TIME DIFFERENCE OPERATORS

$f_s :=$ SAMPLE RATE $\left[f_s := 1/k \right.$
(AUDIO RATE $\quad k =$ TIME STEP (s) $\left. \right]$
48×10^3, 44.1×10^3
96×10^3, ...

$u(t)$ $\longrightarrow t$ (CONTINUOUS

$\overbrace{\quad}^{k}$ $\quad u^n \quad u^{n+1} \quad u^{n+2}$

$n-1 \quad n \quad n+1 \quad n+2$ $\qquad n \in \mathbb{N}$

NYQUIST THEOREM (INFORMATION THEORY)

f_{max} = LARGEST FREQ. IN A SIGNAL
$\hookrightarrow f_s \geq 2 f_{max}$ [f_{max} FOR AUDIO
20×10^3 Hz
LIMIT OF HUMAN HEARING]

u^n = TIME SERIES ; $u(t)$ = CONTINUOUS SOLUTION

$u(t_n) \approx u^n$ $\qquad t_n := nk$

$\uparrow$
APPROXIMATION ... BUT WHAT DOES IT MEAN?

$\begin{cases} \text{TRUNCATION ERROR (TAYLOR SERIES OF OPERATORS)} \\ \text{GLOBAL ERROR (CUMULATIVE ERROR IN A TIME LOOP)} \end{cases}$

DEFINITIONS OF BASIC OPERATORS

u^n

- $1\, u^n = u^n$ — IDENTITY
- $e_{t+}\, u^n = u^{n+1}$ $\quad$ - $e_{t-} u^n = u^{n-1}$ — SHIFT OPS.

- $\delta_{t+} := \dfrac{e_{t+} - 1}{k}$

  ↳ $\underbrace{\delta_{t+} u^n = \dfrac{u^{n+1} - u^n}{k}}_{\text{FORWARD DIFFERENCE OP.}}$ ⇝ THIS APPROXIMATES THE TIME DERIVATIVE!

$$\delta_{t+}\, u(t_n) = \frac{u(t_{n+1}) - u(t_n)}{k} \underset{\text{(TAYLOR SERIES)}}{\approx} \left.\frac{du}{dt}\right|_{t=t_n} + \frac{k}{2}\left.\frac{d^2u}{dt^2}\right|_{t=t_n} = \left.\frac{du}{dt}\right|_{t=t_n} + \underbrace{O(k)}_{\text{BIG-OH}}$$

TRUNCATION ERROR

FIRST-ORDER ACCURATE APPROXIMATION!

$$u(t_{n+1}) \approx u(t_n) + k\left.\frac{du}{dt}\right|_{t=t_n} + \frac{k^2}{2}\left.\frac{d^2u}{dt^2}\right|_{t=t_n} + \dots$$

- $\delta_{t-} := \dfrac{1 - e_{t-}}{k}$ — BACWARD DIFFERENCE OP

  ↳ $\delta_{t-} u^n = \dfrac{u^n - u^{n-1}}{k}$ ⇝ $\delta_{t-} u(t_n) = \left.\dfrac{du}{dt}\right|_{t=t_n} + O(k)$

TRUNCATION ERROR

FIRST-ORDER ACCURATE APPROXIMATION!

$$u(t_{n-1}) \approx u(t_n) - k\left.\frac{du}{dt}\right|_{t=t_n} + \frac{k^2}{2}\left.\frac{d^2u}{dt^2}\right|_{t=t_n} + \dots$$

- $\delta_{t\cdot} := \dfrac{e_{t+} - e_{t-}}{2k}$ — CENTRED DIFF. OP.

  ↳ $\delta_{t\cdot}\, u(t_n) = \dfrac{u(t_{n+1}) - u(t_{n-1})}{2k} \underset{\text{(TAYLOR SERIES)}}{=} \dots = \left.\dfrac{du}{dt}\right|_{t=t_n} + O(k^2)$

TRUNCATION ERROR

SECOND-ORDER ACCURATE APPROXIMATION!

n-1 $\quad$ n $\quad$ n+1

DIFFERENCE OPERATORS

SECOND DIFFERENCE

$$\delta_{tt} := \underbrace{\delta_{t+}\delta_{t-}}_{\text{COMPOSITION}} = \delta_{t-}\delta_{t+}$$

$$\delta_{tt} := \delta_{t+}\delta_{t-}u^n = \delta_{t+}\left[\frac{u^n - e_{t-}u^n}{k}\right] = \frac{1}{k}\left(\delta_{t+}u^n\right) - \frac{1}{k}\delta_{t+}u^{n-1}$$

$$= \frac{1}{k^2}\left[u^{n+1} - u^n\right] - \frac{1}{k^2}\left[u^n - u^{n-1}\right]$$

$$= \frac{u^{n+1} - 2u^n + u^{n-1}}{k^2}$$

CENTRED SECOND DIFF. OP.

TRUNC. ERR.

$$\hookrightarrow \delta_{tt}u(t_n) = \left.\frac{d^2u}{dt^2}\right|_{t=t_n} + O(k^2)$$

SECOND-ORDER ACCURATE APPROXIMATION

$n-1$	n	$n+1$
$+1/k^2$	$-2/k^2$	$+1/k^2$

HIGHER ACCURATE APPROXIMATIONS

HOW? USE INTERPOLATION!

$n-2$	$n-1$	n	$n+1$	$n+2$
(a	b	c	d	e)

$\Rightarrow$ USE A LARGER STENCIL

$$\delta_{t0}u(t_n) := \frac{u(t_{n-2}) - 8u(t_{n-1}) + 8u(t_{n+1}) - u(t_{n+2})}{12k}$$

$$= \left.\frac{du}{dt}\right|_{t=t_n} + O(k^4)$$

PROBLEMS:
1st MORE COEFFICIENTS TO STORE IN MEMORY
2nd STABILITY, ...

MORE NOTATION

- $\mu_{t+} := \frac{e_{t+} + 1}{2}$

$\hookrightarrow \mu_{t+} u^n = \frac{u^{n+1} + u^n}{2}$ FORWARD AVERAGING OPERATOR $\rightsquigarrow \mu_{t+} u(t_n) = u(t_n) + O(k)$ 1st. ORDER ACC.

- $\mu_{t-} := \frac{1 + e_{t-}}{2}$ $\rightsquigarrow \mu_{t-} u^n = \frac{u^n + u^{n-1}}{2}$ BACKWARD AVERAG. OPERATOR $\rightsquigarrow \mu_{t-} u(t_n) = u(t_n) + O(k)$ 1.st ORDER ACCURATE

- $\mu_{t\cdot} := \frac{e_{t+} + e_{t-}}{2}$ $\rightsquigarrow \mu_{t\cdot} u^n = \frac{u^{n+1} + u^{n-1}}{2}$ CENTRED AVERAGING OP. $\rightsquigarrow \mu_{t\cdot} u(t_n) = u(t_n) + O(k^2)$ 2nd ORDER ACCURATE

INTERLEAVED TIME SERIES

$n-1$	$n-1/2$	n	$n+1/2$	$n+1$
□	○	□	○	□

$$\delta_{t+} u^n = \frac{u^{n+1} - u^n}{k} := v^{n+1/2}$$

$$\delta_{t-} \underbrace{\delta_{t+} u^n}_{v^{n+1/2}} = \delta_{tt} u^n = \frac{d^2 u}{dt^2} + O(k^2)$$

$$\hookrightarrow \delta_{t-} v^{n+1/2} = \frac{dv}{dt} + O(k^2)$$

2nd ORDER ACCURATE WHEN APPLIED TO AN INTERLEAVED TIME SERIES...

ENERGY IDENTITIES

$$\frac{du}{dt}\,\frac{d^2u}{dt^2} = \frac{1}{2}\frac{d}{dt}\left(\frac{du}{dt}\right)^2$$

[TRICKS ALLOWING TO RECOVER THE KINETIC AND POTENTIAL ENERGIES OF SHO]

TIME DER. OF KINETIC EN.

$$\frac{du}{dt}\,u = \frac{1}{2}\frac{d}{dt}\,u^2 \rightsquigarrow \geq 0 \text{ (SQUARED)}$$

INVOLVED POTENTIAL ENERGY

CONTINUOUS TIME

DISCRETE TIME

IT'S JUST AN IDENTITY... YOU CAN PROVE IT BY SIMPLE ALGEBRA

- $$\delta_{t\cdot}u^n\,\delta_{tt}u^n = \frac{1}{2}\,\delta_{t+}\left(\delta_{t-}u^n\right)^2 = \frac{1}{2}\,\delta_{t-}\left(\delta_{t+}u^n\right)^2$$

- $$u^n\,\delta_{t\cdot}u^n = \frac{1}{2}\,\delta_{t+}\left(u^n e_{t-}u^n\right) = \frac{1}{2}\,\delta_{t-}\left(u^n e_{t+}u^n\right)$$

THIS IS NOT A SQUARED (≥ 0) QUANTITY

... THIS WILL NEED A STABILITY CONDITION ...

- $$u^n e_{t-}u^n = \underbrace{\left(\mu_{t-}u^n\right)^2}_{\geq 0} - \underbrace{\frac{k^2}{4}\left(\delta_{t-}u^n\right)^2}_{\leq 0}$$

APPROXIMATING THE SHO

$$\frac{d^2u}{dt^2} = -\omega_0^2 u$$

$$\rightarrow \quad \delta_{tt} u^n = -\omega_0^2 u^n$$

$\left[u^0, u^1 \right.$ = INITIAL CONDITIONS $\left. u(0);\ \frac{du}{dt}(0) \right]$

$$u^{n+1} - 2u^n + u^{n-1} = -\omega_0^2 k^2 u^n$$

$$\rightarrow \quad u^{n+1} = (2-\omega_0^2 k^2)\underbrace{u^n}_{\text{known}} - \underbrace{u^{n-1}}_{\text{known}}$$

UPDATE EQUATION

KNOWN FROM PREVIOUS TIME STEPS

PSEUDO CODE

```
for n = 2 : Tsamples
    u^{n+1} = (2 - w0^2 k^2) u^n - u^{n-1};
    u^{n-1} = u^n;  u^n = u^{n+1};
    out[n] = u^n;
end
```

STABILITY

THE SCHEME IS SECOND ORDER ACCURATE..

BUT NOT ALWAYS STABLE!!

$$\delta_{t\cdot} u^n \, \delta_{tt} u^n = -\omega_0^2 u^n (\delta_{t\cdot} u^n)$$

$$\delta_{t+}\left[\frac{1}{2}(\delta_{t-}u^n)^2\right] = \delta_{t+}\left[-\frac{\omega_0^2}{2}(u^n e_{t-} u^n)\right]$$

$$\delta_{t+}\left[\frac{1}{2}(\delta_{t-}u^n)^2 + \frac{\omega_0^2}{2}(u^n e_{t-} u^n)\right] = 0$$

USE IDENTITIES FROM PREVIOUS PAGE

ENERGY BALANCE EQUATION

$$\delta_{t+}\left[\frac{1}{2}(\delta_{t-}u^n)^2 + \frac{\omega_0^2}{2}(\mu_{t-}u^n)^2 - \frac{\omega_0^2 k^2}{8}(\delta_{t-}u^n)^2\right] = 0$$

ENERGY EXPRESSION: AN INTERLEAVED TIME SERIES

$$H^{n-1/2} = \left[1 - \frac{\omega_0^2 k^2}{4}\right]\frac{1}{2}(\delta_{t-}u^n)^2 + \frac{\omega_0^2}{2}(\mu_{t-}u^n)^2 \overset{?}{\geq} 0$$

$$\rightarrow \quad 1 - \frac{\omega_0^2 k^2}{4} \geq 0$$

$$k \leq \frac{2}{\omega_0}$$

STABILITY CONDITION!!

TAKE HOME CONCEPT:

STABILITY OF A TIME-STEPPING SCHEME AMOUNTS TO FINDING CONDITION FOR WHICH A DISCRETE EQUIVALENT OF THE ENERGY IS NON-NEGATIVE