Homework for Math 351-003

Team Homework: Due Monday, March 18

Definition: Let $(a_n)_{n=m}^{\infty}$ be a sequence indexed by a set $\{n \in \mathbf{Z} \mid n \geq m)$ of consecutive integers. If $n \geq m$, one defines the **partial sum**

$$s_n = a_m + a_{m+1} + \cdots + a_n.$$

The associated **infinite series** is the expression $\sum_{n=m}^{\infty} a_n$. If the sequence $(s_n)_{n=m}^{\infty}$ of partial sums converges to $s \in \mathbb{R}$, one writes

$$\sum_{n=m}^{\infty} a_n = s$$

and says that the infinite series $\sum_{n=m}^{\infty} a_n$ **converges** to s.

1. Prove that if $\sum_{n=m}^{\infty} a_n = s$ and $\sum_{n=m}^{\infty} b_n = t$, then

$$\sum_{n=m}^{\infty} (a_n + b_n) = s + t.$$

   (Your proof should use the definition of partial sums above, and not rely on facts about series that you know from previous courses.)

2. a) Prove using the ϵ-δ definition of continuity that the function $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$ is continuous at $x_0 = 3$.

   b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined so that $f(x) = 1 - x^2$ if $x \geq 0$ and $f(x) = -x$ if $x < 0$. Prove using the $\epsilon - \delta$ definition of continuity that f is not continuous at 0.

3. Prove using the ϵ-δ definition of continuity that if $f : \mathbf{D} \to \mathbb{R}$ and $g : \mathbf{D} \to \mathbb{R}$ are both continuous at x_0, then $f + g$ is continuous at x_0.

4. **Definition:** Suppose $f : D \to \mathbb{R}$ is a function, $a \in \mathbb{R}$ and that there exists $\beta > 0$ such that D contains $(a - \beta, a) \cup (a, a + \beta)$. We say that

$$\lim_{x \to a} f(x) = L$$

   if given any sequence (x_n) with values in $D \setminus \{a\}$ such that $\lim x_n = a$ we have $\lim f(x_n) = L$.

   (a) Describe the domain of $\frac{x^2-4}{x-2}$ and prove that $\displaystyle\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$.

   (b) Prove that if $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are functions, $a \in D$, $\displaystyle\lim_{x \to a} f(x) = L$, and $\displaystyle\lim_{x \to a} g(x) = M$, then $\displaystyle\lim_{x \to a} (f + g)(x) = L + M$.