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Homework 3 372

Problem 1 (5 points). Consider the following third order system:

$$G_1(s) = \frac{50}{s(s+5)(s+50)}$$

a) Find the two dominant poles of $G_1(s)$ and approximate $G_1(s)$ with a second order transfer function $G_2(s)$. Verify, in Matlab, by plotting the step responses, that $G_2(s)$ is a good approximation of $G_1(s)$.

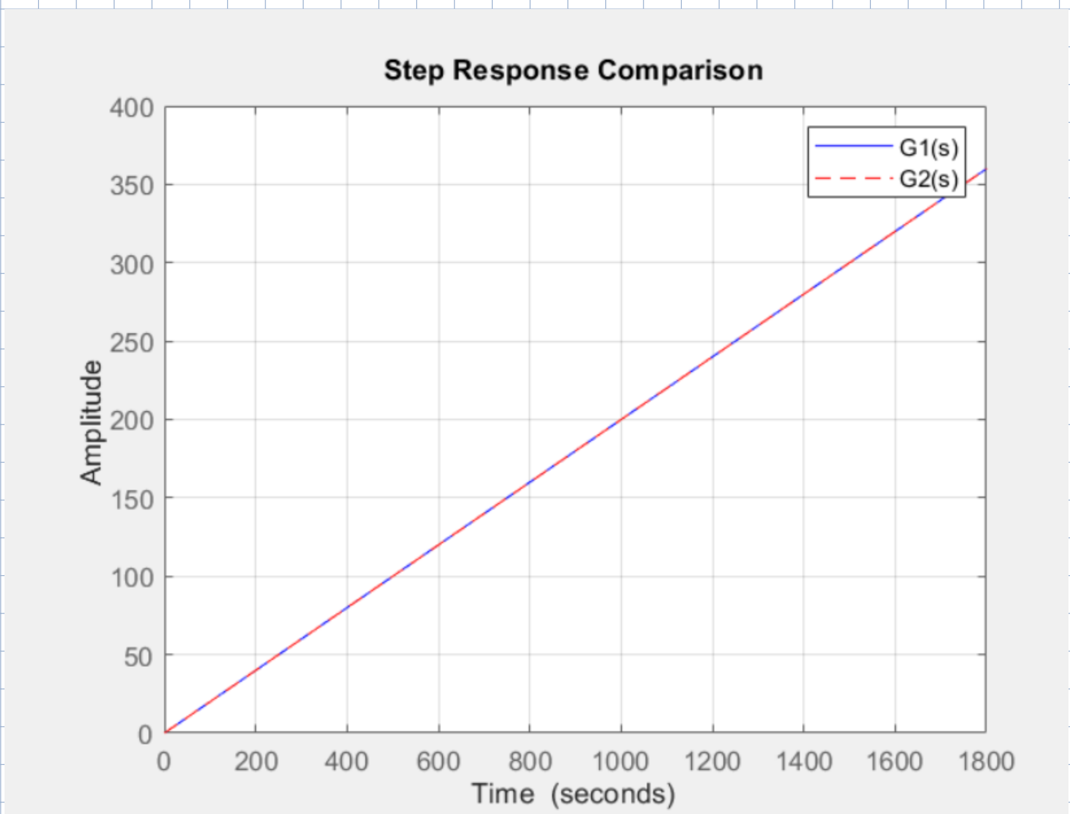
b) Are $G_1(s)$ and $G_2(s)$ stable systems? Are $G_1(s)$ and $G_2(s)$ BIBO stable systems?

a) $p_1 = 0$, $p_2 = -5$, $p_3 = -50$

The dominant poles are the ones closest to the imaginary axis. So they're $p_1 = 0$ and $p_2 = -5$.

Since $p_3 = -50$ is much farther to the left, we can neglect it to get:

$$G_2 \approx \frac{50}{s(s+5)} \frac{1}{50} = \frac{1}{s(s+5)}$$

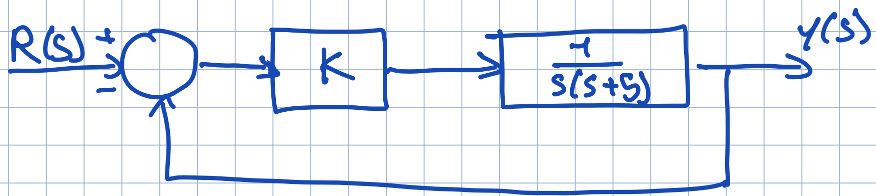


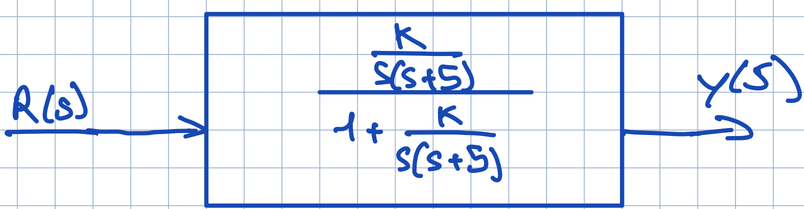
b) For G_1 : $p_1 = 0$, $p_2 = -5$, $p_3 = -50$

and for G_2 : $p_1 = 0$, $p_2 = -5$

For both systems we have one pole at 0 (p_1) and the remaining poles on the left axes. Therefore, both systems are marginally stable and BIBO unstable.

c)





$$W_2(s) = \frac{K}{s(s+5)+K} = \frac{K}{s^2+5s+K}$$

d) $R(s) = \frac{1}{s}$

$$W_2(s) = G(s) = \frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2} = \frac{K}{s^2+5s+K}$$

$\omega_n^2 = K$ and $2\delta\omega_n = 5$

$\omega_n = \sqrt{K}$

$$\delta = \frac{5}{\omega_n} = \frac{5}{2\sqrt{K}}$$

$\delta \leq 0.7$

$$\frac{5}{2\sqrt{K}} \leq 0.7$$

$$\frac{5}{2(0.7)} \leq \sqrt{K} \quad \Leftrightarrow \quad \frac{5}{2(0.7)} \leq \omega_n$$

$$\left(\frac{5}{1.4}\right)^2 \leq \left(\sqrt{K}\right)^2 \qquad \omega_n \geq 3.57$$

$$K \geq 12.76$$

$$\% OS = e^{-\frac{\delta\pi}{\sqrt{1-\delta^2}}} \times 100$$

$$\ln(0.07) \geq -\frac{\delta\pi}{\sqrt{1-\delta^2}}$$

$$\left(\frac{\ln(0.07)}{\pi}\right)^2 \leq \left(\frac{\delta}{\sqrt{1-\delta^2}}\right)^2$$

$$0.7165 \leq \frac{\delta^2}{1-\delta^2}$$

$$\delta^2 \geq 0.7165 - 0.7165\delta^2$$

$$\delta^2(1+0.7165) \geq 0.7165$$

$$\delta^2 \geq 0.4174$$

$$\delta \geq 0.6461$$

$$\delta = \frac{5}{2\sqrt{K}}$$

$$0.6461 \leq \frac{5}{2\sqrt{K}}$$

$$\sqrt{K} \leq \frac{5}{2(0.6461)}$$

$$K \leq 14.974$$

$$12.76 \leq K \leq 14.974$$

e) for $K = 12.76$ (the smallest)

$$W_2(s) = \frac{12.76}{s^2 + 5s + 12.76} = \frac{N(s)}{D(s)}$$

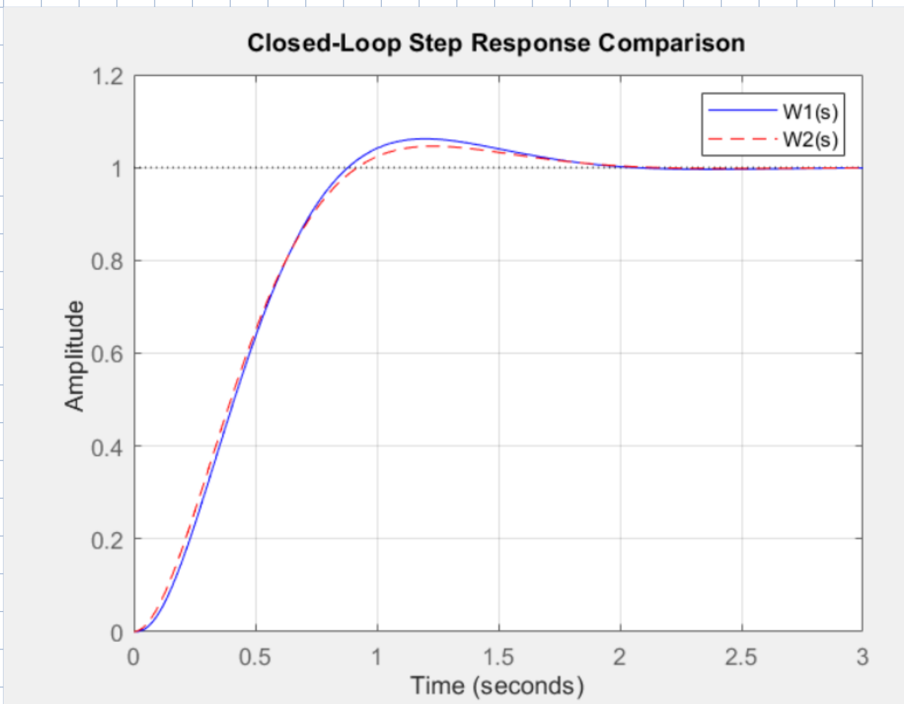
for the 2nd order, without sign change, all the poles are on the negative real part. Therefore the system is BIBO stable.

$$p_1, p_2 = \frac{-5 \pm \sqrt{-26.04}}{2 * 1}$$

$$= \frac{-5 \pm j\sqrt{26.04}}{2}$$

$$= -2.5 \pm j\sqrt{6.51} \qquad \text{Stable}$$

f)




```
>> homework3_question1f
Step Response Information for W1:
        RiseTime: 0.5680
    TransientTime: 1.7079
     SettlingTime: 1.7079
      SettlingMin: 0.9117
      SettlingMax: 1.0619
        Overshoot: 6.1943
       Undershoot: 0
             Peak: 1.0619
         PeakTime: 1.1904

Step Response Information for W2:
         RiseTime: 0.5953
    TransientTime: 1.6738
     SettlingTime: 1.6738
      SettlingMin: 0.9003
      SettlingMax: 1.0460
        Overshoot: 4.6039
       Undershoot: 0
             Peak: 1.0460
         PeakTime: 1.2342

W1 meets the overshoot specification of <= 7%.
W2 meets the overshoot specification of <= 7%.
>>
```

The specification is verified because both $W_1(s)$ and $W_2(s)$ meets the overshoot requirement of $\leq 7\%$. The controller designed using the approximate model $G_2(s)$ effectively controls the original system $G_1(s)$. The similarity in step responses between $W_1(s)$ and $W_2(s)$ indicates that the approximation of $G_2(s)$ is appropriate.

g) Open-Loop without controller (K) and feedback loop.

$$G_1(s) = \frac{50}{s(s+5)(s+50)}$$

$p_1 = 0$, $p_2 = -5$, $p_3 = -50$

we determined from Q1 a) that the system is marginally stable and BIBO unstable.

Closed-Loop

$$W_1(s) = \frac{KG_1(s)}{1 + KG_1(s)}$$

with $K = 12.76$

$$W_1(s) = \frac{\frac{12.76 \times 50}{s(s+5)(s+50)}}{1 + \frac{12.76 \times 50}{s(s+5)(s+50)}} = \frac{638}{s^3 + 55s^2 + 250s + 638}$$

N, S $D(s) = s^3 + 55s^2 + 250s + 638$, BIBO stable iff $\begin{cases} a_3, a_2, a_1, a_0 \text{ same sign} \\ a_2 a_1 > a_0 a_3 \end{cases}$

$a_3 = 1$, $a_2 = 55$, $a_1 = 250$, $a_0 = 638$

same sign

$a_2 a_1 = 55 \times 250 = 13750 > 638 \times 1 = a_0 a$

System is BIBO stable.

With the introduction of controller and feedback loop in this case, the system went from marginally stable and BIBO unstable to BIBO stable with $K = 12.76$ by moving the poles to the left half of the complex plane.

Problem 2 (5 points). Consider the following fourth order system:

$$G(s) = \frac{(s+1)^2}{s^4 + 4s^3 + s^2 - 6s}$$

a) By using the Routh-Hurwitz criterion, analyze the stability and BIBO-stability of $G(s)$. Double-check your result in Matlab, by numerically computing the poles of $G(s)$ using the command *roots*.

$$\begin{array}{c|ccc}
s^4 & & 1 & 1 \\
s^3 & & 4 & -6 \\
s^2 & \frac{10}{4} & 1 & 0 \\
s^1 & & 6 & 0 \\
s^0 & 0 & 6 & 0
\end{array} \qquad \times \frac{4}{10}$$

$$s_1 = -\frac{1}{4}\left| -6 - 4 \right| = +\frac{10}{4}$$

$$s_2 = -\frac{1}{1}\begin{vmatrix} 4 & -6 \\ 1 & 0 \end{vmatrix} = +6$$

$$s_3 = -\frac{1}{6}\begin{vmatrix} 1 & 0 \\ 6 & 0 \end{vmatrix} = 0$$

row of 0 so at least marginally stable or unstable.

$P(s) = 6s + 0, \quad \frac{dP}{ds} = 6 \quad \Rightarrow \quad s_4 = 6$

There is no sign changes therefore the system is marginally stable and BIBO unstable.

```
Poles of G(s):
         0
   -3.0000
   -2.0000
    1.0000

>>
```

b) Consider the feedback control system shown in the figure below, where the controller's transfer function is a static positive gain $K > 0$. Using the Routh-Hurwitz criterion determine for which values of K the closed-loop system is BIBO stable.

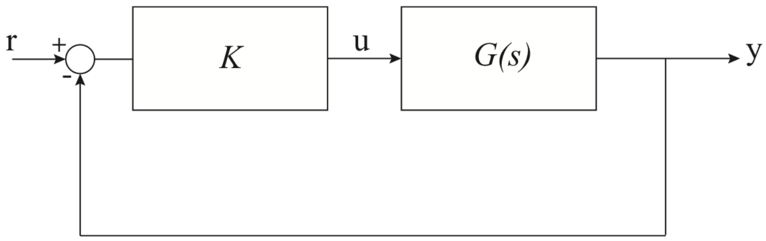


Figure 1: Block Diagram

$$T = \frac{KG}{1+KG} = \frac{K\frac{(s+1)^2}{s^4+4s^3+s^2-6s}}{1+\frac{K(s+1)^2}{s^4+4s^3+s^2-6s}} = \frac{\frac{K(s+1)^2}{\cancel{s^4+4s^3+s^2-6s}}}{\frac{s^4+4s^3+s^2-6s+K(s+1)^2}{\cancel{s^4+4s^3+s^2-6s}}}$$

$$= \frac{K(s+1)^2}{s^4+4s^3+s^2-6s+K(s+1)^2}$$

$$s^4+4s^3+s^2-6s+Ks^2+2Ks+K=0$$

$$s^4+4s^3+(1+K)s^2+(2K-6)s+K=0$$

s^4	1	$1+K$	K	
s^3	4 2	$2K-6$ $K-3$	0	$\div 2$
s^2	$\frac{1K+5}{\cancel{2}\ \cancel{2}}$	$2K$	0	$\times 2$
s^1	$\frac{K^2-2K-15}{K+5}$	0	0	
s^0	$\cancel{2}K$	0	0	$\div 2$

$$a = -\frac{1}{2}\begin{vmatrix} 1 & 1+K \\ 2 & K-3 \end{vmatrix} = -\frac{1}{2}\left(K-3-2(1+K)\right)$$

$$= -\frac{1}{2}\left(K-3-2-2K\right) = -\frac{1}{2}(-K-5)$$

$$a = \frac{1}{2}K+\frac{5}{2}$$

$$b = -\frac{1}{2}\begin{vmatrix} 1 & K \\ 2 & 0 \end{vmatrix} = -\frac{1}{2}(-2K) = K$$

$$c = -\frac{1}{K+5}\begin{vmatrix} 2 & K-3 \\ K+5 & 2K \end{vmatrix} = -\frac{1}{K+5}\left(4K-(K-3)(K+5)\right)$$

$$c = \frac{-1}{K+5}\left(4K-K^2-2K+15\right) = \frac{K^2-2K-15}{K+5}$$

$$d = \frac{-(K+5)}{K^2-2K-15}\begin{vmatrix} K+5 & 2K \\ \frac{K^2-2K-15}{K+5} & 0 \end{vmatrix}$$

$$= \frac{+(\cancel{K+5})}{\cancel{K^2-2K-15}}\left(0 + \frac{2K(\cancel{K^2-2K-15})}{\cancel{K+5}}\right)$$

$$= 2K$$

① $K+5>0 \Rightarrow K>-5$ (not good because $K>0$)

② $\dfrac{K^2-2K-15}{K+5}>0$

$\dfrac{(K+3)(K-5)}{K+5}>0 \Rightarrow K>-3$ and $K>5$ ($K>-3$ not good because $K>0$)

for the closed-loop system to be BIBO stable, $K>5$

c) if $K=5$,

s^4	1	6	5
s^3	2	2	0
s^2	10	10	0
s^1	0 20	0	0
s^0	10	0	0

$$\frac{5^2-2(5)-15}{5+5} = \frac{25-25}{10} = 0$$

row of 0 means we need to test if the system is marginally stable or unstable.

we test with $P(s) = 10s^2+10$

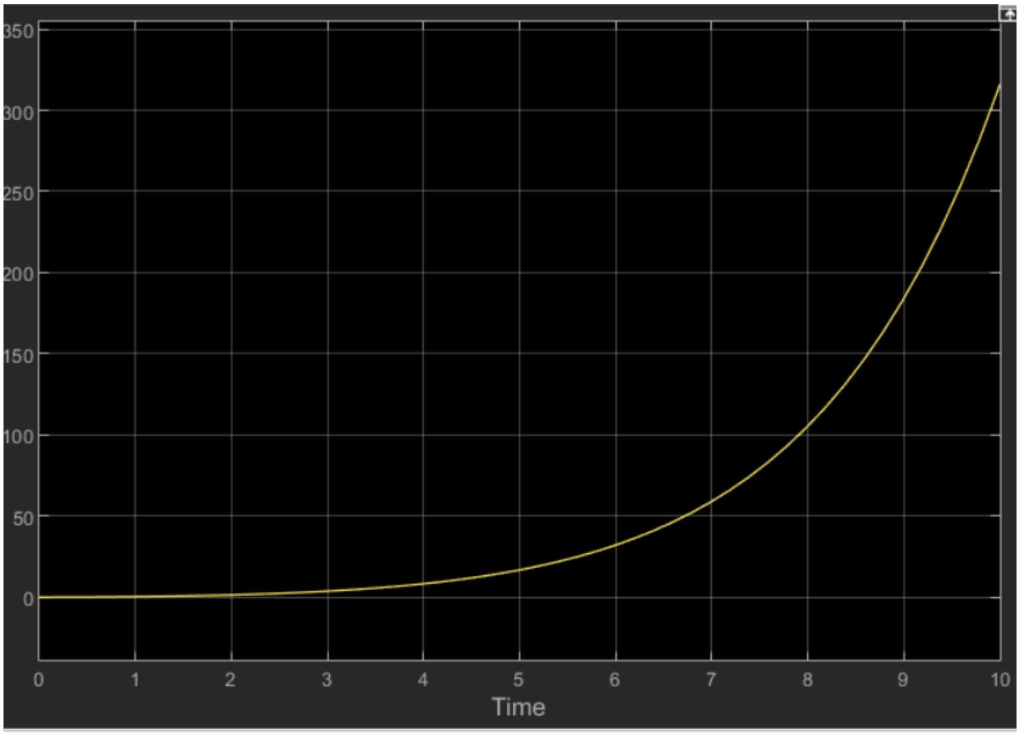
$$\frac{dP}{ds} = 20s$$

$$c = 20$$

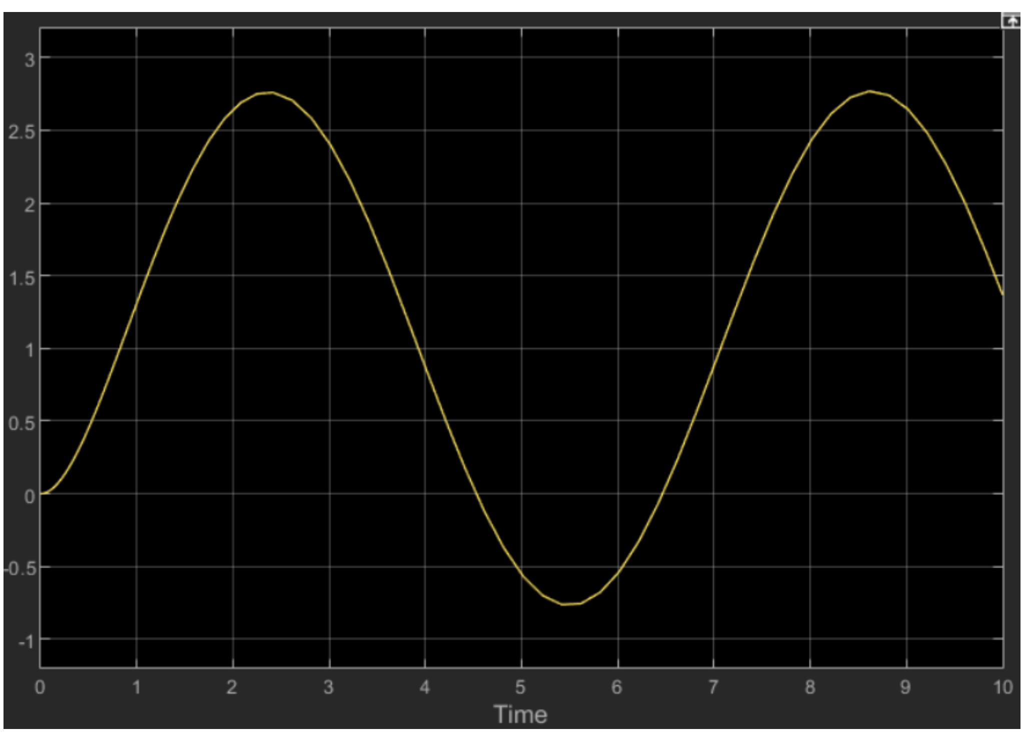
Since there's no sign change, then with $K=5$ the system is now marginally stable.

Question 2 d)

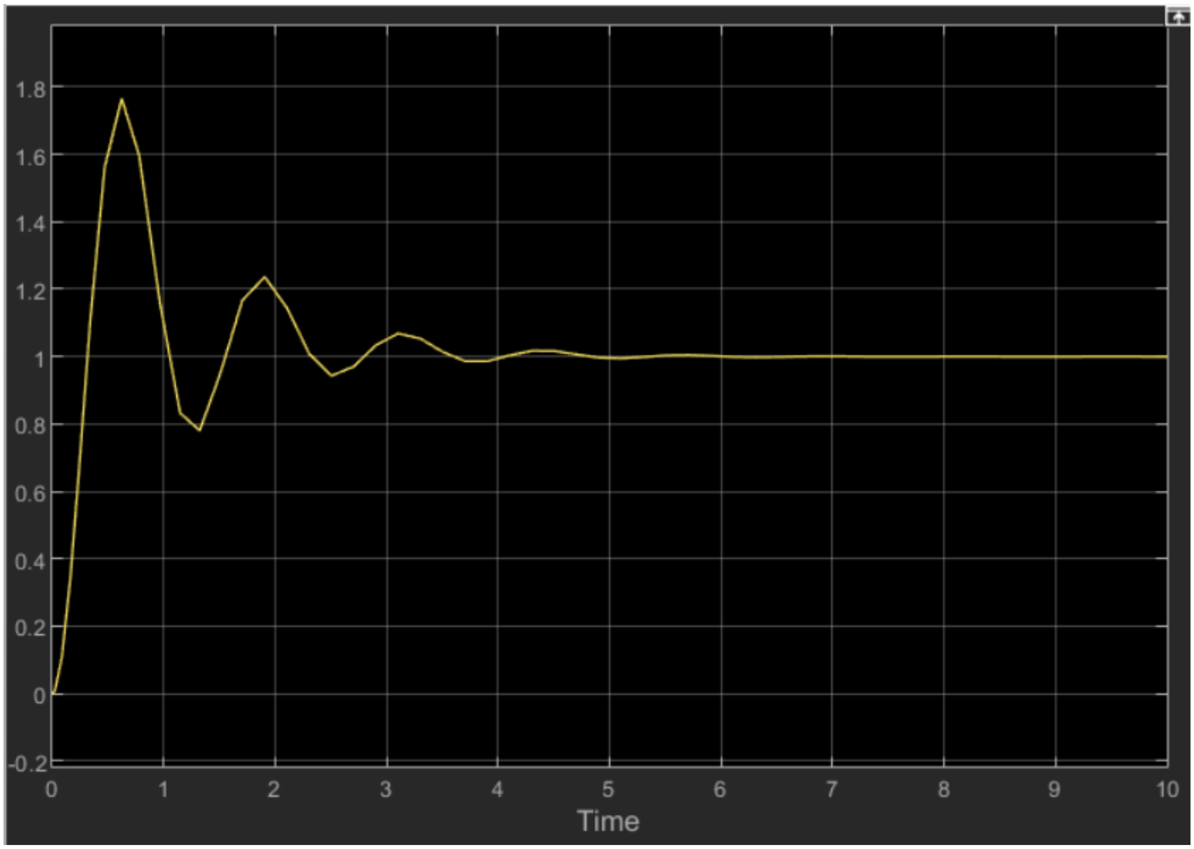
With K = 1, the Simulink plot we get is:

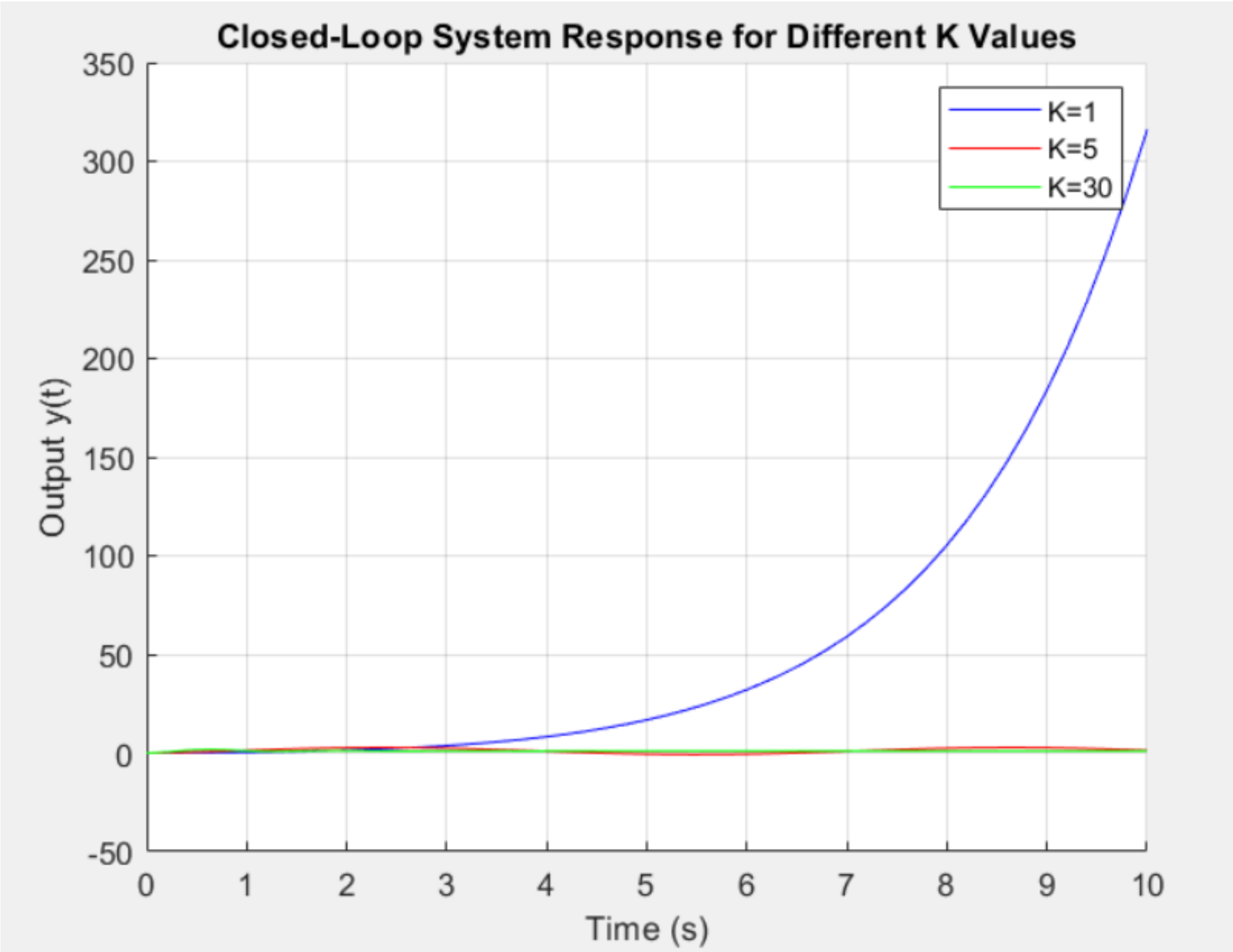


With K = 5, the Simulink plot we get is:



With K = 30, the Simulink plot we get is:

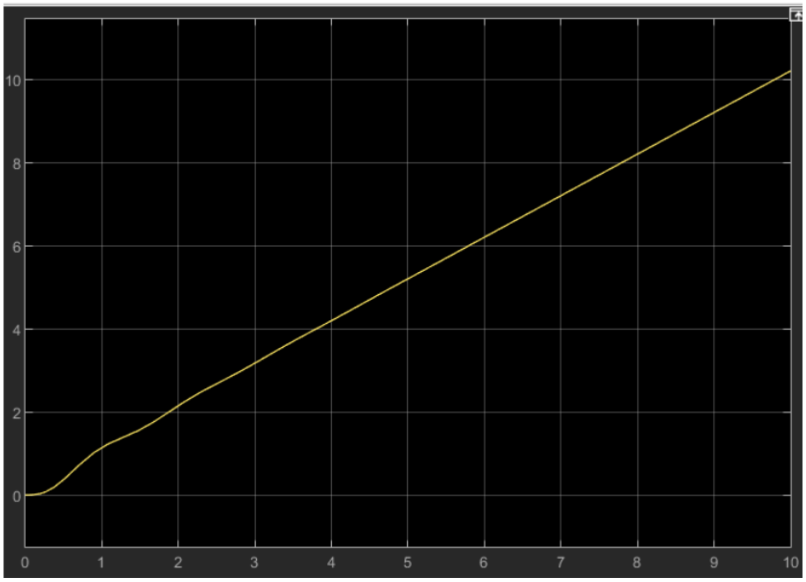


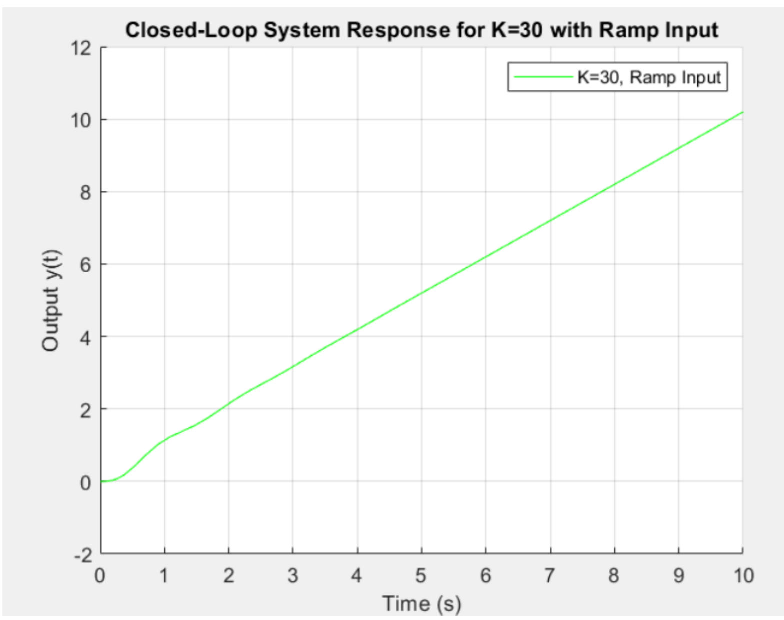


Comment if and why the obtained outputs are consistent with the stability results found in the previous steps.

1. **K=1**: The system is unstable, as indicated by the exponential growth of the output. This is consistent with our Routh-Hurwitz analysis, which predicted instability for K < 5.
2. **K=5**: The system shows a slightly oscillatory response but does not diverge, indicating it is marginally stable. This also aligns with the Routh-Hurwitz analysis, which indicated that K=5.
3. **K=30:** The system is stable, with the output settling quickly and showing no signs of instability. This confirms the Routh-Hurwitz prediction that the system is stable for K > 5.

Question 2 e)





Comment if and why the obtained outputs are consistent with the stability results found in the previous steps.

The output y(t) is following the ramp input r(t)=t with a positive slope, indicating that the system is trying to track the ramp input. There is a noticeable transient response at the beginning, where the output starts from zero and gradually increases to follow the ramp. This is expected for a type 0 system. The output does not diverge or oscillate uncontrollably, indicating that the system remains stable throughout the simulation. The stable response is consistent with our earlier Routh-Hurwitz analysis, which predicted stability for K > 5.

f) $R(s) = \frac{1}{s}$

for $K = 1$, $T(s) = \dfrac{1(s+1)^2}{s^4+4s^3+s^2-6s+1(s+1)^2}$

$$= \frac{(s+1)^2}{s^4+4s^3+2s^2-4s+1}$$

$$e_\infty = \lim_{s \to 0} s\,E(s)$$

$Y(s) = R(s)T(s)$

$$E(s) = R(s) - Y(s)$$

$$E(s) = \frac{1}{s} - \frac{1}{s}T(s)$$

$$= \frac{1}{s}\left(1 - T(s)\right)$$

$$e_\infty = \lim_{s \to 0} \cancel{s} \times \frac{1}{\cancel{s}} \left(1 - \frac{(s+1)^2}{s^4+4s^3+2s^2-4s+1}\right)$$

$$= 1 - \lim_{s \to 0} \frac{(s+1)^2}{s^4+4s^3+2s^2-4s+1} = 1 - \frac{1^2}{0+0+0-0+1}$$

$$= 1 - 1$$

$$e_\infty = 0$$

for $K = 10$, $T(s) = \dfrac{10(s+1)^2}{s^4+4s^3+s^2-6s+10(s+1)^2}$

$$= \frac{10(s+1)^2}{s^4+4s^3+11s^2+14s+10}$$

$$e_\infty = \lim_{s \to 0} \cancel{s} \times \frac{1}{\cancel{s}} \left(1 - \frac{10(s+1)^2}{s^4+4s^3+11s^2+14s+10}\right)$$

$$= 1 - \lim_{s \to 0} \frac{10(s+1)^2}{s^4+4s^3+11s^2+14s+10} = 1 - \frac{10(1)^2}{0+0+0+0+10}$$

$$= 1 - 1$$

$$e_\infty = 0.$$