Exact Real Arithmetic in Haskell

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Numbers

Natural numbers: Counting numbers $\{0, 1, 2, \ldots\}$

Integers: Natural numbers and negatives $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Rational: Fractions $\{\frac{p}{q} : p, q \text{ are integers}\}$

Reals: All values on the continuum

Floating point

$$\begin{pmatrix} 64919121 & -159018721 \\ 41869520.5 & -102558961 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

```haskell
solveAxb :: Fractional t =>
            (t, t, t, t) -> (t, t) -> (t, t)
solveAxb (a11, a12,
          a21, a22)
         (b1,
          b2)
  = (( a22 * b1 - a12 * b2) / det,
     (-a21 * b1 + a11 * b2) / det)
  where det = a11 * a22 - a12 * a21

a :: Fractional t => (t, t, t, t)
a = (64919121,   -159018721,
     41869520.5, -102558961)

b :: Fractional t => (t, t)
b = (1,
     0)
```

Using `Doubles`:

$$x = \begin{pmatrix} 102558961 \\ 41869520.5 \end{pmatrix}$$

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$$x = \begin{pmatrix} 102558961 \\ 41869520.5 \end{pmatrix}$$

Actually...

$$x = \begin{pmatrix} 205117922 \\ 83739041 \end{pmatrix}$$

Almost integers

`Doubles`:

$$\sin(2017\sqrt[5]{2}) = -1$$

Almost integers

`Doubles`:

$$\xcancel{\sin(2017\sqrt[5]{2}) = -1}$$

Actually:

$$\sin(2017\sqrt[5]{2}) = -0.9999999999999999785$$

Arbitrary precision arithmetic

Arbitrary precision *integer* arithmetic comes built in to Haskell (and Python and Ruby and ...)

Can use this to implement arbitrary precision floating point, i.e. $n \times b^c$

Doesn't save us with `sqrt`, `sin`, `pi` ...

Exact arithmetic

Represents any (computable) real number *exactly*

Transcendental functions in `Floating` are no longer approximations

We are able to request any output precision, and the details are handled for us

Cauchy Sequences

Definition

A *Cauchy sequence* is a sequence of rational numbers $\{x_0, x_1, \ldots, x_i, \ldots\}$ such that for any ϵ, there exists an N such that

$$|x_n - x_m| < \epsilon$$

for any $m > N, n > N$.

The real numbers are *defined* to be the set of all Cauchy sequences (where we consider two sequences to be the same if their difference converges to 0)

$$\frac{1}{3} = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots\}$$

$$\pi = \{3, 3.1, 3.14, 3.141, \dots\}$$

Effective Cauchy

Definition

A real number x is represented as an *effective Cauchy sequence* if there is a sequence of rational numbers $\{x_0, x_1, \ldots, x_i, \ldots\}$ such that

$$|x - x_p| < 2^{-p}$$

$$\frac{1}{3} = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{5}{16}, \frac{11}{32}, \ldots\}$$

$$\pi = \{\frac{3}{1}, \frac{6}{2}, \frac{13}{4}, \frac{25}{8}, \frac{50}{16}, \frac{101}{32}, \ldots\}$$

Fast Binary Cauchy

Definition

A real number x is represented as a *Fast Binary Cauchy Sequence* if there is a sequence of *integers* $\{n_0, n_1, \ldots, n_i, \ldots\}$ such that

$$|x - 2^{-p} n_p| < 2^{-p}$$

$$\frac{1}{3} = \{0, 1, 1, 3, 5, 11, \ldots\}$$
$$\pi = \{3, 6, 13, 25, 50, 101, \ldots\}$$

```
type CReal = Natural -> Integer
```

$$\frac{x[p]-1}{2^p} < x < \frac{x[p]+1}{2^p}$$

Easy Stuff

```
fromInteger :: Integer -> CReal
fromInteger n = \p -> n * 2^p

negate :: CReal -> CReal
negate x = \p -> negate (x p)
```

Addition

If:

$$\frac{a[p+2]-1}{2^{p+2}} < a < \frac{a[p+2]+1}{2^{p+2}}$$
$$\frac{b[p+2]-1}{2^{p+2}} < b < \frac{b[p+2]+1}{2^{p+2}}$$

then:

$$\frac{r-1}{2^p} < a+b < \frac{r+1}{2^p}$$

where:

$$r = \lfloor \frac{a[p+2]+b[p+2]}{4} \rceil$$

```
(+) :: CReal -> CReal -> CReal
a + b = \p -> round $ ((a (p+2) + b (p+2)) % 4
```

Transcendental functions

Evaluated using Taylor series:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

If $|x| < 1$, then eventually the terms are very small

Disadvantages

Result arrives all at once

If we later need more precision, we have to start all over

Decimal representation

Consider π:

$$\pi = 3$$

Decimal representation

Consider π:

$$\pi = 3 + 0.1415926\ldots$$

Decimal representation

Consider π:

$$\pi = 3 + \frac{1}{10}(1)$$

Decimal representation

Consider π:

$$\pi = 3 + \frac{1}{10}(1 + 0.415926\ldots)$$

Decimal representation

Consider π:

$$\pi = 3 + \frac{1}{10}(1 + \frac{1}{10}(4))$$

Decimal representation

Consider π:

$$\pi = 3 + \frac{1}{10}(1 + \frac{1}{10}(4 + 0.15926\ldots))$$

Decimal representation

Consider π:

$$\pi = 3 + \frac{1}{10}(1 + \frac{1}{10}(4 + \frac{1}{10}(1)))$$

Decimal representation

Consider π:

$$\pi = 3 + \frac{1}{10}(1 + \frac{1}{10}(4 + \frac{1}{10}(1 + 0.5926\ldots)))$$

Decimal representation

Consider π:

$$\pi = 3 + \frac{1}{10}(1 + \frac{1}{10}(4 + \frac{1}{10}(1 + \frac{1}{10}(5 + \ldots))))$$

Decimal representation

```
decimal :: RealFrac a => a -> [Integer]
decimal a = let d = floor a in
            d : decimal ((a - fromInteger d) * 10)

decimal (1%3) => [0,3,3,3,3,3,3,...
decimal pi    => [3,1,4,1,5,9,2,...
```

Decimal representation

Some problems:

$$\frac{1}{3} = 0.3333333333333333333333333333333333333333333333333333333333333$$

Implementing `Floating` on a stream of decimal digits would be nasty

Why the 10?!

Continued fractions

Consider π:

$$\pi = 3$$

Continued fractions

Consider π:

$$\pi = 3 + 0.1415926\ldots$$

Continued fractions

Consider π:

$$\pi = 3 + \frac{1}{7}$$

Continued fractions

Consider π:

$$\pi = 3 + \frac{1}{7 + 0.0625132\ldots}$$

Continued fractions

Consider π:

$$\pi = 3 + \frac{1}{7 + \frac{1}{15}}$$

Continued fractions

Consider π:

$$\pi = 3 + \cfrac{1}{7 + \frac{1}{15+0.9965996} \dots}$$

Continued fractions

Consider π:

$$\pi = 3 + \cfrac{1}{7 + \frac{1}{15 + \frac{1}{1}}}$$

Continued fractions

Consider π:

$$\pi = 3 + \cfrac{1}{7 + \frac{1}{15 + \frac{1}{1 + 0.0034172\ldots}}}$$

Continued fractions

Consider π:

$$\pi = 3 + \cfrac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \dots}}}}$$

Let us write this as $\pi = [3, 7, 15, 1, 292, \ldots]$

```
type CF = [Integer]

cf :: Fractional a => a -> CF
cf a | fromInteger (floor a) == a = [a]
cf a | otherwise = let d = floor a in
         d : cf (recip (a - fromInteger d))

cf (1%3) => [0,3]
cf pi    => [3,7,15,1,292,...
```

Every real number has a (essentially) unique expansion

This expansion is finite when the number is rational

Arithmetic

Let us consider functions of the form:

$$\frac{ax + b}{cx + d}$$

with a, b, c, d all integers.

```
type Hom = (Integer, Integer,
            Integer, Integer)

hom :: Hom -> [Integer] -> [Integer]
```

Let $x = [x_0, \ldots] = x_0 + \frac{1}{x'}$, then:

$$\frac{ax+b}{cx+d} = \frac{a(x_0 + \frac{1}{x'}) + b}{c(x_0 + \frac{1}{x'}) + d}$$

Let $x = [x_0, ...] = x_0 + \frac{1}{x'}$, then:

$$\begin{aligned}\frac{ax+b}{cx+d} &= \frac{a(x_0 + \frac{1}{x'}) + b}{c(x_0 + \frac{1}{x'}) + d} \\ &= \frac{a(x_0 + \frac{1}{x'}) + b}{c(x_0 + \frac{1}{x'}) + d}\end{aligned}$$

Let $x = [x_0, ...] = x_0 + \frac{1}{x'}$, then:

$$\begin{aligned}
\frac{ax+b}{cx+d} &= \frac{a(x_0 + \frac{1}{x'}) + b}{c(x_0 + \frac{1}{x'}) + d} \\
&= \frac{a(x_0 + \frac{1}{x'}) + b}{c(x_0 + \frac{1}{x'}) + d} \\
&= \frac{ax_0 + a\frac{1}{x'} + b}{cx_0 + c\frac{1}{x'} + d}
\end{aligned}$$

Let $x = [x_0, ...] = x_0 + \frac{1}{x'}$, then:

$$\begin{aligned}
\frac{ax+b}{cx+d} &= \frac{a(x_0 + \frac{1}{x'}) + b}{c(x_0 + \frac{1}{x'}) + d} \\
&= \frac{a(x_0 + \frac{1}{x'}) + b}{c(x_0 + \frac{1}{x'}) + d} \\
&= \frac{ax_0 + a\frac{1}{x'} + b}{cx_0 + c\frac{1}{x'} + d} \\
&= \frac{ax_0x' + a + bx'}{cx_0x' + c + dx'}
\end{aligned}$$

Let $x = [x_0, ...] = x_0 + \frac{1}{x'}$, then:

$$\begin{aligned}
\frac{ax+b}{cx+d} &= \frac{a(x_0 + \frac{1}{x'}) + b}{c(x_0 + \frac{1}{x'}) + d} \\
&= \frac{a(x_0 + \frac{1}{x'}) + b}{c(x_0 + \frac{1}{x'}) + d} \\
&= \frac{ax_0 + a\frac{1}{x'} + b}{cx_0 + c\frac{1}{x'} + d} \\
&= \frac{ax_0x' + a + bx'}{cx_0x' + c + dx'} \\
&= \frac{(ax_0 + b)x' + a}{(cx_0 + d)x' + c}
\end{aligned}$$

```
absorb :: Hom -> Integer -> Hom
absorb (a, b
        c, d) x0 = (a*x0 + b, a,
                     c*x0 + d, c)

hom h (x0:rest) == hom (absorb h x0) rest
```

Let $z = \frac{ax+b}{cx+d}$. As $x \in [0, \infty)$, z must lie between $\frac{a}{c}$ and $\frac{b}{d}$.

So if $\frac{a}{c}$ and $\frac{b}{d}$ have the same integer part q, we know for sure that $z = [q, \ldots]$.

```
tryEmit :: Hom -> Maybe Integer
tryEmit (a, b
         c, d) = if c /= 0 && d /= 0 && r == s then
                    Just r
                  else
                    Nothing
  where r = a `quot` c
        s = b `quot` d
```

Let $z = q + \frac{1}{z'}$, then:

$$z' = (z - q)^{-1}$$

Let $z = q + \frac{1}{z'}$, then:

$$\begin{aligned} z' &= (z - q)^{-1} \\ &= \left(\frac{ax + b}{cx + d} - q \right)^{-1} \end{aligned}$$

Let $z = q + \frac{1}{z'}$, then:

$$\begin{aligned} z' &= (z - q)^{-1} \\ &= \left(\frac{ax+b}{cx+d} - q \right)^{-1} \\ &= \left(\frac{(ax+b) - q(cx+d)}{cx+d} \right)^{-1} \end{aligned}$$

Let $z = q + \frac{1}{z'}$, then:

$$\begin{aligned}
z' &= (z - q)^{-1} \\
&= \left(\frac{ax + b}{cx + d} - q \right)^{-1} \\
&= \left(\frac{(ax + b) - q(cx + d)}{cx + d} \right)^{-1} \\
&= \left(\frac{(a - cq)x + (b - dq)}{cx + d} \right)^{-1}
\end{aligned}$$

Let $z = q + \frac{1}{z'}$, then:

$$
\begin{aligned}
z' &= (z-q)^{-1} \\
&= \left(\frac{ax+b}{cx+d} - q\right)^{-1} \\
&= \left(\frac{(ax+b) - q(cx+d)}{cx+d}\right)^{-1} \\
&= \left(\frac{(a-cq)x + (b-dq)}{cx+d}\right)^{-1} \\
&= \frac{cx+d}{(a-cq)x + (b-dq)}
\end{aligned}
$$

```
emit :: Hom -> Integer -> Hom
emit (a, b
      c, d) q = (c,        d,
                 a - c*q, b - d*q)

hom h cf == q : hom (emit q h) cf
```

```
hom :: Hom -> CF -> CF
hom h (x:xs) = case tryEmit h of
                 Just d -> d : hom (emit h d) (x:xs)
                 Nothing ->    hom (absorb h x) xs

2 * pi == hom (2, 0,
               0, 1) pi
       == CF [6,3,1,1,7,2,146,3,6,1]
```

To do arithmetic, repeat all of the above with

$$\frac{axy + bx + cy + d}{exy + fx + gy + h}$$

```
type Bihom = (Integer, Integer, Integer, Integer,
              Integer, Integer, Integer, Integer)
```

```
bihom :: Bihom -> CF -> CF -> CF
```

Now we can absorb from either x or y, and emit similar to before.

```
(+) = bihom (0, 1, 1, 0,
             0, 0, 0, 1)
(-) = bihom (0, 1, -1, 0,
             0, 0,  0, 1)
(*) = bihom (1, 0, 0, 0,
             0, 0, 0, 1)
(/) = bihom (0, 1, 0, 0,
             0, 0, 1, 0)
```

Transcendental functions

$$e^x = [1, \frac{1}{x}, -2, \frac{3}{x}, 2, \frac{5}{x}, -2, \dots]$$

$$\log x = [0, \frac{1}{x-1}, \frac{2}{1}, \frac{3}{x-1}, \frac{2}{3}, \frac{5}{x-1}, \frac{2}{5}, \dots]$$

Transcendental functions

$$e^x = [1, \frac{1}{x}, -2, \frac{3}{x}, 2, \frac{5}{x}, -2, \ldots]$$

$$\log x = [0, \frac{1}{x-1}, \frac{2}{1}, \frac{3}{x-1}, \frac{2}{3}, \frac{5}{x-1}, \frac{2}{5}, \ldots]$$

```
type Hom a = (a, a, a, a)

hom :: (Num a, ...) => Hom a -> [a] -> CF

cfcf :: [CF] -> CF
cfcf = hom (1, 0, 0, 1)
```

Disadvantages

Much slower than Cauchy sequences

Code

Fast Binary Cauchy:
`http://hackage.haskell.org/package/numbers`

Continued Fractions:
`http://github.com/mvr/cf`

- Ralph W Gosper. "Continued fraction arithmetic". In: *HAKMEM Item 101B, MIT Artificial Intelligence Memo* 239 (1972).
- David R Lester. "Vuillemin's exact real arithmetic". In: *Functional Programming, Glasgow 1991*. Springer, 1992, pp. 225–238.
- Jean E Vuillemin. "Exact real computer arithmetic with continued fractions". In: *Computers, IEEE Transactions on* 39.8 (1990), pp. 1087–1105.