Homework 2/14

1. Prove that every sequence has a monotone subsequence

Solution:

Let (x_n) be a sequence
Case 1: Infinitely many dominant terms
Let (x_{n_k}) be a subsequence of all dominant terms, which we can construct because there are infinitely many of them. Pick some arbitrary point n_k in the subsequence then we know that $x_{n_{k+1}} < x_{n_k}$ since x_{n_k} is a dominant term. Thus for all $x_{n_{k+1}} < x_{n_k}$ for all n_k and (x_{n_k}) is monotone decreasing by definition.
Case 2: Finitely many dominant terms
Since there are finitely many terms, there is a last dominant term. Let x_N be the last dominant term, then let $x_{N+1} = x_{n_1}$ the first terms of the subsequence. We know that x_{N+1} is not a dominant term since x_N was the last dominant term. Thus there exists some $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$. Likewise there exists some $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$. Continuing this process we can construct an increasing subsequence, hence x_{n_k} is a monotone increasing subsequence.
In either case there is a monotone subsequence, thus all sequences have a monotone subsequence.

2. Let $k \in \mathbb{R}$
   (a) Prove that the function $f(x) = kx$ is continious at $x = 5$
   (b) Prove that the function $f(x) = kx$ is continious

Solution:

(a) Let (x_n) be a convergent sequence such that $\{x_n | n \in \mathbb{N}\} \subseteq \mathbb{R}$ and that converges to 5. Then
$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} k \cdot x_n = k \lim_{n\to\infty} x_n = 5k = f(5)$$
Thus by definition $f(x) = kx$ is continious at $x = 5$

(b) Let (x_n) be a convergent sequence such that $\{x_n | n \in \mathbb{N}\} \subseteq \mathbb{R}$ and that converges to x. Then
$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} k \cdot x_n = k \lim_{n\to\infty} x_n = kx = f(x)$$
Thus by definition $f(x) = kx$ is continious.