MATH 351: Principles of Analysis
University of Michigan, Winter 2024

Homework 3/4

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin\left(\frac{1}{x}\right)$ if $x \neq 0$ and $f(0) = 0$. Prove that f is not continious at 0.

Solution:

Let $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Note that $\frac{1}{n} > 0$ for all n. Thus $f(x_n) = \sin\left(\frac{1}{x_n}\right) = \sin(n)$. We proved in class that $\lim_{n\to\infty} \frac{1}{n} = 0$. Now consider

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin(n)$$

which we proved in class diverges. Thus we have that (x_n) converges to 0 but $(f(x_n))$ does not converge to $f(0)$, thus f is not continious at $x = 0$.

2. Suppose that $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$ and $h : D \to \mathbb{R}$ are 3 functions and that

$$f(z) \leq g(z) \leq h(z)$$

for all $z \in D$. Show that if f and h are both continious at $x \in D$ and $f(x) = h(x)$, then g is also continious at x.

Solution:

Let (x_n) be a sequence in D that converges to x. By continuity of f and h and that $f(x) = h(x)$ we know

$$\lim_{n\to\infty} f(x_n) = f(x) = \lim_{n\to\infty} h(x_n) = h(x)$$

by given assumption $f(x_n) \leq g(x_n) \leq h(x_n)$ since $x_n \in D$ for all n. Because $f(x_n)$ and $h(x_n)$ converge to the same value, call it s, by the squeeze theorem we can say $g(x_n)$ converges to that same value s. Furthermore

$$f(x) \leq g(x) \leq h(x)$$
$$s \leq g(x) \leq s$$

Thus $s = g(x)$. Hence we have that (x_n) converges to x and $(g(x_n))$ converges to $g(x)$, thus g is continious at x.

3. Exhibit a continious function $f : (0, 1] \to \mathbb{R}$ so that $f((0, 1])$ is not bounded. Prove all your claims

Solution:

I claim $f(x) = \frac{1}{x}$ works. Let (x_n) be a sequence in $(0, 1]$ that converges to some real number x. Now consider

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x}$$

which we proved in the HW (note that $x_n \in (0,1]$ so all values of x_n are non-zero). So (x_n) converges to x and $f(x_n)$ converges to $f(x)$, thus f is continious over the interval $(0,1]$. To prove f((0, 1]) is unbounded we want to show for any $M \in \mathbb{R}$ there is some $x \in (0,1]$ such that $\frac{1}{x} > M$. Assume by contradiction $f((0,1])$ is bounded, thus there exists some $M_0 \in \mathbb{R}$ such that $\dfrac{1}{x} \leq M_0$ for all $x \in (0,1]$. Note that M_0 must be positive as $\dfrac{1}{x} > 0$ for all $x \in (0,1]$. Choose $x_0 = \dfrac{1}{M_0+1} \in (0,1]$, then we get $\dfrac{1}{x_0} = \dfrac{1}{\frac{1}{M_0+1}} = M_0 + 1 < M_0$, where the last inequality is by assumption. We have reached a contradiction thus, $f((0,1])$ is unbounded.

4. Prove that if $f : D \to \mathbb{R}$ is continous and $C \subset D$ is compact, then $f(C)$ is bounded.

Solution:

Suppose $f(C)$ is not bounded then there exists $x_n \in C$ such that $f(x_n) > n$ for all $n \in \mathbb{N}$. Let $M \in \mathbb{R}$ then there exists some $N \in \mathbb{N}$ such that $N > M$ (by archimedean principle), then for all $n > N$ we have $f(x_n) > n > N > M$. Thus $\lim_{n\to\infty} f(x_n) = \infty$. $x_n \in C$ for all $n \in \mathbb{N}$, since C is compact we know there exists some convergent subsequence (x_{n_k}) that converges to some $x \in C$. By continuity of f, $\lim_{n\to\infty} f(x_{n_k}) = f(x)$. However this contradicts that $\lim_{n\to\infty} f(x_n) = \infty$, as we proved in class that if a sequence diveges then all of its subsequences must also diverge. Thus $f(C)$ must be bounded.