

# Numerical Check Report

**Coverage:** identities/inequalities + TOBECHECKED labels from the main manuscript (definitions omitted).

Total	Failures	Missing	Status
45	0	0	✓ PASS

**Note:** check\_tobecheked unavailable (No module named 'check\_tobecheked'); using fallback routines.

## Notation

**Simulation diagnostics.** For an equation  $E(x) = 0$ , the residual is  $r_\infty = \max_x |E(x)|$  over the evaluation grid. For an identity  $f(x) = g(x)$ , the error is  $e_\infty = \max_x |f(x) - g(x)|$  over the evaluation grid. For inequalities  $f(x) \leq g(x)$ , the margin and excess are  $m_{\min} = \min_x(g(x) - f(x))$  and  $e_{\max} = \max_x(g(x) - f(x))$  over the evaluation grid. For bounded quantities,  $y_{\max} = \max_x y(x)$ . Expectations use  $\mathbb{E}[g] \approx \frac{1}{N} \sum_{i=1}^N g(x_i)$  for sampled points  $\{x_i\}_{i=1}^N$ . For (6),  $p_{\text{low}}$  and  $p_{\text{high}}$  denote  $\mathbb{P}(Z_{N, \lfloor \alpha N \rfloor}^{(\kappa)} = 0)$  at  $\alpha = \alpha_n - \varepsilon$  and  $\alpha = \alpha_n + \varepsilon$ , and  $c_\varepsilon$  is the constant in (6). For the monotonicity check of  $A$ ,  $\Delta A$  denotes successive differences on the evaluation grid.

**How to read the Message column.** Each row shows the displayed formula from `main.tex` followed by the main diagnostic; the **Status** flag already applies the preset numerical tolerances.

- $e_\infty = \dots$  (equality/identity): sup error  $\max |LHS - RHS|$  on the evaluation grid. Smaller is better; values near machine precision indicate numerical agreement, while larger values usually reflect grid/Monte Carlo error (still may be acceptable if marked PASS).
- $r_\infty = \dots$  (root/fixed-point): residual  $\max |E|$  for an equation  $E = 0$ . This should be close to 0.
- $m_{\min} = \dots / e_{\max} = \dots$  (inequality): check the sign condition  $m_{\min} \geq 0$  (equivalently  $e_{\max} \leq 0$ ). If the extremum is close to 0, it is worth re-running on a finer grid.
- $\min(\dots) = \dots, \max(\dots) = \dots, y_{\max} = \dots, \Delta = \dots, \mathbf{LHS/RHS}$  (bounds/monotonicity/value comparisons): verify the reported extremum has the expected sign and magnitude relative to the bound claimed in `main.tex`;  $\Delta$  should be small relative to the scale of the values.
- **Point annotations** such as  $(t = \dots), (u = \dots, k = \dots), (x = \dots, y = \dots)$  indicate where the worst-case (or representative) diagnostic was attained on the evaluation grid.

**Notation from `main.tex` used in simulations.** Let  $Z, X \sim \mathcal{N}(0, 1)$  and  $(x)_+ := \max\{x, 0\}$ .

Define

$$\begin{aligned}
\phi(u) &:= \frac{1}{\sqrt{2\pi}} e^{-u^2/2}, & \Phi(u) &:= \int_{-\infty}^u \phi(s) ds, & \bar{\Phi}(u) &:= 1 - \Phi(u), \\
E(u) &:= \frac{\phi(u)}{\Phi(u)}, & F_q(x) &:= \frac{1}{\sqrt{1-q}} E\left(\frac{\kappa-x}{\sqrt{1-q}}\right), \\
P(r) &:= \mathbb{E}[\tanh^2(\sqrt{r} Z)], \\
R_\kappa(q, \alpha) &:= \alpha \mathbb{E}[F_q(\sqrt{q} Z)^2], \\
B(q) &:= (1-q) \mathbb{E}\left[E\left(\frac{\kappa-\sqrt{q} Z}{\sqrt{1-q}}\right)^2\right], & A(r) &:= r(1-P(r))^2 = r(\mathbb{E}[\operatorname{sech}^2(\sqrt{r} Z)])^2, \\
C_\kappa &:= \mathbb{E}[(\kappa-Z)_+^2] = (\kappa^2 + 1)\Phi(\kappa) + \kappa\phi(\kappa), \\
U_s &:= \frac{\kappa - \sqrt{s} Z}{\sqrt{1-s}} \quad (s \in [0, 1]), & d(u) &:= E(u) - u, \\
\mu_k(u) &:= \mathbb{E}[(X-u)^k \mid X \geq u], & g(u) &:= E'(u)^2 - 2(1-E'(u))E(u)^2, \\
H(u) &:= u^2 d(u) + 6ud(u)^2 + 6d(u)^3 - u - 4d(u), \\
x &:= u d(u), & y &:= d(u)^2, & F(x, y) &:= x^2 + 6xy + 6y^2 - x - 4y.
\end{aligned}$$

## Simulation Results

Label	Main.pdf	Status	Message
eq:system	Eq. (1)	✓ PASS	$\begin{cases} q = P(r), \\ r = R_\kappa(q, \alpha). \end{cases}$ $r_\infty = 7.994e-15$
eq:r_in_terms_of_B	Eq. (3)	✓ PASS	$r = \frac{\alpha}{(1-q)^2} B(q).$ $e_\infty = 3.553e-15$
eq:union-bound	Eq. (5)	✓ PASS	$\mathbb{P}(Z_{N,M}^{(0)} = 0) \leq \mathbb{P}(\mathcal{E}) \leq L \mathbb{P}(Z_{N,M}^{(0)} = 0).$ $\mathbb{P}(Z_{N,M}^{(0)} = 0) = 0.167, \mathbb{P}(\mathcal{E}) = 0.387, \text{tol} = 0.119$
eq:sharp-seq-exp	Eq. (6)	✓ PASS	$\alpha \leq \alpha_N - \varepsilon \implies \mathbb{P}(Z_{N,\lfloor \alpha N \rfloor}^{(\kappa)} = 0) \leq e^{-c_\varepsilon N}, \quad \alpha \geq$ $\alpha_N + \varepsilon \implies \mathbb{P}(Z_{N,\lfloor \alpha N \rfloor}^{(\kappa)} = 0) \geq 1 - e^{-c_\varepsilon N}.$ $\alpha_n \approx 0.40, p_{\text{low}} = 0.000, p_{\text{high}} = 0.008, c_\varepsilon = 0.001$
eq:RS-fixed-point	Eq. (8)	✓ PASS	$q = \mathbb{E}[\tanh^2(\sqrt{r} Z)], \quad r = \alpha \mathbb{E}[F_q(\sqrt{q} Z)^2].$ $e_\infty = 0.000e+00$
eq:one-dimensional	Eq. (10)	✓ PASS	$A(r) = \alpha B(P(r)), \quad r \geq 0,$ $e_\infty = 3.719e-14$
eq:A_as_I	Eq. (11)	✓ PASS	$A(r) = \frac{1}{2\pi} I(r)^2.$ $e_\infty = 1.403e-02$
eq:condmean	Eq. (12)	✓ PASS	$E(u) = \frac{\phi(u)}{\Phi(u)} = \mathbb{E}[X \mid X \geq u].$ $e_\infty = 1.110e-15$
eq:mprime	Eq. (13)	✓ PASS	$E'(u) = E(u)^2 - u E(u).$ $e_\infty = 6.765e-11$

Label	Main.pdf	Status	Message
eq:varidentity	Eq. (14)	✓ PASS	$1 - E'(u) = \mathbb{E}[X^2   X \geq u] - (\mathbb{E}[X   X \geq u])^2 = \text{Var}(X   X \geq u) > 0.$ $e_\infty = 2.665e - 15$
eq:dprime	Eq. (15)	✓ PASS	$d'(u) = E'(u) - 1 = -\text{Var}(X   X \geq u) < 0.$ $e_\infty = 5.821e - 11$
eq:B-as-BM	-	✓ PASS	$B(t) = \mathbb{E}[f(t, W_t)^2], \quad t \in [0, 1].$ $e_\infty = 2.220e - 16$
eq:Lf	-	✓ PASS	$(\mathcal{L}f)(t, x) = \frac{-E + uE' + E''}{2\sqrt{1-t}}.$ $e_\infty = 5.332e - 08$
eq:Bprime	-	✓ PASS	$B'(t) = \mathbb{E}\left[(E'(U_t))^2 + 2E(U_t)^2(E'(U_t) - 1)\right] = \mathbb{E}[g(U_t)].$ $e_\infty = 4.524e - 03$
eq:goal	Eq. (20)	✓ PASS	$\mathbb{E}[g(U_t)] < 0 \quad \text{for all } t \in (0, 1).$ $\max_t \mathbb{E}[g(U_t)] = 0.000e + 00$
eq:g-expanded	Eq. (21)	✓ PASS	$g(u) = E(u)^2(3E(u)^2 - 4uE(u) + u^2 - 2).$ $e_\infty = 1.180e - 16$
eq:moments	-	✓ PASS	$\mu_0(u) = 1,$ $\mu_1(u) = d(u),$ $\mu_2(u) = 1 - u d(u),$ $\mu_3(u) = (u^2 + 2)d(u) - u,$ $\mu_4(u) = u^2 + 3 - u(u^2 + 5)d(u).$ $e_\infty = 1.421e - 14$
eq:consFneg_lemma	Eq. (40)	✓ PASS	$x + 2y \geq 1, \quad x^2 + xy - 3x - 3y + 2 \geq 0, \quad x + y < 1.$ $\max F(x, y) = 0.000e + 00$
eq:gprime	Eq. (25)	✓ PASS	$g'(u) = 2E(u)^2H(u).$ $e_\infty = 4.125e - 11$
eq:detM1	Eq. (26)	✓ PASS	$\det M_1(u) = \mu_1\mu_3 - \mu_2^2 = ud + 2d^2 - 1 \geq 0.$ $\min(ud(u) + 2d(u)^2 - 1) = 4.289e - 02, \quad e_\infty = 4.330e - 15$
eq:detM2	Eq. (27)	✓ PASS	$u^2d^2 + ud^3 - 3d^2 - 3ud + 2 \geq 0.$ $\min(u^2d(u)^2 + ud(u)^3 - 3d(u)^2 - 3ud(u) + 2) = 9.602e - 03, \quad e_\infty = 5.718e - 15$
eq:varY	Eq. (28)	✓ PASS	$\text{Var}(Y_u) = \mu_2 - \mu_1^2 = 1 - ud - d^2 > 0,$ $\min(1 - ud(u) - d(u)^2) = 1.415e - 01, \quad e_\infty = 2.665e - 15$
eq:constraints	Eq. (29)	✓ PASS	$x + 2y \geq 1, \quad x^2 + xy - 3x - 3y + 2 \geq 0, \quad x + y < 1.$ $m_{\min} = 3.967e - 03$
eq:ybound	Eq. (30)	✓ PASS	$0 < y \leq \frac{2}{\pi} < \frac{2}{3}.$ $y_{\max} = 6.366e - 01$
eq:Hxy	Eq. (31)	✓ PASS	$dH(u) = x^2 + 6xy + 6y^2 - x - 4y.$ $e_\infty = 1.865e - 16$
eq:g-critical-r	-	✓ PASS	$g(u_\star) = \frac{r_\star(4 - r_\star)(1 - r_\star)^2}{(r_\star^2 - 6r_\star + 6)^2}, \quad r_\star \in (0, 1).$ $r_\star = 0.778, \quad e_\infty = 6.939e - 18$
eq:split	Eq. (34)	✓ PASS	$\mathbb{E}[g(U_t)] \leq g(0)(1 - p_t) + \frac{1}{18}p_t.$ $e_{\max} = -7.402e - 02$
eq:spin_bd_unif_nos	Eq. (36)	✓ PASS	$-\frac{r_n \varepsilon_n}{2} + \mathbb{E}[\log \cosh(\sqrt{r_n} Z)] \leq \frac{\alpha_n}{2\varepsilon_n} B(q_n).$ $m_{\min} = 4.493e - 01$

Label	Main.pdf	Status	Message
eq:constraint_bd_un	Eq. (37)	✓ PASS	$\mathbb{E}[\log \bar{\Phi}(U_n)] \leq -\frac{A_n}{2\varepsilon_n} + \frac{\Phi(\kappa - \delta)}{2} \log \varepsilon_n + C(\delta),$ $m_{\min} = 2.744e + 00$
eq:BA_bd_unif_noste	Eq. (38)	✓ PASS	$0 \leq B(q_n) - A_n = \varepsilon_n \mathbb{E}[E(U_n)^2 - (U_n)_+^2] \leq C_0 \varepsilon_n.$ $e_{\max} = -5.244e - 02$
eq: two-sided mills	Eq. (39)	✓ PASS	$\bar{\Phi}(u) \geq \frac{\phi(u)}{u} - \frac{\bar{\Phi}(u)}{u^2}, \quad \text{hence} \quad \frac{\phi(u)}{\bar{\Phi}(u)} \leq u + \frac{1}{u}.$ $e_{\max} = -2.439e - 02$
eq:rootsFneg_lemma	Eq. (42)	✓ PASS	$r_{\pm}(y) = \frac{3 - y \pm \sqrt{y^2 + 6y + 1}}{2}.$ $e_{\infty} = 8.882e - 16$
eq:leftEndpoint1Fne	Eq. (44)	✓ PASS	$F(1 - 2y, y) = -2y^2 < 0.$ $e_{\infty} = 5.551e - 17$
eq:leftEndpoint2Fne	Eq. (45)	✓ PASS	$F(0, y) = 6y^2 - 4y = 2y(3y - 2) < 0.$ $e_{\infty} = 0.000e + 00$
eq:rightEndpointFne	Eq. (46)	✓ PASS	$F(r_-(y), y) = \frac{7y^2 + 11y + 2 - (5y + 2)\sqrt{y^2 + 6y + 1}}{2}.$ $e_{\infty} = 0.000e + 00$
eq:squareDiffFneg_1	Eq. (47)	✓ PASS	$(5y + 2)^2(y^2 + 6y + 1) - (7y^2 + 11y + 2)^2 = 8y^3(2 - 3y),$ $e_{\infty} = 0.000e + 00$
lem:A-TOBECHECKED	Lemma 7	✓ PASS	$A(0) = 0, \quad \lim_{r \rightarrow \infty} A(r) = \frac{2}{\pi}.$ $A(0) = 0.000000e + 00, \min \Delta A = 3.955263e - 02, A(200) = 6.340157e - 01, 2/\pi = 6.366198e - 01, \Delta = -2.604094e - 03$
lem:B_endpoints- TOBECHECKED	Lemma 10	✓ PASS	$B(0) = E(\kappa)^2,$ $\lim_{q \uparrow 1} B(q) = C_{\kappa} = \mathbb{E}[(\kappa - Z)_+^2] = (\kappa^2 + 1)\Phi(\kappa) + \kappa\phi(\kappa).$ $e_{\infty,0} = 4.440892e - 16 \quad (\kappa = 1.000, B(0) = 2.326038e + 00, E(\kappa)^2 = 2.326038e + 00), e_{\infty,1} = 8.397408e - 06 \quad (\kappa = 1.000, B(q) = 1.924669e + 00, C_{\kappa} = 1.924660e + 00)$
eq:Bprime- TOBECHECKED	Eq. (18)	✓ PASS	$B'(t) = \mathbb{E}[(E'(U_t))^2 + 2E(U_t)^2(E'(U_t) - 1)] = \mathbb{E}[g(U_t)].$ $e_{\infty} = 4.523982e - 03 \quad (t = 0.300, \text{ LHS} = -2.265129e - 01, \text{ RHS} = -2.219889e - 01)$
eq:moments- TOBECHECKED	Eq. (22)	✓ PASS	$\mu_0(u) = 1,$ $\mu_1(u) = d(u),$ $\mu_2(u) = 1 - u d(u),$ $\mu_3(u) = (u^2 + 2)d(u) - u,$ $\mu_4(u) = u^2 + 3 - u(u^2 + 5)d(u).$ $e_{\infty} = 1.421085e - 14 \quad (u = -0.500, k = 3, \text{ LHS} = 2.770611e + 00, \text{ RHS} = 2.770611e + 00)$
eq:FdefFneg_lemma- TOBECHECKED	Eq. (41)	✓ PASS	$F(x, y) < 0.$ $\max F(x, y) = -8.000000e - 04 \quad (x = 0.96, y = 0.02)$
eq: g-zero-value- TOBECHECKED	Eq. (32)	✓ PASS	$g(0) = \left(\frac{2}{\pi}\right)^2 - 2\left(1 - \frac{2}{\pi}\right)\left(\frac{2}{\pi}\right) = \frac{12}{\pi^2} - \frac{4}{\pi} = -\frac{4(\pi - 3)}{\pi^2}.$ $g(0) = -5.738534e - 02, \text{ expected} = -5.738534e - 02, \Delta = 5.551115e - 17$
eq:g-critical-r- TOBECHECKED	Eq. (33)	✓ PASS	$g(u_{\star}) = \frac{r_{\star}(4 - r_{\star})(1 - r_{\star})^2}{(r_{\star}^2 - 6r_{\star} + 6)^2}, \quad r_{\star} \in (0, 1).$ $u_{\star} = -1.002369e + 00, r_{\star} = 7.775774e - 01, e_{\infty} = 6.938894e - 18$

Label	Main.pdf	Status	Message
eq: pi-estimate-TOBECHECKED	Eq. (35)	✓ PASS	$-\frac{4(\pi - 3)}{\pi^2} < -\frac{1}{18}$ . LHS = $-5.738534e - 02$ , RHS = $-5.555556e - 02$ , $m_{\min} = -1.829785e - 03$
lem: rational function bound-TOBECHECKED	Lemma 16	✓ PASS	$\frac{r(4 - r)(1 - r)^2}{(r^2 - 6r + 6)^2} \leq \frac{1}{18}$ . $\max_{r \in (0,1)} \frac{r(4-r)(1-r)^2}{(r^2-6r+6)^2} = 4.289321e - 02$ ( $r = 0.586$ ), $1/18 = 5.555556e - 02$ , $e_{\max} = -1.266235e - 02$