Homework 4

1. Prove that if (t_n) is a bounded sequence and (s_n) converges to 0, then $(s_n t_n)$ converges to 0.

Solution:

(t_n) is bounded thus there exists some $R \geq 0$ such that $|t_n| \leq R$ for all n.
$(s_n) \to 0$ thus for every $\varepsilon > 0$ there exists some $N_s \in \mathbb{R}$ such that for all $n > N$ we have $|s_n| < \varepsilon/R + 1$. We want to show that $(s_n t_n)$ converges to 0. Fix some $\varepsilon > 0$, then for $n > N_s$

$$|(s_n t_n) - 0| = |s_n||t_n| < \frac{\varepsilon}{R+1} \cdot R < \varepsilon$$

For any $\varepsilon > 0$ we have shown an $N \in \mathbb{R}$ such that for $n > N$ we have $|s_n t_n| < \varepsilon$, thus by definition of convergence $(s_n t_n)$ converges to 0.

2. Prove that if (x_n) converges to $x \neq 0$ and x_n is non-zero for all n, then $\left(\frac{1}{x_n}\right)$ converges to $\frac{1}{x}$.

Solution:

$(x_n) \to 0$ thus for every $\varepsilon > 0$ there exists some $N \in \mathbb{R}$ such that for all $n > N$ we have $|x_n| < \varepsilon$. Fix some $\varepsilon > 0$. We want to show that $\left(\frac{1}{x_n}\right)$ converges to $\frac{1}{x}$, for $n > N$ we have

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| = \frac{|x_n - x|}{|x_n x|} < \frac{\varepsilon}{|x_n x|} < \varepsilon$$

where the last inequality holds becuase $x \neq 0$ and $x_n \neq 0\ \forall n$, thus $|x_n x| > 0$. We have shown for every $\varepsilon > 0$ there exists a $N \in \mathbb{R}$ such that for all $n > N$ we have that $\left|\frac{1}{x_n} - \frac{1}{x}\right| < \varepsilon$. Thus $\left(\frac{1}{x_n}\right)$ converges to $\frac{1}{x}$.

3. Prove that if (x_n) converges to 0 and $x_n > 0$ for all n, then $\frac{1}{x_n}$ diverges to infinity.

Solution:

We want to show that $\forall M \in \mathbb{R}$ there exists some $N \in \mathbb{R}$ such that for all $n > N$ we have that $x_n > M$. (x_n) converges to 0 thus for every $\varepsilon > 0$ there exists some $N \in \mathbb{R}$ such that for all $n > N$ we have

$$|x_n| < \varepsilon \implies -\varepsilon < x_n < \varepsilon$$

If we let $\varepsilon = \frac{1}{M}$ for some $M > 0$ then we get that

$$x_n < \frac{1}{M} \implies \frac{1}{x_n} > M$$

Since ε is arbitrary this holds for all $M > 0$. Now if $M \leq 0$ then $\frac{1}{x_n} > M$ for all n because $x_n > 0$ for all n. Thus we have shown for all $M \in \mathbb{R}$ there is some $N \in \mathbb{R}$ such that for $n > N$, we get that $\frac{1}{x_n} > M$, thus $\left(\frac{1}{x_n}\right)$ diverges to infinity.

4. Prove that if (a_n) is bounded and $\lim_{n\to\infty} b_n = +\infty$, then $\lim_{n\to\infty}(a_n + b_n) = +\infty$.

Solution:

Assume that (a_n) is bounded then there exists some $R \geq 0$ such that $|a_n| \leq R$ for all n and $\lim_{n\to\infty} b_n = +\infty$ then for all $M \in \mathbb{R}$ there exists some $N \in \mathbb{R}$ such that if $n > N$ then $b_n > M$. Since a_n is bounded by R we have

$$a_n \geq -R$$
$$a_n + b_n \geq -R + b_n$$

then if $n > N$ we have that

$$a_n + b_n \geq -R + b_n > -R + M$$

hence we arrive at

$$a_n + b_n > M - R$$

for all $M \in \mathbb{R}$. Because R is a fixed value and M can take on any value we have that $a_n + b_n > k$ for all $k \in \mathbb{R}$ (we just let $M = R + k$). Thus by definition of converging to infinity, $\lim_{n\to\infty}(a_n + b_n) = +\infty$