1. Prove that if (x_n) converges to $x \neq 0$ and x_n is non-zero for all n, then $\left(\frac{1}{x_n}\right)$ converges to $\frac{1}{x}$.

Solution:

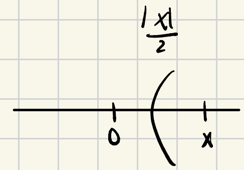
1.)

$$\frac{|x_n - x|}{|x_n x|} < \frac{\varepsilon |x| \cdot m}{\varepsilon |x| m}$$

$|x_n x| = |x_n||x|$

$m|x|$

idea: manipulate $|x_n - x|$ to be close to x but 'away' from 0



$\exists N_1$

$$|x_n - x| < \frac{|x|}{2}$$

$$\frac{|x|}{2} - x < x_n < \frac{|x|}{2} + x$$

$$|x_n| = \underbrace{\min\left\{ \left|\frac{|x|}{2} - x\right|, \left|\frac{|x|}{2} + x\right| \right\}}_{\text{call } m} = m$$

$|x_n| = m$

Proof:

Let (x_n) converge to x, when $x \neq 0$. Then there exists an N_1 s.t. $\forall n > N_1$ $|x_n - x| < \varepsilon = |x|/2$ Since (x_n) converges it is implied that its also bounded. We can construct that bound as follows,

$$|x_n - x| < \frac{|x|}{2}$$

$$\frac{|x|}{2} - x < x_n < \frac{|x|}{2} + x$$

Let $|x_n| = \min\left\{\frac{|x|}{2} - x, \frac{|x|}{2} + x\right\}$ and let's call this value m. Now we have bounds for $|x_n|$ so that $|x_n|$ is still close to x. Now let's consider $\left(\frac{1}{x_n}\right)$. To show $(1/x_n)$ converges to $\frac{1}{x}$ we must find an N s.t. $\forall n > N$ we have,

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| < \varepsilon$$

$\frac{|x|}{2}$

Fix $\varepsilon > 0$. If $n > N$ then,

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| = \frac{|x_n - x|}{|x_n x|} = \frac{|x_n - x|}{|x_n||x|} < \frac{\varepsilon |x| m}{\varepsilon |x| m} < \varepsilon$$

2. (a) Prove that if (s_n) is monotone increasing and unbounded, then $\lim_{n\to\infty} s_n = +\infty$
   (b) Give an example of a sequence that is unbounded but does not diverge to infinity.

Solution:

(a) (s_n) is monotone increasing thus for all n, $s_{n+1} \geq s_n$. Since (s_n) is unbounded for any $M \in \mathbb{R}$ there exists an $N \in \mathbb{R}$ such that for $n > N$ we have that $s_n > M$. combining these two facts we get that for $n > N$, $s_n > M$ thus (s_n) converges to infinity.

(b) $a_n = n(-1)^n$

3. (a) Prove that $[a, b]$ is a closed subset of $\mathbb{R}$
   (b) Prove that $[a, b)$ is not a closed subset of $\mathbb{R}$

Solution:

(a) Suppose (x_n) converges to x and $\{x_n : n \in \mathbb{N}\} \subseteqq [a, b]$. We know that $a \leq x_n \leq b$ for all n. Thus $0 \leq x_n - a$ and $b - x_n \geq 0$ for all n. By limit laws we can say that $(x_n - a)$ converges to $x - a$ and that $(b - x_n)$ converges to $b - x$. By problem 10 on Worksheet 10 which states if (s_n) is a convergent sequence and that $s_n \geq 0$ for all but finitely many values of n, then $\lim_{n\to\infty} \geq 0$, we have that $x - a \geq 0$ and $b - x \geq 0$ thus we have $x \geq a$ and $b \geq x$ combining these inequalities we get that $a \leq x \leq b$. Thus $\lim_{n\to\infty} \in [a, b]$ and hence $[a, b]$ is a closed subset of $\mathbb{R}$.

(b) Consider the sequnece $a_n = b - \frac{b-a}{n}$. Note that $a \leq b - \frac{b-a}{n} < b$ for all n, as $\frac{b-a}{n}$ is always greater than 0 thus $b - \frac{b-a}{n}$ must be less than b. a_n is lower bounded by a for $n = 1$ and for $n > 1$ it is less than b as described above. Using results proven in class

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b - \frac{b-a}{n} = \lim_{n\to\infty} b - \lim_{n\to\infty} \frac{b-a}{n} = b - 0 = b$$

Using results proven in class that $\lim_{n\to\infty} 1/n = 0$ and result from the homework stating $\lim_{n\to\infty} cx_n = c \lim_{n\to\infty} x_n$. Note that $\lim_{n\to\infty} a_n \notin [a, b)$ thus $[a, b)$ is not a closed subset of $\mathbb{R}$.

4. Find the supremum and infimum of $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$ and prove that your answer is correct.

Solution:

$\sup\{1 - \frac{1}{n} : n \in \mathbb{N}\} = 1$. Note that $1 > 1 - \frac{1}{n}$ for all n. Thus 1 is an upper bound for the set. Further, for all $\varepsilon > 0$, by the Archimedean property there exists some $N \in \mathbb{N}$ such that $N > 1/\varepsilon$. Then $1 - 1/N > 1 - \varepsilon$ and thus $1 - \varepsilon$ is not an upper bound for all $\varepsilon > 0$. Thus $\sup\{1 - \frac{1}{n} : n \in \mathbb{N}\} = 1$. Claim $\inf\{1 - \frac{1}{n} : n \in \mathbb{N}\} = 0$. Firstly, $0 < 1 - 1/n$ for all n thus 0 is a lower bound for the set. Moreover, for any $\varepsilon > 0$ we want to show it is not a lower bound of the set. By the Archimedean principle we know there exists a natural number N such that $N > \frac{1}{1+\varepsilon}$ for all $\varepsilon > 0$. Rearranging the inequality we get that $1 - \frac{1}{N} < \varepsilon$. Thus $0 + \varepsilon$ is not a lower bound of the set for all $\varepsilon > 0$. Hence we have that $\inf\{1 - \frac{1}{n} : n \in \mathbb{N}\} = 0$.

5. Suppose that S and T are non-empty subsets of $\mathbb{R}$ and that for any $s \in S$ and $t \in T$, $s \leq t$. Prove that $\sup S \leq \inf T$.

Solution:

Fix some $t \in T$. Then $s \leq t$ for all $s \in S$. Thus t is an upper bound of S, since $\sup S$ is the least upper bound we have that $\sup S \leq t$ for all $t \in T$. Thus $\sup S$ is a lower bound for T, then by definiton of infimum we have that $\sup S \leq \inf T$ because the infinum is the greatest lower bound.