Homework for Math 351-003

Team Homework: Due Monday, April 8

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable functions. Suppose also that $f(0) = g(0)$ and that $f'(x) \leq g'(x)$ for all $x \geq 0$. Prove that $f(x) \leq g(x)$ for all $x \geq 0$.

   a) Prove that if f is continuous on $[a, b]$, f is differentiable on (a, b), and $f'(x) = 0$ if $a < x < b$, then f is constant on $[a, b]$ (i.e. there exists $c \in \mathbf{R}$ so that $f(x) = c$ for all $x \in [a, b]$.)

   b) Suppose that f and g are function which are both continuous on $[a, b]$ and differentiable on (a, b). Prove that if $f'(x) = g'(x)$ for all $x \in (a, b)$, then there exists $c \in \mathbb{R}$ for that $f(x) = g(x) + c$ for all $x \in (a, b)$.

2. a) Suppose S is a non-empty bounded set in $\mathbb{R}$ and that $c \geq 0$. We define the set cS as follows:

$$cS = \{cs | s \in S\}$$

   Prove that $\sup(cS) = c \sup(S)$. Notice that one may similarly prove that $\inf(cS) = c \inf(S)$ (but you don't need to write down a proof of this).

   b) Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Prove that if $c \geq 0$, then

$$U(cf, P) = c\,U(f, P) \quad \text{and} \quad L(cf, P) = c\,L(f, P)$$

   for any partition P of $[a, b]$.

   c) Prove that if $f : [a, b] \to \mathbb{R}$ is a bounded function and $c \geq 0$, then $U(cf) = cU(f)$ and $L(cf) = cL(f)$.

   d) Prove that if f is integrable and $c \geq 0$, then cf is integrable and

$$\int_a^b cf(x)\,dx = c \int_a^b f\,dx.$$

3. Suppose that $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$.

   a) Prove that $U(-f) = -L(f)$ and $L(-f) = -U(f)$. Hint: You may use the fact that if S is a nonempty subset of $\mathbb{R}$, then
   $\inf S = -\sup(-S)$. You do not need to justify this fact.)

   b) Prove that if f is integrable on $[a, b]$, then $-f$ is integrable on $[a, b]$ and

$$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx.$$

   c) Conclude that if $c \in \mathbb{R}$ and f is integrable on $[a, b]$, then cf is integrable on $[a, b]$ and

$$\int_a^b cf(x)dx = c \int_a^b f(x)dx$$