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Change Ringing: The Math Concepts behind the Bells

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What Is Change Ringing?

In the Simplest terms Change Ringing is the act of ringing bells, specifically large tower bells. These bells are rung in precise complex orders/ permutations . These patterns are known as changes. The goal of these changes is to ring all possible permutations of the bells without repetition (this is called an extent) or a subset of them with certain constraints. Alongside the historical significance and team work, We will explore the Mathematical principles involved in Change Ringing, specifically group Theory and graph theory

Bells and their Physical Limitations

In order to properly model change ringing we must consider its Physical limitations The most obvious of the Limitations are the massive sizes of the bell. Due to this it takes about 2 seconds for a bell to swing so Ringers can only swap positions with adjacent bells due to the momentum (This is called a Plane change). Further considering the size of the bells , an extent is only feasible up to 7 bells with the number of permutations being 7! .

Connections to Group Theory

Permutations

Permutations are very important for keeping track of the order in which the bells are ringing. This is done by labeling each bell with a number, with the starting number being the highest pitch and the largest number being the lowest. Ringing these bells in the order from highest to lowest is called rounds. Also, swapping the timing of two adjacent order bells is called a plain change. Plain changes are what allow new permutations to exist, connecting these musical instruments to the set of permutations.

Dihedral groups

This set of permutations for the bells and the changes we make to the order in which they are rung is very similar to the Dihedral group and its symmetries. If we take the order of the bell as the corners of the shape, we could make permutations by using combinations of the rotating or reflecting symmetries of the Dihedral group, we could find all the permutations in the set of permutations, thus making these groups similar. Extra note, we could also use rotating or reflecting combinations that equate to Plain changes and cross changes.

Cosets & Subgroups

With change ringing, a common example is called Plain Hunting, where we change what number is in front and then switch the numbers that are left. When connecting this to cosets and subgroups, we can think of our group being any Sn and then a subgroup would be formed by the certain changes made with the bells. Ringers often use group generators and patterns, like Plain Hunting, to build changes and to move through a group using left cosets. For example, using the group S3 and the Plain Hunting method, we get that 2 changes {(1 2), (2 3)} generate the whole group of S3.

Lagrange Theorem

Lagrange's Theorem states that if G is a finite group and H is a subgroup of G, then the number of distinct cosets of H multiplied by the number of elements in H is the number of elements in G, stating that H is a factor of the group G. Since this is a finite subgroup, then there is a limit on the number of unique changes (permutations) that a method can include. A practical example is the Plain Hunting change, with group S3 with order 6, and if we pick a subgroup {e, (1 2)}, we get exactly 3 cosets.

Conclusion

Change ringing is far more than a centuries-old musical skill—it is a living demonstration of algebra and graph theory in action. Each row is a permutation, hunting forms a subgroup, calls jump between cosets, and an extent is nothing but a Hamiltonian cycle in the Cayley graph of Sn. By listening to bells swap places we are, quite literally, hearing group theory ring out.

References

https://www.ringing.info/beginners/ringing.htm

https://www.oldnorth.com/blog/change-ringing-bells-what-how-and-why/

A First Course in Graph Theory by G. Chartrand and P. Zhang

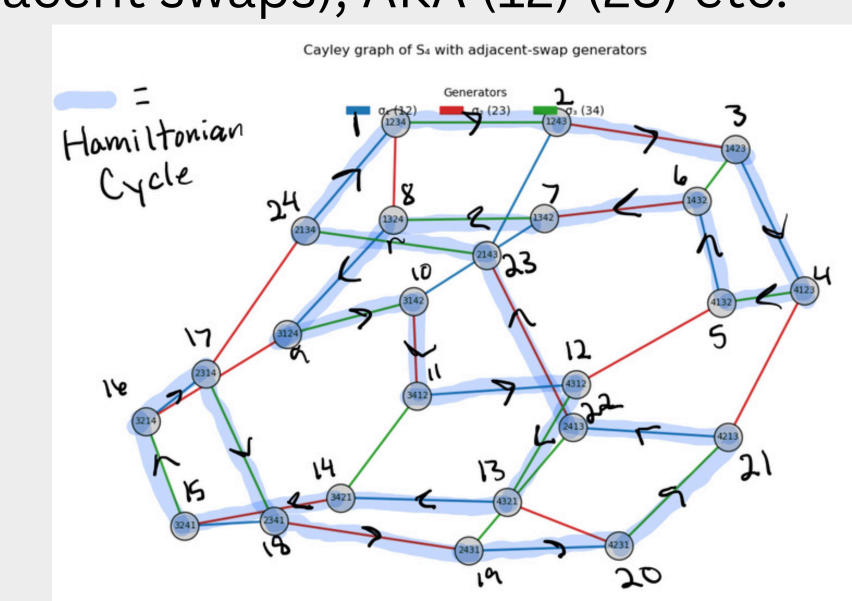
Bell Ringing by Instalments, by Peter D Wenham, https://www.pdg.org.uk/pdw/PDW_RbI_03_Plain_hunt_and_PBD.pdf

Connection to Graph Theory

Graph theory is a branch of mathematics that studies graphs that are structured with vertices and edges. The goal is to visually model relationships in systems.

Cayley Graphs

Cayley graphs will visually represent group structure. In our case the group will be S_4 The vertices will be a permutation of the bells and the generators will be legal moves (adjacent swaps), AKA (12) (23) etc.



Hamiltonian Path/Cycles

A path in a graph is a sequence of edges connecting distinct vertices. A cycle is a path that starts and ends at the same vertex. A Hamiltonian path visits every vertex exactly once. A Hamiltonian cycle is a Hamiltonian path that starts and ends at the same vertex. A complete extent (when all Permutations are used) is a Hamiltonian path through the Cayley graph. If the last change returns to the start then we have a Hamiltonian cycle.