MODIFIED EQUATION TECHNIQUES

$$\delta_{tt} u(t) = \frac{u(t+k) - 2u(t) + u(t-k)}{k^2}$$

$$u(t+k) \approx u(t_n) + k \left.\frac{du}{dt}\right|_{t=t_n} + \frac{k^2}{2}\left.\frac{d^2u}{dt^2}\right|_{t=t_n} + \dots$$

$$u(t-k) \approx \dots +$$

CORRECTION FACTOR

$$\delta_{tt} = \frac{d^2}{dt^2} + \frac{k^2}{12}\frac{d^4}{dt^4} + \frac{k^4}{360}\frac{d^6}{dt^6} + \dots = \sum_{l=1}^{+\infty} \frac{2k^{2(l-1)}}{(2l)!}\frac{d^{2l}}{dt^{2l}}$$

CORRECTION FACTOR

$$\delta_{tt} u^n = -\omega_0^2 u^n$$

$\delta_{tt} \to -\omega_0^2$

$\Rightarrow \delta_{tt}\delta_{tt} \to \omega_0^4$

MODIFY:

$$\left(\delta_{tt} - \frac{k^2}{12}\delta_{tt}\delta_{tt}\right) u^n = -\omega_0^2 u^n$$

$$\left(\delta_{tt} - \frac{k^2}{12}\omega_0^4\right) u^n = -\omega_0^2 u^n$$

$n-1$ $\quad$ n $\quad$ $n+1$

$\Rightarrow \epsilon = O(k^4)$

FOURTH-ORDER ACCURATE...

REWRITE AS

$$\delta_{tt} u^n = \frac{1}{k^2}\left[-\omega_0^2 k^2 + \frac{\omega_0^4 k^4}{12}\right] u^n$$

ADDING AN EXTRA CORRECTION FACTOR

$$\delta_{tt} u^n = \frac{1}{k^2}\left(-\omega_0^2 k^2 + \frac{\omega_0^4 k^4}{12} - \frac{\omega_0^6 k^6}{320}\right) u^n$$

$\epsilon = O(h^6)$

SIXTH-ORDER ACCURATE

$$\frac{1}{k^2}\left(-\omega_0^2 k^2 + \frac{\omega_0^4 k^4}{12} - \frac{\omega_0^6 k^6}{320} + \dots\right)$$

$$= \frac{2}{k^2}\left(-1 + \underbrace{1 - \frac{\omega_0^2 k^2}{2} + \frac{\omega_0^4 k^4}{4!} - \frac{\omega_0^6 k^6}{6!} + \dots}_{\cos(\omega_0 k)}\right)$$

$$\delta_{tt} u^n = \frac{2}{k^2}\left(-1 + \cos(\omega_0 k)\right) u^n$$

$n-1$ $\quad$ n $\quad$ $n+1$

EXACT SCHEME !!