

# Theoretical Guide

## meia noite eu te conto

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## 1 Counting Problems

### 1.1 Burnside's Lemma

Let  $G$  be a group that acts on a set  $X$ . The Burnside Lemma states that the number of distinct orbits is equal to the average number of points fixed by an element of  $G$ .

$$T = \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|$$

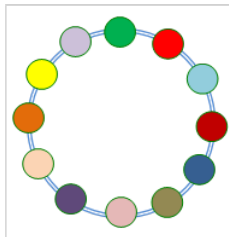
Where a orbit  $\text{orb}(x)$  is defined as

$$\text{orb}(x) = \{y \in X : \exists g \in G \text{ } gx = y\}$$

and  $\text{fix}(g)$  is the set of elements in  $X$  fixed by  $g$

$$\text{fix}(g) = \{x \in X : gx = x\}$$

**Example:** With  $k$  distinct types of beads how many distinct necklaces of size  $n$  can be made? Considering that two necklaces are equal if the rotation of one gives the other.



$$T = \frac{1}{n+1} \sum_{i=0}^n k^{\gcd(i,n)}$$

$$T = \frac{1}{n} \sum_{i=0}^{n-1} k^{\gcd(i,n)}$$

## 2 Bitwise

Turn on bit  $i$   $x \& (1 \ll i)$

Turn off bit  $i$   $x \& (\sim(1 \ll i))$

### 2.1 XOR from 1 to N

$$f(n) = \begin{cases} n & n \equiv 0 \pmod{4} \\ 1 & n \equiv 1 \pmod{4} \\ n+1 & n \equiv 2 \pmod{4} \\ 0 & n \equiv 3 \pmod{4} \end{cases}$$

## 3 C++

```
template<class T> using min_priority_queue = priority_queue<T, vector<T>, greater<T>>();
```

### 3.1 Pragma optimize

```
#pragma GCC optimize("Ofast")
#pragma GCC target("avx,avx2,fma")
```

### 3.2 Ordered set and multiset

```
typedef tree<pair<ll, ll>, null_type, less<pair<ll, ll>>, rb_tree_tag, tree_order_statistics_node_update> ordered_set;
```

To change to multiset switch equal to less\_equal.

### 3.3 Optimized unordered map

```
mp.reserve(8192);
mp.max_load_factor(0.25);
```

### 3.4 Interactive Problems

```
freopen("input.txt", "r", stdin);
freopen("output.txt", "w", stdout);
```

## 4 Identities

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^n \frac{1}{i} \approx \log n \quad \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$$

## 5 Math

### 5.1 Trigonometry

### 5.2 Logarithm

$$\log_b mn = \log_b m + \log_b n \quad \log_b \frac{m}{n} = \log_b m - \log_b n \quad \log_b n^p = p \log_b n$$

$$\log_b \sqrt[q]{n} = \frac{1}{q} \log_b n \quad \log_b n = \log_a n \log_b a \quad b^{\log_b k} = k$$

$$\log_b a = \frac{\log_c a}{\log_c b} \quad \log_b a = \frac{1}{\log_a b} \quad \log_b a \log_a c = \log_b c$$

$$\log_b 1 = 0 \quad \log_b b = 1$$

## 6 Notes

- number of digits in  $n!$

$$\log_b n! = \log_b(1 \times 2 \times 3 \times \dots \times n) = \log_b 1 + \log_b 2 + \log_b 3 + \dots + \log_b n$$

## 7 Constants

```
LLINF = 0x3f3f3f3f3f3f3f3fLL
```

```
MOD = 998'244'353
```

```
PI = acos(-1)
```

```
INT_MIN INT_MAX INT64_MIN INT64_MAX
```

### 7.1 Some Powers of Two

$2^0 \approx 10^0$	$2^1 \approx 10^0$	$2^2 \approx 10^0$	$2^3 \approx 10^0$	$2^4 \approx 10^1$	$2^5 \approx 10^1$
$2^6 \approx 10^1$	$2^7 \approx 10^2$	$2^8 \approx 10^2$	$2^9 \approx 10^2$	$2^{10} \approx 10^3$	$2^{11} \approx 10^3$
$2^{12} \approx 10^3$	$2^{13} \approx 10^3$	$2^{14} \approx 10^4$	$2^{15} \approx 10^4$	$2^{16} \approx 10^4$	$2^{17} \approx 10^5$
$2^{18} \approx 10^5$	$2^{19} \approx 10^5$	$2^{20} \approx 10^6$	$2^{21} \approx 10^6$	$2^{22} \approx 10^6$	$2^{23} \approx 10^6$
$2^{24} \approx 10^7$	$2^{25} \approx 10^7$	$2^{26} \approx 10^7$	$2^{27} \approx 10^8$	$2^{28} \approx 10^8$	$2^{29} \approx 10^8$
$2^{30} \approx 10^9$	$2^{31} \approx 10^9$	$2^{32} \approx 10^9$	$2^{33} \approx 10^9$	$2^{34} \approx 10^{10}$	$2^{35} \approx 10^{10}$
$2^{36} \approx 10^{10}$	$2^{37} \approx 10^{11}$	$2^{38} \approx 10^{11}$	$2^{39} \approx 10^{11}$	$2^{40} \approx 10^{12}$	$2^{41} \approx 10^{12}$
$2^{42} \approx 10^{12}$	$2^{43} \approx 10^{12}$	$2^{44} \approx 10^{13}$	$2^{45} \approx 10^{13}$	$2^{46} \approx 10^{13}$	$2^{47} \approx 10^{14}$
$2^{48} \approx 10^{14}$	$2^{49} \approx 10^{14}$	$2^{50} \approx 10^{15}$	$2^{51} \approx 10^{15}$	$2^{52} \approx 10^{15}$	$2^{53} \approx 10^{15}$
$2^{54} \approx 10^{16}$	$2^{55} \approx 10^{16}$	$2^{56} \approx 10^{16}$	$2^{57} \approx 10^{17}$	$2^{58} \approx 10^{17}$	$2^{59} \approx 10^{17}$
$2^{60} \approx 10^{18}$	$2^{61} \approx 10^{18}$	$2^{62} \approx 10^{18}$	$2^{63} \approx 10^{18}$	$2^{64} \approx 10^{19}$	$2^{65} \approx 10^{19}$
$2^{66} \approx 10^{19}$	$2^{67} \approx 10^{20}$	$2^{68} \approx 10^{20}$	$2^{69} \approx 10^{20}$	$2^{70} \approx 10^{21}$	$2^{71} \approx 10^{21}$

### 7.2 Some Factorials

$6! \approx 10^2$	$7! \approx 10^3$	$8! \approx 10^4$	$9! \approx 10^5$	$10! \approx 10^6$	$11! \approx 10^7$
$12! \approx 10^8$	$13! \approx 10^9$	$14! \approx 10^{10}$	$15! \approx 10^{12}$	$16! \approx 10^{13}$	$17! \approx 10^{14}$
$18! \approx 10^{15}$	$19! \approx 10^{17}$	$20! \approx 10^{18}$	$21! \approx 10^{19}$	$22! \approx 10^{21}$	$23! \approx 10^{22}$

## 8 Number Theory

$$(a + b) \bmod m = (a \bmod m + b \bmod m) \bmod m$$

$$(a - b) \bmod m = (a \bmod m - b \bmod m) \bmod m$$

$$(a \times b) \bmod m = ((a \bmod m) \times (b \bmod m)) \bmod m$$

$$a^b \bmod m = (a \bmod m)^b \bmod m$$

$$a \equiv b \pmod{m} \iff (b - a) | m$$

$$\gcd(a_1, a_2, a_3, a_4) = \gcd(a_1, \gcd(a_2, \gcd(a_3, a_4)))$$

$$\text{lcm}(a, b) \times \gcd(a, b) = a \times b$$

$$\text{lcm}(a, b) = \frac{a \times b}{\gcd(a, b)} = \frac{a}{\gcd(a, b)} \times b$$

### 8.1 Sum of digits of $N$ written in base $b$

$$f(n, b) = \begin{cases} n & n < b \\ f\left(n, \left\lfloor \frac{n}{b} \right\rfloor\right) + (n \bmod b) & n \geq b \end{cases}$$

### 8.2 Some Primes

$$\begin{array}{cccccc} 999999937 & 1000000007 & 1000000009 & 1000000021 & 1000000033 & 10^{18} - 11 \\ & 10^{18} + 3 & 2305843009213693951 & = 2^{61} - 1 & 998244353 & = 119 \times 2^{23} + 1 \\ & 10^6 + 3 & & & & \end{array}$$

### 8.3 Prime counting function - $\pi(x)$

Expected to have  $\frac{x}{\log x}$  primes within  $[1, x]$ . The prime counting function is asymptotic to  $\frac{x}{\log x}$ , by the prime number theorem.

x	10	10 <sup>2</sup>	10 <sup>3</sup>	10 <sup>4</sup>	10 <sup>5</sup>	10 <sup>6</sup>	10 <sup>7</sup>	10 <sup>8</sup>
$\pi(x)$	4	25	168	1 229	9 592	78 498	664 579	5 761 455

### 8.4 Number of Divisors

The number of divisors of  $n$  is about  $\sqrt[3]{n}$ .

$n$	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

### 8.5 Large Prime Gaps

For numbers until  $10^9$  the largest gap is 400.

For numbers until  $10^{18}$  the largest gap is 1500.

### 8.6 Fermat's Theorems

Let  $P$  be a prime number and  $a$  an integer, then:

$$a^p \equiv a \pmod{p}$$

$$a^{p-1} \equiv 1 \pmod{p}$$

**Lemma:** Let  $p$  be a prime number and  $a$  and  $b$  integers, then:

$$(a + b)^p \equiv a^p + b^p \pmod{p}$$

**Lemma:** Let  $p$  be a prime number and  $a$  an integer. The inverse of  $a$  modulo  $p$  is  $a^{p-2}$ :

$$a^{-1} \equiv a^{p-2} \pmod{p}$$

### 8.7 Diophantine Equations