HW 4/3

1. (a) Prove that a constant function $f(x) = c$ for some $c \in \mathbb{R}$ is differentiable at all $a \in \mathbb{R}$ and $f'(a) = 0$

   (b) Prove that a linear function $f(x) = mx + b$, where $m, b \in \mathbb{R}$, is differentiable at all $a \in \mathbb{R}$ and $f'(a) = m$.

Solution:

(a) Fix some $a \in \mathbb{R}$. Let (x_n) be a sequence in $\mathbb{R} \setminus \{a\}$ such that $\lim_{n\to\infty} x_n = a$, then

$$f'(a) = \lim_{n\to\infty} \frac{f(x_n) - f(a)}{x_n - a} = \lim_{n\to\infty} \frac{c - c}{x_n - x} = 0$$

The limit exists, thus f is differentiable for all $a \in \mathbb{R}$ with $f'(a) = 0$

(b) Let $a \in \mathbb{R}$. Let (x_n) be a sequence in $\mathbb{R} \setminus \{a\}$ such that $\lim_{n\to\infty} x_n = a$, then

$$f'(a) = \lim_{n\to\infty} \frac{f(x_n) - f(a)}{x_n - a} = \lim_{n\to\infty} \frac{mx_n + b - (ma + b)}{x_n - a} = \lim_{n\to\infty} \frac{m(x_n - a)}{x_n - a} = m$$

The limit exists, thus f is differentiable for all $a \in \mathbb{R}$ with $f'(a) = m$

2. Suppose $g : D \to \mathbb{R}$ is differentiable at a and that $g(a) \neq 0$. Prove that the function $\frac{1}{g}(x) = \frac{1}{g(x)}$ is differentiable at a and that

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g^2(a)}.$$

(Notice that it follows from the fact that g is continuous at a with $g(a) \neq 0$ that g is non-zero on some open interval about a, so that $\frac{1}{g}$ is defined on an open interval about a.)

Solution:

Assume $g : D \to \mathbb{R}$ is differentiable at a with $g(a) \neq 0$. Let (x_n) be a sequence in $D \setminus \{a\}$ such that $\lim_{n\to\infty} x_n = a$. Then

$$\begin{aligned}
\left(\frac{1}{g}\right)'(a) &= \lim_{n\to\infty} \frac{\frac{1}{g(x_n)} - \frac{1}{g(a)}}{x_n - a} && (1)\\
&= \lim_{n\to\infty} \frac{g(a) - g(x_n)}{g(a) \cdot g(x_n) \cdot (x_n - a)} && (2)\\
&= \lim_{n\to\infty} \frac{g(a) - g(x_n)}{x_n - a} \cdot \lim_{n\to\infty} \frac{1}{g(a) \cdot g(x_n)} && (3)\\
&= -g'(a) \cdot \lim_{n\to\infty} \frac{1}{g(a)} \cdot \lim_{n\to\infty} \frac{1}{g(x_n)} && (4)\\
&= -g'(a) \cdot \frac{1}{g(a)} \cdot \frac{1}{g(a)} && (5)\\
&= -\frac{g'(a)}{g^2(a)} && (6)
\end{aligned}$$

Note since g is differentiable at a, it is continuous at a. Thus $\lim_{n\to\infty} g(x_n) = g(a)$ and it follows from there that $\lim_{n\to\infty} \frac{1}{g(x_n)} = \frac{1}{g(a)}$

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x^2$ for $x \geq 0$ and $f(x) = x$ for $x < 0$. Prove that f is not differentiable at 0.

Solution:

To show a function is not differentiable a point a we need to show the limit for the definitino of the derivative does not exist. We can do this by showing two sequences that converge to a such that their limits under the defintion of the derivative are not equal. Let $x_n = 1/n$ and let $y_n = -1/n$ for all n. We proved in class both of these sequnces converge to 0. Note that $x_n > 0$ for all n,

$$f'(0) = \lim_{n\to\infty} \frac{f(x_n) - f(0)}{x_n - 0} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)^2}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0$$

and note that $y_n < 0$ for all n so,

$$f'(0) = \lim_{n\to\infty} \frac{f(y_n) - f(0)}{y_n - 0} = \lim_{n\to\infty} \frac{1/n}{1/n} = \lim_{n\to\infty} 1 = 1$$

thus since the limit values do not agree we have shown that the derivative does not exist at $x = 0$.

4. Suppose that $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are both uniformly continuous on D. Prove that the function $f + g : D \to \mathbb{R}$ is uniformly continuous on D.

Solution:

Fix $\varepsilon > 0$. Since f is uniformly continuous there exists some $\delta_f > 0$ such that for all $x, y \in D$ if $|x - y| < \delta_f$ then $|f(x) - f(y)| < \varepsilon/2$. Likewise since g is uniformly continuous there exists some $\delta_g > 0$ such that for all $x, y \in D$ if $|x - y| < \delta_g$ then $|g(x) - g(y)| < \varepsilon/2$. Let $\delta = \min\{\delta_f, \delta_g\}$ then for all $x, y \in D$ if $|x - y| < \delta$ we have that

$$\begin{aligned}
|(f+g)(x) - (f+g)(y)| &= |f(x) - f(y) + g(x) - g(y)| \\
&\leq |f(x) - f(y)| + |g(x) - g(y)| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon
\end{aligned}$$

We have shown for all $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $x, y \in D$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. Thus $f + g$ is uniformly continuous on D.