Team HW 4-1

1. Prove that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable at $x_0 \in \mathbb{R}$, then fg is differentiable at x_0 and $(fg)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$.

Solution:

We know that f and g are differentiable at x_0 thus

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \qquad \text{and} \qquad g'(x_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

Let (x_n) be a sequence in $\mathbb{R} \setminus \{x_0\}$ such that $\lim_{n\to\infty} x_n = x_0$. Then by definition of the limit we get that

$$f'(x_0) = \lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \qquad \text{and} \qquad g'(x_0) = \lim_{n \to \infty} \frac{g(x_n) - g(x_0)}{x_n - x_0}$$

Now consider

$$\begin{aligned}
\lim_{n\to\infty} \frac{(fg)(x_n) - (fg)(x_0)}{x - x_0} &= \lim_{n\to\infty} \frac{f(x_n)g(x_n) - f(x_0)g(x_0)}{x - x_0} \\
&= \lim_{n\to\infty} \frac{f(x_n)g(x_n) - f(x_n)g(x_0) + f(x_n)g(x_0) - f(x_0)g(x_0)}{x - x_0} \\
&= \lim_{n\to\infty} \frac{f(x_n)[g(x_n) - g(x_0)] + g(x_0)[f(x_n) - f(x_0)]}{x - x_0} \\
&= \lim_{n\to\infty} \frac{f(x_n)[g(x_n) - g(x_0)]}{x - x_0} + \lim_{n\to\infty} \frac{g(x_0)[f(x_n) - f(x_0)]}{x - x_0} \\
&= \lim_{n\to\infty} f(x_n) \cdot \lim_{n\to\infty} \frac{g(x_n) - g(x_0)}{x - x_0} + \lim_{n\to\infty} g(x_0) \cdot \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x - x_0} \\
&= f(x_0)g'(x_0) + g(x_0)f'(x_0)
\end{aligned}$$

Thus the by definition of the derivative $(fg)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$

2. (a) Prove that the function $f : \mathbb{R} \setminus \{\frac{1}{2}\} \to \mathbb{R}$ given by $f(x) = \frac{3x+4}{2x-1}$ is differentiable at $x_0 = 1$ and evaluate $f'(1)$

   (b) Prove that if $g : \mathbb{R} \to \mathbb{R}$ given by $x^{1/3}$ is not differentiable at $x_0 = 0$

Solution:

(a) Let (x_n) be a sequence in $\mathbb{R} \setminus \{\frac{1}{2}\}$ such that $\lim_{n\to\infty} x_n = x_0$. Then

$$
\begin{aligned}
f'(x_0) &= \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \\
&= \lim_{n\to\infty} \frac{\frac{3x_n+4}{2x_n-1} - \frac{3x_0+4}{2x_0-1}}{x_n - x_0} \\
&= \lim_{n\to\infty} \frac{11(x_0 - x_n)}{(2x_n-1)(2x_0-1)(x_n-x_0)} \\
&= \lim_{n\to\infty} -\frac{11}{(2x_n-1)(2x_0-1)} \\
&= \frac{\lim_{n\to\infty} -11}{\lim_{n\to\infty}(2x_n-1)(2x_0-1)} \\
&= \frac{\lim_{n\to\infty} -11}{\lim_{n\to\infty}(2x_n-1) \cdot \lim_{n\to\infty}(2x_0-1)} \\
&= \frac{-11}{(2x_0-1)^2}
\end{aligned}
$$

Thus f is differentiable at $x_0 = 1$.

$$f'(1) = \frac{-11}{(2(1)-1)^2} = -11$$

(b) In order to prove f is not differentiable at 0, we must show some sequence in $\mathbb{R} \setminus \{0\}$ such that it converges to 0 and $\lim_{n\to\infty} \frac{f(x_n)-f(0)}{x_n-0}$ does not exist. Let $x_n = 1/n$. We proved in class that $\lim_{n\to\infty} x_n = 0$. Consider

$$
\begin{aligned}
\lim_{n\to\infty} \frac{f(x_n) - f(0)}{x_n - 0} &= \lim_{n\to\infty} \frac{f(x_n)}{x_n} \\
&= \lim_{n\to\infty} \frac{(x_n)^{\frac{1}{3}}}{x_n} \\
&= \lim_{n\to\infty} (x_n)^{-\frac{2}{3}} \\
&= \lim_{n\to\infty} \left(\frac{1}{n}\right)^{-\frac{2}{3}} \\
&= \lim_{n\to\infty} n^{2/3}
\end{aligned}
$$

We proved in HW that $\lim_{n\to\infty} \frac{1}{n^p} = 0$ if $p > 0$. Thus $\lim_{n\to\infty} \frac{1}{n^{\frac{2}{3}}} = 0$. We also proved in HW that if a sequence (x_n) converges to 0 and $x_n > 0$ for all n, then $\frac{1}{x_n}$ diverges to infinity. $\frac{1}{n^{\frac{2}{3}}} > 0$ for all n and converges to 0. Thus $\lim_{n\to\infty} n^{\frac{2}{3}}$ diverges to infinity. Thus since the limit diverges, f is not differentiable at 0.

3. Prove that if $f : D \to \mathbb{R}$ is differentiable at a point $a \in D$ then f is continious at a

Solution:

Assume f is differentiable. Let (x_n) be a sequence in $D \setminus \{a\}$ such that $\lim_{n\to\infty} x_n = a$. Then $\lim_{n\to\infty} \frac{f(x_n)-f(a)}{x_n-a}$ exists and is the value of the derivative of f at a. Now consider

$$\begin{aligned}
\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} f(x_n) - f(a) + f(a) \\
&= \lim_{n\to\infty} f(x_n) - f(a) + \lim_{n\to\infty} f(a) \\
&= \lim_{n\to\infty} \frac{f(x_n) - f(a)}{x_n - a} \cdot (x_n - a) + f(a) \\
&= \lim_{n\to\infty} \frac{f(x_n) - f(a)}{x_n - a} \cdot \lim_{n\to\infty} (x_n - a) + f(a) \\
&= f'(a) \cdot (a - a) + f(a) \\
&= 0 + f(a) \\
&= f(a)
\end{aligned}$$

We have shown that for any sequence (x_n) in $D \setminus \{a\}$ that converges to a, the sequence $(f(x_n))$ converges to $f(a)$, thus f is continous at a.

4. (a) Suppose $f : D \to \mathbb{R}$ is a differentiable function, that D contains an open interval (a, b) for some $a < b$, and that $f'(x) > 0$ for all $x \in (a, b)$. Prove that f is strictly increasing on (a, b). That is, prove that if $a < x < y < b$, then $f(x) < f(y)$.

   (b) Suppose $f : D \to \mathbb{R}$ is a differentiable function, that D contains an open interval (a, b) for some $a < b$, and that $f'(x) < 0$ for all $x \in (a, b)$. Prove that f is strictly decreasing on (a, b).

Solution:

(a) Let $x, y \in (a, b)$ such that $x < y$. Note that $[x, y] \subseteq (a, b)$. Thus f is differentiable over (x, y) and continious over $[x, y]$. Thus we can apply we Mean Value Theorem, so there exists some $x_0 \in (x, y)$ such that

$$f'(x_0) = \frac{f(y) - f(x)}{y - x}$$

rearranging we get that

$$f'(x_0) \cdot (y - x) = f(y) - f(x)$$

note that $y - x > 0$ and $f'(x_0) > 0$ by assumption. Thus $f(y) - f(x) > 0$ implying that $f(y) > f(x)$. Hence f is strictly increasing on (a, b)

(b) Let $x, y \in (a, b)$ such that $x < y$. Note that $[x, y] \subseteq (a, b)$. Thus f is differentiable over (x, y) and continious over $[x, y]$. Thus we can apply we Mean Value Theorem, so there exists some $x_0 \in (x, y)$ such that

$$f'(x_0) = \frac{f(y) - f(x)}{y - x}$$

rearranging we get that

$$f'(x_0) \cdot (y - x) = f(y) - f(x)$$

note that $f'(x_0) < 0$ and $y - x > 0$ thus their product $f'(x_0) \cdot (y - x) < 0$, which gives us $f(y) - f(x) < 0$ implying that $f(y) < f(x)$. Hence f is strictly decreasing on (a, b)