MATH 351: Principles of Analysis
University of Michigan, Winter 2024

HW 2

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable functions. Suppose also that $f(0) = g(0)$ and that $f'(x) \le g'(x)$ for all $x \ge 0$. Prove that $f(x) \le g(x)$ for all $x \ge 0$.

   (a) Prove that if f is continuous on $[a, b]$, f is differentiable on (a, b), and $f'(x) = 0$ if $a < x < b$, then f is constant on $[a, b]$ (i.e., there exists $c \in \mathbb{R}$ such that $f(x) = c$ for all $x \in [a, b]$).

   (b) Suppose that f and g are functions which are both continuous on $[a, b]$ and differentiable on (a, b). Prove that if $f'(x) = g'(x)$ for all $x \in (a, b)$, then there exists $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all $x \in (a, b)$.

Solution:

Let $h(x) := g(x) - f(x)$. Then $h'(x) = g'(x) - f'(x) \ge 0$. Thus h is an increasing function (as we proved on the last HW). Then for all $x \ge 0$ we have that

$$h(x) \ge h(0) = g(0) - f(0) = 0$$

$$g(x) - f(x) \ge 0 \implies f(x) \le g(x)$$

(a) Let $x_1, x_2 \in (a, b)$ such that $x_1 \ne x_2$. Then by the Mean Value Theorem there exists some $d \in (x_1, x_2)$ such that

$$f'(d) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \implies f(x_2) = f(x_1)$$

We have shown that two arbitrary points in (a, b) are equal thus f is a constant function over (a, b) and there exists $c \in \mathbb{R}$ such that $f(x) = c$ for all $x \in (a, b)$. Now we must show that the endpoints $f(a) = f(b) = c$. Use continuity argument.

(b) Let $h(x) := f(x) - g(x)$ for all $x \in (a, b)$ then $h'(x) = f'(x) - g'(x) = 0$. By part (a) we know that there exists some $c \in \mathbb{R}$ such that $h(x) = c$ for all $x \in (a, b)$. Plugging back into definition for h we get $c = f(x) - g(x)$ which rearranging yields $f(x) = g(x) + c$ for all $x \in (a, b)$ as desired.

2. (a) Suppose S is a non-empty bounded set in $\mathbb{R}$ and that $c \ge 0$. We define the set cS as follows: $cS = \{cs \mid s \in S\}$. Prove that $\sup(cS) = c\sup(S)$. Notice that one may similarly prove that $\inf(cS) = c\inf(S)$ (but you don't need to write down a proof of this).

   (b) Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Prove that if $c \ge 0$, then
   $U(cf, P) = cU(f, P)$ and $L(cf, P) = cL(f, P)$
   for any partition P of $[a, b]$.

   (c) Prove that if $f : [a, b] \to \mathbb{R}$ is a bounded function and $c \ge 0$, then $U(cf) = cU(f)$ and $L(cf) = cL(f)$.

   (d) Prove that if f is integrable and $c \ge 0$, then cf is integrable and $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

Solution:

(a) We know for all $s \in S$

$$s \leq \sup(S)$$
$$cs \leq c\sup(S)$$

Thus $c\sup(S)$ is an upper bound for cS, thus since the supremum is the least upper bound we get $\sup(cS) \leq c\sup(S)$. Likewise we know that

$$cs \leq \sup(cS)$$
$$s \leq \frac{\sup(cS)}{c}$$

Thus $\frac{\sup(cS)}{c}$ is an upper bound for the set S, then since the supremum is the least upper bound we get that $\sup(S) \leq \frac{\sup(cS)}{c}$ and thus we get $c\sup(S) \leq \sup(cS)$. Combining the fact that $\sup(cS) \leq c\sup(S)$ and $c\sup(S) \leq \sup(cS)$, we get the result $\sup(cS) = c\sup(S)$ as desired.

(b) Let $P = \{t_0, t_1, t_2, \ldots, t_n\}$ be a partition of $[a, b]$ and $c \geq 0$. Then

$$\begin{aligned} U(cf, P) &= \sum_{k=1}^{n} \sup\{cf([t_{k-1}, t_k])\} \cdot (t_k - t_{k-1}) \\ &= c\sum_{k=1}^{n} \sup\{f([t_{k-1}, t_k])\} \cdot (t_k - t_{k-1}) \\ &= c \cdot U(f, P) \end{aligned}$$

Likewise

$$\begin{aligned} L(cf, P) &= \sum_{k=1}^{n} \inf\{cf([t_{k-1}, t_k])\} \cdot (t_k - t_{k-1}) \\ &= c\sum_{k=1}^{n} \inf\{f([t_{k-1}, t_k])\} \cdot (t_k - t_{k-1}) \\ &= c \cdot L(f, P) \end{aligned}$$

(c)

$$\begin{aligned} U(cf) &= \inf\{U(cf, P)\} \\ &= \inf\{cU(f, P)\} \\ &= c\inf\{U(f, P)\} \\ &= cU(f) \end{aligned}$$

Likewise

$$\begin{aligned} L(cf) &= \sup\{L(cf, P)\} \\ &= \sup\{cL(f, P)\} \\ &= c\sup\{L(f, P)\} \\ &= cL(f) \end{aligned}$$

(d) Assume f is integrable, then we know by using part (c)

$$\begin{aligned} U(f) &= L(f) \\ cU(f) &= cL(f) \\ U(cf) &= L(cf) \end{aligned}$$

Thus cf is integrable and $c\int_a^b f = \int_a^b cf$.

3. Suppose that $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$.
   (a) Prove that $U(-f) = -L(f)$ and $L(-f) = -U(f)$. Hint: You may use the fact that if S is a nonempty subset of $\mathbb{R}$, then $\inf S = -\sup(-S)$. You do not need to justify this fact.
   (b) Prove that if f is integrable on $[a, b]$, then $-f$ is integrable on $[a, b]$ and $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$.
   (c) Conclude that if $c \in \mathbb{R}$ and f is integrable on $[a, b]$, then cf is integrable on $[a, b]$ and $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

Solution:

(a) Let $P = \{t_0, t_1, t_2, \ldots, t_n\}$ be a partition of $[a, b]$. Then

$$\begin{aligned} U(-f) &= \inf\{U(-f, P)\} \\ &= \inf\left(\sum_{k=1}^{n} \sup\{-f([t_{k-1}, t_k])\} \cdot (t_k - t_{k-1})\right) \\ &= \inf\left(\sum_{k=1}^{n} -\inf\{f([t_{k-1}, t_k])\} \cdot (t_k - t_{k-1})\right) \\ &= -\sup\left(\sum_{k=1}^{n} \inf\{f([t_{k-1}, t_k])\} \cdot (t_k - t_{k-1})\right) \\ &= -\sup\{L(f, P)\} \\ &= -L(f) \end{aligned}$$

and

$$\begin{aligned} L(-f) &= \sup\{L(-f, P)\} \\ &= \sup\left(\sum_{k=1}^{n} \inf\{-f([t_{k-1}, t_k])\} \cdot (t_k - t_{k-1})\right) \\ &= -\inf\left(\sum_{k=1}^{n} -\inf\{-f([t_{k-1}, t_k])\} \cdot (t_k - t_{k-1})\right) \\ &= -\inf\left(-\sum_{k=1}^{n} -\sup\{f([t_{k-1}, t_k])\} \cdot (t_k - t_{k-1})\right) \end{aligned}$$

$$
\begin{aligned}
&= -\inf\left(\sum_{k=1}^{n}\sup\{f([t_{k-1},t_k])\}\cdot(t_k-t_{k-1})\right)\\
&= -\inf\{U(f,P)\}\\
&= -U(f)
\end{aligned}
$$

(b) This is a specific case of 2 part (d) with $c = -1$.

(c) 3c solution