



# Module 2.4 - Gradients

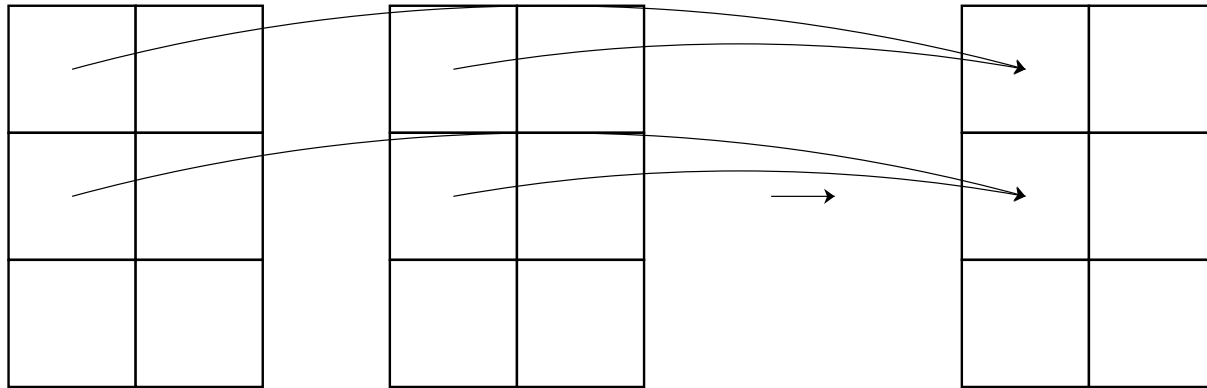


# Rules

- **Rule 1:** Dimension of size 1 broadcasts with anything
- **Rule 2:** Extra dimensions of 1 can be added with `view`
- **Rule 3:** Zip automatically adds starting dims of size 1

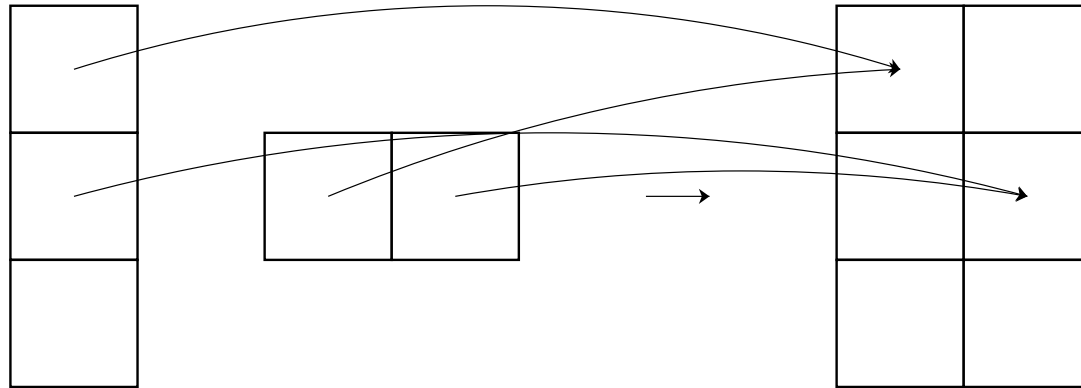


# Zip





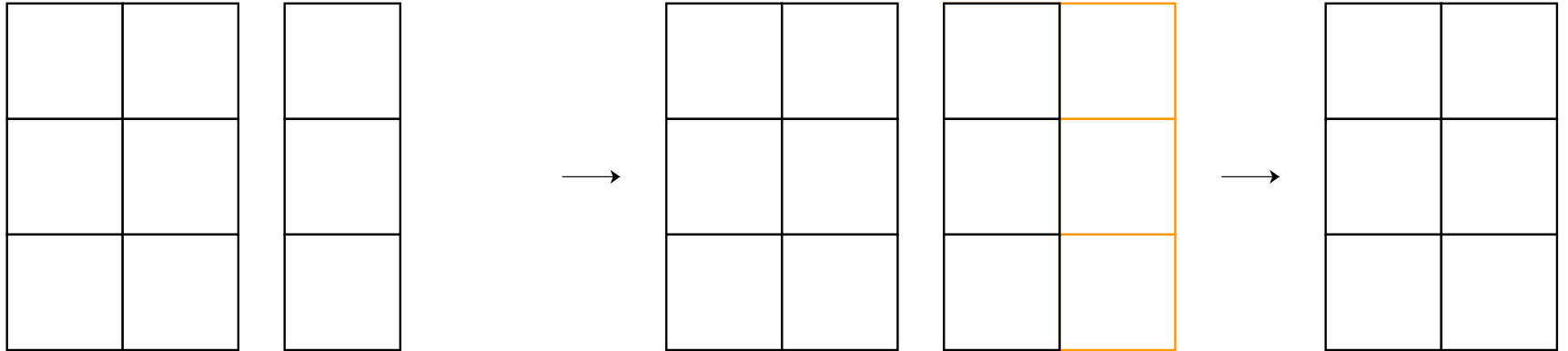
# Zip Broadcasting





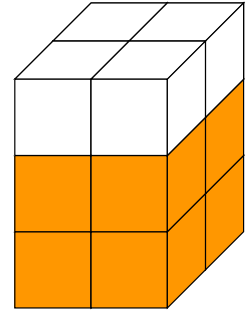
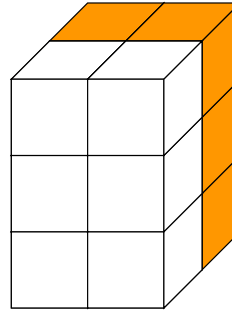
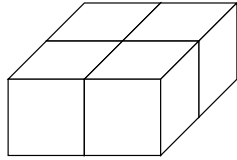
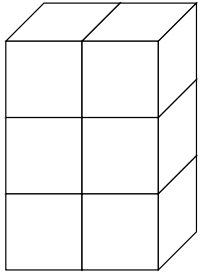


# Matrix-Vector





# Example





# Quiz



# Implementation





# Low-level Operations

- map
- zip
- reduce



# Backends

- Simple backend for debugging
- CPU implementation
- GPU implementation
- ...



# Where is the backend?

- Torch: Stored on the tensor

## Other Options:

- Inferred by environment
- Compiled



# Low-level Operations

```
class TensorOps:
    @staticmethod
    def map(fn: Callable[[float], float]) -> Callable[[Tensor], Tensor]:
        pass

    @staticmethod
    def zip(fn: Callable[[float, float], float]) -> Callable[[Tensor, Tensor],
Tensor]:
        pass

    @staticmethod
    def reduce(
        fn: Callable[[float, float], float], start: float = 0.0
    ) -> Callable[[Tensor, int], Tensor]:
        pass
```





# Constructed Operations

- Stored on tensor `tensor_op.py`

```
self.neg_map = ops.map(operators.neg)
self.sigmoid_map = ops.map(operators.sigmoid)
self.relu_map = ops.map(operators.relu)
self.log_map = ops.map(operators.log)
self.exp_map = ops.map(operators.exp)
self.id_map = ops.map(operators.id)
```



# How to use

```
t1 = minitorch.tensor([1, 2, 3])  
t1.f.neg_map(t1)
```

```
[-1.00 -2.00 -3.00]
```



# Implementation Tips

- Map
- Zip
- Reduce



# Gradients





# Derivatives

- A function with a tensor input is like multiple args
- A function with a tensor output is like multiple functions
- Backward: chain rule from each output to each input.



# Terminology

- Scalar  $\rightarrow$  Tensor
- Derivative  $\rightarrow$  Gradient
- Recommendation: Reason through gradients as many derivatives



# Example

## What is backward?

```
x = minitorch.rand((4, 5), requires_grad=True)
y = minitorch.rand((4, 5), requires_grad=True)
z = x * y
z.sum().backward()
```



# Notation: Gradient

Function from tensor to a scalar

$$f([x_1, x_2, \dots, x_N])$$







# Gradient

$$f'_{x_1}([x_1, x_2, \dots, x_N])$$

$$f'_{x_2}([x_1, x_2, \dots, x_N])$$

...

$$f'_{x_N}([x_1, x_2, \dots, x_N])$$

Each is a standard derivative



# Gradient

$$\begin{aligned} &[f'_{x_1}([x_1, x_2, \dots, x_N]), \\ &f'_{x_2}([x_1, x_2, \dots, x_N]), \\ &\qquad\qquad\qquad \dots \\ &f'_{x_N}([x_1, x_2, \dots, x_N])] \end{aligned}$$

Tensor of derivatives.



# Function to Tensor

Function to a tensor

$$G(x)$$



# Function to Tensor

Think of it as many functions

$$g^1(x), g^2(x), \dots, g^N(x)$$





# Function to Tensor

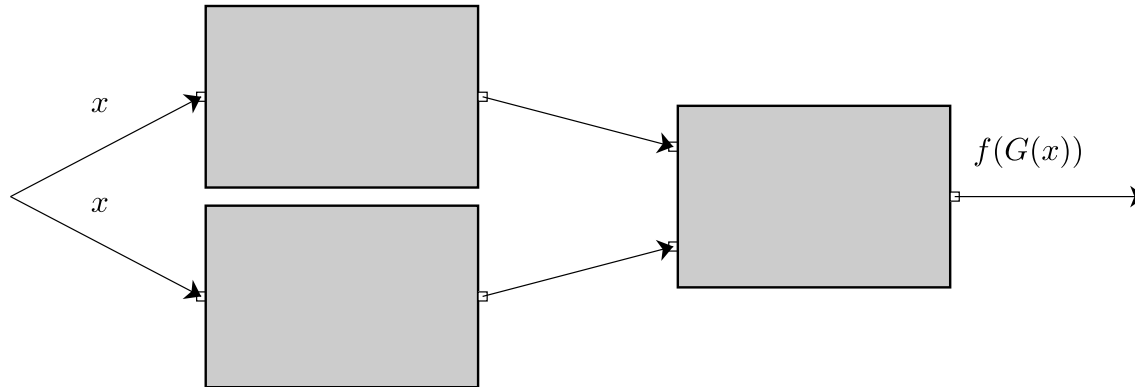
Think of it as many functions

$$G(x) = [g^1(x), g^2(x), \dots, g^N(x)]$$



# Example: Chain Rule For Gradients

- $G(x) = [g^1(x), g^2(x)]$  - scalar to tensor
- $f(x)$  - tensor to scalar

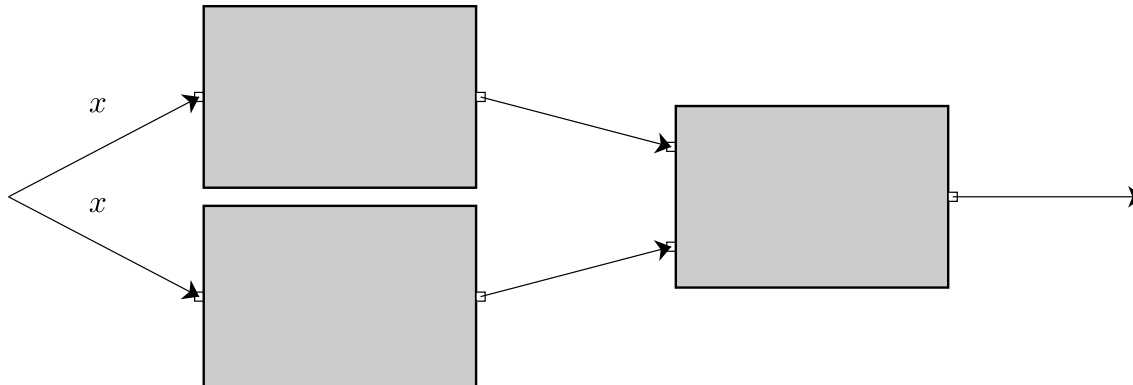




# Review: Chain Rule

$$f(G(x))$$

- $z_1 = g^1(x), z_2 = g^2(x)$
- $d_1 = f'_{z_1}(z_1, z_2), d_2 = f'_{z_2}(z_1, z_2)$
- $f'_x(G(x)) = d_1 g'^1_x(x) + d_2 g'^2_x(x)$

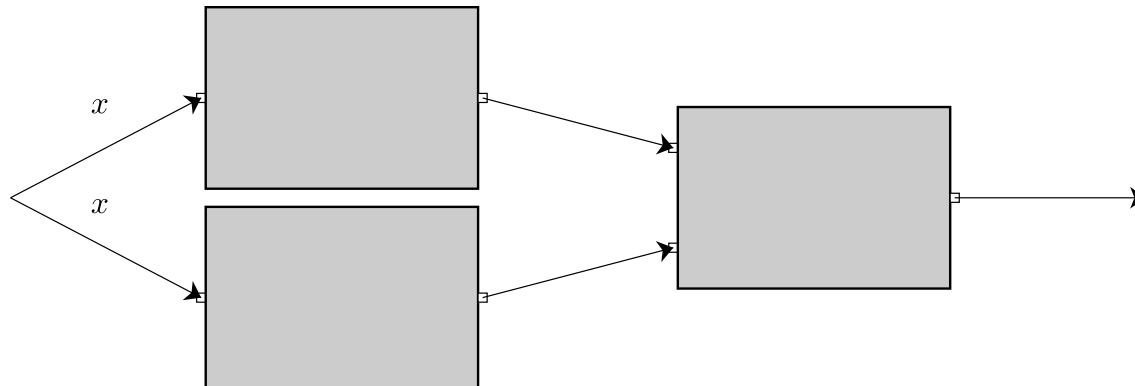




# Review: Chain Rule

$$f(G(x))$$

- $z_1 = g^1(x), z_2 = g^2(x), \dots$
- $d_1 = f'_{z_1}(z), d_2 = f'_{z_2}(z), \dots$
- $f'_{\{x\}}(G(x)) = \sum_i d_i g'^{\{i\}\{x\}}(x)$







# Tensor-to-Tensor

$$G([x_1, \dots, x_N]) = [G^1([x_1, \dots, x_N]), \dots]$$



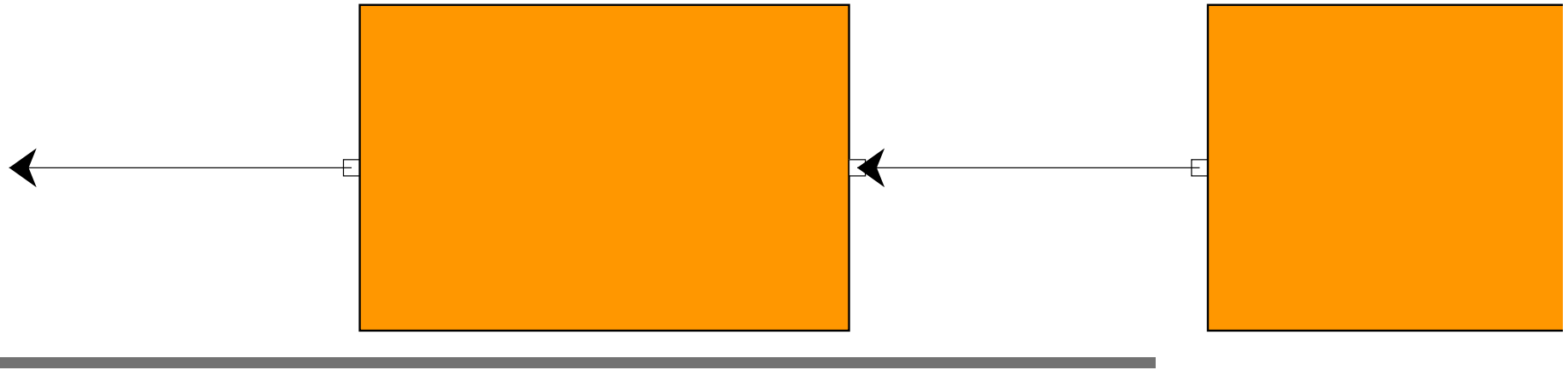
# Chain Rule For Gradients

$$f(G(x))$$

- $z_1 = G^1(x), z_2 = G^2(x), \dots$
- $d_1 = f'_{z_1}(z), d_2 = f'_{z_2}(z), \dots$
- $f'_{x_j}(G(x)) = \sum_i d_i G'^i_{x_j}(x)$



# Chain Rule For Gradients





# Avoiding Gradient Math

- All of this is just notation for scalars
- Can often reason about it with scalars directly





# Special Function: Map

$$G_{x_j}'^i([x_1, \dots, x_N]) \text{ ?}$$

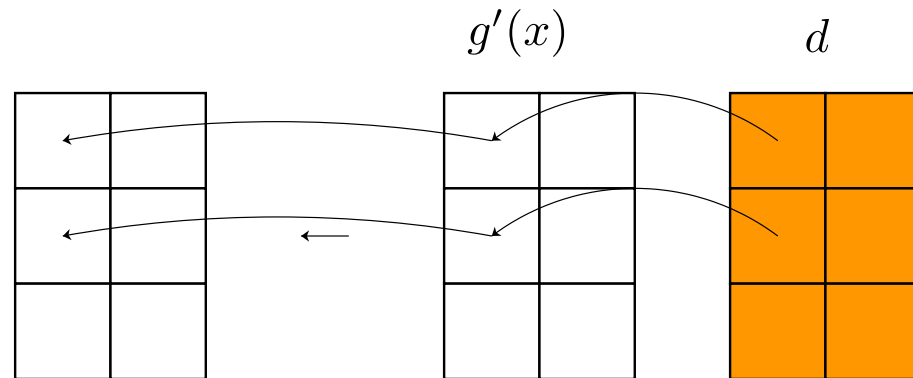


# Special Function: Map

- $G_{x_j}'^i(x) = 0$  if  $i \neq j$
- $f_{x_j}'(G(x)) = d_i g_{x_j}'^j(x)$



# Map Gradient





# Example: Negation

```
class Neg(minitorch.ScalarFunction):  
    @staticmethod  
    def forward(ctx, a: float) -> float:  
        return -a  
  
    @staticmethod  
    def backward(ctx, d: float) -> float:  
        return -d
```





# Example: Tensor Negation

```
class Neg(minitorch.Function):  
    @staticmethod  
    def forward(ctx, t1: Tensor) -> Tensor:  
        return t1.f.neg_map(t1)  
  
    @staticmethod  
    def backward(ctx, d: Tensor) -> Tensor:  
        return d.f.neg_map(d)
```



# Example: Inv

```
class Inv(minitorch.Function):
    @staticmethod
    def forward(ctx, t1: Tensor) -> Tensor:
        ctx.save_for_backward(t1)
        return t1.f.inv_map(t1)

    @staticmethod
    def backward(ctx, d: Tensor) -> Tensor:
        (t1,) = ctx.saved_values
        return d.f.inv_back_zip(t1, d)
```



# Special Function: Zip

$$G_{x_j}^{'i}(x, y) ?$$



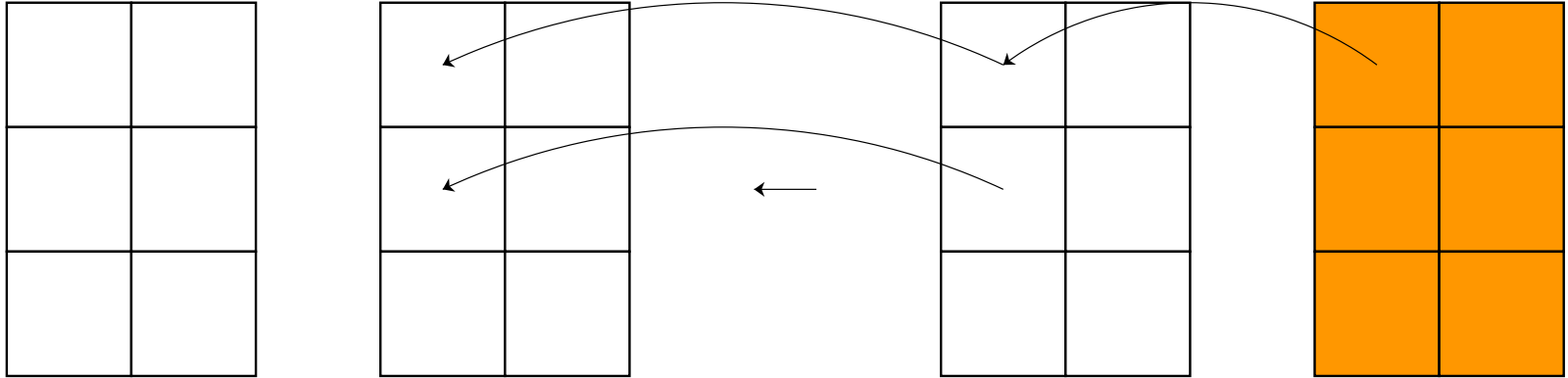
# Special Function: Map

- $G_{x_j}'^i(x) = 0$  if  $i \neq j$
- $f_{x_j}'(G(x)) = d_i g_{x_j}'^j(x, y)$





# Zip Gradient



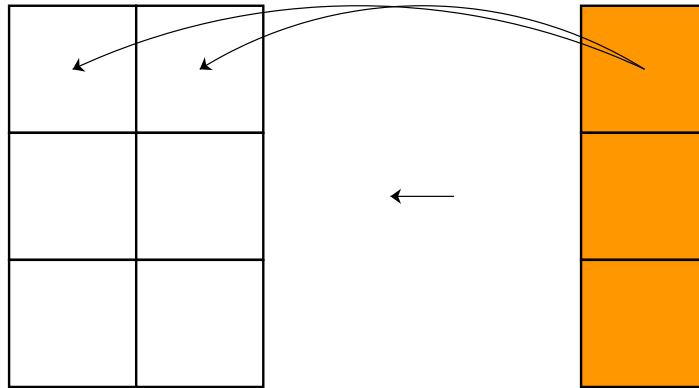


# Example: Add

```
class Add(minitorch.Function):  
    @staticmethod  
    def forward(ctx, t1: Tensor, t2: Tensor) -> Tensor:  
        return t1.f.add_zip(t1, t2)  
  
    @staticmethod  
    def backward(ctx, grad_output: Tensor) -> Tuple[Tensor, Tensor]:  
        return grad_output, grad_output
```



# Reduce Gradient





# Example: Sum

```
class Sum(minitorch.Function):
    @staticmethod
    def forward(ctx, a: Tensor, dim: Tensor) -> Tensor:
        ctx.save_for_backward(a.shape, dim)
        return a.f.add_reduce(a, int(dim.item()))

    @staticmethod
    def backward(ctx, grad_output: Tensor) -> Tuple[Tensor, float]:
        a_shape, dim = ctx.saved_values
        return grad_output, 0.0
```





Q&A

