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MATH 4750 HW 5

Due **Oct 17** at the start of class. **Answers without justification will receive 0 points.**

1. Draw your adopted graph.
   (a) Is your graph Eulerian? Explain how you know.
   (b) Is your graph Hamiltonian? If so, highlight a Hamiltonian cycle in the drawing of your graph. If not, explain why not.

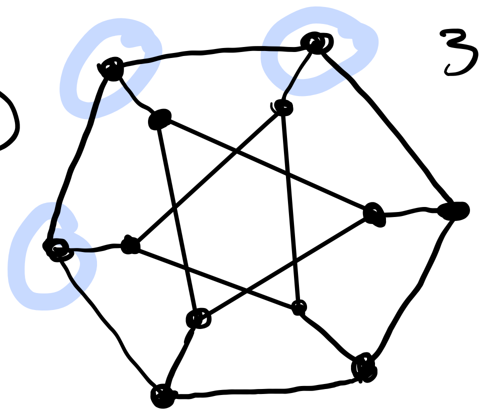
2. Suppose G is a 3-regular graph of order 12 and H is a 4-regular graph of order 11.
   (a) Does $G + H$ have an Eulerian circuit? Why or why not? 3+12, 4+'' same Eulerian Thm
   (b) Does $G + H$ have a Hamiltonian cycle? Why or why not?

3. Let G_1 and G_2 be two Eulerian graphs with no vertex in common. Let v_1 be a vertex in G_1 and let v_2 be a vertex in G_2. Let G be the graph obtained from $G_1 \cup G_2$ by adding the edge v_1v_2.
   (a) Does G have an Eulerian circuit? Why or why not?
   (b) Does G have an Eulerian trail? Why or why not?

4. Prove that the graph $\overline{C_n}$ is Hamiltonian for all $n \geq 5$.

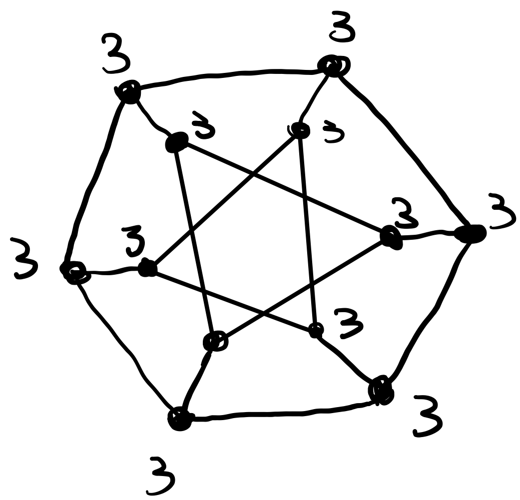
5. Let G be an r-regular graph that is connected but not Eulerian. Prove that if $\overline{G}$ is connected, then $\overline{G}$ is Eulerian.

1\. a) odd degs 3-reg



$\therefore$ Not Eulerian bc more than 2 odd degrees.

b) 3-reg



$$\min \delta(G) \geq n/2$$

$$3 \geq 10/2$$

$\therefore$ Not Hamiltonian since $\min \delta(G) < n/2$.

2. a) $G+H$ 3-reg 4-reg

$3+11 = 14$ deg

$4+12 = 16$ deg

Since all degrees are now even, the graph is Eulerian.

b) min $\deg(G) = 14$

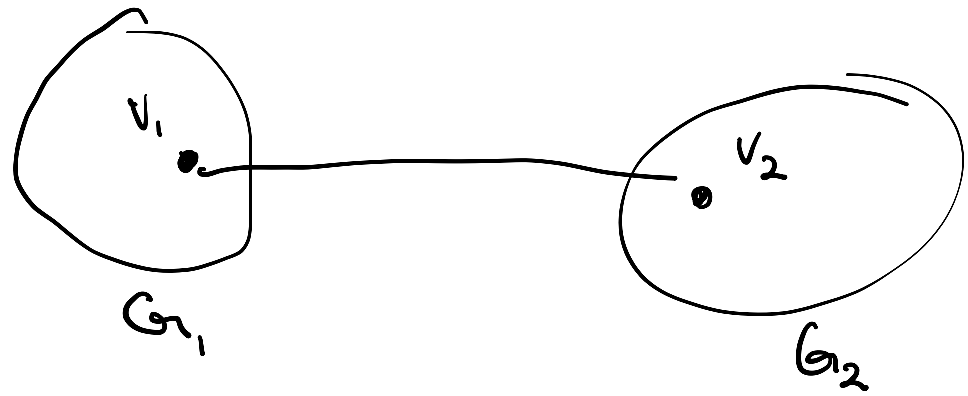
$n = 12+11$
$= 23$

$14 \overset{?}{\geq} n/2$

$14 \geq 23/2$

$\therefore$ Yes the join graph is also Hamiltonian.

3. circuits a) & b)



G_1 and G_2 have all even degrees since they are both Eulerian. When we add an edge between v_1 and v_2, v_1 and v_2 now both have an odd degree. Therefore, the new graph G has no Eulerian circuit, but has an Eulerian trail since there is exactly 2 odd vertices. $\square$

4. Prove: $\bar{C}_n$ is Hamiltonian for all $n \geq 5$.

Proof:

$\bar{C}_n$ is regular w/ degrees equal to degree in K_n - degree in C_n. Since the degree in K_n is $n-1$ and the degree in C_n is 2 (since every cycle is a 2-reg graph), we have the degree of $\bar{C}_n = n-1-2$, which is $n-3$. By Dirac's theorem,

$$n-3 \geq n/2$$

$$2n-6 \geq n$$

$$n \geq 6.$$

Note that when $n=5$, $\bar{C}_5 \cong C_5$, and C_5 is Hamiltonian, so $\bar{C}_5$ is Hamiltonian. Thus by Dirac's theorem, $6 \geq 5$ so $\bar{C}_n$ is Hamiltonian for all $n \geq 5$. $\square$

* and it is connected

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5. Let G be an r-regular graph that is connected but not Eulerian. Prove that if $\overline{G}$ is connected, then $\overline{G}$ is Eulerian.

all deg same

no circuit

allow repeated vxs

An **Eulerian circuit** in a graph (or multigraph) is a walk that

- traverses each edge exactly once, and
- begins and ends at the *same* vertex.

* circuit iff all vxs even deg

Proof:

Since G is not Eulerian, by Eulerian circuit Thm, not all vxs are even. Since this is an r-reg graph, all vxs are odd with r degree. Our $\deg(\overline{G}) = \deg(K) - \deg(G)$, so we have $\deg(\overline{G}) = n-1-r$. We now can consider 2 cases, when n is odd and when n is even. When n is odd, then $n-1$ is even, so $n-1-r$ is even-odd=odd. However, by handshaking lemma, we can't have an odd regular graph with an odd number of vertices. So when n is even, then $n-1$ is odd, and odd-odd=even. So by E-circuit Thm, since $\overline{G}$ is connected w/ all even degrees, $\overline{G}$ is Eulerian. $\square$