Mary Robinson

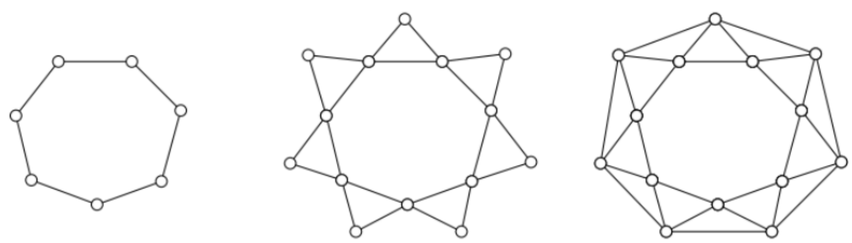
MATH 4750 HW 8

Due **Nov 7** at the start of class. **Answers without justification will receive 0 points.**

✓ 1. Draw your adopted graph. Find a minimal proper vertex coloring of your graph. Explain why the coloring you found uses as few colors as possible.

✓ 2. In class, we saw that the chromatic numbers of cycles have a pattern depending on whether the cycle was even or odd. For this problem, you will investigate patterns in the chromatic numbers for another family of graphs. For $n \geq 3$, let A_n be the graph constructed as follows: (1) Start by drawing a cycle C_n. (2) Next, build a triangle off of each edge of the cycle. (3) Finish by drawing a cycle connecting the points of the triangles.

Example: Step-by-step of how to draw the graph A_7.



A_7

(a) Fill in the blanks: Since A_n contains a triangle, $\chi(A_n) \geq$ 3 .
Since A_n is planar, $\chi(A_n) \leq$ 4 .

(b) Draw the graphs A_n for $3 \leq n \leq 9$ and determine the chromatic number for each. Be sure to exhibit a minimal proper vertex coloring and explain why the coloring you found uses as few colors as possible.

(c) Make a conjecture for the chromatic number of A_n in general: If n is divisible by 3 , then $\chi(A_n) =$ 3. If n is not divisible by 3, then $\chi(A_n) =$ 4.

3. The goal of this problem is to investigate relationships between the chromatic number of the join of two graphs and the chromatic numbers of the individual graphs.

(a) Draw $C_3 + C_4$ and find a minimal proper vertex coloring. Explain why the coloring you found uses as few colors as possible.

(b) Find the chromatic number of the graph $C_p + C_q$. Explain your answer. (Hint: Consider cases depending on the parities of p and q.)

4. Give an example of the following or explain why no such example exists.

Δ = max degrees

(a) a graph G with $\Delta(G) = 2\chi(G)$

(b) a graph G with $\chi(G) = 2\Delta(G)$

(c) a noncomplete graph of order n with chromatic number n

5. The Math Department at Complex U. has eight faculty members, each of whom is teaching three courses this semester. The course numbers are 127, 131, 132, 136, 138, 153, 154, 201, 205, 211. The schedules are:

Agnesi 132, 136, 211

Bernoulli 127, 131, 153

Cauchy 131, 132, 211

Descartes 127, 131, 205

Euler 131, 138, 154

Frobenius 132, 136, 201

Gauss 127, 131, 138

Hamilton 153, 154, 205

It has been decided that each course will have a department-wide final exam such that all sections of a given course will have their finals at the same time. Each professor must proctor their own exam. There is plenty of classroom space available, but everyone would like to get the exams over as soon as possible so they can rush off to winter break and prove lots of new theorems. What is the fewest number of time slots that can be used to give the exams?

✓ (a) To set up this problem as a vertex coloring problem, we can let...

- Vertices represent ____
- Adjacent vertices represent ____
- Colors of vertices represent ____
- The chromatic number represents ____

✓ (b) Draw an appropriate graph and use vertex coloring to find an exam schedule that requires the least number of time slots. Be sure to justify how you know you have found the minimum.

(c) What is the maximum number of courses that can hold their exam at the same time?

independence #, include / exclude

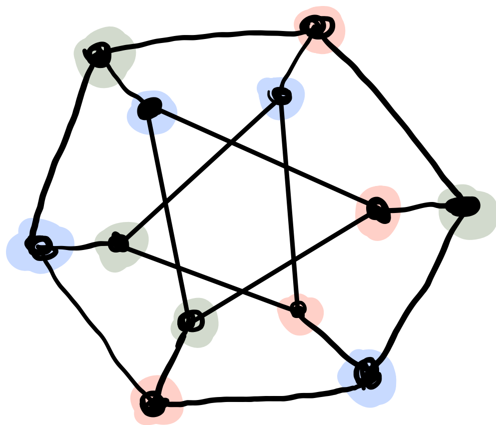
for lower do Ex

* for α! use include/exclude for upper

* for χ use min chromatic foreach G for upper

for lower use ex

1) $\chi(G) \geq 3$



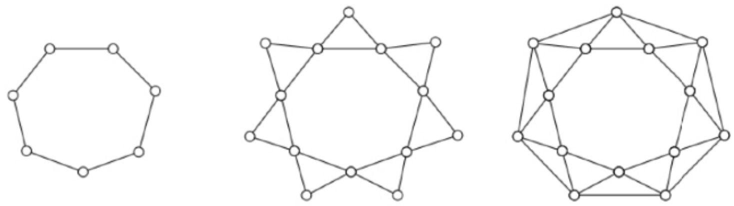
Since there's a subgraph of K_3.

2.

2. In class, we saw that the chromatic numbers of cycles have a pattern depending on whether the cycle was even or odd. For this problem, you will investigate patterns in the chromatic numbers for another family of graphs. For $n \geq 3$, let A_n be the graph constructed as follows: (1) Start by drawing a cycle C_n. (2) Next, build a triangle off of each edge of the cycle. (3) Finish by drawing a cycle connecting the points of the triangles.

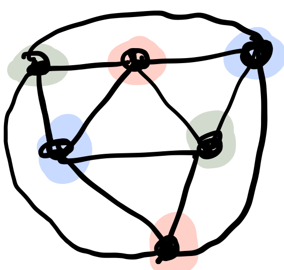
a & c at 1st page

Example: Step-by-step of how to draw the graph A_7.

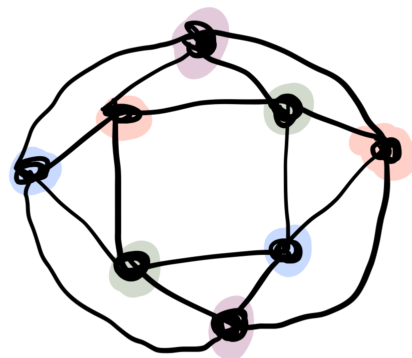


A_7

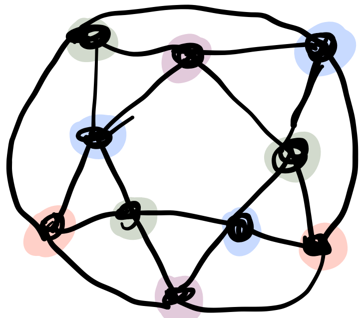
b)



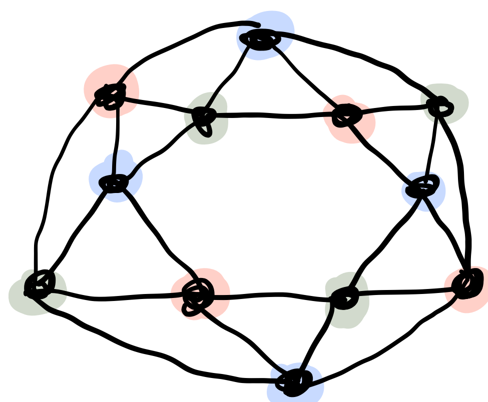
A_3



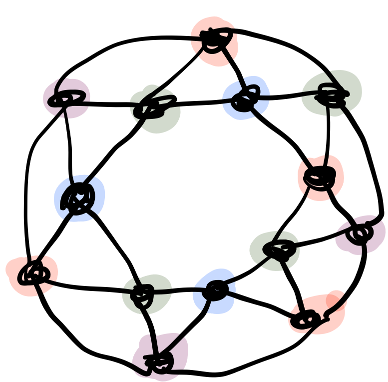
A_4



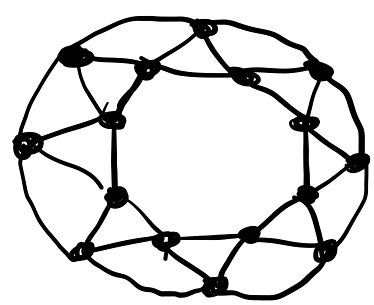
A_5



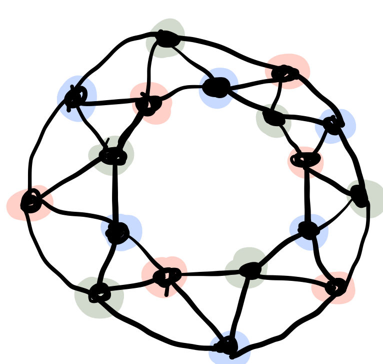
A_6



A_7



A_8



A_9

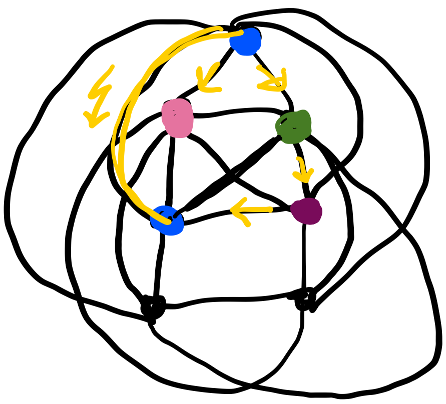
3. The goal of this problem is to investigate relationships between the chromatic number of the join of two graphs and the chromatic numbers of the individual graphs.

(a) Draw $C_3 + C_4$ and find a minimal proper vertex coloring. Explain why the coloring you found uses as few colors as possible.

(b) Find the chromatic number of the graph $C_p + C_q$. Explain your answer. (Hint: Consider cases depending on the parities of p and q.)

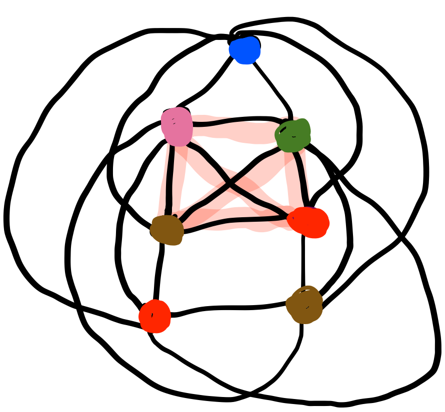
upper: min chromatic # for each C

a) Use an example of → ← for lower bound



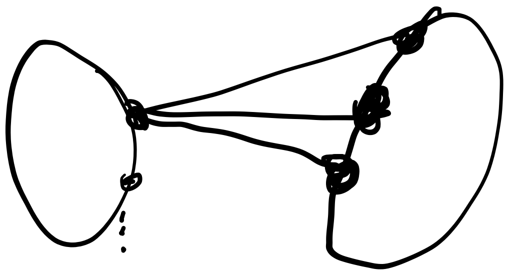
lower bound: Suppose theres a proper 4 coloring (w/ yellow for contradiction). Thus we can't have a proper 4 coloring.

Thus our graph has at least a proper 5 coloring, which an example exists for a proper 5 coloring. Hence $\chi(G) \geq 5$.



Example of a proper 5 coloring.

b) Scratch:



C_p $\quad$ C_q

Bc def of join is all vxs in C_p connected, therefore the proper coloring has to be ~~at least the order of G.~~

Only p chromatic #'s allowed

Only q chromatic #'s allowed

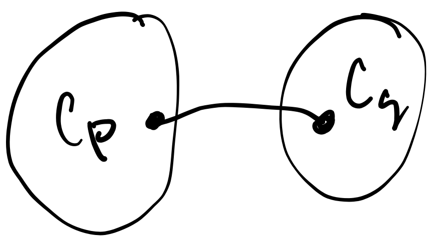
So p+q colors in graph.

Proof: Suppose → ↯

There exists a perfect coloring of $C_p + C_q$ w/ $< \chi(C_p) + \chi(C_q)$. To color C_p, we need $\geq \chi(C_p)$ colors by definition of χ, & we need $\geq \chi(C_q)$ for C_q. But we have total $< \chi(C_p) + \chi(C_q)$. So there exists ≥ 1 color used on both C_p & C_q. That is, there exists more than 1 vx in C_p w/ same colors as ≥ 1 vertex in C_q. But by def of join, those 2 vertices are adj. So this is a monochromatic edge, & the coloring is not proper, thus a ↯.

Now to prove our upper bound:

Fix a <u>Proper</u> $\chi(C_p)$-coloring of C_p w/ colors $\{1, \ldots, \chi(C_p)\}$ & a proper coloring $\chi(C_q)$-coloring w/ colors $\{\chi(C_p)+1, \ldots, \chi(C_p)+\chi(C_q)\}$.



Color $C_p + C_q$ w/ those 2 colorings on C_p and C_q, which means there's no overlap. So by construction, there does not exist a monochromatic edge within C_p or in C_q. $\square$

4 4. Give an example of the following or explain why no such example exists.

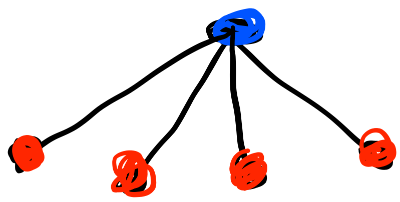
Δ = max degrees

(a) a graph G with $\Delta(G) = 2\chi(G)$

(b) a graph G with $\chi(G) = 2\Delta(G)$

(c) a noncomplete graph of order n with chromatic number n

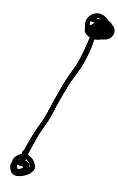
a) Yes! Bipartite



$\Delta(G) = 4$

$\Delta(G) = 2\chi(G)$

b) Yes



$2\Delta(G) = \chi(G)$

c) DNE

G n-vx graph $\neq K_n$ w/ $\chi(G) = n$.
Then $\exists \geq 2$ vxs which are non adjacent. So the coloring which gives $\forall$ vx its own color except those 2 vxs which share a color, is a proper coloring w/ only $n-1$ colors.
Hence $\chi(G) \leq n-1$. $\square$

5.

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Scratch.

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meaning if multiple have 131, all have to be there

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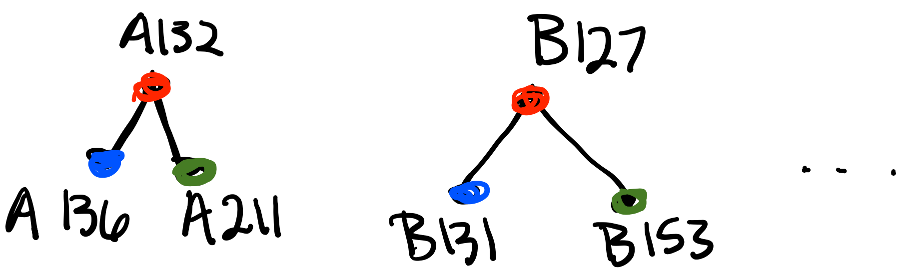
1.



Where

VXS = Specific class w/ Prof.

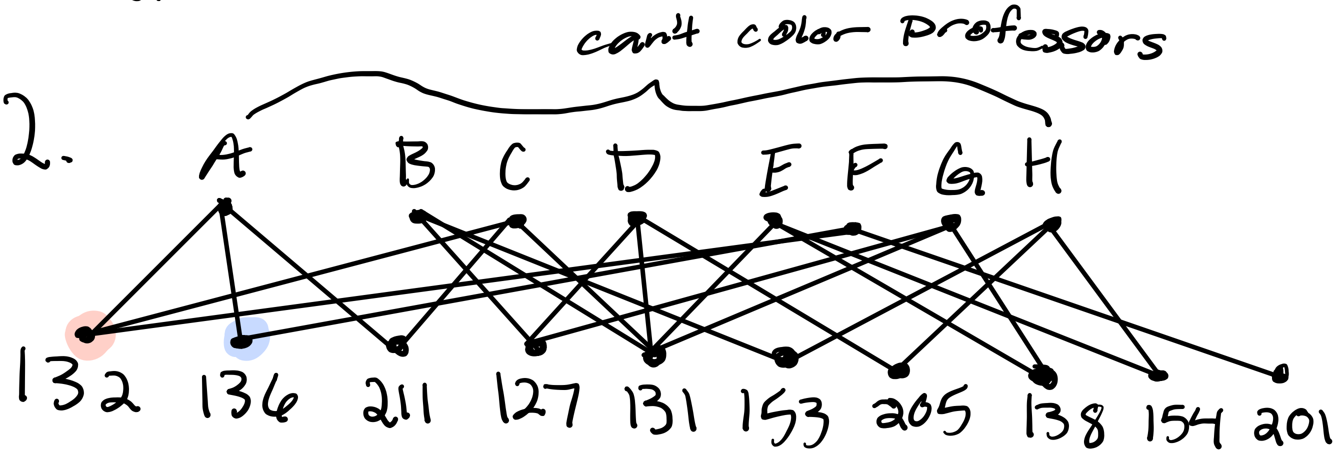
edges = classes that can't be same time

colors = time slots

$\chi(G)$ = fewest time slots

✓ or ~

2.



bipartite?

Where

VXS = Professor/classes

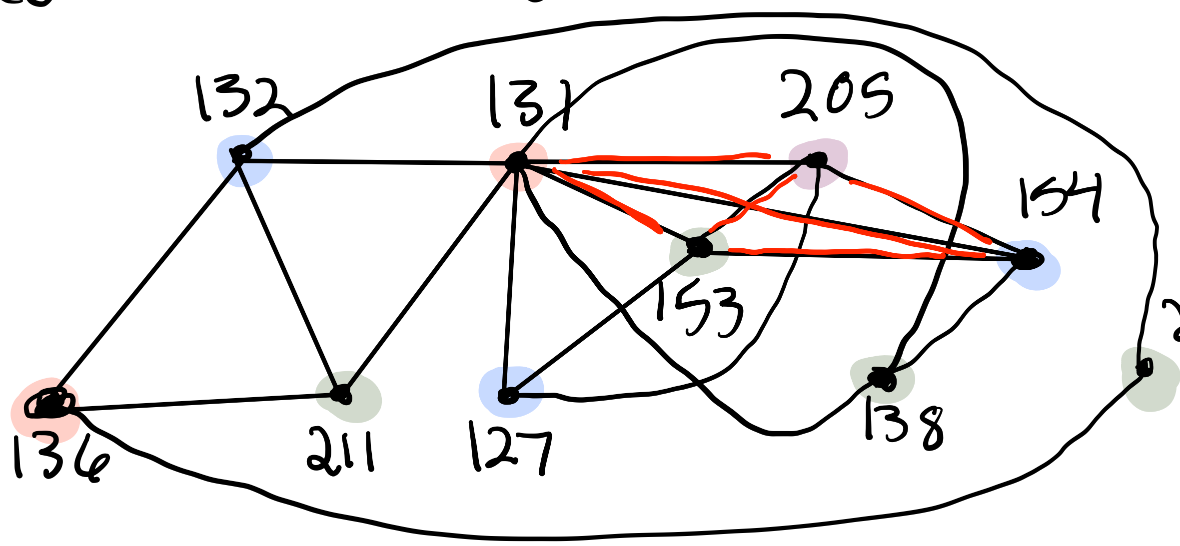
edges = professor's classes

colors = time slots?

$\chi(G)$ = fewest time slots

a) vxs = classes
edges = professors
colors = time slots
$\chi(G)$ = fewest time slots

b)



K_4!

Agnesi 132, 136, 211 ✓
Bernoulli 127, 131, 153 ✓
Cauchy 131, 132, 211 ✓
Descartes 127, 131, 205 ✓
Euler 131, 138, 154 ✓
Frobenius 132, 136, 201 ✓
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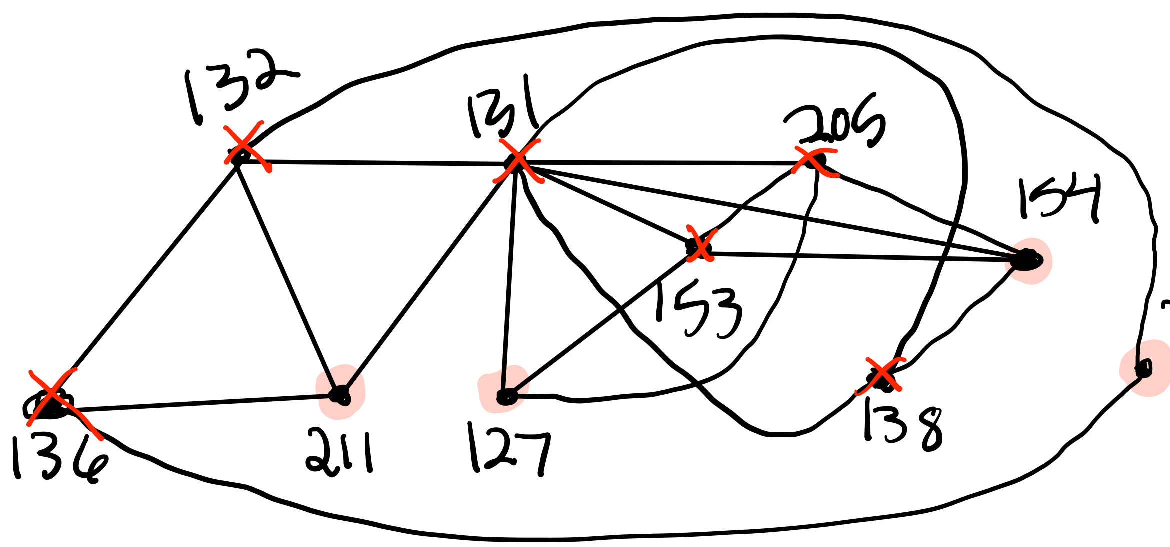
$\chi(G) \geq 4$

c) include/exclude:



max independent #: $\{211, 127, 154, 201\}$

$\alpha(G) \leq 4$