MATH 351: Principles of Analysis
University of Michigan, Winter 2024

Team HW 1

1. Prove that the sequence $(-1)^n$ does not converge.

Solution:

Seeking contradiction assume that $(-1)^n$ converges to some number a

$$\lim_{n\to\infty}(-1)^n = a$$

Fix $\varepsilon = 1/2$ then by definition of convergence there is some $N \in \mathbb{R}$ such that for all $n > N$

$$|(-1)^n - a| < 1/2$$

<u>For odd $n > N$</u>

$$\begin{aligned} |-1-a| &< 1/2 \\ -1/2 < -1-a &< 1/2 \\ 1/2 < -a &< 3/2 \\ -3/2 < a &< -1/2 \end{aligned}$$

<u>For even $n > N$</u>

$$\begin{aligned} |1-a| &< 1/2 \\ -1/2 < 1-a &< 1/2 \\ -3/2 < -a &< -1/2 \\ 1/2 < a &< 3/2 \end{aligned}$$

We have shown $a \in (-3/2, 1/2)$ and that $a \in (1/2, 3/2)$ which is a contradiction. Thus $(-1)^n$ does not converge.

2. Prove that every Cauchy sequence is bounded.

Solution:

Assume that (x_n) is a Cauchy sequence then for every $\varepsilon > 0$ there exists some $N \in \mathbb{R}$ such that for all $n, m > N$ we have $|x_n - x_m| < \varepsilon$. Note that $|x_n| = |x_n - x_m + x_m| \leq |x_n - x_m| + |x_m|$. Fix $\varepsilon = 1$, combining this with the Cauchy criterion we get that for $n, m > N$

$$|x_n| \leq |x_n - x_m| + |x_m| < |x_m| + 1$$

Let $m = N + 1$ then we get that

$$|x_n| < |x_{N+1}| + 1$$

this is true for all $n > N$. This bounds all the terms past the Nth term. For all terms before the Nth terms we can bound it by the maximum of the terms thus for $n \leq N$

$$|x_n| \leq \max\{|x_1|, |x_2|, \cdots, |x_N|\}$$

Thus to bound all terms we can choose the maximum of $|x_{N+1}|+1$ and $\max\{|x_1|, |x_2|, \cdots, |x_N|\}$. Let $R = \max\{|x_1|, |x_2|, \cdots, |x_N|, 1 + |x_{N+1}|\}$. Then $|x_n| \leq R$ for all $n \in \mathbb{N}$ and thus (x_n) is bounded.

3. Prove that if (x_n) and (y_n) are convergent sequenecs, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ and $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $x \leq y$

Solution:

We have that $y_n - x_n \geq 0$ and by limit laws we have that $(y_n - x_n)$ converges to $y - x$. If we can show that $y - x \geq 0$ then we have the desired result. By problem 10 on the worksheet we showed that if (s_n) is a convergent sequence and $s_n \geq 0$ for all but finitely many values of n, then $\lim_{n\to\infty} s_n \geq 0$. Applying this result to $(y_n - x_n)$ we get that $y - x \geq 0$ and thus $x \leq y$.

4. Suppose that $(a_n), (b_n)$ and (s_n) are three sequenecs and that

$$a_n \leq s_n \leq b_n$$

for all $n \in \mathbb{N}$. Prove that if (a_n) and (b_n) both converge to s, then (s_n) also converges to s.

Solution:

Fix $\varepsilon > 0$. (a_n) converges to s thus there exists $N_1 \in \mathbb{R}$ such that $|a_n - s| < \varepsilon$ likewise (b_n) converges to s thus there there exists $N_2 \in \mathbb{R}$ such that $|b_n - s| < \varepsilon$. Let $N = \max\{N_1, N_2\}$ then by combining $a_n \leq s_n \leq b_n$ and a_n converging to s

$$-\varepsilon < a_n - s < \varepsilon$$

and that b_n converges to s

$$-\varepsilon < b_n - s < \varepsilon$$

we have that for $n > N$

$$s_n - s \leq b_n - s < \varepsilon$$

and

$$-\varepsilon < a_n - s \leq s_n - s$$

thus $|s_n - s| < \varepsilon$. Thus for all $\varepsilon > 0$ we have shown an $N \in \mathbb{R}$ such that if $n > N$ then $|s_n - s| < \varepsilon$, thus (s_n) converges to s.