

LAX - MILGRAN LEMMA

Let $(V, (\cdot, \cdot)_V)$ HILBERT

and $\|\cdot\|$ associated norm, given:

- a BILINEAR form

$$A: V \times V \rightarrow \mathbb{R}$$

CONTINUOUS : $\exists \gamma : A(u, v) \leq \gamma \|u\|_V \|v\|_V$

COERCIVE : $\exists \alpha_0 : A(u, u) \geq \alpha_0 \|u\|_V^2$

$$\forall u, v \in V$$

- $F: V \rightarrow \mathbb{R}$ LINEAR form

CONTINUOUS

Then $\exists! u \in V : A(u, v) = F(v)$

WEAK PROBLEM

$$\boxed{\begin{aligned} A(u, v) &= F(v) \\ \forall v \in V \end{aligned}}$$

EXAMPLE POISSON

$$\begin{cases} A(u, v) = (u, v)_1 = \int_0^1 u' v' dx \\ F(v) = \int_0^1 g v dx \\ V = H_0^1(0, 1) \end{cases}$$

PROVE A and F satisfy hypothesis of LM

A BILINEAR

$$A(u, v) = \int_0^1 u' v' dx$$

$$A(u + \bar{u}, v) = \int_0^1 u' v' dx + \int_0^1 \bar{u}' v' dx$$

$$A(u, v + \bar{v}) = \int_0^1 u' v' dx + \int_0^1 u' \bar{v}' dx$$

$$\begin{aligned} A(\alpha u, v) &= \int_0^1 \alpha u' v' dx \\ &= \int_0^1 u' \alpha v' dx = A(u, \alpha v) \\ &= \alpha A(u, v) \quad \checkmark \end{aligned}$$

CONTINUOUS

$$\exists \gamma: A(u, v) \leq \gamma \|u\| \|v\|$$

$$\begin{aligned} \int_0^1 u' v' dx &\stackrel{?}{\leq} \gamma \left(\int_0^1 u'^2 dx \right)^{1/2} \left(\int_0^1 v'^2 dx \right)^{1/2} \\ &= (u, v)_1 \\ &\leq \|u\|_1 \|v\|_1 \quad \checkmark \\ &\text{C.S.} \end{aligned}$$

COERCIVE

$$\exists \alpha_0: A(u, u) \geq \alpha_0 \|u\|^2$$

$$\begin{aligned} \int_0^1 u' u' dx &\stackrel{?}{\geq} \alpha_0 \int_0^1 u'^2 dx \\ &\geq \left[\left(\int_0^1 (u')^2 dx \right)^{1/2} \right]^2 \\ &= \|u\|_1^2 \quad \checkmark \end{aligned}$$

F(r) LINEAR

$$\begin{aligned} F(ar + rw) &= \int_0^1 f(ar + rw) dx \\ &= \int_0^1 (af(r) + f(rw)) dx \\ &= a \int_0^1 f(r) dx + \int_0^1 f(rw) dx \\ &= a F(r) + F(rw) \quad \checkmark \end{aligned}$$

CONTINUOUS

$$\exists \gamma: F(r) \leq \gamma \|f\| \|r\|$$

$$\begin{aligned} \int_0^1 f(r) dx &\stackrel{?}{\leq} \gamma \left(\int_0^1 f^2 dx \right)^{1/2} \left(\int_0^1 r^2 dx \right)^{1/2} \\ &= (f, r) \\ &\leq \|f\|_{L^2} \|r\|_{L^2} \quad \checkmark \end{aligned}$$

C.S.

THEO strong max. principle

let $u \in C^2(\bar{\Omega}) \cap C(\bar{\Omega})$, Ω is OPEN
to include boundaries

HARMONIC in Ω

$$(i) \max_{\bar{\Omega}} u = \max_{\Omega} u$$

(ii) if Ω is CONNECTED and

$$\exists x_0 \in \Omega : u(x_0) = \max_{\bar{\Omega}} u$$

$\Rightarrow u$ is CONSTANT in Ω

(i) EXERCISE

SUPPOSE

$$M = \max_{\bar{\Omega}} u = u(x_0), \quad u(x) \leq M \\ x_0 \notin \partial\Omega \quad \forall x \in \Omega$$

x_0 is LOCAL MAXIMUM $\Rightarrow \frac{\partial^2 u}{\partial x_i^2}(x_0) \leq 0 \quad \forall i$

BUT HARMONIC! $\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = 0$ ABSURD

$\Rightarrow u$ is CONSTANT or M must be on $\partial\Omega$

COROLLARY Uniqueness POISSON

Let $g \in C(\partial\Omega)$, $f \in C(\Omega)$

$\Rightarrow \exists$ at most 1 sol of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

PROOF

Exercise

Suppose

$$\begin{cases} -\Delta \bar{u} = f & \text{in } \Omega \\ \bar{u} = g & \text{in } \partial\Omega \end{cases}$$

$$v = u - \bar{u}$$

$$\begin{aligned} \Rightarrow -\Delta v &= -\Delta(u - \bar{u}) = -\Delta u + \Delta \bar{u} \\ &= f - f = 0 \quad \text{in } \Omega \end{aligned}$$

AND ALSO

$$u = g = \bar{u} \quad \text{in } \partial\Omega \Rightarrow u - \bar{u} = 0$$

v IS HARMONIC

using max. principle $\Rightarrow v \equiv 0$

$$\Rightarrow u = \bar{u} \quad \square$$

TRUNCATION ERROR

$$T_i := \mathcal{L}_h u(x_i) - \mathcal{L}_h u(x_i)$$

$$|T_i| \leq \frac{h^2}{12} \|a\|_e \|u''\|_e + \frac{h^2}{6} \|b\|_e \|u'''\|_e$$

$$\frac{-(a_i - \frac{h}{2} b_i) u_{i+1} + 2 a_i u_i - (a_i + \frac{h}{2} b_i) u_{i-1}}{h^2} = f_i \\ = \mathcal{L}_h u_i$$

PROOF exercise

$$T_i \approx f(x_i) - [a \delta_h^2 u(x_i) + b \delta_h u(x_i)]$$

$$h^2 \delta_h^2 u(x_i) = h^2 u''(x_i) + \frac{h^4}{24} (u''(\xi) + u''(m))$$

$$\xi \in [x_i, x_i + h] \quad m \in [x_i - h, x_i]$$

$$\delta_h^2 u(x_i) = u''(x_i) + \frac{h^2}{24} (u''(\xi) + u''(m))$$

$$h \delta_h u(x_i) = u(x_i + \frac{h}{2}) - u(x_i - \frac{h}{2})$$

$$= \cancel{u(x_i)} + \frac{h}{2} u'(x_i) + \cancel{\frac{h^2}{4} u''(x_i)}$$

$$+ \frac{h^3}{12} u'''(\xi) +$$

$$- \cancel{u(x_i)} + \frac{h}{2} u'(x_i) - \cancel{\frac{h^2}{4} u''(x_i)} +$$

$$+ \frac{h^3}{12} u'''(\xi)$$

$$\delta_h u(x_i) = u'(x_i) + \frac{h^2}{12} (u'''(\xi^+) + u'''(\xi^-))$$

$$\Rightarrow T_i \approx \cancel{f(x_i)} - \left[-a u''(x_i) - a \frac{h^2}{24} (u''(\xi) + u''(y)) \right. \\ \left. + b u'(x_i) + b \frac{h^2}{12} (u'''(\xi^+) + u'''(\xi^-)) \right]$$

$$|T_i| \leq \|a\|_e \frac{h^2}{12} \|u''\|_e + \|b\|_e \frac{h^2}{6} \|u'''\| \\ O(h^2)$$

THEOREM

Let $b = 0$, then

$$|\mu(x_i) - \mu_i| \leq c h^2 \|\mu'\|_e \quad \forall i$$

PROOF exercise we LR for GENERAL CASE

$$e_i = \mu(x_i) - \mu_i$$

$$\mathcal{L}_h e_i = \mathcal{L}_h \mu(x_i) - \underbrace{\mathcal{L}_h \mu_i}_{f_i}$$

$$= \mathcal{L}_h \mu(x_i) - f_i = T_i$$

THANKS TO WHAT HAS BEEN PROVEN IN PREVIOUS PAGE

$$\leq c h^2 \|\mu'\|_e \quad \underline{\forall i}$$

EXERCISE

$$\begin{cases} \alpha + \beta + \gamma = 0 \\ \alpha h_{i+1} - \gamma h_i = 0 \\ \alpha \frac{h_{i+1}^2}{2} + \gamma \frac{h_i^2}{2} = 1 \end{cases}$$

$$\alpha = \frac{\gamma h_i}{h_{i+1}}$$

$$\frac{\gamma h_i}{h_{i+1}} \frac{h_{i+1}^2}{2} + \gamma \frac{h_i^2}{2} = 1$$

$$\gamma \left(\frac{h_i h_{i+1} + h_i^2}{2} \right) = 1$$

$$\gamma = \frac{2}{h_i h_{i+1} + h_i^2} = \frac{2}{h_i(h_{i+1} + h_i)}$$

$$\alpha = \frac{2}{\cancel{h_i}(h_{i+1} + h_i)} \frac{\cancel{h_i}}{h_{i+1}}$$

$$= \frac{2}{h_{i+1}(h_{i+1} + h_i)}$$

$$\beta = -\alpha - \gamma = -\frac{2}{h_{i+1}(h_{i+1} + h_i)} - \frac{2}{h_i(h_{i+1} + h_i)}$$

$$= \frac{-2h_i - 2h_{i+1}}{h_i h_{i+1}(h_{i+1} + h_i)} = -\frac{2(\cancel{h_i} + \cancel{h_{i+1}})}{\cancel{h_i} \cancel{h_{i+1}}(\cancel{h_{i+1}} + h_i)}$$

LEMMA DMP

if $V: \mathcal{L}_h \nabla_{ij} \leq 0$

$\forall (x_i, x_j) \text{ INTERNAL}$

$$\Rightarrow \max_{i,j} \nabla_{ij} = \max_{(x_i, x_j) \in \partial \Omega} \nabla_{ij}$$

PROOF By CONTRADICTION

let $\nabla_{m,m} > \max_{(x_i, x_j) \in \partial \Omega} \nabla_{ij}$

$$\Rightarrow 0 \geq \mathcal{L}_h \nabla_{m,m} = c_{m,m} \nabla_{m,m} - \sum_{k,e} c_{k,e} \nabla_{k,e}$$

$$\nabla_{m,m} \leq \frac{1}{c_{m,m}} \sum_{k,e} c_{k,e} \nabla_{k,e}$$

$$\leq \frac{1}{c_{m,m}} \left(\sum_k c_{k,e} \right) \max_{i,j} \nabla_{ij}$$

$$\leq \max_{i,j} \nabla_{i,j}$$

APPLY ARGUMENT UNTIL BOUNDARY IS REACHED

$$\nabla_{m,m} = \max_{i,j} \nabla_{i,j}$$

VERPA truncation error

IF $u \in C^N(\Omega) \cap C_0(\Omega)$

\Rightarrow TRUNCATION ERROR of 5 POINTS SCHEME

$$|T(x)| \leq \frac{h^2}{12} (M_{xxxx} + M_{yyyy})$$

$$M_{xxxx} = \|u_{xxxx}\|_{C(\bar{\Omega})}$$

$$M_{yyyy} = \|u_{yyyy}\|_{C(\bar{\Omega})}$$

$$T_{ij} = f_{ij} + (\delta_h^x)^2 u_{ij} + (\delta_h^y)^2 u_{ij}$$

$$h \delta_h^x u_{ij} \approx u_{i+h/2,j} - u_{i-h/2,j}$$

$$\begin{aligned} &= \cancel{u_{ij}} + \frac{h}{2} \cancel{u'_{ij}} + \cancel{\frac{h^2}{4} u''_{ij}} + \cancel{\frac{h^3}{12} u'''(\xi)} \\ &\quad - \cancel{u_{ij}} + \cancel{\frac{h}{2} u'_{ij}} - \cancel{\frac{h^2}{4} u''_{ij}} + \cancel{\frac{h^3}{12} u'''(\xi)} \end{aligned}$$

$$\text{MORE PRECISELY} = h u_x(i,j) + \frac{h^3}{12} (u_{xxxx}(\xi) + u_{xxxx}(\eta))$$

$$h^2 (\delta_h^x)^2 u_{ij} \approx h^2 u_{xx}(i,j) + \frac{h^4}{24} (u_{xxxx}(\xi) + u_{xxxx}(\eta))$$

SIMILARLY

$$h^2 (\delta_h^y)^2 u_{ij} \approx h^2 u_{yy}(i,j) + \frac{h^4}{24} (u_{yyyy}(\xi) + u_{yyyy}(\eta))$$

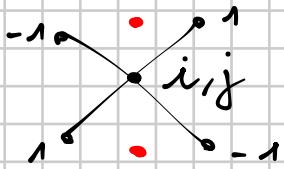
$$\Rightarrow T_{i,j} \approx f_{i,j} + \mu_{xx}(i,j) + \frac{h^2}{24} (\mu_{xxxx}(\xi) + \mu_{xxxx}(\eta)) \\ + \mu_{yy}(i,j) + \frac{h^2}{24} (\mu_{yyyy}(\xi) + \mu_{yyyy}(\eta))$$

$$|T_{ij}| \leq \frac{h^2}{12} (\mu_{xxxx} + \mu_{yyyy})$$

$$u_{x4} \rightarrow S_{2h}^4 S_{2h}^x u$$

show result

STENCIL



$$u_{x4} \sim \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

FOR $i, j+1$

$$u_{x4} \sim \frac{u_{i+1,j+1} - u_{i-1,j+1}}{2h}$$

FOR $i, j-1$

$$u_{x4} \sim \frac{u_{i+1,j-1} - u_{i-1,j-1}}{2h}$$

$$u_{x4} \sim \frac{u_{x4} \sim u_{i,j+1} - u_{x4} \sim u_{i,j-1}}{2h}$$

$$= \frac{1}{4h^2} (u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1})$$

✓

PROVE $\|\cdot\|_1$ is a NORM

$$\|v\|_1 = \left[\int_{\Omega} |\nabla v|^2 dx \right]^{1/2}$$

• $\|v\|_1 \geq 0 \quad \forall v$

BECUSE $|\nabla v|^2 \geq 0$

• $\|v\|_1 = 0 \iff v = 0$

$$\|v\| \leq c_{\Omega} \|v\|_1$$

\hookrightarrow IS A NORM

$$\Rightarrow v = 0$$

• $\|\lambda v\|_1 = |\lambda| \|v\|_1$

FOR LINEARITY OF THE INTEGRAL

• $\|v + w\|_1 \leq \|v\|_1 + \|w\|_1$

$$\left[\int_{\Omega} |\nabla v + \nabla w|^2 dx \right]^{1/2}$$

$$\leq \left[\int_{\Omega} |\nabla v|^2 + |\nabla w|^2 dx \right]^{1/2}$$

PROVE A IS CONTINUOUS

$$A(u, v) = \int_{\Omega} A \nabla u \cdot \nabla v - \int_{\Omega} u \bar{b} \cdot \nabla v + \int_{\Omega} c u v$$

$\exists \gamma: A(u, v) \leq \gamma \|u\| \|v\|?$

1) $|1| \leq \left(\int_{\Omega} A |\nabla u|^2 \right)^{1/2} \left(\int_{\Omega} A |\nabla v|^2 \right)^{1/2}$

CAUCHY-SCHWARZ

SINCE $A < 0$

$$\leq \|A\| \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

2) $|2| \leq \|u\|_{L^2(\Omega)} \|b\|_{L^\infty(\Omega)} \|\nabla v\|_{L^2(\Omega)}$
C.S.

3) $|3| \leq \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$
C.S.

SUMMING UP \Rightarrow CONTINUITY

LEMMA of Céa (CONVERGENCE)

under the assumptions of LAX-MILGRAN

$$\|\mu - \mu_N\|_V \leq \frac{\gamma}{\alpha_0} \inf_{\nu_N \in V_N} \|\mu - \nu_N\|_V$$

CONTINUITY
COERCIVITY

QUASI-OPIMALITY

EXERCISE

[if A symmetric] $\|\mu - \mu_N\|_V \leq \sqrt{\frac{\gamma}{\alpha_0}} \min \|\mu - \nu_N\|_V$

OBS

$$\|\mu - \mu_N\|_A = \min_{\nu_N \in V_N} \|\mu - \nu_N\|_A$$

OPTIMALITY
different norm

$$\alpha_0 \|\mu - \mu_N\|_V^2 \leq A(\mu - \mu_N, \mu - \mu_N)$$

↑
COERC

$$\alpha_0 \|\mu - \mu_N\|_V^2 \leq A(\mu - \mu_N, \mu - \mu_N)$$

$$= \min_{\nu_N \in V_N} A(\mu - \nu_N, \mu - \nu_N)$$

CONTINUITY $\leq \gamma \min_{\nu_N \in V_N} \|\mu - \nu_N\|_V^2$

$$\Rightarrow \|\mu - \mu_N\|_V \leq \sqrt{\frac{\gamma}{\alpha_0}} \min_{\nu_N \in V_N} \|\mu - \nu_N\|_V^2$$

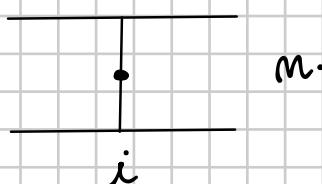
□

CONSISTENCY ASSUMPTION solution regular enough

Θ method

$$T_i^{m+1/2} = \delta_n^+ u(t_{m+1/2}, x_i) - \underline{\Theta} (\delta_n^x)^2 u(t_{m+1}, x_i) + \\ - (1 - \underline{\Theta}) (\delta_n^x)^2 \underline{u}(t_m, x_i)$$

$$|T_i^{m+1/2}| \leq \begin{cases} O(\kappa, h^2) & \text{if } \underline{\Theta} \neq 1/2 \\ O(\kappa^2, h^2) & \text{if } \underline{\Theta} = 1/2 \end{cases}$$



↓
symmetric
cone

EXERCISE

$$\underline{u}(t_{m+1}, x_i) \sim u(t_{m+\frac{1}{2}}, x_i) + \frac{h}{2} u_+(t_{m+\frac{1}{2}}, x_i) +$$

$$+ \frac{h^2}{8} u_{++}(t_{m+\frac{1}{2}}, x_i) + \frac{h^3}{24} u_{+++}(t_{m+\frac{1}{2}}, x_i)$$

$$\underline{u}(t_m, x_i) \sim u(t_{m+\frac{1}{2}}, x_i) - \frac{h}{2} u_+(t_{m+\frac{1}{2}}, x_i) +$$

$$+ \frac{h^2}{8} u_{++}(t_{m+\frac{1}{2}}, x_i) - \frac{h^3}{24} u_{+++}(t_{m+\frac{1}{2}}, x_i)$$

$$\delta_n^+ u(t_{m+\frac{1}{2}}, x_i) = \frac{u(t_{m+1}, x_i) - u(t_m, x_i)}{h}$$

$$\sim \frac{h}{2} u_+(t_{m+\frac{1}{2}}, x_i) + (h^2/24) u_{+++}(t_{m+\frac{1}{2}}, x_i)$$

$$= u_+(t_{m+\frac{1}{2}}, x_i) + \frac{h^2}{24} u_{++}(t_{m+\frac{1}{2}}, x_i)$$

$$(\delta_h^x)^2 u(t_{m+1}, x_i) \sim u_{xx}(t_{m+1}, x_i) + \\ + \frac{h^2}{12} u_{xxxx}(t_{m+1}, x_i)$$

$$\sim u_{xx}(t_{m+\frac{1}{2}}, x_i) + \frac{h}{2} u_{xx+}(t_{m+\frac{1}{2}}, x_i) + \\ + \frac{h^2}{8} u_{xx++}(t_{m+\frac{1}{2}}, x_i) + \frac{h^2}{12} u_{xxxxx}(t_{m+\frac{1}{2}}, x_i) + \\ + \frac{h^2}{12} \cdot \frac{h}{2} u_{xxxxx+}(t_{m+\frac{1}{2}}, x_i) + \frac{h^2}{12} \frac{h^2}{8} u_{xxxxx++}(t_{m+\frac{1}{2}}, x_i)$$

$$(\delta_h^x)^2 u(t_m, x_i) \sim u_{xx}(t_m, x_i) + \frac{h^2}{12} u_{xxxx}(t_m, x_i) \\ \sim u_{xx}(t_{m+\frac{1}{2}}, x_i) - \frac{h}{2} u_{xx+}(t_{m+\frac{1}{2}}, x_i) + \\ + \frac{h^2}{8} u_{xx++}(t_{m+\frac{1}{2}}, x_i) + \frac{h^2}{12} u_{xxxxx}(t_{m+\frac{1}{2}}, x_i) + \\ - \frac{h^2}{12} \cdot \frac{h}{2} u_{xxxxx+}(t_{m+\frac{1}{2}}, x_i) + \frac{h^2}{12} \frac{h^2}{8} u_{xxxxx++}(t_{m+\frac{1}{2}}, x_i)$$

$$\Rightarrow \Theta + (1-\Theta) =$$

$$u_{xx}(t_{m+\frac{1}{2}}, x_i) + \frac{h^2}{12} u_{xxxx}(t_{m+\frac{1}{2}}, x_i) + \\ + \left(\Theta - \frac{1}{2} \right) h \left[u_{xx+}(t_{m+\frac{1}{2}}, x_i) + \frac{h^2}{12} u_{xxxx}(t_{m+\frac{1}{2}}, x_i) \right] \\ + \frac{h^2}{8} \left[u_{xx++}(t_{m+\frac{1}{2}}, x_i) + \frac{h^2}{12} u_{xxxxx++}(t_{m+\frac{1}{2}}, x_i) \right]$$

$$\Rightarrow |T_i^{m+\frac{1}{2}}| = \left| \left(\frac{1}{2} - \Theta \right) h u_{xx+}(t_{m+\frac{1}{2}}, x_i) + \right. \\ \left. - \frac{h^2}{12} u_{xxxx}(t_{m+\frac{1}{2}}, x_i) + \right. \\ \left. + \frac{h^2}{24} u_{xxxxx++}(t_{m+\frac{1}{2}}, x_i) \right| \quad \square$$

STABILITY

\ominus method in C_∞ STABLE if
 $\mu(1-\ominus) \leq 1/2$ *

PROOF

\ominus method with $\alpha=1$

$$\begin{aligned}
 & -\mu \ominus \cdot U_{i+1}^{m+1} + (1+2\mu\ominus) U_i^{m+1} - \mu \ominus U_{i-1}^{m+1} \\
 & \mu(1-\ominus) U_{i+1}^m + (1-2\mu(1-\ominus)) U_i^m - \mu(1-\ominus) U_{i-1}^m \\
 & (1+2\mu\ominus) |U_i^{m+1}| \leq \mu \ominus (|U_{i+1}^{m+1}| + |U_{i-1}^{m+1}|) + \\
 & + \mu(1-\ominus) (|U_{i+1}^m| + |U_{i-1}^m|) + \\
 & + (1-2\mu(1-\ominus)) |U_i^m| \quad * \text{ POSITIVE COEFFIC.} \\
 |U_i^{m+1}| & \leq \frac{\mu \ominus}{1+2\mu\ominus} \left[|U_{i+1}^{m+1}| + |U_{i-1}^{m+1}| \right] + \\
 & + \frac{\mu(1-\ominus)}{1+2\mu\ominus} \left[|U_{i+1}^m| + |U_{i-1}^m| \right] + \\
 & + \frac{1-2\mu(1-\ominus)}{1+2\mu\ominus} |U_i^m|
 \end{aligned}$$

Letting $M = \max_{j, T \in (n, m+1)} \|U_j^+\| = \|U^+\|_\infty$

$$|U_i^{m+1}| \leq \frac{2\mu\ominus + 2\mu(1-\ominus) + 1-2\mu(1-\ominus)M}{1+2\mu\ominus}$$

ITERASIVELY $|U_i^{m+1}| \leq \|U^0\|_\infty$

EXERCISE

$$\lambda(j) = \frac{1 - 4\mu(1-\Theta)}{1 + 4\mu\Theta} \frac{\sin^2(jh/2)}{\sin^2(jh/2)}$$

$$\underline{-\mu\Theta U_{i+1}^{n+1} + (1+2\mu\Theta)U_i^{n+1} - \mu\Theta U_{i-1}^{n+1}} = \underline{\mu(1-\Theta)U_{i+1}^n + (1-2\mu(1-\Theta))U_i^n + \mu(1-\Theta)U_{i-1}^n}$$

$$U_i^n = \lambda(j)^n e^{ij(jih)}$$

$$(1+2\mu\Theta)\lambda^{n+1} e^{ij(jih)} = \mu\Theta\lambda^{n+1} \left[e^{ij(jih)} e^{ijh} + \right.$$

$$+ e^{ij(jih)} e^{-ijh} \left. \right] + \mu(1-\Theta)\lambda^n \left[e^{ij(jih)} e^{ijh} + \right.$$

$$+ e^{ij(jih)} e^{-ijh} \left. \right] + (1-2\mu(1-\Theta))\lambda^n e^{ij(jih)}$$

$$e^{ijh} + e^{-ijh} = 2 \cos(jh)$$

$$(1+2\mu\Theta)\lambda = 2\mu\Theta\lambda \cos(jh) + 2\mu(1-\Theta)\cos(jh) +$$

$$+ (1-2\mu(1-\Theta))\lambda$$

$$\lambda \left[1 + 2\mu\Theta - 2\mu\Theta \cos(jh) \right] = 2\mu(1-\Theta)\cos(jh) +$$

$$+ (1-2\mu(1-\Theta))\lambda$$

$$\lambda = \frac{1 - 2\mu(1-\Theta) + 2\mu(1-\Theta)\cos(jh)}{1 + 2\mu\Theta - 2\mu\Theta \cos(jh)}$$

$$\cos(jh) = 1 - 2 \sin^2(jh/2)$$

$$\therefore 1 - 2\mu(1-\Theta) + 2\mu(1-\Theta)(1 - 2 \sin^2(jh/2))$$

$$= 1 - 4\mu(1-\Theta) \sin^2(jh/2)$$

$$\therefore 1 + 2\mu\Theta(1 - 2 \sin^2(jh/2)) = 1 + 4\mu\Theta \sin^2(jh/2)$$

$$\Rightarrow \lambda = \frac{1 - 4\mu(1-\Theta) \sin^2(jh/2)}{1 + 4\mu\Theta \sin^2(jh/2)}$$

WAVE EQ

$$M_{++} - M_{xx} = 0 \quad (w)$$

is equivalent to system

$$M_+ + N_x = 0 \quad (1)$$

$$M_x + N_+ = 0 \quad (2)$$

PROOF

$$① \quad M_+ + N_x = 0$$

$$M_{++} + (N_+)_x = 0$$

$$M_{++} - M_{xx} = 0 \quad \text{using } ②$$

similarly N has also to satisfy (w)

PROOF

$$② \quad N_x + N_{++} = 0$$

$$(N_+)_x + N_{++} = 0$$

$$\text{using } ①$$

$$-(N_x)_x + N_{++} = 0$$

$$N_{++} - N_{xx} = 0 \quad \checkmark$$

for (w) $\mu_{tt} - \mu_{xx} = 0$

$$\frac{1}{2} \int_R (\mu_t)^2 + (\mu_x)^2 dx \text{ is conserved}$$

hint test with μ_t

PROOF

$$\int_R \mu_{tt} \mu_t dx - \int_R \mu_{xx} \mu_t dx = 0$$

$$\frac{1}{2} \frac{\partial}{\partial t} \int_R \mu_t^2 dx - \int_R \mu_{xx} \mu_t dx = 0$$

↓ BY PART INTEGRATIONS

$$\frac{1}{2} \frac{\partial}{\partial t} \int_R \mu_t^2 dx + \int_R \mu_x \mu_{xt} dx = 0$$

$$\frac{1}{2} \frac{\partial}{\partial t} \left[\int_R \mu_t^2 + \mu_x^2 dx \right] = 0 \quad \checkmark$$

CONSTANT

LAX-WENDROFF TRUNCATION ERROR

$$|T_i^h| \leq \frac{h^2}{2} M_{TTT} |a| \frac{h^2}{3} M_{XXX}$$

$$\frac{u_i^{m+1} - u_i^m}{h} + a_i^m \frac{u_i^m - u_{i-1}^m}{2h} - \frac{h(a_i^m)^2}{2} \frac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{h^2}$$

$$u_i^{m+1} = u(t_m, x_i) + h u_+(t_m, x_i) + \frac{h^2}{2} M_{TT}(t_m, x_i) \\ + \frac{h^3}{6} M_{TTT}(t, x_i), \quad t \in (t_m, t_{m+1})$$

$$u_{i\pm 1}^m = u(t_m, x_i) \pm h u_x(t_m, x_i) + \frac{h^2}{2} M_{XX}(t_m, x_i) \\ \pm \frac{h^3}{6} M_{XXX}(t_m, \eta^\pm), \quad \eta^+ \in (x_i, x_{i+1}) \\ \eta^- \in (x_{i-1}, x_i)$$

~~$$u_+(t_m, x_i) + \frac{h}{2} M_{TT}(t_m, x_i) + \frac{h^2}{6} M_{TTT}(t, x_i) \\ + a u_x(t_m, x_i) + a \frac{h^2}{12} [M_{XXX}(t_m, \eta^+) + M_{XXX}(t_m, \eta^-)] \\ - \frac{h a^2}{2} M_{XX}(t_m, x_i) - \frac{h a^2}{12} h [M_{XXX}(t_m, \eta^+) - M_{XXX}(t_m, \eta^-)]$$~~

$$u_+ + a u_x = 0 \Rightarrow M_{TT} + a M_{XT} = 0 \\ \Rightarrow M_{TT} = -a M_{XX}$$

$$\frac{h^2}{6} M_{TT}(t, x_i) + \frac{a h}{12} (h - ah) M_{XXX}(t_m, \eta^+) + \\ + \frac{ah}{12} (h + ah) M_{XXX}(t_m, \eta^-)$$

$$\text{NORM on } h \pm \alpha h \quad \text{if } |\alpha| \sqrt{h} \leq 1$$
$$\Rightarrow |h \pm \alpha h| \leq 2h$$

$$|\bar{T}_i^n| \leq \frac{\kappa^2}{6} M_{++} + \frac{|\alpha| h^2}{3} M_{XXX}$$

LEAP FROG good for WAVE EQUATION

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0, x) = u_0(x) \\ u_t(0, x) = v_0(x) \\ u(t, 0) = 0 = u(t, 1) \end{cases}$$

$(0, T] \times (0, 1)$
 $x \in (0, 1)$

CENTERED DIFF. in x & t

$$\frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{h^2} - \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} = 0$$

CFL / VON NEUMANN: $\sqrt{\lambda} \leq 1$

CONSISTENCY $O(h^2, h^2)$

ex

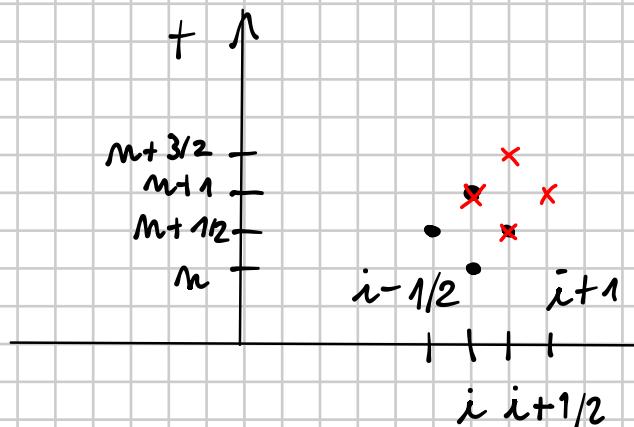
• write wave eq. as 1st order system (u, v)

• discretize (U, V) with LEAP FROG

• SHOW that the resulting scheme for U in the LF METHOD ABOVE

[trick: USE STAGGERED GRIDS

• for U
x for V



PROOF

$$\begin{aligned}
 & \left\{ \begin{array}{l} M_i + N_i = 0 \\ M_i + N_{i+1} = 0 \end{array} \right. \\
 & \bullet \frac{U_i^{m+1/2} - U_i^{m-1/2}}{h} + \frac{V_{i+1/2}^m - V_{i-1/2}^m}{h} = 0 \\
 & \quad \frac{U_{i+1}^{m+1/2} - U_i^{m+1/2}}{h} + \frac{V_{i+1/2}^{m+1} - V_{i+1/2}^m}{h} = 0 \\
 & \text{if } U_i^{m+1/2} = \frac{M_i^{m+1} - M_i^m}{h} \quad U_i^{m-1/2} = \frac{M_i^m - M_i^{m-1}}{h} \\
 & \quad V_{i+1/2}^m = - \frac{(M_{i+1}^m - M_i^m)}{h} \quad V_{i-1/2}^m = + \frac{(M_i^m - M_{i-1}^m)}{h} \\
 & \Rightarrow \frac{M_i^{m+1} - M_i^m}{h^2} - \frac{M_i^m - M_i^{m-1}}{h^2} + \\
 & \quad - \frac{M_{i+1}^m - M_i^m}{h^2} - \frac{M_i^m - M_{i-1}^m}{h^2} = 0 \\
 & \bullet \frac{M_i^{m+1}}{h^2} - 2 \frac{M_i^m}{h^2} + \frac{M_i^{m-1}}{h^2} - \frac{M_{i+1}^m}{h^2} + 2 \frac{M_i^m}{h^2} + \frac{M_{i-1}^m}{h^2} = 0
 \end{aligned}$$