Mathematical References

This section describes the methods that I found useful while making this game.

Process of finding the Cartesian position of an object as a function of time:

1. Determine the vectors v, r, h, and e
2. Find a, w, and i
3. Calculate the True Anomaly, Eccentric Anomaly and Mean Anomaly
4. Increment time and calculate new Mean Anomaly
5. Determine Eccentric Anomaly through Newton's method
6. Find Eccentric Anomaly and True Anomaly
7. Determine the position and velocity

Determining the Orbit of the Spacecraft from State Vectors

State vectors are the displacement vector $\boldsymbol{r}$ and velocity vector $\boldsymbol{v}$.

Given the vectors: $\boldsymbol{r}, \boldsymbol{v}$

First, determine the specific relative angular momentum, $\boldsymbol{h}$**:**

$$\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v}$$

Which, in a 2D plane:

$$\boldsymbol{h} \begin{cases} 0_x \\ 0_y \\ (r_x v_y - r_y v_x)_z \end{cases}$$

The value h_Z determines the direction of the orbit. If h_Z is positive, then the orbit is anticlockwise, else it is clockwise.

Then, calculate the vector $\boldsymbol{e}$ (eccentricity):

$$\boldsymbol{e} = \frac{1}{\mu}(v^2 - \frac{\mu}{r}\boldsymbol{r} - \boldsymbol{v}(\boldsymbol{r} \cdot \boldsymbol{v}))$$

Which, in a 2D plane yields:

$$\boldsymbol{e} \begin{cases} \frac{1}{\mu}(v^2 - \frac{\mu}{r} r_x - v_y(\boldsymbol{r} \cdot \boldsymbol{v}))_x \\ \frac{1}{\mu}(v^2 - \frac{\mu}{r} r_x - v_y(\boldsymbol{r} \cdot \boldsymbol{v}))_y \\ 0_z \end{cases}$$

Now, a, the semi major axis, is given by:

$$a = \frac{h^2}{\mu(1-e^2)}$$

Next, determine the argument of periapsis (ω), which is the angle between the x axis and vector $\boldsymbol{e}$. Knowing this:

$$cos\omega = \frac{\langle 0, \boldsymbol{e} \rangle}{\|0\|\|\boldsymbol{e}\|}$$

Or,

$$tan\omega = \frac{e_y}{e_x}$$

However, in an anticlockwise orbit,

$$\tan(-\omega) = \frac{e_y}{e_x}$$

Again, the True Anomaly is the angle between the $\boldsymbol{e}$ vector and the $\boldsymbol{r}$ vector. So:

$$\theta = \cos^{-1} \frac{\langle \boldsymbol{r}, \boldsymbol{e} \rangle}{\|\boldsymbol{r}\|\|\boldsymbol{e}\|}$$

Eccentric anomaly is given by:

Elliptical orbit:

$$E = 2\tan^{-1}\sqrt{\frac{\theta(1-e)}{2(1+e)}}$$

Hyperbolic orbit:

$$F = \cosh^{-1}(\frac{e + cos\theta}{1 + ecos\theta})$$

Mean anomaly is given by:

Elliptical orbit:

$$M = E - e\sin E$$

Hyperbolic orbit:

$$M = e\sinh F - F$$

The angle i is the inclination of the orbit. Since we are doing this in a 2D plane, the inclination is either $i = 0$ or $i = \pi$.

Therefore:

$$i = \begin{cases} 0\,; h_z \geq 0 \\ \pi\,; h_z < 0 \end{cases}$$

All the orbital elements are now defined. The shape of the ellipse can be determined through the equation of a conic in polar coordinates:

$$r(\theta) = \frac{a(1-e^2)}{1 + e\cos(\theta + \omega)}$$

Now, if we wish to find the position as a function of time:

Let the time at epoch be t_0 and variable time be t.

First, find the mean anomaly at time t.

$$M = \sqrt{\frac{\mu}{|a^3|}}(t - t_0) + M_0$$

Where M_0 is the mean anomaly at t_0, the epoch time.

Now, determine the eccentric anomaly.

Elliptical orbit:

$$M = E - e\sin E$$

Hyperbolic orbit:

$$M = e\sinh F - F$$

Notice that there is no closed solution. Therefore, an iteration method has to be used.

Elliptical orbit:

$$E_n = M + e\sin E_{n-1}$$

Hyperbolic orbit:

$$F_n = F_{n-1} + \frac{M - e\sinh F_{n-1} + F_{n-1}}{e\cosh F_{n-1} - 1}$$

Repeat until the solution satisfies the equation within margin.

Now, simply determine the true anomaly:

Elliptical orbit:

$$\theta = 2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right)$$

Hyperbolic orbit:

$$\theta = 2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tanh\frac{E}{2}\right)$$

Now determine the new position using the conic section distance formula:

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos(\theta+\omega)}$$

And turn it into Cartesian coordinates.
Determine the velocity in the local coordinate vectors **P,Q**

$$\boldsymbol{v_{PQ}}\begin{cases}\sqrt{\dfrac{\mu}{|a(1-e^2)|}}(-\sin\theta)\boldsymbol{P}\\ \sqrt{\dfrac{\mu}{|a(1-e^2)|}}(-\sin\theta)\boldsymbol{Q}\end{cases}$$

However, we must also account for the "tilt" of the orbit. Therefore:

$$\boldsymbol{v_{xy}}\begin{cases}|v|\cos(\tan^{-1}(\dfrac{v_Q}{v_P}+\omega))_x\\ |v|\sin(\tan^{-1}(\dfrac{v_Q}{v_P}+\omega))_y\end{cases}$$

Or, for clockwise orbits:

$$\boldsymbol{v_{xy}}\begin{cases}|v|\cos(\tan^{-1}(\dfrac{v_Q}{v_P}+\omega))_x\\ -|v|\sin(\tan^{-1}(\dfrac{v_Q}{v_P}+\omega))_y\end{cases}$$

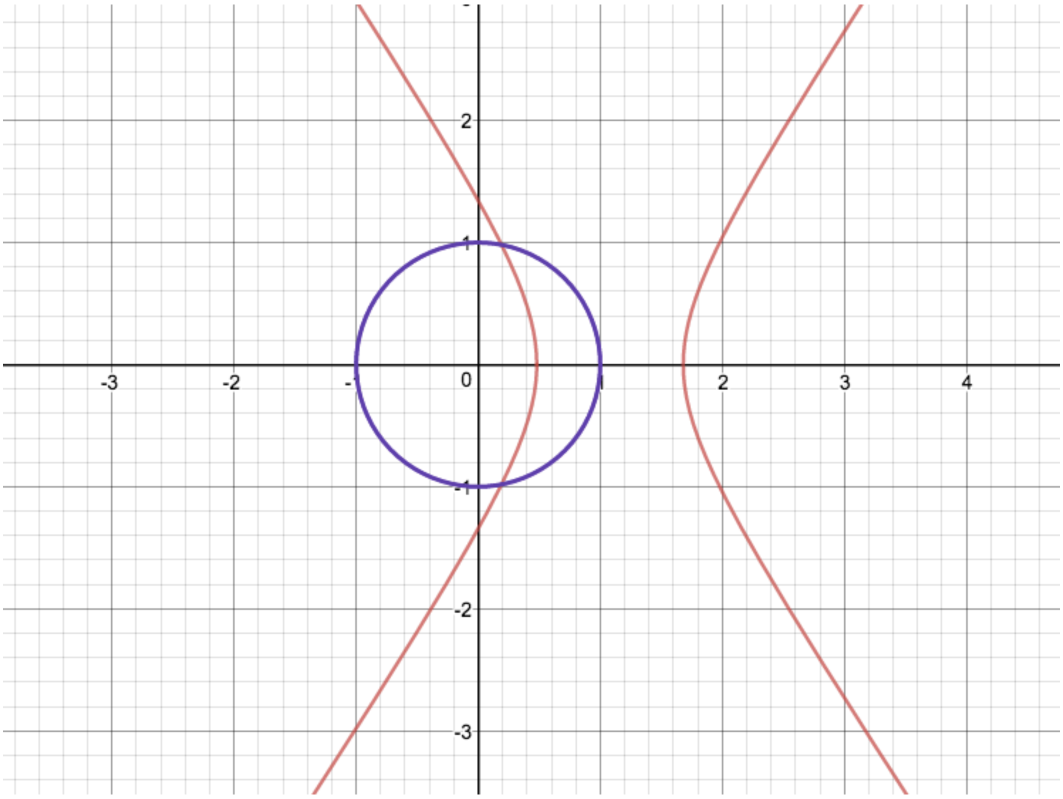
And thus, the Cartesian vectors are obtained. This process can be repeated to calculate for the change in shape of the orbit when a burn is made.

Finding Position and Velocity at Escape

In a two body Keplerian orbit, the spacecraft is considered to be free from any gravitational influence of another mush more massive body when it is outside of its "sphere of influence".

In order to compute the path of the spacecraft at escape, we most know its Cartesian state vectors. Doing so, we need to find a few things:

1. True anomaly at escape
2. Position and velocity at escape.



Now, let's consider the situation:

We know that the ship scapes the influence of the body once it reaches outside the sphere of influence. Therefore, if we consider the hyperbolic/elliptical path of the spacecraft, the escape must happen at the intercept of the path and a circle centred at the body with the radius of the sphere of influence.

Let:

$$r_{soi} = a(\frac{m}{M})^{\frac{2}{5}}$$

Where r_{soi} is the radius of the sphere of influence; a is the semi major axis of the targeted body and m, M are the respective mass of the targeting body and its parent.

Now, consider the polar equation of a circle and a conic section:

$$\begin{cases} r(\theta) = r_{soi} \ (1) \\ r(\theta) = \dfrac{a(1-e^2)}{1+e\cos(\theta)} \ (2) \end{cases}$$

Notice that we neglect the argument of periapsis ω, as we are trying to find the true anomaly.

Solving (1) and (2) simultaneously:

$$\frac{a(1-e^2)}{1+e\cos(\theta)} = r_{soi}$$

$$\theta = \pm\cos^{-1}\left(\frac{1}{e}\left(\frac{a(1-e^2)}{r_{soi}} - 1\right)\right)$$

Since the mean anomaly is always increasing (no matter clockwise or anticlockwise), the escape true anomaly must be the positive case.

And thus, this can be used to calculate the position and velocity at time of escape.

Time between two locations
Recall:

$$M = \sqrt{\frac{\mu}{|a^3|}}(t - t_0) + M_0$$

And:

$$M = E - e\sin E,\ M = e\sinh F - F$$

First, find the eccentric anomaly at current location and escape true anomaly through these equations:

Elliptical orbit:

$$E = 2\tan^{-1}\sqrt{\frac{\theta(1-e)}{2(1+e)}}$$

Hyperbolic orbit:

$$F = \cosh^{-1}(\frac{e + cos\theta}{1 + ecos\theta})$$

Now,

$$E - e\sin E = \sqrt{\frac{\mu}{|a^3|}}(t - t_0) + (E_0 - e\sin E_0)$$

And,

$$e\sinh F - F = \sqrt{\frac{\mu}{|a^3|}}(t - t_0) + (e\sinh F_0 - F_0)$$

Rearranging yields:

$$t - t_0 = \sqrt{\frac{|a^3|}{\mu}}\big((E - e\sin E) - (E_0 - e\sin E_0)\big)$$

And,

$$t - t_0 = \sqrt{\frac{|a^3|}{\mu}}\big((e\sinh F - F) - (e\sinh F_0 - F_0)\big)$$

Hence, the time taken to travel from one true anomaly to another is found.

Calculating burn time

This equation is derived from the Tsiolkovsky rocket equation:

$$\Delta V = v_e \ln\frac{m_0}{m_f}$$

Where ΔV is the change in velocity, v_e is the exhaust velocity, m_0 is mass before burn and m_f is the mass after burn.

Now, $v_e = I_{sp}g_0$ and $F = \dot{m}I_{sp}g_0$. ($\dot{m}$ is the rate of change of mass)So:

$$\Delta V = \frac{F}{\dot{m}}\ln\frac{m_0}{m_0 - \dot{m}t_0}$$

Where t_0 is the time taken for the burn and F is thrust. Simply rearranging this equation yields:

$$t_0 = \frac{m_0}{\dot{m}}\left(1 - e^{-\frac{\dot{m}}{F}\Delta V}\right)$$

This equation can also be derived analytically. Consider this scenario:

“A rocket with arbitrary mass contains an initial fuel mass of m_0 kg and has an initial velocity of v_0 ms^{-1}. After t seconds, the fuel mass is now m_f kg and the rocket is travelling at v ms^{-1}. Let the rate of change in mass be $\dot{m}$ and the full thrust of the rocket be T N”

Consider the mass of the rocket as the fuel burns:

$$m = m_0 - \dot{m}t$$

Now, according to Newton’s Second law, F=ma. So:

$$\dot{v} = \frac{T}{m_0 - \dot{m}t}$$

Simply integrate to find the velocity:

$$
\begin{aligned}
v &= \int \frac{T}{m_0 - \dot{m}t} dt \\
&= -\frac{T}{\dot{m}} \ln(m_0 - \dot{m}t) + c
\end{aligned}
$$

At $t = 0$, $v = v_0$. So:

$$
\begin{aligned}
v_0 &= -\frac{T}{\dot{m}} \ln(m_0) + c \\
c &= v_0 + \frac{T}{\dot{m}} \ln(m_0)
\end{aligned}
$$

Substituting:

$$
\begin{aligned}
v &= -\frac{T}{\dot{m}} \ln(m_0 + \dot{m}t) + v_0 + \frac{T}{\dot{m}} \ln(m_0) \\
v - v_0 &= -\frac{T}{\dot{m}} \ln(m_0 + \dot{m}t) + \frac{T}{\dot{m}} \ln(m_0) \\
v - v_0 &= \frac{T}{\dot{m}} \ln\left(\frac{m_0}{m_0 - \dot{m}t}\right)
\end{aligned}
$$

Rearranging:

$$
\begin{aligned}
&\frac{\dot{m}}{T}(v - v_0) = \ln\left(\frac{m_0}{m_0 - \dot{m}t}\right) \\
&e^{\frac{\dot{m}}{T}(v - v_0)} = \frac{m_0}{m_0 - \dot{m}t} \\
&\dot{m}t = m_0 - \frac{m_0}{e^{\frac{\dot{m}}{T}(v - v_0)}} \\
&t = \frac{m_0}{\dot{m}}\left(1 - \frac{1}{e^{\frac{\dot{m}}{T}(v - v_0)}}\right)
\end{aligned}
$$