Numerical Check Report

Total	Failures	Missing	Status
50	0	0	✓ PASS

Notation

1. $e_\infty = \cdots$ (equality/identity): sup error $\max|\text{LHS} - \text{RHS}|$ on the evaluation grid. Smaller is better; values near machine precision indicate numerical agreement, while larger values usually reflect grid/Monte Carlo error (still may be acceptable if marked PASS).
2. $r_\infty = \cdots$ (root/fixed-point): residual $\max|E|$ for an equation $E = 0$. This should be close to 0.
3. $m_{\min} = \cdots$ / $e_{\max} = \cdots$ (inequality): check the sign condition $m_{\min} \geq 0$ (equivalently $e_{\max} \leq 0$). If the extremum is close to 0, it is worth re-running on a finer grid.
4. $\min(\cdots) = \cdots$, $\max(\cdots) = \cdots$, $y_{\max} = \cdots$, $\Delta = \cdots$, **LHS/RHS** (bounds/monotonicity/value comparisons): verify the reported extremum has the expected sign and magnitude relative to the bound claimed in `main.tex`; Δ should be small relative to the scale of the values.
5. **Point annotations** such as $(t = \cdots)$, $(u = \cdots, k = \cdots)$, $(x = \cdots, y = \cdots)$ indicate where the worst-case (or representative) diagnostic was attained on the evaluation grid.

Notation from the main latex file. Let $Z, X \sim \mathcal{N}(0,1)$ and $(x)_+ := \max\{x, 0\}$. Define

$$
\begin{aligned}
\phi(u) &:= \frac{1}{\sqrt{2\pi}} e^{-u^2/2}, \qquad \Phi(u) := \int_{-\infty}^{u} \phi(s)\, ds, \qquad \overline{\Phi}(u) := 1 - \Phi(u),\\
E(u) &:= \frac{\phi(u)}{\overline{\Phi}(u)}, \qquad F_q(x) := \frac{1}{\sqrt{1-q}}\, E\left(\frac{\kappa - x}{\sqrt{1-q}}\right),\\
P(r) &:= \mathbb{E}\big[\tanh^2(\sqrt{r}\, Z)\big],\\
R_\kappa(q, \alpha) &:= \alpha\, \mathbb{E}\big[F_q(\sqrt{q}\, Z)^2\big],\\
B(q) &:= (1-q)\, \mathbb{E}\left[E\left(\frac{\kappa - \sqrt{q}\, Z}{\sqrt{1-q}}\right)^2\right], \qquad A(r) := r(1 - P(r))^2 = r\big(\mathbb{E}[\operatorname{sech}^2(\sqrt{r}\, Z)]\big)^2,\\
C_\kappa &:= \mathbb{E}\big[(\kappa - Z)_+^2\big] = (\kappa^2 + 1)\Phi(\kappa) + \kappa\phi(\kappa),\\
U_s &:= \frac{\kappa - \sqrt{s}\, Z}{\sqrt{1-s}} \quad (s \in [0,1)), \qquad d(u) := E(u) - u,\\
\mu_k(u) &:= \mathbb{E}\big[(X-u)^k \mid X \geq u\big], \qquad g(u) := E'(u)^2 - 2(1 - E'(u))E(u)^2,\\
H(u) &:= u^2 d(u) + 6u d(u)^2 + 6 d(u)^3 - u - 4d(u),\\
x &:= u\, d(u), \qquad y := d(u)^2, \qquad F(x,y) := x^2 + 6xy + 6y^2 - x - 4y.
\end{aligned}
$$

Simulation Results

Main.pdf	Status	Message
Eq. (1)	✓ PASS	$q = P(r), \qquad r = R_\kappa(q,\alpha).$ $r_\infty = 7.994e-15$
Eq. (2)	✓ PASS	$B(q) = (1-q)\,\mathbb{E}\left[E\left(\frac{\kappa-\sqrt{q}\,Z}{\sqrt{1-q}}\right)^2\right], \qquad q\in[0,1).$ $e_\infty = 4.441e-16$
Eq. (3)	✓ PASS	$r = \frac{\alpha}{(1-q)^2}\,B(q).$ $e_\infty = 3.553e-15$
Eq. (4)	✓ PASS	$\mathcal{F}_\kappa(\alpha;q,r) = -\frac{r(1-q)}{2} + \mathbb{E}\big[\log(2\cosh(\sqrt{r}\,Z))\big] + \alpha\,\mathbb{E}\left[\log\overline{\Phi}\left(\frac{\kappa-\sqrt{q}\,Z}{\sqrt{1-q}}\right)\right].$ $\mathcal{F}_\kappa = 1.354e-01, -3.517e-01$ on the sampled grid
Eq. (7)	✓ PASS	$\mathbb{P}(Z^{(0)}_{N,M}=0) \le \mathbb{P}(\mathcal{E}) \le L\,\mathbb{P}(Z^{(0)}_{N,M}=0).$ $\mathbb{P}(Z^{(0)}_{N,M}=0) = 0.167,\ \mathbb{P}(\mathcal{E}) = 0.387,\ \mathrm{tol} = 0.119$
Eq. (8)	✓ PASS	$\alpha \le \alpha_N - \varepsilon \implies \mathbb{P}\Big(Z^{(\kappa)}_{N,\lfloor\alpha N\rfloor}=0\Big) \le e^{-c_\varepsilon N}, \quad \alpha \ge \alpha_N+\varepsilon \implies$ $\mathbb{P}\Big(Z^{(\kappa)}_{N,\lfloor\alpha N\rfloor}=0\Big) \ge 1-e^{-c_\varepsilon N}.$ $\alpha_n \approx 0.40,\ p_{\mathrm{low}} = 0.000,\ p_{\mathrm{high}} = 0.008,\ c_\varepsilon = 0.001$
Eq. (5)	✓ PASS	$\mathrm{RS}(\alpha;U;q,r) = -\frac{r(1-q)}{2} + \mathbb{E}\Big[\log\big(2\cosh(\sqrt{r}\,Z)\big)\Big] + \alpha\,\mathbb{E}\big[L_q(\sqrt{q}\,Z)\big].$ $e_\infty = 7.760e-03$
Eq. (6)	✓ PASS	$q = \mathbb{E}\big[\tanh^2(\sqrt{r}\,Z)\big], \qquad r = \alpha\,\mathbb{E}\big[F_q(\sqrt{q}\,Z)^2\big].$ $e_\infty = 8.882e-16$
Eq. (9)	✓ PASS	$A(r) = r\big(1-P(r)\big)^2 = r\Big(\mathbb{E}\big[\operatorname{sech}^2(\sqrt{r}\,Z)\big]\Big)^2.$ $e_\infty = 2.776e-17$
Eq. (10)	✓ PASS	$A(r) = \alpha\,B\big(P(r)\big), \qquad r \ge 0.$ $e_\infty = 3.719e-14$
Eq. (11)	✓ PASS	$A(r) = \frac{1}{2\pi}\,I(r)^2.$ $e_\infty = 1.403e-02$
Eq. (12)	✓ PASS	$E(u) = \frac{\phi(u)}{\overline{\Phi}(u)} = \mathbb{E}[X \mid X \ge u].$ $e_\infty = 1.110e-15$
Eq. (13)	✓ PASS	$E'(u) = E(u)^2 - u\,E(u).$ $e_\infty = 4.545e-11$
Eq. (14)	✓ PASS	$1 - E'(u) = \mathbb{E}[X^2 \mid X \ge u] - \big(\mathbb{E}[X \mid X \ge u]\big)^2 = \mathrm{Var}(X \mid X \ge u) > 0.$ $e_\infty = 2.665e-15$
Eq. (15)	✓ PASS	$d'(u) = E'(u) - 1 = -\mathrm{Var}(X \mid X \ge u) < 0.$ $e_\infty = 5.821e-11$
–	✓ PASS	$B(t) = \mathbb{E}\big[f(t,W_t)^2\big], \qquad t\in[0,1).$ $e_\infty = 0.000e+00$
–	✓ PASS	$(\mathcal{L}f)(t,x) = \frac{-E+uE'+E''}{2\sqrt{1-t}}.$ $e_\infty = 5.332e-08$
–	✓ PASS	$B'(t) = \mathbb{E}\Big[(E'(U_t))^2 + 2E(U_t)^2\big(E'(U_t)-1\big)\Big] = \mathbb{E}\big[g(U_t)\big].$ $e_\infty = 4.524e-03$
Eq. (16)	✓ PASS	$U_t = \frac{\kappa-\sqrt{t}\,Z}{\sqrt{1-t}}, \qquad t\in(0,1).$ Definitional consistency check

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Eq. (19)	✓ PASS	$g(u) = E'(u)^2 - 2\big(1 - E'(u)\big)E(u)^2.$ Definitional consistency check
Eq. (20)	✓ PASS	$\mathbb{E}\big[g(U_t)\big] < 0 \qquad \text{for all } t \in (0,1).$ $\max_t \mathbb{E}[g(U_t)] = 0.000e+00$
Eq. (21)	✓ PASS	$g(u) = E(u)^2\big(3E(u)^2 - 4u\,E(u) + u^2 - 2\big).$ $e_\infty = 1.166e-15$
–	✓ PASS	$\mu_0(u) = 1,$ $\mu_1(u) = d(u),$ $\mu_2(u) = 1 - u\,d(u),$ $\mu_3(u) = (u^2+2)\,d(u) - u,$ $\mu_4(u) = u^2 + 3 - u(u^2+5)\,d(u).$ $e_\infty = 1.421e-14$
Eq. (40)	✓ PASS	$x + 2y \geq 1, \qquad x^2 + xy - 3x - 3y + 2 \geq 0, \qquad x + y < 1.$ $\max F(x,y) = 0.000e+00$
–	✓ PASS	$F(x,y) < 0$ on the feasible region. $\max F(x,y) = 0.000e+00$
Eq. (25)	✓ PASS	$g'(u) = 2\,E(u)^2\,H(u).$ $e_\infty = 4.125e-11$
Eq. (26)	✓ PASS	$\det M_1(u) = u\,d(u) + 2d(u)^2 - 1 \geq 0.$ $\min(ud(u) + 2d(u)^2 - 1) = 4.289e-02,\ e_\infty = 4.330e-15$
Eq. (27)	✓ PASS	$u^2d(u)^2 + ud(u)^3 - 3d(u)^2 - 3ud(u) + 2 \geq 0.$ $\min(u^2d(u)^2 + ud(u)^3 - 3d(u)^2 - 3ud(u) + 2) = 9.602e-03,\ e_\infty = 5.718e-15$
Eq. (28)	✓ PASS	$\mathrm{Var}(Y_u) = \mu_2 - \mu_1^2 = 1 - ud(u) - d(u)^2 > 0.$ $\min(1 - ud(u) - d(u)^2) = 1.415e-01,\ e_\infty = 2.665e-15$
Eq. (29)	✓ PASS	$x + 2y \geq 1, \qquad x^2 + xy - 3x - 3y + 2 \geq 0, \qquad x + y < 1.$ $m_{\min} = 3.967e-03$
Eq. (30)	✓ PASS	$0 < y \leq \frac{2}{\pi} < \frac{2}{3}.$ $y_{\max} = 6.366e-01$
Eq. (31)	✓ PASS	$d\,H(u) = x^2 + 6xy + 6y^2 - x - 4y.$ $e_\infty = 1.865e-16$
–	✓ PASS	$g(u_\star) = \frac{r_\star(4-r_\star)(1-r_\star)^2}{(r_\star^2 - 6r_\star + 6)^2}, \qquad r_\star \in (0,1).$ $r_\star = 0.778,\ e_\infty = 6.939e-18$
Eq. (34)	✓ PASS	$\mathbb{E}[g(U_t)] \leq g(0)\,(1-p_t) + \frac{1}{18}\,p_t.$ $e_{\max} = -7.402e-02$
Eq. (36)	✓ PASS	$-\frac{r_n\varepsilon_n}{2} + \mathbb{E}\big[\log\cosh(\sqrt{r_n}\,Z)\big] \leq \frac{\alpha_n}{2\varepsilon_n}\,B(q_n).$ $m_{\min} = 4.493e-01$
Eq. (37)	✓ PASS	$\mathbb{E}\big[\log\overline{\Phi}(U_n)\big] \leq -\frac{A_n}{2\varepsilon_n} + \frac{\Phi(\kappa-\delta)}{2}\log\varepsilon_n + C(\delta).$ $m_{\min} = 2.744e+00$
Eq. (38)	✓ PASS	$0 \leq B(q_n) - A_n = \varepsilon_n\,\mathbb{E}\big[E(U_n)^2 - (U_n)_+^2\big] \leq C_0\,\varepsilon_n.$ $e_{\max} = -5.244e-02$
Eq. (39)	✓ PASS	$\overline{\Phi}(u) \geq \frac{\phi(u)}{u} - \frac{\overline{\Phi}(u)}{u^2}, \qquad \text{hence} \qquad \frac{\phi(u)}{\overline{\Phi}(u)} \leq u + \frac{1}{u}.$ $e_{\max} = -2.439e-02$

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Eq. (42)	✓ PASS	$r_\pm(y) = \dfrac{3 - y \pm \sqrt{y^2+6y+1}}{2}$. $e_\infty = 8.882e-16$
Eq. (43)	✓ PASS	$I_y = \big[\max\{0, 1-2y\},\, r_-(y)\big]$. $x_{\text{lo}} = 2.000e-01,\ r_-(y) = 3.570e-01$
Eq. (44)	✓ PASS	$F(1-2y, y) = -2y^2 < 0$. $e_\infty = 5.551e-17$
Eq. (45)	✓ PASS	$F(0, y) = 6y^2 - 4y = 2y(3y-2) < 0$. $e_\infty = 0.000e+00$
Eq. (46)	✓ PASS	$F(r_-(y), y) = \dfrac{7y^2 + 11y + 2 - (5y+2)\sqrt{y^2+6y+1}}{2}$. $e_\infty = 0.000e+00$
Eq. (47)	✓ PASS	$(5y+2)^2(y^2+6y+1) - (7y^2+11y+2)^2 = 8y^3(2-3y)$. $e_\infty = 0.000e+00$
–	✓ PASS	$C_\kappa = (\kappa^2+1)\Phi(\kappa) + \kappa\phi(\kappa)$. $e_\infty = 5.154e-03$
Lemma 7	✓ PASS	$A(0) = 0, \qquad \lim_{r\to\infty} A(r) = \dfrac{2}{\pi}$. $A(0) = 0.000000e+00,\ \min \Delta A = 3.955263e-02,\ A(200) = 6.340157e-01,\ 2/\pi = 6.366198e-01,\ \Delta = -2.604094e-03$
Lemma 10	✓ PASS	$B(0) = E(\kappa)^2$, $\lim_{q\uparrow 1} B(q) = C_\kappa = (\kappa^2+1)\Phi(\kappa) + \kappa\phi(\kappa)$. $e_{\infty,0} = 4.440892e-16$ $(\kappa = 1.000,\ B(0) = 2.326038e+00,\ E(\kappa)^2 = 2.326038e+00)$, $e_{\infty,1} = 8.397408e-06$ $(\kappa = 1.000,\ B(q) = 1.924669e+00,\ C_\kappa = 1.924660e+00)$
Eq. (32)	✓ PASS	$g(0) = \left(\dfrac{2}{\pi}\right)^2 - 2\left(1 - \dfrac{2}{\pi}\right)\left(\dfrac{2}{\pi}\right) = -\dfrac{4(\pi-3)}{\pi^2}$. $g(0) = -5.738534e-02,\ \text{expected} = -5.738534e-02,\ \Delta = 5.551115e-17$
Eq. (35)	✓ PASS	$-\dfrac{4(\pi-3)}{\pi^2} < -\dfrac{1}{18}$. $\text{LHS} = -5.738534e-02,\ \text{RHS} = -5.555556e-02,\ m_{\min} = -1.829785e-03$
Lemma 16	✓ PASS	$\dfrac{r(4-r)(1-r)^2}{(r^2-6r+6)^2} \le \dfrac{1}{18}$. $\max_{r\in(0,1)} \frac{r(4-r)(1-r)^2}{(r^2-6r+6)^2} = 4.289321e-02$ $(r = 0.586)$, $1/18 = 5.555556e-02,\ e_{\max} = -1.266235e-02$