Fibonacci and Lucas numbers and socially distanced seatings

With the COVID-19 pandemic, every person suddenly learned the meaning of "social distancing." This project explores how people can sit in chairs, if there must be at least one empty chair between any two chairs with people. We'll call such a seating arrangment a "socially distanced seating arrangment" or "SDSA".

We will think about two types of seating arrangements. First, imagine a long line of seats, such as at a row of computer monitors facing a wall. The second type of seating is in a circle, like in a literature class where you are going to discuss the latest novel.

1. Suppose you have a line of n chairs. In how many ways can people occupy the chairs in an SDSA? For example, if there are 3 chairs, the occupied chairs could be 13, 1, 2, 3, or $\emptyset$. Thus there are 5 SDSA in a line of 3 chairs. Figure 1a (next page) shows the 5 SDSA for $n = 3$. Let $s_n =$ the number of SDSA in a line of n chairs.

   (a) Enumerate the SDSA for 4 chairs and for 5 chairs, and find the values of s_4 and s_5.

   (b) Explain why $s_n = s_{n-1} + s_{n-2}$. (Hint: think about whether the last chair is occupied.)

   (c) Show that s_n is a Fibonacci number. Which Fibonacci number is it?

2. Suppose you have a circle of n chairs. Let c_n be the number of SDSA in a circle of n chairs.

   (a) Enumerate the SDSA for a circle of 3 chairs and for a circle of 4 chairs. Find the values of c_3 and c_4. Figure ab shows that $c_5 = 11$.

   (b) Explain why $c_n = F_{n-2} + F_n$. (Hint: You might want to start by expressing c_n in terms of s_n.)

   (c) The Lucas (Loo-CAH) numbers are defined by $L_1 = 2, L_2 = 1$ and $L_n = L_{n-1} + L_{n-2}$. Notice that $L_2 = F_0 + F_2$ and $L_3 = F_1 + F_3$. In fact, Theorem 4.2.9 in Mazur states that $L_n = F_{n-2} + F_n$ for all $n \geq 2$. Thus the number of SDSA in n chairs is the nth Lucas number!

3. Use OGFs and the recurrence relation $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ to find a formula for L_n similar to Theorem 4.2.10.

The journal *Fibonacci Quarterly* has many articles about Fibonacci and Lucas numbers. There is also a particularly interesting article by Benjamin and Quinn about combinatorial proofs of Fibonacci identities:

A. Benjamin and J. Quinn. Recounting Fibonacci and Lucas identities. *The College Mathematics Journal*, 30(5): 359:366, 1999.

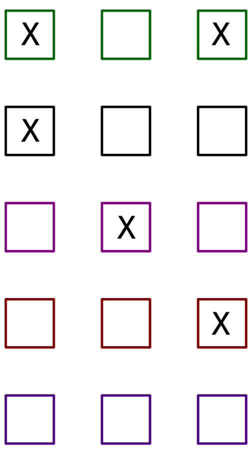

Figure 1a

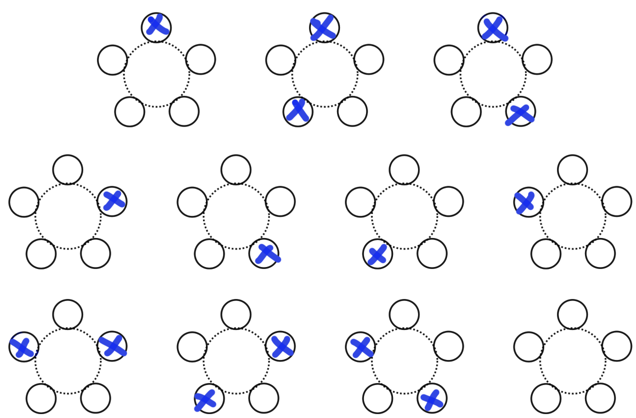

Figure 1b

Figure 1: $s_3 = 5$ and $c_5 = 11$