Fibonacci and Lucas Numbers with Socially Distanced Seatings

by Mary Robinson

1. a) 2 chairs

xo
ox
oo

= 3 ways

3 chairs

xox
xoo
oxo
oox
ooo

= 5 ways

4 chairs

oxox
oxoo
ooxo
ooox
oooo
xoxo
xoox
xooo

= 8 ways

5 chairs

ooxox
ooxoo
oooxo
oooox
ooooo
oxoxo
oxoox
oxooo
xoxox
xoxoo
xooxo
xooox
xoooo

= 13 ways

b) Why $S_n = S_{n-1} + S_{n-2}$?

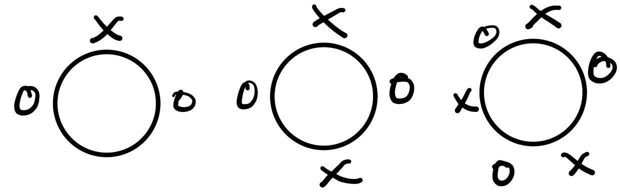
If we take the two previous seating charts (S_{n-1} and S_{n-2}), we can make S_n from adding n to S_{n-1} and 2n to S_{n-2}. For this instance, we add one empty seat to S_{n-1}, since we don't want two people potentially sitting next to each other, and a filled seat and empty seat get added to S_{n-2}. Thus, we get S_n.

c) $\boxed{S_n = F_{n+1}}$

$S_2 = 3 \rightarrow F_3 = 3$

$S_3 = 5 \rightarrow F_4 = 5$

2. a) examples:



C_2

ox
xo
oo

C_3

oxo
ooo
oox
xoo

$= 4$

C_4

ooxo
oooo
ooox
oxoo
oxox
xoxo
oxoo

$= 7$

b) $C_n = S_n - 1$

$= S_{n-1} + S_{n-2} - 1$

$= F_n + F_{n-1} - 1$

$= F_n + F_{n-2}$

c) For Lucas numbers:

$L_1 = 2$, $L_2 = 1$, $L_n = L_{n-1} + L_{n-2}$

$L_2 = F_0 + F_2$ and $L_3 = F_1 + F_3$

so $L_n = F_{n-2} + F_n$ for all $n \geq 2$! Cool!

3. $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$

Let $L(x) = \sum_{n \geq 0} L_n x^n$. So $l_n = [L(x)]_{x^n}$

$$\sum_{n=2}^{\infty} l_n x^n = \sum_{n=2}^{\infty} l_{n-1} x^n + \sum_{n=2}^{\infty} l_{n-2} x^n$$

So,

$$L(x) - l_1 x - l_0 = x(L(x) - l_0) + x^2 L(x)$$

$$L(x) \cancel{-x} - 2 = x L(x) \cancel{-x} + x^2 L(x)$$

$$L(x)(1 - x - x^2) = 2$$

$$L(x) = \frac{2}{1-x-x^2} = 2 \cdot \frac{1}{1-x-x^2} = 2 \cdot \frac{-1}{(c_1-x)(c_2-x)}$$

$$c_1, c_2 = \frac{-1 \pm \sqrt{5}}{2}$$

So,

$$L(x) = \frac{-1}{(c_1-x)(c_2-x)} = \frac{A}{c_1-x} + \frac{B}{c_2-x}$$

$$-1 = A(c_2 - x) + B(c_1 - x)$$

Let $x = c_1$

$$A = \frac{-1}{c_2 - c_1}$$

Let $x = c_2$

$$B = \frac{-1}{c_1 - c_2}$$

$$c_1 = \frac{-1+\sqrt{5}}{2}$$

$$c_2 = \frac{-1-\sqrt{5}}{2}$$

$$A = \frac{-1}{-\sqrt{5}} = \frac{1}{\sqrt{5}} \qquad B = \frac{-1}{\sqrt{5}}$$

$$c_1 - c_2 = \frac{-1+\sqrt{5}+1\sqrt{5}}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5}$$

$$L(x) = 2\left[\frac{1}{\sqrt{5}} \cdot \frac{1}{(c_1 - x)} - \frac{1}{\sqrt{5}} \cdot \frac{1}{c_2 - x}\right]$$

$$l_n = \left[2 \cdot \left[\frac{1}{\sqrt{5}} \cdot \frac{1}{c_1 - x} + \frac{-1}{\sqrt{5}} \cdot \frac{1}{c_2 - x}\right]\right] x^n$$

$$= 2 \cdot \left[\frac{1}{\sqrt{5}} \left[\frac{1}{c_1\left(1 - \frac{x}{c_1}\right)}\right] x^n + \frac{-1}{\sqrt{5}} \left[\frac{1}{c_2\left(1 - \frac{x}{c_2}\right)}\right] x^n\right.$$

$$= 2 \cdot \left[\frac{1}{\sqrt{5}} \cdot \frac{1}{c_1}\left(\frac{1}{c_1}\right)^n - \frac{1}{\sqrt{5}}\, \frac{1}{c_2}\left(\frac{1}{c_2}\right)^n\right]$$

$$l_n = 2\left[\frac{1}{\sqrt{5}}\left(\frac{1}{c_1}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1}{c_2}\right)^{n+1}\right]$$

$$c_1 = \frac{-1+\sqrt{5}}{2} \qquad \frac{1}{c_1} = \frac{2}{-1+\sqrt{5}} \cdot \frac{-1-\sqrt{5}}{-1-\sqrt{5}} = \frac{1+\sqrt{5}}{2}$$

$$\frac{1}{c_2} = \frac{1-\sqrt{5}}{2}$$

$$l_n = 2 \cdot \left[\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right]$$

Thus,

$$\boxed{l_n = \frac{2}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{2}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}}$$

QED $\square$