



TIME DIFFERENCE OPERATORS

$f_s := \text{SAMPLE RATE}$ [$f_s := 4/k$
 (AUDIO RATE) $4.8 \times 10^3, 44.1 \times 10^3$ $k = \text{TIME STEP (s)}$]
 $u(t)$ $\xrightarrow{\quad}$ t (CONTINUOUS)
 $\boxed{\quad} \quad \boxed{n \kappa} \quad \boxed{u^n} \quad \boxed{u^{n+1}} \quad \boxed{u^{n+2}}$
 $n-1 \quad n \quad n+1 \quad n+2$ $n \in \mathbb{N}$

NYQUIST THEOREM (INFORMATION THEORY)

$f_{\max} = \text{LARGEST FREQ. IN A SIGNAL}$

$\hookrightarrow f_s \geq 2f_{\max}$ [f_{\max} FOR AUDIO
 $20 \times 10^3 \text{ Hz}$
 LIMIT OF HUMAN HEARING]

$u^n = \text{TIME SERIES}; u(t) = \text{CONTINUOUS SOLUTION}$

$$u(t_n) \approx u^n \quad t_n := nk$$

↑
APPROXIMATION ...

BUT WHAT DOES IT MEAN?
 { TRUNCATION ERROR (TAYLOR SERIES OF OPERATORS)
 GLOBAL ERROR (CUMULATIVE ERROR IN A TIME LOOP)

DEFINITIONS OF BASIC OPERATORS

48

- $\underbrace{1 u^n}_\text{IDENTITY} = u^n$

IDENTITY

$$\underbrace{e_t + u^n}_{\text{SHIFT OPS.}} = u^{n+2} \quad - e_t - u^n = u^{n-2}$$

SHIFT OPS.

$$S_{t+} := \frac{e_{t+} - 1}{k}$$

$\Delta t + u^n = \frac{u^{n+1} - u^n}{k}$ is THIS APPROXIMATES
THE TIME DERIVATIVE.
FORWARD DIFFERENCE OP.

$$St + u(t_n) = \frac{u(t_{n+1}) - u(t_n)}{\Delta t} \stackrel{(TAYLOR SERIES)}{\approx} \frac{du}{dt} \Big|_{t=t_n} + \frac{k}{2} \frac{d^2 u}{dt^2} \Big|_{t=t_n} = \frac{du}{dt} \Big|_{t=t_n} + O(k)$$

(TAYLOR SERIES)

$$u(t_{n+1}) \approx u(t_n) + k \frac{du}{dt} \Big|_{t=t_n} + \frac{k^2}{2} \frac{d^2u}{dt^2} \Big|_{t=t_n} + \dots$$

TRUNCATION ERR

Big-OH

二

FIRST-ORDER
ACCURATE

APPROXIMATION

TRUNCATION ERROR

$$\text{St-} := \frac{1 - e^{-t}}{k} \quad \text{BACKWARD DIFFERENCE OF}$$

$$\hookrightarrow St - u^n = \frac{u^n - u^{n-1}}{k} \quad \rightsquigarrow St - u(t_n) = \left. \frac{du}{dt} \right|_{t=t_n}$$

$$u(t_{n+1}) \approx u(t_n) - k \frac{du}{dt} \Big|_{t=t_n} + \frac{k^2}{2} \frac{d^2u}{dt^2} \Big|_{t=t_n} + \dots$$

FIRST-ORDER ACCURATE APPROXIMATION

$$\therefore S_t := \frac{e_{t+} - e_{t-}}{2k} \quad \text{CENTRED DIFF. OP.}$$

$$\Rightarrow \text{St. } u(t_{n_k}) = \frac{u(t_{n+1}) - u(t_{n-1})}{2k} = \dots$$

(TAYLOR SERIES)

$m-3$ n $n+1$

四

4

$$\frac{du}{dt} \Big|_{t=t_0} + \mathcal{O}(k^2)$$

SECOND-ORDER
ACCURATE
APPROXIMATION!

DIFFERENCE

OPERATORS

SECOND DIFFERENCE

$$S_{tt} := S_{t+} S_{t-} = \underbrace{S_t}_{\text{COMPOSITION}} - \underbrace{S_{t+}}$$

$$\begin{aligned} S_{tt} &:= S_{t+} S_{t-} - u^n = S_{t+} \left[\frac{u^n - e_t - u}{k} \right] = \frac{1}{k} (S_t + u^n) - \frac{1}{k} S_{t+} u^{n-1} \\ &\quad \text{TRUNC. ERR.} = \frac{1}{k^2} [u^{n+1} - u^n] - \frac{1}{k^2} [u^n - u^{n-1}] \\ (\Rightarrow S_{tt} u(t_n)) &= \frac{d^2 u}{dt^2} \Big|_{t=t_n} + O(k^2) = \frac{u^{n+2} - 2u^n + u^{n-1}}{k^2} \quad \text{CENTRED SECOND DIFF. OP.} \\ &\quad \text{SECOND- ORDER ACCURATE APPROXIMATION} \end{aligned}$$

HIGHER ACCURATE APPROXIMATIONS

How,²

USE INTERPOLATION!

$$\begin{matrix} n-2 & n-2 & n & n+1 & n+2 \\ \square & \square & 0 & \square & \square \\ (a) & b & c & d & e \end{matrix} \rightsquigarrow$$

USE A LARGE STENCIL

$$\begin{aligned} S_{t+} u(t_n) &:= \frac{u(t_{n-2}) - 8u(t_{n-1}) + 8u(t_{n+1}) - u(t_{n+2})}{12k} \\ &= \frac{du}{dt} \Big|_{t=t_n} + O(k^4) \end{aligned}$$

PROBLEMS:

- [1st MORE COEFFICIENTS TO STORE IN MEMORY]
- 2nd STABILITY, --

MORE NOTATION

$$\mu_{t+} := \frac{e^{t+} + 1}{2}$$

$$\hookrightarrow \mu_{t+} u^n = \frac{u^{n+1} + u^n}{2}$$

FORWARD AVERAGING
OPERATOR

1st. ORDER
ACC.

$$\Rightarrow \mu_{t+} u(t_n) = u(t_n) + O(h)$$

$$\mu_{t-} := \frac{1 + e^{t-}}{2}$$

$$\Rightarrow \mu_{t-} u^n = \frac{u^n + u^{n-1}}{2}$$

BACKWARD
AVERAG.
OPERATOR

1st ORDER
ACCURATE

$$\Rightarrow \mu_{t-} u(t_n) = u(t_n) + O(h)$$

$$\mu_t := \frac{e^{t+} + e^{t-}}{2}$$

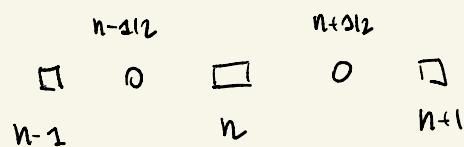
$$\Rightarrow \mu_t u^n = \frac{u^{n+1} + u^{n-1}}{2}$$

CENTRED
AVERAGING
OP.

$$\Rightarrow \mu_t u(t_n) = u(t_n) + O(h^2)$$

2nd ORDER
ACCURATE

INTERLEAVED TIME SERIES



$$S_t + u^n = \frac{u^{n+1} - u^n}{h} := v^{n+1/2}$$

$$S_t - S_t + u^n = S_{t+} u^n = \underbrace{\frac{d^2 u}{dt^2}}_{v^{n+1/2}} + O(h^2)$$

$$\hookrightarrow S_t - v^{n+1/2} = \frac{d v}{dt} + O(h^2)$$

2nd ORDER
ACCURATE
WHEN APPLIED

To AN INTERLEAVED TIME SERIES --

ENERGY IDENTITIES

$$\frac{du}{dt} \frac{d^2 u}{dt^2} = \frac{1}{2} \frac{d}{dt} \left(\frac{du}{dt} \right)^2$$

TIME DER. OF KINETIC EN.

Tricks allowing to recover the kinetic and potential energies of step]

$$\frac{du}{dt} u = \frac{1}{2} \frac{d}{dt} u^2 \Rightarrow \geq 0 \text{ (SQUARED)}$$

INVOLVED POTENTIAL ENERGY

CONTINUOUS TIME



DISCRETE TIME



$$\underbrace{St \cdot u^n}_{\sim} \underbrace{St u^n}_{\sim} = \frac{1}{2} St + (\delta t - u^n)^2 \\ = \frac{1}{2} St - (\delta t + u^n)^2$$

IT'S JUST AN IDENTITY...
YOU CAN PROVE IT BY SIMPLE ALGEBRA

$$\underbrace{u^n}_{\sim} \underbrace{St \cdot u^n}_{\sim} = \frac{1}{2} St + \underbrace{(u^n \delta t - u^n)}_{\sim} = \frac{1}{2} St - (u^n \delta t + u^n)$$

THIS IS NOT A SQUARED (≥ 0) QUANTITY
... THIS WILL NEED A STABILITY CONDITION --

$$\underbrace{u^n \delta t - u^n}_{\geq 0} = (\mu t - u^n)^2 - \frac{k^2}{4} (St - u^n)^2 \leq 0$$

APPROXIMATING THE SHO

$$\frac{d^2 u}{dt^2} = -\omega_0^2 u$$

$\downarrow S_{tt} u^n = -\omega_0^2 u^n$

$\left[\begin{array}{l} u^0 \\ u^1 \end{array} \right] = \text{INITIAL CONDITIONS}$
 $u(0); \frac{du(0)}{dt}$

$u^{n+1} - 2u^n + u^{n-1} = -\omega_0^2 k^2 u^n$

$\downarrow u^{n+1} = (2 - \omega_0^2 k^2) u^n - u^{n-1}$

UPDATE EQUATION

$\underbrace{\omega_0^2}_{\text{known}} \quad \underbrace{k^2}_{\text{known}}$
 $\underbrace{u^n}_{\text{FROM PREVIOUS TIME STEPS}}$

STABILITY

THE SCHEME IS SECOND ORDER ACCURATE..
 BUT NOT ALWAYS STABLE!!

$$\underbrace{S_t \cdot u^n}_{\sim} \underbrace{S_{tt} u^n}_{\sim} = -\omega_0^2 \underbrace{u^n}_{\sim} (\underbrace{S_t \cdot u^n}_{\sim})$$

$$S_t + \left[\frac{1}{2} (S_t - u^n)^2 \right] = S_t + \left[-\frac{\omega_0^2}{2} (u^n \cdot t - u^n) \right]$$

$$S_t + \left[\frac{1}{2} (S_t - u^n)^2 + \frac{\omega_0^2}{2} (u^n \cdot t - u^n) \right] = 0$$

ENERGY BALANCE EQUATION

$$S_t + \left[\frac{1}{2} (S_t - u^n)^2 + \frac{\omega_0^2}{2} (u^n \cdot t - u^n)^2 - \frac{\omega_0^2 k^2}{8} (S_t - u^n)^2 \right] = 0$$

USE IDENTITIES
FROM PREVIOUS PAGE

PSEUDO CODE

for $n = 2 : T_{samples}$

$$u^{n+1} = (2 - \omega_0^2 k^2) u^n - u^{n-1};$$

$$u^{n-1} = u^n; u^n = u^{n+1};$$

$\Rightarrow \text{out}[n] = u^n -$

end

ENERGY EXPRESSION: AN INTERLEAVED TIME SERIES

$$H^{n+1/2} = \left[1 - \frac{\omega_0^2 k^2}{4} \right] \frac{1}{2} (S_t - u^n)^2 + \frac{\omega_0^2}{2} (u^n \cdot t - u^n)^2$$

$$> 0 \quad \frac{1 - \omega_0^2 k^2}{4} > 0$$

$$k < \frac{2}{\omega_0}$$

STABILITY
CONDITION!!

TAKE HOME CONCEPT:

STABILITY OF A TIME-STEPPING SCHEME AMOUNTS TO FINDING CONDITION FOR WHICH A DISCRETE EQUIVALENT OF THE ENERGY IS NON-NEGATIVE