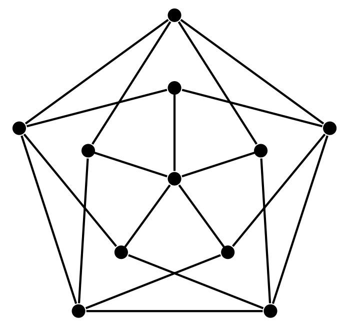
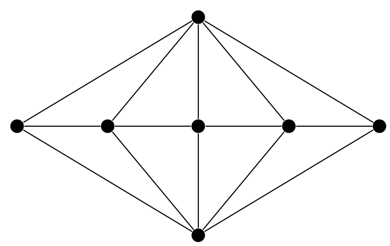
MATH 4750 HW 9

Due **Nov 14** at the start of class. **Answers without justification will receive 0 points.**

1. Prove that the chromatic index of the graph below is $\chi' = 5$.



2. Determine the chromatic number χ and the chromatic index χ' of the graph below.



3. Prove that if G is an r-regular graph of odd order, then $\chi'(G) = r + 1$.

4. In this problem, we will discover the chromatic index of the Cartesian product $G \Box K_2$ for a nonempty graph G.

   (a) For $3 \leq n \leq 7$, give a plane drawing of each of the graphs $C_n \Box K_2$. Find a minimal proper edge coloring of each.

   (b) Make a conjecture: The edge coloring number of $C_n \Box K_2$ is ______

   (c) Recall that the chromatic index of C_n depends on whether n is even or odd. Does your conjecture for $\chi'(C_n \Box K_2)$ depend on whether n is even or odd?

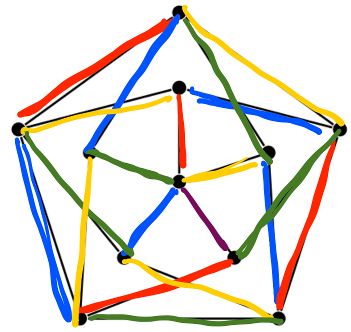
   (d) Prove that for every nonempty graph G, the graph $G \Box K_2$ will have chromatic index $1 + \Delta(G)$.

5. Seven softball teams from Atlanta, Boston, Chicago, Denver, Louisville, Miami, and Nashville have been invited to participate in tournaments, where each team is scheduled to play a certain number of the other teams (given below). No team is allowed to play more than one game each day. Set up a schedule of games over the smallest number of days possible.

   Atlanta: Boston, Chicago, Miami, Nashville
   Boston: Atlanta, Chicago, Nashville
   Chicago: Atlanta, Boston, Denver, Louisville
   Denver: Chicago, Louisville, Miami, Nashville
   Louisville: Chicago, Denver, Miami
   Miami: Atlanta, Denver, Louisville, Nashville
   Nashville: Atlanta, Boston, Denver, Miami

1. Prove $\chi' = 5$

For the upper bound proof: Here, we show our $\chi' = 5$.



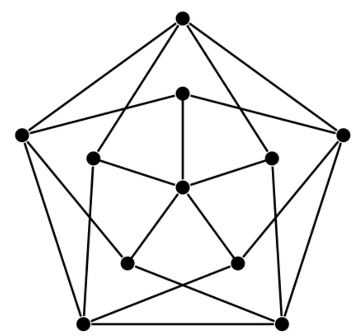
Upper bound: give example

Lower bound: min chromatic #

Lower bound:

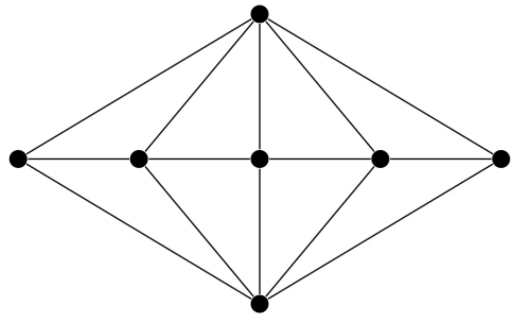
Suppose our $\chi' = 4$.

Then we would have 4 colors to use for our edge coloring. As we see, we have a deg 5 vertex in the center, thus we need a $\chi' \geq 5$.



$\square$

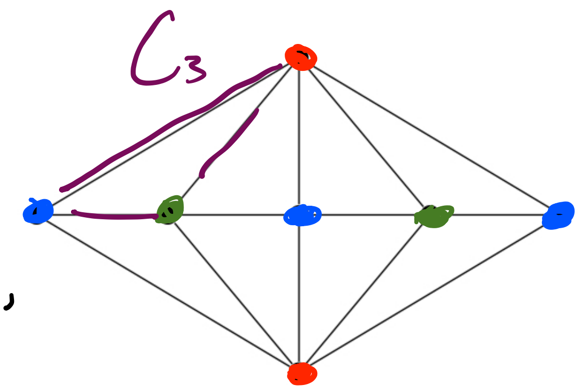
2. χ & χ'



χ: Upper max chromatic #, lower ex

Since the graph contains a subgraph C_3, then $\chi \leq 3$. There is an example of a proper 3 vertex coloring, so $\chi = 3$.

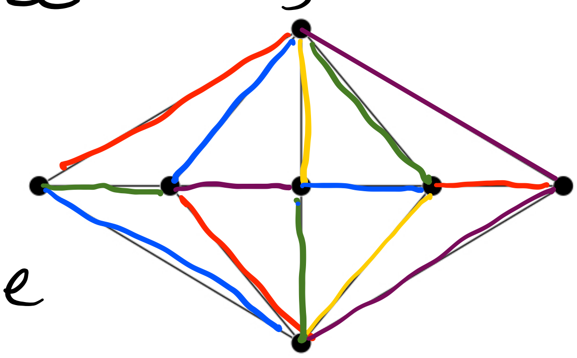
ex:



χ': Upper ex, lower min chromatic #

The max degree of the graph is 5, so $\chi' \geq 5$, and an example exists, thus $\chi' = 5$.

ex:



3. Prove if G is r-regular, then $\chi'(G) = r+1$.

Suppose we have G, an r-regular graph. By VIT, $r \leq \chi'(G) \leq r+1$. Assume $\chi' = r$. There exists a proper edge coloring of G with r colors. Since it's a matching, this means that the size $\leq \alpha'(G) \leq \frac{n-1}{2}$ with the total number of edges being $\leq r\left(\frac{n-1}{2}\right)$. When we use handshaking lemma, we get a contradiction since $r\left(\frac{n-1}{2}\right)$ is odd.

$$\frac{1}{2}\sum \deg(G) = \frac{r \cdot n}{2}. \ \lightning$$

Thus, $\chi'(G) = r+1$. $\square$

4. In this problem, we will discover the chromatic index of the Cartesian product $G \square K_2$ for a nonempty graph G.

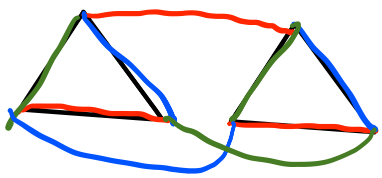
   (a) For $3 \le n \le 7$, give a plane drawing of each of the graphs $C_n \square K_2$. Find a minimal proper edge coloring of each.

   (b) Make a conjecture: The edge coloring number of $C_n \square K_2$ is 3

   (c) Recall that the chromatic index of C_n depends on whether n is even or odd. Does your conjecture for $\chi'(C_n \square K_2)$ depend on whether n is even or odd?

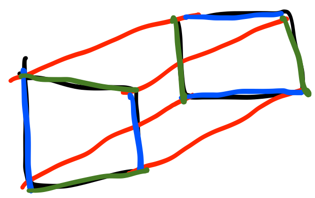
   (d) Prove that for every nonempty graph G, the graph $G \square K_2$ will have chromatic index $1+\Delta(G)$.

a) $C_3 \square K_2$



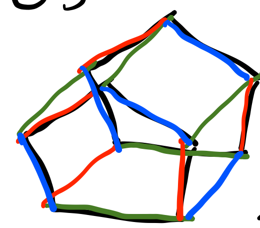
3

$C_4 \square K_2$



3

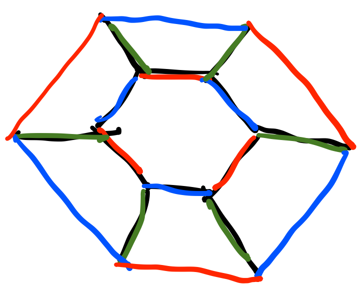
$C_5 \square K_2$



3

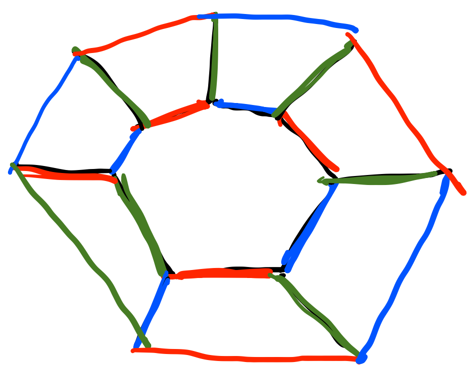
$C_6 \square K_2$

3



$C_7 \square K_2$

3



For all $C_n \square K_2$ ($3 \le n \le 7$), the min degree is 3. Since we also have examples of each, the $\chi' = 3$ for these graphs.

c) It doesn't depend on if n is even or odd since all $C_n \square K_2$ are a 3-regular graph where $n \geq 3$.

d) Proof:

By VIT, $\Delta(G \square K_2) \leq \chi'(G \square K_2) \leq \Delta(G \square K_2)+1$. We know that $\Delta(G \square K_2) = \Delta(G)+1$. Then $\Delta(G)+1 \leq \chi'(G \square K_2) \leq \Delta(G)+2$.

There exists a proper edge coloring of the graph G with $\Delta(G)+1$ colors. For $G \square K_2$, assume both copies of G have a proper edge coloring of $\Delta G+1$. Each vtx has diff color edges than $\Delta(G)+1$. Therefore there will be one edge that will not be colored. Now we can consider the extra edge w/ the last color. Now we conclude $\exists$ a proper edge coloring of $\Delta(G)+1$ for $G \square K_2$. Hence $\chi'(G) = \Delta(G)+1$ $\square$

5. Seven softball teams from Atlanta, Boston, Chicago, Denver, Louisville, Miami, and Nashville have been invited to participate in tournaments, where each team is scheduled to play a certain number of the other teams (given below). No team is allowed to play more than one game each day. Set up a schedule of games over the smallest number of days possible.

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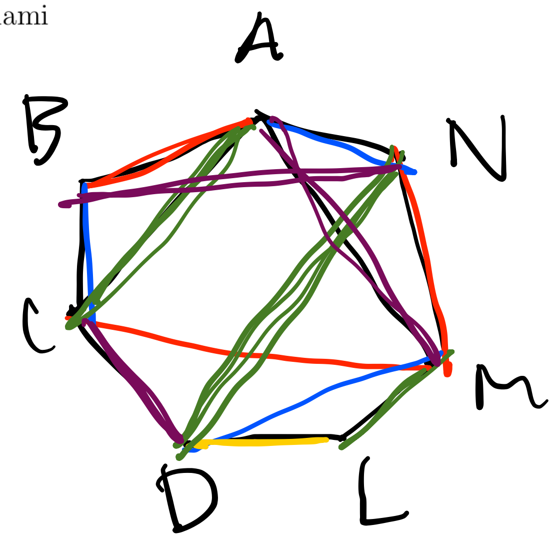
Min coloring
5



Proof:

Assume $\chi'(G) = \Delta$. Then the proper edge coloring exists and since it's a matching, then size $\le \alpha'(G) \le \frac{n-1}{2}$ where total colored edges are $\le \Delta \cdot \frac{n-1}{2}$ $= \frac{\Delta n - \Delta}{2}$. But the number of edges in G is $m^{\le}$, Hence $\chi'(G) = \Delta + 1$.

$\square$