Homework 3

1. Prove that the sequence $(\sin(\frac{n\pi}{3}))$ does not converge

Solution:

skip

2. Prove that if (a_n) and (b_n) are two convergent sequences such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $(a_n b_n)$ converges to ab. (This is the Multiplication Limit Law for sequences.)

Solution:

To prove the Multiplication Limit Law for sequences, I will first prove that convergent sequences are bounded.

Proof. Assume that (a_n) is a converges to a. By the definition of convergence (with $\varepsilon = 1$) there is some N where

$$|a_n - a| < 1$$

for all $n > N$. That is, $a - 1 < a_n < a + 1$ for all $n > N$. Let

$$U = \max\{a_1, a_2, \ldots, a_N, a + 1\}$$

and

$$L = \min\{a_1, a_2, \ldots, a_N, a - 1\}$$

Note that if $n \leq N$, then $L \leq a_n \leq U$, since each a_n is included in the sets we are taking the minimum and maximum of. And if $n > N$, then we already have shown that $a - 1 < a_n < a + 1$, which implies that

$$L \leq a - 1 < a_n < a + 1 \leq U$$

and hence we have that

$$L \leq a_n \leq U$$

for all n. And thus by definition, (a_n) is bounded. $\square$

Fix $\varepsilon > 0$. Since (a_n) converges we know by above that it is bounded, and by the **Conseqeunce of Class Exercise** from worksheet 1.4 we know there exists some $R \geq 0$ such that $|b_n| \leq R$ for all $n \in \mathbb{N}$. Let $\varepsilon_1 = \frac{\varepsilon}{2|b|+1}$ and $\varepsilon_2 = \frac{\varepsilon}{2R+1}$. Since $\varepsilon_1 > 0$ we know there exists some N_1 such that $|a_n - a| < \varepsilon_1$. Likewise, since $\varepsilon_2 > 0$ we know there exists N_2 such that $|b_n - b| < \varepsilon_2$. Let $N = \max\{N_1, N_2\}$. Then for any $n > N$,

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &= |a_n(b_n - b) + b(a_n - a)| \\ &\leq |a_n(b_n - b)| + |b(a_n - a)| \\ &= |a_n| \cdot |b_n - b| + |b| \cdot |a_n - a| \end{aligned}$$

$$
\begin{aligned}
&< |a_n| \cdot \varepsilon_2 + |b| \cdot \varepsilon_1 \\
&= |a_n| \cdot \frac{\varepsilon}{2R+1} + |b| \cdot \frac{\varepsilon}{2|b|+1} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

That is, for $n > N$ we have $|a_n b_n - ab| < \varepsilon$. Therefore $(a_n b_n)$ converges to ab. □

3. Prove that every convergent sequence is Cauchy.

Solution:

Asssume that (x_n) converges to x. Fix $\varepsilon > 0$. Notice that $\varepsilon/2 > 0$, there exists some $N \in \mathbb{R}$ such that for every $n > N$ we have

$$|x_n - x| < \varepsilon/2$$

Then, for any n, $m > N$,

$$
\begin{aligned}
|x_n - x_m| &= |x_n - a + a - x_m| \\
&\leq |x_n - a| + |x_m - a| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon
\end{aligned}
$$

That is, for all $\varepsilon > 0$ there exists an $N \in \mathbb{R}$ such that for any $n, m > N$ we have $|x_n - x_m| < \varepsilon$, thus (x_n) is Cauchy. □