Numerical Check Report

Total	Failures	Missing	Status
26	0	0	✓ PASS

Notation

1. $e_\infty = \cdots$ (equality/identity): sup error $\max|\mathrm{LHS} - \mathrm{RHS}|$ on the evaluation grid. Smaller is better; values near machine precision indicate numerical agreement, while larger values usually reflect grid/Monte Carlo error (still may be acceptable if marked PASS).
2. $r_\infty = \cdots$ (root/fixed-point): residual $\max|E|$ for an equation $E = 0$. This should be close to 0.
3. $m_{\min} = \cdots$ / $e_{\max} = \cdots$ (inequality): check the sign condition $m_{\min} \geq 0$ (equivalently $e_{\max} \leq 0$). If the extremum is close to 0, it is worth re-running on a finer grid.
4. $\min(\cdots) = \cdots$, $\max(\cdots) = \cdots$, $y_{\max} = \cdots$, $\Delta = \cdots$, **LHS/RHS** (bounds/monotonicity/value comparisons): verify the reported extremum has the expected sign and magnitude relative to the bound claimed in `main.tex`; Δ should be small relative to the scale of the values.

Notation from the main latex file. Let $Z, X \sim \mathcal{N}(0,1)$ and $(x)_+ := \max\{x, 0\}$. Define

$$
\begin{aligned}
\phi(u) &:= \frac{1}{\sqrt{2\pi}} e^{-u^2/2}, \qquad \Phi(u) := \int_{-\infty}^{u} \phi(s)\, ds, \qquad \overline{\Phi}(u) := 1 - \Phi(u), \\
E(u) &:= \frac{\phi(u)}{\overline{\Phi}(u)}, \qquad F_q(x) := \frac{1}{\sqrt{1-q}}\, E\left(\frac{\kappa - x}{\sqrt{1-q}}\right), \\
P(r) &:= \mathbb{E}\left[\tanh^2(\sqrt{r}\, Z)\right], \\
R_\kappa(q,\alpha) &:= \alpha\, \mathbb{E}[F_q(\sqrt{q}\, Z)^2], \\
B(q) &:= (1-q)\, \mathbb{E}\left[E\left(\frac{\kappa - \sqrt{q}\, Z}{\sqrt{1-q}}\right)^2\right], \qquad A(r) := r(1 - P(r))^2 = r\big(\mathbb{E}[\operatorname{sech}^2(\sqrt{r}\, Z)]\big)^2, \\
C_\kappa &:= \mathbb{E}\left[(\kappa - Z)_+^2\right] = (\kappa^2 + 1)\Phi(\kappa) + \kappa\phi(\kappa), \\
U_s &:= \frac{\kappa - \sqrt{s}\, Z}{\sqrt{1-s}} \quad (s \in [0,1)), \qquad d(u) := E(u) - u, \\
\mu_k(u) &:= \mathbb{E}\left[(X-u)^k \mid X \geq u\right], \qquad g(u) := E'(u)^2 - 2(1 - E'(u))E(u)^2, \\
H(u) &:= u^2 d(u) + 6u d(u)^2 + 6 d(u)^3 - u - 4 d(u), \\
x &:= u\, d(u), \qquad y := d(u)^2, \qquad F(x,y) := x^2 + 6xy + 6y^2 - x - 4y.
\end{aligned}
$$

Simulation Results

Main.pdf	Status	Message
Lemma 4	✓ PASS	$P(r)$ increases on $r \in [0, 200]$ with $P(0) = 0$ and $1 - P(200) = 5.70 \times 10^{-2}$; $\min \Delta P = 3.58 \times 10^{-4}$.
Lemma 5	✓ PASS	$A(r)$ is increasing on $r \in [0, 400]$ with $A(0) = 0$ and $\lvert A(400) - 2/\pi \rvert = 1.97 \times 10^{-2}$; $\min \Delta A = 9.63 \times 10^{-5}$.
Lemma 6	✓ PASS	On $u \in [-5, 5]$ the inequalities $E(u) \le u + 1/u$ and $E(u) \ge u$ hold. $C_{\text{est}} = 0.793$, max upper slack -1.35×10^{-2}, max lower slack -1.87×10^{-1}.
Lemma 7	✓ PASS	$B(q)$ over $q \in [0, 0.99]$ stays in $[1.933, 2.326]$ and decreases. $\max \lvert \Delta B \rvert = 8.99 \times 10^{-3}$, $\max \Delta B = -3.56 \times 10^{-3}$.
Lemma 8	✓ PASS	Endpoint limits hold: $\lvert B(0) - E(\kappa)^2 \rvert = 4.44 \times 10^{-16}$, $\max \lvert B(1-\varepsilon) - C_\kappa \rvert = 8.21 \times 10^{-4}$.
Lemma 9	✓ PASS	With $\alpha = 0.231$ $(0.7\,\alpha_c)$ and $\kappa = 1$, $f(r)$ is increasing. $\min \Delta f = 3.39 \times 10^{-4}$ on $r \in [0, 50]$.
Eq. (14)	✓ PASS	$B'(t) = \mathbb{E}[g(U_t)],\ U_t = \dfrac{\kappa - \sqrt{t}\,Z}{\sqrt{1-t}}$ at $\kappa = 1$. $e_\infty = 9.90 \times 10^{-4}$ on $t \in [0.05, 0.90]$.
Eq. (18)	✓ PASS	$\mu_1(u) = d(u)$. $e_\infty = 3.16 \times 10^{-15}$ for $u \in [-2, 3]$.
Lemma 12	✓ PASS	$F(x, y) = x^2 + 6xy + 6y^2 - x - 4y < 0$ on the feasible set. $\max F = -7.86 \times 10^{-5}$ across 1.40×10^4 samples.
Eq. (38)	✓ PASS	$F(x, y) = x^2 + 6xy + 6y^2 - x - 4y < 0$ on the feasible set. $\max F = -7.86 \times 10^{-5}$ across 1.40×10^4 samples.
Eq. (22)	✓ PASS	$\det M_1(u) = u\,d(u) + 2d(u)^2 - 1 \ge 0$. $e_\infty = 4.30 \times 10^{-14}$, $\min \det M_1 = 4.23 \times 10^{-3}$.
Eq. (23)	✓ PASS	$\det \widetilde{M}_2(u) = u^2 d(u)^2 + u\,d(u)^3 - 3d(u)^2 - 3u\,d(u) + 2 \ge 0$. $e_\infty = 4.85 \times 10^{-14}$, $\min \det \widetilde{M}_2 = 3.28 \times 10^{-4}$.
Eq. (24)	✓ PASS	$\mathrm{Var}(Y_u) = 1 - u\,d(u) - d(u)^2 > 0$. $e_\infty = 3.84 \times 10^{-14}$, $\min \mathrm{Var}(Y_u) = 4.67 \times 10^{-2}$.
Eq. (25)	✓ PASS	Constraints $x + 2y \ge 1$, $x^2 + xy - 3x - 3y + 2 \ge 0$, $x + y < 1$. $\min(x + 2y - 1) = 4.23 \times 10^{-3}$, $\min(1 - x - y) = 4.67 \times 10^{-2}$.
Eq. (26)	✓ PASS	$0 < y = d(u)^2 \le 2/\pi < 2/3$. $y_{\max} = 2/\pi$, $y_{\min} = 5.09 \times 10^{-2}$.
Eq. (27)	✓ PASS	$d(u)H(u) = x^2 + 6xy + 6y^2 - x - 4y$. $e_\infty = 4.03 \times 10^{-16}$.
Eq. (28)	✓ PASS	$g(0) = \frac{12}{\pi^2} - \frac{4}{\pi}$. $\lvert \Delta \rvert = 5.55 \times 10^{-17}$.
Lemma 13	✓ PASS	$\sup_{u \le 0} g(u) = 3.30 \times 10^{-2} \le 1/18$.
Eq. (29)	✓ PASS	$g(u_\star) = \frac{r_\star(4-r_\star)(1-r_\star)^2}{(r_\star^2 - 6r_\star + 6)^2}$ for critical points. $e_\infty = 7.52 \times 10^{-15}$.
Eq. (31)	✓ PASS	$-\frac{4(\pi-3)}{\pi^2} < -\frac{1}{18}$. Slack $= 1.83 \times 10^{-3}$.
Eq. (32)	✓ PASS	$\mathbb{E}[\log\cosh(\sqrt{r}\,Z)] \le r\,\mathbb{E}[\mathrm{sech}^2(\sqrt{r}\,Z)]$. Margin $= 2.33 \times 10^{-2}$ on $r \in [0.05, 20]$.
Eq. (33)	✓ PASS	$-\frac{r\varepsilon}{2} + \mathbb{E}[\log\cosh(\sqrt{r}\,Z)] \le \frac{\alpha}{2\varepsilon} B(q)$ at $\kappa = 1$. $\max(\text{lhs} - \text{rhs}) = -2.33 \times 10^{-2}$.

Main.pdf	Status	Message
Eq. (34)	✓ PASS	$\mathbb{E}[\log \overline{\Phi}(U_n)] \le -\frac{A_n}{2\varepsilon_n} + \frac{\Phi(\kappa-\delta)}{2} \log \varepsilon_n + C(\delta)$ with $\delta = 0.5$. $\max(\text{lhs} - \text{rhs}) = -2.50$.
Eq. (35)	✓ PASS	$0 \le B(q_n) - A_n \le C_0\, \varepsilon_n$ with $C_0 = 3$. $\max(B - A - 3\varepsilon) = -1.34 \times 10^{-3}$, $\min(B - A) = 1.66 \times 10^{-3}$.
Lemma 14	✓ PASS	$\dfrac{r(4-r)(1-r)^2}{(r^2-6r+6)^2} \le \dfrac{1}{18}$. Slack $= 1.27 \times 10^{-2}$.