MATH 351: Principles of Analysis
University of Michigan, Winter 2024 **HW 3/20**

1. Prove using the ε-δ definition of continuity, that $h(x) = 2x^3$ is continious at 1.

Solution:

Notice that if $|x-1| < \delta$ then $1-\delta < x < 1+\delta$ from which we get

$$(1-\delta)^2 < x^2 < (1+\delta)^2$$

and

$$2-\delta < x+1 < 2+\delta$$

adding both of these inequalities we get that

$$(2-\delta) + (1-\delta)^2 < x^2 + x + 1 < (1+\delta)^2 + 2 + \delta$$

then if $\delta < 1$

$$x^2 + x + 1 < (1+1)^2 + 2 + 1 = 7$$

Proof. Fix $\varepsilon > 0$. Let $\delta = \min\{1, \frac{\varepsilon}{14}\}$. Then if $|x-1| < \delta$ we have

$$\begin{aligned}|f(x) - f(1)| &= |2x^3 - 2| \\ &= 2|x^3 - 1| \\ &= 2|x-1||x^2+x+1| \\ &< 2 \cdot \frac{\varepsilon}{14} \cdot 7 \\ &= \varepsilon\end{aligned}$$

$\square$

2. Suppose that $f : D \to \mathbb{R}$ is continious at $x_0 \in D$ in the ε-δ definition of continuity. Prove that if (x_n) is a sequence in D converginh to x_0, then $(f(x_n))$ converges to $f(x_0)$.

Solution:

$f : D \to \mathbb{R}$ is continious at $x_0 \in D$ then by definition we know for all $\varepsilon > 0$ there exists some $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$. Now let (x_n) be a sequence in D that converges to $x_0 \in D$. Then by definition of sequence convergence we know for all $\varepsilon > 0$ there exists some $N \in \mathbb{R}$ such that $|x_n - x_0| < \varepsilon$ for all $n > N$. We want to show show that $(f(x_n))$ converges to $f(x_0)$ or in other words that for all $\varepsilon > 0$ there exists some $N \in \mathbb{R}$ such that $|f(x_n) - f(x_0)| < \varepsilon$ for all $n > N$.

Proof. Fix $\varepsilon > 0$. Assume $f : D \to \mathbb{R}$ is continious at x_0 then we know there exists some $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$. Now since (x_n) converges to x_0 we can find some $N \in \mathbb{R}$ such that for $n > N$ the distance between x_n and x_0 is any value we want. Let this distance be the aforementioned δ (we are setting the ε in the definition for sequence convergence to δ). So for some $N_f \in \mathbb{R}$ if $n > N_f$ then $|x_n - x_0| < \delta$ and by assumption that f is continous this implies that $|f(x_n) - f(x_0)| < \varepsilon$. Thus $f(x_n)$ converges to $f(x_0)$. $\square$

3. Prove that if a function $f : D \to \mathbb{R}$ does not satisfy the ε-δ definition of continuity at some $x_0 \in D$, then it does not satisfy the sequence definition of continuity at x_0

Solution:

skip

4. Prove that if $\sum_{n=m}^{\infty} a_n = s$ for some real number s, then $\lim_{n\to\infty} a_n = 0$

Solution:

Assume $\sum_{n=m}^{\infty} a_n = s$ for some real number s. Then by definition the sequence of the partial sums $(s_n)_{n=m}^{\infty}$ converges to s. Notice that

$$s_k = \sum_{n=m}^{k} a_n = \sum_{n=m}^{k-1} a_n + a_k$$

and that

$$s_{k-1} = \sum_{n=m}^{k-1} a_n$$

thus combining the two equations we get

$$s_k = s_{k-1} + a_k \implies a_k = s_k - s_{k-1}$$

Thus

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} s_k - s_{k-1} = \lim_{k\to\infty} s_k - \lim_{k\to\infty} s_{k-1} = s - s = 0$$