MATH 351: Principles of Analysis
University of Michigan, Winter 2024 Homework 3-18

Definition: Let $(a_n)_{n=m}^{\infty}$ be a sequence indexed by a set $\{n \in \mathbb{Z} \mid n \geq m\}$ of consecutive integers. If $n \geq m$, one defines the **partial sum**

$$s_n = a_m + a_{m+1} + \cdots + a_n$$

The associated **infinite series** is the expression $\sum_{n=m}^{\infty} a_n$. If the sequence $(s_n)_{n=m}^{\infty}$ of partial sums converges to $s \in \mathbb{R}$, one writes

$$\sum_{n=m}^{\infty} a_n = s$$

and says that the infinite series $\sum_{n=m}^{\infty} a_n$ **converges** to s.

1. Prove that if $\sum_{n=m}^{\infty} a_n = s$ and $\sum_{n=m}^{\infty} b_n = t$, then

$$\sum_{n=m}^{\infty} (a_n + b_n) = s + t$$

Solution:

Since $\sum_{n=m}^{\infty} a_n$ converges to s we know the sequence of partial sums $(s_n)_{n=m}^{\infty}$ converges to s. Likewise since $\sum_{n=m}^{\infty} b_n$ converges to t we know the sequence of partial sums $(t_n)_{n=m}^{\infty}$ converges to t. Thus by the addition limit law

$$(s_n + t_n)_{n=m}^{\infty} = s + t$$

and hence by definition of series convergence $\sum_{n=m}^{\infty}(a_n + b_n) = s + t$

2. (a) Prove using the ε-δ definition of continuity that the function $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given $g(x) = \frac{1}{x}$ is continious at $x_0 = 3$.
   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined so that $f(x) = 1 - x^2$ if $x \geq 0$ and $f(x) = -x$ if $x < 0$. Prove using the ε-δ definition of continuity that f is not continious at 0.

Solution:

(a) Fix $\varepsilon > 0$ and $\delta = \min\left\{\frac{9\varepsilon}{2}, \frac{3}{2}\right\}$. Then for $|x - 3| < \delta$ we have that $x > \frac{3}{2}$. Now consider

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \left|\frac{x-3}{3x}\right| < \left|\frac{2|x-3|}{9}\right| = \frac{2}{9}|x-3| < \frac{2}{9} \cdot \frac{9\varepsilon}{2} = \varepsilon$$

Thus by definition $g(x) = \frac{1}{x}$ is continous at $x = 3$.

(b) We want to show there exits some $\varepsilon > 0$ such that for all $\delta > 0$ there exists some $x \in \mathbb{R}$ such that $|x| < \delta$ and $|f(x) - 1| \geq \varepsilon$. Let $\varepsilon = 1/2$. Fix $\delta > 0$ then let $x = \max\{-1/2, -\delta/2\}$, then for $|x| < \delta$ we have that if $-1/2 > -\delta/2$

$$|f(x) - 1| = | - x - 1|$$

$$
\begin{aligned}
&= |\frac{1}{2} - 1| \\
&= |-1/2| \\
&= 1/2 \geq 1/2
\end{aligned}
$$

and if $-1/2 < -\delta/2$ then we have

$$
\begin{aligned}
|f(x) - 1| &= |-x - 1| \\
&= |\delta/2 - 1| \\
&> |1/2| \\
&= 1/2 \geq 1/2
\end{aligned}
$$

So in either case we have that if $|x - 0| < \delta$ then $|f(x) - f(0)| \geq 1/2$, thus f is not continious at 0.

3. Prove using the ϵ-δ definition of continuity that if $f : \mathbf{D} \to \mathbb{R}$ and $g : \mathbf{D} \to \mathbb{R}$ are both continous at x_0, then $f + g$ is continious at x_0.

Solution:

Fix $\varepsilon > 0$. Then there exists some $\delta_1 > 0$ such that if $|x - x_0| < \delta_1$ then $|f(x) - f(x_0)| < \varepsilon/2$. Likewise there exists some $\delta_2 > 0$ such that if $|x - x_0| < \delta_2$ then $|g(x) - g(x_0)| < \varepsilon/2$. Let $\delta = \min\{\delta_1, \delta_2\}$ then if $|x - x_0| < \delta$

$$
\begin{aligned}
|(f+g)(x) - (f+g)(x_0)| &= |f(x) + g(x) - f(x_0) - g(x_0)| \\
&= |(f(x) - f(x_0)) + (g(x) - g(x_0))| \\
&\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon
\end{aligned}
$$

Thus by definition, $f + g$ is continous at x_0

4. **Definition:** Suppose $f : D \to \mathbb{R}$ is a function, $a \in \mathbb{R}$ and there exists $\beta > 0$ such that D contains $(a - \beta, a) \cup (a, a + \beta)$. We say that

$$\lim_{x \to a} f(x) = L$$

if given any sequence (x_n) with values in $D \setminus \{a\}$ such that $\lim x_n = a$ we have $\lim f(x_n) = L$.

(a) Describe the domain of $\frac{x^2-4}{x-2}$ and prove that $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$

(b) Prove that if $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are functions, $a \in D$, $\lim_{x\to a} f(x) = L$, and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a}(f + g)(x) = L + M$

Solution:

(a) Domain $D = \{x \mid x \in (-\infty, 2) \cup (2, \infty)\}$. Let (x_n) be a sequence in D such that $\lim_{n\to\infty} x_n = 2$. Let $f(x) = \frac{x^2-4}{x-2}$ where $x \in D$. Then we have that

$$\lim f(x_n) = \lim_{n\to\infty} \frac{x_n^2 - 4}{x_n - 2} = \lim_{n\to\infty} \frac{(x_n - 2)(x_n + 2)}{x_n - 2} = \lim_{n\to\infty} x_n + 2 = 2 + 2 = 4$$

(b) Let (x_n) be a sequence in D such that $\lim_{n\to\infty} x_n = a$. Since $\lim_{x\to a} f(x) = L$ then $\lim f(x_n) = L$ and since $\lim_{x\to a} g(x) = M$ then $\lim g(x_n) = M$. Thus we get that

$$\lim(f + g)(x_n) = \lim f(x_n) + \lim g(x_n) = L + M$$

thus

$$\lim_{x\to a}(f + g)(x) = L + M$$