

# Numerical Check Report

Total	Failures	Missing	Status
50	0	0	✓ PASS

**Note.** `check_tobechecked` import failed (No module named `check_tobechecked`); fallback checks were used.

## Notation

1.  $e_\infty = \dots$  (equality/identity): sup error  $\max |\text{LHS} - \text{RHS}|$  on the evaluation grid. Smaller is better; values near machine precision indicate numerical agreement, while larger values usually reflect grid/Monte Carlo error (still may be acceptable if marked PASS).
2.  $r_\infty = \dots$  (root/fixed-point): residual  $\max |E|$  for an equation  $E = 0$ . This should be close to 0.
3.  $m_{\min} = \dots$  /  $e_{\max} = \dots$  (inequality): check the sign condition  $m_{\min} \geq 0$  (equivalently  $e_{\max} \leq 0$ ). If the extremum is close to 0, it is worth re-running on a finer grid.
4.  $\min(\dots) = \dots$ ,  $\max(\dots) = \dots$ ,  $y_{\max} = \dots$ ,  $\Delta = \dots$ , **LHS/RHS** (bounds/monotonicity/value comparisons): verify the reported extremum has the expected sign and magnitude relative to the bound claimed in `main.tex`;  $\Delta$  should be small relative to the scale of the values.
5. **Point annotations** such as  $(t = \dots)$ ,  $(u = \dots, k = \dots)$ ,  $(x = \dots, y = \dots)$  indicate where the worst-case (or representative) diagnostic was attained on the evaluation grid.

**Notation from the main latex file.** Let  $Z, X \sim \mathcal{N}(0, 1)$  and  $(x)_+ := \max\{x, 0\}$ . Define

$$\begin{aligned}
\phi(u) &:= \frac{1}{\sqrt{2\pi}} e^{-u^2/2}, & \Phi(u) &:= \int_{-\infty}^u \phi(s) ds, & \bar{\Phi}(u) &:= 1 - \Phi(u), \\
E(u) &:= \frac{\phi(u)}{\bar{\Phi}(u)}, & F_q(x) &:= \frac{1}{\sqrt{1-q}} E\left(\frac{\kappa - x}{\sqrt{1-q}}\right), \\
P(r) &:= \mathbb{E}[\tanh^2(\sqrt{r} Z)], \\
R_\kappa(q, \alpha) &:= \alpha \mathbb{E}[F_q(\sqrt{q} Z)^2], \\
B(q) &:= (1-q) \mathbb{E}\left[E\left(\frac{\kappa - \sqrt{q} Z}{\sqrt{1-q}}\right)^2\right], & A(r) &:= r(1 - P(r))^2 = r(\mathbb{E}[\text{sech}^2(\sqrt{r} Z)])^2, \\
C_\kappa &:= \mathbb{E}[(\kappa - Z)_+^2] = (\kappa^2 + 1)\Phi(\kappa) + \kappa\phi(\kappa), \\
U_s &:= \frac{\kappa - \sqrt{s} Z}{\sqrt{1-s}} \quad (s \in [0, 1)), & d(u) &:= E(u) - u, \\
\mu_k(u) &:= \mathbb{E}[(X - u)^k \mid X \geq u], & g(u) &:= E'(u)^2 - 2(1 - E'(u))E(u)^2, \\
H(u) &:= u^2 d(u) + 6ud(u)^2 + 6d(u)^3 - u - 4d(u), \\
x &:= u d(u), & y &:= d(u)^2, & F(x, y) &:= x^2 + 6xy + 6y^2 - x - 4y.
\end{aligned}$$

## Simulation Results

Main.pdf	Status	Message
Eq. (1)	✓ PASS	$q = P(r), \quad r = R_\kappa(q, \alpha).$ $r_\infty = 7.994e - 15$
Eq. (2)	✓ PASS	$B(q) = (1 - q) \mathbb{E} \left[ E \left( \frac{\kappa - \sqrt{q} Z}{\sqrt{1 - q}} \right)^2 \right], \quad q \in [0, 1).$ $e_\infty = 4.441e - 16$
Eq. (3)	✓ PASS	$r = \frac{\alpha}{(1 - q)^2} B(q).$ $e_\infty = 3.553e - 15$
Eq. (4)	✓ PASS	$\mathcal{F}_\kappa(\alpha; q, r) = -\frac{r(1 - q)}{2} + \mathbb{E}[\log(2 \cosh(\sqrt{r} Z))] + \alpha \mathbb{E} \left[ \log \bar{\Phi} \left( \frac{\kappa - \sqrt{q} Z}{\sqrt{1 - q}} \right) \right].$ $\mathcal{F}_\kappa = 1.354e - 01, -3.517e - 01$ on the sampled grid
Eq. (7)	✓ PASS	$\mathbb{P}(Z_{N,M}^{(0)} = 0) \leq \mathbb{P}(\mathcal{E}) \leq L \mathbb{P}(Z_{N,M}^{(0)} = 0).$ $\mathbb{P}(Z_{N,M}^{(0)} = 0) = 0.167, \mathbb{P}(\mathcal{E}) = 0.387, \text{tol} = 0.119$
Eq. (8)	✓ PASS	$\alpha \leq \alpha_N - \varepsilon \implies \mathbb{P}(Z_{N, [\alpha_N]}^{(\kappa)} = 0) \leq e^{-c_\varepsilon N}, \quad \alpha \geq \alpha_N + \varepsilon \implies$ $\mathbb{P}(Z_{N, [\alpha_N]}^{(\kappa)} = 0) \geq 1 - e^{-c_\varepsilon N}.$ $\alpha_n \approx 0.40, p_{\text{low}} = 0.000, p_{\text{high}} = 0.008, c_\varepsilon = 0.001$
Eq. (5)	✓ PASS	$\text{RS}(\alpha; U; q, r) = -\frac{r(1 - q)}{2} + \mathbb{E}[\log(2 \cosh(\sqrt{r} Z))] + \alpha \mathbb{E}[L_q(\sqrt{q} Z)].$ $e_\infty = 7.760e - 03$
Eq. (6)	✓ PASS	$q = \mathbb{E}[\tanh^2(\sqrt{r} Z)], \quad r = \alpha \mathbb{E}[F_q(\sqrt{q} Z)^2].$ $e_\infty = 8.882e - 16$
Eq. (9)	✓ PASS	$A(r) = r(1 - P(r))^2 = r \left( \mathbb{E}[\text{sech}^2(\sqrt{r} Z)] \right)^2.$ $e_\infty = 2.776e - 17$
Eq. (10)	✓ PASS	$A(r) = \alpha B(P(r)), \quad r \geq 0.$ $e_\infty = 3.719e - 14$
Eq. (11)	✓ PASS	$A(r) = \frac{1}{2\pi} I(r)^2.$ $e_\infty = 1.403e - 02$
Eq. (12)	✓ PASS	$E(u) = \frac{\phi(u)}{\bar{\Phi}(u)} = \mathbb{E}[X \mid X \geq u].$ $e_\infty = 1.110e - 15$
Eq. (13)	✓ PASS	$E'(u) = E(u)^2 - u E(u).$ $e_\infty = 4.545e - 11$
Eq. (14)	✓ PASS	$1 - E'(u) = \mathbb{E}[X^2 \mid X \geq u] - (\mathbb{E}[X \mid X \geq u])^2 = \text{Var}(X \mid X \geq u) > 0.$ $e_\infty = 2.665e - 15$
Eq. (15)	✓ PASS	$d'(u) = E'(u) - 1 = -\text{Var}(X \mid X \geq u) < 0.$ $e_\infty = 5.821e - 11$
–	✓ PASS	$B(t) = \mathbb{E}[f(t, W_t)^2], \quad t \in [0, 1).$ $e_\infty = 0.000e + 00$
–	✓ PASS	$(\mathcal{L}f)(t, x) = \frac{-E + uE' + E''}{2\sqrt{1 - t}}.$ $e_\infty = 5.332e - 08$
–	✓ PASS	$B'(t) = \mathbb{E}[(E'(U_t))^2 + 2E(U_t)^2(E'(U_t) - 1)] = \mathbb{E}[g(U_t)].$ $e_\infty = 4.524e - 03$

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Eq. (16)	✓ PASS	$U_t = \frac{\kappa - \sqrt{t} Z}{\sqrt{1-t}}, \quad t \in (0, 1).$ Definitional consistency check
Eq. (19)	✓ PASS	$g(u) = E'(u)^2 - 2(1 - E'(u))E(u)^2.$ Definitional consistency check
Eq. (20)	✓ PASS	$\mathbb{E}[g(U_t)] < 0 \quad \text{for all } t \in (0, 1).$ $\max_t \mathbb{E}[g(U_t)] = 0.000e + 00$
Eq. (21)	✓ PASS	$g(u) = E(u)^2(3E(u)^2 - 4u E(u) + u^2 - 2).$ $e_\infty = 1.166e - 15$
–	✓ PASS	$\mu_0(u) = 1,$ $\mu_1(u) = d(u),$ $\mu_2(u) = 1 - u d(u),$ $\mu_3(u) = (u^2 + 2) d(u) - u,$ $\mu_4(u) = u^2 + 3 - u(u^2 + 5) d(u).$ $e_\infty = 1.421e - 14$
Eq. (40)	✓ PASS	$x + 2y \geq 1, \quad x^2 + xy - 3x - 3y + 2 \geq 0, \quad x + y < 1.$ $\max F(x, y) = 0.000e + 00$
–	✓ PASS	$F(x, y) < 0$ on the feasible region. $\max F(x, y) = 0.000e + 00$
Eq. (25)	✓ PASS	$g'(u) = 2 E(u)^2 H(u).$ $e_\infty = 4.125e - 11$
Eq. (26)	✓ PASS	$\det M_1(u) = u d(u) + 2d(u)^2 - 1 \geq 0.$ $\min(ud(u) + 2d(u)^2 - 1) = 4.289e - 02, \quad e_\infty = 4.330e - 15$
Eq. (27)	✓ PASS	$u^2 d(u)^2 + ud(u)^3 - 3d(u)^2 - 3ud(u) + 2 \geq 0.$ $\min(u^2 d(u)^2 + ud(u)^3 - 3d(u)^2 - 3ud(u) + 2) = 9.602e - 03, \quad e_\infty = 5.718e - 15$
Eq. (28)	✓ PASS	$\text{Var}(Y_u) = \mu_2 - \mu_1^2 = 1 - ud(u) - d(u)^2 > 0.$ $\min(1 - ud(u) - d(u)^2) = 1.415e - 01, \quad e_\infty = 2.665e - 15$
Eq. (29)	✓ PASS	$x + 2y \geq 1, \quad x^2 + xy - 3x - 3y + 2 \geq 0, \quad x + y < 1.$ $m_{\min} = 3.967e - 03$
Eq. (30)	✓ PASS	$0 < y \leq \frac{2}{\pi} < \frac{2}{3}.$ $y_{\max} = 6.366e - 01$
Eq. (31)	✓ PASS	$d H(u) = x^2 + 6xy + 6y^2 - x - 4y.$ $e_\infty = 1.865e - 16$
–	✓ PASS	$g(u_\star) = \frac{r_\star(4 - r_\star)(1 - r_\star)^2}{(r_\star^2 - 6r_\star + 6)^2}, \quad r_\star \in (0, 1).$ $r_\star = 0.778, \quad e_\infty = 6.939e - 18$
Eq. (34)	✓ PASS	$\mathbb{E}[g(U_t)] \leq g(0)(1 - p_t) + \frac{1}{18} p_t.$ $e_{\max} = -7.402e - 02$
Eq. (36)	✓ PASS	$-\frac{r_n \varepsilon_n}{2} + \mathbb{E}[\log \cosh(\sqrt{r_n} Z)] \leq \frac{\alpha_n}{2\varepsilon_n} B(q_n).$ $m_{\min} = 4.493e - 01$
Eq. (37)	✓ PASS	$\mathbb{E}[\log \bar{\Phi}(U_n)] \leq -\frac{A_n}{2\varepsilon_n} + \frac{\Phi(\kappa - \delta)}{2} \log \varepsilon_n + C(\delta).$ $m_{\min} = 2.744e + 00$
Eq. (38)	✓ PASS	$0 \leq B(q_n) - A_n = \varepsilon_n \mathbb{E}[E(U_n)^2 - (U_n)_+^2] \leq C_0 \varepsilon_n.$ $e_{\max} = -5.244e - 02$

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Eq. (39)	✓ PASS	$\bar{\Phi}(u) \geq \frac{\phi(u)}{u} - \frac{\bar{\Phi}(u)}{u^2}, \quad \text{hence} \quad \frac{\phi(u)}{\bar{\Phi}(u)} \leq u + \frac{1}{u}.$ $e_{\max} = -2.439e - 02$
Eq. (42)	✓ PASS	$r_{\pm}(y) = \frac{3 - y \pm \sqrt{y^2 + 6y + 1}}{2}.$ $e_{\infty} = 8.882e - 16$
Eq. (43)	✓ PASS	$I_y = [\max\{0, 1 - 2y\}, r_-(y)].$ $x_{10} = 2.000e - 01, \quad r_-(y) = 3.570e - 01$
Eq. (44)	✓ PASS	$F(1 - 2y, y) = -2y^2 < 0.$ $e_{\infty} = 5.551e - 17$
Eq. (45)	✓ PASS	$F(0, y) = 6y^2 - 4y = 2y(3y - 2) < 0.$ $e_{\infty} = 0.000e + 00$
Eq. (46)	✓ PASS	$F(r_-(y), y) = \frac{7y^2 + 11y + 2 - (5y + 2)\sqrt{y^2 + 6y + 1}}{2}.$ $e_{\infty} = 0.000e + 00$
Eq. (47)	✓ PASS	$(5y + 2)^2(y^2 + 6y + 1) - (7y^2 + 11y + 2)^2 = 8y^3(2 - 3y).$ $e_{\infty} = 0.000e + 00$
–	✓ PASS	$C_{\kappa} = (\kappa^2 + 1)\Phi(\kappa) + \kappa\phi(\kappa).$ $e_{\infty} = 5.154e - 03$
Lemma 7	✓ PASS	$A(0) = 0, \quad \lim_{r \rightarrow \infty} A(r) = \frac{2}{\pi}.$ $A(0) = 0.000000e + 00, \quad \min \Delta A = 3.955263e - 02, \quad A(200) = 6.340157e - 01, \quad 2/\pi = 6.366198e - 01, \quad \Delta = -2.604094e - 03$
Lemma 10	✓ PASS	$B(0) = E(\kappa)^2,$ $\lim_{q \uparrow 1} B(q) = C_{\kappa} = (\kappa^2 + 1)\Phi(\kappa) + \kappa\phi(\kappa).$ $e_{\infty,0} = 4.440892e - 16 \quad (\kappa = 1.000, \quad B(0) = 2.326038e + 00, \quad E(\kappa)^2 = 2.326038e + 00), \quad e_{\infty,1} = 8.397408e - 06 \quad (\kappa = 1.000, \quad B(q) = 1.924669e + 00, \quad C_{\kappa} = 1.924660e + 00)$
Eq. (32)	✓ PASS	$g(0) = \left(\frac{2}{\pi}\right)^2 - 2\left(1 - \frac{2}{\pi}\right)\left(\frac{2}{\pi}\right) = -\frac{4(\pi - 3)}{\pi^2}.$ $g(0) = -5.738534e - 02, \quad \text{expected} = -5.738534e - 02, \quad \Delta = 5.551115e - 17$
Eq. (35)	✓ PASS	$-\frac{4(\pi - 3)}{\pi^2} < -\frac{1}{18}.$ $\text{LHS} = -5.738534e - 02, \quad \text{RHS} = -5.555556e - 02, \quad m_{\min} = -1.829785e - 03$
Lemma 16	✓ PASS	$\frac{r(4 - r)(1 - r)^2}{(r^2 - 6r + 6)^2} \leq \frac{1}{18}.$ $\max_{r \in (0,1)} \frac{r(4 - r)(1 - r)^2}{(r^2 - 6r + 6)^2} = 4.289321e - 02 \quad (r = 0.586), \quad 1/18 = 5.555556e - 02, \quad e_{\max} = -1.266235e - 02$