PDEs

PARTIAL DIFF. EQS.

$u(x,t)$

u SOME PHYSICAL VARIABLE (DISPLACEMENT) [m] ; $x \in \mathcal{I} \subset \mathbb{R}$ SPATIAL COORDINATE ; $t \subset \mathbb{R}^+_0$ time

$\mathcal{I} = [0, L]$ (CLOSED INT.)

VIBRATION! IS A CONSEQUENCE OF NEWTON'S LAW WITH RESTORING FORCES (OPPOSITE TO MOTION...)

$$F = ma \, ,$$

$m=$ mass of an ideal point [Kg]

$a=$ acceleration $[m/s^2]$ $= \dfrac{d^2u}{dt^2}$

$F=$ force [N]

Force is conservative!

$$F = -\frac{d\phi}{du} \, ; \quad \phi = \phi(u) : \mathbb{R} \to \mathbb{R}$$

[ϕ IS A SUFFICIENTLY SMOOTH FUNCTION ...]

$$\begin{cases} m \dfrac{d^2u}{dt^2} = -\dfrac{d\phi}{du} \\ u(t=0) = u_0 \\ \dfrac{du}{dt}(t=0) = v_0 \end{cases}$$

INITIAL CONDITION...

BASIC MODEL FOR VIBRATION

ODE

ORDINARY DIFF. EQUATION

IVB = INITIAL VALUE PROBLEM (ODE + ICs ...)

$$u_3 = \underbrace{a_1 u_1 + a_2 u_2}$$

SOLUTIONS!

$$m \frac{du_3}{dt} = -\frac{d\phi}{du_3} \overset{?}{=}$$

$$\overset{?}{=} a_1 \frac{d\phi(u_1)}{du_1} + a_2 \frac{d\phi(u_2)}{du_2}$$

$\phi = c u^2$ $\rightsquigarrow$ QUADRATIC POTENT.

$\Rightarrow$ LINEAR SYSTEM...

ENERGY CONSERVATION

$$W_{[\text{WORK}]} = \int_0^u F \, du$$

$$= \int_0^u m \frac{d^2u}{dt^2} \, du = - \int_0^u \frac{d\phi}{du} \, du$$

$$du = \frac{du}{dt} \, dt$$

$$\int_0^t m \frac{d^2u}{dt^2} \frac{du}{dt} \, dt = - \int_0^t \frac{d\phi}{du} \frac{du}{dt} \, dt$$

CHAIN RULE...

$$\int_0^t \frac{d}{dt} \left(\frac{m}{2} \left(\frac{du}{dt} \right)^2 \right) dt = - \int_0^t \frac{d\phi}{dt} \, dt$$

$$\int_0^t \frac{d}{dt} \underbrace{\left(\frac{m}{2} \left(\frac{du}{dt} \right)^2 + \phi \right)}_{= H} dt = 0$$

$= H$ (CONSTANT = ENERGY OF THE SYSTEM..)

$$H = \underbrace{\frac{m}{2} \left(\frac{du}{dt} \right)^2}_{H_K = \text{KINETIC ENERGY}} + \underbrace{\phi}_{H_P = \text{POTENTIAL ENERGY}} = \underbrace{H_0 := \frac{m}{2} v_0^2 + \phi(u_0)}_{\text{ENERGY AT TIME ZERO}}$$

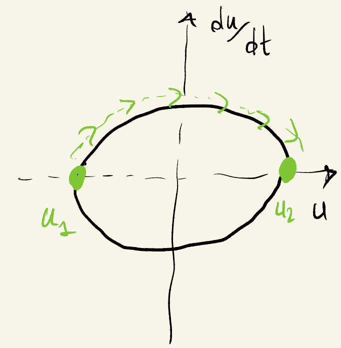
PERIODICITY

$$H_0 = \frac{m}{2} \left(\frac{du}{dt} \right)^2 + \phi$$

$$\left(\frac{du}{dt} \right)^2 = \frac{2}{m} \left[H_0 - \phi \right]$$

$$\frac{du}{dt} = \sqrt{\frac{2}{m}} \left[H_0 - \phi \right]^{1/2}$$

$$dt = \sqrt{\frac{m}{2}} \frac{du}{\sqrt{H_0 - \phi}}$$

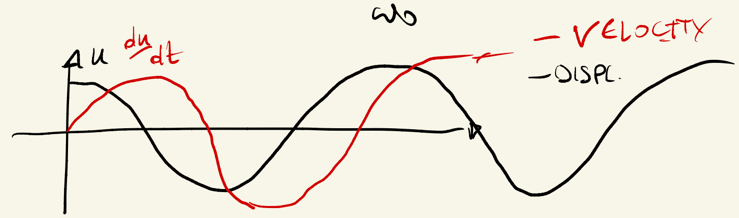


τ = PERIOD OF VIB.

$$= 2 \sqrt{\frac{m}{2}} \int_{u_1}^{u_2} \frac{du}{\sqrt{H_0 - \phi}}$$

(due to symmetry in Phase-Space)

$$u(t) = u_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)$$



du/dt = ---- ⇝ PHASE SHIFT

$\frac{du}{dt}$ IS SHIFTHED BY $\pi/2$

$$H > H_0 = \frac{K U^2}{2} = \frac{m V^2}{2}$$

U = LARGEST AMPLITUDE

V = LARGEST VELOCITY