MATH 351: Principles of Analysis
University of Michigan, Winter 2024

Homework 2

1. Use the definition of sequence convergence to prove that

$$\lim_{n\to\infty} \frac{3n+4}{5n-1} = \frac{3}{5}$$

> **Solution:**
>
> Fix $\varepsilon > 0$. Let $N = \frac{23}{25\varepsilon} + \frac{1}{5}$. If $n > N$ then we have
>
> $$\left|\frac{3n+4}{5n-1} - \frac{3}{5}\right| = \left|\frac{23}{5(5n-1)}\right| = \frac{23}{5(5n-1)} < \frac{23}{5(5N-1)} = \frac{23}{5(5\left(\frac{23}{25\varepsilon}+\frac{1}{5}\right)-1)} = \varepsilon$$
>
> Thus for any ε greater than 0 we have exhibited an $N \in \mathbb{R}$ such that if $n > N$ then $\left|\frac{3n+4}{5n-1} - \frac{3}{5}\right| < \varepsilon$. Hence by definition of sequence convergence $\lim_{n\to\infty} \frac{3n+4}{5n-1} = \frac{3}{5}$ $\square$

2. Prove that if (a_n) converges to a and k is a real number, then the sequence (ka_n) converges to ka.

> **Solution:**
>
> Assume that (a_n) converges to a and $k \in \mathbb{R}$. If $k = 0$ then $ka_n = 0$ for all n and as proved in the previous HW constant sequences converge to the constant thus (ka_n) converges to 0 in this case. Since (a_n) converges to a we know for all $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that for $n > N$ and $k \neq 0$, $|a_n - a| < \frac{\varepsilon}{|k|}$.
>
> Fix $\varepsilon > 0$, then for $|k| \neq 0$ and $n > N$ we have
>
> $$|(ka_n) - (ka)| = |k(a_n - a)| = |k||a_n - a| < |k|\frac{\varepsilon}{|k|} = \varepsilon$$
>
> We have shown for any $\varepsilon > 0$ there exists an $N \in R$ so that if $n > N$ then $|(ka_n) - (ka)| < \varepsilon$ hence (ka_n) converges to ka. $\square$

3. Prove that $\lim_{n\to\infty} \frac{2n+1}{n^2} = 0$.

> **Solution:**
>
> $$\lim_{n\to\infty} \frac{2n+1}{n^2} = \lim_{n\to\infty} \frac{2}{n} + \frac{1}{n^2} = \lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} \frac{1}{n^2}$$
>
> where the last equality is by the properties of sequences proved in Worksheet 1.2. We proved in class that $\lim_{n\to\infty} \frac{1}{n^2} = 0$ and that $\lim_{n\to\infty} \frac{1}{n} = 0$. By using the result proven in problem 2
>
> $$\lim_{n\to\infty} \frac{2}{n} = \lim_{n\to\infty} 2 \cdot \frac{1}{n} = 2 \cdot 0 = 0$$

Combining the results we get

$$\lim_{n\to\infty} \frac{2n+1}{n^2} = \lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} \frac{1}{n^2} = 0 + 0 = 0 \quad \square$$

4. Prove that $\lim_{n\to\infty} \frac{1}{n^p} = 0$ if $p > 0$.

Solution:

Fix $\varepsilon > 0$. Let $N = \frac{1}{\sqrt[p]{\varepsilon}}$. If $n > N$ then we have

$$\left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} < \frac{1}{N^p} = \frac{1}{\left(\frac{1}{\sqrt[p]{\varepsilon}}\right)^p} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

Thus for any $\varepsilon > 0$ we have exhibited an $N \in \mathbb{R}$ such that if $n > N$ then $\left|\frac{1}{n^p} - 0\right| < \varepsilon$. Thus if $p > 0$ the sequence $\left(\frac{1}{n^p}\right)$ converges to 0 $\square$