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MHD turbulence

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ABSTRACT

Quick historical review on MHD turbulence theories. Theory is heavily influeced by Schekochihin 2018 (MHD Turbulence: A Biased Review). Verma, and Bresnyak

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1. STATISTICAL DESCRIPTION OF TURBULENCE

Single realization of a turbulent flow is chaotic and unpredictable. Order can be gained by describing the system statistically with averaged quantities. According to ergodic hypothesis statistical ensemble measurements can be replaced by time- and volume-averaging; this is under the assumption that turbulence is volume-filling and persistent process.

Second order structure function of a physical variable $\boldsymbol{X}(\boldsymbol{\ell})$ (like $\boldsymbol{V}$, $\boldsymbol{B}$, etc.) is defined as

$$S(\boldsymbol{\ell})^2 = \langle [X(\boldsymbol{\ell} - \delta\boldsymbol{\ell}) - X(\boldsymbol{\ell})]^2 \rangle, \quad (1)$$

where $\langle \cdot \rangle$ denotes averaging (time, volume, statistical; no stance taken on what the exact nature actually is). This quantity captures spatial correlation of the field $\boldsymbol{X}$. For velocity it reduces to kinetic energy, $S(\boldsymbol{v}) = 4\mathcal{E}_V$ on a scale ℓ.

Fourier transformed S can be related to the so-called energy spectrum $E(k)$. It describes energy in the wavenumber space such that $dE = E(k)dk$ is the energy at that particular wavenumber (or scale). Total energy is obtained by summing over all scales, $\mathcal{E} = \int E(k)dk$. For a statistically self-similar system we expect a power-law scaling for $E(k)$.

Fourier transform of a turbulent quantity u is

$$u(\boldsymbol{\ell}) = \int e^{i\boldsymbol{k}x} d\hat{u}(\boldsymbol{k}). \quad (2)$$

Square of the transform is called power spectrum,

$$P(\boldsymbol{k})d\boldsymbol{k} = \langle |d\hat{u}(\boldsymbol{k})|^2 \rangle. \quad (3)$$

For isotropic system, i.e., if $P(\boldsymbol{k})$ depends only on $|k|$, we can integrate over the shells in k-space to get

$$E(k) = 4\pi k^2 P(k) \quad (4)$$

Spectra and structure functions have one-to-one correspondence

$$S(\ell)^2 = 2\int_0^\infty \left(1 - \frac{\sin kr}{kr}\right) E(k)dk \quad (5)$$

If spectrum is a power-law, $E(k) \propto k^\alpha$, then by substitution of $k = x/\ell$, we have $S(\ell)^2 \propto \ell^{-1-\alpha}$.

From a statistical viewpoint, if the turbulence is self-similar (i.e., has a single-fractal structure) higher-order structure functions are all connected as

$$[S(\ell)^n)]^{1/n} \sim [S(\ell)^m)]^{1/m}, \quad (6)$$

for any arbitrary orders n and m.

2. BASIC THEORETICAL BACKGROUND OF TURBULENT CASCADES

Energy dissipation for incompressible flow can be defined per unit mass as ε with units $\mathrm{cm^2\,s^{-3}}$. Kolmogorov model assumes that the statistical properties of turbulence are uniquely determined by the amount of energy available in a stationary homogenous system, i.e. by ε alone. Furthermore, it assumes that energy self-similarly cascades through series of scales known as the inertial range. Cascade means that energy is being transferred from one scale to another without dissipation.

Kolmogorov model can be deduced from dimensional analysis alone. If the spectrum, $E(k)$, has units of $\mathrm{cm^3\,s^{-2}}$ and wavenumber, $k = 2\pi/\ell$, has units of $\mathrm{cm^{-1}}$ then we must have

$$E(k) = C_K \varepsilon^{2/3} k^{-5/3}, \quad (7)$$

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where C_K is a dimensionless Kolmogorov constant. The $3D$ powerspectrum $P(k) \sim E(k)k^{-2} \sim k^{-11/3}$.

The physics of this derivation is the following. Characteristic velocity on scale ℓ is u_ℓ and if the energy rate is constant for all scales then

$$\frac{u_\ell^2}{t_{\rm cascd}} = \varepsilon, \tag{8}$$

where $t_{\rm cascd}$ is a cascading time, time it takes for nonlinearities to remove energy from scale ℓ and transfer it to smaller scales.

For hydrodynamic cascade we assume that $t_{\rm cascd}$ is a dynamical time on a particular scale, so that

$$t_{\rm cascd} \approx \frac{\ell}{u_\ell}. \tag{9}$$

By substitituion we then have

$$\varepsilon \sim \frac{u_\ell^2}{t_{\rm cascd}} \sim \frac{u_\ell^3}{\ell} \tag{10}$$

that results in

$$u_\ell \sim (\varepsilon \ell)^{1/3} \sim \varepsilon^{1/3} k^{-1/3}. \tag{11}$$

From the definition of the energy spectrum we have that $E(k)k \sim u_\ell^2$ so that

$$E(k) \sim \frac{u_\ell^2}{k} \sim \frac{(\varepsilon^{1/3}k^{-1/3})^2}{k} \sim \varepsilon^{2/3}k^{-5/3}. \tag{12}$$

3. PHENOMENOLOGICAL MHD TURBULENCE MODELS

Before we start with phenomenological models, let us make a remark of the actual observations. Especially the hydrodynamical turbulence is a well-studied problem. Our best understanding of this system is still roughly a Kolmogorov-like system that predicts $-5/3$ scaling.

In reality, physical and numerical experiments have never agreed with this theoretical scaling. It is a solid fact that the scaling is more close to -1.7 CHECK and not -1.66 as would be predicted. The situation is even worse for MHD cascades since we can not perform detailed physical experiments of plasma turbulence nor can we simulate large enough MHD systems.[1] Therefore, one should not get too carried away by the different theoretical models; none of them is actually correct.

3.1. *Kolmogorovian theory*

Energy is pumped into a homogenous conducting medium with a fixed rate ϵ. Dimensionless analysis gives for the energy spectrum (?)

$$E(k) \sim \epsilon^{2/3} k^{-5/3}. \tag{13}$$

Same can be written in terms of average velocity increments

$$\delta u_\lambda \sim (\epsilon\lambda)^{1/3}. \tag{14}$$

3.2. *Iroshnikov-Kraichan theory*

If the $\boldsymbol{B}$ field has an important role in energy transfer, than similar dimensionless analysis gives Iroshnikov Kraichnan 1965

$$E(k) \sim (\epsilon v_A)^{1/2} k^{-3/2} \tag{15}$$

and

$$\delta u_\lambda \sim (\epsilon v_A \lambda)^{1/4}, \tag{16}$$

for the Alfv́en speed with density ρ

$$v_A = \frac{B}{\sqrt{4\pi\rho}}. \tag{17}$$

The reasoning is based on the fact that Alfv́en time, $\tau_A \sim 1/kv_A$ is the time which interactions occur so the energy must come with a combination ϵv_A.

The theory is incorrect because it assumes a uniform scale k whereas in reality in a presence of a strong guide field $\boldsymbol{B}_0$ the scales split into $k_\parallel$ and $k_\perp$.

3.3. *Goldreich-Sridhar critical balance*

In a strong magnetic field $k_\parallel \ll k_\perp$. Parallel direction variation propagation velocity corresponds to Alfv́en waves with

$$\tau_A = \frac{l_\parallel}{v_A}, \tag{18}$$

whereas perpendicular variation is governed by nonlinear intearctions with characteristic times

$$\tau_{\rm nl} \sim \frac{l_\perp}{\delta u_\lambda}. \tag{19}$$

For Alfv́enic perturbations $\delta u_\lambda \sim \delta b_\lambda$. The two times, τ_A and $\tau_{\rm nl}$, are assumed to be equal. The natural "cascade time" must also be of same order, $\tau_c \sim \tau_A \sim \tau_{\rm nl}$ This gives

$$\frac{\delta u_\lambda^2}{\tau_c} \sim \epsilon \quad \text{and} \quad \tau_c \sim \tau_{\rm nl} \sim \frac{\lambda}{\delta u_\lambda}, \tag{20}$$

so that

$$\delta u_\lambda \sim (\epsilon\lambda)^{1/3} \tag{21}$$

[1] The problem is computationally very demanding. From the simulation perspective, there is a roughly an one order of magnitude of settling scales below the injection scale ℓ_0 so that inertial range begins at scales of $\lesssim l_0/10$. Similarly, dissipation (and grid scale) physics (and numerics) are present at roughly one order of magnitude above the dissipation scale ℓ_ν so that a clean inertial range extends to $\gtrsim 10\ell_\nu$. In order to perform scaling measurements with a inertial range of length at least 2 decades, we then need to simulate a turbulent system of a size of at least $(10 \times 10 \times 10^2)^3 \sim (10^4)^3$. Current simulations are only starting to probe these scales whereas such a system is only the minimum viable numerical measurement experiment.

and equally (Goldreich & Sridhar 1995; ?)

$$E(k_\perp) \sim \epsilon^{2/3} k_\perp^{-5/3}, \tag{22}$$

yielding a Kolmogov scaling for the perpendicular scales. Simultaneously, along the field the velocity increment satisfy

$$\frac{\delta u_\parallel^2}{\tau_c} \sim \epsilon \quad \text{and} \quad \tau_c \sim \tau_A \sim \frac{l_\parallel}{v_A} \tag{23}$$

so that

$$\delta u_{l\parallel} \sim \left(\frac{\epsilon l_\parallel}{v_A}\right)^{1/2}. \tag{24}$$

From here it follows

$$l_\parallel \sim v_A \epsilon^{-1/3} \lambda^{2/3}. \tag{25}$$

Physically $l_\parallel$ is the distance an Alfvénic pulse travels along the field at speed v_A over time $\tau_{\rm nl}$.

3.4. *Reduced MHD and Elsasser fields*

Elsasser fields are given as $\boldsymbol{Z} = \boldsymbol{u} \pm \boldsymbol{b}$. Here $b = B_0/\sqrt{4\pi\rho}$.

3.5. *Weak turbulence*

Weak turbulence theory stems from a perturbation in a (assumedly) small ratio $\tau_A/\tau_{\rm nl}$. WT scaling originates from

$$\delta Z_\lambda \sim \left(\frac{\epsilon}{\tau_A}\right)^{1/4} \lambda^{1/2} \tag{26}$$

where δZ_λ is perturbed Elsasser field. This gives a scaling

$$E(k_\perp) \sim \left(\frac{\epsilon}{\tau_A}\right)^{1/2} k_\perp^{-2}. \tag{27}$$

Eventually weak turbulence will transition to strong turbulence. For balanced turbulence this happens when the perturbation parameter becomes of order unity,

$$\frac{\tau_A}{\tau_{\rm nl}} \sim \frac{\tau_A^{3/4}\epsilon^{1/4}}{\lambda^{1/2}} \sim 1 \tag{28}$$

corresponding to a scale (assuming critical balance)

$$\lambda_{\rm CB} \sim \epsilon^{1/2} \tau_A^{3/2}. \tag{29}$$

3.6. *Critical balance*

In a strong turbulence, $2D$ structures with $\tau_{\rm nl} \ll \tau_A$ are unsustainable because of causality: Information propagates along $\boldsymbol{B}$ at v_A so no structure longer than $l_\parallel \sim v_A \tau_{\rm nl}$ can be kept coherent (?).

Alfvén wave is a basic element of MHD motion. Because of this, strong magnetic perturbations would "want" to resemble Alfvv́en waves as closely as possible. Critical balance relates to this: An Alfvénic perturbation decorrelates in roughly one wave period.

3.7. *Dynamical alignement*

Dynamic alignment derives from the same assumption of MHD tendency towards Alfvénic nature. In an Alfvén wave $\boldsymbol{u}_\perp$ and $\boldsymbol{b}_\perp$ are the same, i.e., plasma flows drag field with them or the field accelerates flows to relax under tension. However, if the two fields are exactly parallel, there would be no non-linearity.

Boldyrev's theory on dynamical alignement states that the angle, θ_λ between the two fields can not be known more precisely than

$$\sin\theta_\lambda \sim \theta_\lambda \sim \frac{\delta b}{v_A} \ll 1 \tag{30}$$

Then the non-linear time is modified to be

$$\tau_{\rm nl} \sim \frac{\lambda}{\delta Z_\lambda^\mp \sin\theta_\lambda} \tag{31}$$

This leads to Kraichnan-type of 3/2 scaling

$$E(k_\perp) \sim (\epsilon v_A)^{1/2} k_\perp^{-3/2} \tag{32}$$

and

$$l_\parallel \sim v_A^{3/2} \epsilon^{-1/2} \lambda^{1/2}. \tag{33}$$

Yet another way to rewrite the scaling relations in terms of the critical balance scale, $\lambda_{\rm CB}$, is

$$E(k_\perp) \sim \epsilon^{2/3} \lambda_{\rm CB}^{1/6} k_\perp^{-3/2}, \tag{34}$$

that follows the prediction for the spectrum by Perez et al 2012, 2014b. This is sometimes known as aligned turbulence.

3.8. *Intermittency*

Classical turbulence theories rely on self-similarity of the structures. Intermittency means that this assumption is broken and instead we introduce all three length scale directions: perpendicular λ, parallel $l_\parallel$, and fluctuation direction ζ.

Intermittency states that eddies are not completely space-filling, but have more rare fluctuations on top of the "typica" ones. In hydrodynamic turbulence, corrections to K41 theory come in powers of λ/L. Similar can be found in MHD turbulence as the self-similarity is broken with the appeareance of outer-scale size $L_\parallel$ in to the scaling equations.

Mallet & Schekochihin conjectured that $l_\parallel \sim \lambda^\alpha$, i.e., $l_\parallel/\lambda^\alpha$ has a scale-invariant distribution. A typical (second) conjecture is that most intense structures are sheets transverse to the local perpendicular direction.

3.9. *Recap*

Therefore, in reality, the scaling are expected to be fully $3D$. From critical balance and from the assumption of dynamical alignment it then follows that that field's spectra depends as: -2 in $l_{\parallel}$ direction, $-3/2$ in λ direction, and $-5/3$ in ζ direction.

In addition, we have

$$\zeta \sim l_{\parallel} \frac{\delta Z_{\lambda}}{v_A} \sim l_{\parallel} \frac{\delta b_{\lambda}}{v_A} a, \tag{35}$$

i.e., ζ is the typical displacement of a fluid element and a typical perpendicular distance a field line wanders within a structure that is coherent on the parallel scale $l_{\parallel}$. Fluctuations must therefore preseve cohenre in their respective direction at least on scale ζ. They are not constrained in the third λ direction. Finally, they are expected to have an angular uncertainty of the order of the angle θ_{λ} between the two fields.

Finally, we note that this is picture is not solid yet. On the contrast, it relies on the refined dynamical alignment conjecture that itself is not proven to be correct or incorrect yet. Another way to think about this conjecture is, that it is used to describe the intermittency. For a solid theory of MHD turbulence, a complete theory of intermittenty is therefore needed.

4. LITERATURE WORK-IN-PROGRESS

Objects themselves:

- PWN: Woosley 1993
- jets from AGNs: Reynolds 1996
- GRBs: Wardle 1998

4.1. *Non-thermal particles from turbulence*

- Melrose 1980;
- Petrosian 2012;
- Lazarian 2012;

4.2. *Turbulence in astrophysics*

turbulence in stellar coronae: Matthaeus 1999, Cranmer 2007

ISM: Armstrong 1995, Lithwick & Goldreich 2001

SNRs: Weiler & Sramek 1988, Roy 2009

PWN: Porth 2014, Lyutikov 2019

BH disks: Balbus & Hawley 1998, Brandenburg & Subramanian 2005

jets from AGNs: Marscher 2008, MacDonald & Marscher 2018 (MacDonald & Marscher 2018)

radio lobes: Vogt & Ensslin 2005, O'Sullivan 2009

GRBs: Wardle 1998 Piran 2004, (Kumar & Narayan 2009)

Galaxy clusters: Zweibel & Heiles 1997, Subramanian 2006

Laser laboratory plasma: Sarri 2015

4.3. *Magnetically-dominated turbulence*

Sustained relativistic turbulence (force-free) (Thompson & Blaes 1998): extension of Goldreich & Sridhar 1995 to exterme relativistic limit (no plasma inertia; force-free MHD). Anisotropic cascade is formed, dissipation occurs at the scale of current starvation (when not enough charge carriers in plasma to maintain currents from Alfén waves).

MHD: (Cho 2005) Inoue 2011 (Cho & Lazarian 2014) (Zrake & East 2016)

Relativistic MHD: (Zrake & MacFadyen 2012) (Zrake 2014)

4.4. *Bright non-thermal synchrotron and inverse Compton signatures*

Pulsar magnetospheres and winds: Buhler & Blandford 2014

Jets from AGNs: Begelman 1984

Coronae of accretion disks: (Yuan & Narayan 2014)

4.5. *Kinetic turbulence*

(Zhdankin et al. 2017b) Letter
(Zhdankin et al. 2017a) Paper
(Zhdankin et al. 2018) System size convergence
(Zhdankin et al. 2019a) electron-proton plasma
(Zhdankin et al. 2019b) radiative turbulence
(Comisso & Sironi 2018) acceleration
(Wong et al. 2019) acceleration
(Nättilä 2019) Runko and turbulence
(Comisso & Sironi 2019) acceleration

4.6. *Radiative turbulence*

4.6.1. *Analytic work on radiative turbulence*

(Uzdensky 2018);
(?) (GRBs)
(Sobacchi & Lyubarsky 2019);

4.6.2. *PIC simulations*

(Zhdankin et al. 2019b) radiative turbulence

4.6.3. *Fokker-Planck equation in momentum space with radiative cooling term*

(Schlickeiser 1984, 1985); not in original list (Schlickeiser 1989)
(Stawarz & Petrosian 2008)

4.7. *radiative PIC simulations*

4.7.1. *Reconnection*

(Jaroschek & Hoshino 2009)
(Cerutti et al. 2013, 2014b,a)
(Kagan et al. 2016b,a)
(Hakobyan et al. 2019)
(Werner et al. 2019)
(Schoeffler et al. 2019)

4.7.2. *Decay of magnetostatic equilibria*

(Yuan et al. 2016)
(Nalewajko et al. 2018)

4.7.3. *Pulsar wind*

(Cerutti & Philippov 2017)

4.7.4. *Pulsar magnetospheres*

(Cerutti et al. 2016)
(Philippov & Spitkovsky 2018)

4.7.5. *Synchrotron and jitter radiative signatures of collisionless shocks*

(Medvedev & Spitkovsky 2009)
(Sironi & Spitkovsky 2009)
(Kirk & Reville 2010)
(Nishikawa et al. 2011)

4.7.6. *Radiative turbulence*

(Zhdankin et al. 2019c)

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