# Poisson Process

Lecture: Weeks 7-9

# Why study Poisson process?


- The Poisson process is a counting process for the number of events that have occurred up to a particular time.
- Sometimes called a "jump" process because it jumps up to a higher state each time an event occurs.
- It is also a special case of a continuous Markov process.
- It has potential applications in insurance e.g.
  - total insurance claims consists usually of a sum of individual claim amounts.
  - the number of claims is usually assumed to occur according to a Poisson process.

# The exponential distribution

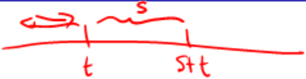
- The exponential distribution plays a very important role in Poisson process partly because the time between events or jumps follow an exponential distribution.
- Random variable $X$ is said to have an exponential distribution if density has the form

X ~ Exp(λ)

$$f_X(x) = \lambda e^{-\lambda x}, \quad \text{for } x \geq 0.$$

- Some important results:
  - Mean is $\mathrm{E}(X) = 1/\lambda$ and variance is $\mathrm{Var}(X) = 1/\lambda^2$.
  - Moment generating function: $M_X(t) = \mathrm{E}(e^{Xt}) = \dfrac{\lambda}{\lambda - t}$, $t < \lambda$.
  - Cumulative distribution function (cdf): $F_X(x) = 1 - e^{-\lambda x}$, $x \geq 0$. P(X≤x)
  - Survival function: $S_X(x) = e^{-\lambda x}$, $x \geq 0$. P(X>x)
  - (Constant) hazard function: $\mu(x) = \dfrac{f_X(x)}{1 - F_X(x)} = \lambda$. constant hazard

# The "no memory" property




- $X$ is said to have no memory, or memoryless, property if for all $s, t \geq 0$, we have

$$P(X > s + t | X > t) = P(X > s),$$

or equivalently,

$$P(X > s + t) = P(X > t)P(X > s).$$

- It is straightforward to show that the Exponential distribution satisfies the memoryless property.
- Indeed, it is the only distribution possessing this property.
- The Exponential distribution is often used for modeling lifelengths, but not a very realistic assumption for modeling human lifetime.

$$P(X > s+t \mid X > t) = \frac{P(X > s+t, X > t)}{P(X > t)}$$

$$= \frac{P(X > s+t)}{P(X > t)}$$

$$= \frac{e^{-\lambda(s+\cancel{t})}}{\cancel{e^{-\lambda t}}} = e^{-\lambda s}$$

$$= P(X > s)$$

$\Rightarrow$ memoryless

only distribution with this property.

# Other interesting properties of the exponential

Consider $n$ independent exponential random variables $X_1, X_2, \ldots, X_n$ with rate parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively.

- If all the rates are equal, $\lambda_i = \lambda$ for all $i = 1, 2, \ldots, n$, then the sum $S = X_1 + X_2 + \cdots + X_n$ has a $\text{Gamma}(n, \lambda)$ distribution with density

$$f_S(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}, \quad x \geq 0.$$

n integer

- The minimum $\min(X_1, X_2, \ldots, X_n)$ also has an exponential distribution with rate parameter

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

- The probability that among $X_1, X_2, \ldots, X_n$, $X_j$ is the smallest, is given by

$$P(X_j = \min(X_1, \ldots, X_n)) = \frac{\lambda_j}{\sum_{i=1}^{n} \lambda_i}.$$

Proof: $P(X > x) = P(\min(X_1, \dots, X_n) > x)$

$$= \underbrace{P(X_1 > x, X_2 > x \dots X_n > x)}$$

$$= \underbrace{P(X_1 > x)}_{e^{-\lambda_1 x}} \cdots \underbrace{P(X_n > x)}_{e^{-\lambda_n x}}$$

$$= e^{-(\lambda_1 + \dots + \lambda_n) x}$$

$$X \sim \underbrace{Exp(\lambda_1 + \dots + \lambda_n)}$$

$$E(X) = \frac{1}{\lambda_1 + \dots + \lambda_n}$$

**Proof:** $\underbrace{X_1 \sim Exp(\lambda_1) \quad X_2 \sim Exp(\lambda_2)}_{\text{independent}}$

$$P(X_1 = \min(X_1, X_2)) = \underbrace{P(X_1 < X_2)}$$

$$= \int_0^{\infty} \underbrace{P(X_1 < X_2 \mid X_1 = x)}_{P(X_2 > x)} f_{X_1}(x)\, dx$$

law of total probability

Remember

MLC

$$= \int_0^{\infty} e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x}\, dx$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2} \underbrace{\int_0^{\infty} (\lambda_1+\lambda_2) e^{-(\lambda_1+\lambda_2)x}\, dx}_{=1} = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

$$P(X_j = \min(X_1, \dots, X_n)) = P(\underbrace{X_j}_{\sim Exp(\lambda_j)} < \underbrace{\min(X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_n)}_{\sim Exp(\sum_{k \neq j} \lambda_k)})$$

$$= \frac{\lambda_j}{\lambda_j + \sum_{k \neq j} \lambda_k} = \frac{\lambda_j}{\sum_{k=1}^{n} \lambda_k}$$

# Convolution of exponentials

- Let $X_1, X_2, \ldots, X_n$ be $n$ independent exponentials with rate $\lambda_i$, respectively for $i = 1, 2, \ldots, n$.
- Suppose $\lambda_i \neq \lambda_j$ for $i \neq j$. Then the sum $S = X_1 + X_2 + \cdots + X_n$ is said to have a hyperexponential distribution with density

$$f_S(x) = \sum_{i=1}^{n} C_{i,n} \lambda_i e^{-\lambda_i x},$$

hyperexponential

where $C_{i,n} = \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}$.

Proof by induction

$n=2$ ✓

$n=3$ ✓

⋮

$n=k \Rightarrow n=k+1$

- In the case where $n = 2$, we have

$$\boxed{f_{X_1+X_2}(x) = \frac{\lambda_1}{\lambda_1 - \lambda_2} \lambda_2 e^{-\lambda_2 x} + \frac{\lambda_2}{\lambda_2 - \lambda_1} \lambda_1 e^{-\lambda_1 x}.}$$

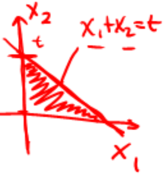
**Proof** $n=2$ $\quad X_1 + X_2 = S_2$

convolution formula:

$$f_{S_2}(t) = \int_0^t f_{X_1}(s)\, f_{X_2}(t-s)\, ds \quad \leftarrow \text{convolution formula.}$$

$$F_{S_2}(t) = P(X_1 + X_2 \le t)$$

$$= \int_0^t \int_0^{t-x_2} f_{X_1}(s)\, f_{X_2}(t)\, ds\, dt$$




$$\frac{dF_{S_2}(t)}{dt} = f_{S_2}(t) = \text{use Leibnitz rule}$$

$$f_{S_2}(t) = \int_0^t \lambda_1 e^{-\lambda_1 s} \lambda_2 e^{-\lambda_2 (t-s)} \, ds$$

$$= \lambda_1 \lambda_2 e^{-\lambda_2 t} \underbrace{\int_0^{t} e^{(\lambda_2 - \lambda_1) s} \, ds}$$

$$= \frac{\lambda_1 \lambda_2 e^{-\lambda_2 t}}{\lambda_2 - \lambda_1} \left[ e^{(\lambda_2 - \lambda_1) t} - 1 \right]$$

$$= \frac{\lambda_2}{\lambda_2 - \lambda_1} \lambda_1 e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1 - \lambda_2} \lambda_2 e^{-\lambda_2 t}$$

$$f_{S_3}(t) = f_{S_2 + X_3}(t)$$

$$f_{S_3}(t) = \int_0^t f_{S_2}(s) \underbrace{f_{X_3}(t-s)}_{\text{exponential}} \, ds$$

$$f_{S_3}(t) = \frac{\lambda_1}{\lambda_1-\lambda_3}\cdot\frac{\lambda_2}{\lambda_2-\lambda_3}\,\underline{\underline{\lambda_3 e^{-\lambda_3 t}}} +$$

$$\frac{\lambda_1}{\lambda_1-\lambda_2}\,\frac{\lambda_3}{\lambda_3-\lambda_2}\,\underline{\lambda_2} e^{-\lambda_2 t} +$$

$$\frac{\lambda_2}{\lambda_2-\lambda_1}\,\frac{\lambda_3}{\lambda_3-\lambda_1}\,\underline{\lambda_1} e^{-\lambda_1 t}$$

$\underline{\underline{\text{mgf}}}$ $X \sim Exp(\lambda)$ $M_X(t) = \lambda/\lambda-t = E(e^{Xt})$

$$M_{\underline{X_1+X_2}}(t) = \underbrace{M_{X_1}(t)}_{\frac{\lambda_1}{\lambda_1-t}}\cdot\underbrace{M_{X_2}(t)}_{\frac{\lambda_2}{\lambda_2-t}}$$

$$= \frac{\lambda_1}{\lambda_1-\lambda_2}\,\underbrace{\frac{\lambda_2}{\lambda_2-t}}_{Exp(\lambda_2)} + \frac{\lambda_2}{\lambda_2-\lambda_1}\,\underbrace{\frac{\lambda_1}{\lambda_1-t}}_{Exp(\lambda_1)}$$

$$M_{X_1+X_2+X_3}(t) = M_{X_1}(t)\,M_{X_2}(t)\,M_{X_3}(t) = \underbrace{\frac{\lambda_1}{\lambda_1-t}\,\frac{\lambda_2}{\lambda_2-t}\,\frac{\lambda_3}{\lambda_3-t}}$$

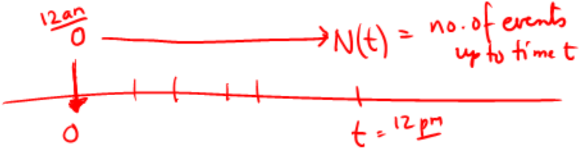
# Counting process

A stochastic process $\{N(t),\ t \geq 0\}$ is a counting process if it represents the number of events that occur up to time $t$.

To illustrate, consider:

- If $N(t)$ is the number of customers who have entered McDonald's for service at or prior to time $t$, then $\{N(t),\ t \geq 0\}$ is a counting process in which the event corresponds to the number of customers entering McDonald's.

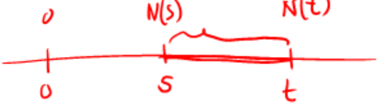
- However, if $N(t)$ represents the number of customers in McDonald's at time $t$, then $\{N(t),\ t \geq 0\}$ is not a counting process.



# - continued

$N(t)-N(s)$

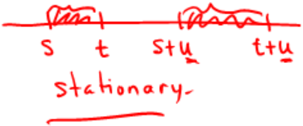
A counting process $\{N(t),\ t \geq 0\}$ must satisfy:

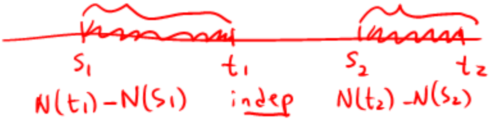



1. $N(t) \geq 0$.
2. $N(t)$ is integer-valued.
3. $N(s) \leq N(t)$ for any $s < t$, i.e. it must be non-decreasing.
4. For $s < t$, $N(t) - N(s)$ is the number of events that have occurred in the interval $(s, t]$.

# Independent and stationary increments

- A counting process $\{N(t),\ t \geq 0\}$ has independent increments if the number of events that occur between time $s$ and $t$, $N(t) - N(s)$, is independent of the number of events that occur up to time $s$.
    - In other words, the number of events that occur in disjoint time intervals are independent.
- A counting process $\{N(t),\ t \geq 0\}$ has stationary increments if the distribution of the number of events occurring in any interval depends only on the length of the interval.
    - In other words, $\{N(t),\ t \geq 0\}$ has stationary increments if $N(t_2 + s) - N(t_1 + s)$ has the same distribution as $N(t_2) - N(t_1)$ for all $t_1 < t_2$ and $s > 0$.



# Definition of a Poisson process

simpler

A counting process $\{N(t),\ t \geq 0\}$ is a Poisson process with rate $\lambda$, for $\lambda > 0$, if:

1. $N(0) = 0$; counting
2. it has independent increments; and
3. the number of events in any interval of length $t$ has a Poisson distribution with mean $\lambda t$. That is, for all $s, t \geq 0$,

Stationarity

$$P\left(N(t+s) - N(s) = n\right) = e^{-\lambda t}\frac{(\lambda t)^n}{n!},$$

for $n = 0, 1, 2, \ldots$

$N(t+s) - N(t) \sim \text{Poisson}(\lambda t)$

$\lambda$ rate of events per unit of time

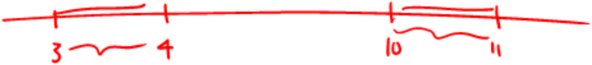
$\Rightarrow$ stationary

$\lambda t$

$s$ $s+t$

$t$

$N(4) - N(3)$ $\qquad$ $N(11) - N(10)$




$\sim \text{Poisson}(\lambda \cdot 1)$ $\qquad$ $\sim \text{Poisson}(\lambda \cdot 1)$

also <u>indep</u>

# Alternative definition

A counting process $\{N(t),\ t \geq 0\}$ is a Poisson process with rate $\lambda$, for $\lambda > 0$, if:

1 event


1. $N(0) = 0$;
2. it has independent increments;
3. it has stationary increments; and
4. it has unit jumps, that is,

"very small"

$$P\left(N(t+h) - N(t) = 1\right) = \lambda h + o(h)$$

and

"very small"

$$P\left(N(t+h) - N(t) \geq 2\right) = o(h).$$

Here, a function $f$ is said to be $o(h)$ ("little oh" of $h$) if

$$\lim_{h \to 0} f(h)/h = 0.$$

$o(h) \to$ very small for tiny $h$

Note: The two definitions of a Poisson process above are equivalent.

Review: $N(t) \sim$ Poisson process at $\lambda$

(1) $N(0) = 0$

(2) $\underbrace{N(t) - \overbrace{N(0)}^{=0}}_{\text{\# of events between } 0 \,\&\, t} \sim \text{Poisson}(\lambda \cdot t)$


$$P(N(t) = n) = e^{-\lambda t} (\lambda t)^n / n!$$

(3) $N(t) - N(s) \sim \text{Poisson}(\lambda \cdot (t-s))$


$$P(N(t) - N(s) = n) = e^{-\lambda(t-s)} (\lambda(t-s))^n / n!$$

(4) stationary

independent

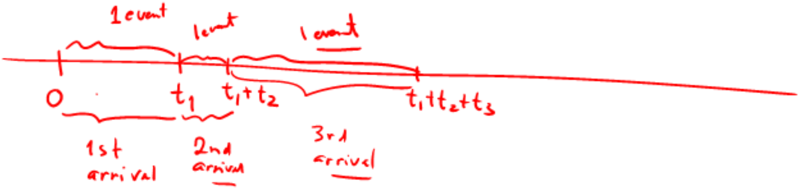
# Inter-arrival and waiting times

- Denote by $T_n$ the elapsed time between the $(n-1)$-th and $n$-th events. Thus, $T_1$ is the arrival time of the first event, $T_2$ is the arrival time of the next event from the time of arrival of the first event, etc.
- These are called more precisely inter-arrival times.
- Result: The inter-arrival times $T_1, T_2, \ldots$ are i.i.d. exponential random variables with mean $1/\lambda$.
  - Proof: in class.
- The arrival time, sometimes called waiting time, of the $n$-th event then is the sum of the first $n$ interarrival times:

$$S_n = X_1 + X_2 + \cdots + X_n$$

  - Clearly, $S_n$ has a Gamma distribution.

$\lambda = 4$ in an hour
each arrival will
every $\frac{1}{4}$ or 15 minutes




$T_1$ $\quad T_2$ $\quad T_3$ ... inter arrival times

① independent $T_1, T_2, \ldots$

② identically distributed $T_i \sim Exp(\lambda)$

$E(T_i) = 1/\lambda$

**Proof:** $T_1$ = 1st arrival time

$P(A|B) = P(A)$

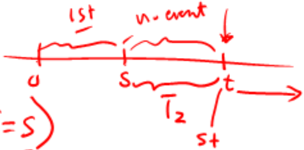
$$P(\underbrace{T_1 > t}) = P(N(t) - \underbrace{\cancel{N(0)}}_{=0} = 0)$$

$N(t) \sim \text{Poisson}(\lambda t)$

$$= e^{-\lambda t}$$


$$f_{T_1}(t) = \lambda e^{-\lambda t}, \quad t > 0 \implies T_1 \text{ is } \mathrm{Exp}(\lambda).$$

$T_2$ = 2nd arrival time




$$\underbrace{P(T_2 > t \mid T_1 = s)}_{P(T_2 > t)} = P(\underbrace{N(t+s) - N(s) = 0} \mid \underbrace{T_1 = s}_{N(s) - N(0) = 1})$$

$$= P(\underbrace{N(t+s) - N(s) = 0})$$

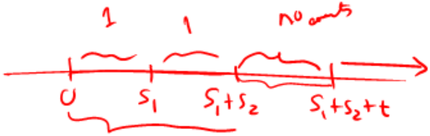
$$= e^{-\lambda t} \implies T_2 \text{ is } \mathrm{Exp}(\lambda)$$

$T_1$ & $T_2$ are independent!

$P(T_3 > t \mid \underbrace{T_1 = s_1, T_2 = s_2}) =$

etc.

etc.




$N(s_1+s_2) - N(0) = 2$

3 $\overset{\sim Exp(\lambda)}{T_1} + \overset{\sim Exp(\lambda)}{T_2} + \dots + T_n = S_n$ = arrival time

waiting time

time you wait for the $n^{th}$ arrival

$\boxed{S_n \sim Gamma(n, \lambda)} \longrightarrow$ at least density -

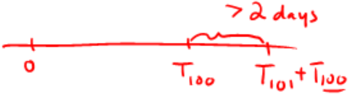
$P(S_n \le s)$ = not easy to evaluate -

approximated by a Normal -

# Example 5.13 variant

Suppose that people immigrate to a particular territory at a Poisson rate of $\lambda = 1.5$ per day.

1. What is the expected time until the 100-th immigrant arrives?
2. What is the probability that the elapsed time between the 100-th immigrant and the next immigrant's arrival exceeds 2 days?
3. What is the probability that the 100-th immigrant will arrive after one year? You may approximate this probability with a Normal distribution and you may assume there are 365 days in a year.




$P(T_{101} > 2) = ?$

$P(T_1 + T_2 + \dots + T_{100} > 365) = ?$


(a) $E(T_1 + \dots + T_{100}) = ?$

$T_1, \dots, T_{100} \sim$ iid Exp($\lambda = 1.5$/day)

(a) $E(T_1 + \dots + T_{100}) = \underbrace{E(T_1) + \qquad + E(T_{100})}_{100 * \frac{1}{1.5}} = \boxed{66.7 \text{ days}}$

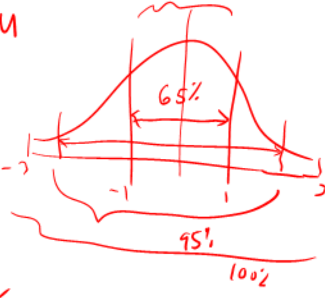
(b) $P(T_{101} > 2) = e^{-1.5(2)} = e^{-3} \approx .04979$

(c) $P(T_1 + \dots + T_{100} > 365)$ very very small

$T_1 + \dots + T_{100} \approx \text{Normal}\left(\frac{200}{3}, \frac{400}{9}\right)$

$Var(T_1 + \dots + T_{100}) = 100 * \frac{1}{1.5^2}$

$$\approx P\left(\underset{\substack{\downarrow \\ \text{standard normal}}}{Z} > \frac{365 - \frac{200}{3}}{\underbrace{\sqrt{\frac{400}{9}}}_{4.475}}\right) \approx 0$$



# Example 1

The time elapsed between the claims processed is modeled such that $T_k$ represents the time elapsed between processing the $(k-1)$-th and $k$-th claim where $T_1$ is the time until the first claim is processed, etc.

You are given:

- $T_1, T_2, \ldots$ are mutually independent; and
- The pdf of each $T_k$ is

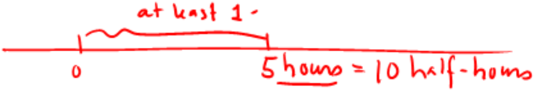
$$f(t) = 0.1 e^{-0.1t}, \quad \text{for } t > 0,$$

$\sim \text{Exp}(\lambda = 0.1)$

where $t$ is measured in half-hours.

Calculate the probability that at least one claim will be processed in the next 5 hours.

Number of claims processed as a Poisson process




$$1 - P(T_1 > 10) = 1 - e^{-0.1(10)} = 1 - e^{-1} \approx .6321$$

63% chance at least 1 claim processed

What is the probability that at least 3 claims processed within 5 hours?

$N(t)$ is a Poisson process at the rate of $\lambda = \frac{1}{10}$ per half-hour

$$N(10) \sim \text{Poisson}\left(10 \cdot \tfrac{1}{10}\right) = \text{Poisson}(1)$$

$$P(N(10) \geq 3) = 1 - P(N(10)=0) - P(N(10)=1) - P(N(10)=2)$$

$$= 1 - e^{-1} - e^{-1}\frac{1}{1!} - e^{-1}\frac{1^2}{2!} = 1 - \frac{5}{2}e^{-1} \approx \underline{\ ?\ }$$

# Example 2

Students for classes at the Math Science Building (MSB) can enter from one of two entrances: one from the ABC street and the other from the XYZ street. The flows of students arriving to the building from these two entrances are independent Poisson processes with rates $\lambda_1 = 0.5$ per minute and $\lambda_2 = 1.5$ per minute, respectively.

1. What is the probability that no students will enter the building during a fixed three-minute time interval?
2. What is the mean time between arrivals of students to the building?
3. What is the probability that a given student actually entered from XYZ street?


$N_{ABC}(t) \sim$ Poisson process $\lambda_1 = 1/2$ per minute

$N_{XYZ}(t) \sim$ Poisson process $\lambda_2 = 3/2$ per minute

$N_{ABC}(t) + N_{XYZ}(t) = N(t) \sim$ Poisson process at rate $\lambda = \frac{1}{2} + \frac{3}{2}$

$= 2$ per minute.

(1)


$P(T_1 > 3 \text{ minutes})$

$= e^{-6} \approx .0025$.

$T_1 \sim Exp(2)$

$E(T_1) = \frac{1}{2}$ minutes

(2) Mean arrival time is clearly

$$E(T_k) = \frac{1}{2} \text{ minute}.$$

(3) Let $T_{ABC}$ interarrival time entering from ABC.

$T_{XYZ}$ " " " " XYZ.

$P(T_{XYZ} < T_{ABC}) =$ prob that a student entered from XYZ.

# Sums of independent Poisson processes

$T_{XYZ} \sim Exp(3/2)$

$T_{ABC} \sim Exp(1/2)$

- Consider $\{N_1(t),\ t \geq 0\}$ and $\{N_2(t),\ t \geq 0\}$ which are two independent Poisson processes with rates $\lambda_1$ and $\lambda_2$ respectively.

$P(T_{XYZ} < T_{ABC}) = \dfrac{\lambda_{XYZ}}{\lambda_{XYZ} + \lambda_{ABC}} = \dfrac{3/2}{3/2 + 1/2} = 3/4 = .75$

- Result: $\{N(t),\ t \geq 0\}$ where

$$N(t) = N_1(t) + N_2(t),$$

is also a Poisson process with rate $\lambda_1 + \lambda_2$.

$N(t) = N_1(t) + N_2(t) + \ldots + N_m(t)$

$\lambda_1 \quad \lambda_2 \quad \lambda_m$

independent

$\sim$ Poisson process at rate $\lambda_1 + \ldots + \lambda_m$

10 questions

6 MD

4 MS

MC

# Example 3

At a subway station, eastbound trains and northbound trains arrive independently, both according to a Poisson process. On average, there is one eastbound train every 12 minutes and one northbound train every 8 minutes.

Proof:

Suppose you arrive at the subway station at a certain point in time and start observing trains.

very straightforward

→ add parameters.

1. What is the probability that exactly 2 eastbound trains will arrive in the first 24 minutes and exactly 3 northbound trains will arrive in the first 36 minutes?

e.g. $N_1(t)$ = # of McDonald's customers in Vernon

$N_2(t)$ in Manchester

In general:

$N(t) = N_1(t) + N_2(t) + \dots + N_m(t)$

$\lambda_1 \quad \lambda_2 \quad \lambda_m$

2. What is the expected waiting time, in minutes, until the first train (of either type) arrives?

$N(t)$ = Vernon + Manchester

e.g. $N_1(t)$ = # of claims processed in agency 1

$N_2(t)$ 2

~ Poisson process

3. What is the probability that it will take at least 20 minutes for 2 northbound trains to arrive?

$\lambda = \lambda_1 + \lambda_2 + \dots + \lambda_m$

100

+) $N_{100}(t)$

- independent -

$N(t)$ ~ Poisson process

$X \sim \text{Exp}(\alpha) \Rightarrow EX = 1/\alpha$

$N_e(t)$ eastbound trains arrive $\sim$ PP with $\lambda_e = \frac{1}{12}$

$N_n(t)$ northbound trains arrive $\sim$ PP " $\lambda_n = \frac{1}{8}$


① $\underbrace{P\left(N_e(24) = 2,\ N_n(36) = 3\right)}$

$$= P\left(\underbrace{N_e(24) = 2}_{\sim \text{Poisson}(2)}\right) P\left(\underbrace{N_n(36) = 3}_{\sim \text{Poisson}(4.5)}\right)$$

$$= e^{-2}\frac{2^2}{2!} \quad e^{-4.5}\frac{4.5^3}{3!} = 30.375\, e^{-6.5} \approx .04567$$

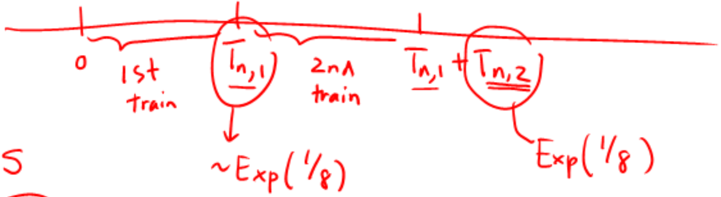
② $T_{e,1}$ = arrival time of 1st eastbound train $\sim \text{Exp}\left(\frac{1}{12}\right)$

$T_{n,1}$ = arrival " " " northbound train $\sim \text{Exp}\left(\frac{1}{8}\right)$

$$\min(T_{e,1}, T_{n,1}) \sim \text{Exp}\left(\frac{1}{12} + \frac{1}{8} = \frac{5}{24}\right)$$

$$E\left(\underbrace{\min(T_{e,1}, T_{n,1})}\right) = \frac{24}{5} = 4.8 \text{ minutes}$$

(3)




$$P(\overbrace{T_{n,1}+T_{n,2}}^{S} \geq 20)$$

$$T_{n,1}+T_{n,2} \sim Gamma(\overset{n}{2}, \overset{\lambda}{1/8})$$

$$f_S(x) = \lambda e^{-\lambda x}(\lambda x)^{n-1}/(n-1)!$$

$$= \frac{1}{8} e^{-x/8}\left(\frac{x}{8}\right)^1/1! = \overbrace{\frac{1}{64} x e^{-x/8}}, \quad \underline{x>0}$$

$$= \int_{20}^{\infty} \frac{1}{64} x e^{-x/8}\,dx \qquad \begin{matrix} u=x & dv=\frac{1}{8}e^{-x/8}dx \\ du=dx & v=-e^{-x/8} \end{matrix}$$

$$= \left[-\frac{1}{8} x e^{-x/8}\Big|_{20}^{\infty} + \underbrace{\int_{20}^{\infty}\frac{1}{8}e^{-x/8}\,dx}\right] = +\frac{20}{8}e^{-20/8} + e^{-20/8}$$

$$= \frac{7}{2}e^{-5/2}$$

# Thinning of Poisson processes

number of events - claims young. old Mcdonalds

- Consider a Poisson process $\{N(t),\ t \geq 0\}$ with rate $\lambda$.
- Suppose that each time an event occurs, it is classified as either:
  - a Type I event with probability $p$, or
  - a Type II event with probability $1-p$, independently of all other events.

$N(t) = N_1(t) + N_2(t)$

- Result: If $N_1(t)$ and $N_2(t)$ denote respectively the Type I and Type II events occurring in $[0,t]$, then:
  - $\{N_1(t),\ t \geq 0\}$ is a Poisson process with rate $\lambda p$;
  - $\{N_2(t),\ t \geq 0\}$ is a Poisson process with rate $\lambda(1-p)$; and
  - The two Poisson processes are also independent.
- This result extends to several types, say type $1, 2, \ldots, r$.

**Proof:** $N_1(t)$ $N_2(t)$ — joint probability

$$P(N_1(t)=n, N_2(t)=m)$$

$$= \underbrace{\sum_{k=0}^{\infty} \overbrace{P(N_1(t)=n, N_2(t)=m \mid N(t)=k)}\, P(N(t)=k)}_{\text{law of total probability}}$$

$N_1(t)+N_2(t)$

$$= \underbrace{P(N_1(t)=n, N_2(t)=m \mid N(t)=n+m)}\; \underbrace{P(N(t)=n+m)}$$

* all other probabilities are zero

$p$ $\quad$ $1-p$

$\sim \text{binomial}(n+m, p) = \text{binomial}(n+m, 1-p)$ $\qquad \boxed{e^{-\lambda t}\dfrac{(\lambda t)^{n+m}}{(n+m)!}}$

$$= \binom{n+m}{n} p^n (1-p)^m$$

$$\binom{n+m}{m} (1-p)^m p^n$$

$$= \frac{\cancel{(n+m)!}}{n!\, m!} p^n (1-p)^m e^{-\lambda t} \cancel{(\lambda t)^{n+m}} \Big/ \cancel{(n+m)!}$$

$$= \frac{(\lambda p t)^n}{n!} e^{-\lambda p t} \cdot \frac{(\lambda(1-p)t)^m}{m!} e^{-\lambda(1-p)t}$$

$$P(N_1(t)=n) = \sum_{m=0}^{\infty} P(N_1(t)=n,\ \underbrace{N_2(t)=m})$$ getting marginals

$$= \frac{(\lambda p t)^n e^{-\lambda p t}}{n!} \underbrace{\sum_{m=0}^{\infty} \frac{(\lambda(1-p)t)^m e^{-\lambda(1-p)t}}{m!}}_{=1}$$

$\sim \text{Poisson}(\lambda p)$

also, $N_2(t) \sim \text{Poisson}(\lambda(1-p)t)$

- independent

$$\underbrace{N_1(t)}_{\substack{\sim \text{P.P.} \\ \downarrow \\ \lambda p}} + \underbrace{N_2(t)}_{\substack{\sim \text{P.P.} \\ \downarrow \\ \lambda(1-p)}} = N(t) \sim \text{P.P.}$$

$\doteq$

# Example 4

$N(t)$ = no. of claims $\sim$ P.P. at rate $(\lambda = 9/\text{day})$

Consider an insurance company that has two types of policy: Policy A and Policy B.

Total claims from the company arrive according to a Poisson process at the rate of 9 per day.

2/3 for policy B -

A randomly selected claim has a 1/3 chance that it is of policy A.

1. Calculate the probability that claims from policy A will be fewer than 2 on a given day.
2. Calculate the probability that claims policy B will be fewer than 2 on a given day.
3. Calculate the probability that total claims from the company will be fewer than 2 on a given day.

$N_A(t)$ = number of claims of policy A $\sim$ P.P. at $\lambda \cdot p = 3/\text{day}$

+) $N_B(t)$ = " " " " B $\sim$ P.P. at $\lambda(1-p) = 6/\text{day}$

$N(t)$ = total number $\sim$ P.P. at $\lambda = 9/\text{day}$

① $P(N_A(1) < 2) = P(N_A(1)=0) + P(N_A(1)=1)$

$$e^{-3}\frac{3^0}{0!} + e^{-3}\frac{3^1}{1!} = 4e^{-3} \approx .19915$$

② $P(N_B(1)<2) = e^{-6}\frac{6^0}{0!} + e^{-6}\frac{6^1}{1!} = 7e^{-6} \approx .01735$

③ $P(N(1)<2) = e^{-9}\frac{9^0}{0!} + e^{-9}\frac{9^1}{1!} = 10e^{-9} \approx .0123$

# Example 4 - continued

1/3

A randomly selected claim from policy A has a 2/3 probability of being over \$10,000 while a randomly selected claim from policy B has probability 2/9 of being over \$10,000.

2/3

1. Determine the expected number of claims over \$10,000 on a given day.
2. Calculate the probability that on a given day, fewer than 2 claims will be over \$10,000.

Let $N^*(t)$ = no. of claims over \$10,000

$N_A^*(t)$ or $N_B^*(t)$

$N_A^*(t)$ = no. of claims of type A, over \$10,000

$\sim$ P.P. with rate $9 \times \underset{\substack{\checkmark \\ \text{policy} \\ \text{A}}}{\frac{1}{3}} \times \underset{\substack{\checkmark \\ \text{over} \\ 10{,}000}}{\frac{2}{3}} = \underline{2/\text{day}}$

$N_B^*(t)$ = no. of claims of type B, over \$10,000

+) $\sim$ P.P. with rate $9 \times \frac{2}{3} \times \frac{2}{9} = \underline{\frac{4}{3}/\text{day}}$

---

$N^*(t) \sim$ P.P. with rate $\left(2 + \frac{4}{3}\right)/\text{day} = \frac{10}{3}/\text{day}$

① $E(N^*(1)) = 10/3$

② $P(N^*(1) < 2) = \underbrace{P(N^*(1)=0)}_{e^{-10/3}\left(\frac{10}{3}\right)^0/0!} + \underbrace{P(N^*(1)=1)}_{e^{-10/3}\left(\frac{10}{3}\right)^1/1!} = \frac{13}{3}e^{-10/3} \approx .15459$

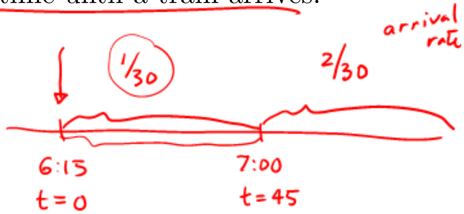
# Example 5

wait times are exponential -

Suppose that you arrive at a subway station at 6:15 AM:

- Until 7:00 AM, trains arrive at a Poisson rate of 1 train per 30 minutes.
- Starting at 7:00 AM, they arrive at a Poisson rate of 2 trains per 30 minutes.

Calculate your expected waiting time until a train arrives.



$$T_w = \text{wait time} = \begin{cases} \sim \text{Exp}(1/30), & T_w \le 45 \\ \sim \text{Exp}(1/15), & T_w > 45 \end{cases}$$

in minutes

$E(T_w)$ = average wait time


$$= \underbrace{E(\min(45, T_b))} + \underbrace{E(T_a)}_{15} * \underbrace{P(N(45)=0)}$$

$N(45) \sim$ Poisson at the rate of $1/30$

$$e^{-45/30}$$

$$\int_0^{45} t \, \frac{1}{30} e^{-t/30} \, dt + \int_{45}^{\infty} 45 \cdot \frac{1}{30} e^{-t/30} \, dt + 15 e^{-1.5}$$

$$= 30(1-e^{-1.5}) + 15e^{-1.5} = 30 - 15e^{-1.5} \approx 26.65$$

Alternative solution

$$T_w = \begin{cases} \mathrm{Exp}(1/30), & T_w \le 45 \\ \underline{\mathrm{Exp}(1/15)}, & T_w > 45 \end{cases}$$

$$E(T_w) = \underbrace{E(T_w \mid T_w \le 45)}\ \underbrace{P(\cancel{T_w} \le 45)} + \underbrace{E(T_w \mid T_w > 45)}\ \underbrace{P(T_w > 45)}$$

$$P(T_w \le 45) \to (1-e^{-1.5})$$

$$E(T_w \mid T_w \le 45) = \int t \cdot \underbrace{f_{T_w \mid T_w \le 45}(t)}\, dt$$

$$E(T_w \mid T_w > 45) \to \int t\, f_{T_w \mid T_w > 45}(t)\, dt$$

$$P(T_w > 45) \to P(N(45)=0) \to e^{-1.5}$$

$$\vdots$$

$$P(T_w \le t \mid T_w \le 45) = \frac{P(T_w \le t, T_w \le 45)}{P(T_w \le 45)} = \frac{P(T_w \le t)}{P(\cancel{T_w} \le 45)} = \frac{1-e^{-t/30}}{1-e^{-1.5}}$$

$$= \int_0^{45} t \cdot \underline{\tfrac{1}{30} e}^{-t/30}\, dt + \int_{45}^{\infty} t \cdot \tfrac{1}{15} e^{-t/15}\, dt$$

$$\underbrace{\int_0^{45} t\,\frac{1}{30}e^{-t/30}\,dt}_{\text{use by parts}} + \underbrace{\int_{45}^{\infty} t\,\frac{1}{15}e^{-t/15}\,dt}$$

$$= 30 - 75e^{-1.5} \quad + \quad 60e^{-1.5}$$

$$= 30 - 15e^{-1.5} \approx \underline{26.65 \text{ minutes}}$$

$$\underbrace{E(T_w \mid T_w > 45)}$$

$$\downarrow$$

$$\frac{1}{15} e^{-t/15} \Big/ e^{-1.5} \longleftarrow f_{T_w \mid T_w > 45}$$

$$\downarrow$$

$$\frac{P(T_w \le t \mid T_w > 45)}{P(T_w > 45)}$$

$$= \frac{P(T_w \le t, T_w > 45)}{P(T_w > 45)}$$

$$= \frac{P(45 < T_w \le t)}{P(T_w > 45)}$$

$$= \frac{\int_{45}^{t} \frac{1}{15} e^{-x/15}\, dx}{e^{-1.5}} = \frac{e^{-3} - e^{-t/15}}{e^{-1.5}}$$

# Generalizations to the non-homogeneous case

The counting process $\{N(t),\ t \geq 0\}$ is said to be a non-homogeneous Poisson process with intensity function $\lambda(t)$, for $t \geq 0$ if:

1. $N(0) = 0$;
2. it has independent increments; and
3. it has unit jumps, that is,

$$P\left(N(t+h) - N(t) = 1\right) = \lambda(t)h + o(h)$$

and

$$P\left(N(t+h) - N(t) \geq 2\right) = o(h).$$

Some remarks:

- In the non-homogeneous case, the rate parameter $\lambda(t)$ now depends on $t$.
- When $\lambda(t) = \lambda$, constant, then it reduces to the homogeneous case.

$N(t)$ is Poisson at rate $\lambda$

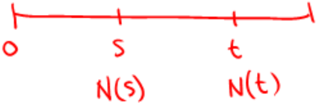
independent of time $t$

homogeneous Poisson.

$N(t)$ is Poisson process at the rate of $\lambda(t)$ intensity function

depends on time $t$

non-homogeneous Poisson process.




$N(t) - N(s) \sim$ what is this?

$m(t)$ = mean value function of a non-homogeneous Poisson process

$$= \int_0^{t} \lambda(y)\,dy$$

$N(t) - N(s) \sim$ Poisson distribution with mean $m(t) - m(s)$, $s < t$

$\lambda(t) = \lambda$

$N(t) = N(t) - \underbrace{N(0)}_{=0} \sim$ Poisson distributed with mean $m(t)$ =

# Some remarks

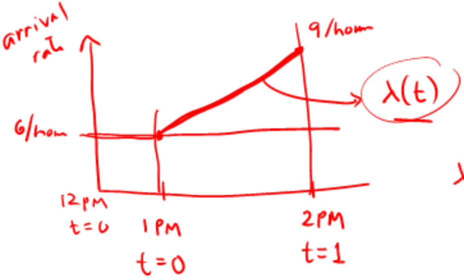
- The mean value function of a non-homogeneous Poisson process is defined by
$$m(t) = \int_0^t \lambda(y)dy.$$
- $N(s+t) - N(s)$ has a Poisson distribution with mean $m(s+t) - m(s)$.
- $N(t)$ is a Poisson random variable with mean $m(t)$.
- If $\{N(t),\ t \geq 0\}$ is a non-homogeneous with mean value function $m(t)$, then $\{N\left(m^{-1}(t)\right),\ t \geq 0\}$ is homogeneous with intensity $\lambda = 1$.
  - This result follows because $N(t)$ is Poisson random variable with mean $m(t)$ and if we let $X(t) = N\left(m^{-1}(t)\right)$, then $X(t)$ is Poisson with mean
$$m\left(m^{-1}(t)\right) = t.$$

# Example 6

Assume that the customers in a department store arrive at a Poisson rate that increases linearly from 6 per hour at 1 PM, to 9 per hour at 2 PM.

Calculate the probability that exactly 2 customers arrive between 1 PM and 2 PM.

in hours




linear

$$\lambda(t) = a + bt = 6 + 3t$$

$\lambda(0) = 6$

$\lambda(1) = 9$

$N(1) \sim$ Poisson with mean

$$m(1) = \int_0^1 (6+3y)\,dy = 15/2$$

$$P(N(1)=2) = e^{-15/2} \cdot \frac{\left(\frac{15}{2}\right)^2}{2!} \approx .016$$

Take $t=0$ at 12 noon

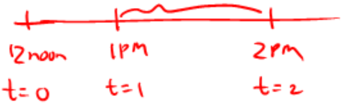
$\lambda(t) = a + bt = 3 + 3t$

$\lambda(1) = 6 \qquad a+b = 6$

$\lambda(2) = 9 \qquad a+2b = 9$

$b = 3$

$a = 3$



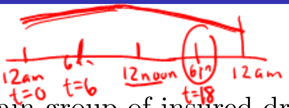

$N(2) - N(1) \sim$ Poisson distributed with mean

$$m(2) - m(1) = \int_1^2 (3+3y)\,dy = 15/2$$

$$\int_0^2 - \int_0^1$$

$$P(N(2)-N(1) = 2) = e^{-15/2} \left(\tfrac{15}{2}\right)^2 / 2!$$

# Example 7




An insurance company finds that for a certain group of insured drive~~s~~ rs, the number of accidents over each 24-hour period rises from midnight to noon, and then declines until the following midnight.

Suppose that the number of accidents can be modeled by a non-homogeneous Poisson process where the intensity at time $t$ is given by

$$\boxed{\lambda(t) = \frac{1}{6} - \frac{(12-t)^2}{1152}},$$

in hours

where $t$ is the number of hours since midnight.

24 hours

1. Calculate the expected number of daily accidents. $E[N(24)]$
2. Calculate the probability that there will be exactly one accident between 6:00 AM and 6:00 PM.

$P(N(18) - N(6) = 1) = ?$

$N(24) \sim$ Poisson distributed with mean

$$m(24) = \int_0^{24} \underbrace{\frac{1}{6} - \frac{(12-y)^2}{1152}}_{\lambda(y)} \, dy \qquad m(t) = \int_0^t \lambda(y)\,dy$$

$$= 3 \text{ accidents}$$

① $E(N(24)) = 3$ accidents

② $P(\underbrace{N(18) - N(6)}_{\sim \text{Poisson with mean } m(18)-m(6)} = 1)$ $\qquad \int_0^{18} - \int_0^{6}$

$$\underbrace{m(18) - m(6)}_{\int_6^{18} \frac{1}{6} - \frac{(12-y)^2}{1152} dy}$$

$$= 1.875$$

$$= e^{-1.875} \frac{1.875^1}{1!} \approx .2875$$