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Relativistic plasma physics

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ABSTRACT

Basic plasma parameters are presented and succintly discussed here.

Key words. Plasmas – Turbulence – Methods: numerical

1. Theory

1.1. Plasma parameters

Two important frequencies in a plasma are the plasma frequency,

$$\omega_p \equiv \sqrt{\frac{4\pi n q^2}{\gamma_0 m_e}}, \tag{1}$$

and cyclotron frequency,

$$\omega_B \equiv \frac{qB}{\gamma_0 m_e c}. \tag{2}$$

Particle Lorentz factor is defined as

$$\gamma = \sqrt{\frac{1}{1-\beta^2}} = \sqrt{1+\frac{u^2}{c^2}}, \tag{3}$$

where β is the particle coordinate velocity and u is the (spatial component of) four-velocity. Additionally, $u/c = \gamma\beta$. The relativistic Lorentz factor, γ_0, present in above formulae can either be some bulk Lorentz factor of the flow ($\gamma_0 = \Gamma$) or the mean thermal Lorentz factor of the plasma ($\gamma_0 = \langle\gamma\rangle$), depending on the problem at hand.

The frequencies can be used to define two characteristic length scales, skin depth

$$\lambda_p \equiv \frac{c}{\omega_p}, \tag{4}$$

and Larmor radius (gyrorotation orbit)

$$\lambda_L \equiv \frac{c}{\omega_B}. \tag{5}$$

Additionally, we note that the magnetization, defined more rigorously later on, is related to the above frequencies and length scales as

$$\sigma = \frac{\omega_B^2}{\omega_p^2} = \frac{\lambda_p^2}{\lambda_L^2}. \tag{6}$$

Note also that relativistic time dilatation and length stretching work such that $\omega_{p,\mathrm{rel}} = \omega_p/\sqrt{\gamma_0}$, $\lambda_{p,\mathrm{rel}} = \lambda_p\sqrt{\gamma_0}$, and $\omega_{B,\mathrm{rel}} = \omega_B/\gamma_0$, where X_rel are the "true" relativistic counterparts to the classical plasma values.

1.2. Hot plasmas

Plasma temperature in units of electron rest mass is

$$\theta = \frac{kT}{m_e c^2}. \tag{7}$$

Plasma is considered hot when $\theta \gtrsim 1$.

Particle distribution of a relativistic plasma in thermodynamic equilibrium follows Maxwell-Jüttner distribution,

$$M(\boldsymbol{u}) = \frac{1}{4\pi K_2(\theta^{-1})} \exp\left(-\frac{\sqrt{1+\gamma^2\beta^2}}{\theta}\right), \tag{8}$$

where K_2 is the modified Bessel function of the second kind. In terms of Lorentz factor it is

$$M(\gamma) = \frac{\gamma^2\beta}{K_2(\theta^{-1})\theta} \exp\left(-\frac{\gamma}{\theta}\right). \tag{9}$$

Maxwell-Jüttner distribution has a few useful averages that have a physical correspondence. First, we note that

$$\kappa_{ij}(x) = \frac{K_i(x)}{K_j(x)} = \begin{cases} 1+\frac{5}{2}\theta, & \text{for } \theta \to 0 \text{ (non-relativistic)} \\ 4\theta, & \text{for } \theta \to \infty \text{ (ultra-relativistic)} \end{cases}, \tag{10}$$

where K_n is the modified Bessel function of the nth kind.

A mean Lorentz factor is

$$\langle\gamma\rangle = \kappa_{32}(\theta^{-1}) - \theta^{-1} = \begin{cases} 1+\frac{3}{2}\theta, & \text{for } \theta \to 0 \text{ (non-relativistic)} \\ 3\theta, & \text{for } \theta \to \infty \text{ (ultra-relativistic)} \end{cases}. \tag{11}$$

Note that $\langle\gamma\rangle \approx 1+3\theta$, which approximatively captures both limits and therefore mimicks the thermal Lorentz factor of a plasma particle. Similarly,

$$\langle\gamma^2\beta^2\rangle = 3\kappa_{32}(\theta^{-1}) = \begin{cases} 3\theta, & \text{for } \theta \to 0 \text{ (non-relativistic)} \\ 12\theta^2, & \text{for } \theta \to \infty \text{ (ultra-relativistic)} \end{cases}. \tag{12}$$

Average Larmor radius of a thermal plasma is

$$\lambda_L' = \lambda_L \sqrt{2\theta\kappa_{32}(\theta^{-1})} \tag{13}$$

Thermal pressure is

$$P = \frac{1}{3} n \langle \boldsymbol{v} \cdot \gamma m \boldsymbol{v} \rangle = n\theta m_e c^2. \tag{14}$$

Enthalpy is

$$h = \frac{n\langle \gamma m c^2 \rangle + P}{nmc^2} = \kappa_{32}(\theta^{-1}) \tag{15}$$

$$= \begin{cases} 1 + \frac{5}{2}\theta, & \text{for } \theta \to 0 \text{ (non-relativistic)} \\ 4\theta, & \text{for } \theta \to \infty \text{ (ultra-relativistic)} \end{cases}. \tag{16}$$

Adiabatic exponent,

$$\Gamma_{\mathrm{ad}} = 1 + \frac{P}{n\langle \gamma - 1 \rangle mc^2} = 1 + \left(\frac{1}{\theta}\kappa_{32}(\theta^{-1}) - \theta^{-1} - 1 \right)^{-1} \tag{17}$$

$$= \begin{cases} \frac{5}{3}, & \text{for } \theta \to 0 \text{ (non-relativistic)} \\ \frac{4}{3}, & \text{for } \theta \to \infty \text{ (ultra-relativistic)} \end{cases}. \tag{18}$$

A plasma moving with a bulk drift coordinate-velocity $\boldsymbol{V}_0$ (with Lorentz factor Γ_0) to the y direction ($\boldsymbol{V}_0 = V_0\hat{\boldsymbol{y}}$), on the other hand, has

$$M(\boldsymbol{u})' = \frac{1}{4\pi K_2(\theta^{-1})\Gamma_0} \exp\left(-\Gamma_0 \frac{-V_0^2\Gamma_0 + \sqrt{1+\gamma^2\beta^2}}{\theta} \right), \tag{19}$$

and number density in that frame is $n = \Gamma_0 n_0$. Mean four-velocity in this case is

$$\langle \gamma \boldsymbol{v} \rangle = \kappa_{32}(\theta^{-1})\Gamma_0 \boldsymbol{V}_0 \tag{20}$$

$$= \begin{cases} \Gamma_0 \boldsymbol{V}_0, & \text{for } \theta \to 0 \text{ (non-relativistic)} \\ 4\theta\Gamma_0 \boldsymbol{V}_0, & \text{for } \theta \to \infty \text{ (ultra-relativistic)} \end{cases} \tag{21}$$

and mean Lorentz factor is

$$\langle \gamma \rangle = \kappa_{32}(\theta^{-1})\Gamma_0 - (\Gamma_0\theta)^{-1} \tag{22}$$

$$= \begin{cases} \Gamma_0, & \text{for } \theta \to 0 \text{ (non-relativistic)} \\ 4\theta\Gamma_0 - (\Gamma_0\theta)^{-1}, & \text{for } \theta \to \infty \text{ (ultra-relativistic)} \end{cases} \tag{23}$$

. Note that the perpendicular to the drift direction, the mean four-velocity fluxes (related to pressure) are

$$\langle u_x v_x \rangle = \langle u_z v_z \rangle = \theta/\Gamma_0, \tag{24}$$

whereas to the drift direction we have

$$\langle u_y v_y \rangle = \theta/\Gamma_0 + \Gamma_0 V_0^2 \kappa_{32}(\theta^{-1}). \tag{25}$$

1.3. Magnetized plasmas

Magnetic field energy density is

$$U_B \equiv \frac{B^2}{8\pi} \tag{26}$$

and electric field energy density is

$$U_E \equiv \frac{E^2}{8\pi}. \tag{27}$$

Similarly, internal energy content density of the plasma can be quantified with the enthalpy density

$$w \equiv \gamma n_e m_e c^2. \tag{28}$$

For magnetized plasmas it is useful to define the magnetization parameter that is the ratio of the magnetic energy density to the plasma enthalpy density. For a cold, non-relativistic plasma it reduces to

$$\sigma_c = \frac{B^2}{4\pi n m_e c^2}. \tag{29}$$

A hot plasma ($\theta \equiv kT/m_e c^2 \gtrsim 1$) carries a relativistic energy content and so the true "hot" magnetization is

$$\sigma = \frac{B^2}{4\pi n \gamma_{\mathrm{th}} m_e c^2} \approx \frac{B^2}{4\pi n(1+3\theta)m_e c^2}, \tag{30}$$

where $\gamma_{\mathrm{th}} = \langle \gamma \rangle$ is the average Lorentz factor of a hot plasma. Similarly, a plasma with a considerable relativistic bulk motion ($\Gamma > 1$) has a true "relativistic" magnetization of

$$\sigma = \frac{B^2}{4\pi n \Gamma m_e c^2}. \tag{31}$$

2. Plasma instabilities

TODO:

2.1. Plasma dispersion relations

2.2. Two-stream instability

2.3. Current filamentation instability

2.4. Weibel instability

2.5. Tearing mode instability

Tearing mode dispersion relation from the relativistic pair-plasma fluid equations (e.g., Koide 2009)
by applying the standard tearing mode analysis
Furth 1963
Coppi 1976
Ara 1978
non-relativistic one Porcelli 1991

3. Literature

On PIC: Birdsall & Langdon (1985). On useful plasma parameters: Melzani et al. (2014)

References

Birdsall, C. & Langdon, A. 1985, Plasma physics via computer simulation, The Adam Hilger series on plasma physics (McGraw-Hill, New York)

Melzani, M., Walder, R., Folini, D., Winisdoerffer, C., & Favre, J. M. 2014, Astronomy & Astrophysics, 570, A112

Appendix A: Runko code units

Time in Runko code units is related to the plasma frequency as

$$\hat{\omega}_p \Delta t = \sqrt{\frac{\hat{n}\hat{q}}{\gamma_0 \hat{m}}}, \tag{A.1}$$

where $\hat{q}$ is the particle charge normalization (`qe` or `qi`), $\hat{n}$ is the (numerical) total particle number density (typically 2×`ppc`), and $\hat{m} = m/m_e$ is the mass in units of electron masses. Note that here $4\pi = 1$ and $q_e/m_e = 1$ by definition. Cyclotron frequency is similarly

$$\hat{\omega}_B \Delta t = \frac{\hat{B}}{\gamma_0 \hat{m} \hat{C}}, \tag{A.2}$$

where $\hat{B}$ is the magnetic field in code units and $\hat{C}$ is the numerical speed of light in the simulation (CFL number; `cfl`). Magnetization is

$$\sigma = \frac{\hat{\omega}_B^2}{\hat{\omega}_p^2}. \tag{A.3}$$

Typical number density per "pixel" is

$$\hat{n}_0 = 2 n_{\mathrm{ppc}} \mathcal{S}^2, \tag{A.4}$$

where n_{ppc} is the particle per cell per species (`ppc`), and $\mathcal{S}$ is the output striding factor (`stride`). A standard charge density normalization is then

$$\hat{\rho}_0 = \hat{q}_e \hat{n}_0. \tag{A.5}$$

Similarly, the current density can be normalized with

$$\hat{J}_0 = \hat{q}_e \hat{n}_0 \hat{C}^2, \tag{A.6}$$

Resulting in current density units of $q_e n_0 c$. Note that the extra $\hat{C}$ factor in the above equation originates from the fact that the numerical current stored in the memory is actually $\hat{J}\Delta t$.

Fields can be normalized with

$$\hat{B}_0 = \frac{\hat{m}}{\hat{q}_e} \hat{C}^2 \mathcal{R}_\lambda, \tag{A.7}$$

where grid resolution in units of skin depth is $\mathcal{R}_\lambda = \lambda_p/\Delta x$ (`c_omp`). This gives fields units of $\omega_p m_e/q_e$.