Numerical Check Report

Total	Failures	Missing	Status
50	0	0	✓ PASS

Note. `check_tobechecked` import failed (No module named `check_tobechecked`); fallback checks were used.

Notation

1. $e_\infty = \cdots$ (equality/identity): sup error $\max |\text{LHS} - \text{RHS}|$ on the evaluation grid. Smaller is better; values near machine precision indicate numerical agreement, while larger values usually reflect grid/Monte Carlo error (still may be acceptable if marked PASS).
2. $r_\infty = \cdots$ (root/fixed-point): residual $\max |E|$ for an equation $E = 0$. This should be close to 0.
3. $m_{\min} = \cdots$ / $e_{\max} = \cdots$ (inequality): check the sign condition $m_{\min} \geq 0$ (equivalently $e_{\max} \leq 0$). If the extremum is close to 0, it is worth re-running on a finer grid.
4. $\min(\cdots) = \cdots$, $\max(\cdots) = \cdots$, $y_{\max} = \cdots$, $\Delta = \cdots$, **LHS/RHS** (bounds/monotonicity/value comparisons): verify the reported extremum has the expected sign and magnitude relative to the bound claimed in `main.tex`; Δ should be small relative to the scale of the values.
5. **Point annotations** such as $(t = \cdots)$, $(u = \cdots, k = \cdots)$, $(x = \cdots, y = \cdots)$ indicate where the worst-case (or representative) diagnostic was attained on the evaluation grid.

Notation from the main latex file. Let $Z, X \sim \mathcal{N}(0, 1)$ and $(x)_+ := \max\{x, 0\}$. Define

$$\phi(u) := \frac{1}{\sqrt{2\pi}} e^{-u^2/2}, \qquad \Phi(u) := \int_{-\infty}^{u} \phi(s)\, ds, \qquad \overline{\Phi}(u) := 1 - \Phi(u),$$

$$E(u) := \frac{\phi(u)}{\overline{\Phi}(u)}, \qquad F_q(x) := \frac{1}{\sqrt{1-q}} E\left(\frac{\kappa - x}{\sqrt{1-q}}\right),$$

$$P(r) := \mathbb{E}\big[\tanh^2(\sqrt{r}\, Z)\big],$$

$$R_\kappa(q, \alpha) := \alpha\, \mathbb{E}\big[F_q(\sqrt{q}\, Z)^2\big],$$

$$B(q) := (1-q)\, \mathbb{E}\left[E\left(\frac{\kappa - \sqrt{q}\, Z}{\sqrt{1-q}}\right)^2\right], \qquad A(r) := r(1 - P(r))^2 = r\big(\mathbb{E}[\text{sech}^2(\sqrt{r}\, Z)]\big)^2,$$

$$C_\kappa := \mathbb{E}\big[(\kappa - Z)_+^2\big] = (\kappa^2 + 1)\Phi(\kappa) + \kappa\phi(\kappa),$$

$$U_s := \frac{\kappa - \sqrt{s}\, Z}{\sqrt{1-s}} \quad (s \in [0, 1)), \qquad d(u) := E(u) - u,$$

$$\mu_k(u) := \mathbb{E}\big[(X - u)^k \mid X \geq u\big], \qquad g(u) := E'(u)^2 - 2(1 - E'(u))E(u)^2,$$

$$H(u) := u^2 d(u) + 6u d(u)^2 + 6d(u)^3 - u - 4d(u),$$

$$x := u\, d(u), \qquad y := d(u)^2, \qquad F(x, y) := x^2 + 6xy + 6y^2 - x - 4y.$$

Simulation Results

Label	Main.pdf	Status	Message
`eq:system`	Eq. (1)	✓ PASS	$q = P(r), \quad r = R_\kappa(q, \alpha).$ $r_\infty = 7.994e - 15$
`eq:defB`	Eq. (2)	✓ PASS	$B(q) = (1 - q)\, \mathbb{E}\left[E\left(\frac{\kappa - \sqrt{q}\, Z}{\sqrt{1 - q}} \right)^2 \right], \quad q \in [0, 1).$ $e_\infty = 4.441e - 16$
`eq:r_in_terms_of_B`	Eq. (3)	✓ PASS	$r = \dfrac{\alpha}{(1 - q)^2}\, B(q).$ $e_\infty = 3.553e - 15$
`eq:RSfunctional`	Eq. (4)	✓ PASS	$\mathcal{F}_\kappa(\alpha; q, r) =$ $-\dfrac{r(1 - q)}{2} + \mathbb{E}\left[\log(2 \cosh(\sqrt{r}\, Z)) \right] + \alpha\, \mathbb{E}\left[\log \overline{\Phi}\left(\frac{\kappa - \sqrt{q}\, Z}{\sqrt{1 - q}} \right) \right].$ $\mathcal{F}_\kappa = 1.354e - 01, -3.517e - 01$ on the sampled grid
`eq:union-bound`	Eq. (7)	✓ PASS	$\mathbb{P}(Z^{(0)}_{N,M} = 0) \le \mathbb{P}(\mathcal{E}) \le L\, \mathbb{P}(Z^{(0)}_{N,M} = 0).$ $\mathbb{P}(Z^{(0)}_{N,M} = 0) = 0.167, \ \mathbb{P}(\mathcal{E}) = 0.387, \ \text{tol} = 0.119$
`eq:sharp-seq-exp`	Eq. (8)	✓ PASS	$\alpha \le \alpha_N - \varepsilon \implies \mathbb{P}\left(Z^{(\kappa)}_{N,\lfloor \alpha N \rfloor} = 0 \right) \le e^{-c_\varepsilon N}, \quad \alpha \ge$ $\alpha_N + \varepsilon \implies \mathbb{P}\left(Z^{(\kappa)}_{N,\lfloor \alpha N \rfloor} = 0 \right) \ge 1 - e^{-c_\varepsilon N}.$ $\alpha_n \approx 0.40, \ p_{\text{low}} = 0.000, \ p_{\text{high}} = 0.008, \ c_\varepsilon = 0.001$
`eq:RS-functional`	Eq. (5)	✓ PASS	$\mathrm{RS}(\alpha; U; q, r) =$ $-\dfrac{r(1 - q)}{2} + \mathbb{E}\left[\log\left(2 \cosh(\sqrt{r}\, Z) \right) \right] + \alpha\, \mathbb{E}\left[L_q(\sqrt{q}\, Z) \right].$ $e_\infty = 7.760e - 03$
`eq:RS-fixed-point`	Eq. (6)	✓ PASS	$q = \mathbb{E}\left[\tanh^2(\sqrt{r}\, Z) \right], \quad r = \alpha\, \mathbb{E}\left[F_q(\sqrt{q}\, Z)^2 \right].$ $e_\infty = 8.882e - 16$
`eq:defA`	Eq. (9)	✓ PASS	$A(r) = r\left(1 - P(r) \right)^2 = r\left(\mathbb{E}\left[\operatorname{sech}^2(\sqrt{r}\, Z) \right] \right)^2.$ $e_\infty = 2.776e - 17$
`eq:one-dimensional`	Eq. (10)	✓ PASS	$A(r) = \alpha\, B\left(P(r) \right), \quad r \ge 0.$ $e_\infty = 3.719e - 14$
`eq:A_as_I`	Eq. (11)	✓ PASS	$A(r) = \dfrac{1}{2\pi}\, I(r)^2.$ $e_\infty = 1.403e - 02$
`eq:condmean`	Eq. (12)	✓ PASS	$E(u) = \dfrac{\phi(u)}{\overline{\Phi}(u)} = \mathbb{E}[X \mid X \ge u].$ $e_\infty = 1.110e - 15$
`eq:mprime`	Eq. (13)	✓ PASS	$E'(u) = E(u)^2 - u\, E(u).$ $e_\infty = 4.545e - 11$
`eq:varidentity`	Eq. (14)	✓ PASS	$1 - E'(u) = \mathbb{E}[X^2 \mid X \ge u] - \left(\mathbb{E}[X \mid X \ge u] \right)^2 = \mathrm{Var}(X \mid X \ge u) > 0.$ $e_\infty = 2.665e - 15$
`eq:dprime`	Eq. (15)	✓ PASS	$d'(u) = E'(u) - 1 = -\mathrm{Var}(X \mid X \ge u) < 0.$ $e_\infty = 5.821e - 11$
`eq:B-as-BM`	–	✓ PASS	$B(t) = \mathbb{E}\left[f(t, W_t)^2 \right], \quad t \in [0, 1).$ $e_\infty = 0.000e + 00$
`eq:Lf`	–	✓ PASS	$(\mathcal{L}f)(t, x) = \dfrac{-E + uE' + E''}{2\sqrt{1 - t}}.$ $e_\infty = 5.332e - 08$

Label	Main.pdf	Status	Message
`eq:Bprime`	–	✓ PASS	$B'(t) = \mathbb{E}\Big[(E'(U_t))^2 + 2E(U_t)^2\big(E'(U_t) - 1\big)\Big] = \mathbb{E}\big[g(U_t)\big].$ $e_\infty = 4.524e - 03$
`eq:U-def`	Eq. (16)	✓ PASS	$U_t = \dfrac{\kappa - \sqrt{t}\,Z}{\sqrt{1-t}}, \qquad t \in (0,1).$ Definitional consistency check
`eq:g-def`	Eq. (19)	✓ PASS	$g(u) = E'(u)^2 - 2\big(1 - E'(u)\big)E(u)^2.$ Definitional consistency check
`eq:goal`	Eq. (20)	✓ PASS	$\mathbb{E}\big[g(U_t)\big] < 0 \qquad$ for all $t \in (0,1).$ $\max_t \mathbb{E}[g(U_t)] = 0.000e + 00$
`eq:g-expanded`	Eq. (21)	✓ PASS	$g(u) = E(u)^2\big(3E(u)^2 - 4u\,E(u) + u^2 - 2\big).$ $e_\infty = 1.166e - 15$
`eq:moments`	–	✓ PASS	$\mu_0(u) = 1,$ $\mu_1(u) = d(u),$ $\mu_2(u) = 1 - u\,d(u),$ $\mu_3(u) = (u^2 + 2)\,d(u) - u,$ $\mu_4(u) = u^2 + 3 - u(u^2 + 5)\,d(u).$ $e_\infty = 1.421e - 14$
`eq:consFneg_lemma`	Eq. (40)	✓ PASS	$x + 2y \geq 1, \qquad x^2 + xy - 3x - 3y + 2 \geq 0, \qquad x + y < 1.$ $\max F(x,y) = 0.000e + 00$
`eq:FdefFneg_lemma`	–	✓ PASS	$F(x,y) < 0$ on the feasible region. $\max F(x,y) = 0.000e + 00$
`eq:gprime`	Eq. (25)	✓ PASS	$g'(u) = 2\,E(u)^2\,H(u).$ $e_\infty = 4.125e - 11$
`eq:detM1`	Eq. (26)	✓ PASS	$\det M_1(u) = u\,d(u) + 2d(u)^2 - 1 \geq 0.$ $\min(ud(u) + 2d(u)^2 - 1) = 4.289e - 02,\ e_\infty = 4.330e - 15$
`eq:detM2`	Eq. (27)	✓ PASS	$u^2 d(u)^2 + ud(u)^3 - 3d(u)^2 - 3ud(u) + 2 \geq 0.$ $\min(u^2 d(u)^2 + ud(u)^3 - 3d(u)^2 - 3ud(u) + 2) = 9.602e - 03,\ e_\infty = 5.718e - 15$
`eq:varY`	Eq. (28)	✓ PASS	$\mathrm{Var}(Y_u) = \mu_2 - \mu_1^2 = 1 - ud(u) - d(u)^2 > 0.$ $\min(1 - ud(u) - d(u)^2) = 1.415e - 01,\ e_\infty = 2.665e - 15$
`eq:constraints`	Eq. (29)	✓ PASS	$x + 2y \geq 1, \qquad x^2 + xy - 3x - 3y + 2 \geq 0, \qquad x + y < 1.$ $m_{\min} = 3.967e - 03$
`eq:ybound`	Eq. (30)	✓ PASS	$0 < y \leq \dfrac{2}{\pi} < \dfrac{2}{3}.$ $y_{\max} = 6.366e - 01$
`eq:Hxy`	Eq. (31)	✓ PASS	$d\,H(u) = x^2 + 6xy + 6y^2 - x - 4y.$ $e_\infty = 1.865e - 16$
`eq:g-critical-r`	–	✓ PASS	$g(u_\star) = \dfrac{r_\star(4 - r_\star)(1 - r_\star)^2}{(r_\star^2 - 6r_\star + 6)^2}, \qquad r_\star \in (0,1).$ $r_\star = 0.778,\ e_\infty = 6.939e - 18$
`eq:split`	Eq. (34)	✓ PASS	$\mathbb{E}[g(U_t)] \leq g(0)\,(1 - p_t) + \dfrac{1}{18}\,p_t.$ $e_{\max} = -7.402e - 02$
`eq:spin_bd_unif_nos`	Eq. (36)	✓ PASS	$-\dfrac{r_n \varepsilon_n}{2} + \mathbb{E}\big[\log \cosh(\sqrt{r_n}\,Z)\big] \leq \dfrac{\alpha_n}{2\varepsilon_n}\,B(q_n).$ $m_{\min} = 4.493e - 01$
`eq:constraint_bd_un`	Eq. (37)	✓ PASS	$\mathbb{E}\big[\log \overline{\Phi}(U_n)\big] \leq -\dfrac{A_n}{2\varepsilon_n} + \dfrac{\Phi(\kappa - \delta)}{2}\,\log \varepsilon_n + C(\delta).$ $m_{\min} = 2.744e + 00$

Label	Main.pdf	Status	Message
`eq:BA_bd_unif_noste`	Eq. (38)	✓ PASS	$0 \leq B(q_n) - A_n = \varepsilon_n \, \mathbb{E}\big[E(U_n)^2 - (U_n)_+^2\big] \leq C_0 \, \varepsilon_n.$ $e_{\max} = -5.244e - 02$
`eq: two-sided mills`	Eq. (39)	✓ PASS	$\overline{\Phi}(u) \geq \dfrac{\phi(u)}{u} - \dfrac{\overline{\Phi}(u)}{u^2}, \qquad \text{hence} \qquad \dfrac{\phi(u)}{\overline{\Phi}(u)} \leq u + \dfrac{1}{u}.$ $e_{\max} = -2.439e - 02$
`eq:rootsFneg_lemma`	Eq. (42)	✓ PASS	$r_\pm(y) = \dfrac{3 - y \pm \sqrt{y^2 + 6y + 1}}{2}.$ $e_\infty = 8.882e - 16$
`eq:IyFneg_lemma`	Eq. (43)	✓ PASS	$I_y = \big[\max\{0, 1 - 2y\}, \, r_-(y)\big].$ $x_{\mathrm{lo}} = 2.000e - 01, \;\; r_-(y) = 3.570e - 01$
`eq:leftEndpoint1Fne`	Eq. (44)	✓ PASS	$F(1 - 2y, y) = -2y^2 < 0.$ $e_\infty = 5.551e - 17$
`eq:leftEndpoint2Fne`	Eq. (45)	✓ PASS	$F(0, y) = 6y^2 - 4y = 2y(3y - 2) < 0.$ $e_\infty = 0.000e + 00$
`eq:rightEndpointFne`	Eq. (46)	✓ PASS	$F(r_-(y), y) = \dfrac{7y^2 + 11y + 2 - (5y + 2)\sqrt{y^2 + 6y + 1}}{2}.$ $e_\infty = 0.000e + 00$
`eq:squareDiffFneg_l`	Eq. (47)	✓ PASS	$(5y + 2)^2(y^2 + 6y + 1) - (7y^2 + 11y + 2)^2 = 8y^3(2 - 3y).$ $e_\infty = 0.000e + 00$
`unlabeled:C_kappa_c`	–	✓ PASS	$C_\kappa = (\kappa^2 + 1)\Phi(\kappa) + \kappa\phi(\kappa).$ $e_\infty = 5.154e - 03$
`lem:A-TOBECHECKED`	Lemma 7	✓ PASS	$A(0) = 0, \qquad \lim_{r \to \infty} A(r) = \dfrac{2}{\pi}.$ $A(0) = 0.000000e + 00, \; \min \Delta A = 3.955263e - 02, \; A(200) = 6.340157e - 01, \; 2/\pi = 6.366198e - 01, \; \Delta = -2.604094e - 03$
`lem:B_endpoints-TOBECHECKED`	Lemma 10	✓ PASS	$B(0) = E(\kappa)^2,$ $\lim_{q \uparrow 1} B(q) = C_\kappa = (\kappa^2 + 1)\Phi(\kappa) + \kappa\phi(\kappa).$ $e_{\infty,0} = 4.440892e - 16 \; (\kappa = 1.000, \; B(0) = 2.326038e + 00, \; E(\kappa)^2 = 2.326038e + 00), \; e_{\infty,1} = 8.397408e - 06 \; (\kappa = 1.000, \; B(q) = 1.924669e + 00, \; C_\kappa = 1.924660e + 00)$
`eq: g-zero-value-TOBECHECKED`	Eq. (32)	✓ PASS	$g(0) = \left(\dfrac{2}{\pi}\right)^2 - 2\left(1 - \dfrac{2}{\pi}\right)\left(\dfrac{2}{\pi}\right) = -\dfrac{4(\pi - 3)}{\pi^2}.$ $g(0) = -5.738534e - 02, \; \text{expected} = -5.738534e - 02, \; \Delta = 5.551115e - 17$
`eq: pi-estimate-TOBECHECKED`	Eq. (35)	✓ PASS	$-\dfrac{4(\pi - 3)}{\pi^2} < -\dfrac{1}{18}.$ $\text{LHS} = -5.738534e - 02, \; \text{RHS} = -5.555556e - 02, \; m_{\min} = -1.829785e - 03$
`lem: rational function bound-TOBECHECKED`	Lemma 16	✓ PASS	$\dfrac{r(4 - r)(1 - r)^2}{(r^2 - 6r + 6)^2} \leq \dfrac{1}{18}.$ $\max_{r \in (0,1)} \dfrac{r(4-r)(1-r)^2}{(r^2-6r+6)^2} = 4.289321e - 02 \; (r = 0.586), \; 1/18 = 5.555556e - 02, \; e_{\max} = -1.266235e - 02$