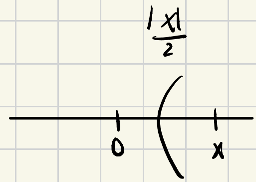
1.)

$$\frac{|x_n - x|}{|x_n x|} < \frac{\varepsilon |x| \cdot m}{\varepsilon |x| m}$$

$|x_n x| = |x_n||x|$

$m|x|$



idea: manipulate $|x_n - x|$ to be close to x but 'away' from 0

$\exists N_1$

$$|x_n - x| < \frac{|x|}{2}$$

$$\frac{|x|}{2} - x < x_n < \frac{|x|}{2} + x$$

$$\underbrace{|x_n| = \min\left\{ \left|\frac{|x|}{2} - x\right|, \left|\frac{|x|}{2} + x\right| \right\}}_{\text{call } m} = m$$

$|x_n| = m$

Proof:

Let (x_n) converge to x, when $x \neq 0$. Then there exists an N_1 s.t. $\forall n > N_1$ $|x_n - x| < \varepsilon = |x|/2$. Since (x_n) converges it is implied that its also bounded. We can construct that bound as follows,

$$|x_n - x| < \frac{|x|}{2}$$

$$\frac{|x|}{2} - x < x_n < \frac{|x|}{2} + x$$

Let $|x_n| = \min\left\{ \frac{|x|}{2} - x, \frac{|x|}{2} + x \right\}$ and let's call this value m.
Now we have bounds for $|x_n|$ so that $|x_n|$ is still close to x.
Now let's consider $\left(\frac{1}{x_n}\right)$. To show $(1/x_n)$ converges to $\frac{1}{x}$ we must find an N s.t. $\forall n > N$ we have,

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| < \varepsilon$$

$\frac{|x|}{2}$

Fix $\varepsilon > 0$. If $n > N$ then,

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| = \frac{|x_n - x|}{|x_n \cdot x|} = \frac{|x_n - x|}{|x_n||x|} < \frac{\varepsilon |x| m}{\varepsilon |x| m} < \varepsilon$$