracy of the numerical procedure employed *in fine*. With regard to the remaining functions, some results will be given for the sake of comparison with other works but with only brief indications on their derivation.

The functions F_{pq} and G_p describe the expansion of the SIFs in the case of a straight ($a^* = 0$, $C^* = 0$) extension. Since they are of universal value, they can be determined by

† The necessity to consider such singular shapes for describing the actual propagation of cracks is established notably in Cotterell and Rice (1980).

‡ Although use will be made of the results obtained in Leblond (1989), no detailed reading of this paper is necessary; the material needed is entirely recalled here.

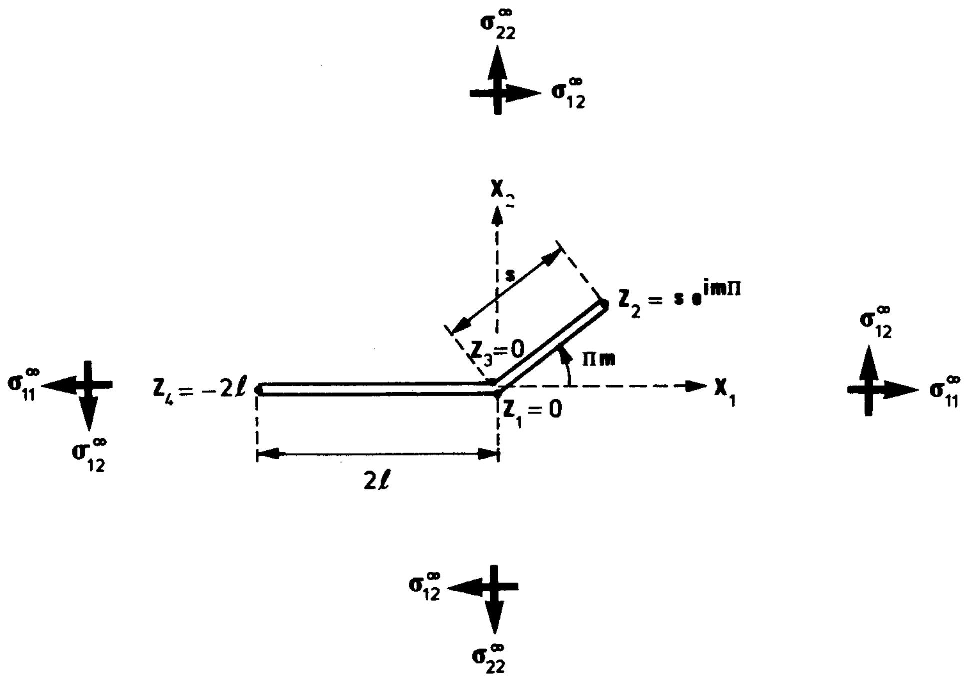

Fig. 2. Particular case considered for the determination of the functions F_{pq} and G_p.

studying the particular case of a crack composed of two straight branches placed in an infinite body loaded by uniform forces at infinity (Fig. 2). On the other hand, the (again universal) functions $H_{pq}(m)$, $K_p(m)$, $M_{pq}(m)$ relate to the influence of the curvature parameters of the extension. The simplest way to determine them is to study the same particular case as before, but with a curved extension (Fig. 3).

In the case of a secondary branch of *finite length*, the first problem (Fig. 2) was considered in many papers, all of which will not be quoted here. The most remarkable of these papers, in the authors' view, are due to Dudukalenko and Romalis (1973), Hussain *et al.* (1974) and Chatterjee (1975). Using Muskhelishvili's method (Muskhelishvili, 1953), the first two groups of authors established an integral equation governing the complex potentials of the problem, but mistakes in the resolution of this equation led to erroneous

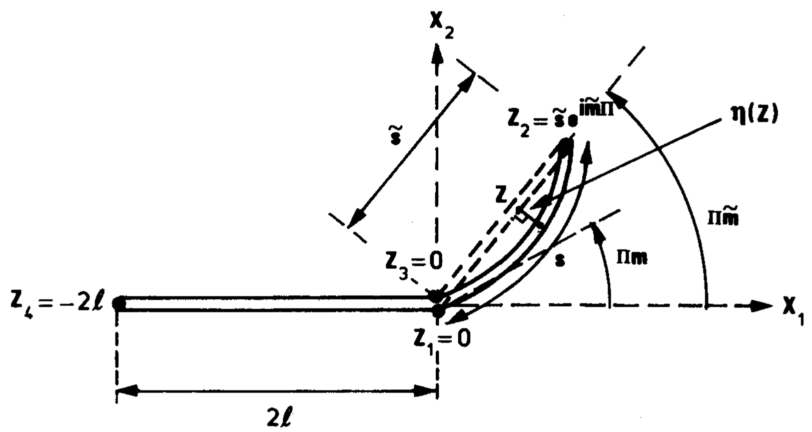

Fig. 3. Particular case considered for the determination of the functions H_{pq}, K_p and M_{pq}.

results, as analyzed by Amestoy (1987). The same integral equation was also derived, and solved correctly for the first time, by Chatterjee.

Despite the interest in this problem, the results obtained are not of general value and cannot be applied to other, more complex situations (non-uniform forces at infinity, finite body, etc.) since $k_p(s)$, for finite s, is not a universal quantity (this results from the non-universality of the term Z_p in eqn (5)). The case of an *infinitesimal extension* is more interesting, since it yields the universal functions F_{pq} and G_p. Chatterjee was well aware of the importance of such an asymptotic study, as appears in the following sentence: "... it is necessary to obtain some asymptotic solutions to the integral equations presented here, for small values of r_2^*" (s in the present notation). "It is hoped that such asymptotic solutions will be taken up in a future study." This problem was studied most convincingly by Bilby and Cardew (1975) and Bilby *et al.* (1977) using a previous work by Khrapkov (1971) based on the Mellin transform, and Wu (1978a,b, 1979) using conformal mapping [the results of this author were confirmed by Amestoy *et al.* (1979) and Amestoy and Leblond (1985)]. Both groups of works yielded the functions F_{pq}; on the other hand the functions G_p were studied only by Bilby and Cardew and in a very incomplete manner (there is little doubt, however, that a thorough study would have been possible). The only drawbacks of these works were that they did not really fulfil Chatterjee's wish because their treatments were based on the hypothesis of infinitesimal length from the beginning to the end, no connection with the case of a finite length being made, and also that the universality of the results obtained was presumably not well realized, let alone proved.

The situation is more critical with regard to the second problem, which involves a curved extension (Fig. 3). The only published results, due to Karihaloo *et al.* (1981), Sumi *et al.* (1983) and Sumi (1986, 1991), are the first order (with respect to m) expressions of the H_{pq}s, K_ps and M_{pq}s and the zeroth order expressions of the L_{pq}s; the question of the universality of these results was raised and partially solved in Sumi's works. The method used was a perturbative procedure with respect to the parameters m, a^* and C^* characterizing the deviations of the crack from straightness; this procedure was proposed by Cotterell and Rice (1980), following and extending an earlier work of Banichuk (1970). In this approach the original problem is reduced to a new one involving a fictitious straight crack having the same tips as the real, kinked and curved one (Fig. 4).

Of course, this method also yields low order (with respect to m) expressions of the functions F_{pq} related to the case of a straight extension. It was remarked by Sumi (1991) that the second order formula obtained in that way for F_{22} does not agree with the exact result derived by Wu (1979) and confirmed by Amestoy and Leblond (1985). In Sumi's terms, "... some second order terms of the perturbation solution are simply very good approximations of the exact asymptotic behaviour given by Wu (1979) and Amestoy and Leblond (1985). The slight difference of the representations may arise from the fact that the stress singularities at the branched corner are disregarded in the perturbation analysis". In more precise terms, the procedure involves a shift in the cut of the complex potentials

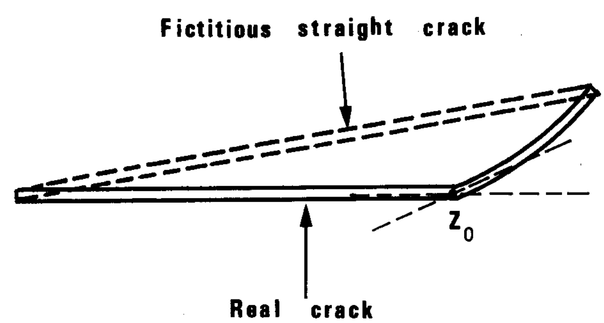

Fig. 4. The real and fictitious cracks in the Banichuk–Cotterell–Rice procedure.

from the original crack to the fictitious straight one; the latter crack being for instance supposed to lie above the real one as on Fig. 4, this means that the original values of the potentials in the region comprised between the cracks are eliminated and replaced by values obtained by analytic continuation from the lower half-plane through the original crack. The problem is that such an analytic continuation is *not* possible. Indeed if the domain of definition of the complex potentials could be extended beyond the angular point Z_0, they would be regular at that point, i.e. admit a representation as a power series of $Z-Z_0$ for small values of this variable, in contradiction with Williams' (1952) results on stress singularities at the apex of corners and notches. Because of that fundamental drawback, the entire Banichuk–Cotterell–Rice procedure becomes illicit if the crack contains an angular point, or more generally any geometric singularity generating a non-$\mathscr{C}^{\infty}$ stress field, such as a point of discontinuous curvature for instance. Even though, as will be seen, many of the results obtained by this procedure turn out to be correct, they cannot be accepted as such without another, rigorous analysis.

Part A of this paper is devoted to the calculation of the F_{pq}s and G_ps through the consideration of the particular case sketched in Fig. 2. Though a notable part of the material here is not new, this presentation is deemed necessary because the more original Part B makes an essential use of the method and results expounded. The integral equation governing the complex potentials is first derived in the case of an extension of finite length, using conformal mapping. Then integral equations allowing for the calculation of the F_{pq}s and G_ps are obtained by letting the length of the extension tend toward zero through suitable changes of variables and functions, which establishes the desired connection between the cases of a finite extension and of an infinitesimal one for the first time. Solutions of these integral equations in the form of series and accurate, high order expansions of the F_{pq}s and G_ps are finally provided.

The particular case of Fig. 3 is studied in Part B in order to derive the values of the H_{pq}s, K_ps and M_{pq}s. Since in the expressions (4) and (5) of $k_p^{(1/2)}$ and $k_p^{(1)}$, these functions appear in terms which are *linear* with respect to the curvature parameters a^* and C^*, they can be calculated *exactly* by using a first order perturbative procedure with respect to these parameters. Following the method sketched in Leblond and Amestoy (1989), the fictitious reference crack is taken to consist of two straight branches, the secondary one extending between the angular point and the tip of the original curved extension (see Fig. 3). There are two essential advantages in this approach with respect to the classical Banichuk–Cotterell–Rice procedure: first, analytic continuation of the potentials from the lower half-plane becomes possible since the angular, singular point is not crossed in the process; second, the treatment is not perturbative with respect to the kink angle πm so that the latter can take arbitrary values instead of being restricted to small ones. Once the reduction to a kinked-but-not-curved crack problem is achieved, the solution is obtained by the same techniques as in Part A: conformal mapping, integral equations and solutions in the form of series. Values of the H_{pq}s, K_ps and M_{pq}s are finally calculated through numerical evaluation of these series.

In conclusion, the expansion of the stress intensity factors in powers of the crack extension length studied in Part I and here is combined with Goldstein and Salganik's (1974) *principle of local symmetry* in order to derive the expressions of the geometric quantities (kink angle, curvature parameters) characterizing future propagation of the crack; in particular, a general equation is given for the curvature of the crack in its regular ($\mathscr{C}^{\infty}$) part. A strategy for numerical predictions of crack paths based on these results is finally discussed.

PART A: THE CASE OF A STRAIGHT EXTENSION

A.1. *Presentation of the problem and reduction to an integral equation*

We consider the problem depicted on Fig. 2. Use is made of the complex variable $Z = X_1 + iX_2$. The X_1-axis is taken to be collinear to the main crack branch. The tips of the crack are located at $Z_2 = s\,e^{im\pi}$ and $Z_4 = -2\ell$ (2ℓ = length of the main branch) and the angular points at $Z_1 = 0$ and $Z_3 = 0$. The stress tensor at infinity is denoted σ^{∞}.

The problem consists of finding Muskhelishvili's potentials Φ and Ψ, which are analytic everywhere except on the crack and subject to the following conditions:

$$\Phi(Z)+Z\overline{\Phi'(Z)}+\overline{\Psi(Z)} = Cst \qquad \text{on the crack;} \tag{8}$$

$$\Phi(Z) = \Gamma Z+0(1); \quad \Psi(Z) = \Gamma' Z+0(1) \qquad \text{at infinity,} \tag{9}$$

where Γ and Γ' are given in terms of the stress tensor at infinity by

$$\Gamma = \tfrac{1}{4}(\sigma_{11}^{\infty}+\sigma_{22}^{\infty}); \quad \Gamma' = \tfrac{1}{2}(\sigma_{22}^{\infty}-\sigma_{11}^{\infty})+i\sigma_{12}^{\infty}. \tag{10}$$

The exterior of the crack $Z_1Z_2Z_3Z_4Z_1$ can be mapped onto the exterior Ω^- of the unit circle $\mathscr{U}$ in a new z-plane (Fig. 5), by defining [see for instance Dudukalenko and Romalis (1973)]:

$$Z = \omega(z) = R\, e^{im\alpha}\frac{(z-e^{i\alpha})(z-e^{-i\alpha})}{z}\left(\frac{z-e^{-i\alpha}}{z-e^{i\alpha}}\right)^m, \tag{11}$$

where R and α are constants connected to the lengths 2ℓ and s of the crack branches and the kink angle πm by the relations

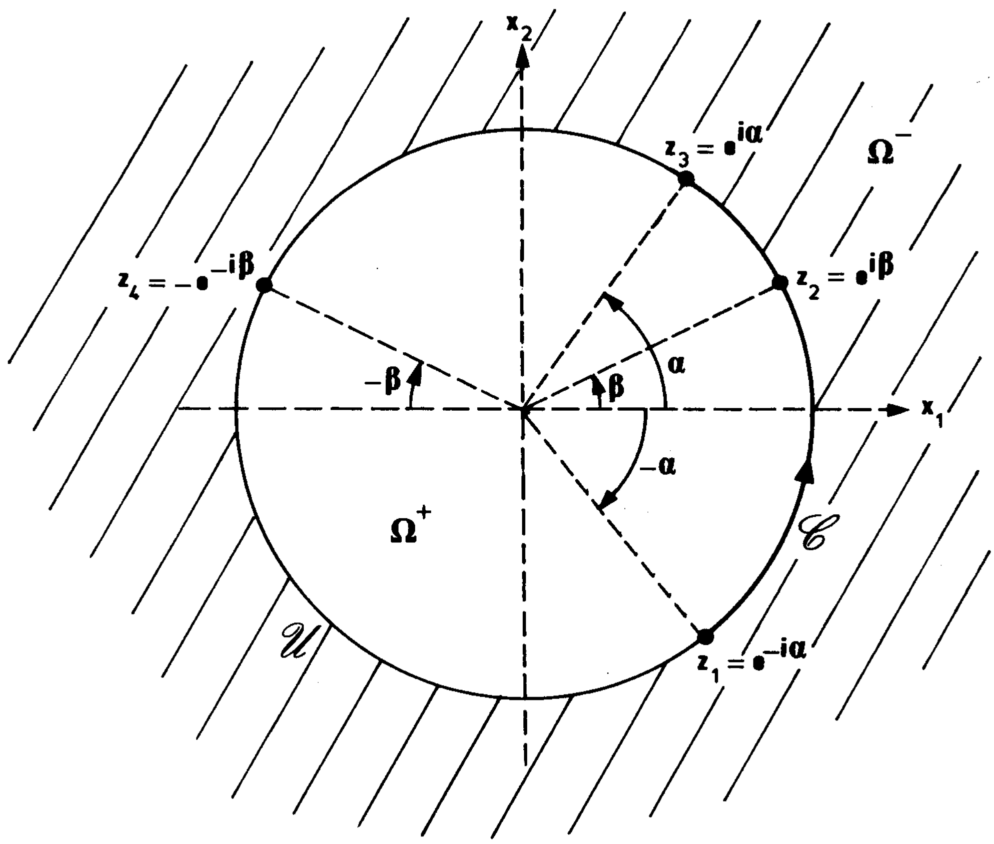

Fig. 5. The z-plane.

$$\ell = 2R\left(\cos\frac{\alpha+\beta}{2}\right)^{1-m}\left(\cos\frac{\alpha-\beta}{2}\right)^{1+m}; \tag{12}$$

$$s = 4R\left(\sin\frac{\alpha+\beta}{2}\right)^{1+m}\left(\sin\frac{\alpha-\beta}{2}\right)^{1-m}; \tag{13}$$

$$\sin\beta \equiv m\sin\alpha. \tag{14}$$

The determination of the function $((z-e^{-i\alpha})/(z-e^{i\alpha}))^m$ to be used in eqn (11) is such that the cut be located along any line connecting the points $e^{i\alpha}$ and $e^{-i\alpha}$ within the interior Ω^+ of the unit circle, and that its limit for $z\to\infty$ be equal to unity. The images of the crack tips in the z-plane are $z_2 = e^{i\beta}$ and $z_4 = -e^{-i\beta}$, and those of the corner points, $z_1 = e^{-i\alpha}$ and $z_3 = e^{i\alpha}$ (see Fig. 5).

Equations (8), (9) read, for the potentials $\varphi(z) = \Phi(Z)$ and $\psi(z) = \Psi(Z)$ in the new plane:

$$\varphi(z) + \frac{\omega(z)}{\overline{\omega'(z)}}\overline{\varphi'(z)} + \overline{\psi(z)} = Cst \qquad \text{for} \quad z\in\mathscr{U}; \tag{15}$$

$$\varphi(z) = \Gamma R\, e^{im\alpha}z + 0(1); \quad \psi(z) = \Gamma' R\, e^{im\alpha}z + 0(1) \quad \text{at infinity.} \tag{16}$$

The quantity $\omega(z)/\overline{\omega'(z)}$ is readily shown to be equal to $-e^{2im\pi}Q(z)$ for $z\in\mathscr{C}$ and to $-Q(z)$ for $z\in\mathscr{U}-\mathscr{C}$, where $\mathscr{C}$ is the arc $z_1z_2z_3$ (see Fig. 5) and $Q(z)$ is defined by

$$Q(z) = \frac{(z-e^{i\alpha})(z-e^{-i\alpha})}{z(z-e^{i\beta})(z+e^{-i\beta})}. \tag{17}$$

This can be written

$$\frac{\omega(z)}{\overline{\omega'(z)}} = [-1 + (1-e^{2im\pi})I(z)]Q(z), \tag{18}$$

where I is the characteristic function of the arc $\mathscr{C}$:

$$I(z) = 1 \quad \text{if} \quad z\in\mathscr{C}, \qquad 0 \quad \text{if} \quad z\in\mathscr{U}-\mathscr{C}. \tag{19}$$

It follows that eqn (15) reads

$$\varphi(z) - Q(z)\overline{\varphi'(z)} + (1-e^{2im\pi})I(z)Q(z)\overline{\varphi'(z)} + \overline{\psi(z)} = Cst \quad \text{for} \quad z\in\mathscr{U}. \tag{20}$$

The resolution of eqns (16), (20) will require the knowledge of the behaviour of the potentials in the vicinity of the points $e^{i\alpha}$, $e^{-i\alpha}$, $e^{i\beta}$ and $-e^{-i\beta}$. Let us consider the points $e^{i\beta}$ and $-e^{-i\beta}$ first. Since $\sigma_{11}+\sigma_{22} = 4\mathrm{Re}\,\Phi'(Z)$ and the stresses admit expansions in powers of $|Z-Z_2|$ and $|Z-Z_4|$ with exponents $-1/2$, 0, 1/2,... near Z_2 and Z_4, $\Phi(Z)$ admits expansions in powers of $Z-Z_2$ and $Z-Z_4$ with exponents 0, 1/2, 1,... near these points. Now it is clear from eqns (17), (18) that $\omega'(e^{i\beta}) = \omega'(-e^{-i\beta}) = 0$ and one can check that $\omega''(e^{i\beta}) \neq 0$ and $\omega''(-e^{-i\beta}) \neq 0$; this implies that $Z-Z_2 = 0((z-e^{i\beta})^2)$ and $Z-Z_4 = 0((z+e^{-i\beta})^2)$. Thus $\varphi(z)$ admits expansions in powers of $z-e^{i\beta}$ and $z+e^{-i\beta}$ with exponents 0, 1, 2,..., which implies that φ *is indefinitely differentiable at* $e^{i\beta}$ and $-e^{-i\beta}$. Then eqns (17), (20) show that, in contrast, ψ *has simple poles at these points.*

Let us now consider the points $e^{\pm i\alpha}$. Williams' (1952) results imply that the stresses are $0(|Z-Z_1|^{c_1})$ and $0(|Z-Z_3|^{c_3})$ near the angular points Z_1 and Z_3, where the exponents c_1 and c_3 are greater than $-1/2$. Thus $\Phi(Z)-\Phi(Z_1)$ and $\Phi(Z)-\Phi(Z_3)$ are $0((Z-Z_1)^{c_1+1})$ and $0((Z-Z_3)^{c_3+1})$, the exponents c_1+1 and c_3+1 being greater than 1/2, and *a fortiori* positive. Since by eqn (11), $Z-Z_1 = 0((z-e^{-i\alpha})^{c'_1})$ and $Z-Z_3 = 0((z-e^{i\alpha})^{c'_3})$ where c'_1 and c'_3 are positive constants, $\varphi(z)-\varphi(e^{-i\alpha})$ and $\varphi(z)-\varphi(e^{i\alpha})$ are also

$0((z-e^{-i\alpha})^{c_1''})$ and $0((z-e^{i\alpha})^{c_3''})$ with positive exponents c_1'' and c_3''. It follows that *the function* φ' *is* $0((z-e^{-i\alpha})^{c_1''-1})$ *and* $0((z-e^{i\alpha})^{c_3''-1})$ *near* $e^{-i\alpha}$ *and* $e^{i\alpha}$ *respectively, where the constants* $c_1''-1$ *and* $c_3''-1$ *are greater than* -1; this implies in particular that it verifies the conditions

$$\lim_{z \to e^{\pm i\alpha}} (z-e^{\pm i\alpha})\varphi'(z) = 0.$$

Such a function will be termed *weakly singular* (at the points $e^{i\alpha}$ and $e^{-i\alpha}$) in the sequel. Equations (17), (20) imply that ψ' *is also weakly singular at these points.*

The reduction of eqns (16), (20) to an integral equation is based on the following lemma, the (elementary) proof of which is given in Appendix A:

Lemma. Let f and g be complex functions defined and continuous on $\Omega^- \cup \mathscr{U}$, analytic on Ω^- including at the point at infinity,† and such that

$$f(z) = \overline{g(z)} \quad \text{for} \quad z \in \mathscr{U}. \tag{21}$$

Then f and g are constant and conjugate to each other.

To put eqn (20) in the form (21), we define

$$\chi(z) = \frac{1-e^{2im\pi}}{2i\pi} \int_{\mathscr{C}} \frac{Q(t)\overline{\varphi'(t)}\,dt}{t-z}, \tag{22}$$

where the arc $\mathscr{C}$ is oriented from $e^{-i\alpha}$ to $e^{i\alpha}$. In the above integral, the pole $e^{i\beta}$ of the function Q (see eqn (17)), which lies on the integration path, is "slightly displaced" toward Ω^-, i.e. $\chi(z)$ must in fact be understood as the limit, for $\varepsilon > 0$, $\varepsilon \to 0$, of the same integral but with $e^{i\beta}$ replaced by $e^{i\beta^-} = e^{i(\beta - i\varepsilon)}$; this is not indicated explicitly in eqn (22) because the notation would become too awkward. On the other hand, the points $e^{\pm i\alpha}$ do not raise any convergence problems for the integral, since eqn (17) and the condition of weak singularity of φ' imply that the integrand vanishes at these points. Then, if $\chi(z^\pm)$ denotes the limit of $\chi(t)$ for $t \in \Omega^\pm \to z \in \mathscr{U}$, $\chi(z^+) - \chi(z^-)$ is equal to $(1-e^{2im\pi})Q(z)\overline{\varphi'(z)}$ if $z \in \mathscr{C}$ by Plemelj's formula, and to 0 if $z \in \mathscr{U} - \mathscr{C}$. Thus the third term in the left-hand side of eqn (20) is equal to $\chi(z^+) - \chi(z^-)$. Furthermore, if we define, following Muskhelishvili, the analytic function $\chi_*(z) = \overline{\chi(1/\bar{z})}$, the term $\chi(z^+)$ in this expression can be replaced by $\chi(1/\overline{z^-}) = \overline{\chi_*(z^-)}$. Finally $Q(z)$ can also be replaced by $\overline{Q_*(z)}$ in the second term of the left-hand side of eqn (20). This equation then takes the form

$$\varphi(z) - \chi(z^-) = \overline{Q_*(z)}\;\overline{\varphi'(z) - \chi_*(z^-) - \psi(z)} + Cst \quad \text{for} \quad z \in \mathscr{U}. \tag{23}$$

Most of the hypotheses of the lemma stated above are verified in this equation. Indeed the left-hand side and the conjugate of the right-hand side are analytic on Ω^-. Moreover, let us show that the left-hand side is finite on $\mathscr{U}$. The only points which raise problems are $e^{\pm i\alpha}$, $e^{i\beta}$ and $-e^{-i\beta}$. Because of the behaviour of φ at these points (see above), it will suffice to prove that $\chi(z)$ has finite limits for $z \to e^{\pm i\alpha}$ or $e^{i\beta}$ in Ω^-. $\chi(e^{\pm i\alpha})$ is finite because the term $t - e^{\pm i\alpha}$ in the denominator of the integrand in eqn (22) cancels out with that in the numerator of $Q(t)$ (see eqn (17)), so that the weakly singular behaviour of $\varphi'(t)$ near $e^{\pm i\alpha}$ ensures the convergence of the integral. To show that $\chi(z)$ is finite for $z \to e^{i\beta}$ in Ω^-, write the integrand in eqn (22) in the form $f(t)/((t-e^{i\beta^-})(t-z))$ where f is $\mathscr{C}^\infty$ on $\mathscr{C}$ and split the integral into two parts:

$$\int_{\mathscr{C}} \frac{Q(t)\overline{\varphi'(t)}\,dt}{t-z} = \int_{\mathscr{C}} \frac{f(t)-f(e^{i\beta})}{(t-e^{i\beta})(t-z)}\,dt + f(e^{i\beta}) \int_{\mathscr{C}} \frac{dt}{(t-e^{i\beta^-})(t-z)}.$$

† It is recalled that a holomorphic function is said to be *analytic at infinity* if it has a finite limit for $z \to \infty$. It then admits a representation as a power series of the variable $1/z$ for sufficiently large values of z.

The first integral in the right-hand side has a finite limit for $z \to e^{i\beta}$ in Ω^- since the function $(f(t)-f(e^{i\beta}))/(t-e^{i\beta})$ is differentiable on $\mathscr{C}$ [see Muskhelishvili (1953)], and so has the second one since the integration path can be displaced toward Ω^+ without crossing the poles $e^{i\beta^-}$ and z $(\in \Omega^-)$. Thus the left-hand side of eqn (23) is finite on $\mathscr{U}$, and likewise for the right-hand side since they are equal on $\mathscr{U}$. Both of them are, therefore, continuous on $\Omega^- \cup \mathscr{U}$.

There remains the condition of analycity at infinity. This condition is not fulfilled because of the behaviour of the potentials at infinity (eqns (16)). More precisely, eqns (16_1) and (22) imply that

$$\varphi(z)-\chi(z) = \Gamma R\, e^{im\alpha} z + 0(1) \quad \text{for} \quad z \to \infty.$$

Moreover, calculation of $Q_*(z)$ yields

$$Q_*(z) = -\frac{z(z-e^{i\alpha})(z-e^{-i\alpha})}{(z-e^{i\beta})(z+e^{-i\beta})}, \tag{24}$$

which implies, together with eqn (16_2), that

$$Q_*(z)\varphi'(z)-\chi_*(z)-\psi(z) = -(\Gamma+\Gamma')R\, e^{im\alpha} z + 0(1) \quad \text{for} \quad z \to \infty.$$

Therefore, if we add $-\Gamma R\, e^{im\alpha} z + (\Gamma+\overline{\Gamma'})R\, e^{-im\alpha}/z$ to the left-hand side of eqn (23) and $-\overline{\Gamma R\, e^{-im\alpha}/z} + \overline{(\Gamma+\Gamma')R\, e^{im\alpha} z}$ to the right-hand side (those expressions are equal for $z \in \mathscr{U}$), this equation becomes

$$\begin{aligned}\varphi(z)-\chi(z^-)-\Gamma R\, e^{im\alpha} z + (\Gamma+\overline{\Gamma'})R\, e^{-im\alpha}/z \\ = \overline{Q_*(z)}\,\overline{\varphi'(z)} - \overline{\chi_*(z^-)} - \overline{\psi(z)} - \overline{\Gamma R\, e^{-im\alpha}/z} + \overline{(\Gamma+\Gamma')R\, e^{im\alpha} z} + Cst \quad \text{for} \quad z \in \mathscr{U}\end{aligned} \tag{25}$$

and now both the left-hand side and the conjugate of the right-hand side are analytic at infinity.

Applying the lemma and using eqn (17), we get

$$\varphi(z) = \Gamma R\, e^{im\alpha} z - \frac{(\Gamma+\overline{\Gamma'})R\, e^{-im\alpha}}{z} + \frac{1-e^{2im\pi}}{2i\pi}\int_{\mathscr{C}} \frac{(t-e^{i\alpha})(t-e^{-i\alpha})\overline{\varphi'(t)}\, dt}{t(t-e^{i\beta^-})(t+e^{-i\beta})(t-z)} + Cst;$$

upon differentiation this yields

$$\varphi'(z) = \varphi^{0\prime}(z) + \mathscr{L}\varphi'(z) \tag{26}$$

where the function $\varphi^{0\prime}$ and the operator $\mathscr{L}$ are defined by

$$\varphi^{0\prime}(z) = \Gamma R\, e^{im\alpha} + \frac{(\Gamma+\overline{\Gamma'})R\, e^{-im\alpha}}{z^2}; \tag{27}$$

$$\mathscr{L}f(z) = \frac{1-e^{2im\pi}}{2i\pi}\int_{\mathscr{C}} \frac{(t-e^{i\alpha})(t-e^{-i\alpha})\overline{f(t)}\, dt}{t(t-e^{i\beta^-})(t+e^{-i\beta})(t-z)^2}, \tag{28}$$

for any analytic function f. Equation (26) is the integral equation obtained by Dudukalenko and Romalis (1973), Hussain *et al.* (1974) and Chatterjee (1975). The lemma also yields an expression of ψ in terms of φ, which is not needed here.

Once the function φ' is known through the resolution of eqn (26), the SIFs $k_1(s)$, $k_2(s)$ at the tip of the crack extension can be obtained by using Andersson's (1969) formula:

$$k_1(s)-ik_2(s) = 2\sqrt{\pi}\varphi'(e^{i\beta})e^{-i\delta/2}[\dot{\omega}''(e^{i\beta})]^{-1/2}, \tag{29}$$

where δ is the angle between the X_1-axis and the tangent to the crack at its tip, which is equal to πm in the present case. The determination of the square root to be used in eqn (29) is easily found by considering the special case of a straight crack ($m = 0$).

Equation (26) was solved numerically by Chatterjee (1975) through discretization of the integration path and transformation into a system of linear equations. It was also shown by Amestoy (1987) that the solution can be expressed under the form of a series. The results obtained for the stress intensity factors will not be repeated here since our prime interest does not focus on the case of an extension of finite length as considered in this section, but on that of an infinitesimal extension.

A.2. *Expansion in powers of the crack extension length*

The asymptotic expressions, for $s \to 0$, of the constants R, α, β defined by eqns (12), (13), (14) are easily found to be

$$R = \frac{\ell}{2} + 0(s); \tag{30}$$

$$\alpha = \sqrt{\frac{2}{(1-m^2)\ell}}\left(\frac{1-m}{1+m}\right)^{m/2}\sqrt{s} + 0(s^{3/2}); \tag{31}$$

$$\beta = m\alpha + 0(s^{3/2}). \tag{32}$$

Equation (31) implies that $\alpha = 0(\sqrt{s})$. Therefore an expansion in powers of s is equivalent to an expansion in powers of α. Such an expansion will be achieved through the following changes of variable and function:

$$z = e^{i\alpha\zeta}; \quad \varphi'(z) = e^{-i\alpha\zeta}[\sqrt{\ell}U(\zeta) + \alpha\ell V(\zeta) + 0(\alpha^2)].\dagger \tag{33}$$

The change of variable defined by eqn (33_1) maps the domain of definition Ω^- of the complex potentials in the z-plane onto the domain $\mathrm{Im}\,\zeta < 0$, $-\pi/\alpha < \mathrm{Re}\,\zeta \leqslant \pi/\alpha$ in the ζ-plane, which becomes identical to the entire lower half-plane Π^- in the limit $\alpha \to 0$ (Fig. 6). The images of the points $z_1 = e^{-i\alpha}$, $z_2 = e^{i\beta}$, $z_3 = e^{i\alpha}$ and $z_4 = -e^{-i\beta}$ are $\zeta_1 = -1$,

$$\zeta_2 = \frac{\beta}{\alpha} = m + 0(s), \quad \zeta_3 = 1 \quad \text{and} \quad \zeta_4 = \frac{\pi-\beta}{\alpha} = 0(1/\sqrt{s}) \quad \text{or} \quad -\frac{\pi+\beta}{\alpha} = 0(-1/\sqrt{s})$$

depending on the sign of m. It is thus clear that the effect of this change of variable is to "scale up" the vicinity of the arc $\mathscr{C}$ corresponding to the crack extension in the z-plane, as desired for an asymptotic study of infinitesimal extensions.

Since the derivative of the function involved in the change of variable (33_1) is non-zero, the behaviour of the function φ' at the points $e^{i\beta}$ and $e^{\pm i\alpha}$ is preserved in the transformation, which means that *the functions U and V are indefinitely differentiable at the point m and weakly singular at the points* ± 1.

Inserting eqns (30), (32) and (33) into eqns (26)–(28), putting $t = e^{i\alpha\lambda}$ and expanding in powers of α, one obtains after a tedious but straightforward calculation

$$\frac{U(\zeta)}{\sqrt{\ell}} + \alpha V(\zeta) = \Gamma + \frac{\overline{\Gamma'}}{2} - i\alpha\frac{\overline{\Gamma'}}{2}(\zeta + m)$$

$$+ \frac{1-e^{2im\pi}}{4i\pi}\int_{-1}^{+1}\frac{(\lambda^2-1)[\overline{U(\lambda)}/\sqrt{\ell} + \alpha\overline{V(\lambda)}]\,d\lambda}{(\lambda-m^-)(\lambda-\zeta)^2} + 0(\alpha^2),$$

where $m^- = m - i\varepsilon$, $\varepsilon > 0$, $\varepsilon \to 0$. (Note that the weakly singular behaviour of U and V at

† The reason for the introduction of the seemingly unnecessary term $e^{-i\alpha\zeta}$ in eqn (33_2) will be explained below.

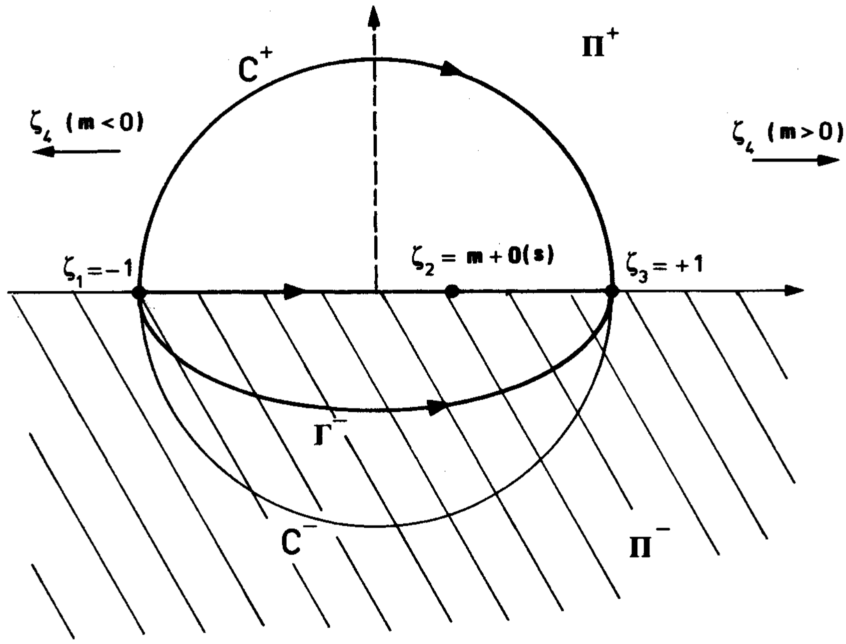

Fig. 6. The ζ-plane.

± 1 ensures the convergence of the integral.) Introducing then, following Muskhelishvili, the analytic functions $\bar{U}(\zeta) = \overline{U(\bar{\zeta})}$ and $\bar{V}(\zeta) = \overline{V(\bar{\zeta})}$, replacing $\overline{U(\lambda)}$ and $\overline{V(\lambda)}$ by $\bar{U}(\lambda)$ and $\bar{V}(\lambda)$ on the real interval $]-1, +1[$, deforming the integration path away from the pole m^- onto the semi-circle (denoted C^+) $|\lambda| = 1$, $\mathrm{Im}\,\lambda > 0$, oriented from -1 to $+1$, and identifying terms of order $\alpha^0 = 1$ and $\alpha^1 = \alpha$, we get

$$U(\zeta) = U^0(\zeta) + \mathscr{A}U(\zeta); \quad V(\zeta) = V^0(\zeta) + \mathscr{A}V(\zeta), \tag{34}$$

where the functions U^0 and V^0 and the operator $\mathscr{A}$ are defined by

$$U^0(\zeta) = \left(\Gamma + \frac{\overline{\Gamma'}}{2}\right)\sqrt{\ell}; \quad V^0(\zeta) = -\frac{\mathrm{i}\overline{\Gamma'}}{2}(\zeta + m);$$

$$\mathscr{A}f(\zeta) = \frac{1 - \mathrm{e}^{2\mathrm{i}m\pi}}{4\mathrm{i}\pi}\int_{C^+} \frac{(\lambda^2 - 1)\bar{f}(\lambda)\,\mathrm{d}\lambda}{(\lambda - m)(\lambda - \zeta)^2}, \tag{35}$$

for any analytic function f. U^0 can be put in a more interesting form by noting that $\Gamma + \overline{\Gamma'}/2 = \frac{1}{2}(\sigma_{22}^{\infty} - \mathrm{i}\sigma_{12}^{\infty}) = (k_1 - \mathrm{i}k_2)/(2\sqrt{\pi\ell})$ where k_1 and k_2 are the SIFs at the tip of the original crack of length 2ℓ:

$$U^0(\zeta) = \frac{k_1 - \mathrm{i}k_2}{2\sqrt{\pi}}. \tag{36}$$

Also, $\mathrm{Re}\,\Gamma' = \frac{1}{2}(\sigma_{22}^{\infty} - \sigma_{11}^{\infty}) = -T/2$ where T is the non-singular stress at the original crack tip, and $\mathrm{Im}\,\Gamma' = \sigma_{12}^{\infty}$; therefore V^0 can be written as

$$V^0(\zeta) = \left(-\frac{\sigma_{12}^{\infty}}{2} + \frac{\mathrm{i}T}{4}\right)(\zeta + m). \tag{37}$$

Equation (34_1) is Wu's integral equation for an infinitesimal extension (Wu, 1978a, b), obtained here from that for a finite extension for the first time. On the other hand, to the best knowledge of the authors, eqn (34_2) is new.

The expression of the SIFs in terms of the functions U and V is obtained by expanding Andersson's equation (29) in powers of α. Calculation of $\omega''(e^{i\beta})$ to the first order yields first

$$\omega''(e^{i\beta}) = \ell\, e^{im\pi}\left(\frac{1+m}{1-m}\right)^m (1-2im\alpha)+0(\alpha^2)\,; \tag{38}$$

insertion of this expression and eqns (31)–(33) into eqn (29) gives

$$k_1^* - ik_2^* = 2\sqrt{\pi}\, e^{-im\pi}\left(\frac{1-m}{1+m}\right)^{m/2} U(m)\,; \tag{39}$$

$$k_1^{(1/2)} - ik_2^{(1/2)} = 2\sqrt{\frac{2\pi}{1-m^2}}\, e^{-im\pi}\left(\frac{1-m}{1+m}\right)^m V(m), \tag{40}$$

where the k_p^*s and $k_p^{(1/2)}$s are defined by eqn (2).

Equations (34)–(37), (39) and (40) show that the k_p^*s and $k_p^{(1/2)}$s can be determined quite independently: the k_p^*s depend only on the function U which can be found from eqns (34_1), (35_3) and (36) where the function V does not appear; similarly, the $k_p^{(1/2)}$s depend only on V, and eqns (34_2), (35_3) and (37) for this function do not involve U. This remarkable property can be evidenced only by introducing the seemingly unnecessary term $e^{-i\alpha\zeta}$ in the change of function (33_2); omitting it would result in the introduction of a term proportional to U in the expression (37) of V^0, and of another analogous term in eqn (40) for the $k_p^{(1/2)}$s, so that the independence property would not appear clearly.

Since the function U^0 depends on the *three* components of the stress tensor at infinity only through *two* parameters, namely the SIFs at the initial crack tip, the same is true of U and the k_p^*s (this property does not hold for the SIFs $k_p(s)$ at the tip of an extension of finite length, which are easily seen from eqns (26)–(29) to depend on all three components of σ^∞). This is an illustration of the universality result (3). On the other hand, since V^0 includes terms proportional to T *and* σ_{12}^∞, the same is true of V, and it seems therefore that the $k_p^{(1/2)}$s should not be simply proportional to T as predicted by eqn (4) (with $a^* = 0$). In fact the contradiction is only apparent; indeed it can be shown (see Appendix B) that that part of V which arises from the σ_{12}^∞ term in eqn (37) (i.e. the function V corresponding to a zero T) admits the following expression:

$$[V(\zeta)]_{T=0} = \frac{\sigma_{12}^\infty}{2}(m-\zeta)\left(\frac{\zeta+1}{\zeta-1}\right)^m \quad (\zeta \in \Pi^-), \tag{41}$$

where the determination of the function $((\zeta+1)/(\zeta-1))^m$ is such that the cut be located along any arc connecting the points ± 1 in the upper half-plane Π^+ and that its limit for $\zeta \to \infty$ be equal to unity. Equation (41) implies that $[V(\zeta)]_{T=0}$ is zero at the point $\zeta = m$, so that σ_{12}^∞ does not generate any contribution in the expression (40) of the $k_p^{(1/2)}$s, in agreement with eqn (4). This feature represents a good check on the correctness of the approach used.

It is remarkable that such a simple, explicit solution can be found to a complicated integral equation. However, the authors have failed to discover similar explicit expressions for the function U and that part of V which is proportional to T; this is unfortunate since it is these potentials that lead to the determination of the functions F_{pq} and G_p which are sought. The best the authors have achieved concerning the analytic expression of these functions is to have obtained formulae that allow for the determination of their exact expansion up to an arbitrary order in m (see below).

A.3. *Solutions under the form of series and high order expansions of the function F_{pq} and G_p*

In the right-hand side of eqn (35_3) defining $\mathscr{A}$, the only values of f involved are those on the conjugate C^- of the arc C^+ (see Fig. 6). $\mathscr{A}$ can therefore be viewed as acting on the space of functions f defined and continuous on C^-. Furthermore, let $\mathscr{E}$ denote the subspace of functions (defined and continuous on C^-) verifying the condition that the quantity

$$\|f\| = \underset{\zeta \in C^-}{\mathrm{Max}} |(\zeta^2-1)f(\zeta)| \tag{42}$$

be finite. The functions U and V do lie in this space because of the condition of weak singularity at ± 1. We shall show that if $\mathscr{E}$ is endowed with the norm $\| \quad \|$ defined by eqn (42), $\mathscr{A}$ is a *contractant* operator on this space, i.e. there exists a constant c smaller than 1 such that

$$\|\mathscr{A}f\| \leqslant c\|f\|, \tag{43}$$

for every f in $\mathscr{E}$.

To prove (43), let us note that

$$\|\mathscr{A}f\| \leqslant \frac{\sin|m\pi|}{2\pi} \cdot \|f\| \cdot \underset{\zeta \in C^-}{\mathrm{Max}} \left\{ |\zeta^2-1| \int_{C^+} \frac{|\mathrm{d}\lambda|}{|\lambda-m||\lambda-\zeta|^2} \right\}.$$

Putting then $\zeta = \mathrm{e}^{\mathrm{i}\gamma}$ $(-\pi < \gamma < 0)$, $\lambda = \mathrm{e}^{\mathrm{i}\theta}$ $(0 < \theta < \pi)$, and noting that $|\lambda - m| \geqslant 1-|m|$, we get

$$|\zeta^2-1| \int_{C^+} \frac{|\mathrm{d}\lambda|}{|\lambda-m||\lambda-\zeta|^2} \leqslant -\frac{2\sin\gamma}{1-|m|} \int_0^\pi \frac{\mathrm{d}\theta}{4\sin^2\left(\dfrac{\theta-\gamma}{2}\right)} = \frac{2}{1-|m|}.$$

It follows that $\|\mathscr{A}f\|$ verifies an inequality of the form (43), with

$$c = \frac{\sin|m\pi|}{\pi(1-|m|)}.$$

This quantity is smaller than 1 for $-1 < m < +1$, which concludes the proof.

It follows immediately from the contractant nature of $\mathscr{A}$ that the solutions of the integral equations (34) are given by $U = \Sigma_{n=0}^{+\infty} \mathscr{A}^n U^0$ and $V = \Sigma_{n=0}^{+\infty} \mathscr{A}^n V^0$, where the series of functions converge in the sense defined by the norm $\| \quad \|$ introduced above. This implies that for every $\zeta \in C^-$,

$$U(\zeta) = \sum_{n=0}^{+\infty} \mathscr{A}^n U^0(\zeta); \quad V(\zeta) = \sum_{n=0}^{+\infty} \mathscr{A}^n V^0(\zeta), \tag{44}$$

the convergence of the numerical series being uniform on every subset of C^- not containing the points ± 1 in its closure. Once these equations have provided the values of U and V on C^-, these functions can be calculated on their whole domain of definition Π^- (or even, by extension, on the entire complex plane except on C^+) by re-applying eqns (34), now with $\zeta \in \Pi^-$ (or $\mathbb{C} - C^+$).

Equations (44) can be used to numerically compute the functions F_{pq} and G_p [see Amestoy (1987)]. The results obtained for the F_{pq}s are in complete agreement with those of Wu (1978b) and Bilby *et al.* (1975, 1977). Concerning the G_ps, they confirm the only information available in the literature concerning numerical values of these functions, namely Bilby and Cardew's (1975) finding that the zero of G_2 occurs close to 98°. All these results will not be repeated here because, as will now be seen, eqns (44) can be used to derive more convenient, and considerably more accurate, high order expansions of the F_{pq}s and G_ps.

Indeed, it is shown in Appendix C that eqn (44_1) yields the following expression of $k_1^* - ik_2^*$ (from which the F_{pq}s readily follow):

$$k_1^* - ik_2^* = \left(\frac{1-m}{1+m}\right)^{m/2} \sum_{n=0}^{+\infty} u_n, \tag{45}$$

where

$$u_n = \begin{cases} (k_1 - ik_2)\left(\dfrac{\sin m\pi}{2\pi}\right)^n e^{-im\pi} x_n & \text{if } n \text{ is even} \\ -(k_1 + ik_2)\left(\dfrac{\sin m\pi}{2\pi}\right)^n x_n & \text{if } n \text{ is odd;} \end{cases} \tag{46}$$

the x_ns in these expressions are given by

$$x_n = \sum_{p=0}^{2n} \sum_{q=0}^{n} a_{pq}^{(n)} \frac{d^p}{dm^p}\left[P_q^{(n)}\left(-\frac{1}{2i\pi}\log^- \frac{m-1}{m+1}\right)\right], \tag{47}$$

where $\log^-$ is the logarithm function defined on $\mathbb{C} - i\mathbb{R}^+$ by $\log^-(\rho\, e^{i\theta}) = \ln \rho + i\theta$ with $-3\pi/2 < \theta < \pi/2$ and the $a_{pq}^{(n)}$s and $P_q^{(n)}(X)$s coefficients and polynomials verifying the following induction formulae:

$$P_q^{(0)}(X) = X^q; \quad P_q^{(n)}(X) = (-1)^q \sum_{r=0}^{q} C_q^r B_{q-r} P_r^{(n-1)}(X) \quad (n \geqslant 1) \tag{48}$$

(B_q = qth Bernoulli number);

$$a_{00}^{(0)} = 1; \tag{49}$$

$$a_{pq}^{(n)} = \frac{(-1)^n 2i\pi}{q}\left[(p-1)\overline{a_{p,q-1}^{(n-1)}} + \frac{2m(p-1)}{p}\overline{a_{p-1,q-1}^{(n-1)}} + \frac{m^2-1}{p}\overline{a_{p-2,q-1}^{(n-1)}}\right] + \frac{2}{p}\overline{a_{p-1,q}^{(n-1)}}$$

$$\text{for } 2 \leqslant p \leqslant 2n,\ 1 \leqslant q \leqslant n; \tag{50}$$

$$a_{1q}^{(n)} = 2\overline{a_{0q}^{(n-1)}} \quad \text{for } 0 \leqslant q \leqslant n; \tag{51}$$

$$a_{0q}^{(n)} = \frac{(-1)^{n-1} 2i\pi}{q}\overline{a_{0,q-1}^{(n-1)}} \quad \text{for } 1 \leqslant q \leqslant n; \tag{52}$$

$$a_{p0}^{(n)} = \frac{2}{p}\overline{a_{p-1,0}^{(n-1)}} + \sum_{r=p}^{2n} \sum_{s=0}^{n-1} \frac{(-1)^{n-1} 2i\pi}{p(s+1)} C_{r-2}^{p-2} \overline{a_{r-2,s}^{(n-1)}} \frac{d^{r-p}}{dm^{r-p}}\left[(m^2-1) P_{s+1}^{(n)}\left(-\frac{1}{2i\pi}\log^- \frac{m-1}{m+1}\right)\right]$$

$$\text{for } 2 \leqslant p \leqslant 2n; \tag{53}$$

$$a_{00}^{(n)} = (-1)^n 2i\pi \sum_{s=0}^{n-1} \frac{P_{s+1}^{(n)}(0)}{s+1}\overline{a_{0s}^{(n-1)}} \tag{54}$$

(where the coefficients $a_{pq}^{(n-1)}$ are to be considered as nil if $p > 2n-2$ or $q > n-1$).

$G_1 - iG_2$ is also given by similar formulae:

$$G_1 - \mathrm{i}G_2 = \mathrm{i}\sqrt{\frac{\pi}{2(1-m^2)}}\left(\frac{1-m}{1+m}\right)^m \sum_{n=0}^{+\infty} v_n\,; \tag{55}$$

$$v_n = \begin{cases} \left(\dfrac{\sin m\pi}{2\pi}\right)^n \mathrm{e}^{-\mathrm{i}m\pi} y_n & \text{if } n \text{ is even} \\[2ex] \left(\dfrac{\sin m\pi}{2\pi}\right)^n y_n & \text{if } n \text{ is odd}\,; \end{cases} \tag{56}$$

$$y_n = \sum_{p=0}^{2n} \sum_{q=0}^{n} b_{pq}^{(n)} \frac{\mathrm{d}^p}{\mathrm{d}m^p}\left[P_q^{(n)}\left(-\frac{1}{2\mathrm{i}\pi}\log^- \frac{m-1}{m+1}\right)\right]; \tag{57}$$

$$b_{-1,0}^{(0)} = 1\,; \quad b_{00}^{(0)} = 2m\,; \tag{58}$$

$$b_{pq}^{(n)} = \frac{(-1)^n 2\mathrm{i}\pi}{q}\left[(p-1)\overline{b_{p,q-1}^{(n-1)}} + \frac{2m(p-1)}{p}\overline{b_{p-1,q-1}^{(n-1)}} + \frac{m^2-1}{p}\overline{b_{p-2,q-1}^{(n-1)}}\right] + \frac{2}{p}\overline{b_{p-1,q}^{(n-1)}}$$

$$\text{for} \quad 2 \leqslant p \leqslant 2n,\ 1 \leqslant q \leqslant n\,; \tag{59}$$

$$b_{1q}^{(n)} = 2\overline{b_{0q}^{(n-1)}} \quad \text{for} \quad 0 \leqslant q \leqslant n\,; \tag{60}$$

$$b_{0q}^{(n)} = \frac{(-1)^{n-1} 2\mathrm{i}\pi}{q}\left[\overline{b_{0,q-1}^{(n-1)}} + 2m\overline{b_{-1,q-1}^{(n-1)}}\right] + 2\overline{b_{-1,q}^{(n-1)}} \quad \text{for} \quad 1 \leqslant q \leqslant n\,; \tag{61}$$

$$b_{-1,q}^{(n)} = \frac{(-1)^{n-1} 4\mathrm{i}\pi}{q}\overline{b_{-1,q-1}^{(n-1)}} \quad \text{for} \quad 1 \leqslant q \leqslant n\,; \tag{62}$$

$$b_{p0}^{(n)} = \frac{2}{p}\overline{b_{p-1,0}^{(n-1)}} + \sum_{r=p}^{2n} \sum_{s=0}^{n-1} \frac{(-1)^{n-1} 2\mathrm{i}\pi}{p(s+1)} C_{r-2}^{p-2} \overline{b_{r-2,s}^{(n-1)}} \frac{\mathrm{d}^{r-p}}{\mathrm{d}m^{r-p}}\left[(m^2-1)P_{s+1}^{(n)}\left(-\frac{1}{2\mathrm{i}\pi}\log^- \frac{m-1}{m+1}\right)\right]$$

$$\text{for} \quad 2 \leqslant p \leqslant 2n\,; \tag{63}$$

$$b_{00}^{(n)} = (-1)^n 2\mathrm{i}\pi \sum_{s=0}^{n-1} \frac{P_{s+1}^{(n)}(0)}{s+1}\left[\overline{b_{0s}^{(n-1)}} + 2m\overline{b_{-1,s}^{(n-1)}}\right] + 2\sum_{s=1}^{n} P_s^{(n)}(0)\overline{b_{-1,s}^{(n-1)}} + 4\overline{b_{-1,0}^{(n-1)}}\,; \tag{64}$$

$$b_{-1,0}^{(n)} = (-1)^n 4\mathrm{i}\pi \sum_{s=0}^{n-1} \frac{P_{s+1}^{(n)}(0)}{s+1}\overline{b_{-1,s}^{(n-1)}} \tag{65}$$

(with $b_{pq}^{(n-1)} \equiv 0$ if $p > 2n-2$ or $q > n-1$).

Since by eqns (46) and (56), u_n and v_n are $0(m^n)$, the expansion of the F_{pq}s and G_ps up to any order n can be obtained by retaining only the first $n+1$ terms of the series (45) and (55) and treating all functions of m in eqns (45)–(65) as polynomials of suitable degree. This has been done analytically up to order six for the F_{pq}s and three for the G_ps, and numerically up to order 20 for all functions. In the latter computation, the coefficients of the expansions have been derived under a purely numerical form, despite the fact that they are in reality polynomial expressions of π with rational coefficients (except for an additional multiplicative $\sqrt{\pi/2}$ factor in the case of the G_ps). The results are as follows:

$$F_{11}(m) = 1 - \frac{3\pi^2}{8}m^2 + \left(\pi^2 - \frac{5\pi^4}{128}\right)m^4 + \left(\frac{\pi^2}{9} - \frac{11\pi^4}{72} + \frac{119\pi^6}{15360}\right)m^6 + 5.07790m^8$$
$$-2.88312m^{10} - 0.0925m^{12} + 2.996m^{14} - 4.059m^{16} + 1.63m^{18} + 4.1m^{20} + 0(m^{22})\,;$$

$$F_{12}(m) = -\frac{3\pi}{2}m + \left(\frac{10\pi}{3} + \frac{\pi^3}{16}\right)m^3 + \left(-2\pi - \frac{133\pi^3}{180} + \frac{59\pi^5}{1280}\right)m^5 + 12.313906m^7$$
$$-7.32433m^9 + 1.5793m^{11} + 4.0216m^{13} - 6.915m^{15} + 4.21m^{17} + 4.56m^{19} + 0(m^{21})\,;$$

$$F_{21}(m) = \frac{\pi}{2}m - \left(\frac{4\pi}{3} + \frac{\pi^3}{48}\right)m^3 + \left(-\frac{2\pi}{3} + \frac{13\pi^3}{30} - \frac{59\pi^5}{3840}\right)m^5 - 6.176023m^7$$
$$+4.44112m^9 - 1.5340m^{11} - 2.0700m^{13} + 4.684m^{15} - 3.95m^{17} - 1.32m^{19} + 0(m^{21});$$
$$F_{22}(m) = 1 - \left(4 + \frac{3\pi^2}{8}\right)m^2 + \left(\frac{8}{3} + \frac{29\pi^2}{18} - \frac{5\pi^4}{128}\right)m^4$$
$$+\left(-\frac{32}{15} - \frac{4\pi^2}{9} - \frac{1159\pi^4}{7200} + \frac{119\pi^6}{15360}\right)m^6 + 10.58254m^8 - 4.78511m^{10}$$
$$-1.8804m^{12} + 7.280m^{14} - 7.591m^{16} + 0.25m^{18} + 12.5m^{20} + 0(m^{22}); \quad (66)$$
$$G_1(m) = (2\pi)^{3/2}m^2 - 47.933390m^4 + 63.665987m^6 - 50.70880m^8 + 26.66807m^{10}$$
$$-6.0205m^{12} - 7.314m^{14} + 10.947m^{16} - 2.85m^{18} - 13.7m^{20} + 0(m^{22});$$
$$G_2(m) = -2\sqrt{2\pi}m + 12\sqrt{2\pi}m^3 - 59.565733m^5 + 61.174444m^7 - 39.90249m^9$$
$$+15.6222m^{11} + 3.0343m^{13} - 12.781m^{15} + 9.69m^{17} + 6.62m^{19} + 0(m^{21}). \quad (67)$$

The coefficients have in fact been calculated much more accurately than these equations seem to indicate; many significant digits have been omitted here because the resulting error in the value of the function expressed is smaller than that arising from the discarded ($0(m^{21})$ or $0(m^{22})$) term. Since the coefficients of the last few terms appear to be $\lesssim 10$, these formulae can be estimated to provide values of the F_{pq}s and G_ps with an accuracy better than 10^{-6} in the interval [0°, 80°] of practical interest (80° being about the maximum observable kink angle); this is much smaller than the errors which result from the use of conventional numerical tables.

As far as the second order expression of the F_{pq}s is concerned, the present results agree with those derived by Wu (1979). On the other hand, the expression of F_{22} is in conflict with that obtained by Sumi (1991) using the Banichuk–Cotterell–Rice perturbative procedure, which reads

$$\boldsymbol{F_{22}(m) = 1 - \frac{7\pi^2}{8}m^2 + 0(m^4).}\dagger$$

As noted by Sumi himself and detailed in the Introduction, this discrepancy stems from the basic inadequacy of the Banichuk–Cotterell–Rice procedure for dealing with non-$\mathscr{C}^\infty$ cracks. Karihaloo *et al.* (1981) and Sumi (1991) also derived second order expressions for the G_ps which agree with eqns (67).

A.4. *Application to the problem of the conjectural coincidence of the maximum-energy-release-rate criterion and the principle of local symmetry*

As an interesting application, we shall now consider the problem of the possible identity of two classical criteria for predicting the kink angle, namely the *maximum-energy-release-rate* criterion (Erdogan and Sih, 1963) and the *principle of local symmetry* (Goldstein and Salganik, 1974). Among the various criteria that have been proposed, those two have always aroused particular interest; especially intriguing is the almost perfect coincidence of their predictions which has been evidenced by numerical calculations of the functions F_{pq}, in terms of which they are expressed [see for instance Bilby and Cardew (1975)]. In fact, the difference is so tiny that it falls within the numerical errors; thus deciding whether it is real or not can only be achieved by analytical means. The expansions derived in the preceding section will now be seen to provide one way of settling the question.

† In order to avoid any ambiguity, erroneous results (or at least deemed so by the present authors!) are indicated in bold.

The energy-release-rate for non-collinear crack growth is given by the following extension of the classical Irwin formula:

$$\mathscr{G} = \frac{1-\nu^2}{E}(k_1^{*2}+k_2^{*2}),$$

where E and ν are Young's modulus and Poisson's ratio (Ichikawa and Tanaka, 1982).† It follows that the kink angle predicted by the maximum-energy-release-rate criterion is such that

$$k_1^*\frac{\partial k_1^*}{\partial m}+k_2^*\frac{\partial k_2^*}{\partial m} = 0.$$

On the other hand, the principle of local symmetry stipulates that the second stress intensity factor must be zero just after the kink, which reads

$$k_2^* = 0.$$

Therefore, coincidence of the two criteria would require that

$$k_2^* = 0 \overset{?}{\Rightarrow} k_1^*\frac{\partial k_1^*}{\partial m}+k_2^*\frac{\partial k_2^*}{\partial m} = 0,$$

and consequently, since k_1^* and k_2^* can easily (for instance numerically) be verified not to vanish simultaneously, that

$$k_2^* = 0 \overset{?}{\Rightarrow} \frac{\partial k_1^*}{\partial m} = 0.$$

In terms of the functions F_{pq}, this reads

$$F_{21}(m)k_1+F_{22}(m)k_2 = 0 \overset{?}{\Rightarrow} F'_{11}(m)k_1+F'_{12}(m)k_2 = 0.$$

Writing these equations in the form $F_{21}(m)/F_{22}(m) = -k_2/k_1$; $F'_{11}(m)/F'_{12}(m) = -k_2/k_1$, one sees that for their solutions (in m) to be identical for all values of k_2/k_1, the functions on the left-hand sides must be equal; equivalently, the equation

$$\frac{F'_{11}(m)}{F_{21}(m)} \overset{?}{=} \frac{F'_{12}(m)}{F_{22}(m)}$$

must hold for all values of m. Now eqns (66) imply that

$$\frac{F'_{11}(m)}{F_{21}(m)} = -\frac{3\pi}{2}+\left(4\pi-\frac{3\pi^3}{8}\right)m^2+\left(10\pi-\frac{41\pi^3}{30}+\frac{\pi^5}{32}\right)m^4+0(m^6);$$

$$\frac{F'_{12}(m)}{F_{22}(m)} = -\frac{3\pi}{2}+\left(4\pi-\frac{3\pi^3}{8}\right)m^2+\left(10\pi-\frac{23\pi^3}{18}+\frac{\pi^5}{32}\right)m^4+0(m^6).$$

These expressions show that the above identity does not hold and therefore that *the two criteria are definitely distinct*. The difference only appears when the expansion of the F_{pq}s

† This formula is classically established for a *straight* (deviated) extension of a *straight* crack. However it is easily verified to be also applicable if the crack and its extension are curved.

is carried out up to order six, that is four orders higher than in the most accurate expansion presently available in the literature (Wu, 1979).

PART B: THE CASE OF A CURVED EXTENSION

B.1. *New expression of the stress intensity factors*

We now consider the problem defined in Fig. 3, which will be solved by a first order perturbative procedure with respect to the curvature parameters a^* and C^*, as sketched in the Introduction. We denote by $\tilde{s}$ and $\pi\tilde{m}$ the length of the fictitious straight extension and the angle it makes with the X_1-axis. Using eqn (1), one easily shows that $\tilde{s}$ and $\pi\tilde{m}$ are related to the length s of the curved extension and the angle πm between the X_1-axis and its tangent at the angular point (see Fig. 3) by the relations

$$\tilde{s} = s+0(s^2)\,; \quad \tilde{m} = m+\frac{a^*\sqrt{s}}{\pi}+\frac{C^*s}{2\pi}+0(s^{3/2}). \tag{68}$$

The perturbative procedure will yield an expression for the SIFs which will appear as the sum of that for a straight extension of length $\tilde{s}$ in the direction $\pi\tilde{m}$ and some corrective terms due to curvature. This expression will not be directly comparable to eqns (2)–(5) which define the functions H_{pq}, K_p and M_{pq} which are sought, since the extension length and kink angle involved in these equations are s and πm instead of $\tilde{s}$ and $\pi\tilde{m}$. It is therefore necessary to re-express these equations in terms of $\tilde{s}$ and $\tilde{m}$ using eqns (68):

$$\begin{aligned} k_p(s) &= F_{pq}(m)k_q+[G_p(m)T+a^*H_{pq}(m)k_q]\sqrt{s} \\ &\quad +[Z_p(m)+I_{pq}(m)b_q+CJ_{pq}(m)k_q+a^*K_p(m)T+C^*M_{pq}(m)k_q]s+0(s^{3/2}) \\ &= F_{pq}(\tilde{m})k_q+\left[G_p(\tilde{m})T+a^*H_{pq}(\tilde{m})k_q-\frac{a^*}{\pi}F'_{pq}(\tilde{m})k_q\right]\sqrt{\tilde{s}}+\Big[Z_p(\tilde{m})+I_{pq}(\tilde{m})b_q \\ &\quad +CJ_{pq}(\tilde{m})k_q+a^*K_p(\tilde{m})T+C^*M_{pq}(\tilde{m})k_q-\frac{C^*}{2\pi}F'_{pq}(\tilde{m})k_q-\frac{a^*}{\pi}G'_p(\tilde{m})T\Big]\tilde{s}+0(\tilde{s}^{3/2}) \end{aligned}$$

(where second order terms with respect to a^* and C^* have been disregarded). This expression is of the form

$$k_p(s) = [k_p(\tilde{s})]^{\pi\tilde{m}}_{a^*=C^*=0}+a^*\tilde{H}_{pq}(\tilde{m})k_q\sqrt{\tilde{s}}+[a^*\tilde{K}_p(\tilde{m})T+C^*\tilde{M}_{pq}(\tilde{m})k_q]\tilde{s}+0(\tilde{s}^{3/2}), \tag{69}$$

where $[k_p(\tilde{s})]^{\pi\tilde{m}}_{a^*=C^*=0}$ denotes the pth SIF at the tip of a straight extension of length $\tilde{s}$ in the direction $\pi\tilde{m}$, and the $\tilde{H}_{pq}$s, $\tilde{K}_p$s and $\tilde{M}_{pq}$s functions defined by

$$\tilde{H}_{pq} = H_{pq}-\frac{F'_{pq}}{\pi} \Leftrightarrow H_{pq} = \tilde{H}_{pq}+\frac{F'_{pq}}{\pi}; \tag{70}$$

$$\tilde{K}_p = K_p-\frac{G'_p}{\pi} \Leftrightarrow K_p = \tilde{K}_p+\frac{G'_p}{\pi}; \tag{71}$$

$$\tilde{M}_{pq} = M_{pq}-\frac{F'_{pq}}{2\pi} \Leftrightarrow M_{pq} = \tilde{M}_{pq}+\frac{F'_{pq}}{2\pi}. \tag{72}$$

The expression (69) of the SIFs, which is exact to the first order with respect to a^* and C^*, is precisely of the form which will result from the perturbative procedure. It will therefore be easy to identify the functions $\tilde{H}_{pq}$, $\tilde{K}_p$, $\tilde{M}_{pq}$ and the H_{pq}s, K_ps, M_{pq}s will follow from eqns (70_2), (71_2) and (72_2).

B.2. *Reduction of the problem to integral equations*

The equations of the problem in the physical Z-plane take the same form as in the case of a straight extension (eqns (8) and (9)), Φ and Ψ denoting the *real* potentials, which are discontinuous across the real, curved extension.

We associate with Φ and Ψ some new potentials Φ^a and Ψ^a by shifting the cut, through analytic continuation, from the curved extension to the fictitious straight one, hereafter simply denoted $Z_1Z_2Z_3$. Furthermore, Z being an arbitrary point on $Z_1Z_2Z_3$, we denote by $\eta(Z)$ the gap between the two extensions, and we expand Φ^a and Ψ^a in powers of η:

$$\Phi^a = \Phi_0+\Phi_1+0(\eta^2); \quad \Psi^a = \Psi_0+\Psi_1+0(\eta^2). \tag{73}$$

If Z is on $Z_1Z_2Z_3$, $Z+\eta(Z)$ is on the real extension, and $\Phi[Z+\eta(Z)] = \Phi^a(Z)+\Phi^{a\prime}(Z)\eta(Z)+0(\eta^2) = \Phi_0(Z)+\Phi_1(Z)+\Phi_0'(Z)\eta(Z)+0(\eta^2)$; similar equations hold for Φ' and Ψ. Thus the boundary condition on the deviated branch can be written as

$$\begin{aligned}\Phi_0(Z)+\Phi_1(Z)+\Phi_0'(Z)\eta(Z)+[Z+\eta(Z)][\overline{\Phi_0'(Z)}+\overline{\Phi_1'(Z)}+\overline{\Phi_0''(Z)}\,\overline{\eta(Z)}]\\ +\overline{\Psi_0(Z)}+\overline{\Psi_1(Z)}+\overline{\Psi_0'(Z)}\,\overline{\eta(Z)}+0(\eta^2) = Cst \quad \text{for} \quad Z\in Z_1Z_2Z_3.\end{aligned}$$

The same equation also holds for $Z\in Z_3Z_4Z_1$, putting $\eta(Z) = 0$ in that case. Identifying terms of order $\eta^0 = 1$ and $\eta^1 = \eta$, we get the following boundary conditions for the potentials Φ_0, Ψ_0, Φ_1, Ψ_1:

$$\begin{gathered}\Phi_0(Z)+Z\overline{\Phi_0'(Z)}+\overline{\Psi_0(Z)} = Cst;\\ \Phi_1(Z)+Z\overline{\Phi_1'(Z)}+\overline{\Psi_1(Z)}+[\Phi_0'(Z)+\overline{\Phi_0'(Z)}]\eta(Z)+[Z\overline{\Phi_0''(Z)}+\overline{\Psi_0'(Z)}]\overline{\eta(Z)} = Cst\\ \text{for} \quad Z\in Z_1Z_2Z_3Z_4Z_1.\end{gathered} \tag{74}$$

The conditions at infinity are also easily obtained by expanding eqns (9) (for Φ^a and Ψ^a) in powers of η; since Γ and Γ' are independent of η, one gets

$$\Phi_0(Z) = \Gamma Z+0(1); \quad \Psi_0(Z) = \Gamma' Z+0(1); \quad \Phi_1(Z) = 0(1); \quad \Psi_1(Z) = 0(1) \quad \text{at infinity.} \tag{75}$$

Equations (75_3) and (75_4) mean that in contrast to Φ_0 and Ψ_0, Φ_1 and Ψ_1 are analytic at infinity.

We now introduce the same conformal mapping as in Part A (eqns (11)–(14)), except for the replacement of s and m by $\tilde{s}$ and $\tilde{m}$. In the new z-plane, the equations for the potentials $\varphi_0(z) = \Phi_0(Z)$, $\psi_0(z) = \Psi_0(Z)$ are the same as in the case of a straight extension (eqns (15) and (16)) (except for the replacement of m, φ and ψ by $\tilde{m}$, φ_0 and ψ_0), and those for the potentials $\varphi_1(z) = \Phi_1(Z)$, $\psi_1(z) = \Psi_1(Z)$ read

$$\begin{aligned}\varphi_1(z)+\frac{\omega(z)}{\overline{\omega'(z)}}\overline{\varphi_1'(z)}+\overline{\psi_1(z)}+\left[\frac{\varphi_0'(z)}{\omega'(z)}+\frac{\overline{\varphi_0'(z)}}{\overline{\omega'(z)}}\right]\eta(z)\\ +\left[\frac{\omega(z)\overline{\varphi_0''(z)}}{\overline{\omega'(z)}^2}-\frac{\omega(z)\overline{\omega''(z)}\,\overline{\varphi_0'(z)}}{\overline{\omega'(z)}^3}+\frac{\overline{\psi_0'(z)}}{\overline{\omega'(z)}}\right]\overline{\eta(z)} = Cst \quad \text{for} \quad z\in\mathscr{U};\end{aligned} \tag{76}$$

$$\varphi_1(z) = 0(1); \quad \psi_1(z) = 0(1) \quad \text{at infinity.} \tag{77}$$

With regard to the zeroth order potentials, the method of solution is identical to that described in Part A and leads to the same integral equation (26) for φ_0' (except for the usual replacements). The expression of ψ_0 will also be needed here; it is obtained by applying the lemma of Part A to the right-hand side of eqn (25):

$$\psi_0(z) = (\Gamma+\Gamma')R\,e^{i\tilde{m}\alpha}z - \Gamma R\,e^{-i\tilde{m}\alpha}/z + \overline{Q_*(z)}\overline{\varphi_0'(z)} - \chi_{0*}(z) + Cst$$

(where χ_0 is defined in terms of φ_0' by eqn (22) with m and φ' replaced by $\tilde{m}$ and φ_0'). To get $\chi_{0*}(z) = \overline{\chi_0(1/\bar{z})}$, use eqn (22) and replace $\bar{t}$, $d\bar{t}$ and $\overline{Q(t)}$ by $1/t$, $-dt/t^2$ and $Q_*(t)$ in the integral over $\mathscr{C}$:

$$\chi_{0*}(z) = \frac{1-e^{-2i\tilde{m}\pi}}{2i\pi}\int_{\mathscr{C}} \frac{zQ_*(t)\varphi_0'(t)\,dt}{t(z-t)}.$$

In the above integral the pole $e^{i\beta}$ of the function $Q_*(z)$ (see eqn (24)) must in fact be understood as $e^{i\beta^+} = e^{i(\beta+i\varepsilon)}$, $\varepsilon > 0$, $\varepsilon \to 0$. Inserting this result into the preceding expression of ψ_0 and using eqn (24), one obtains

$$\psi_0(z) = (\Gamma+\Gamma')R\,e^{i\tilde{m}\alpha}z - \frac{\Gamma R\,e^{-i\tilde{m}\alpha}}{z} - \frac{z(z-e^{i\alpha})(z-e^{-i\alpha})}{(z-e^{i\beta})(z+e^{-i\beta})}\overline{\varphi_0'}(z)$$
$$+\frac{1-e^{-2i\tilde{m}\pi}}{2i\pi}\int_{\mathscr{C}} \frac{z(t-e^{i\alpha})(t-e^{-i\alpha})\varphi_0'(t)\,dt}{(t-e^{i\beta^+})(t+e^{-i\beta})(z-t)} + Cst. \quad (78)$$

Solving eqns (76) and (77) for the first order potentials will require to know their behaviour near the points $e^{\pm i\alpha}$, $e^{i\beta}$ and $-e^{-i\beta}$, and also that of the bracketed terms in eqn (76) near $e^{\pm i\alpha}$ and $e^{i\beta}$. The potentials φ^a and ψ^a satisfy the properties mentioned in Section A.1 for all values of η, so that these properties apply separately to the zeroth and first order potentials: therefore *φ_1 is indefinitely differentiable, and ψ_1 has simple poles, at $e^{i\beta}$ and $-e^{-i\beta}$; φ_1' and ψ_1' are weakly singular at $e^{\pm i\alpha}$*. The behaviour of the expression $[\ldots]\eta(z)+\overline{[\ldots]\eta(z)}$ in eqn (76) near $e^{i\beta}$ is easily deduced from that of $\varphi_1(z)+(\omega(z)/\overline{\omega'(z)})\,\overline{\varphi_1'(z)}+\overline{\psi_1(z)}$: using the properties of φ_1, ψ_1 just mentioned and eqns (17) and (18), one concludes that *both quantities have simple poles at* $e^{i\beta}$. As for the behaviour of $[\ldots]\eta(z)+\overline{[\ldots]\eta(z)}$ near $e^{\pm i\alpha}$, the simplest reasoning consists in noting that this expression arises from the analogous one $[\ldots]\eta(Z)+\overline{[\ldots]\eta(Z)}$ in eqn (74). Since by Williams' (1952) results, $\Phi_0'(Z)$ and $\Psi_0'(Z)$ are $0[(Z-Z_1)^{c_1}]$ and $0[(Z-Z_3)^{c_3}]$ near Z_1 and Z_3 respectively, where the constants c_1 and c_3 are greater than $-1/2$, and since $\eta(Z)$ has simple zeros at these points (see Fig. 3), $[\ldots]\eta(Z)+\overline{[\ldots]\eta(Z)}$ is $0(|Z-Z_1|^{c_1+1})$ and $0(|Z-Z_3|^{c_3+1})$ where c_1+1 and c_3+1 are greater than $1/2$ and *a fortiori* positive. $Z-Z_1$ and $Z-Z_3$ being $0((z-e^{-i\alpha})^{c_1'})$ and $0((z-e^{i\alpha})^{c_3'})$ with positive exponents c_1' and c_3', it follows that $[\ldots]\eta(z)+\overline{[\ldots]\eta(z)}$ *is also* $0(|z-e^{-i\alpha}|^{c_1''})$ *and* $0(|z-e^{i\alpha}|^{c_3''})$ *near* $e^{-i\alpha}$ *and* $e^{i\alpha}$ *respectively for some positive constants c_1'' and c_3''.*

We now define a function χ_1 by eqn (22), with m and φ' replaced by $\tilde{m}$ and φ_1'. We also introduce the function

$$\varphi_1^0(z) = \frac{1}{2i\pi}\int_{\mathscr{C}}\left\{\left[\frac{\varphi_0'(t)}{\omega'(t)}+\frac{\overline{\varphi_0'(t)}}{\overline{\omega'(t)}}\right]\eta(t)+\left[\frac{\omega(t)\overline{\varphi_0''(t)}}{\overline{\omega'(t)}^2}-\frac{\omega(t)\overline{\omega''(t)}\,\overline{\varphi_0'(t)}}{\overline{\omega'(t)}^3}+\frac{\overline{\psi_0'(t)}}{\overline{\omega'(t)}}\right]\overline{\eta(t)}\right\}\frac{dt}{t-z}.$$
$$(79)$$

The integrand here has a simple pole at $e^{i\beta}$, which is slightly displaced toward Ω^-; this means that the integrand being put under the form $f(t)/(t-e^{i\beta})$ where f is $\mathscr{C}^\infty$ on $\mathscr{C}$, $e^{i\beta}$ is replaced by $e^{i\beta^-}$. On the other hand the integral is convergent at the end points $e^{\pm i\alpha}$ since the integrand vanishes there. Following then the same kind of reasoning as in Part A, based on eqn (18) and Plemelj's formula, we transform eqn (76) into

$$\varphi_1(z)-\chi_1(z^-)-\varphi_1^0(z^-) = \overline{Q_*(z)}\,\overline{\varphi_1'(z)}-\overline{\chi_{1*}(z^-)}-\overline{\psi_1(z)}-\overline{\varphi_{1*}^0(z^-)}+Cst \quad \text{for} \quad z\in\mathscr{U}.$$

It is now easy, following the same lines as in Part A, to show that the left-hand side and the conjugate of the right-hand side are analytic on Ω^- and continuous on $\Omega^-\cup\mathscr{U}$.

Moreover, all functions in the left-hand side, and all conjugates of the functions in the right-hand side, are analytic at infinity (for the term $Q_*(z)\varphi_1'(z)$, this results from the fact that $Q_*(z)$ is $0(z)$ at infinity by eqn (24), and that $\varphi_1'(z)$ is $0(1/z^2)$ by eqn (77_1)). Applying then the lemma of Part A and differentiating, we get the following integral equation for the function φ_1':

$$\varphi_1'(z) = \varphi_1^{0\prime}(z) + \mathscr{L}\varphi_1'(z), \tag{80}$$

where $\mathscr{L}$ is the integral operator defined by eqn (28) (with m replaced by $\tilde{m}$). The expression of ψ_1 can also be obtained but will not be needed.

The solution can therefore be obtained as follows: solve the integral equation (26) for φ_0'; evaluate ψ_0 from eqn (78); calculate φ_1^0 from eqn (79); solve the integral equation (80) for φ_1'; finally get the SIFs from Andersson's formula (29) with φ' replaced by $\varphi^{a\prime} = \varphi_0' + \varphi_1' + 0(\eta^2)$.

B.3. *Expansion up to order* 1/2 *with respect to the crack extension length*

We shall now expand the preceding equations up to order $\sqrt{\tilde{s}}$, or equivalently up to order α, as in Section A.2. This will yield the functions H_{pq}. As remarked in the Introduction, these functions will be calculated *exactly* despite the fact that the method used is a first order perturbative procedure with respect to the curvature parameters, since they appear, in eqn (4), only in a term *linear* with respect to a^*.

We introduce changes of variables and functions analogous to those defined by eqns (33):

$$\begin{gathered} z = \mathrm{e}^{\mathrm{i}\alpha\zeta};\quad \varphi_0'(z) = \mathrm{e}^{-\mathrm{i}\alpha\zeta}[\sqrt{\ell}\,U_0(\zeta) + \alpha\ell V_0(\zeta) + 0(\alpha^2)]; \\ \psi_0'(z) = \mathrm{e}^{-\mathrm{i}\alpha\zeta}[\sqrt{\ell}\,X_0(\zeta) + 0(\alpha)];\quad \varphi_1'(z) = \mathrm{e}^{-\mathrm{i}\alpha\zeta}[\sqrt{\ell}\,U_1(\zeta) + \alpha\ell V_1(\zeta) + 0(\alpha^2)]. \end{gathered} \tag{81}$$

The properties of the new functions at the singular points are the same as those of the old ones: U_0, V_0, U_1, V_1 *are indefinitely differentiable, and* X_0 *has a double*† *pole, at the point* $\tilde{m}$; *all five functions are weakly singular at* ± 1.

Expansion of the integral equation (26) for φ_0' up to order $\alpha^1 = \alpha$ leads to the integral equations (34) of Section A.2, with U, V and m replaced by U_0, V_0 and $\tilde{m}$. The expressions of U_0' and X_0 will also be needed here. The first one is easily obtained by differentiating eqn (34_1):

$$U_0'(\zeta) = \frac{1-\mathrm{e}^{2\mathrm{i}\tilde{m}\pi}}{2\mathrm{i}\pi}\int_{C^+}\frac{(\lambda^2-1)\overline{U_0}(\lambda)\,\mathrm{d}\lambda}{(\lambda-\tilde{m})(\lambda-\zeta)^3}. \tag{82}$$

The second one is derived by differentiating and expanding eqn (78):

$$X_0(\zeta) = \frac{k_1+\mathrm{i}k_2}{2\sqrt{\pi}} - \frac{\zeta^2-2\tilde{m}\zeta+1}{2(\zeta-\tilde{m})^2}U_0(\zeta) - \frac{\zeta^2-1}{2(\zeta-\tilde{m})}U_0'(\zeta) - \frac{1-\mathrm{e}^{-2\mathrm{i}\tilde{m}\pi}}{4\mathrm{i}\pi}\int_{\Gamma^-}\frac{(\lambda^2-1)U_0(\lambda)\,\mathrm{d}\lambda}{(\lambda-\tilde{m})(\lambda-\zeta)^2}. \tag{83}$$

In this expression the integral, originally obtained in the form

$$\int_{-1}^{+1}\frac{(\lambda^2-1)U_0(\lambda)\,\mathrm{d}\lambda}{(\lambda-\tilde{m}^+)(\lambda-\zeta)^2},$$

has been transformed by deforming the integration path $]-1, +1[$ away from the pole $\tilde{m}^+$ onto the semi-ellipse $\Gamma^-: \lambda = \cos\theta + (\mathrm{i}/2)\sin\theta$, $-\pi < \theta < 0$ (see Fig. 6). This is allowed

† Because it is connected with the *derivative* of ψ_0.

provided the point ζ lies below Γ^- ; in practice only points belonging to the semi-circle C^- will be used.

We shall now expand the integral equation (80) for φ_1' in powers of α. The first task is to get the expansions of $\omega(z)$, $\omega'(z)$, $\omega''(z)$. Using eqns (11), (30) and (33_1) (with the usual replacements), one obtains

$$\omega(z) = -\frac{\ell\alpha^2}{2}(\zeta^2-1)\left(\frac{\zeta+1}{\zeta-1}\right)^{\tilde{m}}+0(\alpha^3);\quad \omega'(z) = \mathrm{i}\ell\alpha(\zeta-\tilde{m})\left(\frac{\zeta+1}{\zeta-1}\right)^{\tilde{m}}+0(\alpha^2);$$

$$\omega''(z) = \ell\frac{\zeta^2-2\tilde{m}\zeta+2\tilde{m}^2-1}{\zeta^2-1}\left(\frac{\zeta+1}{\zeta-1}\right)^{\tilde{m}}+0(\alpha). \tag{84}$$

The function $((\zeta+1)/(\zeta-1))^{\tilde{m}}$ here has its cut along any arc connecting the points ± 1 in the upper half-plane Π^+, and its limit for $\zeta\to\infty$ is unity.

Next one must derive the expression of the gap η between the two crack extensions. Using eqn (1), the fact that $|Z| = |\omega(z)|$ and eqns (31) and (84_1), one gets

$$\begin{aligned}\eta &= -\mathrm{i}a^*\,\mathrm{e}^{\mathrm{i}\tilde{m}\pi}|Z|\,(\sqrt{\tilde{s}}-\sqrt{|Z|})+0(\tilde{s}^2)\\ &= -\mathrm{i}a^*\left(\frac{\ell}{2}\right)^{3/2}\alpha^3\,\mathrm{e}^{\mathrm{i}\tilde{m}\pi}(\zeta^2-1)\left(\frac{1+\zeta}{1-\zeta}\right)^{\tilde{m}}g(\zeta)+0(\alpha^4),\end{aligned} \tag{85}$$

where g is the function defined on $]-1,+1[$ by

$$g(\zeta) = \sqrt{1-\zeta^2}\left(\frac{1+\zeta}{1-\zeta}\right)^{\tilde{m}/2}-\sqrt{1-\tilde{m}^2}\left(\frac{1+\tilde{m}}{1-\tilde{m}}\right)^{\tilde{m}/2}. \tag{86}$$

All necessary elements are now gathered for expanding $\varphi_1^{0\prime}$ and the integral equation (80). Equation (79) yields, after a lengthy but straightforward calculation:

$$\begin{aligned}\varphi_1^{0\prime}(\zeta) = \frac{a^*\ell\alpha}{4\pi\sqrt{2}}\int_{-1}^{+1}\frac{(\lambda^2-1)g(\lambda)}{(\lambda-\tilde{m}^-)(\lambda-\zeta)^2}\Bigg[U_0(\lambda)+\frac{\mathrm{e}^{2\mathrm{i}\tilde{m}\pi}}{2}\frac{-3\lambda^2+6\tilde{m}\lambda-4\tilde{m}^2+1}{(\lambda-\tilde{m}^-)^2}\overline{U_0(\lambda)}\\ +\frac{\mathrm{e}^{2\mathrm{i}\tilde{m}\pi}}{2}\frac{\lambda^2-1}{\lambda-\tilde{m}^-}\overline{U_0'(\lambda)+X_0(\lambda)}\Bigg]\mathrm{d}\lambda+0(\alpha^2).\end{aligned}$$

Since this expression is $0(\alpha)$, it does not yield any contribution in the term of order zero of the expansion of eqn (80). This equation yields therefore, to the lowest order, an integral equation for U_1 analogous to eqns (34) for U_0 and V_0 but with a zero "second member": $U_1(\zeta) = \mathscr{A}U_1(\zeta)$, where $\mathscr{A}$ is the operator defined by eqn (35_3). Because of the contractant nature of $\mathscr{A}$ (see Section A.3), this implies that U_1 is zero. Since U_1 describes the (first order) effect of curvature in the limit $\alpha\to 0$ (see eqn (81_4)) or equivalently $s\to 0$, this result means that the asymptotic solution for an infinitesimal extension is independent of curvature, in agreement with the universality result (3).

Expanding eqn (80) up to order $\alpha^1 = \alpha$ using the previous expression of $\varphi_1^{0\prime}$, one gets

$$V_1(\zeta) = V_1^0(\zeta)+\mathscr{A}V_1(\zeta) \tag{87}$$

where

$$V_1^0(\zeta) = \frac{a^*}{4\pi\sqrt{2}}\left\{\int_{\Gamma^-}\frac{(\lambda^2-1)g(\lambda)U_0(\lambda)\,\mathrm{d}\lambda}{(\lambda-\tilde{m})(\lambda-\zeta)^2}+\int_{C^+}\frac{(\lambda^2-1)g(\lambda)}{(\lambda-\tilde{m})(\lambda-\zeta)^2}\right.$$
$$\left.\times\left[\frac{\mathrm{e}^{2\mathrm{i}\tilde{m}\pi}}{2}\,\frac{-3\lambda^2+6\tilde{m}\lambda-4\tilde{m}^2+1}{(\lambda-\tilde{m})^2}\,\overline{U_0}(\lambda)+\frac{\mathrm{e}^{2\mathrm{i}\tilde{m}\pi}}{2}\,\frac{\lambda^2-1}{\lambda-\tilde{m}}\,\overline{U_0'}(\lambda)+\overline{X_0}(\lambda)\right]\mathrm{d}\lambda\right\}. \quad (88)$$

In the second integral $\overline{U_0(\lambda)}$, $\overline{U_0'(\lambda)}$ and $\overline{X_0(\lambda)}$ have been replaced by the analytic functions $\overline{U_0}(\lambda)$, $\overline{U_0'}(\lambda)$ and $\overline{X_0}(\lambda)$ and the integration path has been deformed away from the pole $\tilde{m}^-$ onto the semi-circle C^+. In the first integral it has been deformed onto the semi-ellipse Γ^-. This is allowed provided ζ lies below Γ^- (and in particular for $\zeta \in C^-$); indeed no pole is then crossed in the process, the quantity $g(\lambda)/(\lambda-\tilde{m})$ being finite at $\lambda=\tilde{m}$ (see eqn (86)). The determination of the function g here is such that this function be analytic on the unit disc and reduce to formula (86) on the real interval $]-1,+1[$.

The expansion of Andersson's equation (29) is obtained following the same lines as in Section A.2; using eqn (38) (m being replaced by $\tilde{m}$), paying attention to the fact that here the angle δ between the X_1-axis and the tangent to the crack extension at its tip is not πm but $\pi m+\frac{3}{2}a^*\sqrt{s}+0(s) = \pi\tilde{m}+(a^*/2)\sqrt{\tilde{s}}+0(\tilde{s})$, and identifying the terms proportional to U_0 and V_0 but not to a^* with $[k_p(\tilde{s})]^{\pi\tilde{m}}_{a^*=C^*=0}$, one gets

$$k_1(s)-\mathrm{i}k_2(s) = [k_1(\tilde{s})-\mathrm{i}k_2(\tilde{s})]^{\pi\tilde{m}}_{a^*=C^*=0}+\sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}\tilde{m}\pi}\left[-\frac{\mathrm{i}a^*}{2}\left(\frac{1-\tilde{m}}{1+\tilde{m}}\right)^{\tilde{m}/2}U_0(\tilde{m})\right.$$
$$\left.+2\sqrt{\frac{2}{1-\tilde{m}^2}}\left(\frac{1-\tilde{m}}{1+\tilde{m}}\right)^{\tilde{m}}V_1(\tilde{m})\right]\sqrt{\tilde{s}}+0(\tilde{s}). \quad (89)$$

The quantity $V_1(\tilde{m})$ here can be evaluated, once V_1 is known on C^- through resolution of the integral equation (87), by re-applying this equation with $\zeta=\tilde{m}$. However, when calculating the term $V_1^0(\tilde{m})$ in this equation, one must pay attention to the fact that this quantity is *not* given by eqn (88) with $\zeta=\tilde{m}$, since this point lies above Γ^- (see above). Putting $\zeta=\tilde{m}^-$ and evaluating the residue at $\lambda=\tilde{m}^-$ using eqn (86), one easily gets

$$V_1^0(\tilde{m}) = \begin{pmatrix}\text{Same expression as in}\\ \text{eqn (88), with } \zeta=\tilde{m}\end{pmatrix}-\frac{\mathrm{i}a^*}{4}\sqrt{\frac{1-\tilde{m}^2}{2}}\left(\frac{1+\tilde{m}}{1-\tilde{m}}\right)^{\tilde{m}/2}U_0(\tilde{m}). \quad (90)$$

U_0 is known to be proportional to the k_ps (see Section A.2); the same is true of X_0 by eqn (83). Equations (87), (88) and (90) imply then that V_1^0 and $V_1(\tilde{m})$ are proportional to a^* and the k_ps. The term $\sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}\tilde{m}\pi}[\ldots]\sqrt{\tilde{s}}$ in eqn (89) is therefore also proportional to these quantities. This equation agrees thus with the form predicted for $k_p(s)$ by eqn (69) up to order $\sqrt{\tilde{s}}$ (i.e. basically with the universality result (4)) and allows for an easy evaluation of the functions $\tilde{H}_{pq}(\tilde{m})$.

B.4. *Expansion up to order one with respect to the crack extension length*

The calculations necessary to obtain the expansion of the SIFs up to order $\tilde{s}$, or equivalently α^2, are quite analogous to those presented in the preceding section, though notably heavier; only the basic equations and results will be given here.

Change of functions:

$$\varphi_0'(z) = \mathrm{e}^{-\mathrm{i}\alpha\zeta}[\sqrt{\ell}U_0(\zeta)+\alpha\ell V_0(\zeta)+\alpha^2\ell^{3/2}W_0(\zeta)+0(\alpha^3)];$$
$$\psi_0'(z) = \mathrm{e}^{-\mathrm{i}\alpha\zeta}[\sqrt{\ell}X_0(\zeta)+\alpha\ell Y_0(\zeta)+0(\alpha^2)];$$
$$\varphi_1'(z) = \mathrm{e}^{-\mathrm{i}\alpha\zeta}[\alpha\ell V_1(\zeta)+\alpha^2\ell^{3/2}W_1(\zeta)+0(\alpha^3)]. \quad (91)$$

Integral equation for W_0: necessary to obtain the term proportional to $\tilde{s}$ in the expression of $[k_p(\tilde{s})]^{\pi\tilde{m}}_{a^*=C^*=0}$, but not needed here.

Expressions of V'_0 and Y_0:

$$V'_0(\zeta) = -\frac{\sigma_{12}^{\infty}}{2}+\frac{iT}{4}+\frac{1-e^{2i\tilde{m}\pi}}{2i\pi}\int_{C^+}\frac{(\lambda^2-1)\overline{V_0}(\lambda)\,d\lambda}{(\lambda-\tilde{m})(\lambda-\zeta)^3}; \tag{92}$$

$$Y_0(\zeta) = -\left(\frac{\sigma_{12}^{\infty}}{2}+\frac{iT}{4}\right)(\zeta+\tilde{m})-\frac{\zeta^2-2\tilde{m}\zeta+1}{2(\zeta-\tilde{m})^2}V_0(\zeta)-\frac{\zeta^2-1}{2(\zeta-\tilde{m})}V'_0(\zeta)$$
$$-\frac{1-e^{-2i\tilde{m}\pi}}{4i\pi}\int_{\Gamma^-}\frac{(\lambda^2-1)V_0(\lambda)\,d\lambda}{(\lambda-\tilde{m})(\lambda-\zeta)^2}\quad(\zeta \text{ below } \Gamma^-). \tag{93}$$

Integral equation for W_1:

$$W_1(\zeta) = W_1^0(\zeta)+\mathscr{A}W_1(\zeta), \tag{94}$$

where

$$W_1^0(\zeta) = \frac{a^*}{4\pi\sqrt{2}}\left\{\int_{\Gamma^-}\frac{(\lambda^2-1)g(\lambda)V_0(\lambda)\,d\lambda}{(\lambda-\tilde{m})(\lambda-\zeta)^2}+\int_{C^+}\frac{(\lambda^2-1)g(\lambda)}{(\lambda-\tilde{m})(\lambda-\zeta)^2}\right.$$
$$\left.\times\left[\frac{e^{2i\tilde{m}\pi}}{2}\,\frac{-3\lambda^2+6\tilde{m}\lambda-4\tilde{m}^2+1}{(\lambda-\tilde{m})^2}\overline{V_0}(\lambda)+\frac{e^{2i\tilde{m}\pi}}{2}\,\frac{\lambda^2-1}{\lambda-\tilde{m}}\overline{V'_0}(\lambda)+\overline{Y_0}(\lambda)\right]d\lambda\right\}$$
$$+\frac{C^*}{16\pi}\left\{\int_{\Gamma^-}\frac{(\lambda^2-1)h(\lambda)U_0(\lambda)\,d\lambda}{(\lambda-\tilde{m})(\lambda-\zeta)^2}+\int_{C^+}\frac{(\lambda^2-1)h(\lambda)}{(\lambda-\tilde{m})(\lambda-\zeta)^2}\right.$$
$$\left.\times\left[\frac{e^{2i\tilde{m}\pi}}{2}\,\frac{-3\lambda^2+6\tilde{m}\lambda-4\tilde{m}^2+1}{(\lambda-\tilde{m})^2}\overline{U_0}(\lambda)+\frac{e^{2i\tilde{m}\pi}}{2}\,\frac{\lambda^2-1}{\lambda-\tilde{m}}\overline{U'_0}(\lambda)+\overline{X_0}(\lambda)\right]d\lambda\right\}$$
$$(\zeta \text{ below } \Gamma^-). \tag{95}$$

In this equation h is the function defined by

$$h(\zeta) = (1-\zeta^2)\left(\frac{1+\zeta}{1-\zeta}\right)^{\tilde{m}}-(1-\tilde{m}^2)\left(\frac{1+\tilde{m}}{1-\tilde{m}}\right)^{\tilde{m}}. \tag{96}$$

Expression of the stress intensity factors:

$$k_1(s)-ik_2(s) = \begin{pmatrix}\text{Same expression}\\ \text{as in eqn (89)}\end{pmatrix}+\sqrt{\pi}\,e^{-i\tilde{m}\pi}\left[-\frac{iC^*}{2}\left(\frac{1-\tilde{m}}{1+\tilde{m}}\right)^{\tilde{m}/2}U_0(\tilde{m})\right.$$
$$\left.-\frac{ia^*}{\sqrt{2(1-\tilde{m}^2)}}\left(\frac{1-\tilde{m}}{1+\tilde{m}}\right)^{\tilde{m}}V_0(\tilde{m})+\frac{4}{1-\tilde{m}^2}\left(\frac{1-\tilde{m}}{1+\tilde{m}}\right)^{3\tilde{m}/2}W_1(\tilde{m})\right]\tilde{s}+0(\tilde{s}^{3/2}), \tag{97}$$

$W_1(\tilde{m})$ being given by eqn (94) with

$$W_1^0(\tilde{m}) = \begin{pmatrix}\text{Same expression as in}\\ \text{eqn (95), with } \zeta=\tilde{m}\end{pmatrix}-\frac{ia^*}{4}\sqrt{\frac{1-\tilde{m}^2}{2}}\left(\frac{1+\tilde{m}}{1-\tilde{m}}\right)^{\tilde{m}/2}V_0(\tilde{m})$$
$$-\frac{iC^*}{8}(1-\tilde{m}^2)\left(\frac{1+\tilde{m}}{1-\tilde{m}}\right)^{\tilde{m}}U_0(\tilde{m}). \tag{98}$$

It can be verified that the term proportional to σ_{12}^{∞} in V_0 (eqn (41), with $[V(\zeta)]_{T=0}$ and m replaced by $[V_0(\zeta)]_{T=0}$ and $\tilde{m}$) does not generate any contribution to the function W_1^0, nor consequently to the function W_1. It follows that the expression $\sqrt{\pi}\,e^{-i\tilde{m}\pi}[\ldots]\tilde{s}$ in eqn (97) contains terms proportional to a^* and T on the one hand, and to C^* and the k_ps on the other hand, in agreement with eqn (69) (i.e. fundamentally with eqn (5), except for the term $a^{*2}L_{pq}(m)k_q$ disregarded in the first order perturbative analysis). Like the vanishing of $[V(\zeta)]_{T=0}$ at $\zeta = m$ (see Section A.2), this represents a good test of the correctness of the approach employed.

B.5. *Solutions under the form of series and numerical calculation of the functions H_{pq}, K_p and M_{pq}*

It would of course be desirable to express the functions V_1 and W_1 under the form of series analogous to (44):

$$V_1(\zeta) = \sum_{n=0}^{+\infty} \mathscr{A}^n V_1^0(\zeta); \quad W_1(\zeta) = \sum_{n=0}^{+\infty} \mathscr{A}^n W_1^0(\zeta) \quad (\forall\, \zeta \in C^-). \tag{99}$$

Since the operator $\mathscr{A}$ is contractant on the space $\mathscr{E}$ (see Section A.3), establishing these formulae only requires to prove that V_1^0 and W_1^0 lie in this space. Let us consider V_1^0 for instance (eqn (88)). In the integral over Γ^-, let us deform the integration path onto C^+. (This is feasible since there is no pole at $\lambda = \tilde{m}$ (see Section B.3) and U_0 can be extended to the whole complex plane except on C^+ (see Section A.3); the value of U_0 to be used at $\lambda \in C^+$ in the integral is then the limit of $U_0(\mu)$ for $\mu \to \lambda$ from below.) Using the fact that the term in factor of $1/((\lambda-\tilde{m})(\lambda-\zeta)^2)$ is bounded on C^+ in both integrals and inequalities analogous to those of Section A.3, one shows that $\mathrm{Max}_{\zeta \in C^-}\, |(\zeta^2-1)V_1^0(\zeta)|$ is finite, i.e. that V_1^0 lies in $\mathscr{E}$.

The expansions of the H_{pq}s, K_ps and M_{pq}s up to a given order in m, just like those of the F_{pq}s and G_ps, can be derived from eqns (99) through truncation of the series and analytical evaluation of the integrals. However the calculations implied are even more enormous than for the F_{pq}s and G_ps, so that the authors have not attempted to obtain high order expansions analogous to eqns (66) and (67) for the latter functions and limited their calculations to the first order; the main objective here is not to derive accurate values of the functions (which will be achieved through numerical computation of the series (99), see below) but to compare the results with those of other authors. Thus one gets

$$H_{11}(m) = -\frac{9\pi}{8}m + 0(m^3); \quad H_{12}(m) = -\frac{9}{4} + 0(m^2);$$

$$H_{21}(m) = \frac{3}{4} + 0(m^2); \quad H_{22}(m) = \left(-8 + \frac{3\pi}{8}\right)m + 0(m^3);$$

$$K_1(m) = \frac{11}{2}\sqrt{2\pi}m + 0(m^3); \quad K_2(m) = -\frac{3}{4}\sqrt{2\pi} + 0(m^2);$$

$$M_{11}(m) = -\frac{3\pi}{4}m + 0(m^3); \quad M_{12}(m) = -\frac{3}{2} + 0(m^2);$$

$$M_{21}(m) = \frac{1}{2} + 0(m^2); \quad M_{22}(m) = \left(\frac{\pi}{4} - \frac{16}{\pi}\right)m + 0(m^3).$$

Similar formulae were derived by Karihaloo *et al.* (1981), Sumi *et al.* (1983) and Sumi (1986, 1991), using the Banichuk–Cotterell–Rice perturbative procedure; their results are the same as those above (after correction of some algebraic errors in the first work), except for the values of H_{22} and M_{22} which were calculated only by Sumi (1991) and found by this author to be

$$H_{22}(m) = \left(2-\frac{21\pi}{8}\right)m+0(m^3); \quad M_{22}(m) = -\frac{5\pi}{4}m+0(m^3).$$

These discrepancies, just like that observed for the function F_{22}, illustrate the fundamental inapplicability of the Banichuk–Cotterell–Rice procedure to non-$\mathscr{C}^\infty$ cracks (see the Introduction).

Some results were also obtained by Sumi (1991) concerning the other functions involved in eqn (5). These functions are considered of secondary interest by the present authors, for reasons explained in the Introduction. Calculations analogous to those of Sumi are nevertheless presented in Appendix D in order to complete the comparison with this author's work.

Equations (99) also allow for a numerical evaluation of the functions H_{pq}, K_p and M_{pq}. For the H_{pq}s for instance, the procedure is as follows: deduce $\mathscr{A}U_0^0$ from $U_0^0 \equiv (k_1-\mathrm{i}k_2)/(2\sqrt{\pi})$, $\mathscr{A}^2U_0^0$ from $\mathscr{A}U_0^0$, $\mathscr{A}^3U_0^0$ from $\mathscr{A}^2U_0^0\ldots$ through Gaussian integration over the arc C^+, using the values of U_0^0, $\mathscr{A}U_0^0$, $\mathscr{A}^2U_0^0\ldots$ at the conjugates of the Gauss points of the arc C^+, and varying ζ among the same points (in this way all functions are calculated at the same points); calculate U_0 on C^- through suitable truncation of the series (44_1), then at the point $\tilde{m}$ and at some Gauss points of Γ^- by re-using eqn (34_1); evaluate U'_0 (eqn (82)) on C^-; calculate X_0 (eqn (83)) on C^-; use these results to compute V_1^0 on C^- (eqn (88)); evaluate the (suitably truncated) series (99_1) on C^- in the same way as the series (44_1); calculate $V_1^0(\tilde{m})$ (eqn (90)) and $V_1(\tilde{m})$ (eqn (87)); compute the term $\sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}\tilde{m}\pi}[\ldots]\sqrt{\tilde{s}}$ due to curvature in eqn (89), and consequently derive the values of the $\tilde{H}_{p1}$s (for $k_1 = 1$, $k_2 = 0$) or $\tilde{H}_{p2}$s (for $k_1 = 0$, $k_2 = 1$); finally get the H_{pq}s from eqn (70_2).

The accuracy of the method can be very simply assessed by applying it to the computation of the term proportional to σ_{12}^∞ in W_1 ($=0$ in theory, see Section B.4) and examining the smallness of the result.

However, this method raises a non-trivial numerical problem; indeed the accuracy obtained by using ordinary Gaussian integration, estimated as indicated above, turns out to be poor ($\sim 10^{-1}$). This is because many successive integrations are required, so that small numerical errors made at each step add up and result in a large final inaccuracy. The main errors arise from the evaluation of integrals of the form

$$\int_{C^+,\Gamma^-} \frac{f(\lambda)\,\mathrm{d}\lambda}{(\lambda-\zeta)^{2,3}}$$

for ζ close to ± 1: the function in the denominator then varies very quickly when $\lambda \to \pm 1$ so that the integration is inaccurate near these points. Simple strategies aimed at improving the evaluation of such integrals, such as increasing the number of Gauss points in each element or the number of elements near ± 1, prove to be relatively inefficient; this is because the set of ζ points used is the conjugate of the set of λ points, so that the accuracy gained by using more λ points near ± 1 is degraded by requiring the calculation of more integrals. The solution which was adopted consisted of accounting for the quick variation of the function in the denominator by replacing $f(\lambda)$ by its average value in each interval of integration $]\lambda_i, \lambda_{i+1}[$ (evaluated by standard Gaussian integration), extracting it from the $\int$ symbol and calculating the remaining integral

$$\int_{\lambda_i}^{\lambda_{i+1}} \frac{\mathrm{d}\lambda}{(\lambda-\zeta)^{2,3}}$$

exactly.

Tables 1, 2 *and* 3 display the results obtained with 100 Gauss points distributed among 50 elements; the lengths of the latter decrease toward the singular points ± 1 according to a geometric progression the parameter of which is adjusted for every kink angle in order to optimize the accuracy (estimated as explained above). These tables are easily supplemented for negative kink angles since simple symmetry considerations show that H_{12}, H_{21}, K_2, M_{12}, M_{21} are even, and H_{11}, H_{22}, K_1, M_{11}, M_{22} odd, functions of m. The (absolute) accuracy is of the order of 10^{-3}.

Table 1. The H_{pq} functions

Kink angle (°)	H_{11}	H_{12}	H_{21}	H_{22}
0	0	−2.250	0.750	0
5	−0.098	−2.236	0.746	−0.189
10	−0.194	−2.196	0.731	−0.374
15	−0.288	−2.129	0.707	−0.553
20	−0.377	−2.037	0.675	−0.723
25	−0.461	−1.922	0.635	−0.879
30	−0.538	−1.786	0.587	−1.021
35	−0.608	−1.631	0.533	−1.145
40	−0.669	−1.460	0.474	−1.250
45	−0.721	−1.276	0.410	−1.334
50	−0.763	−1.082	0.344	−1.396
55	−0.796	−0.881	0.276	−1.436
60	−0.819	−0.677	0.207	−1.454
65	−0.833	−0.472	0.139	−1.450
70	−0.837	−0.270	0.072	−1.424
75	−0.832	−0.073	0.009	−1.378
80	−0.818	0.115	−0.052	−1.313

Table 2. The K_p functions

Kink angle (°)	K_1	K_2	Kink angle (°)	K_1	K_2
0	0	−1.879	45	2.291	0.043
5	0.381	−1.850	50	2.290	0.377
10	0.751	−1.763	55	2.228	0.701
15	1.101	−1.622	60	2.108	1.006
20	1.419	−1.428	65	1.935	1.284
25	1.698	−1.192	70	1.712	1.527
30	1.930	−0.917	75	1.448	1.728
35	2.109	−0.613	80	1.150	1.883
40	2.230	−0.290			

Table 3. The M_{pq} functions

Kink angle (°)	M_{11}	M_{12}	M_{21}	M_{22}
0	0	−1.500	0.500	0
5	−0.065	−1.491	0.497	−0.119
10	−0.130	−1.464	0.488	−0.236
15	−0.192	−1.420	0.474	−0.349
20	−0.252	−1.358	0.454	−0.457
25	−0.307	−1.282	0.429	−0.556
30	−0.359	−1.192	0.399	−0.645
35	−0.405	−1.089	0.366	−0.724
40	−0.446	−0.975	0.328	−0.791
45	−0.481	−0.853	0.289	−0.845
50	−0.510	−0.724	0.247	−0.885
55	−0.532	−0.590	0.205	−0.911
60	−0.548	−0.454	0.162	−0.924
65	−0.557	−0.318	0.119	−0.922
70	−0.560	−0.184	0.077	−0.907
75	−0.557	−0.053	0.037	−0.879
80	−0.548	0.073	−0.001	−0.840

CONCLUSION

The detailed expansion of the stress intensity factors being now available, it only remains to indicate how it can be combined with a propagation criterion for crack path predictions. The criterion that will be considered consists of two parts:

(i) the principle of local symmetry, which gives the kink angle through the equation $k_2^* = 0$ ($\Leftrightarrow k_2 = 0$ in the regular, kink-free part of the crack, where the SIFs are continuous);

(ii) the Griffith postulate, which stipulates the intensity of the loading necessary to effectively promote propagation *via* the equation $\mathscr{G} = \mathscr{G}_c \Leftrightarrow k_1^* = k_{1c}$ ($\Leftrightarrow k_1 = k_{1c}$ in the regular part of the crack).†

It now becomes necessary to introduce the possibility of a variable loading, otherwise the Griffith postulate cannot be perpetually satisfied as the crack extends. Here we shall make the simplifying assumption that the loading is *proportional*, i.e. that it varies only through multiplication by a time-dependent scalar $\mu(t)$; extending the reasoning to non-proportional loadings is straightforward. Equations (2), (3), (6) and (7) are valid for a constant loading, but this restriction can easily be removed by noting that since the solution of an elasticity problem depends only on the *current* geometry and loading, the $k_p(s)$s can be evaluated by prescribing a constant loading corresponding to the *final* value of $\mu(t)$; this just means multiplying the expression of the $k_p(s)$s given by eqns (2), (3), (6) and (7) by this final value. Writing $\mu(t)$ as

$$\mu(t) \equiv 1+\mu^{(1/2)}\sqrt{s}+\mu^{(1)}s+0(s^{3/2}),\ddagger \tag{100}$$

we get

$$\begin{aligned} k_p(s) = F_{pq}(m)k_q &+ \{[k_p^{(1/2)}]^{\pi m}_{a^*=0} + a^* H_{pq}(m)k_q + \mu^{(1/2)}F_{pq}(m)k_q\}\sqrt{s} \\ &+ \{[k_p^{(1)}]^{\pi m, a^*}_{C^*=0} + C^* M_{pq}(m)k_q + \mu^{(1/2)}([k_p^{(1/2)}]^{\pi m}_{a^*=0} + a^* H_{pq}(m)k_q) \\ &\qquad + \mu^{(1)}F_{pq}(m)k_q\}s + 0(s^{3/2}), \end{aligned}$$

where the notations k_q, $[k_p^{(1/2)}]^{\pi m}_{a^*=0}$ and $[k_p^{(1)}]^{\pi m, a^*}_{C^*=0}$ refer to the original loading (prior to multiplication by $\mu(t)$).

Using the above criterion, one must equate the successive coefficients of the expansions of $k_1(s)$ and $k_2(s)$ to k_{1c} (for the first coefficient of the former expansion) or zero (for the other coefficients). Thus one gets at the various orders:

Order 0:

$$\frac{F_{21}(m)}{F_{22}(m)} = -\frac{k_2}{k_1}; \quad F_{1q}(m)k_q = k_{1c}. \tag{101}$$

Order 1/2:

$$a^* = -\frac{[k_2^{(1/2)}]^{\pi m}_{a^*=0}}{H_{2q}(m)k_q}; \quad \mu^{(1/2)} = -\frac{[k_1^{(1/2)}]^{\pi m}_{a^*=0} + a^* H_{1q}(m)k_q}{k_{1c}}. \tag{102}$$

† The first equivalence arises from the fact that $\mathscr{G}$ is proportional to $k_1^{*2}+k_2^{*2}$ with $k_2^* = 0$.

‡ Since in practice μ will be a differentiable function of time, such a $\sqrt{s}$-dependence may seem at first sight to be experimentally unachievable. This is not so however, because it only requires that s vary proportionally to t^2 just after the kink, which is not unreasonable.

Order 1 :

$$C^* = -\frac{[k_2^{(1)}]^{\pi m, a^*}_{C^*=0}}{M_{2q}(m)k_q}; \quad \mu^{(1)} = (\mu^{(1/2)})^2 - \frac{[k_1^{(1)}]^{\pi m, a^*}_{C^*=0} + C^* M_{1q}(m)k_q}{k_{1c}}. \tag{103}$$

These equations provide the geometric parameters of the crack extension, the intensity of the initial loading and the coefficients of the expansion (100) of $\mu(t)$. (Inverting this equation, one gets s as a function of time, which is physically more meaningful.) The expressions for m, a^* and C^* are seen not to involve $\mu^{(1/2)}$ and $\mu^{(1)}$, which means that *the variation of the loading has no influence on the shape of the path followed by the crack.* This property would not subsist for a non-proportional loading; it arises from the fact that when one applies the principle of local symmetry at the order $\sqrt{s}$ for instance, the term $\mu^{(1/2)}F_{2q}(m)k_q$ automatically vanishes as a result of its previous application at the order $s^0 = 1$.

As a particular case, one may consider the regular ($\mathscr{C}^\infty$) part of the crack. Then $[k_2^{(1/2)}]^{\pi m}_{a^*=0} = G_2(0)T = 0$ (see eqn (67_2)) so that eqn (102_1) yields $a^* = 0$. Furthermore $[k_2^{(1)}]^{\pi m, a^*}_{C^*=0}$ can be identified with *the derivative* $[\mathrm{d}k_2/\mathrm{d}s]_{\text{straight}}$ *of* $k_2(s)$ *with respect to* s *along a straight extension in the direction of the tangent* (for a constant loading); indeed the term proportional to $\sqrt{s}$ in the expansion of $k_2(s)$ along such an extension is zero. Since in addition $M_{21}(0) = 1/2$ (see Table 3), $k_1 = k_{1c}$ and $k_2 = 0$, eqn (103_1) takes the following simple form:

$$C \equiv C^* = -\frac{2}{k_{1c}}\left[\frac{\mathrm{d}k_2}{\mathrm{d}s}\right]_{\text{straight}}, \tag{104}$$

no distinction between the initial and subsequent curvatures C and C^* being necessary here since the curvature is continuous. This remarkable formula may be regarded as the *general equation of the propagating crack* in that it provides the value of the curvature at any regular point. It could in fact be inferred from other works such as those of Karihaloo *et al.* (1981) and Sumi *et al.* (1983), but it was not presented there in the form (104) and, more importantly, it was only obtained under certain restrictive hypotheses, among which the straightness of the initial crack. (As a consequence, the expression found for $k_p^{(1)}$, analogous to (5), did not include the term $CJ_{pq}(m)k_q$ proportional to the initial crack curvature and was therefore invalid for a curved initial crack.)

Equations (101_1), (102_1), (103_1) and (104) allow the envisaging of crack path predictions over arbitrary long distances by step-by-step methods, each step involving numerical computations of stress intensity factors, calculations of geometric parameters of the future extension, and remeshing operations. One possible strategy consists in: computing the initial values of k_1 and k_2; evaluating m from eqn (101_1); adding a short straight extension to the crack in the direction πm through remeshing; getting $[k_2^{(1/2)}]^{\pi m}_{a^*=0}$ from the value of k_2 at the tip of this extension; using eqn (102_1) to obtain the value of a^*; suppressing the preceding extension, replacing it by another one having an a^* equal to the value determined but a zero C^*, and obtaining C^* in a manner analogous to a^*; suppressing this second extension and replacing it by a third, final one having the values of a^* and C^* determined; stopping the extension at an arbitrary but small distance from the original crack tip, and reiterating the procedure. The intensity of the loading at each step, if desired, can also be obtained *via* the value of $\mu(t)$ which is equal to the ratio k_{1c}/k_1, k_1 being calculated for the original, reference loading.

Several variants are possible. One may think for instance of skipping the calculation of C^*. However, neglecting this parameter is only possible at the first step which involves a large, initial kink. Indeed the subsequent kink angles, which arise from the fact that eqns (101_1), (102_1) and (103_1) ensure the vanishing of $k_2(s)$ up to order s only, are very small, so that the values of a^* (which vanishes together with m, as was seen above) are also small and C^* becomes the major parameter governing the crack shape. This feature suggests disregarding m and a^*, instead of C^*, after the initial kink. In that case the path determined numerically would involve only one large, initial kink; the curvature at all subsequent steps could be determined from eqn (104), $[\mathrm{d}k_2/\mathrm{d}s]_{\text{straight}}$ being evaluated by comparing the original value of k_2 with that at the tip of a small, straight extension in the direction of the tangent. One drawback of the method would be that in the absence of a kink, k_2 would not be obliged to vanish at each step and could thus become relatively large beyond a certain distance.

The method proposed corresponds, in essence, to that employed by Sumi for numerical studies of crack paths in situations of practical interest [see for instance Sumi (1991)]. However there are two notable differences. First, unlike eqns (2), (3), (6) and (7), the expansion of the $k_p(s)$s used by Sumi was valid only under restrictive hypotheses, as explained above. Second, the quantities $[k_2^{(1/2)}]^{\pi m}_{a^*=0}$, $[k_2^{(1)}]^{\pi m, a^*}_{C^*=0}$, $[\mathrm{d}k_2/\mathrm{d}s]_{\text{straight}}$ were not expressed in that form but in terms of the coefficients T, b_1, b_2 of the terms proportional to $r^0 = 1$ and $\sqrt{r}$ in the stress expansion near the original crack tip. As a result, in contrast to the method proposed here, that of Sumi did not only require the calculation of stress intensity factors but also that of these coefficients.

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APPENDIX A: PROOF OF THE LEMMA OF SECTION A.1

With the notations and hypotheses of the lemma, the function $g_*(z) = \overline{g(1/\bar{z})}$ is continuous on $\Omega^+ \cup \mathscr{U}$ and analytic on Ω^+. Furthermore $z = 1/\bar{z}$ on $\mathscr{U}$. Thus eqn (21) can be written as

$$f(z) = g_*(z) \quad \text{for} \quad z \in \mathscr{U}.$$

Using then Muskhelishvili's (1953) formulae (70.1′) (for f) and (70.2) (for g_*), we get, if $z \in \Omega^-$:

$$\frac{1}{2i\pi}\int_{\mathscr{U}} \frac{f(t)\,\mathrm{d}t}{t-z} = -f(z)+f(\infty) = \frac{1}{2i\pi}\int_{\mathscr{U}} \frac{g_*(t)\,\mathrm{d}t}{t-z} = 0,$$

where the arc $\mathscr{U}$ is oriented anticlockwise; it follows that f is a constant. Furthermore Muskhelishvili's formulae (70.2′) (for f) and (70.1) (for g_*) yield, for $z \in \Omega^+$:

$$\frac{1}{2i\pi}\int_{\mathscr{U}} \frac{f(t)\,\mathrm{d}t}{t-z} = f(\infty) = \frac{1}{2i\pi}\int_{\mathscr{U}} \frac{g_*(t)\,\mathrm{d}t}{t-z} = g_*(z),$$

so that g is also constant and conjugate to f.

APPENDIX B: PROOF OF EQUATION (41)

Proving eqn (41) requires that it is shown that the function $w(\zeta)$ defined by the right-hand side verifies the integral equation (34_2) with the "second member" $V^0(\zeta) = -(\sigma_{12}^\infty/2)(\zeta+m)$, i.e. that

$$\mathscr{A}w(\zeta) = \frac{\sigma_{12}^\infty}{2}\left[\zeta+m+(m-\zeta)\left(\frac{\zeta+1}{\zeta-1}\right)^m\right] \quad (\text{for } \zeta \in \Pi^-), \tag{B1}$$

the determination of the function $((\zeta+1)/(\zeta-1))^m$ being such that the cut be located on any arc connecting the points ± 1 in Π^+ and that its limit for $\zeta \to \infty$ be equal to unity.

Using eqn (35_3), deforming the integration path back to the real interval $]-1, +1[$, and noting that $\bar{w}(\lambda)$ is given by the same formula (41) as $w(\zeta)$, but with the cut of the function $((\lambda+1)/(\lambda-1))^m$ along some arc connecting the points ± 1 in Π^- instead of Π^+ so that $((\lambda+1)/(\lambda-1))^m = e^{-im\pi}((1+\lambda)/(1-\lambda))^m$ for $\lambda \in]-1, +1[$, one gets

$$\mathscr{A}w(\zeta) = \frac{\sigma_{12}^\infty \sin m\pi}{4\pi}\int_{-1}^{+1} \frac{\lambda^2-1}{(\lambda-\zeta)^2}\left(\frac{1+\lambda}{1-\lambda}\right)^m \mathrm{d}\lambda.$$

The change of variable $u = (1+\lambda)/(1-\lambda)$ leads, after a bit of algebra, to the following expression:

$$\mathscr{A}w(\zeta) = -\frac{2\sigma_{12}^\infty \sin m\pi}{\pi(\zeta-1)^2}\int_0^{+\infty} \frac{u^{m+1}\,\mathrm{d}u}{(u+1)^2\left(u+\dfrac{\zeta+1}{\zeta-1}\right)^2}. \tag{B2}$$

The value of a closely related integral can be found in Gradshteyn and Ryzhik (1965) (formula (3.223.1)):

$$\int_0^{+\infty} \frac{u^{\mu-1}\,\mathrm{d}u}{(u+a)(u+b)} = \frac{\pi}{\sin \mu\pi}\,\frac{a^{\mu-1}-b^{\mu-1}}{b-a}.$$

The determination of the power functions in this formula is the usual one with the cut along the line of negative reals. Differentiating this expression with respect to a and b, we get

$$\int_0^{+\infty} \frac{u^{\mu-1}\,\mathrm{d}u}{(u+a)^2(u+b)^2} = \frac{\pi}{\sin \mu\pi}\left[-(\mu-1)\frac{a^{\mu-2}+b^{\mu-2}}{(a-b)^2}+2\frac{a^{\mu-1}-b^{\mu-1}}{(a-b)^3}\right].$$

Inserting this result with $\mu = m+2$, $a = 1$ and $b = (\zeta+1)/(\zeta-1)$ into eqn (B2), one obtains, after a few manipulations, eqn (B1) for $\mathscr{A}w(\zeta)$, with the desired determination of the function $((\zeta+1)/(\zeta-1))^m$.

The function $w(\zeta) = [V(\zeta)]_{T=0}$ defined by eqn (41) possesses a nice mechanical interpretation which explains its lack of influence upon the SIFs. Indeed it is well known from the work of Muskhelishvili that adding a term of the form $\mathrm{i}AZ$ (A being real) to the potential $\Phi(Z)$ does not change the stresses, because it represents a rigid rotation. Let us transform Φ in that way; then we add $\mathrm{i}A\omega(z)$ to $\varphi(z)$, and therefore $\mathrm{i}A\omega'(z)$ to $\varphi'(z)$. Expressing this quantity in terms of ζ using eqn (33_1), and expanding the result in powers of α, one finds that

$$\mathrm{i}A\omega'(z) = \ell A\alpha(m-\zeta)\left(\frac{\zeta+1}{\zeta-1}\right)^m + 0(\alpha^2).$$

Equation (33_2) then shows that a term $A(m-\zeta)((\zeta+1)/(\zeta-1))^m$ is added to $V(\zeta)$, and this is precisely the form of $[V(\zeta)]_{T=0}$. This function therefore only represents a rigid rotation, which is why it has no effect on the SIFs.

APPENDIX C: PROOF OF EQUATIONS (45)–(65)

The proof of eqns (45)–(54) will be presented in detail, but only brief comments will be given *in fine* on that of eqns (55)–(65), which follows essentially similar lines. Our starting point is eqns (35_3), (36), (39) and (44_1). The last equation was proved only for $\zeta \in C^-$, but it is easily verified to be also applicable to the calculation of $U(m)$, and in fact to that of $U(\zeta)$ at any point $\zeta \in \mathbb{C} - C^+$.

Step 1. In order to concentrate on the major difficulties, we first "extract" some uninteresting factors from the equations just mentioned by introducing the following set of functions, defined on $\mathbb{C} - C^+$:

$$x_0(\zeta) = 1; \quad x_n(\zeta) = \int_{C^+} \frac{(\lambda^2-1)\overline{x_{n-1}}(\lambda)\,\mathrm{d}\lambda}{(\lambda-m)(\lambda-\zeta)^2} \quad (n \geqslant 1). \tag{C1}$$

It is then easy to show by induction that

$$\mathscr{A}^n U^0(\zeta) = \begin{cases} \dfrac{k_1 - \mathrm{i}k_2}{2\sqrt{\pi}}\left(\dfrac{\sin m\pi}{2\pi}\right)^n x_n(\zeta) & \text{if } n \text{ is even} \\ -\dfrac{k_1+\mathrm{i}k_2}{2\sqrt{\pi}}\left(\dfrac{\sin m\pi}{2\pi}\right)^n \mathrm{e}^{\mathrm{i}m\pi} x_n(\zeta) & \text{if } n \text{ is odd.} \end{cases}$$

Combining this result with eqns (39) and (44_1), one gets eqns (45) and (46), with $x_n \equiv x_n(m)$. The problem is thus reduced to calculating the $x_n(\zeta)$s.

Step 2. One discovers, upon study of the first $x_n(\zeta)$s, that calculation of these functions essentially requires the evaluation of the integrals

$$\Theta_q(\zeta) = \int_{C^+} \left(\log^+ \frac{\lambda-1}{\lambda+1}\right)^q \frac{\mathrm{d}\lambda}{\lambda-\zeta} \quad (q = 0, 1, 2, \ldots), \tag{C2}$$

where $\log^+$ is the logarithm function defined on $\mathbb{C} - \mathrm{i}\mathbb{R}^-$ by $\log^+(\rho \mathrm{e}^{\mathrm{i}\theta}) = \ln\rho + \mathrm{i}\theta$ with $-\pi/2 < \theta < 3\pi/2$. Having easily calculated $\Theta_0(\zeta)$ and encountering the integral $\Theta_1(\zeta)$, Wu (1979) wrote that "unfortunately the appearance of the logarithmic term [...] makes the explicit evaluation of the higher order terms impossible". This statement was obviously motivated by the fact that functions of the form $(\log x)/(x-a)$ must be integrated to get $\Theta_1(\zeta)$ and that *indefinite* integrals of this kind are not expressible in terms of elementary functions. It will be seen, however, to be over-pessimistic: indeed $\Theta_1(\zeta)$, and more generally the $\Theta_q(\zeta)$s, are *definite*, *calculable* integrals.

We shall begin by evaluating the real integrals

$$\mathscr{I}_q = \int_{-1}^{+1} \left(\ln \frac{1-\lambda}{1+\lambda}\right)^q \mathrm{d}\lambda. \tag{C3}$$

First, it is obvious that

$$\mathscr{I}_0 = 2; \quad \mathscr{I}_{2k+1} = 0 \quad (k = 0, 1, 2, \ldots). \tag{C4}$$

$\mathscr{I}_{2k}$ $(k \geqslant 1)$ remains to be calculated. Writing it in the form $2\int_0^1 \ldots$, putting $t = (1-\lambda)/(1+\lambda)$, and integrating by parts using $1 - 1/(1+t)$ as an indefinite integral of $1/(1+t)^2$, we get

$$\mathscr{I}_{2k} = 4\int_0^1 \frac{(\ln t)^{2k}\,\mathrm{d}t}{(1+t)^2} = 4\left[\left(1 - \frac{1}{1+t}\right)(\ln t)^{2k}\right]_{t=0}^{t=1} - 8k\int_0^1 \frac{(\ln t)^{2k-1}\,\mathrm{d}t}{1+t}.$$

The term $[\ldots]_{t=0}^{t=1}$ is zero and the value of the last integral is $(-1)^{k-1}((1-2^{2k-1})/2k)\pi^{2k}B_{2k}$ where B_q is the qth Bernoulli number (Gradshteyn and Ryzhik, 1965, formula 4.271.2). Therefore

$$\mathscr{I}_{2k} = 4(-1)^k(1-2^{2k-1})\pi^{2k}B_{2k} \quad (k \geqslant 1). \tag{C5}$$

Since $B_0 = 1$, $1 - 2^{q-1} = 0$ for $q = 1$ and $B_q = 0$ for q odd $\geqslant 3$ [see Gradshteyn and Ryzhik (1965)], eqns (C4) and (C5) can be condensed into the following single formula:

$$\mathscr{I}_q = 4(1-2^{q-1})(-\mathrm{i}\pi)^q B_q \quad (q = 0, 1, 2, \ldots). \tag{C6}$$

Step 3. We shall now calculate the integrals

$$\mathscr{J}_q = \int_{C^+} \left(\log^+ \frac{\lambda-1}{\lambda+1}\right)^q \mathrm{d}\lambda, \tag{C7}$$

where $\log^+$ is the function introduced above. First we deform the integration path onto the real interval $]-1, +1[$. This does not raise any problem since $\log^+$ is defined on $\mathbb{C}-\mathrm{i}\mathbb{R}^-$ and $(\lambda-1)/(\lambda+1) \in \mathrm{i}\mathbb{R}^-$ only if $\lambda \in C^-$, which does not occur during the deformation. We then note that if $\lambda \in]-1, +1[$, $(\lambda-1)/(\lambda+1) \in]-\infty, 0[$ so that $\log^+((\lambda-1)/(\lambda+1)) = \ln((1-\lambda)/(1+\lambda)) + \mathrm{i}\pi$. Using Newton's formula and eqn (C6), we thereby obtain

$$\begin{aligned}\mathscr{J}_q &= \sum_{r=0}^{q} C_q^r (\mathrm{i}\pi)^{q-r} \mathscr{I}_r = 4(\mathrm{i}\pi)^q \sum_{r=0}^{q} C_q^r (-1)^r (1-2^{r-1}) B_r \\ &= 4(-\mathrm{i}\pi)^q \left[\sum_{r=0}^{q} C_q^r (-1)^{q-r} B_r - 2^{q-1} \sum_{r=0}^{q} C_q^r \left(-\frac{1}{2}\right)^{q-r} B_r\right].\end{aligned}$$

Now the first Σ is nothing other than $B_q(-1)$, where $B_q(X)$ denotes the qth Bernoulli polynomial [see Gradshteyn and Ryzhik (1965)]. Similarly, the second Σ is equal to $B_q(-1/2)$. Furthermore Gradshteyn and Ryzhik's formula (9.624) yields $B_q(-1) - 2^{q-1} B_q(-1/2) = 2^{q-1} B_q$ (since $B_q(0) = B_q$). It follows that

$$\mathscr{J}_q = 2(-2\mathrm{i}\pi)^q B_q. \tag{C8}$$

Step 4. A differential equation for the functions $\Theta_q(\zeta)$ will now be exhibited by calculating the integral

$$\mathscr{K} = \int_{C^+} \frac{\lambda^2-1}{(\lambda-\zeta)^2} \left(\log^+ \frac{\lambda-1}{\lambda+1}\right)^q \mathrm{d}\lambda$$

in two different ways. First, using the identity $\lambda^2 - 1 = (\lambda-\zeta)^2 + 2\zeta(\lambda-\zeta) + \zeta^2 - 1$, we get

$$\mathscr{K} = \mathscr{J}_q + 2\zeta\Theta_q(\zeta) + (\zeta^2-1)\Theta_q'(\zeta).$$

Secondly, assuming $q \geqslant 1$, integrating by parts and writing $2\lambda = 2(\lambda-\zeta) + 2\zeta$, we obtain

$$\begin{aligned}\mathscr{K} = \left[-\frac{\lambda^2-1}{\lambda-\zeta}\left(\log^+ \frac{\lambda-1}{\lambda+1}\right)^q\right]_{\lambda=-1}^{\lambda=+1} + \int_{C^+} \frac{2\lambda}{\lambda-\zeta}\left(\log^+ \frac{\lambda-1}{\lambda+1}\right)^q \mathrm{d}\lambda \\ + \int_{C^+} \left(\log^+ \frac{\lambda-1}{\lambda+1}\right)^{q-1} \frac{2q\,\mathrm{d}\lambda}{\lambda-\zeta} = 2\mathscr{J}_q + 2\zeta\Theta_q(\zeta) + 2q\Theta_{q-1}(\zeta).\end{aligned}$$

Comparison between these results yields

$$\Theta_q'(\zeta) = \frac{\mathscr{J}_q + 2q\Theta_{q-1}(\zeta)}{\zeta^2-1} \quad (q \geqslant 1). \tag{C9}$$

Step 5. Equation (C9) will now be used to prove inductively that

$$\Theta_q(\zeta) = \frac{(-2\mathrm{i}\pi)^{q+1}}{q+1}\left[B_{q+1}\left(-\frac{1}{2\mathrm{i}\pi}\log^- \frac{\zeta-1}{\zeta+1}\right) - B_{q+1}\right] \quad (\zeta \in \mathbb{C} - C^+), \tag{C10}$$

where $B_q(X)$ denotes the qth Bernoulli polynomial as above and $\log^-$ the logarithm function defined on $\mathbb{C}-\mathrm{i}\mathbb{R}^+$ by $\log^-(\rho e^{\mathrm{i}\theta}) = \ln\rho + \mathrm{i}\theta$ with $-3\pi/2 < \theta < \pi/2$.

For $q = 0$, eqn (C2) immediately yields $\Theta_0(\zeta) = \log((\zeta-1)/(\zeta+1))$, and the determination of the logarithm is readily verified to be that of the function $\log^-$ defined above. This result agrees with eqn (C10), since $B_1(X) = X - 1/2$ and $B_1 = -1/2$ [see Gradshteyn and Ryzhik (1965)]. Let eqn (C10) now be assumed to hold for $q-1$ $(q \geqslant 1)$. To prove it for q, let us call $\Xi_q(\zeta)$ the function defined by the right-hand side and evaluate $\Xi_q'(\zeta)$ using the property $B_{q+1}'(X) = (q+1)B_q(X)$ (Gradshteyn and Ryzhik, 1965, formula 9.623.3) and eqns (C8) and (C10) (for $q-1$):

$$\begin{aligned}\Xi_q'(\zeta) &= \frac{2(-2\mathrm{i}\pi)^q}{\zeta^2-1} B_q\left(-\frac{1}{2\mathrm{i}\pi}\log^- \frac{\zeta-1}{\zeta+1}\right) \\ &= \frac{2}{\zeta^2-1}[q\Theta_{q-1}(\zeta) + (-2\mathrm{i}\pi)^q B_q] = \frac{\mathscr{J}_q + 2q\Theta_{q-1}(\zeta)}{\zeta^2-1}.\end{aligned}$$

Comparison with eqn (C9) shows that $\Xi_q'(\zeta) = \Theta_q'(\zeta)$ and thus that $\Xi_q(\zeta) = \Theta_q(\zeta) + Cst$. The constant is easily identified by letting ζ tend to infinity: then $\Theta_q(\zeta) \to 0$ by eqn (C2), and $\log^-((\zeta-1)/(\zeta+1)) \to \log^-(1) = 0$ so that $\Xi_q(\zeta) \to 0$; therefore the constant is zero. It follows that $\Xi_q(\zeta) = \Theta_q(\zeta)$, which establishes eqn (C10) for q.

Step 6. The expression for the $\Theta_q(\zeta)$s now being known, it only remains to connect the $x_n(\zeta)$s with them. With trial and error, it is found that the integrals defining the $x_n(\zeta)$s can be most naturally expressed in terms of certain combinations of the $\Theta_q(\zeta)$s of the form

$$\int_{C^+} P_q^{(n)}\left(\frac{1}{2i\pi}\log^+\frac{\lambda-1}{\lambda+1}\right)\frac{d\lambda}{\lambda-\zeta}$$

and their derivatives, where the $P_q^{(n)}(X)$s are polynomials defined by eqns (48) of the text.†

We shall first establish inductively the following elementary properties of this set of polynomials:

$$P_0^{(n)}(X) = 1\,;\quad P_q^{(n)\prime}(X) = (-1)^n q P_{q-1}^{(n)}(X)\quad (q \geqslant 1). \tag{C11}$$

The proof of eqn ($C11_1$) is trivial. Equation ($C11_2$) for $n = 0$ results immediately from eqn (48_1). If it is true for $n-1$, eqn (48_2) yields

$$P_q^{(n)\prime}(X) = (-1)^q \sum_{r=1}^{q} C_q^r B_{q-r}(-1)^{n-1} r P_{r-1}^{(n-1)}(X).$$

Since $rC_q^r = qC_{q-1}^{r-1}$, this can be written, putting $s = r-1$:

$$P_q^{(n)\prime}(X) = (-1)^n q(-1)^{q-1}\sum_{s=0}^{q-1} C_{q-1}^s B_{q-1-s} P_s^{(n-1)}(X) = (-1)^n q P_{q-1}^{(n)}(X),$$

which establishes eqn ($C11_2$) for n.

Step 7. We shall now prove inductively that

$$\int_{C^+} P_q^{(n)}\left(\frac{1}{2i\pi}\log^+\frac{\lambda-1}{\lambda+1}\right)\frac{d\lambda}{\lambda-\zeta} = \frac{(-1)^n 2i\pi}{q+1}\left[P_{q+1}^{(n+1)}\left(-\frac{1}{2i\pi}\log^-\frac{\zeta-1}{\zeta+1}\right) - P_{q+1}^{(n+1)}(0)\right]\quad (\zeta \in \mathbb{C}-C^+). \tag{C12}$$

For $n = 0$, this equation reduces to eqn (C10), because $P_q^{(1)}(X) = (-1)^q B_q(X)$ by the definition of the Bernoulli polynomials. Let us assume it to be correct for $n-1$ ($n \geqslant 1$). For the index n, the integral considered is equal, by eqn (48_2), to

$$(-1)^q \sum_{r=0}^{q} C_q^r B_{q-r}\frac{(-1)^{n-1}2i\pi}{r+1}\left[P_{r+1}^{(n)}\left(-\frac{1}{2i\pi}\log^-\frac{\zeta-1}{\zeta+1}\right) - P_{r+1}^{(n)}(0)\right].$$

Since $C_q^r/(r+1) = C_{q+1}^{r+1}/(q+1)$, this quantity can be expressed, putting $s = r+1$, as

$$\frac{(-1)^n 2i\pi}{q+1}(-1)^{q+1}\sum_{s=0}^{q+1} C_{q+1}^s B_{q+1-s}\left[P_s^{(n)}\left(-\frac{1}{2i\pi}\log^-\frac{\zeta-1}{\zeta+1}\right) - P_s^{(n)}(0)\right]$$

(the term ($s = 0$) in the above sum being zero by eqn ($C11_1$)), which is identical to the right-hand of eqn (C12) by eqn (48_2).

Differentiating eqn (C12) with respect to ζ using eqn ($C11_2$), we also obtain

$$\int_{C^+} P_q^{(n)}\left(\frac{1}{2i\pi}\log^+\frac{\lambda-1}{\lambda+1}\right)\frac{d\lambda}{(\lambda-\zeta)^2} = \frac{2}{\zeta^2-1}P_q^{(n+1)}\left(-\frac{1}{2i\pi}\log^-\frac{\zeta-1}{\zeta+1}\right)\quad (\zeta \in \mathbb{C}-C^+). \tag{C13}$$

Step 8. The trickiest results are now established; the two remaining steps are somewhat tedious but straightforward. First we calculate an integral analogous to that in eqn (C12) but with $(\lambda^2-1)/((\lambda-m)(\lambda-\zeta)^2)$ instead of $1/(\lambda-\zeta)$. This is easily done be decomposing this rational function into partial fractions and using eqns (C12) and (C13):

$$\frac{\lambda^2-1}{(\lambda-m)(\lambda-\zeta)^2} = \frac{\dfrac{m^2-1}{(\zeta-m)^2}}{\lambda-m} + \frac{1+\dfrac{1-m^2}{(\zeta-m)^2}}{\lambda-\zeta} + \frac{\dfrac{\zeta^2-1}{\zeta-m}}{(\lambda-\zeta)^2} \Rightarrow \int_{C^+}\frac{\lambda^2-1}{(\lambda-m)(\lambda-\zeta)^2}P_q^{(n)}\left(\frac{1}{2i\pi}\log^+\frac{\lambda-1}{\lambda+1}\right)d\lambda$$

$$\begin{aligned} &= \frac{(-1)^n 2i\pi}{q+1}\left\{\left[1+\frac{1-m^2}{(\zeta-m)^2}\right]P_{q+1}^{(n+1)}\left(-\frac{1}{2i\pi}\log^-\frac{\zeta-1}{\zeta+1}\right)\right. \\ &\quad \left.+\frac{m^2-1}{(\zeta-m)^2}P_{q+1}^{(n+1)}\left(-\frac{1}{2i\pi}\log^-\frac{m-1}{m+1}\right) - P_{q+1}^{(n+1)}(0)\right\} \\ &\quad + \frac{2}{\zeta-m}P_q^{(n+1)}\left(-\frac{1}{2i\pi}\log^-\frac{\zeta-1}{\zeta+1}\right)\quad (\zeta \notin C^+, \zeta \neq m). \end{aligned} \tag{C14}$$

† These polynomials offer strong similarities with the so-called *generalized Bernoulli polynomials* $B_q^{(n)}(X)$ [see for instance Fletcher *et al.* (1962), p. 69], which verify the same induction formula (48_2) except for the $(-1)^q$ factor. In fact these sets of polynomials can be shown to be tied by the relations $P_q^{(n)}(X) = B_q^{(n)}(X+n/2)$ for n even, $B_q^{(n)}[(n+1)/2-X]$ for n odd.

Differentiating this equation p times with respect to m and dividing by $p!$, we also get

$$\int_{C^+} \frac{\lambda^2-1}{(\lambda-m)^{p+1}(\lambda-\zeta)^2} P_q^{(n)}\left(\frac{1}{2i\pi}\log^+\frac{\lambda-1}{\lambda+1}\right) d\lambda$$

$$= \frac{(-1)^n 2i\pi}{q+1}\left\{\left[\frac{(p+1)(1-m^2)}{(\zeta-m)^{p+2}} - \frac{2mp}{(\zeta-m)^{p+1}} + \frac{1-p}{(\zeta-m)^p}\right] P_{q+1}^{(n+1)}\left(-\frac{1}{2i\pi}\log^-\frac{\zeta-1}{\zeta+1}\right)\right.$$

$$\left. + \sum_{r=0}^{p} \frac{r+1}{(p-r)!(\zeta-m)^{r+2}} \frac{d^{p-r}}{dm^{p-r}}\left[(m^2-1)P_{q+1}^{(n+1)}\left(-\frac{1}{2i\pi}\log^-\frac{m-1}{m+1}\right)\right]\right\}$$

$$+ \frac{2}{(\zeta-m)^{p+1}} P_q^{(n+1)}\left(-\frac{1}{2i\pi}\log^-\frac{\zeta-1}{\zeta+1}\right) \quad (p \geqslant 1, \zeta \notin C^+, \zeta \neq m). \tag{C15}$$

Step 9. We shall now use eqns (C14) and (C15) to prove inductively that the functions $x_n(\zeta)$ admit the following expression:

$$x_n(\zeta) = \sum_{p=0}^{2n} \sum_{q=0}^{n} \frac{p!a_{pq}^{(n)}}{(\zeta-m)^p} P_q^{(n)}\left(-\frac{1}{2i\pi}\log^-\frac{\zeta-1}{\zeta+1}\right) \quad (\zeta \notin C^+, \zeta \neq m), \tag{C16}$$

where the $a_{pq}^{(n)}$s are coefficients which depend on m. This equation is obviously true for $n = 0$ with $a_{00}^{(0)} = 1$, which is identical to eqn (49). Let it be assumed to hold for $n-1$ ($n \geqslant 1$). One then gets, using eqn ($C1_2$):

$$x_n(\zeta) = \sum_{p=0}^{2n-2} \sum_{q=0}^{n-1} p!\overline{a_{pq}^{(n-1)}} \int_{C^+} \frac{\lambda^2-1}{(\lambda-m)^{p+1}(\lambda-\zeta)^2} \overline{P_q^{(n-1)}\left(-\frac{1}{2i\pi}\log^-\frac{\bar{\lambda}-1}{\bar{\lambda}+1}\right)} d\lambda.$$

Now the coefficients of the polynomial $P_q^{(n-1)}(X)$ being real,

$$\overline{P_q^{(n-1)}\left(-\frac{1}{2i\pi}\log^-\frac{\bar{\lambda}-1}{\bar{\lambda}+1}\right)} = P_q^{(n-1)}\left(\frac{1}{2i\pi}\overline{\log^-\frac{\bar{\lambda}-1}{\bar{\lambda}+1}}\right),$$

and it is easy to check that $\overline{\log^-((\bar{\lambda}-1)/(\bar{\lambda}+1))} = \log^+((\lambda-1)/(\lambda+1))$ for $\lambda \in C^+$. Therefore the above integrals are precisely of the form of those on the left-hand sides of eqns (C14) and (C15). Using these equations, one finds that $x_n(\zeta)$ admits an expression of the type (C16), the coefficients $a_{pq}^{(n)}$ being given in terms of the $a_{pq}^{(n-1)}$s by eqns (50)–(54).

Since eqn (C16) does not apply for $\zeta = m$, it remains to derive the expression for $x_n(m) \equiv x_n$. This is easily done by expanding eqn (C16) in powers of $\zeta-m$, which is allowable for sufficiently small values of this quantity:

$$x_n(\zeta) = \sum_{p=0}^{2n} \sum_{q=0}^{n} \frac{p!a_{pq}^{(n)}}{(\zeta-m)^p} \sum_{r=0}^{+\infty} \frac{1}{r!} \frac{d^r}{dm^r}\left[P_q^{(n)}\left(-\frac{1}{2i\pi}\log^-\frac{m-1}{m+1}\right)\right] \cdot (\zeta-m)^r.$$

If one takes the limit $\zeta \to m$, the divergent terms ($r < p$) must be zero since $x_n(\zeta)$ is regular at $\zeta = m$ (this is obvious from eqn ($C1_2$)), the terms ($r > p$) vanish, and one therefore gets eqn (47). This concludes the proof.

With regard to the functions G_p (eqns (55)–(65)), one is led to introducing another set of functions $y_n(\zeta)$ which verify the same induction formula ($C1_2$) as the $x_n(\zeta)$s, but with $y_0(\zeta) = \zeta+m$ instead of 1. These functions can be expressed under a form similar to eqns (C16) and (47), but with p now ranging from -1 (instead of 0) to n in the expression of $y_n(\zeta)$, $\zeta \neq m$:

$$y_n(\zeta) = \sum_{p=-1}^{2n} \sum_{q=0}^{n} \frac{p!b_{pq}^{(n)}}{(\zeta-m)^p} P_q^{(n)}\left(-\frac{1}{2i\pi}\log^-\frac{\zeta-1}{\zeta+1}\right) \quad (\zeta \notin C^+, \zeta \neq m)$$

(where $(-1)! \equiv 1$ by definition);

$$y_n \equiv y_n(m) = \sum_{p=0}^{2n} \sum_{q=0}^{n} b_{pq}^{(n)} \frac{d^p}{dm^p}\left[P_q^{(n)}\left(-\frac{1}{2i\pi}\log^-\frac{m-1}{m+1}\right)\right],$$

where the $b_{pq}^{(n)}$s verify eqns (58)–(65). The integrals needed in the proof include those above plus a new one,

$$\int_{C^+} P_q^{(n)}\left(\frac{1}{2i\pi}\log^+\frac{\lambda-1}{\lambda+1}\right) d\lambda,$$

which is easily evaluated by letting ζ tend to infinity in eqn (C12) and identifying terms of order ζ^{-1}.

APPENDIX D: COMPARISON WITH SUMI'S RESULTS FOR THE NON-UNIVERSAL TERM Z_p AND THE FUNCTIONS $I_{pq}(m)$ AND $L_{pq}(m)$

In addition to the results mentioned in Sections A.3 and B.5, Sumi (1991) obtained the following formulae for the non-universal quantities Z_p involved in eqn (5):

$$
\begin{aligned}
Z_1 &= \left[\left(1-\frac{7\pi^2 m^2}{8}\right)\bar{k}_{11} - \pi m\bar{k}_{12} - \frac{3\pi m}{2}\bar{k}_{21} + \frac{3\pi^2 m^2}{2}\bar{k}_{22}\right]k_1 \\
&\quad + \left[-\pi m\bar{k}_{11} + \left(1-\frac{5\pi^2 m^2}{8}\right)\bar{k}_{12} + \frac{3\pi^2 m^2}{2}\bar{k}_{21} - \frac{3\pi m}{2}\bar{k}_{22}\right]k_2; \\
Z_2 &= \left[\frac{\pi m}{2}\bar{k}_{11} - \frac{\pi^2 m^2}{2}\bar{k}_{12} + \left(1-\frac{11\pi^2 m^2}{8}\right)\bar{k}_{21} - \pi m\bar{k}_{22}\right]k_1 \\
&\quad + \left[-\frac{\pi^2 m^2}{2}\bar{k}_{11} + \frac{\pi m}{2}\bar{k}_{12} - \pi m\bar{k}_{21} + \left(1-\frac{9\pi^2 m^2}{8}\right)\bar{k}_{22}\right]k_2. \qquad \text{(D1)}
\end{aligned}
$$

In these equations the $\bar{k}_{pq}$s are coefficients which depend on the *whole* geometry of the body and the initial crack and on which portions of the boundary have tractions versus displacements prescribed (they are *non-universal* quantities), but are independent of the geometric parameters m, a^* and C^* of the crack extension and also of the loading. Sumi also derived the following low order expressions of the universal functions $I_{pq}(m)$ and $L_{pq}(m)$:

$$
\begin{aligned}
&I_{11}(m) = \frac{1}{2} + \frac{5\pi^2 m^2}{16} + 0(m^4); \quad I_{12}(m) = -\frac{5\pi m}{4} + 0(m^3); \\
&I_{21}(m) = -\frac{\pi m}{4} + 0(m^3); \quad I_{22}(m) = \frac{1}{2} - \frac{31\pi^2 m^2}{16} + 0(m^4); \qquad \text{(D2)} \\
&L_{11}(m) = -\frac{27}{32} + 0(m^2); \quad L_{12}(m) = 0(m); \\
&L_{21}(m) = 0(m); \quad L_{22}(m) = -\frac{63}{32} + \frac{2}{\pi} + 0(m^2). \qquad \text{(D3)}
\end{aligned}
$$

With regard to the Z_ps, the following formula (in matrix notation) was derived by Leblond (1989) from the Bueckner–Rice weight function theory:

$$
[Z] = [F(m)][\bar{k}][F(m)]^{\mathrm{T}}[F(m)][k], \qquad \text{(D4)}
$$

where $[X]^{\mathrm{T}}$ denotes the transpose of $[X]$. Inserting the second order expressions of the F_{pq}s derived by Wu (1979) and confirmed here [see eqns (66)] into this formula, one obtains

$$
\begin{aligned}
Z_1 &= \left\{\left[1-\frac{7\pi^2 m^2}{8}\right]\bar{k}_{11} - \pi m\bar{k}_{12} - \frac{3\pi m}{2}\bar{k}_{21} + \frac{3\pi^2 m^2}{2}\bar{k}_{22}\right\}k_1 \\
&\quad + \left\{-\pi m\bar{k}_{11} + \left[1+\left(-8+\frac{9\pi^2}{8}\right)m^2\right]\bar{k}_{12} + \frac{3\pi^2 m^2}{2}\bar{k}_{21} - \frac{3\pi m}{2}\bar{k}_{22}\right\}k_2; \\
Z_2 &= \left\{\frac{\pi m}{2}\bar{k}_{11} - \frac{\pi^2 m^2}{2}\bar{k}_{12} + \left[1-\left(4+\frac{7\pi^2}{8}\right)m^2\right]\bar{k}_{21} - \pi m\bar{k}_{22}\right\}k_1 \\
&\quad + \left\{-\frac{\pi^2 m^2}{2}\bar{k}_{11} + \frac{\pi m}{2}\bar{k}_{12} - \pi m\bar{k}_{21} + \left[1+\left(-12+\frac{9\pi^2}{8}\right)m^2\right]\bar{k}_{22}\right\}k_2. \qquad \text{(D5)}
\end{aligned}
$$

The term proportional to $\bar{k}_{12}k_2$ in Z_1, and those proportional to $\bar{k}_{21}k_1$ and $\bar{k}_{22}k_2$ in Z_2, are different from those in eqns (D1). These discrepancies are not very surprising since eqn (D4) involves the function F_{22} and Sumi's result for this function differs from that given by Wu (1979) and confirmed here (see Section A.3). It must be noted however that the differences cannot be completely explained in that way; indeed, inserting Sumi's value for F_{22} into eqn (D4), one still does not obtain eqns (D1). (In other words, Sumi's results are incompatible with the Bueckner–Rice weight function theory, which is the basis of eqn (D4)).

It conspicuously appears in eqn (5) that the functions $I_{pq}(m)$ relate to the value of the $k_p^{(1)}$s for a *straight extension* of a *straight initial crack* ($C = a^* = C^* = 0$). One way to determine their second order expressions is to pursue the expansion in powers of s presented in Section A.2 up to order s, and to expand the resulting formulae up to the second order with respect to m. In that way one derives the values of the quantities $Z_p + I_{pq}(m)b_q$. It remains to subtract the Z_ps. This can be done by using eqns (D5), provided the values of the non-universal coefficients $\bar{k}_{pq}$ are known for the case under study of a straight crack in an infinite body loaded by uniform forces at infinity. To derive these values, one can use the fact that the $\bar{k}_{pq}$s are connected with the derivative of the stress field with respect to the length of the crack, when the latter is extended in the direction of its tangent [see Leblond (1989)]; it is easy to obtain this derivative in the case considered since in the absence of a kink, the solution is quite classical. The values of the $I_{pq}(m)$s derived in this manner are identical to those given by eqns (D2), except for the value of $I_{22}(m)$ which reads

$$
I_{22}(m) = \frac{1}{2} + \left(-22 + \frac{5\pi^2}{16}\right)m^2 + 0(m^4).
$$

Finally, the zeroth order expressions of the $L_{pq}(m)$s can be obtained by putting $\pi m = 0$ and extending the perturbative analysis of Section B.2 up to the second order with respect to a^*. The results are the same as those of Sumi (eqns (D3)) except for the expression of $L_{22}(m)$, which is found to be

$$L_{22}(m) = -\frac{55}{32} + 0(m^2).$$

Once again, the reason for all these discrepancies lies in Sumi's use of the Banichuk–Cotterell–Rice perturbative procedure for dealing with a non-$\mathscr{C}^\infty$ crack.