Numerical Check Report

Total	Failures	Missing	Status
45	0	0	✓ PASS

Notation

- $e_\infty = \cdots$ (equality/identity): sup error $\max|\mathrm{LHS} - \mathrm{RHS}|$ on the evaluation grid. Smaller is better; values near machine precision indicate numerical agreement, while larger values usually reflect grid/Monte Carlo error (still may be acceptable if marked PASS).
- $r_\infty = \cdots$ (root/fixed-point): residual $\max|E|$ for an equation $E = 0$. This should be close to 0.
- $m_{\min} = \cdots$ / $e_{\max} = \cdots$ (inequality): check the sign condition $m_{\min} \geq 0$ (equivalently $e_{\max} \leq 0$). If the extremum is close to 0, it is worth re-running on a finer grid.
- $\min(\cdots) = \cdots$, $\max(\cdots) = \cdots$, $y_{\max} = \cdots$, $\Delta = \cdots$, **LHS/RHS** (bounds/monotonicity/value comparisons): verify the reported extremum has the expected sign and magnitude relative to the bound claimed in `main.tex`; Δ should be small relative to the scale of the values.
- **Point annotations** such as $(t = \cdots)$, $(u = \cdots, k = \cdots)$, $(x = \cdots, y = \cdots)$ indicate where the worst-case (or representative) diagnostic was attained on the evaluation grid.

Notation from the main latex file. Let $Z, X \sim \mathcal{N}(0,1)$ and $(x)_+ := \max\{x, 0\}$. Define

$$\begin{aligned}
\phi(u) &:= \frac{1}{\sqrt{2\pi}} e^{-u^2/2}, \qquad \Phi(u) := \int_{-\infty}^{u} \phi(s)\,ds, \qquad \overline{\Phi}(u) := 1 - \Phi(u),\\
E(u) &:= \frac{\phi(u)}{\overline{\Phi}(u)}, \qquad F_q(x) := \frac{1}{\sqrt{1-q}}\, E\left(\frac{\kappa - x}{\sqrt{1-q}}\right),\\
P(r) &:= \mathbb{E}\big[\tanh^2(\sqrt{r}\,Z)\big],\\
R_\kappa(q,\alpha) &:= \alpha\,\mathbb{E}\big[F_q(\sqrt{q}\,Z)^2\big],\\
B(q) &:= (1-q)\,\mathbb{E}\left[E\left(\frac{\kappa - \sqrt{q}\,Z}{\sqrt{1-q}}\right)^2\right], \qquad A(r) := r(1-P(r))^2 = r\big(\mathbb{E}[\mathrm{sech}^2(\sqrt{r}\,Z)]\big)^2,\\
C_\kappa &:= \mathbb{E}\big[(\kappa - Z)_+^2\big] = (\kappa^2+1)\Phi(\kappa) + \kappa\phi(\kappa),\\
U_s &:= \frac{\kappa - \sqrt{s}\,Z}{\sqrt{1-s}} \quad (s \in [0,1)), \qquad d(u) := E(u) - u,\\
\mu_k(u) &:= \mathbb{E}\big[(X-u)^k \mid X \geq u\big], \qquad g(u) := E'(u)^2 - 2(1-E'(u))E(u)^2,\\
H(u) &:= u^2 d(u) + 6u d(u)^2 + 6d(u)^3 - u - 4d(u),\\
x &:= u\,d(u), \qquad y := d(u)^2, \qquad F(x,y) := x^2 + 6xy + 6y^2 - x - 4y.
\end{aligned}$$

Simulation Results

Label	Main.pdf	Status	Message
`eq:r_in_terms_of_B`	Eq. (3)	✓ PASS	$r = \dfrac{\alpha}{(1-q)^2}\, B(q).$ $e_\infty = 3.553e-15$
`eq:union-bound`	Eq. (5)	✓ PASS	$\mathbb{P}(Z^{(0)}_{N,M}=0) \le \mathbb{P}(\mathcal{E}) \le L\,\mathbb{P}(Z^{(0)}_{N,M}=0).$ $\mathbb{P}(Z^{(0)}_{N,M}=0) = 0.167,\ \mathbb{P}(\mathcal{E}) = 0.387,\ \mathrm{tol} = 0.119$
`eq:sharp-seq-exp`	Eq. (6)	✓ PASS	$\alpha \le \alpha_N - \varepsilon \implies \mathbb{P}\Big(Z^{(\kappa)}_{N,\lfloor\alpha N\rfloor} = 0\Big) \le e^{-c_\varepsilon N}, \quad \alpha \ge \alpha_N + \varepsilon \implies \mathbb{P}\Big(Z^{(\kappa)}_{N,\lfloor\alpha N\rfloor} = 0\Big) \ge 1 - e^{-c_\varepsilon N}.$ $\alpha_n \approx 0.40,\ p_{\text{low}} = 0.000,\ p_{\text{high}} = 0.008,\ c_\varepsilon = 0.001$
`eq:RS-fixed-point`	Eq. (8)	✓ PASS	$q = \mathbb{E}\big[\tanh^2(\sqrt{r}\,Z)\big], \qquad r = \alpha\,\mathbb{E}\big[F_q(\sqrt{q}\,Z)^2\big].$ $e_\infty = 0.000e+00$
`eq:one-dimensional`	Eq. (10)	✓ PASS	$A(r) = \alpha\,B\big(P(r)\big), \qquad r \ge 0,$ $e_\infty = 3.719e-14$
`eq:A_as_I`	Eq. (11)	✓ PASS	$A(r) = \dfrac{1}{2\pi}\, I(r)^2.$ $e_\infty = 1.403e-02$
`eq:condmean`	Eq. (12)	✓ PASS	$E(u) = \dfrac{\phi(u)}{\overline{\Phi}(u)} = \mathbb{E}[X \mid X \ge u].$ $e_\infty = 1.110e-15$
`eq:mprime`	Eq. (13)	✓ PASS	$E'(u) = E(u)^2 - u\,E(u).$ $e_\infty = 6.765e-11$
`eq:varidentity`	Eq. (14)	✓ PASS	$1 - E'(u) = \mathbb{E}[X^2 \mid X \ge u] - \big(\mathbb{E}[X \mid X \ge u]\big)^2 = \mathrm{Var}(X \mid X \ge u) > 0.$ $e_\infty = 2.665e-15$
`eq:dprime`	Eq. (15)	✓ PASS	$d'(u) = E'(u) - 1 = -\mathrm{Var}(X \mid X \ge u) < 0.$ $e_\infty = 5.821e-11$
`eq:B-as-BM`	–	✓ PASS	$B(t) = \mathbb{E}\big[f(t, W_t)^2\big], \qquad t \in [0,1).$ $e_\infty = 2.220e-16$
`eq:Lf`	–	✓ PASS	$(\mathcal{L}f)(t,x) = \dfrac{-E + uE' + E''}{2\sqrt{1-t}}.$ $e_\infty = 5.332e-08$
`eq:Bprime`	–	✓ PASS	$B'(t) = \mathbb{E}\Big[(E'(U_t))^2 + 2E(U_t)^2\big(E'(U_t) - 1\big)\Big] = \mathbb{E}\big[g(U_t)\big].$ $e_\infty = 4.524e-03$
`eq:goal`	Eq. (20)	✓ PASS	$\mathbb{E}\big[g(U_t)\big] < 0 \qquad \text{for all } t \in (0,1).$ $\max_t \mathbb{E}[g(U_t)] = 0.000e+00$
`eq:g-expanded`	Eq. (21)	✓ PASS	$g(u) = E(u)^2\big(3E(u)^2 - 4u\,E(u) + u^2 - 2\big).$ $e_\infty = 1.180e-16$
`eq:moments`	–	✓ PASS	$\mu_0(u) = 1,$ $\mu_1(u) = d(u),$ $\mu_2(u) = 1 - u\,d(u),$ $\mu_3(u) = (u^2+2)\,d(u) - u,$ $\mu_4(u) = u^2 + 3 - u(u^2+5)\,d(u).$ $e_\infty = 1.421e-14$
`eq:consFneg_lemma`	Eq. (40)	✓ PASS	$x + 2y \ge 1, \qquad x^2 + xy - 3x - 3y + 2 \ge 0, \qquad x + y < 1.$ $\max F(x,y) = 0.000e+00$
`eq:gprime`	Eq. (25)	✓ PASS	$g'(u) = 2\,E(u)^2\,H(u).$ $e_\infty = 4.125e-11$

Label	Main.pdf	Status	Message
`eq:detM1`	Eq. (26)	✓ PASS	$\det M_1(u) = \mu_1\mu_3 - \mu_2^2 = ud + 2d^2 - 1 \geq 0.$ $\min(ud(u) + 2d(u)^2 - 1) = 4.289e-02,\ e_\infty = 4.330e-15$
`eq:detM2`	Eq. (27)	✓ PASS	$u^2d^2 + ud^3 - 3d^2 - 3ud + 2 \geq 0.$ $\min(u^2d(u)^2 + ud(u)^3 - 3d(u)^2 - 3ud(u) + 2) = 9.602e-03,\ e_\infty = 5.718e-15$
`eq:varY`	Eq. (28)	✓ PASS	$\mathrm{Var}(Y_u) = \mu_2 - \mu_1^2 = 1 - ud - d^2 > 0,$ $\min(1 - ud(u) - d(u)^2) = 1.415e-01,\ e_\infty = 2.665e-15$
`eq:constraints`	Eq. (29)	✓ PASS	$x + 2y \geq 1, \qquad x^2 + xy - 3x - 3y + 2 \geq 0, \qquad x + y < 1.$ $m_{\min} = 3.967e-03$
`eq:ybound`	Eq. (30)	✓ PASS	$0 < y \leq \frac{2}{\pi} < \frac{2}{3}.$ $y_{\max} = 6.366e-01$
`eq:Hxy`	Eq. (31)	✓ PASS	$d\,H(u) = x^2 + 6xy + 6y^2 - x - 4y.$ $e_\infty = 1.865e-16$
`eq:g-critical-r`	–	✓ PASS	$g(u_\star) = \frac{r_\star(4 - r_\star)(1 - r_\star)^2}{(r_\star^2 - 6r_\star + 6)^2}, \qquad r_\star \in (0, 1).$ $r_\star = 0.778,\ e_\infty = 6.939e-18$
`eq:split`	Eq. (34)	✓ PASS	$\mathbb{E}[g(U_t)] \leq g(0)\,(1 - p_t) + \frac{1}{18}\,p_t.$ $e_{\max} = -7.402e-02$
`eq:spin_bd_unif_nos`	Eq. (36)	✓ PASS	$-\frac{r_n\varepsilon_n}{2} + \mathbb{E}\big[\log\cosh(\sqrt{r_n}\,Z)\big] \leq \frac{\alpha_n}{2\varepsilon_n}\,B(q_n).$ $m_{\min} = 4.493e-01$
`eq:constraint_bd_un`	Eq. (37)	✓ PASS	$\mathbb{E}\big[\log\overline{\Phi}(U_n)\big] \leq -\frac{A_n}{2\varepsilon_n} + \frac{\Phi(\kappa - \delta)}{2}\log\varepsilon_n + C(\delta),$ $m_{\min} = 2.744e+00$
`eq:BA_bd_unif_noste`	Eq. (38)	✓ PASS	$0 \leq B(q_n) - A_n = \varepsilon_n\,\mathbb{E}\big[E(U_n)^2 - (U_n)_+^2\big] \leq C_0\,\varepsilon_n.$ $e_{\max} = -5.244e-02$
`eq: two-sided mills`	Eq. (39)	✓ PASS	$\overline{\Phi}(u) \geq \frac{\phi(u)}{u} - \frac{\overline{\Phi}(u)}{u^2}, \qquad \text{hence} \qquad \frac{\phi(u)}{\overline{\Phi}(u)} \leq u + \frac{1}{u}.$ $e_{\max} = -2.439e-02$
`eq:rootsFneg_lemma`	Eq. (42)	✓ PASS	$r_\pm(y) = \frac{3 - y \pm \sqrt{y^2 + 6y + 1}}{2}.$ $e_\infty = 8.882e-16$
`eq:leftEndpoint1Fne`	Eq. (44)	✓ PASS	$F(1 - 2y, y) = -2y^2 < 0.$ $e_\infty = 5.551e-17$
`eq:leftEndpoint2Fne`	Eq. (45)	✓ PASS	$F(0, y) = 6y^2 - 4y = 2y(3y - 2) < 0.$ $e_\infty = 0.000e+00$
`eq:rightEndpointFne`	Eq. (46)	✓ PASS	$F(r_-(y), y) = \frac{7y^2 + 11y + 2 - (5y + 2)\sqrt{y^2 + 6y + 1}}{2}.$ $e_\infty = 0.000e+00$
`eq:squareDiffFneg_l`	Eq. (47)	✓ PASS	$(5y + 2)^2(y^2 + 6y + 1) - (7y^2 + 11y + 2)^2 = 8y^3(2 - 3y),$ $e_\infty = 0.000e+00$
`lem:A-TOBECHECKED`	Lemma 7	✓ PASS	$A(0) = 0, \qquad \lim_{r\to\infty} A(r) = \frac{2}{\pi}.$ $A(0) = 0.000000e+00,\ \min \Delta A = 3.955263e-02,\ A(200) = 6.340157e-01,\ 2/\pi = 6.366198e-01,\ \Delta = -2.604094e-03$

Label	Main.pdf	Status	Message
`lem:B_endpoints-TOBECHECKED`	Lemma 10	✓ PASS	$B(0) = E(\kappa)^2,$ $\lim_{q\uparrow 1} B(q) = C_\kappa = \mathbb{E}\big[(\kappa - Z)_+^2\big] = (\kappa^2+1)\Phi(\kappa) + \kappa\phi(\kappa).$ $e_{\infty,0} = 4.440892e-16$ $(\kappa = 1.000,\ B(0) = 2.326038e+00,\ E(\kappa)^2 = 2.326038e+00),$ $e_{\infty,1} = 8.397408e-06$ $(\kappa = 1.000,\ B(q) = 1.924669e+00,\ C_\kappa = 1.924660e+00)$
`eq:Bprime-TOBECHECKED`	Eq. (18)	✓ PASS	$B'(t) = \mathbb{E}\Big[(E'(U_t))^2 + 2E(U_t)^2\big(E'(U_t) - 1\big)\Big] = \mathbb{E}\big[g(U_t)\big].$ $e_\infty = 4.523982e-03$ $(t = 0.300,\ \mathrm{LHS} = -2.265129e-01,\ \mathrm{RHS} = -2.219889e-01)$
`eq:moments-TOBECHECKED`	Eq. (22)	✓ PASS	$\mu_0(u) = 1,$ $\mu_1(u) = d(u),$ $\mu_2(u) = 1 - u\,d(u),$ $\mu_3(u) = (u^2+2)\,d(u) - u,$ $\mu_4(u) = u^2 + 3 - u(u^2+5)\,d(u).$ $e_\infty = 1.421085e-14$ $(u = -0.500,\ k = 3,\ \mathrm{LHS} = 2.770611e+00,\ \mathrm{RHS} = 2.770611e+00)$
`eq:FdefFneg_lemma-TOBECHECKED`	Eq. (41)	✓ PASS	$F(x,y) < 0.$ $\max F(x,y) = -8.000000e-04$ $(x = 0.96,\ y = 0.02)$
`eq: g-zero-value-TOBECHECKED`	Eq. (32)	✓ PASS	$g(0) = \left(\frac{2}{\pi}\right)^2 - 2\left(1 - \frac{2}{\pi}\right)\left(\frac{2}{\pi}\right) = \frac{12}{\pi^2} - \frac{4}{\pi} = -\frac{4(\pi-3)}{\pi^2}.$ $g(0) = -5.738534e-02,$ $\mathrm{expected} = -5.738534e-02,$ $\Delta = 5.551115e-17$
`eq:g-critical-r-TOBECHECKED`	Eq. (33)	✓ PASS	$g(u_\star) = \frac{r_\star(4-r_\star)(1-r_\star)^2}{(r_\star^2 - 6r_\star + 6)^2}, \qquad r_\star \in (0,1).$ $u_\star = -1.002369e+00,$ $r_\star = 7.775774e-01,$ $e_\infty = 6.938894e-18$
`eq: pi-estimate-TOBECHECKED`	Eq. (35)	✓ PASS	$-\frac{4(\pi-3)}{\pi^2} < -\frac{1}{18}.$ $\mathrm{LHS} = -5.738534e-02,$ $\mathrm{RHS} = -5.555556e-02,$ $m_{\min} = -1.829785e-03$
`lem: rational function bound-TOBECHECKED`	Lemma 16	✓ PASS	$\frac{r(4-r)(1-r)^2}{(r^2-6r+6)^2} \le \frac{1}{18}.$ $\max_{r\in(0,1)} \frac{r(4-r)(1-r)^2}{(r^2-6r+6)^2} = 4.289321e-02$ $(r = 0.586),$ $1/18 = 5.555556e-02,$ $e_{\max} = -1.266235e-02$