LAX - MILGRAN LEMMA

Let $(V, (\cdot,\cdot)_V)$ HILBERT

and $\|\cdot\|$ associated norm, given:

- a BILINEAR form

$$A : V \times V \to \mathbb{R}$$

CONTINUOUS : $\exists\, \gamma : A(u,v) \leq \gamma \|u\|_V \|v\|_V$

COERCIVE : $\exists\, \alpha_0 : A(u,u) \geq \alpha_0 \|u\|_V^2$

$\forall\, u, v \in V$

- $F : V \to \mathbb{R}$ LINEAR form CONTINUOUS

Then $\exists!\, u \in V$: $\boxed{A(u,v) = F(v) \quad \forall\, v \in V}$ WEAK PROBLEM

EXAMPLE POISSON

$$\begin{cases} A(u,v) = (u,v)_1 = \int_0^1 u' v' \, dx \\ F(v) = \int_0^1 f v \, dx \\ V = H_0^1(0,1) \end{cases}$$

PROVE A and F satisfy hypotesis of LM

A BILINEAR

$$A(u,v) = \int_0^1 u' v' \, dx$$

$$A(u+\bar{u}, v) = \int_0^1 u' v' \, dx + \int_0^1 \bar{u}' v' \, dx$$

$$A(u, v+\bar{v}) = \int_0^1 u' v' \, dx + \int_0 u' \bar{v}' \, dx$$

$$A(\alpha u, v) = \int_0^1 \alpha u' v' \, dx$$

$$= \int_0^1 u' \alpha v' \, dx = A(u, \alpha v)$$

$$= \alpha A(u,v) \quad \checkmark$$

CONTINUOUS

$$\exists \gamma : A(u,v) \leq \gamma \|u\| \|v\|$$

$$\int_0^1 u' v' \, dx \overset{?}{\leq} \gamma \left(\int_0^1 u'^2 \, dx\right)^{1/2} \left(\int_0^1 v'^2 \, dx\right)^{1/2}$$

$$= (u,v)_1$$

$$\underset{C.S.}{\leq} \|u\|_1 \|v\|_1 \quad \checkmark$$

COERCIVE

$$\exists\, \alpha_0 : \quad A(u,u) \geq \alpha_0 \|u\|^2$$

$$\int_0^1 u' u' \, dx \overset{?}{\geq} \alpha_0 \int_0^1 u'^2 \, dx$$

$$\geq \left[\left(\int_0^1 (u')^2 \, dx \right)^{1/2} \right]^2$$

$$= \|u\|_1^2 \qquad \checkmark$$

$F(v)$ LINEAR

$$F(av+w) = \int_0^1 f\,(av+w)\, dx$$

$$= \int_0^1 (a f v + f w)\, dx$$

$$= a\int_0^1 f v \, dx + \int_0^1 f w \, dx$$

$$= a\, F(v) + F(w) \qquad \checkmark$$

CONTINUOUS

$$\exists\, \gamma : \; F(v) \leq \gamma \|f\| \, \|v\|$$

$$\int_0^1 f v \, dx \overset{?}{\leq} \gamma \left(\int_0^1 f^2 \, dx \int_0^1 v^2 dx \right)^{1/2}$$

$$= (f, v)$$

$$\leq \|f\|_{L^2} \, \|v\|_{L^2} \qquad \checkmark$$

C.S.

THEO strong max. principle

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$, Ω is OPEN

to include boundaries

HARMONIC in Ω

(i) $\max\limits_{\overline{\Omega}} u = \max\limits_{\Omega} u$

(ii) if Ω is CONNECTED and

$\exists\, x_0 \in \Omega : u(x_0) = \max\limits_{\overline{\Omega}} u$

$\Rightarrow u$ is CONSTANT in Ω

(i) EXERCISE

SUPPOSE

$M = \max\limits_{\overline{\Omega}} u = u(x_0)$, $u(x) \leq M$

$x_0 \notin \partial\Omega$ $\qquad \forall x \in \Omega$

x_0 IS LOCAL MAXIMUM $\Rightarrow \dfrac{\partial^2 u(x_0)}{\partial x_i} \leq 0 \quad \forall i$

BUT HARMONIC! $\displaystyle\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0$ ABSURD

$\Rightarrow u$ IS CONSTANT or M must be on $\partial\Omega$

COROLLARY Uniqueness POISSON

Let $g \in C(\partial\Omega)$, $f \in C(\Omega)$

$\Rightarrow \exists$ AT MOST 1 SOL of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

PROOF exercise

suppose $\begin{cases} -\Delta \bar{u} = f & \text{in } \Omega \\ \bar{u} = g & \text{in } \partial\Omega \end{cases}$

$v = u - \bar{u}$

$\Rightarrow -\Delta v = -\Delta(u - \bar{u}) = -\Delta u + \Delta \bar{u}$
$= f - f = 0$ in Ω

AND ALSO

$u = g = \bar{u}$ in $\partial\Omega$ $\Rightarrow u - \bar{u} = 0$

v IS HARMONIC

using max. principle $\Rightarrow v \equiv 0$

$\Rightarrow u = \bar{u}$ $\square$

TRUNCATION ERROR

$$T_i := \mathcal{L} u(x_i) - \mathcal{L}_h u(x_i)$$

$$|T_i| \leq \frac{h^2}{12} \|a\|_e \|u^{IV}\|_e + \frac{h^2}{6} \|b\|_e \|u'''\|_e$$

$$\frac{-\left(a_i - \frac{h}{2} b_i\right) u_{i+1} + 2 a_i u_i - \left(a_i + \frac{h}{2} b_i\right) u_{i-1}}{h^2} = f_i$$

$$= \mathcal{L}_h u_i$$

PROOF exercise

$$T_i \simeq f(x_i) - \left[- a\, \delta_h^2 u(x_i) + b\, \delta_h u(x_i)\right]$$

$$h^2 \delta_h^2 u(x_i) = h^2 u''(x_i) + \frac{h^4}{24}\left(u^{IV}(\xi) + u^{IV}(\eta)\right)$$

$$\xi \in [x_i, x_i + h] \quad \eta \in [x_i - h, x_i]$$

$$\delta^2 u(x_i) = u''(x_i) + \frac{h^2}{24}\left(u^{IV}(\xi) + u^{IV}(\eta)\right)$$

$$h\, \delta_h u(x_i) = u\left(x_i + \frac{h}{2}\right) - u\left(x_i - \frac{h}{2}\right)$$

$$= \cancel{u(x_i)} + \frac{h}{2} u'(x_i) + \cancel{\frac{h^2}{4} u''(x_i)}$$

$$+ \frac{h^3}{12} u'''(\xi^+) +$$

$$- \cancel{u(x_i)} + \frac{h}{2} u'(x_i) - \cancel{\frac{h^2}{4} u''(x_i)} +$$

$$+ \frac{h^3}{12} u'''(\xi^-)$$

$$\delta_h u(x_i) = u'(x_i) + \frac{h^2}{12}\left(u'''(\xi^+) + u'''(\xi^-)\right)$$

$$\Rightarrow T_i \approx \cancel{f(x_i)} - \left[\cancel{-a u''(x_i)} - a\frac{h^2}{24}\left(u^{IV}(\xi) + u^{IV}(\eta)\right) \right.$$

$$\left. \cancel{+ b u'(x_i)} + b\frac{h^2}{12}\left(u'''(\xi^+) + u'''(\xi^-)\right) \right]$$

$$|T_i| \leq \|a\|_e \frac{h^2}{12}\|u^{IV}\|_e + \|b\|_e \frac{h^2}{6}\|u'''\|$$

$$O(h^2)$$

THEOREM

Let $b=0$, then

$$|u(x_i)-u_i| \leq c\,h^2 \,\|u^{IV}\|_e \qquad \forall i$$

PROOF exercise see LT for GENERAL CASE

$$e_i = u(x_i) - u_i$$

$$\mathcal{L}_h e_i = \mathcal{L}_h\, u(x_i) - \underbrace{\mathcal{L}_h\, u_i}_{f_i}$$

$$= \mathcal{L}_h\, u(x_i) - f_i = T_i$$

THANKS TO WHAT HAS BEEN PROVEN IN PREVIOUS PAGE

$$\leq c\,h^2\,\|u^{IV}\|_e \qquad \forall i$$

EXERCISE

$$\begin{cases} \alpha + \beta + \gamma = 0 \\ \alpha h_{i+1} - \gamma h_i = 0 \\ \alpha \dfrac{h_{i+1}^2}{2} + \gamma \dfrac{h_i^2}{2} = 1 \end{cases}$$

$$\begin{cases} \alpha = \dfrac{\gamma h_i}{h_{i+1}} \\ \dfrac{\gamma h_i}{\cancel{h_{i+1}}} \dfrac{h_{i+1}^{\cancel{2}}}{2} + \gamma \dfrac{h_i^2}{2} = 1 \end{cases}$$

$$\gamma \left(\frac{h_i h_{i+1} + h_i^2}{2} \right) = 1$$

$$\gamma = \frac{2}{h_i h_{i+1} + h_i^2} = \frac{2}{h_i (h_{i+1} + h_i)}$$

$$\alpha = \frac{2}{\cancel{h_i}(h_{i+1} + h_i)} \frac{\cancel{h_i}}{h_{i+1}}$$

$$= \frac{2}{h_{i+1}(h_{i+1} + h_i)}$$

$$\beta = -\alpha - \gamma = -\frac{2}{h_{i+1}(h_{i+1} + h_i)} - \frac{2}{h_i(h_{i+1} + h_i)}$$

$$= \frac{-2h_i - 2h_{i+1}}{h_i h_{i+1}(h_{i+1} + h_i)} = \frac{-2\cancel{(h_i + h_{i+1})}}{h_i h_{i+1} \cancel{(h_{i+1} + h_i)}}$$

LEMMA DMP

if V: $\mathcal{L}_h v_{ij} \leq 0$
$\forall (x_i, x_j)$ INTERNAL

$$\Rightarrow \max_{ij} v_{ij} = \max_{(x_i,x_j)\in\partial\Omega} v_{ij}$$

PROOF BY CONTRADDICTION

let $$v_{m,n} > \max_{(x_i,x_j)\in\partial\Omega} v_{ij}$$

$$\Rightarrow 0 \geq \mathcal{L}_h v_{m,n} = c_{m,n} v_{m,n} - \sum_{k,l} c_{kl} v_{kl}$$

$$v_{m,n} \leq \frac{1}{c_{m,n}} \sum_{k,l} c_{k,l} v_{k,l}$$

$$\leq \frac{1}{c_{m,n}} \left(\sum_{k} c_{k,l}\right) \max_{i,j} v_{ij}$$

$$\leq \max_{i,j} v_{i,j}$$

APPLY ARGUMENT UNTIL BOUNDARY IS REACHED

$$v_{m,n} \equiv \max_{i,j} v_{i,j}$$

LEMMA truncation error

IF $u \in C^{IV}(\Omega) \cap C_0(\Omega)$

$\Rightarrow$ TRUNCATION ERROR of 5 POINTS SCHEME

$$|T(x)| \leq \frac{h^2}{12}\left(M_{xxxx} + M_{yyyy}\right)$$

$$M_{xxxx} = \| u_{xxxx} \|_{C(\bar{\Omega})} \qquad M_{yyyy} = \| u_{yyyy} \|_{C(\bar{\Omega})}$$

$$T_{ij} = f_{ij} + (\delta_h^x)^2 u_{ij} + (\delta_h^y)^2 u_{ij}$$

$$h\delta_h^x u_{ij} \simeq u_{i+h/2,j} - u_{i-h/2,j}$$

$$= \cancel{u_{i,j}} + \frac{h}{2} u'_{i,j} + \cancel{\frac{h^2}{4} u''_{i,j}} + \frac{h^3}{12} u'''(\xi^+)$$

$$- \cancel{u_{i,j}} + \frac{h}{2} u'_{i,j} - \cancel{\frac{h^2}{4} u''_{i,j}} + \frac{h^3}{12} u'''(\xi^-)$$

MORE PRECISELY $= h\, u_x(i,j) + \frac{h^3}{12}\left(u_{xxx}(\xi^+) + u_{xxx}(\xi^-)\right)$

$$h^2 (\delta_h^x)^2 u_{ij} \simeq h^2 u_{xx}(i,j) + \frac{h^4}{24}\left(u_{xxxx}(\xi) + u_{xxxx}(\eta)\right)$$

SIMILARLY

$$h^2 (\delta_h^y)^2 u_{ij} \simeq h^2 u_{yy}(i,j) + \frac{h^4}{24}\left(u_{yyyy}(\xi) + u_{yyyy}(\eta)\right)$$

$$\Longrightarrow T_{i,j} \approx \cancel{f_{i,j}} + \cancel{u_{xx}(i,j)} + \frac{h^2}{24}\Big(u_{xxxx}(\xi) + u_{xxxx}(\eta)\Big)$$

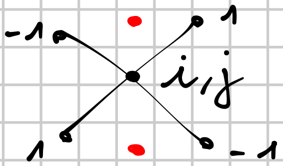
$$+ \cancel{u_{yy}(i,j)} + \frac{h^2}{24}\Big(u_{yyyy}(\xi) + u_{yyyy}(\eta)\Big)$$

$$|T_{ij}| \leq \frac{h^2}{12}\left(M_{xxxx} + M_{yyyy}\right)$$

$u_{xy} \longrightarrow \delta_{2h}^{y}\,\delta_{2h}^{x}\, u$

show results

STENCIL



$$u_{x\;i,j} \sim \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

FOR $i, j+1$

$$u_{x\;i,j+1} \sim \frac{u_{i+1,j+1} - u_{i-1,j+1}}{2h}$$

FOR $i, j-1$

$$u_{x\;i,j-1} \sim \frac{u_{i+1,j-1} - u_{i-1,j-1}}{2h}$$

$$u_{xy\;i,j} \approx \frac{u_{x\;i,j+1} - u_{x\;i,j-1}}{2h}$$

$$= \frac{1}{4h^2}\left(u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}\right)$$

√

PROVE $\|\cdot\|_1$ is a NORM

$$\|v\|_1 = \left[\int_\Omega |\nabla v|^2 \, dx\right]^{1/2}$$

- $\|v\|_1 \geq 0 \quad \forall v$

  BECAUSE $|\nabla v|^2 \geq 0$

- $\|v\|_1 = 0 \iff v = 0$

"<=" $v \equiv 0$ implies $\nabla v = 0$ and so, the norm is equal to 0

"=>" $\|v\| \leq C_\Omega \|v\|_1$

  $\hookrightarrow$ IS A NORM

  $\Rightarrow v = 0$

- $\|\lambda v\|_1 = \lambda \|v\|_1$

  FOR LINEARITY OF THE INTEGRAL

- $\|v + w\|_1 \leq \|v\|_1 + \|w\|_1$

$$\left[\int_\Omega |\nabla v + \nabla w|^2 \, dx\right]^{1/2} \leq \left[\int |\nabla v|^2 + |\nabla w|^2 \, dx\right]^{1/2}$$

PROVE A IS CONTINUOUS

$$\mathcal{A}(u,v) = \int_\Omega A \nabla u \nabla v - \int_\Omega u \bar{b} \cdot \nabla v + \int_\Omega c u v$$

with the three terms numbered 1, 2, 3.

$$\exists \gamma : \mathcal{A}(u,v) \leq \gamma \|u\| \|v\| \ ?$$

1) $$|1| \leq \left(\int_\Omega A|\nabla u|^2\right)^{1/2} \left(\int_\Omega A|\nabla v|^2\right)^{1/2}$$

CAUCHY-
SCHWARZ

SINCE $A < \infty$

$$\leq \|A\| \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

2) $$|2| \underset{\text{C.S.}}{\leq} \|u\|_{L^2(\Omega)} \|b\|_{L^\infty(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

3) $$|3| \underset{\text{C.S.}}{\leq} \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

SUMMING UP $\Longrightarrow$ CONTINUITY

LEMMA of Céa (CONVERGENCE)

under the assumptions of LAX-MILGRAM

$$\|u-u_N\|_V \le \frac{\gamma}{\alpha_0} \inf_{v_N \in V_N} \|u-v_N\|_V$$

$\gamma \to$ CONTINUITY

$\alpha_0 \to$ COERCIVITY

QUASI-OPTIMALITY

EXERCISE

[if A SYMMETRIC $\quad \|u-u_N\|_V \le \sqrt{\frac{\gamma}{\alpha_0}} \min \|u-v_N\|_V$]

OBS

$$\|u-u_N\|_A = \min_{v_N \in V_N} \|u-v_N\|_A$$

OPTIMALITY different norm

$$\alpha_0 \|u-u_N\|_V^2 \le A(u-u_N, u-u_N)$$

COERC

$$\alpha_0 \|u-u_N\|_V^2 \le A(u-u_N, u-u_N)$$

$$= \min_{v_N \in V_N} A(u-v_N, u-v_N)$$

CONTINUITY $\quad \le \gamma \min\limits_{v_N \in V_N} \|u-v_N\|_V^2$

$$\Longrightarrow \|u-u_N\|_V \le \sqrt{\frac{\gamma}{\alpha_0}} \min_{v_N \in V_N} \|u-v_N\|_V^2 \qquad \square$$

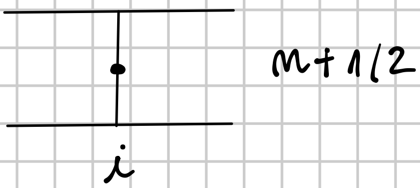
CONSISTENCY ASSUMPTION solution regular enough

Θ method

$$T_i^{n+1/2} = \delta_k^t u(t_{n+1/2}, x_i) - \Theta (\delta_h^x)^2 \underline{u(t_{n+1}, x_i)} + - (1-\Theta)(\delta_h^x)^2 \underline{u(t_n, x_i)}$$

$$|T_i^{n+1/2}| \leq \begin{cases} O(k, h^2) & \text{if } \Theta \neq 1/2 \\ O(k^2, h^2) & \text{if } \Theta = 1/2 \end{cases}$$

$\Theta = 1/2 \rightarrow$ symmetric one



EXERCISE

$$\underline{u(t_{n+1}, x_i)} \sim u(t_{n+\frac{1}{2}}, x_i) + \frac{k}{2} u_t(t_{n+\frac{1}{2}}, x_i) + + \frac{k^2}{8} u_{tt}(t_{n+\frac{1}{2}}, x_i) + \frac{k^3}{24} u_{ttt}(t_{n+\frac{1}{2}}, x_i)$$

$$\underline{u(t_n, x_i)} \sim u(t_{n+\frac{1}{2}}, x_i) - \frac{k}{2} u_t(t_{n+\frac{1}{2}}, x_i) + + \frac{k^2}{8} u_{tt}(t_{n+\frac{1}{2}}, x_i) - \frac{k^3}{24} u_{ttt}(t_{n+\frac{1}{2}}, x_i)$$

$$\delta_k^t u(t_{n+\frac{1}{2}}, x_i) = \frac{u(t_{n+1}, x_i) - u(t_n, x_i)}{k}$$

$$\sim \frac{\cancel{k}\, u_t(t_{n+\frac{1}{2}}, x_i) + (k^{\cancel{3}\,2}/24)\, u_{ttt}(t_{n+\frac{1}{2}}, x)}{\cancel{k}}$$

$$= u_t(t_{n+\frac{1}{2}}, x_i) + \frac{k^2}{24} u_{ttt}(t_{n+\frac{1}{2}}, x_i)$$

$$\left(\delta_h^x\right)^2 u\left(t_{n+1}, x_i\right) \sim u_{xx}\left(t_{n+1}, x_i\right) + \frac{h^2}{12} u_{xxxx}\left(t_{n+1}, x_i\right)$$

$$\sim u_{xx}\left(t_{n+\frac{1}{2}}, x_i\right) + \frac{k}{2} u_{xxt}\left(t_{n+\frac{1}{2}}, x_i\right) + \frac{k^2}{8} u_{xxtt}\left(t_{n+\frac{1}{2}}, x_i\right) + \frac{h^2}{12} u_{xxxx}\left(t_{n+\frac{1}{2}}, x_i\right) + \frac{h^2}{12} \cdot \frac{k}{2} u_{xxxxt}\left(t_{n+\frac{1}{2}}, x_i\right) + \frac{h^2}{12} \frac{k^2}{8} u_{xxxxtt}\left(t_{n+\frac{1}{2}}, x_i\right)$$

$$\left(\delta_h^x\right)^2 u\left(t_n, x_i\right) \sim u_{xx}\left(t_n, x_i\right) + \frac{h^2}{12} u_{xxxx}\left(t_n, x_i\right)$$

$$\sim u_{xx}\cancel{\left(t_{n+\frac{1}{2}}, x_i\right)} - \frac{k}{2} u_{xxt}\left(t_{n+\frac{1}{2}}, x_i\right) + \cancel{\frac{k^2}{8} u_{xxtt}\left(t_{n+\frac{1}{2}}, x_i\right)} + \frac{h^2}{12} \cancel{u_{xxxx}\left(t_{n+\frac{1}{2}}, x_i\right)} - \frac{h^2}{12} \cdot \frac{k}{2} u_{xxxxt}\left(t_{n+\frac{1}{2}}, x_i\right) + \cancel{\frac{h^2}{12} \frac{k^2}{8}} u_{xxxxtt}\left(t_{n+\frac{1}{2}}, x_i\right)$$

$$\Longrightarrow \Theta \, \boxed{} + \left(1 - \Theta\right) \boxed{} =$$

$$u_{xx}\cancel{\left(t_{n+\frac{1}{2}}, x_i\right)} + \frac{h^2}{12} u_{xxxx}\left(t_{n+\frac{1}{2}}, x_i\right) + \left(\Theta - \frac{1}{2}\right) k \left[u_{xxt}\left(t_{n+\frac{1}{2}}, x_i\right) + \frac{h^2}{12} u_{xxxx}\left(t_{n+\frac{1}{2}}, x_i\right)\right] + \frac{k^2}{8}\left[u_{xxtt}\left(t_{n+\frac{1}{2}}, x_i\right) + \frac{h^2}{12} u_{xxxxtt}\left(t_{n+\frac{1}{2}}, x_i\right)\right]$$

$$\Longrightarrow \left|T_i^{n+\frac{1}{2}}\right| = \left|\left(\frac{1}{2} - \Theta\right) k u_{xxt}\left(t_{n+\frac{1}{2}}, x_i\right) + - \frac{h^2}{12} u_{xxxx}\left(t_{n+\frac{1}{2}}, x_i\right) + + \frac{k^2}{24} u_{ttt}\left(t_{n+\frac{1}{2}}, x_i\right)\right| \qquad \square$$

STABILITY

Θ method in ℓ_∞ STABLE if

$$\mu(1-\theta) \le 1/2 \quad ★$$

PROOF

Θ method with $a=1$

$$-\mu\theta \cdot U_{i+1}^{n+1} + (1+2\mu\theta) U_i^{n+1} - \mu\theta U_{i-1}^{n+1} =$$

$$\mu(1-\theta) U_{i+1}^{n} + (1-2\mu(1-\theta)) U_i^{n} - \mu(1-\theta) U_{i-1}^{n}$$

$$(1+2\mu\theta)|U_i^{n+1}| \le \mu\theta|(U_{i+1}^{n+1} + U_{i-1}^{n+1})| +$$

$$+\mu(1-\theta)|(U_{i+1}^{n} + U_{i-1}^{n})| +$$

$$+(1-2\mu(1-\theta))|U_i^n| \quad ★ \text{ POSITIVE COEFFIC.}$$

$$|U_i^{n+1}| \le \frac{\mu\theta}{1+2\mu\theta}\left[|U_i^{n+1}| + |U_{i-1}^{n+1}|\right] +$$

$$+\frac{\mu(1-\theta)}{1+2\mu\theta}\left[|U_{i+1}^{n}| + |U_{i}^{n}{}_{-1}|\right] +$$

$$+\frac{1-2\mu(1-\theta)}{1+2\mu\theta}|U_i^n|$$

~~letting $\mathcal{U} = \max_{j,T\in(n,n+1)} U_j^T = \|U^T\|_\infty$~~

$$|U_i^{n+1}| \le \frac{2\mu\theta + \cancel{2\mu(1-\theta)} + 1 - 2\mu(1-\theta)}{\cancel{1+2\mu\theta}}\mathcal{U}$$

Italic U is ||U^n||_infinity

ITERATIVELY $|U_i^{n+1}| \le \|U^0\|_\infty$

EXERCISE

$$\lambda(j) = \frac{1-4\mu(1-\Theta)\,\text{sen}^2(jh/2)}{1+4\mu\Theta\,\text{sen}^2(jh/2)}$$

$$-\mu\Theta U_{i+1}^{n+1} + (1+2\mu\Theta)U_i^{n+1} - \mu\Theta U_{i-1}^{n+1}$$

$$=$$

$$\mu(1-\Theta)U_{i+1}^{n} + (1-2\mu(1-\Theta))U_i^{n} + \mu(1-\Theta)U_{i-1}^{n}$$

$$U_i^n = \lambda(j)^n e^{ij(ih)}$$

$$(1+2\mu\Theta)\cancel{\lambda}^{n+1}\cancel{e^{ij(ih)}} = \mu\Theta\cancel{\lambda}^{n+1}\left[\cancel{e^{ij(ih)}}e^{ijh} + \cancel{e^{ij(ih)}}e^{-ijh}\right] + \mu(1-\Theta)\cancel{\lambda}^{n}\left[\cancel{e^{ij(ih)}}e^{ijh} + \cancel{e^{ij(ih)}}e^{-ijh}\right] + (1-2\mu(1-\Theta))\cancel{\lambda}^{n}\cancel{e^{ij(ih)}}$$

$$e^{ijh} + e^{-ijh} = 2\cos(jh)$$

$$(1+2\mu\Theta)\lambda = 2\mu\Theta\lambda\cos(jh) + 2\mu(1-\Theta)\cos(jh) + (1-2\mu(1-\Theta))$$

$$\lambda\left[1+2\mu\Theta - 2\mu\Theta\cos(jh)\right] = 2\mu(1-\Theta)\cos(jh) + (1-2\mu(1-\Theta))$$

$$\lambda = \frac{1-2\mu(1-\Theta)+2\mu(1-\Theta)\cos(jh)}{1+2\mu\Theta-2\mu\Theta\cos(jh)}$$

$$\cos(jh) = 1 - 2\,\mathrm{sen}^2(jh/2)$$

N: $1 - 2\mu(1-\theta) + 2\mu(1-\theta)\left(1 - 2\,\mathrm{sen}^2(jh/2)\right)$

$$= 1 - 4\mu(1-\theta)\,\mathrm{sen}^2(jh/2)$$

D: $1 + 2\mu\theta\left(1 - 2\,\mathrm{sen}^2(jh/2)\right) = 1 + 4\mu\theta\,\mathrm{sen}^2(jh/2)$

$$\Rightarrow \lambda = \frac{1 - 4\mu(1-\theta)\,\mathrm{sen}^2(jh/2)}{1 + 4\mu\theta\,\mathrm{sen}^2(jh/2)}$$

- **WAVE EQ**

$$u_{tt} - u_{xx} = 0 \quad (W)$$

is equivalent to system
$$\begin{cases} u_t + v_x = 0 & (1) \\ u_x + v_t = 0 & (2) \end{cases}$$

PROOF (1)
$$u_{tt} + v_{xt} = 0$$
$$u_{tt} + (v_t)_x = 0$$
$$u_{tt} - u_{xx} = 0 \quad \text{using } (2)$$

similarly v has also to satisfy (W)

PROOF

(2)
$$u_{xt} + v_{tt} = 0$$
$$(u_t)_x + v_{tt} = 0$$
using (1)
$$-(v_x)_x + v_{tt} = 0$$
$$v_{tt} - v_{xx} = 0 \quad \checkmark$$

for (W) $u_{tt} - u_{xx} = 0$

$$\frac{1}{2}\int_{\mathbb{R}} (u_t)^2 + (u_x)^2 \, dx$$ is CONSERVED

<u>hint</u> test with u_t

PROOF

$$\int_{\mathbb{R}} u_{tt} \, u_t dx - \int_{\mathbb{R}} u_{xx} \, u_t dx = 0$$

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{R}} u_t^2 \, dx - \int_{\mathbb{R}} u_{xx} \, u_t \, dx = 0$$

↓ BY PART INTEGRATIONS

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{R}} u_t^2 \, dx + \int_{\mathbb{R}} u_x \, u_{xt} \, dx = 0$$

$$\frac{1}{2} \frac{\partial}{\partial t} \left[\int_{\mathbb{R}} u_t^2 + u_x^2 \, dx \right] = 0 \qquad \checkmark$$

CONSTANT

LAX - WENDROFF TRUNCATION ERROR

$$|T_i^k| \le \frac{k^2}{2} M_{ttt} + |a| \frac{h^2}{3} M_{xxx}$$

$$\underbrace{\frac{u_i^{n+1} - u_i^n}{k} + a_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2h} - \frac{k(a_i^n)^2}{2} \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}}_{=0}$$

$$u_i^{n+1} = u(t_n, x_i) + k\, u_t(t_n, x_i) + \frac{k^2}{2} u_{tt}(t_n, x_i) + \frac{k^3}{6} u_{ttt}(\tau, x_i), \quad \tau \in (t_n, t_{n+1})$$

$$u_{i\pm 1}^n = u(t_n, x_i) \pm h\, u_x(t_n, x_i) + \frac{h^2}{2} u_{xx}(t_n, x_i) \pm \frac{h^3}{6} u_{xxx}(t_n, \eta^\pm), \quad \eta^+ \in (x_i, x_{i+1}), \quad \eta^- \in (x_{i-1}, x_i)$$

$$\cancel{u_t(t_n, x_i)} + \frac{k}{2} \cancel{u_{tt}(t_n, x_i)} + \frac{k^2}{6} u_{ttt}(\tau, x_i)$$
$$+ \cancel{a u_x(t_n, x_i)} + a \frac{h^2}{12} \left[u_{xxx}(t_n, \eta^+) + u_{xxx}(t_n, \eta^-) \right]$$
$$- \frac{k}{2} \cancel{a^2 u_{xx}(t_n, x_i)} - \frac{k a^2 h}{12} \left[u_{xxx}(t_n, \eta^+) - u_{xxx}(t_n, \eta^-) \right]$$

$$u_t + a u_x = 0 \implies u_{tt} + a u_{xt} = 0$$
$$\implies u_{tt} = a^2 u_{xx}$$

$$\frac{k^2}{6} u_{ttt}(\tau, x_i) + \frac{ah}{12}(h - ak)\, u_{xxx}(t_n, \eta^+) + \frac{ah}{12}(h + ak)\, u_{xxx}(t_n, \eta^-)$$

NOTE on $h \pm ah$ if $|a|\nu \leq 1$

$$\Rightarrow |h \pm ak| \leq 2h$$

$$|T_i^n| \leq \frac{k^2}{6} M_{ttt} + \frac{|a|h^2}{3} M_{xxx}$$

LEAP FROG good for WAVE EQUATION

$$\begin{cases} u_{tt} - u_{xx} = 0 & (0,T] \times (0,1) \\ u(0,x) = u_0(x) & x \in (0,1) \\ u_t(0,x) = v_0(x) \\ u(t,0) = 0 = u(t,1) \end{cases}$$

CENTERED DIFF. in x & t

$$\frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{k^2} - \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} = 0$$

CFL / VON NEUMANN: $\nu \leq 1$

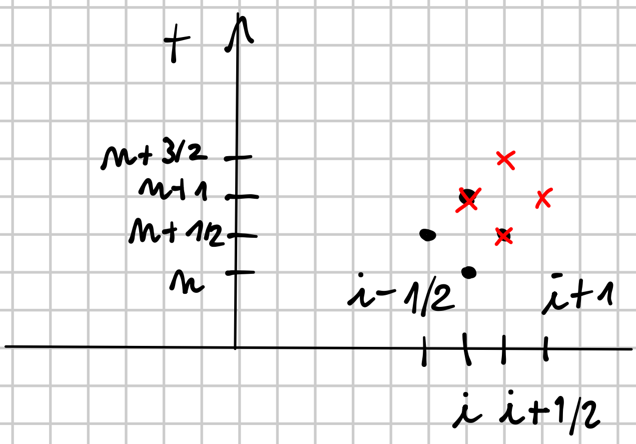
CONSISTENCY $O(h^2, k^2)$

ex

- write wave eq. as 1^{st} order system (u, v)
- discretize (U, V) with LEAP FROG
- SHOW that the RESULTING SCHEME for U is the LF METHOD ABOVE

[trick: USE STAGGER GRIDS

- • for U
- × for V



PROOF

- $$\begin{cases} u_t + v_x = 0 \\ u_x + v_t = 0 \end{cases}$$

- $$\frac{U_i^{n+1/2} - U_i^{n-1/2}}{k} + \frac{V_{i+1/2}^{n} - V_{i-1/2}^{n}}{h} = 0$$

$$\frac{U_{i+1}^{n+1/2} - U_i^{n+1/2}}{h} + \frac{V_{i+1/2}^{n+1} - V_{i+1/2}^{n}}{k} = 0$$

if $$U_i^{n+1/2} = \frac{u_i^{n+1} - u_i^{n}}{k} \qquad U_i^{n-1/2} = \frac{u_i^{n} - u_i^{n-1}}{k}$$

$$V_{i+1/2}^{n} = \frac{-(u_{i+1}^{n} - u_i^{n})}{h} \qquad V_{i-1/2}^{n} = + \frac{(u_i^{n} - u_{i-1}^{n})}{h}$$

$$\Longrightarrow \frac{u_i^{n+1} - u_i^{n}}{k^2} - \frac{u_i^{n} - u_i^{n-1}}{k^2} +$$

$$- \frac{u_{i+1}^{n} - u_i^{n}}{h^2} - \frac{u_i^{n} - u_{i-1}^{n}}{h^2} = 0$$

- $$\frac{u_i^{n+1} - 2u_i^{n} + u_i^{n-1}}{k^2} - \frac{u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{h^2} = 0$$