MATH 351: Principles of Analysis
University of Michigan, Winter 2024

Homework 5

1. Prove or disprove the following claim: *There exists a subset of* $\mathbb{R}$ *that is closed and not bounded.*

> **Solution:**
>
> The statement is true. Consider $\mathbb{R}$, which is a subset of $\mathbb{R}$. Let (x_n) be a convergent sequence that converges to x such that $x_n \in \mathbb{R}$ for all n. By definition $x \in \mathbb{R}$, thus $\mathbb{R}$ is closed. Seeking contradiction, assume that $\mathbb{R}$ is bounded and that α is its upper bound. Then clearly $\alpha > 0$, but we also have that $1 + \alpha > \alpha$ where $1 + \alpha \in \mathbb{R}$, but this contradicts that α is an upper bound. Thus it must be that $\mathbb{R}$ is unbounded. Therefore we have shown a subset of $\mathbb{R}$ (namely $\mathbb{R}$) that is closed and unbounded.

2. Prove that if S is a non-empty subset of a bounded set T, then

$$\inf T \leq \inf S \leq \sup S \leq \sup T$$

> **Solution:**
>
> $s \leq \sup S$ for all $s \in S$ and $t \leq \sup T$ for all $t \in T$ by definition of supremum. Since $S \subseteq T$ for all $s \in S$, $s \in T$ hence $s \leq \sup T$ for all $s \in S$. Notice that $\sup T$ is an upper bound for S, thus $\sup S \leq \sup T$ since $\sup S$ is the *least* upper bound. $\inf S \leq s$ for all $s \in S$ and $\inf T \leq t$ for all $t \in T$ by definition of infimum. Since $S \subseteq T$ for all $s \in S$, $s \in T$ thus $\inf T \leq s$. Notice that $\inf T$ is a lower bound for S, thus $\inf T \leq \inf S$ since $\inf S$ is the *greatest* lower bound. Combining the inequalites we get that
>
> $$\inf T \leq \inf S \leq s \leq \sup S \leq \sup T$$

3. Let S be a nonempty subset of $\mathbb{R}$. Prove that $-\sup S = \inf -S$ where $-S = \{-s | s \in S\}$

> **Solution:**
>
> For all $s \in S$ we have
>
> $$s \leq \sup S \implies -s \geq -\sup S$$
>
> Thus $-\sup S$ is a lower bound for $-S$. So we have that
>
> $$-\sup S \leq \inf -S$$
>
> since $\inf -S$ is the greatest lower bound. For all $-s \in -S$ we have that
>
> $$\inf -S \leq -s \implies -\inf -S \geq s$$
>
> Notice that $-\inf -S$ is an upper bound for S. Thus
>
> $$\sup S \leq -\inf -S \implies -\sup S \geq \inf -S$$
>
> So we have shown that $-\sup S \leq \inf -S$ and that $-\sup S \geq \inf -S$ thus
>
> $$-\sup S = \inf -S$$

4. Use results we have already established to prove that if (s_n) and (t_n) are convergent, $s = \lim s_n$, $t = \lim t_n \neq 0$, and t_n is non-zero for all n, then $\left(\frac{s_n}{t_n}\right)$ converges to $\frac{s}{t}$

Solution:

$$\lim_{n\to\infty} \frac{s_n}{t_n} = \lim_{n\to\infty} s_n \cdot \frac{1}{t_n} = \lim_{n\to\infty} s_n \cdot \lim_{n\to\infty} \frac{1}{t_n} = s \cdot \frac{1}{t} = \frac{s}{t}$$

We proved multiplication limit law in class and from previous HW we proved that if (x_n) converges to $x \neq 0$ and x_n is non-zero for all n, then $\left(\frac{1}{x_n}\right)$ converges to $\frac{1}{x}$. Which I applied to $\left(\frac{1}{t_n}\right)$