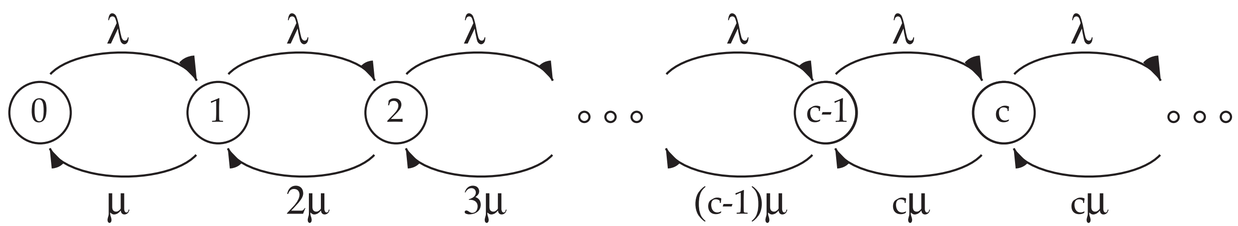
The $M/M/c$ queue

Again, $X(t)$ the number of individuals in the queueing system can be modeled as a birth & death process.

The transition state diagram for the $X(t)$ is:



Clearly, the critical thing here in terms of whether or not a steady state exists is whether or not $\lambda/(c\mu) < 1$.

Let $a = \lambda/\mu$ and $\rho = a/c = \lambda/(c\mu)$.

The balance equations for steady state are:

The $M/M/c$ queue: balance equations

$$\begin{aligned} p_1 &= a p_0 \\ p_2 &= \frac{a^2}{2\cdot 1} p_0 \\ p_3 &= \frac{a^3}{3!} p_0 \\ &\cdots \\ p_c &= \frac{a^c}{c!} p_0 \end{aligned}$$

$$\begin{aligned} p_{c+1} &= \rho \cdot \frac{a^c}{c!} p_0 \\ &\cdots \\ p_n &= \rho^{n-c} \cdot \frac{a^c}{c!} p_0 \qquad \text{for } n \geq c. \end{aligned}$$

In order to get an expression for p_0, we use the condition, that the overall sum of probabilities must be 1. This gives:

$$1 = \sum_{k=0}^{\infty} p_k = p_0 \left(\sum_{k=0}^{c-1} \frac{a^k}{k!} + \frac{a^c}{c!} \sum_{k=c}^{\infty} \rho^{k-c} \right) = p_0 \left(\underbrace{\sum_{k=0}^{c-1} \frac{a^k}{k!} + \frac{a^c}{c!} \frac{1}{1-\rho}}_{=:S} \right).$$

This system has a steady state, if $\rho < 1$; in that case,

$$p_0 = S^{-1}.$$

The other probabilities p_n are given as:

$$p_n = \begin{cases} \frac{a^n}{n!} p_0 & \text{for } 0 \leq n \leq c-1 \\ \frac{a^n}{c! c^{n-c}} p_0 & \text{for } n \geq c \end{cases}$$

A key descriptor for the system is the probability that an entering customer must queue for service - this is equal to the probability that all servers are busy.

The formula for this probability is known as Erlang's C formula or Erlang's delay formula and written as $C(c, a)$.

The $M/M/c$ queue: Erlang's C Formula

Obviously, in a $M/M/c$ queue, an entering individual must queue for service exactly when c or more individuals are already in the system.

$$\begin{aligned}\lim_{t\to\infty} P(X(t) \geq c) &= C(c,a) = \sum_{k=c}^{\infty} p_k = 1 - \sum_{k=0}^{c-1} p_k = \\ &= p_0\left(\frac{1}{p_0} - \sum_{k=0}^{c-1}\frac{a^k}{k!}\right) = \\ &= \underline{p_0\frac{a^c}{c!(1-\rho)}}.\end{aligned}$$

The $M/M/c$ queue: properties

The steady state mean number of individuals in the queue L_q is

$$\begin{aligned}
L_q &= \sum_{k=c}^{\infty}(k-c)p_k = \sum_{k=c}^{\infty}(k-c)\frac{a^k}{c!c^{k-c}}p_0 = \\
&= p_0\frac{a^c}{c!} \underbrace{\sum_{k=1}^{\infty} k\rho^k}_{\rho\left(\sum_{k=1}^{\infty}\rho^k\right)'} = p_0\frac{a^c}{c!}\frac{\rho}{(1-\rho)^2} \\
&= \underline{\frac{\rho}{1-\rho}C(c,a).}
\end{aligned}$$

By Little's Law, the mean waiting time in the queue W_q is

$$W_q = L_q/\lambda = \frac{1}{\lambda} \cdot \frac{\rho}{1-\rho} C(c,a) = \frac{1}{c\mu(1-\rho)} C(c,a).$$

Thus the overall time in system is then

$$W = W_q + W_s = W_q + \frac{1}{\mu},$$

and the overall number of individuals in the system is on average

$$L = W \cdot \lambda = a + \frac{\rho}{1-\rho} C(c,a).$$

Example

Bank: A bank has three tellers. Customers arrive at a rate of 1 per minute and stay in a single queue. Each teller needs on average 2 min to deal with a customer. What are the specifications of this queue?

For this queue, $\lambda = 1$, $\mu = 0.5$, $c = 3$, $a = \frac{\lambda}{\mu} = 2$, and $\rho = \frac{a}{c} = 2/3$

The probability that no customer is in the bank then is

$$p_0 = \left(\sum_{k=0}^{c-1} \frac{a^k}{k!} + \frac{a^c}{c!} \frac{1}{1-\rho} \right)^{-1} = \left(1 + 2 + \frac{4}{2} + \frac{2^3}{3!} \cdot \frac{1}{1-\rho} \right)^{-1} = \frac{1}{9}.$$

Thus length of the queue is $L_q = p_0 \cdot \frac{a^c}{c!} \cdot \frac{\rho}{(1-\rho)^2} = 8/9$

Calculate the waiting time in the queue: $W_q = L_q/\lambda = 8/9$ min.

Average service time is $W_s = \frac{1}{\mu} = 2$ minutes using the service time distribution.

This gives the total waiting time as $W = W_s + W_q = 26/9$ min.

Hence the average number of people in the bank is $L = W\lambda = 26/9$