MATH 423

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Some remark.

Remark.

> Some more remark.

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Figure 1.1: Waterloo, ON

Chapter 2

Options

2.1 Introduction to Options (Lecture 21)

Definition 2.1.1: Derivative

A product that you can issue and sell, however payout is not pre-agreed upon that depends on the value of another underlying asset

Definition 2.1.2: European Call Option

When issued, what is specified is:

- An underlying asset with value, let's say, S_t at any time $t \geq 0$
- A future time T, called maturity
- A price K called "strike"

2.1.1 European Call Option

The option is sold after issuance, and the person who holds it has the *right* to but at time T, the underlying asset from the issuer of the option, at the price K. Thus if $S_T > K$, exercising the right of the option saves $S_T - K$ dollars.

On the other hand, unlike a long forward position the option does not go with obligation to make the above purchase, so if $S_T \leq K$ a rational holder of the option will simply not exercise. In this case the option payout is 0. Thus, in any case, payoff at time T is:

$$(S_T - K)^+ = \max(S_T - K, 0)$$

2.1.2 European Put Option

Works like European Call option, but holder has right to *sell* the underlying asset to the issuer for K dollars. Thus they save $K - S_T$ dollars if $K > S_T$ and don't save or lose anything otherwise. The payoff at time T is

$$(K - S_T)^+ = \max(K - S_T, 0)$$

Notation

- $C_e(t, T, K)$: the value at time $t \leq T$ of European Call Option with maturity T and strike K
- $P_e(t, T, K)$ the value at time $t \leq T$ of European Put Option with maturity T and strike K

so clearly for $t = T$:

$$C_e(t, T, K) = (S_T - K)^+$$

$$P_e(t, T, K) = (K - S_T)^+$$

We want to be able to compute the prices of these options for any time $t \leq T$, but this requires a certain model for now the price S_t of the underlying asset evolves with time t. However, some properties can be obtained even without a model. The first being

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K$$

which is equivalently:

$$C_e(T, T, K) - P_e(T, T, K) = S_T - K$$

which can be extended to *any* time $t \leq T$ as:

$$C_e(t, T, K) - P_e(t, T, K) = S_T - K \cdot B(t, T)$$

This is the Put-Call Parity

Lemma 2.1.3: Higher payoff Higher price Lemma

Consider 2 products with values $V_1(t)$ and $V_2(t)$ at time t respectively. We say that if $V_1(T) > V_2(T)$ for any $\omega \in \Omega$ then $V_1(t) > V_2(t)$ at any previous time $t \leq T$

Proof. Suppose $V_1(T) > V_2(T)$ for all $\omega \in \Omega$ but $V_1(t) \leq V_2(t)$ (so $V_1(t)$ is undervalued at time t and $V_2(t)$ is overvalued)
Then at time t:

- Short sell product 2 to earn $V_2(t)$ dollars
- Buy product 1 by paying $V_1(t)$ dollars
- Invest the difference in bonds

At time T:

- Earn $V_1(t)$
- Pay $V_2(t)$ to buy product 2 and deliver it to close short selling position
- Receive $\frac{V_2(t)-V_1(t)}{B(t,T)}$ dollars from bonds

So our balance is

$$V_1(T) - V_2(T) + \frac{V_2(t) - V_1(t)}{B(t,T)} > 0$$

for all ω, which is arbitrage! Contradiction! □

Corollary 2.1.4

If $V_1(T) = V_2(T)$ for all $\omega \in \Omega$ we can apply above lemma twice to get $V_1(t) = V_2(t)$ for all $t \leq T$ and for all $\omega \in \Omega$

Theorem 2.1.5: Put-Call Parity

$$C_e(t,T,K) - P_e(t,T,K) = S_T - K \cdot B(t,T)$$

Proof. We showed that

$$C_e(t,T,K) - P_e(t,T,K) = S_T - K$$

where the right side is the value of a long forward position at delivery T, which equals the payoff So we apply the previous result for the products:

1. A portfolio with 1 call and -1 Put, both with strike K and maturity T
2. A long forward position with forward price K and delivery T

so the above equality is $V_1(T) = V_2(T)$ so by the previous corollary $V_1(t) = V_2(t)$ or in other words $C_e(t,T,K) - P_e(t,T,K) = S_t - K \cdot B(t,T)$ □

2.2 Properties of European and American Options (Lecture 22)

Last week we proved the "higher payoff-higher price" lemma and we used it to prove the Put-Call Parity:

$$C_e(t,T,K) - P_e(t,T,K) = S_t - K \cdot B(t,T)$$

where the right hand side is the long forward position at time t, which agrees with $(F(t,T) - F(t_0,T)) \cdot B(t,T)$ which we already knew when forward contract is made at time $t_0 \leq t$. To verify that, recall then $F(t,T) = \frac{S_t}{B(t,T)}$ and $F(t_0,T) = K$: forward price used in the contract of this forward position. We can also prove that

$$C_e(t,T,K) < S_t$$

Indeed, for $t = T$ this reduces to

$$C_e(T,T,K) = (S_T - K)^+ < (S_T)^+ = S_T$$

because $f(x) = x^+$ is increasing. Then we just apply the "higher payoff-higher price" for one product being the stock and the other the call option to get the same result for any $t \leq T$:

$$C_e(t,T,K) < S_t$$

Another inequality that holds is

$$C_e(t,T,K) \geq (S_t - K \cdot B(t,T))^+$$

Proof. By the Put-Call parity we know that

$$\begin{aligned} C_e(t,T,K) - P_e(t,T,K) &= S_t - K \cdot B(t,T) \\ C_e(t,T,K) &= S_t - K \cdot B(t,T) + P_e(t,T,K) \\ &\geq S_t - K \cdot B(t,T) \end{aligned}$$

since also $C_e(t,T,K) \geq 0$ it follows that

$$\begin{aligned} C_e(t,T,K) &\geq \max\{0, S_t - K \cdot B(t,T)\} \\ &= (S_t - K \cdot B(t,T))^+ \end{aligned}$$

□

Definition 2.2.1: American Option

An American Option is like an European Option but its holder can exercise it at ANY time before maturity T

Remark.

> $C_A(t,T,K)$: American Call Option at time $t \leq T$, when K is the strike and T is the maturity.

Properties similar to those of $P_e(t,T,K)$ and $C_e(t,T,K)$ hold for $P_A(t,T,K)$ and $C_A(t,T,K)$, but the "higher payoff-higher price" lemma cannot be used to prove those, since the payoffs of Americans are not certainly paid at time T. Thus we prove such properties by the standard no-arbitrage arguments

Example.

> We have something similar to the Put-Call Parity:
>
> $$S_t - K \cdot B(t,T) \geq C_A(t,T,K) - P_A(t,T,K) \geq S_t - K$$

Let's prove the above statement in the example

Proof. We only prove the right part

$$C_A(t,T,K) - P_A(t,T,K) \geq S_t - K$$

First we rearrange

$$C_A(t,T,K) + K \geq S_t + P_A(t,T,K)$$

Suppose for the sake of contradiction we have

$$C_A(t,T,K) + K < S_t + P_A(t,T,K)$$

$C_A(t,T,K)$ is American Call Price at time t, K is the value at time t of $\frac{K}{B(t,T)}$ units of bond that pays 1 dollar at time T, each worth $B(t,T)$. $P_A(t,T,K)$ American Put Price as time t and S_t is the stock price at time t. C_A and K are on the small side, so undervalued and the other products are on the big side so they are overvalued. Now we just follow the standard procedure.
<u>At time t</u>

- Issue and sell an American Put to get $P_A(t,T,K)$ dollars
- Short sell the stock to get S_t dollars
- Buy an American Call by paying $C_A(t,T,K)$
- Buy $\frac{K}{B(t,T)}$ units of bond each for $B(t,T)$ dollars, paying K dollars in total

<u>Our Balance</u>

$$M = P_A(t,T,K) + S_t - (C_A(t,T,K) + K) > 0$$

<u>Optimal action</u>: Invest M in bonds, by buying $\frac{M}{B(t,T)}$ units, each worth $B(t,T)$ dollars so we pay M dollars. So now our balance becomes 0.
Normally we would go to time T, but there is a possibility that the American option we sold is exercised before that! <u>Consider 2 cases:</u>
<u>Case 1:</u> American Put we shold is exercised at some $t_0 \in [t,T]$. Then we go to time t_0.

At time t_0: We have the obligation to buy from the option holder, the stock for K dollars. Thus, borrow K dollars by issuing and selling $\frac{K}{B(t_0,T)}$ that pays 1 dollar at time T, each worth $B(t_0,T)$ since we are now at time t_0, so we receive K dollars. Then, give those dollars to option holder who exercised it to get the stock.
At time T:

- We have a stock received at time t_0 but also an open short-selling position on the stock at time t. Thus we just deliver the stock to close the position.
- From bonds bought at time t, earn
$$\frac{M}{B(t,T)} + \frac{K}{B(t,T)} \text{ dollars}$$
- From bonds sold at time t_0 we must pay $\frac{K}{B(t_0,T)}$ dollars

Nothing else happens so check balance:

$$\frac{M}{B(t,T)} + \frac{K}{B(t,T)} - \frac{K}{B(t_0,T)}$$

$$\frac{M}{B(t,T)} + K\left(\frac{1}{B(t,T)} - \frac{1}{B(t_0,T)}\right)$$

where $\frac{1}{B(t,T)} - \frac{1}{B(t_0,T)} \geq 0$ because $B(t,T) = (1+r_e)^{t-T} \leq (1+r_e)^{t_0-T} = B(t_0,T)$ since $t_0 \geq t$

Case 2 The American Option we sold is never exercised. Thus we go directly to time T
At time T:
Again, we must clos short-selling made at time t, but now we do not posses a stock in this second case. Thus we exercise the American call we hold (which we didn't have to do in case 1) to get the stock for K dollars, and we deliver the stock to close short-selling. Then, we also earn from bonds bought at time t:

$$\frac{M}{B(t,T)} + \frac{K}{B(t,T)} \text{ dollars}$$

So our balance is

$$\frac{M}{B(t,T)} + \frac{K}{B(t,T)} - K \text{ dollars}$$

$$\frac{M}{B(t,T)} + K\left(\frac{1}{B(t,T)} - 1\right) > 0$$

because $B(tT) = (1+r_e)^{t-T} < 1$ since $t < T$. Thus started with 0 money and ended up with positive balance which is arbitrage → Contradiction. □

Theorem 2.2.2

Consider 2 options A, B with prices $A(t)$ and $B(t)$ respectively at any time t. If the holder of A can also use it exactly as B, then $A(t) \geq B(t)$ for all t

Sketch of proof. Suppose $A(t) < B(t)$. So we issue and sell B, buy A and invest the positive balance $B(t)-A(t)$ in bonds (all at time t). Now when B is exercised, we exercise A using it exactly as B, so we earn from A exactly what we must pay to the one who exercises B. Thus we are only left with our investments in bonds, which leads to arbitrage.

Consequences of the above theorem

1. $C_A(t,T,K) \geq C_e(t,T,K)$ and $P_A(t,T,K) \geq P_e(t,T,K)$ because an American Option can be used exactly as the corresponding European Option (just by deciding to exercise at time T)

2. American Options with more distant maturities have higher prices $C_A(t,T,K) \geq C_A(t,T',K)$ and $P_A(t,T,K) \geq P_A(t,T',K)$ whenever $T \geq T'$ because an american with maturity T can be used as the

2.3 Option Results & Binomial Model

While we have
$$C_A(t,T,K) \geq C_e(t,T,K)$$
when the stock pays no dividends we actually have

$$C_A(t,T,K) = C_e(t,T,K)$$

and it is optimal for an investor to exercise the American at time T

We have obtained various identities and bounds for option prices, but to explicitly compute those prices we need to know how the price S_t of the underlying stock evolves with time t.

We focus on 2 models:

1. The Binomial model (Discrete time)

2. The Black-Scholes model (Continuous time)

2.3.1 The Binomial Model

Given a time period $[0,T]$, a stock S with price S_t at time $t \in [0.T]$, and a bond that pays 1 dollar at time T with price $B(t,T)$ at time $t \in [0.T]$

Everything (changes in the stock price and interest compounding which changes bond prices) happen at the following times:

$t=0, t=h, t=2h, \cdots t=Nh=T$ i.e prices change every h units of time and there are $N=\frac{T}{h}$ steps in total. Let $S(n)=S_{nh}$ (stock price at the n-th step).

Stock return:
$$R(n+1) = \frac{S(n+1)-S(n)}{S(n)}$$

Under the binomial model, the returns at different steps are independent random variables, with:

$R(n) =$
U with probability p
D with probability 1-p
for all $n \in [1, 2, \cdots, N]$, where: $-1 < D < U$

using the equality for returns the previous equality becomes:
$S(n+1) = \begin{cases} S(n)(1+U) & \text{with probability } p \\ S(n)(1+D) & \text{with probability } 1-p \end{cases}$
for $n \in \{0, 1, \cdots N-1\}$
We can now compute the price:

$$\begin{aligned} S(n) &= S(n-1) \cdot (1 + R(n)) \\ &= S(n-2) \cdot (1 + R(n-1)) \cdot (1 + R(n)) \\ &\vdots \\ &= S(0) \cdot \prod_{i=1}^{n} (1 + R(i)) \\ &= S(0) \cdot (1+U)^k \cdot (1+D)^{n-k} \end{aligned}$$

where k is the number of the returns $R(i)$ that equal U, so remaining $n-k$ are equal to D.

Notation: If we observe the returns $R(1), R(2), \cdots, R(n)$ until the n-th step, we write: $S^{R(1)R(2)\cdots R(n)}(n)$ for the price of the stock at the n-th step under those observed returns.

Example.

> $S^{UUD}(3)$ the price of the stock at the 3rd step when $R(1) = U$, $R(2) = U$, and $R(3) = D$ which is equal $S(0) \cdot (1+U)^2 \cdot (1+D)$

If Ω is the underlying sample space, each evolution scenario for the market scenario for the market corresponds to a $\omega \in \Omega$, which can be identified with the sequence of returns $R(i)(\omega)$ under that ω

$$\omega = R(1)(\omega)R(2)(\omega) \cdots R(N)(\omega)$$

Consider now one possible price at step n:

$$S(0)(1+U)^k \cdot (1+D)^{n-k}$$

The probability of the price at time n being exactly the above for fixed n and k is precisely:

$$\Pr(S(n) = S(0)(1+U)^k \cdot (1+D)^{n-k}) = \binom{n}{k} p^k (1-p)^{n-k}$$

Bond Prices:
Consider a bond that pays 1 dollar at step m (time $m \cdot h$). Its price at step $n \leq m$ (time

$n \cdot h$) is then

$$\begin{aligned} B(nh, mh) &= (1 + r_e)^{-(mh - nh)} \\ &= ((1 + r_e)^h)^{(n-m)} \end{aligned}$$

so that $B(nh, mh) = (1 + r_h)^{n-m}$. For a single step ($m = n + 1$), $B(nh, (n + 1)h) = \frac{1}{1+r_n}$. The bond's return in $[nh, (n + 1)h]$ is then:

$$\frac{B((n+1)h, (n+1)h - B(nh, (n+1)h}{B(nh, n(+1)h} = r_n \to \text{interest rate b/w consecutive steps}$$

Theorem 2.3.1

Under the Binomial model, to have no arbitrage we require:

$$D < r_h < U$$

between 2 consecutive steps, the bond's return must be strictly between the 2 possible returns of the stock.

Proof. We prove $r_h < U$. Suppose for contradiction $r_h \geq U$ so that $r_h \geq U > D$ this means that the bond has for sure a better return compared to the stock, so we buy it by short selling the stock.
At time 0: Short sell the stock to earn $S(0)$ which we invest in buying bonds that pay 1 dollar at the next step, each bond is worth $B(0, h)$, so we by $\frac{S(0)}{B(0,h)}$ units.
At time h (step 1): We earn $\frac{S(0)}{B(0,h)} = S(0) \cdot (1 + r_h)$ dollars. We must close short selling, so we pay $S(1)$ to buy the stock and we deliver it to close short selling. Our balance at this point is now

$$\begin{aligned} S(0) \cdot (1 + r_h) - S(1) &= S(0)(1 + r_h) - S(0)(1 + R(1)) \\ &= S(0)(r_h - R(1)) \\ &= S(0) \begin{cases} r_h - U & \text{with probability } p \\ r_h - D & \text{with probability } 1 - p \end{cases} \end{aligned}$$

which is certainly ≥ 0 since $r_h \geq U > D$ and with positive probability $1 - p > 0$ it is a strictly positive profit of $r_h - D > 0$. This is arbitrage, thus we have reached a contradiction. □

2.4 Lecture 24

2.4.1 Self Financing Property

Suppose at the n-th step we hold $\alpha_S(n)$ units of stock (each unit workh $S(n)$) and $\alpha_B(n)$ units of bond that pays 1 dollar at step N (each unit worth $B(nh, T)$), so our portfolio has value

$$V(n) = \alpha_S(n)S(n) + \alpha_B(n)B(nh, T) \tag{2.1}$$

At the next step, values of products change the stock $S(n) \longrightarrow S(n+1)$ and the bond $B(nh,T) \longrightarrow B((n+1)h,T) = B(nh,T)(1+r_n)$ so our portfolio value becomes

$$V(n+1) = \alpha_S(n)S(n+1) + \alpha_B(n)B((n+1)h,T) \tag{2.2}$$

Then, we can rebalance: sell stocks and use money earned to buy bonds or the opposite, so amounts of assets we hold change: $\alpha_S(n) \to \alpha_S(n+1)$ and $\alpha_B(n) \to \alpha_B(n+1)$ However if no extra money is added no money is put aside the value of the portfolio remains the same after $\alpha_S(n)$ and $\alpha_B(n)$ change (self-financing property)

$$V(n+1) = \alpha_S(n+1)S(n+1) + \alpha_B(n+1)B((n+1)h,T) \tag{2.3}$$

Subtracting 2.3 from 2.2 we get the Self-financing property

$$0 = (\alpha_S(n+1) - \alpha_S(n)) \cdot S(n+1) + (\alpha_B(n+1) - \alpha_B(n)) \cdot B((n+1)h,T) \tag{2.4}$$

Alternatively we can subtract 2.2 from 2.1 to find

$$V(n+1) - V(n) = \alpha_S(n) \cdot (S(n+1) - S(n)) + \alpha_B(n) \cdot (B((n+1)h,T) - B(nh,T))$$

2.4.2 Hedging and Pricing

Suppose that aprt from bond and stock we have an option, whose value at step $n+1$ is:

$$P(n+1) = H_{n+1}(S(n+1))$$

e.g if the option is a European Call maturing at step $n+1$: $H_{n+1}(x) = (x-K)^+$

How to find option price $P(n)$ at the previous step n?

The idea is to construct a portfolio at step n, whose price at step $n+1$ will always coincide with the option price at step $n+1$, by approximately selecting the $\alpha_S(n)$ and $\alpha_B(n)$ that do the work. There are two scenarios

(a) with probability p
the portfolio value becomes

$$\begin{aligned} V(n+1) &= \alpha_S(n)S(n+1) + \alpha_B(n)B((n+1)h,T) \\ &= \alpha_S(n)S(n)(1+U) + \alpha_B(n)B((n+1)h,T) \end{aligned}$$

the option value becomes

$$P(n+1) = H_{n+1}(S(n+1)) = H_{n+1}(S(n)(1+U))$$

(b) with probability $1-p$

the portfolio value becomes

$$\begin{aligned} V(n+1) &= \alpha_S(n)S(n+1) + \alpha_B(n)B((n+1)h,T) \\ &= \alpha_S(n)S(n)(1+D) + \alpha_B(n)B((n+1)h,T) \end{aligned}$$

the option value becomes

$$P(n+1) = H_{n+1}(S(n+1)) = H_{n+1}(S(n)(1+D))$$

We want the portfolio value to coincide with the option value at step $n+1$:

$$V(n+1) = P(n+1)$$

under both scenarios, so that by the "Higher payoff- higher price" lemma we will then have equality of values at step n

$$P(n) = V(n)$$

Requiring that $V(n+1) = P(n+1)$ under both scenarios gives a 2x2 system which solving yields

$$\alpha_S(N) = \frac{1}{S(n)} \cdot \frac{H_{n+1}(S(n)(1+U)) - H_{n+1}(S(n)(1+D))}{U-D}$$

and

$$\alpha_B(N) = \frac{1}{S(n)} \cdot \frac{H_{n+1}(S(n)(1+U)) - H_{n+1}(S(n)(1+D))}{(D-U)(1+r_n)^{n+1-N}}$$

as expected those only depend on $S(n)$ and deterministic quantities, so they are known at step n. Now, by what we said above, our option price at step n is

$$P(n) = V(n) = \alpha_S(n)S(n) + \alpha_B(n)B(nh,T)$$

where we can plug the $\alpha_S(n)$, and $\alpha_B(n)$ we computed to find: $P(n) = \cdots = H_n(S(n))$

so again it is a function of the stock price at the step at which the option price is computed. However at this step n:

$$H_n(x) = \frac{1}{1+r_n}\left(\frac{U-r_n}{U-D}H_{n+1}((1+D)x) + \frac{r_n-D}{U-D}H_{n+1}((1+U)x)\right)$$

which differs from the $H_{n+1}(x)$ that we have the next step $n+1$

Now we write:

$$P(n) = \frac{1}{1+r_n}\mathbb{E}^{p^*}(H_{n+1}(S(n+1))|\mathcal{F}_n)$$

where expectation $\mathbb{E}^{p*}$ is taken in an "artificial" regime where:

$$S(n+1) = \begin{cases} S(n)(1+U) & \text{with probability } p^* = \frac{r_n-D}{U-D} \\ S(n)(1+D) & \text{with probability } 1-p^* \end{cases}$$

Thus we deduce that we can rewrite the above as

$$P(n) = \frac{1}{1+r_n}\mathbb{E}^{p^*}(P(n+1)|\mathcal{F}_n)$$

Dividing now by (1_{r_n}) and defining $\tilde{P}(k) = \frac{P(k)}{(1+r_n)^k}$ we get which is the price of the option at time k discounter to time 0 where the discount factor is $(1+r_n)^k = B(0, kh)$: we get from the last equality:

$$\tilde{P}(n) = \mathbb{E}^{p^*}(\tilde{P}(n+1)|\mathcal{F}_n)$$

This is the martingale property of the discounted price $\tilde{P}(n)$ and because we also have:

$$\tilde{P}(n-1) = \mathbb{E}^{p^*}(\tilde{P}(n)|\mathcal{F}_{n-1})$$

we can get:

$$\tilde{P}(n-1) = \mathbb{E}^{p^*}(\mathbb{E}^{p^*}(\tilde{P}(n+1)|\mathcal{F}_n)|\mathcal{F}_{n-1})$$

notice here we are conditioning on $\mathcal{F}_{n-1}$ plus things observed at step n so by the tower rule the inner expectation disappears and we get:

$$\tilde{P}(n-1) = \mathbb{E}^{p^*}(\tilde{P}(n+1)|\mathcal{F}_{n-1})$$

which can be extended to any step k:

$$\tilde{P}(n-k) = \mathbb{E}^{p^*}(\tilde{P}(n+1)|\mathcal{F}_{n-k})$$

for any $k \in \{0, 1, \cdots, n\}$ giving the option price at ANY step before $n+1$

2.5 Lecture 25

skip for now

2.6 Lecture 26

Went over but lots of diagrams so lowkey dont want to type it up

2.7 Brownian motion and Ito's calculus (Lecture 27)

In Continuous time models we will only cover the Black-Scholes model. In this lecture we develop the necessary mathematical tools.

Start with a probability space

$$(\Omega, \mathcal{F}, \mathbb{P})$$

Definition 2.7.1: Continuous Time Stochastic Process

A continuous time stochastic process is a function X that maps any (t, ω) for $t \geq 0$ and $\omega \in \Omega$ to the value $X(t, \omega)$ which describes the value of a randomly evolving quantity (e.g a stock price) at time t under scenario ω

- For a fixed outcome ω: $X(\cdot, \omega)$ is a function of the time variable t, which maps any time t to the value $X(t, \omega)$ (describes the evolution of the process with time, under a fixed scenario/outcome ω)
- For a time t: $X(t, \cdot)$ is a random variable that describes the possible values of the process at a fixed time t, under different scenarios/outcomes ω

Definition 2.7.2: Brownian Motion

A stochastic process W that maps any time t and outcome ω to $W(t, \omega) = W_t$ is called a Brownian motion when:

1. $W_0 = W(0, \omega) = 0$
2. The function $W(\cdot, \omega)$, which is a function of t, is a *continuous function* of t under any possible scenario ω
3. For $t_1 \leq t_2 \leq \cdots \leq t_n$, the increments $W(t_n, \cdot) - W(t_{n-1}, \cdot)$, $W(t_{n-1}, \cdot) - W(t_{n-2}, \cdot)$, $\cdots$, $W(t_2, \cdot) - W(t_1, \cdot)$ (increments over non-lapping intervals) are *independent* random variables.
4. For $t \geq s$
$$W(t, \cdot) - W(s, \cdot) \sim N(0, t-s)$$

Calculus of Brownian Motion

Let's try to compute the differential

$$dW_t = W_{t+dt} - W_t$$

By property 4 of Brownian Motion $dW_t \sim N(0, dt)$ so:

- $\mathbb{E}(dW_t) = 0$, $\mathrm{Var}(dW_t) = dt$
- $\mathbb{E}((dW_t)^2) = \mathrm{Var}(dW_t) + \mathbb{E}^2(dW_t) = dt$

Moreover $\mathrm{Var}((dW_t)^2)$ is proportional to $(dt)^2$ and thus negligible so:

$$dW_t^2 = \mathbb{E}((dW_t)^2) = dt$$

Recap of ordinary calculus

Consider $z = f(x, y)$, we want to compute dz, a small change in z that occurs from a small change dx in x and a small change dy in y

$$dz = df(x, y) = f(x + dx, y + dy) - f(x, y)$$

By taylor expanding we get

$$f(x+dx, y+dy) = f(x, y) + f_x(x, y)dx + f_y(x, y)dy + \frac{1}{2}f_{xx}(x, y)(dx)^2 + \frac{1}{2}f_{yy}(x, y)(dy)^2 + f_{xy}(x, y)dxdy$$

so we can easily see that

$$dz = f_x(x,y)dx + f_y(x,y)dy + \frac{1}{2}f_{xx}(x,y)(dx)^2 + \frac{1}{2}f_{yy}(x,y)(dy)^2 + f_{xy}(x,y)dxdy$$

Assume now that $x = x(t)$ and $y = y(t)$, so small changes in those occure from a small chnage dt in t. Then:

$$dz = f_x(x(t),y(t))dx(t) + f_y(x(t),y(t))dy(t) + \frac{1}{2}f_{xx}(x(t),y(t))(dx(t))^2 + \frac{1}{2}f_{yy}(x(t),y(t))(dy(t))^2 + f_{xy}(x(t),y(t))dx(t)dy(t)$$

Now if $x(t)$, $y(t)$ are differntiable in t: $dx(t) = x'(t)dt + O((dt)^2)$ and $dy(t) = y'(t)dt + O((dt)^2)$ which can be plugged into the above equation and we can observe that all terms that are accompanied by a factor of $(dt)^2$, $(dt)^3$ or higher are regarded as O because $(dt)^n$ for $n > 1$ is much smaller than dt. Then it is easy to see that ALL terms involving 2nd order derivativees will vanish in the equation and it becomes

$$dz = f_x(x(t),y(t))x'(t)dt + f_y(x(t),y(t))y'(t)dt$$

which is just the standard chain rule of ordinary calculus.

Now we can consider if $x(t) = t$ but $y(t) = W_t$ is a Brownian motion (non-differentiable). We cannot plug into $dy(t) = y'(t)dt + O((dt)^2)$ because $y'(t)$ does not exist. Actually the double derivative with resepect to y term becomes

$$\frac{1}{2}f_{yy}(x(t),y(t))(dy(t))^2 = \frac{1}{2}f_{yy}(t,W_t)(dW_t)^2 = \frac{1}{2}f_{yy}(t,W_t)dt$$

Remark.

> other 2nd order derivatives will still vanish as the f_{xy} term goes with $dW_t dt$ which is also very much smaller than dt and thus is regarded as 0

So our differential becomes

Definition 2.7.3: Itô's Formula

$$dz = df(t,W_t) = f_x(t,W_t)dt + f_y(t,W_t)dW_t + \frac{1}{2}f_{yy}(t,W_t)dt$$

If we integrate the above in $[s,t]$

$$\begin{aligned} f(t,W_t) - f(s,W_s) &= \int_s^t df(r,W_r) \\ &= \int_s^t f_x(r,W_r)dr + \int_s^t \frac{1}{2}f_{yy}(r,W_r)dr + \int_s^t f_y(r,W_r)dW_r \end{aligned}$$

where the first two integrals are ordinary integrals and the last one is an Itô integral

Properties of Itô integrals

Lets compute

$$\mathbb{E}\left[\int_a^b f(r)dW_r\right] = \int_a^b \mathbb{E}[f(r)dW_r]$$

so if we assume that the values of f until any time s are <u>known</u> provided that the values of W until that same time are also known then

$$\mathbb{E}[f(r)dW_r] = \mathbb{E}[f(r)(W_{r+dr} - W_r)]$$

where $W_{r+dr} - W_r$ is independent of $W_r^{'}$ for $r' \leq r$ and because the latter fully determine the value of $f(r)$, $W_{r+dr} - W_r$ is independent of $f(r)$

$$\begin{aligned}
\mathbb{E}[f(r)(W_{r+dr} - W_r)] &= \mathbb{E}[f(r)]\mathbb{E}[W_{r+dr} - W_r] \\
&= \mathbb{E}[f(r)] \cdot \mathbb{E}[N(0, dr)] \\
&= \mathbb{E}[f(r)] \cdot 0 \\
&= 0
\end{aligned}$$

Thus we have that $\mathbb{E}\left[\int_a^b f(r)dW_r\right] = 0$ which extends to

$$\mathbb{E}\left[\int_s^t f(r)dW_r|\mathcal{F}_s\right] = 0$$

where $\mathcal{F}_s$ represents everything observable until time s and from that we can obtain the following theorem

Theorem 2.7.4: Martingale Property of the Ito Integral

Provided that f is W-adapted (has known values until any time which the Brownian motion is observed)

$$\begin{aligned}
\mathbb{E}\left[\int_0^t f(r)dW_r|\mathcal{F}_s\right] &= \mathbb{E}\left[\int_s^t f(r)dW_r + \int_0^s f(r)dW_r|\mathcal{F}_s\right] \\
&= \int_0^s f(r)dW_r
\end{aligned}$$

The first integral was shown to have 0 expectation above and the second one is known given $\mathcal{F}_s$. Compute $\mathbb{E}\left[W_3^2 \mid W_1 = 5\right]$ We need to express W_t^2 as a combination of integrals and use the above properties of expecations. We use Ito's formula on $f(x, y) = y^2$ so that $f(t, W_t) = W_t^2$. $f_x(x, y) = 0$, $f_y(x, y) = 2y$ and $f_{yy}(x, y) = 2$ so by Itô's formula (integral form):

$$
\begin{aligned}
f(t, W_t) - f(s, W_s) &= \int_s^t f_x(r, W_r)dr + \int_s^t f_y(r, W_r)dW_r + \frac{1}{2}\int_s^t f_{yy}(r, W_r)dr \\
&= 0 + \int_s^t 2W_r dW_r + \int_s^t 1dr \\
&= \int_s^t 2W_r dW_r + (t-s)
\end{aligned}
$$

then taking expectations given $\mathcal{F}_s$ we get

$$
\begin{aligned}
\mathbb{E}[W_t^2 \mid \mathcal{F}_s] - \mathbb{E}[W_s^2 \mid \mathcal{F}_s] &= \mathbb{E}\left[\int_s^t 2W_r dW_r \mid \mathcal{F}_s\right] + (t-s) \\
&= (t-s) \\
&\vdots \\
\mathbb{E}[W_t^2 \mid \mathcal{F}_s] &= \mathbb{E}[W_s^2 \mid \mathcal{F}_s] + (t-s) \\
&= W_s^2 + (t-s)
\end{aligned}
$$

where the expectation of the first integral is 0 by property of the Itô integral and the expecation of W_s^2 given $\mathcal{F}_s$ is just W_s^2 because it is known and since $\mathbb{E}[W_3^2 \mid W_1 = 5] = \mathbb{E}[W_2^2 \mid \mathcal{F}_1]$ when $W_1 = 5$ is observed, all we need is to plug in $t = 3$, $s = 1$ and $W_1 = 5$ into the above equation to get what we need

$$\mathbb{E}[W_3^2 \mid W_1 = 5] = 5^2 + (3-1) = 25 + 2 = 27$$

2.8 Lecture 28

Consider a Binomial model with step $h > 0$. We study a regime with prices changing very frequently, h is small. We are in the time interval of $[[\frac{s}{h}]h, ([\frac{s}{h}]+1)h]$ where s is a time in the interval. so the price at time t is

$$S_t = S_{[\frac{t}{h}]\cdot h} = S\left(\left[\frac{t}{h}\right]\right) = S\left(\left[\frac{t}{h} - 1\right]\right) \cdot \left(1 + R\left(\left[\frac{t}{h}\right]\right)\right)$$

continuing this backward computation we get

$$S_t = S_s \cdot e^{X_1 + X_2 + \cdots + X_n}$$

where $X_i = \ln\left(1 + R\left(\left[\frac{s}{h}\right] + i\right)\right)$ and $n = \left[\frac{t}{h}\right] - \left[\frac{s}{h}\right] \to \infty$ as $h \to 0$. The X_i are independent random variables with, lets say, mean μ_h and variance σ_h^2.

so by the central limit theorem:

$$X_1 + X_2 + \cdots + X_n \sim n\mu_h + \sqrt{n\sigma_h^2} \cdot N(0,1)$$

when n is large (so when h is small). So our stock price becomes

$$\begin{aligned} S_t &= S_s \cdot e^{X_1+X_2+\cdots+X_n} \\ &= S_s \cdot e^{n\mu_h+\sqrt{n\sigma_h^2}\cdot N(0,1)} \end{aligned}$$

It is natural to define $\mu_h = \mu \cdot h$ and $\sigma_h^2 = \sigma^2 \cdot h$ in which case

$$\begin{aligned} S_t &= S_s \cdot e^{nh\mu+\sqrt{nh\sigma^2}\cdot N(0,1)} \\ &\overset{h\to 0}{=} S_s \cdot e^{\mu(t-s)+\sigma\sqrt{t-s}\cdot N(0,1)} \\ &= S_s \cdot e^{\mu(t-s)+\sigma(W_t-W_s)} \end{aligned}$$

for $t \geq s$ where W_t is a Brownian motion, which then reduces to:

Definition 2.8.1: Black Scholes Model

$$S_t = S_0 \cdot e^{\mu t+\sigma W_t}$$

for all $t \geq 0$

under the binomail model we worked with a risk-nuetral probability (artificial, not the real-world probability) under which the stock price discounted to time 0 was martingale.

$$\begin{aligned} \frac{S(n)}{(1+r_h)^n} &= \frac{(1+U)S(n)p^* + (1+D)S(n)(1-p^*)}{(1+r_h)^{n+1}} \\ &= \frac{\mathbb{E}^{p^*}(S(n)(1+R(n+1)) \mid \mathcal{F}_n)}{(1+r_h)^{n+1}} \\ &= \mathbb{E}^{p^*}\left(\frac{S(n+1)}{(1+r_h)^{n+1}} \mid \mathcal{F}_n\right) \end{aligned}$$

We want to extend this to the Black-Scholes model, so that

$$e^{-rt}S_t = S_0 \cdot e^{(\mu-r)t+\sigma W_t} = f(t, W_t)$$

is martingale under some risk-neutral probability where in the above $f(x, y) = S_0 e^{(\mu-r})x + \sigma y$ We can easily check $f_x(x, y) = (\mu - r)f(x, y)$, $f_y(x, y) = \sigma f(x, y)$ and $f_{yy}(x, y) = \sigma^2 f(x, y)$ so by Itô's formula we have

$$\begin{aligned} e^{-rt}S_t &= S_0 + \int_0^t f_x(s, W_s)ds + \int_0^t f_y(s, W_s)dW_s + \frac{1}{2}\int_0^t f_{yy}(s, W_s)ds \\ &= S_0 + \int_0^t (\frac{\sigma^2}{2} + \mu - r)f(s, W_s)ds + \sigma\int_0^t f(s, W_s)dW_s \\ &= S_0 + \sigma\int_0^t S_s dW_s + \int_0^t (\frac{\sigma^2}{2} + \mu - r)S_s ds \end{aligned}$$

the first two termas are martingale as it is a constant plus an Itô integral which are martingale. The last term is not a martingale because of the ds term, so we must choose the risk-nuetral probability such that this term disappears. We can rewrite the above as

$$\begin{aligned} e^{-rt}S_t &= S_0 + \sigma \int_0^t S_s \cdot d\left(W_s + \left(\frac{\sigma}{2} + \frac{\mu - r}{\sigma}\right) s\right) \\ &= S_0 + \sigma \int_0^t S_s \cdot dW_s^* \end{aligned}$$

where $W_s^* = W_s + \left(\frac{\sigma}{2} + \frac{\mu-r}{\sigma}\right) s$, while this is NOT a Brownian motion under the real world probability we can take an "artificial" risk nuetral probability under which it is a Brownian motion. Then under the risk-nuetral probability we have

$$e^{-rt}S_t = S_0 + \sigma \int_0^t S_s \cdot dW_s^*$$

which is a martingale as we wanted.