MATH 351: Principles of Analysis
University of Michigan, Winter 2024 **HW 4-10**

1. (Note: for this problem, you may use the fact that between any two real numbers there exists both a rational number and an irrational number. The former is proved in the "Additional Reading" section at the end of the course notes. The latter you may use without proof.)

   Define $f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational} \end{cases}$.

   For a partition P of the interval $[a, b]$, find $U(f, P)$ and $L(f, P)$, and then find $U(f)$ and $L(f)$. Is f integrable?

Solution:

$$U(f,P) = \sum_{k=1}^{n} \sup\{f([t_{k-1}, t_k])\} \cdot (t_k - t_{k-1})$$

Per the note there exists an irrational number between t_{k-1} and t_k .As f can only take on the values 0 and 1 we can say $\sup\{f([t_{k-1}, t_k])\} = 1$. Thus our upper Darboux Sum becomes

$$U(f,P) = \sum_{k=1}^{n} 1 \cdot (t_k - t_{k-1}) = \sum_{k=1}^{n} (t_1 - a) + (t_2 - t_1) + \ldots + (b - t_{n-1}) = b - a$$

notice the sum telescopes to $b - a$. Thus the Upper Darboux Integral is

$$U(f) = \inf\{U(f,P)\} = \inf\{1\} = 1$$

On the other hand

$$L(f,P) = \sum_{k=1}^{n} \inf\{f([t_{k-1}, t_k])\} \cdot (t_k - t_{k-1})$$

Per the note there exists an rational number between t_{k-1} and t_k, and as f can only take on the values 0 and 1 we can say that $\inf\{f([t_{k-1}, t_k])\} = 0$. Thus our lower Darboux Sum becomes

$$L(f,P) = \sum_{k=1}^{n} 0 \cdot (t_k - t_{k-1}) = 0$$

So the Lower Darboux Integral is

$$L(f) = \sup\{L(f,P)\} = \sup\{0\} = 0$$

We have that $L(f) = 0$ and $U(f) = 1$, thus since the values do not agree f is not integrable.

2. Prove that if f is bounded and monotone increasing on $[a, b]$ and $[t_{k-1}, t_k] \subset [a, b]$, then

$$\sup\{f([t_{k-1}, t_k])\} - \inf\{f([t_{k-1}, t_k])\} = f(t_k) - f(t_{k-1}).$$

Solution:

In order to prove the given statement, I will show $f(t_k) = \sup\{f([t_{k-1}, t_k])\}$ and that $f(t_{k-1}) = \inf\{f([t_{k-1}, t_k])\}$. Let $x \in [t_{k-1}, t_k]$, then since $x \leq t_k$ and f is monotone increasing we have that $f(x) \leq f(t_k)$. Since x was arbitrary we have shown $f(t_k)$ is an upper bound for $f([t_{k-1}, t_k])$. Now we must show it is the supremum, let α be an upper bound for $f([t_{k-1}, t_k])$. Then by definition of upper bound $f(x) \leq \alpha$ for all $x \in [t_{k-1}, t_k]$. Since $t_k \in [t_{k-1}, t_k]$ we have that $f(t_k) \leq \alpha$. Hence $\sup\{f([t_{k-1}, t_k])\} = f(t_k)$. Similarly let $y \in [t_{k-1}, t_k]$, thus we know $y \geq t_{k-1}$ and since f is monotone increasing $f(y) \geq f(t_{k-1})$. Thus $f(t_{k-1})$ is a lower bound for the set $f([t_{k-1}, t_k])$. Now we must show that is the infimum. Let β be a lower bound for $f([t_{k-1}, t_k])$. Then $\beta \leq f(x)$ for all $x \in [t_{k-1}, t_k]$, by definition of lower bound. Since $t_{k-1} \in [t_{k-1}, t_k]$ we can say $\beta \leq f(t_{k-1})$. Thus $\inf\{f([t_{k-1}, t_k])\} = f(t_{k-1})$. Subtracting the supremum and infimum we get

$$\sup\{f([t_{k-1}, t_k])\} - \inf\{f([t_{k-1}, t_k])\} = f(t_k) - f(t_{k-1})$$

as desired.