Theoretical Guide
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1 Counting Problems

1.1 Burnside's Lemma

Let G be a group that acts on a set X. The Burnside Lemma states that the number of distinct orbits is equal to the average number of points fixed by an element of G.

$$T = \frac{1}{|G|} \sum_{g \in G} |\texttt{fix}(g)|$$

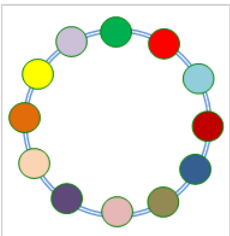
Where a orbit $\texttt{orb}(x)$ is defined as

$$\texttt{orb}(x) = \{y \in X : \exists g \in G \ gx = y\}$$

and $\texttt{fix}(g)$ is the set of elements in X fixed by g

$$\texttt{fix}(g) = \{x \in X : gx = x\}$$

Example: With k distinct types of beads how many distinct necklaces of size n can be made? Considering that two necklaces are equal if the rotation of one gives the other.



$$T = \frac{1}{n+1} \sum_{i=0}^{n} k^{gcd(i,n)} \qquad T = \frac{1}{n} \sum_{i=0}^{n-1} k^{gcd(i,n)}$$

2 Bitwise

Turn on bit i x & (1 << i)
Turn off bit i x & (~(1 << i))

2.1 XOR from 1 to N

$$f(n) = \begin{cases} n & n \equiv 0 \pmod 4 \\ 1 & n \equiv 1 \pmod 4 \\ n+1 & n \equiv 2 \pmod 4 \\ 0 & n \equiv 3 \pmod 4 \end{cases}$$

3 C++

```
template<class T> using min_priority_queue = priority_queue<T, vecto
```

```
string(1, 'a')
```

3.1 Pragma optimize

```
#pragma GCC optimize("Ofast")
#pragma GCC target("avx,avx2,fma")
```

3.2 Ordered set and multiset

```
typedef tree<pair<ll, ll>, null_type, less<pair<ll, ll>>, rb_tree_ta
```

To change to multiset switch equal to less_equal.

3.3 Optimized unordered map

```
mp.reserve(8192);
mp.max_load_factor(0.25);
```

3.4 Interactive Problems

```
freopen("input.txt", "r", stdin);
freopen("output.txt", "w", stdout);
```

4 Identities

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} \frac{1}{i} \approx \log n \qquad \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$$

5 Math

5.1 Trigonometry

5.2 Logarithm

$$\log_b mn = \log_b m + \log_b n \qquad log_b \frac{m}{n} = \log_b m - \log_n n \qquad \log_b n^p = p \log_b n$$

$$\log_b \sqrt[q]{n} = \frac{1}{q} \log_b n \qquad \log_b n = \log_a n \log_b a \qquad b^{\log_b k} = k$$

$$\log_b a = \frac{\log_c a}{\log_c b} \qquad \log_b a = \frac{1}{\log_a b} \qquad \log_b a \;\; \log_a c = \log_b c$$

$$\log_b 1 = 0 \qquad \log_b b = 1$$

6 Notes

- number of digits in $n!$

$$\log_b n! = \log_b(1 \times 2 \times 3 \times ... \times n) = \log_b 1 + \log_b 2 + \log_b 3 + ... + \log_b n$$

7 Geometry

Fórmulas de geometria plana

8 Constants

```
LLINF = 0x3f3f3f3f3f3f3f3fLL
```

```
MOD = 998'244'353
```

```
PI = acos(-1)
```

9 Number Theory

$$(a+b) \bmod m = (a \bmod m + b \bmod m) \bmod m$$
$$(a-b) \bmod m = (a \bmod m - b \bmod m) \bmod m$$
$$(a \times b) \bmod m = ((a \bmod m) \times (b \bmod m)) \bmod m$$
$$a^b \bmod m = (a \bmod m)^b \bmod m$$
$$a \equiv b \pmod m \iff (b-a)|m$$

$$\gcd(a_1, a_2, a_3, a_4) = \gcd(a_1, \gcd(a_2, \gcd(a_3, a_4)))$$
$$\operatorname{lcm}(a,b) \times \gcd(a,b) = a \times b$$
$$\operatorname{lcm}(a,b) = \frac{a \times b}{\gcd(a,b)} = \frac{a}{\gcd(a,b)} \times b$$

9.1 Sum of digits of N written in base b

$$f(n,b) = \begin{cases} n & n < b \\ f\left(n, \left\lfloor \frac{n}{b} \right\rfloor + (n \mod b)\right) & n \geq b \end{cases}$$

9.2 Some Primes

999999937 1000000007 1000000009 1000000021 1000000033 $10^{18}-11$ $10^{18}+3$ $2305843009213693951 = 2^{61}-1$ $998244353 = 119 \times 2^{23}+1$ 10^6+3

9.3 Prime counting function - $\pi(x)$

Expected to have $\frac{x}{\log x}$ primes within $[1, x]$. The prime counting function is asymptotic to $\frac{x}{\log x}$, by the prime number theorem.

x	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\pi(x)$	4	25	168	1 229	9 592	78 498	664 579	5 761 455

9.4 Number of Divisors

The number of divisors of n is about $\sqrt[3]{n}$.

n	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

9.5 Large Prime Gaps

For numbers until 10^9 the largest gap is 400.
For numbers until 10^{18} the largest gap is 1500.

9.6 Fermat's Theorems

Let P be a prime number and a an integer, then:

$$a^p \equiv a \pmod p$$

$$a^{p-1} \equiv 1 \pmod p$$

Lemma: Let p be a prime number and a and b integers, then:

$$(a+b)^p \equiv a^p + b^p \pmod p$$

Lemma: Let p be a prime number and a an integer. The inverse of a modulo p is a^{p-2}:

$$a^{-1} \equiv a^{p-2} \pmod p$$

9.7 Diophantine Equations