Higher order Partial Diffe. Equation.

General linear P.D.E of order n is

$$\left(K_0 \frac{\partial^n z}{\partial x^n} + K_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + K_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + K_n \frac{\partial^n z}{\partial y^n}\right) +$$

$$\cdots + \left[L_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + L_1 \frac{\partial^{n-1} z}{\partial x^{n-2}\partial y} + \cdots + L_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}}\right]$$

$$+ \cdots + \left(M_0 \frac{\partial z}{\partial x} + M_1 \frac{\partial z}{\partial x}\right) + N_0 z = f(x,y) \quad \text{--- (1)}$$

where $K_0, K_1, K_2 \sim K_n, L_0, L_1, L_2 - - L_{n-1}$

$\cdots M_0, M_1, N_0$ are functions of x and y.

Note (1) if $K_0, K_1, K_2 \cdot K_n$; $L_0, L_1, \cdots L_{n-1}$ M_0, M_1 ; N_0 are constants.

then eqn (1) is called linear Partial equation with con. cofficent of order n.

Note (2) If Partial derivatives of same order n occur in (1).

i.e. $\left[K_0 \frac{\partial^n z}{\partial x^n} + K_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + K_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + K_n \frac{\partial^n z}{\partial y^n}\right]$

$$= f(x,y) \quad \text{--- (2)}$$

then eqn (2) is called linear homogenous P.D.E of order n

Note (3) If $K_0, K_1, K_2 \cdots K_n$ are constant in (2) then ② is called linear Homogenous P.D.E. with constant cofficent of order n.

e.g $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} = e^{x+y}$ — linear Homogenous P.D.E with constant cofficent of ②.

for example then

First of all we study

$$K_0 \frac{\partial^n z}{\partial x^n} + K_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + K_n \frac{\partial^n z}{\partial y^n} = f(x,y)$$

It's genral solⁿ is

$$z = z_c + z_p$$

* Consider the linear Homogenous Partial D.E. with constant cofficent of order n with right side zero.

$$K_0 \frac{\partial^n z}{\partial x^n} + K_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + K_n \frac{\partial^n z}{\partial y^n} = 0$$

$$[K_0 D_x^n + K_1 D_x^{n-1} D_y + \cdots + K_n D_y^n] z = 0$$

$$D = D_x = \frac{\partial}{\partial x}$$

$$D' = D_y = \frac{\partial}{\partial y}$$

Scanned by CamScanner

134

x / x

Case I If $K_0 \neq 0$

then * can be written as

$$(D_x - \lambda_1 D_y)(D_x - \lambda_2 D_y) \cdots (D_x - \lambda_n D_y) Z = 0$$

Subcase I $\phi(D_x, D_y) Z = 0$

Put $D_x = \lambda$, $D_y = 1$

we have a polynomial of degree n in λ which is $\phi(\lambda) = 0$ (A.E).

Say $\lambda = \lambda_1, \lambda_2, \ldots \lambda_n$

the general solⁿ of * is

$$Z = \phi_1(y + \lambda_1 x) + \phi_2(y + \lambda_2 x) + \ldots + \phi_n(y + \lambda_n x)$$

where $\phi_1, \phi_2 \ldots \phi_n$ are arbitrary functio

Subcase II – when value of λ value are repeat

say $\lambda = \lambda_1, \lambda_1, \lambda_3, \lambda_4 \ldots \lambda_n$

then general soln is

$$Z = [\phi_1(y + \lambda_1 x) + x \cdot \phi_2(y + \lambda_2 x)] + \phi_3[y + \lambda_3 x] + \ldots + \phi_n(y + \lambda_n x)$$

where $\phi_1, \phi_2 \ldots \phi_n$ are arbitrary fn

or

$$Z = [\phi_1(y + \lambda_1 x) + y\,\phi_2(y + \lambda_1 x)] + \phi_3(y + \lambda_3 x) + \ldots + \phi_n(y + \lambda_n x)$$

Provided $\lambda_1 \neq 0$

Case II $K_0 = 0$

$(K_1 D_x^{n-1} D_y - -)Z = 0$

then D_y^r mus be the factor of $\phi(D_x, D_y)Z = 0$ -(*)
$(r \geq 1)$

$D_y^r (K_r D_x^{n-r} + K_{r+1} D_x^{n-r-1} D_y + \cdots + K_n D_y^{n-r}]Z = 0$

Solution corresponding to $D_y^r Z = 0$ is

$Z = \phi_1(x) + y\phi_2(x) + \cdots + y^{r-1}\phi_r(x)$ — (A)

where $\phi_1, \phi_2 \cdots \phi_r$ are arbitrary function.

Put $D_x = \lambda$, $D_y = 1$, in given equation (**)

we have soln of

$(K_r D_x^{n-r} + K_{r+1} D_x^{n-r-1} \cdot D_y + \cdots + K_n D_y^{n-r}) Z = 0$

$Z = \phi_{r+1}(y + \lambda_{r+1} x) + \phi_{r+2}(y + \lambda_{r+2} x) + \cdots + \phi_n(y + \lambda_n x$ — (B)

Combine (A) and (B)

we get general soln (**)

e.g. $(D_x^3 - D_y^3) Z = 0$

$K_0 = 1 \neq 0$

Ans Put $D_x = \lambda$, $D_y = 1$

$(\lambda^3 - 1) Z = 0$ (Case I)

$\lambda = 1, \omega, \omega^2$ (subcase I)

general solution

$Z = \phi_1(y + x) + \phi_2(y + \omega x) + \phi_3(y + \omega^2 x)$

$(\partial x)^2$ $\cancel{+ \frac{\partial}{\partial x}}$ ∂x

Question solve $((D_x - 2D_y)^2 D_x) Z = 0$

$$\boxed{D_x (D_x^2 + 4D_y^2 - 4D_x D_y) Z = 0} \quad -(1)$$

$K_0 = 1 \neq 0$

Case I

Subcase II

Put $D_x = \lambda$, $D_y = 1$

$\lambda = 0, 2, 2$

general solution.

$$Z = \phi_1(y) + \phi_2(\underline{y+2x}) + x \cdot \phi_3(\underline{y+2x}).$$

or

$$Z = \phi_1(y) + \phi_2(y+2x + y \cdot \phi_3(y+2x).$$

$(\because \lambda = \text{Repeated})$

Ques Solve: $(D_x^3 - D_x^2 D_y) Z = 0$

$K_0 = 1 \neq 0$

case I

Put $D_3 = \lambda$, $D_y = 1$

$\lambda^3 - \lambda^2 = 0$

$\lambda = 0, 0, 1$

then gen sol^n $\rightarrow$

$$\boxed{Z = [\phi_1(y) + x\phi_2(y)] + \phi_3(y+x)} \text{ Ans}$$

~~Second~~

Question $(D_x D_y^3 + D_y^3) Z = 0$

Ans $K_0 \neq 0$

case II

$D_y^3 (D_x + D_y) Z = 0$ — (1)

Now general sol^n corr. to $D_y^3 Z = 0$ $\quad \frac{\partial^3 Z}{\partial y^3} = 0$

$Z = \phi_1(x) + y\phi_2(x) + y^2\phi_3(x)$ — (2)

Now general solution

$(D_x + D_y) Z = 0$

put $D_x = d$, $D_y = 1$

A.E. is $d + 1 = 0$

$d = -1$

general solution corresponding to $(D_x + D_y) Z = 0$ is

$Z = \phi_4(y - x)$ — (3)

Combine (2) and (3)

$$\boxed{Z = \phi_1(x) + y\phi_2(x) + y^2\phi_3(x) + \phi_4(y - x)}$$

general sol^n in (1)

Question $(D_x^3 D_y^2 - D_x^2 D_y^3) Z = 0$

$D_x^4 +$

K_0.

$K_0 = 0$

Case II $\quad D_y^2 (D_x^3 - D_y^2 D_x^2) Z = 0$

$D_y^2 =$

corresponding to $D_y^2 z = 0$

$$z = \phi(x) + y\phi_2(x) - ②$$

corresponding $(D_x^3 - D_y D_x^2)$ -

$$D_x = d$$

$$d^3 - d^2 = 0,$$

$$d = 0, 0, 1$$

$$z = \phi_3(y) + x\phi_4(y) + \phi_5(y+x). - ③.$$

$$Z = \phi_1(x) + y\phi_2(x) + \phi_3(y) + x\phi_4(y) + \phi_5(y+x)$$

Linear Homogenous P.D.E with constant cofficients

$\phi(D_x D_y) y = f(x,y)$

* Particular Integral

$f(x,y) = \psi(\alpha x + \beta y)$

$$P.I = Z_p = \frac{1}{\phi(D_x, D_y)} \psi(\alpha x + \beta y)$$

$$\Rightarrow Z_p = \frac{1}{\phi(\alpha,\beta)} \underbrace{\int\int \cdots \int}_{n\text{times}} \psi(u) \underbrace{du\,du \ldots du}_{n\text{times}}$$

where $u = \alpha x + \beta y$

Provided $\phi(\alpha, \beta) \neq 0$

e.g $(D_x^2 + D_x D_y - 2D_y^2) Z = \sqrt{x+2y}$

Ans $\psi(\alpha x + \beta y) = \sqrt{x+2y}$

$\alpha = 1$, $\beta = 2$

$\phi(D_x, D_y) = D_x^2 + D_x D_y - 2D_y^2$

$\phi(\alpha, \beta) = \phi(1,2) = 1 + 1\cdot 2 - 2\cdot 2^2 = -5 \neq 0$

$$Z_p = \frac{1}{\phi(\alpha,\beta)} \int\int \psi(u)\,du\,du$$

where $u = \alpha x + \beta y = x + 2y$

$$Z_p = \frac{1}{-5} \int\int \sqrt{u}\,du\,du$$

$$= -\frac{1}{5} \int \frac{u^{3/2}}{3/2}\,du$$

$$Z_p = -\frac{1}{5} \frac{u^{5/2}}{3/2 \cdot 5/2} = \frac{-4}{75} u^{5/2}$$

$$\boxed{Z_p = -\frac{4}{75} (x+2y)^{5/2}}$$

Q. $(D_x^2 + D_x D_y - 2D_y^2)Z = 0$

$K_0 \neq 1$

$D_x = \lambda \qquad D_y = 1$

$\lambda^2 + \lambda - 2 = 0$

$\lambda = 1, -2$

So $\quad Z_C = \phi_1(y+x) + \phi_2(y-2x).$

general solution is

$Z = Z_C + Z_p$

$= \phi_1(y+x) + \phi_2(y-2x) - \frac{4}{75}(x+2y)^{5/2}$

Case of failure

$\phi(D_x, D_y)Z = \varphi(\alpha x + \beta y)$

$Z_p = \frac{1}{\phi(D_x \cdot D_y)} \varphi(\alpha x + \beta y)$

If $\phi(\alpha, \beta) = 0.$

$\phi(D_x, D_y) = (\beta D_x - \alpha D_y)^r \cdot g(D_x \cdot D_y), \; r \geq 1$

where $g(\alpha, \beta) \neq 0$

$Z_p = \frac{1}{(\beta D_x - \alpha D_y)^r \, g(D_x, D_y)} \varphi(\alpha x + \beta y)$

$= \frac{x^r}{(\beta^r)\lfloor r} \left[\frac{1}{g(D_x, D_y)} \varphi(\alpha x + \beta y) \right]$ "n" β^r L.I

Scanned by CamScanner

$$= \frac{x^r}{(\beta)^r \lfloor r} \left[\frac{1}{g(\alpha,\beta)} \iint \cdots \int \underbrace{\varphi(u)\, du\, du \cdots du}_{n-r} \right]$$

where $u = \alpha x + \beta y$

Or $$Z_p = \frac{(y)^r}{(-\alpha)^r \lfloor r} \left[\frac{1}{g(D_x, D_y)} \varphi(\alpha x + \beta y) \right]$$

e.g. $(3D_x^2 - 5D_x D_y + 2D_y^2) Z = \log(2x+3y)$

$\alpha = 2, \quad \beta = 3$

$\phi(\alpha,\beta) = \phi(2,3) = 3\cdot 2^2 - 5\cdot 2\cdot 3 + 2\cdot 3^2 = 0$

case fails.

$(3D_x^2 - 5D_x D_y + 2D_y^2) = 3D_x^2 - 6D_x D_y + D_x D_y + 2D_y^2$

$= 3D_x(D_x - 2D_y) + D_y(D_x - 2D_y)$

$= (3D_x - D_y)(D_x - 2D_y)$

$$Z_p = \frac{1}{\phi(D_x, D_y)} \varphi(\alpha x + \beta y)$$

$$Z_p = \frac{1}{(3D_x - 2D_y)(D_x - D_y)} \log(2x+3y)$$

$$= \frac{x^1}{(3)^1 \lfloor 1} \left[\frac{1}{(D_x - D_y)} \log(2x+3y) \right]$$

$$= \frac{x}{3} \left[\frac{1}{(2-3)} \int \log u \, du \right]$$ where $u = 2x+3y$

$$= -\frac{x}{3}\int (\log u)\cdot 1\, du$$

$$= -\frac{x}{3}\int \left[u\log u - u\right]$$

$$Z_p = -\frac{x}{3}\left[(2x+3y)\log(2x+3y)-(2x+3y)\right]$$

Particular Integral.

Question : $(D_x^2 - 4D_xD_y + 4D_y^2)Z = \tan(2x+y)$

Ans $\phi(D_x, D_y) = D_x^2 - 4D_xD_y + 4D_y^2$

$\alpha = 2,\ \beta = 1$

$\phi(2,1) = 4-8+4 = 0$

case fails.

$$\phi(D_x, D_y) = (D_x - 2D_y)^2$$

$$Z_p = \frac{1}{\phi(D_x, D_y)}\varphi(\alpha x+\beta y) = \frac{1}{(D_x-2D_y)^2}\tan(2x+y)$$

$$Z = \frac{x^2}{(1)^2 \lfloor 2}\tan(2x+y)$$

or

$$Z_p = \frac{y^2}{(-2)^2 \lfloor 2}\tan(2x+y)$$

\# $Z_p = \frac{1}{\phi(D_x, D_y)}\varphi(\alpha x+\beta y)$ $e^{\alpha x+\beta y}$

Special case I - $\varphi(\alpha x+\beta y) = e^{\alpha x+\beta y}$

$$Z_p = \frac{1}{\phi(D_x, D_y)} \cdot e^{\alpha x + \beta y}$$

$$Z_p = \frac{1}{\phi(\alpha, \beta)} e^{\alpha x + \beta y} \quad \text{Provided } \phi(\alpha,\beta) \neq 0$$

Special case II $\quad \psi(\alpha x + \beta y) = \sin(\alpha x + \beta y)$ or $\cos(\alpha x + \beta y)$

$$Z_p = \frac{1}{\phi(D_x D_y)} \sin(\alpha x + \beta y) \text{ or } \cos(\alpha x + \beta y)$$

Replace D_x^2 by $-\alpha^2$, D_y^2 by $-\beta^2$
$D_x D_y$ by $-\alpha\beta$

Provided the denominator is non zero.

Question Find P.I for
$(3D_x^2 - 10 D_x D_y + 3D_y^2) Z = 10 \sin(2x+y) - e^{2x-y}$

$$Z_p = 10 \left[\frac{1}{3D_x^2 - 10 D_x D_y + 3D_y^2}\right] \sin(2x+y)$$

$\alpha = 2, \beta = 1$
$D_x^2 = -2, D_y^2 = -1$
$D_x D_y = -2 \cdot 1$

$$\left[\frac{-1}{3D_x^2 - 10 D_x D_y + 3D_y^2} e^{2x-y}\right]$$

$\alpha = 2, \beta = -1$

$D_x D_y$, $\alpha = 2$
D_y by $\beta = -1$

$$Z_p = \frac{10}{5} \sin(2x+y) - \frac{1}{35} e^{2x-y}$$

$Z_p = \frac{1}{\phi(D_x, D_y)} x^m y^n, \quad m, n \in \mathbb{N}$

If $m > n$ then we expand the binomial the binomial the asending power of D_y

If $m \leq n$ the we expand the binomial the binomial the asending power of D_x

If $m = n$ then we can expand the Binomial in asensideng power of both D_x or D_y.

2013

e.g. Solve

$(D_x^3 - D_y^3) Z = x^2 y$

$m = 2, \quad n = 1$

$m > n$

$$Z_p = \frac{1}{(D_x^3 - D_y^3)} x^2 y$$

$$= \frac{1}{D_x^3 \left[1 - \frac{D_y^3}{D_x^3}\right]} x^2 y$$

$$Z_p = \frac{1}{D_x^3} \left[1 - \frac{D_y^3}{D_x^3}\right]^{-1} x^2 y$$

$$= \frac{1}{D_x^3}$$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \text{---}$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \text{---}$

$$Z_p = \frac{1}{D_x^3}\left[1 - \frac{D_y^3}{D_x^3}\right]^{-1} x^2 y$$

$$= \frac{1}{D_x^3}\left[1 + \frac{D_y^3}{D_x^3}\right] x^2 y$$

{Nelglect the term having power > 3} x

$$= \frac{1}{D_x^3}\left[x^2 y + 0\right]$$

$$Z_p = \frac{1}{D_x^3}(x^2 y) = \frac{1}{D_x^2}\left[\frac{x^3}{3} y\right] = \frac{x^5 y}{3 \cdot 4 \cdot 5} = \frac{x^5}{60} y$$

Question $(D_x^2 - D_y^2) Z = x^2 y^3$

$$Z_p = \frac{1}{(D_x^2 - D_y^2) Z}(x^2 y^3)$$

$$= \frac{1}{-D_y^2\left[1 - \frac{D_x^2}{D_y^2}\right]} = \frac{-1}{D_y^2}\left[1 - \frac{D_x^2}{D_y}\right]^{-1} x^2 y^3$$

$$= \frac{-1}{D_y^2}\left[1 + \frac{D_x^2}{D_y^2} + \frac{D_x^4}{D_y^4}\right] x^2 y^3$$

$$= \frac{-1}{D_y^2}\left[x^2 y^3 + \frac{1}{D_y^2}\left[2y^3\right] + 0\right]$$

$$= \frac{-1}{D_y^2}\left[x^2 y^3 + \frac{2 \cdot y^5}{4 \cdot 5}\right] = \frac{-1}{D_y^2}\left[x^2 y^3 + \frac{1}{10} y^5\right]$$

$$\boxed{Z_p = \frac{-x^2 y^5}{20} - \frac{y^7}{420} \text{ Ans}}$$

genral Method for finding P. I.

$$Z_p = \frac{1}{\phi(D_x, D_y)} f(x,y)$$

let $\phi(D_x, D_y) = (D_x - \alpha, D_y)(D_x - \alpha D_y) \cdots (D_x - \alpha_n D_y)$

$$Z_p = \frac{1}{(D_x - \alpha D_y)(D_x - \alpha D_y) \cdots (D_x - \alpha D_y} f(x,y)$$

$$\frac{1}{(D_x - \alpha D_y)} f(x,y) = \int f(x, c - \alpha x)$$

where $y = c - \alpha x$

Ques Solve $(D_x^2 + D_x D_y - 6 D_y^2) Z = y \sin x$

Ans
$$Z_p = \frac{1}{(D_x^2 + D_x D_y - 6 D_y^2)} y \sin x$$

$$= \frac{1}{(D_x - 2D_y)(D_x + 3 D_y)} y \sin x$$

$$Z_p = \frac{1}{(D_x - 2 D_y)} \left[\frac{1}{D_x + 3 D_y} y \sin x \right]$$

$\alpha = -3$

$$= \frac{1}{(D_x - 2D_y)} \left[\frac{1}{D_x - (-3 D_y)} y \sin x \right]$$

$\alpha = -3, \ f(x,y) = y \sin x$

$$= \frac{1}{(D_x - 2D_y)} \int f(x, c - \alpha x) \, dx$$

$$= \frac{1}{D_x - 2D_y} \int f(x, c+3x)\, dx$$

$$= \frac{1}{D_x - 2D_y} \int (c+3x) \cdot \sin x\, dx$$

$$= \frac{1}{D_x - 2D_y} \left[(c+3x) - \cos x + 3\sin x \right]$$

$$= \frac{1}{D_x - 2D_y} \left[(c+3x) - \cos x + 3\sin x \right]$$

$$z_p = \frac{1}{D_x - 2D_y} \left[-y\cos x + 3\sin x \right]$$

$$\alpha = 2, \quad f(x,y) = -y\cos x + 3\sin x$$

$$z_p = \int f(x, c - \alpha x)\, dx$$

$$z_p = \int f(x, c-2x)\, dx$$

$$z_p = \int -(c-2x)\cos x + 3\sin x)\, dx$$

$$z_p = \int (c-2x)\cos x\, dx - 3\cos x$$

$$\boxed{z_p = -y\sin x - \cancel{2}\cos x - 3\cos x}$$

$$z_p = -y\sin x - \cos x$$

Dec 2012 4.75

* Consider $p+q=pq$ then which of the following is correct.

(I) The charpit's A.E for given eqn

$$\frac{dx}{1-q} = \frac{dy}{1-p} = \frac{dz}{pq} = \frac{dp}{p+q} = \frac{dq}{0}$$

(II) A solution of charpits eqn is $q=b$

(III) The corresponding value of p is $p = \frac{b}{p-1}$

(IV) A solution of the given $z = \frac{b}{b-1}x + by + a$

Important class 31-09-2014 one question 4.75

Linear Non-Homogenous P.D.E with constant coff.

eg. $r - p = \sin(2x+3y)$ → linear non-Homogenous P.D.E with constant coff. of order 2.

Consider $\phi(D_x, D_y)z = f(x,y)$ a linear non-Homogenous P.D.E with constant cofficents.

general solution is

$$z = z_c + z_p$$

First of all we solve $\phi(D_x, D_y)z = 0$

we have two cases.

* case I $\phi(D_x, D_y)$ can be factorized linearly.

* case II $\phi(D_x, D_y)$ can not be factorized linearly.

case I $\phi(D_x, D_y)$ can factorized linearly.

Subcase I - when $\phi(D_x, D_y)$ has distinct linear factors.

$$\phi(D_x, D_y) = (\alpha_1 D_x + \beta_1 y + \gamma_1)(\alpha_2 D_x + \beta_2 D_y + \gamma_2) \cdots (\alpha_n D_x + \beta_n D_y + \gamma_n)$$

where α_i is β_i's and V_i's are constant.

$$\phi(D_x, D_y) Z = 0$$

consider $(\alpha_1 D_x + \beta_1 D_y + \gamma_1) Z = 0$

$$\alpha_1 \frac{\partial z}{\partial x} + \beta_1 \frac{dz}{\partial y} = -\gamma_1 z$$

$$\frac{dx}{\alpha_1} = \frac{dy}{\beta_1} = \frac{dz}{-\gamma_1 z}$$

$$\beta_1 x - \alpha_1 x = c_1$$

$$\frac{dx}{\alpha_1} = \frac{dz}{-\gamma_1 z}$$

$$-\gamma_1 x = \alpha_1 \log z + \log c_2$$

$$c_2 = -\gamma_1 x + \alpha_1 \log z$$

$$e^{-\gamma_1/\alpha_1 x} = z c_2$$

$$\Rightarrow z e^{\gamma_1/\alpha_1 x} = \frac{1}{c_2} = c_2' \quad - (2)$$

general solution

$$z e^{\gamma_1/\alpha_1} = \phi_1(\beta_1 x - \alpha_1 y)$$

$$z = e^{-\gamma_1/\alpha_1 x} \phi_1(\beta_1 x - \alpha_1 y)$$

genral solution is

$$Z = e^{-\gamma_1/\alpha_1 x}\phi_1(\beta_1 x - \alpha_1 y) + e^{-\gamma_2/\alpha_2 x}\phi_2(\beta_2 x - \alpha_2 y)$$

$$+ \cdots + e^{-\gamma_n/\alpha_n x}\phi_n(\beta_n x - \alpha_n y)$$

or

$$Z = e^{-\gamma_1/\beta_1 y}\phi_1(\beta_1 x - \alpha_1 y) + e^{-\gamma_2/\beta_2 y}\phi_2(\beta_2 x - \alpha_2 y)$$

$$+ \cdots + e^{-\gamma_n/\beta_n y}\phi_n(\beta_n x - \alpha_n y)$$

Ans

Question which of the following is correct ?

$$(3D - D' + 2)(D - 2D' + 1)Z = 0$$

Then genral solution is $Z =$

(I) $e^{3x}\phi_1(x + 3y) + e^{-x}(2x + y)$ ✗

(II) $Z = e^{2y}\phi_1(x + 3y) + e^{-x}\phi_2(2x + y)$

(III) $Z = e^{-2/3x}\phi_1(x + 3y) + e^{1/2 y}\phi_2(2x + y)$

(IV) None of them

Ans $\alpha_1 = 3/2$, $\beta_1 = -1$, $\gamma_1 = 2$

$\alpha_2 = 1$, $\beta_2 = -2$, $\gamma_2 = 1$

(II) $Z = e^{2y}\phi_1(-x - 3y) + e^{-x}\phi_2(-2x - y)$

$$Z = e^{2y}\phi(x + 3y) + e^{-x}\phi_2(2x + y)$$

(V) $Z = e^{-2/3 x}(\phi_1(x + 3y) + e^{-x}\phi_2(2x + y))$

(VI) $Z = e^{2y}\phi_1(x + 3y) + e^{1/2 y}\phi_2(2x + y)$

(VII) $Z = e^{2y}\phi_1(e^{x + 3y}) + e^{1/2 y}\phi_2(\log(2x + y))$

Solve $s + p = q + z$

$$\cancel{D_x D_y + D_x^2 = D_y^2 + z}$$

$$D_x D_y + D_x = D_y + z$$

$$D_x(D_y + 1) - D_y + z = 0$$

$$(D_x D_y + D_x - D_y - 1)\, z = 0$$

$$D_x(D_y + 1) - 1(D_y + 1) z = 0$$

$$(D_x - 1)(D_y + 1) = 0$$

$\gamma_1 = -1$, $\alpha = 1$, $\gamma_2 = -1$, $\beta_1 = 0$, $\alpha_2 = 0$

general sol

$$z = e^{x} \phi_1(y) + e^{-y} \phi_2(x) \quad \text{Ans}$$

Subcase II when $\phi(D_x, D_y)$ has repeated factors.

$$(\alpha D_x + \beta D_y + \gamma)^2 z = 0$$

$$\boxed{\begin{gathered} z = e^{-\gamma/\alpha\, x}\left[\phi_1(\beta x - \alpha y) + x\,\phi_2(\beta x - \alpha y)\right] \\ \text{or} \\ z = e^{-\gamma/\beta\, y}\left[\phi_1(\beta x - \alpha y) + y\,\phi_2(\beta x - \alpha y)\right] \end{gathered}}$$

EXAMPLE 29.2. Solve the second order partial differential equation

$(r + 2s + t) = e^{2x+3y}$ where symbols have their usual meaning.

SOLUTION: The given equation is $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = e^{2x+3y}$

or $(D^2 + 2DD' + D'^2) z = e^{2x+3y}$

The auxiliary equation for C.F. is $m^2 + 2m + 1 = 0$ giving $m = -1, -1$.

$\therefore$ C.F. $= \phi_1 (y-x) + x\phi_2 (y-x)$

and, on using short cur method (i), we get

$$\text{P.I.} = \frac{1}{D^2 + 2DD' + D'^2} e^{2x+3y} = \frac{1}{2^2 + 2(2)(3) + 3^2} e^{2x+3y} = \frac{1}{25} e^{2x+3y}$$

Therefore, the complete solution of the given equation, is

$$z = \phi_1 (y - x) + x\phi_2 (y - x) + \frac{1}{25} e^{2x+3y}$$

Ans.

where ϕ_1 and ϕ_2 are arbitrary functions.

EXAMPLE 29.3. Solve $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3) z = e^{y-2x} + e^{y+2x} + e^{y+x}$,

where $D \equiv \dfrac{\partial}{\partial x}$, $D' \equiv \dfrac{\partial}{\partial y}$.

SOLUTION: For C.F., the auxiliary equation is $m^3 - 4m^2 + 5m - 2 = 0$ $\therefore$ $m = 1, 1, 2.$

Therefore, C.F. $= \phi_1 (y + 2x) + \phi_2 (y + x) + x\phi_3 (y + x)$ where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

Next, $\text{P.I.} = \dfrac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3} (e^{y-2x} + e^{y+2x} + e^{y+x})$

$$= \frac{1}{(D - D')^2 (D - 2D')} e^{y-2x} + \frac{1}{(D - D')^2 (D - 2D')} e^{y+2x} + \frac{1}{(D - D')^2 (D - 2D')} e^{y+x}$$

Here $\dfrac{1}{(D - D')^2 (D - 2D')} e^{y-2x} = \dfrac{1}{(-2 - 1)^2 (-2 - 2\cdot 1)} e^{y-2x} = \dfrac{-1}{36} e^{y-2x}$,

and $\dfrac{1}{(D - D')^2 (D - 2D')} e^{y+2x} = \dfrac{1}{(2 - 1)^2} \cdot \dfrac{1}{(D - 2D')} e^{y+2x} = 1 \cdot \dfrac{x}{1} \cdot e^{y+2x}$.

and $$\frac{1}{(D-D')^2(D-2D')}e^{y+x} = \frac{1}{(1-2\cdot 1)}\cdot\frac{1}{(D-D')^2}e^{y+x} = (-1)\cdot\frac{x}{2(D-D')}e^{y+x}$$

$$= (-1)\frac{x\cdot x}{2\cdot 1}e^{y+x} = -\frac{x^2}{2}e^{y+x}.$$

Thus, the complete solution of the given equation, is

$$z = \phi_1(y+2x) + \phi_2(y+x) + x\phi_3(y+x) - \frac{1}{36}e^{y-2x} + xe^{y+2x} - \frac{x^2}{2}e^{y+x}.$$

EXAMPLE 29.4. Solve $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$ where $D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$

SOLUTION: For C.F. the auxiliary equation is $m^2 - 6m + 9 = 0$ hence $m = 3, 3$.

Therefore C.F. $= \phi_1(y+3x) + x\phi_2(y+3x)$.

Next, $$\text{P.I} = \frac{1}{D^2 - 6DD' + 9D'^2}(12x^2 + 36xy)$$

$$= \frac{1}{(D-3D')^2}(12x^2+36xy) = \frac{1}{D^2}\left[1 - \frac{3D'}{D}\right]^{-2}(12x^2+36xy).$$

$$= \frac{1}{D^2}\left[1 + 2\cdot\left(\frac{3D'}{D}\right) + \frac{(-2)(-3)}{2\cdot 1}\cdot\left(\frac{3D'}{D}\right)^2 + \ldots\right](12x^2+36xy)$$

$$= \frac{1}{D^2}\left[1 + \frac{6D'}{D} + \frac{27D'^2}{D^2}\right](12x^2+36xy) \quad \text{(retaining terms upto } D'^2\text{)}$$

$$= \frac{1}{D^2}\left[(12x^2+36xy) + \frac{6}{D}(0+36x) + 0\right] = \frac{1}{D^2}(12x^2 + 36xy) + \frac{1}{D^3}(6\times 36x)$$

$$= x^4 + 6x^3y + 6\cdot(36)\frac{x^4}{2\cdot 3\cdot 4} = 10x^4 + 6x^3y$$

Thus, the complete solution is $z = \phi_1(y+3x) + x\phi_2(y+3x) + 10x^4 + 6x^3y$.

EXAMPLE 29.5. Solve $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3y^3$.

SOLUTION: The given equation can be written as: $(D^3 - D'^3)z = (xy)^3$.

For complementary function, the auxiliary equation is $m^3 - 1 = 0$ whose roots are $m = 1, \omega, \omega^2$ where ω and ω^2 are the complex cube roots of unity.

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y+\omega x) + \phi_3(y+\omega^2 x)$

where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions and $\omega = \left(-1+i\sqrt{3}\right)/2$ and $\omega^2 = \left(-1-i\sqrt{3}\right)/2$

Next, $$\text{P.I.} = \frac{1}{D^3 - D'^3}x^3y^3 = \frac{1}{D^3}\left[1 - \frac{D'^3}{D^3}\right]^{-1}(x^3y^3) = \frac{1}{D^3}\left[1 + \frac{D'^3}{D^3} + \frac{D'^6}{D^6} + \ldots\right](x^3y^3)$$

$$= \frac{1}{D^3}\left[1 + \frac{D'^3}{D^3}\right](x^3y^3) + 0 = \frac{1}{D^3}\left[x^3y^3 + \frac{x^6}{4\cdot 5\cdot 6}(3\cdot 2\cdot 1)\right]$$

$$= \frac{x^6y^3}{4\cdot 5\cdot 6} + \frac{1}{20}\cdot\frac{x^9}{7\cdot 8\cdot 9} = \frac{x^6y^3}{120} + \frac{x^9}{10080}$$

Therefore, the complete solution is

$$z = \phi_1(y+x) + \phi_2(y+\omega x) + \phi_3(y+\omega^2 x) + \frac{x^6y^3}{120} + \frac{x^9}{10080}. \quad \textbf{Ans.}$$

EXAMPLE 29.6. Solve the third order homogeneous partial differential equation

$$\frac{\partial^3 z}{\partial x^3} - 7\frac{\partial^3 z}{\partial x \partial y^2} - 6\frac{\partial^3 z}{\partial y^3} = \sin(x+2y) + x^2 y.$$

SOLUTION: The given equation can be written as

$$(D^3 - 7DD'^2 - 6D'^3)\, z = \sin(x+2y) + x^2 y.$$

The auxiliary equation is $m^3 - 7m - 6 = 0$ with roots $-1, -2, 3$.

$\therefore$ C.F. $= \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$ where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

Next, P.I $= \dfrac{1}{(D+D')(D+2D')(D-3D')}\{\sin(x+2y) + x^2y\}$.

Evaluating it term by term, we get

$$\frac{1}{(D+D')(D+2D')(D-3D')}\sin(x+2y) = \frac{1}{(D+D')(D^2 - DD' - 6D'^2)}\sin(x+2y)$$

$$= \frac{1}{(D+D')\{-1+1\cdot 2 - 6(-4)\}}\sin(x+2y)$$

$$= \frac{1}{25}\cdot\frac{1}{D+D'}\sin(x+2y) = \frac{1}{25}\cdot\frac{D-D'}{D^2-D'^2}\sin(x+2y) = \frac{1}{25}\cdot\left(\frac{(D-D')}{-1-(-4)}\right)\sin(x+2y)$$

$$= \frac{1}{75}[D\sin(x+2y) - D'\sin(x+2y)] = -\frac{1}{75}\cos(x+2y).$$

and

$$\frac{1}{(D+D')(D+2D')(D-3D')}(x^2y) = \frac{1}{D^3 - 7DD'^2 - 6D'^3}(x^2y)$$

$$= \frac{1}{D^3}\left[1 - \left(7\frac{D'^2}{D^2} + 6\frac{D'^3}{D^3}\right)\right]^{-1}(x^2y) = \frac{1}{D^3}\left[1 + \left(7\frac{D'^2}{D^2} + 6\frac{D'^3}{D^3} + \ldots\right)\right](x^2y)$$

$$= \frac{1}{D^3}[x^2y + 0 + 0 + \ldots] = \iiint x^2 y\, dx\, dx\, dx = \frac{x^5 y}{3\cdot 4\cdot 5}$$

Thus, P.I. $= \dfrac{-1}{75}\cos(x+2y) + \dfrac{x^5 y}{60}$

Therefore, the complete solution is

$$z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) + \frac{x^5y}{60} - \frac{1}{75}\cos(x+2y).$$ **Ans.**

EXAMPLE 29.7. Solve $(D^3 + D^2D' - DD'^2 - D'^3)\, z = e^x \cos 2y$ where $D \equiv \dfrac{\partial}{\partial x}$, $D' \equiv \dfrac{\partial}{\partial y}$.

SOLUTION: For C.F., the auxiliary equation is $m^3 + m^2 - m - 1 = 0$ whose roots are $1, -1, -1$.

$\therefore$ C.F. $= \phi_1(y-x) + x\phi_2(y-x) + \phi_3(y+x)$ where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

Next, P.I. $= \dfrac{1}{(D+D')^2(D-D')}e^x\cos 2y$

$$= e^x\frac{1}{(D+1+D')^2(D+1-D')}\cos 2y \qquad \text{(replacing D by D + 1)}$$

$$= e^x\frac{1}{[(D+1)^2 - D'^2](D+D'+1)}\cos 2y = e^x\frac{1}{(D+D'+1)(D^2 - D'^2 + 2D + 1)}\cos 2y$$

$$= e^x \frac{1}{(D+D'+1)} \cdot \frac{1}{(0-(-4)+2D+1)} \cos 2y = e^x \frac{1}{(D+D'+1)(2D+5)} \cos 2y$$

$$= e^x \frac{1}{2D^2+2DD'+7D+5D'+5} \cos 2y = e^x \frac{1}{0+0+7D+5D'+5} \cos 2y$$

$$= e^x \frac{7D+5D'-5}{(7D+5D')^2-25} \cos 2y = e^x \frac{7D+5D'-5}{49D^2+70DD'+25D'^2-25} \cos 2y$$

$$= e^x \frac{7D+5D'-5}{0+0+25(-4)-25} \cos 2y = e^x \frac{7D+5D'-5}{-125} \cos 2y$$

$$= \frac{e^x}{-125}(0-10\sin 2y - 5\cos 2y) = \frac{e^x}{25}(2\sin 2y + \cos 2y)$$

Thus the complete solution is $z = \phi_1(y-x) + x\phi_2(y-x) + \phi_3(y+x) + \frac{e^x}{25}(2\sin 2y + \cos 2y)$ **Ans.**

EXAMPLE 29.8. Solve $(D^2 - DD' - 2D'^2)z = (y-1)e^x$ *[GGSIPU III Sem End Term 2012]*

SOLUTION: For C.F. the A.E. is $m^2 - m - 2 = 0 \quad \therefore \quad m = -1, 2$

$\therefore$ C.F. $= \phi_1(y+2x) + \phi_2(y-x)$ where ϕ_1 and ϕ_2 are arbitrary functions

and P.I. $= \dfrac{1}{D^2 - DD' - 2D'^2}(y-1)e^x = \dfrac{1}{(D-2D')(D+D')}(y-1)e^x$

$$= \frac{1}{3D'}\left[\frac{1}{D-2D'} - \frac{1}{D+D'}\right](y-1)e^x = \frac{1}{3D'}\left[\frac{1}{D-2D'}(y-1)e^x - \frac{1}{D+D'}(y-1)e^x\right]$$

$$= \frac{1}{3D'}\left[\int_{y \text{ constt}} (y-2x-1)e^x dx - \int_{y \text{ constt.}} (y+x-1)e^x dx\right]$$

$$= \frac{1}{3D'}\left[\left\{(y-2x-1)e^x - \int(-2)e^x dx\right\} - \left\{(y+x-1)e^x - \int 1\, e^x dx\right\}\right]$$

$$= \frac{1}{3D'}\left[\left\{(y-2x-1)e^x + 2e^x\right\}_{y \to y+2x} - \left\{(y+x-1)e^x - e^x\right\}_{y \to y-x}\right]$$

$$= \frac{1}{3D'}\left[(y+2x-2x-1+2)e^x - (y-x+x-1-1)e^x\right]$$

$$= \frac{1}{3D'}[(y+1)e^x - (y-2)e^x] = \frac{1}{3D'} \cdot 3e^x = y\, e^x$$

$\therefore$ The solution is $z = \phi_1(y+2x) + \phi_2(y-x) + y\, e^x$. **Ans.**

EXAMPLE 29.9. Solve $(D^2 + 2DD' - 8D'^2)z = \sqrt{2x+3y}$.

SOLUTION: The auxiliary equation is $m^2 + 2m - 8 = 0$ whose roots are $m = -4$ and 2.

$\therefore$ C.F. $= \phi_1(y+2x) + \phi_2(y-4x)$ where ϕ_1, ϕ_2 are arbitrary functions.

Next, P.I. $= \dfrac{1}{(D-2D')(D+4D')}(2x+3y)^{\frac{1}{2}} = \dfrac{1}{(D-2D')}\left[\dfrac{1}{(D+4D')}(2x+3y)^{\frac{1}{2}}\right]$

$$= \frac{1}{(D-2D')}\int [2x+3\{y-(-4)x\}]^{\frac{1}{2}}\,dx = \frac{1}{(D-2D')}\int [14x+3y]^{\frac{1}{2}}\,dx$$

$$= \frac{1}{(D-2D')}\left[\frac{2}{3\cdot 14}(14x+3y)^{\frac{3}{2}}\right]_{(y\to y-4x)}$$

$$= \frac{1}{21}\cdot\frac{1}{(D-2D')}[14x+3(y-4x)]^{\frac{3}{2}} = \frac{1}{21}\cdot\frac{1}{D-2D'}(2x+3y)^{\frac{3}{2}}$$

$$= \frac{1}{21}\int [2x+3(y-2x)]^{\frac{3}{2}}\,dx = \frac{1}{21}\int (3y-4x)^{\frac{3}{2}}\,dx = \frac{1}{21}\cdot\left[\frac{2\,(3y-4x)^{\frac{5}{2}}}{5(-4)}\right]_{(y\to y+2x)}$$

$$= -\frac{1}{210}[3(y+2x)-4x]^{\frac{5}{2}} = -\frac{1}{210}(2x+3y)^{\frac{5}{2}}$$

Therefore the general solution is

$$z = \phi_1(y+2x)+\phi_2(y-4x)-\frac{1}{210}(2x+3y)^{\frac{5}{2}}.$$ **Ans.**

EXAMPLE 29.10. Solve the equation $(D^3-7DD'^2-6D'^3)z = \sin(x+2y)+e^{2x+y}$.

[GGSIPU III Sem End Term 2006]

SOLUTION: For complementary function we have the auxiliary equation of the given equation as $m^3-7m-6=0$ whose roots are $m=-1,-2,3$.

$\therefore$ $C.F. = \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$ where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

Next, $$P.I. = \frac{1}{D^3-7DD'^2-6D'^3}[\sin(x+2y)+e^{2x+y}]$$

$$= \frac{1}{-1^2D-7D(-4)-6(-4)D'}\sin(x+2y)+\frac{1}{2^3-7.2.1^2-6.1^3}e^{2x+y}$$

$$= \frac{1}{27D+24D'}\sin(x+2y)-\frac{1}{12}e^{2x+y}$$

$$= \frac{1}{3}\left(\frac{9D-8D'}{81D^2-64D'^2}\right)\sin(x+2y)-\frac{e^{2x+y}}{12}$$

$$= \frac{9D-8D'}{3[81(-1)-64(-4)}\sin(x+2y)-\frac{e^{2x+y}}{12}$$

$$= \frac{9\cos(x+2y)-16\cos(x+2y)}{3(175)}-\frac{e^{2x+y}}{12}$$

or $$P.I. = \frac{-1}{75}\cos(x+2y)-\frac{1}{12}e^{2x+y}$$

$\therefore$ Complete solution is $z = \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)-\frac{1}{75}\cos(x+2y)-\frac{1}{12}e^{2x+y}$. **Ans.**

EXERCISE 29A

1. Solve $(D^2 - 3DD' + 2D'^2)z = e^{2x+3y} + e^{x+y} + \sin(x - 2y)$

2. Find the particular integral of $p + 3q = \cos(2x + y)$ where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

3. Solve $(D - 2D')^2 (D + 3D')z = e^{2x+y}$

4. Solve $(D^3 - 7DD'^2 - 6D'^3)z = \cos(x - y) + x^2 + xy^2 + y^3$.

5. Solve $(D - D')^2 z = x + \phi(x + y)$.

6. Solve $r + s - 6t = y \cos x$ where symbols carry their usual meaning.

7. Solve $(D^2 + 2DD' + D'^2)z = 2\cos y - x \sin y$.

8. Solve $(D^2 - 2DD')z = e^{2x} + x^2 y$

9. Solve $(D^3 - 2D^2D')z = 2e^{2x} + 3x^2 y$

10. Solve $(D^3 - 3DD'^2 - 2D'^3)z = \cos(x + 2y) - e^y(y + 2x)$

11. Solve $\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x^2 \partial y} + 4\frac{\partial^3 u}{\partial y^3} = e^{x+2y}$

12. Solve the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x$$

13. Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y$

[GGSIPU III Sem End Term 2012]

14. Solve $2\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 5\sin(2x + y)$

EXAMPLE 29.12. Solve $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} - 3\dfrac{\partial z}{\partial x} + 3\dfrac{\partial z}{\partial y} = xy + e^{x+2y}$.

SOLUTION: Given equation can be written as $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$

or $\quad (D - D')(D + D' - 3)\, z = xy + e^{x+2y}$

$\therefore \quad \text{C.F.} = e^{0x}\phi_1(y+x) + e^{3x}\phi_2(y-x).$

and $\quad \text{P.I.} = \dfrac{1}{(D-D')(D+D'-3)}(xy) + \dfrac{1}{(D-D')(D+D'-3)}e^{x+2y}$

Now
$$\frac{1}{(D-D')(D+D'-3)}e^{x+2y} = e^x \frac{1}{(D+1-D')(D+1+D'-3)}e^{2y}$$
$$= e^x \frac{1}{(D+D'-2)}\left[\frac{1}{(D-D'+1)}e^{2y}\right]$$
$$= e^x \frac{1}{(D+D'-2)}\,\frac{1}{(0-2+1)}e^{2y}$$
$$= -e^x \frac{1}{D+D'-2}e^{2y} \qquad \text{(case of failure)}$$
$$= -e^x \frac{y}{\frac{\partial}{\partial D'}(D+D'-2)}e^{2y} = -e^x y\cdot\frac{1}{1}e^{2y} = -y\,e^{x+2y}.$$

and
$$\frac{1}{(D-D')(D+D'-3)}xy = \frac{-1}{3(D-D')}\left[1-\frac{D+D'}{3}\right]^{-1}(xy)$$
$$= \frac{-1}{3(D-D')}\left[1+\frac{1}{3}(D+D')+\frac{1}{9}(D+D')^2+\dots\right]xy$$
$$= \frac{-1}{3(D-D')}\left[xy+\frac{1}{3}D(xy)+\frac{1}{3}D'(xy)+\frac{D^2}{9}(xy)+\frac{2}{9}DD'(xy)+\frac{D'^2}{9}(xy)+\dots\right]$$
$$= \frac{-1}{3(D-D')}\left[xy+\frac{y}{3}+\frac{x}{3}+\frac{2}{9}\right] = \frac{-1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left[xy+\frac{x}{3}+\frac{y}{3}+\frac{2}{9}\right]$$
$$= \frac{-1}{3D}\left(1+\frac{D'}{D}+\frac{D'^2}{D^2}+\dots\right)\left[xy+\frac{x}{3}+\frac{y}{3}+\frac{2}{9}\right]$$
$$= \frac{-1}{3D}\left(xy+\frac{x}{3}+\frac{y}{3}+\frac{2}{9}+\frac{x^2}{2}+\frac{x}{3}\right) = \frac{-1}{3}\left[\frac{x^2}{2}y+\frac{x^2}{6}+\frac{xy}{3}+\frac{2x}{9}+\frac{x^3}{6}+\frac{x^2}{6}\right]$$
$$= -\left[\frac{1}{6}x^2y+\frac{x^2}{9}+\frac{xy}{9}+\frac{2x}{27}+\frac{x^3}{18}\right]$$

$\therefore$ The complete solution is

$$z = \phi_1(y+x) + e^{2x}\phi_2(y-x) - ye^{x+2y} - \left[\frac{1}{6}x^2y+\frac{x^2}{9}+\frac{xy}{9}+\frac{2x}{27}+\frac{x^3}{18}\right] \qquad \textbf{Ans.}$$

EXAMPLE 29.13. Solve $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + \sin(2x+y) + xy.$

SOLUTION: The given equation can be written as

$$(D + D')(D - 2D' + 2)z = e^{2x+3y} + \sin(2x+y) + xy.$$

Hence $\quad$ C.F. $= e^{0x}\phi_1(y - x) + e^{-2x}\phi_2(y + 2x).$

Next, the P.I. corresponding to e^{2x+3y}, is

$$= \frac{1}{(D+D')(D-2D'+2)}e^{2x+3y} = \frac{1}{(2+3)(2-6+2)}e^{2x+3y} = -\frac{1}{10}e^{2x+3y}.$$

and $\quad$ P.I. corresponding to $\sin(2x+y)$, is

$$= \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'}\sin(2x+y) = \frac{1}{-4+2(1)-2(-1)+2D+2D'}\sin(2x+y)$$

$$= \frac{1}{2(D+D')}\sin(2x+y) = \frac{1}{2}\frac{D-D'}{D^2-D'^2}\sin(2x+y)$$

$$= \frac{1}{2}\frac{(D-D')}{(-4+1)}\sin(2x+y) = -\frac{1}{6}[2\cos(2x+y) - \cos(2x+y)] = -\frac{1}{6}\cos(2x+y).$$

and finally, P.I. corresponding to xy, is

$$= \frac{1}{(D+D')(D-2D'+2)}(xy) = \frac{1}{2D}\left(1+\frac{D'}{D}\right)^{-1}\left(1+\frac{1}{2}D - D'\right)^{-1}(xy)$$

$$= \frac{1}{2D}\left(1+\frac{D'}{D}\right)^{-1}\left[1-\frac{D}{2}+D'+\left(\frac{1}{4}D^2 + D'^2 - DD'\right)\ldots\right](xy)$$

$$= \frac{1}{2D}\left(1+\frac{D'}{D}\right)^{-1}\left[xy - \frac{y}{2} + x - 1\right] = \frac{1}{2D}\left[1 - \frac{D'}{D} + \frac{D'^2}{D^2}\ldots\right]\left(xy - \frac{y}{2} + x - 1\right)$$

$$= \frac{1}{2D}\left(1 - \frac{D'}{D}\right)\left(xy - \frac{y}{2} + x - 1\right) = \frac{1}{2D}\left[xy - \frac{y}{2} + x - 1 - \frac{x^2}{2} + \frac{x}{2}\right]$$

$$= \frac{1}{2}\left[\frac{x^2y}{2} - \frac{xy}{2} + \frac{x^2}{2} - x - \frac{x^3}{6} + \frac{x^2}{4}\right] = \frac{1}{2}\left[\frac{1}{2}x^2y - \frac{1}{2}xy - \frac{x^3}{6} + \frac{3}{4}x^2 - x\right]$$

$\therefore$

$$\text{P. I.} = \frac{-1}{10}e^{2x+3y} - \frac{1}{6}\cos(2x+y) + \frac{1}{2}\left[\frac{1}{2}x^2y - \frac{1}{2}xy - \frac{x^3}{6} + \frac{3}{4}x^2 - x\right]$$

$\therefore$ Complete solution is $\quad z = \text{C.F.} + \text{P.I.}$ **Ans.**

EXERCISE 29B

1. Solve $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} - 2\dfrac{\partial^2 z}{\partial y^2} + 2\dfrac{\partial z}{\partial x} - 4\dfrac{\partial z}{\partial y} = 0$

2. Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial z}{\partial y} - z = e^{x} + e^{-x}$

3. Solve $\dfrac{2\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} - 3\dfrac{\partial z}{\partial y} = 3\cos(3x - 2y)$

4. Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} - 3\dfrac{\partial z}{\partial x} + 3\dfrac{\partial z}{\partial y} + 2z = e^{2x - y}$

5. Solve $\dfrac{\partial^2 z}{\partial x^2} + 3\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} - 4\dfrac{\partial z}{\partial x} - 5\dfrac{\partial z}{\partial y} + 3z = 2x + 3y$

6. Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial z}{\partial y} - z = x^2 y$

7. Solve $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial z}{\partial x} + 3\dfrac{\partial z}{\partial y} - 2z = e^{x - y} - x^2 y$

8. Solve $(D - 3D' - 2)^2 z = 2e^{2x}\tan(y + 3x)$ where $D = \dfrac{\partial}{\partial x}$ and $D' = \dfrac{\partial}{\partial y}$.

Applications of Partial Differential Equations

→ Method of separation of Variables -

we put $\boxed{Z = X(x)Y(y)}$

$$\frac{\partial z}{\partial x} = X'(x)Y(X), \quad \frac{\partial z}{\partial y} = Y'(y)X(x)$$

Question solve $3\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = 0, \quad z(x,0) = 4e^{-x}$

→ By Method of separation of variables

$$z = X(x)\,Y(y)$$

$$3\frac{dX}{dx}Y + 2X\frac{dY}{dy} = 0$$

$$3X'Y + 2XY' = 0$$

$$\frac{3X'}{X} = -\frac{2Y'}{Y} = K$$

$$\Rightarrow \frac{3X'}{X} = K \Rightarrow X = Ae^{\frac{Kx}{3}}$$

and $\frac{Y'}{Y} = \frac{-K}{2} \Rightarrow Y = Be^{-Ky/2}$

$$z = XY = ce^{\frac{Kx}{3}}e^{\frac{-Ky}{2}}$$ where c is some constant of integration.

$$z(x,0) = 4e^{-x}$$

$$ce^{\frac{Kx}{3}} = 4e^{-x}$$

$$c = 4 \quad \text{and} \quad K = -3$$

$$z = 4e^{-x} \cdot e^{3y/2}$$ **Ans**

Scanned by CamScanner

Using the Method of sepration of variables

solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, $u(x,0) = 6e^{-3x}$

Sol-

$$u(x,t) = X(x)\,T(t)$$

$$X'T = 2XT' + XT$$

$$(X'-X)T = 2XT'$$

$$\frac{(X'-X)}{2X} = \frac{T'}{T} = k$$

$X' - X - 2kX = 0$ or $\frac{X'}{X} = 1+2k$ —(I)

and $\frac{T'}{T} = k$ —(II)

$$\log X = (1+2k)x + \log c$$

$$X = c\,e^{(1+2k)x}$$

Form II

$$\log T = \cancel{c'e^{kt}}$$

$$\log T = kt + \log c'$$

$$T = c'e^{kt}$$

$$u(x,t) = XT = cc'e^{(1+2k)x}\,e^{kt}$$

$$6e^{-3x} = u(x,0) = cc'e^{(1+2k)x}$$

$cc' = 6$, $1+2k = -3$, $k = -2$

$$u = 6e^{-3x}e^{-2t}$$

$$u = 6e^{-(3x+2t)}$$

Scanned by CamScanner

HW

1. Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

2. Solve $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} = 3u$, $u = 3e^{-y} - e^{-5y}$ when $x = 0$

3. $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + u$, $u(x,0) = 3e^{-4x}$

4. $\cancel{x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = 0}$

Physical Applications

(Heat).

1. The diffusion Equation in one dimension

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

2. Laplace Equation in two dimensions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0$$

3. In the conduction of heat in a plate in steady state.

The equation is also valid in electrostatic potential problems.

4. Wave equation.

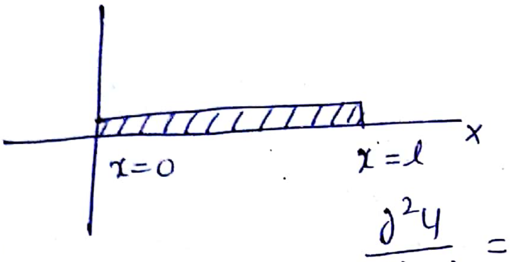
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

→ Solution of one dimensional Heat flow equation

Solve $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, c^2 = constant of diffusity of material

Subject to condition
$u(0,t) = u(l,t) = 0$ and $u(x,0) = f(x)$ at $t=0$

Solution by sepration of variable



$$u(x,t) = X(x)\,T(t) \quad -(1)$$

$$\frac{\partial^2 u}{\partial x^2} = X''T, \quad \frac{\partial u}{\partial t} = XT'$$

$$XT' = c^2 X''T$$

$$\Rightarrow \frac{T'}{c^2T} = \frac{X''}{X} = K \quad -(2)$$

$$\Rightarrow \frac{T'}{c^2T} = K \quad -(3)$$

and $\frac{X''}{X} = K$ $-(4)$ $\Rightarrow X'' - KX = 0$ $-(4)$

There are three cases aries for constant K.

Case I when $K=0$ then

$$T' = 0, \Rightarrow T = C_1$$

$$X'' = 0 \Rightarrow X' = C_2 \Rightarrow X = C_2x + C_3$$

Hence the complete solution

$$u(x,t) = C_1(C_2x + C_3)$$

<u>Case 2</u> when K is positive $K = p^2$
(where p is any number)

$$\frac{T'}{T} = p^2c^2 \Rightarrow \log T = p^2c^2t + c_1$$

$$T = e^{p^2c^2t + c_1} = e^{c_1}e^{p^2c^2 \cdot t}$$

or $\boxed{T = e^{p^2c^2t} \cdot c'}$

and $X'' - p^2X = 0$

$\Rightarrow (D^2 - p^2)X = 0$

A.E. $m^2 - p^2 = 0$

$\Rightarrow m = \pm p$

$m = c_1e^{px} + c_2e^{-px}$

Hence complete solution of (1)

$$u(x,t) = c'e^{p^2c^2t}(c_2e^{px} + c_2e^{-px}) \quad -(2)$$

<u>Case 3</u> when K is negative

$K = -p^2$ (p is any number)

then from (3) and (4)

$$\frac{T'}{T} = -p^2c^2 \Rightarrow \text{integrating } \log T = -p^2c^2t + c_1$$

$$\Rightarrow T = e^{-p^2c^2t} \cdot c'$$

and $X'' + p^2X = 0 \Rightarrow (D^2 + p^2)X = 0$

A.E. $(m^2 + p^2) = 0 \Rightarrow m = \pm ip$

$$\boxed{X = C_2\cos px + C_3\sin px}$$

$$u(x,t) = c'e^{-p^2c^2t}(C_2\cos pn + C_3\sin pn) \quad -(A)$$

Appling condition put $x=0$

$$u(0,t) = c_1 e^{-p^2c^2t}(c_2\cdot 1+0) = 0$$

$$\Rightarrow c_2 = 0$$

$$u(x,t) = c' e^{-p^2c^2t} c_3 \sin px$$

$$= c c_3 e^{-p^2c^2t} \sin px = A e^{-p^2c^2t} \sin px \quad (7)$$

Again put $x = l$ into

$$u(l,t) = A e^{-p^2c^2t} \sin pl = 0$$

$$\sin pl = 0$$

$$\Rightarrow \cancel{pl} \quad pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Hence complete solution of (1)

$$u(x,t) = A e^{-\frac{n^2\pi^2c^2}{l^2}t} . \sin \frac{n\pi x}{l} \quad -(7B)$$

By principle of superposition

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2c^2t}{l^2}} . \sin \frac{n\pi x}{l} \quad -(7C)$$

Put $t=0$ into $7(C)$

or $$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

Multiplying both side by $\sin\left(\frac{m\pi x}{l}\right)$ and then integrating w.r.t x b/w the limit $x=0$ to $x=l$

$$\int_0^l f(x) \sin \frac{m\pi x}{l} dx = \sum_{n=1}^{\infty} A_n \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx$$

$$\therefore \int_0^l \sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx = \begin{cases} \frac{l}{2} & m=n \\ 0 & m\neq n \end{cases}$$

$$\int_0^l f(x)\sin\left(\frac{m\pi x}{l}\right)dx = A_m \frac{l}{2}$$

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{m\pi x}{l}\,dx$$

Hence the ~~genral~~ final genral Solution of Heat flow equation

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2 c^2 t}{l^2}} \sin\left(\frac{n\pi x}{l}\right)$$

where

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Example
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = 3\sin n\pi x$$

$$u(0,t) = 0, \quad u(1,t) = 0 \qquad 0<x<1$$
$$t>0$$

Sol -
$$u(x,t) = 3\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\sin n\pi x$$

$$u(x,t) = 3\sum_{n=1}^{\infty}\sin(n\pi)x\; e^{-n^2\pi^2 \cdot t}$$

#
$$u_{xx} = \frac{1}{c}u_t \qquad 0<x<l,\ t>0$$
$$u_x(0,t) = u_x(l,t) = 0$$
$$u(x,0) = f(x)$$
$$u(x,t) = \sum_{n=1}^{\infty} E_n \cos\frac{n\pi x}{l}\, e^{-\frac{\alpha^2\pi^2 n^2}{l} t} + \frac{E_0}{2}$$
$$E_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\,dx\right), \quad E_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

Heat Equation

$\rightarrow \quad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

$$u(x,t) = e^{c^2 m^2 t}(c_1 e^{mx} + c_2 e^{-mx})$$

WAVE EQUATION

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$\rightarrow \quad u(x,t) = X(x)T(t)$

$$\frac{\partial u}{\partial x} = X'(x)T(t), \quad \frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$$

$$\frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$$

$$X(x)T''(t) = c^2 X''(x)T(t)$$

$$c^2 \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = d$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = d$$

$$\frac{X''(x)}{X(x)} = d \quad \Rightarrow \quad X''(x) = d\,X(x)$$

$$\Rightarrow \quad X''(x) - d\,X(x) = 0$$

Case I $\quad d = 0$

$$X''(x) = 0, \quad X''(x) = 0$$

$$\Rightarrow \quad \boxed{X(x) = c_1 x + c_2}$$

$$T''(t) = 0 \quad \Rightarrow \quad T(t) = d_1 t + d_2$$

$$u(x,t) = (c_1 x + c_2)(d_1 t + d_2)$$

Scanned by CamScanner

Case 2 $\alpha = p^2$

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = p^2$$

$$\frac{X''(x)}{X(x)} = p^2 \Rightarrow X''(x) = p^2 X(x)$$

$$\Rightarrow X''(x) - p^2 X(x) = 0$$

$$m^2 - p^2 = 0, \Rightarrow m = \pm p$$

$$X(x) = c_1 e^{px} + c_2 e^{-px}$$

$$T''(t) = c^2 p^2 T(t)$$

$$T''(t) - c^2 p^2 T(t) = 0$$

$$m^2 - c^2p^2 = 0 \qquad m = \pm cp$$

$$T(t) = d_1 e^{-cpt} + d_2 e^{+cpt}$$

$$T(t) = d_1 e^{-cpt} + d_2 e^{cpt}$$

$$u(x,t) = (c_1 e^{px} + c_2 e^{-px})(d_1 e^{-cpt} + d_2 e^{cpt})$$

Case 3 $\alpha = -p^2$

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -p^2$$

$$\frac{X''(x)}{X(x)} = -p^2 \Rightarrow X''(x) + p^2 X(x) = 0$$

$$X(x) = c_1 \sin px + c_2 \cos px$$

$$\frac{T''(t)}{c^2 T(t)} = -p^2$$

$$T''(t) + p^2 c^2 T(t) = 0$$

$$T(t) = d_1 \sin cpt + d_2 \cos cpt$$

$$u(x,t) = (c_1 \sin px + c_2 \cos px)(d_1 \sin cpt + d_2 \cos cpt)$$

Laplace Equation (Boundary condition)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(x,y) = X(x) Y(y)$$

$$X''(x) Y(y) + X(x) Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = d$$

$$\cancel{\frac{X''(x)}{X(x)} = d} \Rightarrow \cancel{X(x) =}$$

(1) if $d = 0$

$$\frac{X''(x)}{X(x)} = 0 \Rightarrow X''(x) = 0 \Rightarrow X(x) = c_1 x + c_2$$

$$\frac{-Y''(y)}{Y(y)} = 0$$

$$Y(y) = d_1 y + d_2$$

$$u(x,y) = (c_1 x + c_2)(d_1 y + d_2)$$

(II) if $\alpha = p^2$

$$\frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = p^2$$

$$X''(x) - p^2 X(x) = 0$$

$$X(x) = c_1 e^{px} + c_2 e^{-px}$$

$$-\frac{Y''(y)}{Y(y)} = p^2 \Rightarrow Y''(y) + p^2 Y(y) = 0$$

$$Y(y) = d_1 \cos py + d_2 \sin py$$

$$u(x,y) = (c_1 e^{px} + c_2 e^{-px})(d_1 \cos py + d_2 \sin py)$$

(III) if $\alpha = -p^2$

$$\frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = -p^2$$

$$\frac{X''(x)}{X(x)} + p^2 = 0$$

$$X''(x) + p^2 X(x) = 0$$

$$X(x) = c_1 \cos px + c_2 \sin px$$

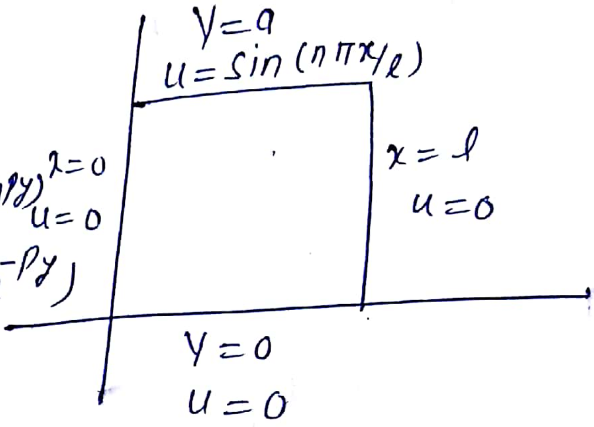
$$\frac{-Y''(y)}{Y(y)} = -p^2 \Rightarrow Y''(y) + p^2 Y(y) = 0$$

$$Y(y) = d_1 e^{py} + d_2 e^{-py}$$

$$u(x,y) = (d_1 e^{py} + d_2 e^{-py})(c_1 \cos px + c_2 \sin px)$$

Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions $u(0,y) = u(l,y) = u(x,0) = 0$, and $u(x,a) = \sin n\pi x\, l)$

<u>Sol</u> $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$



Trivial

$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$

$u = (c_9 x + c_{10})(c_{11} y + c_{12})$

Trivial

$$u(x,y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$$

$$c_5 (c_7 e^{py} + c_8 e^{-py}) = 0$$

$$c_5 = 0$$

$$u = c_6 \sin px (c_7 e^{py} + c_8 e^{-py})$$

$$c_6 \sin pl\, (c_7 e^{py} + c_8 e^{-py}) = 0$$

$$c_6 = 0 \quad \text{or} \quad \sin pl = 0$$

$$\sin pl = 0, \quad p = \frac{n\pi}{l}, \quad n = 0, \text{——}$$

$$u = c_6 \sin\left(\frac{n\pi x}{l}\right)\left(c_1 e^{\frac{n\pi y}{l}} + c_8 e^{\frac{-n\pi y}{l}}\right)$$

$$0 = c_6 \sin \frac{n\pi x}{l} \cdot (c_7 + c_8)$$

$$c_8 = -c_7$$

$$u(x,y) = b_n \sin\frac{n\pi x}{l}\left(e^{-n\pi y/l} - e^{-n\pi y/l}\right)$$

$$b_n = C_6 C_7$$

$$u(x,a) = \sin\frac{n\pi x}{l} = b_n \sin\frac{n\pi x}{l}\left(e^{\frac{n\pi a}{l}} - e^{\frac{-n\pi a}{l}}\right)$$

$$b_n = \frac{1}{\left(e^{n\pi a/l} - e^{-n\pi a/l}\right)}$$

$$u(x,y) = \frac{e^{n\pi y/l} - e^{-n\pi y/l}}{e^{n\pi a/l} - e^{-n\pi a/l}} \sin\frac{n\pi x}{l}$$

$$u(x,y) = \frac{\sinh(n\pi y/l)}{\sinh(n\pi a/l)} \sin\frac{n\pi x}{l}$$

EXAMPLE 29.19. (a) Solve the partial differential equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ representing the vibration of a string of length l, fixed at both ends, subject to the boundary conditions $y(0, t) = 0$, $y(l, t) = 0$ and initial conditions $y = y_0 \sin \frac{\pi x}{l}$ and $\frac{\partial y}{\partial t} = 0$ at $t = 0$. [GGSIPU III Sem End Term 2010]

$y(x,0) = y_0 \sin \frac{\pi x}{l}$

$y_t(x,0) = 0$

(b) Solve the vibrating string problem with

(i) $u(0, t) = 0 = u(l, t)$

(ii) $u(x, 0) = \begin{cases} x , & 0 < x < l/2 \\ l - x, & l/2 < x < l \end{cases}$

(iii) $u_t(x, 0) = x(l - x),\ 0 < x < l.$ [GGSIPU III Sem End Term 2013]

SOLUTION: **(a)** We use the method of separation of variables here.

Let us take $u = X(x) \cdot T(t)$ then the boundary conditions $y(0, t) = y(l, t) = 0$ suggest the form of solution to be $y = (c_1 \cos px + c_2 \sin px)(c_3 \cos apt + c_4 \sin apt)$ ...(1)

Applying the boundary condition $y(0, t) = 0$, gives

$$0 = (c_1 + 0)(c_3 \cos apt + c_4 \sin apt) \Rightarrow c_1 = 0.$$

$\therefore$ (1) becomes $y = \sin px\,(c'_3 \cos apt + c'_4 \sin apt)$ where c'_3 and c'_4 are arbitrary constants. ...(2)

Now using the boundary condition $y(l, t) = 0$, we get

$$0 = \sin pl\,(c'_3 \cos apt + c'_4 \sin apt)$$

which implies that $\sin pl = 0$ hence $p = n\pi/l$ where n is any integer.

Thus (2) becomes $$y = \sin \frac{n\pi x}{l}\left(c'_3 \cos \frac{an\pi t}{l} + c'_4 \sin \frac{an\pi t}{l}\right). \quad ...(3)$$

From (3) we can write $$\frac{\partial y}{\partial t} = \sin \frac{n\pi x}{l}\left[-\frac{an\pi}{l} c'_3 \sin \frac{an\pi t}{l} + \frac{an\pi}{l} c'_4 \cos \frac{an\pi t}{l}\right]$$

Since $\frac{\partial y}{\partial t} = 0$ at $t = 0$, we get $0 = c'_4 \frac{an\pi}{l} \sin \frac{n\pi x}{l} \Rightarrow c'_4 = 0$

$\therefore$ (3) becomes $y = c'_3 \sin \frac{n\pi x}{l} \cos \frac{an\pi t}{l}$.

From the initial condition $y = y_0 \sin \frac{\pi x}{l}$ at $t = 0$, we have

$$y_0 \sin \frac{\pi x}{l} = c'_3 \sin \frac{n\pi x}{l}.$$ It implies that $c'_3 = y_0$ and $n = 1$.

Therefore, the solution is $y = y_0 \sin \frac{\pi x}{l} \cos \frac{a\pi t}{l}$. **Ans.**

(b) The vibration of the string follows the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$ where $u(x, t)$ is the displacement function.

Here $u(0, t) = 0$ and $u(l, t) = 0$ and the initial displacement $u(x, 0) = f(x) = \begin{cases} x, & 0 < x < l/2 \\ l - x, & l/2 < x < l \end{cases}$

and the initial velocity $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) = x(l - x),\ 0 < x < l.$

The solution is $$u(x, t) = \sum_1^{\infty}\left[A_n \cos\frac{an\pi t}{l} + B_n \sin\frac{an\pi t}{l}\right]\sin\frac{n\pi x}{l},$$

where $$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2}{l}\left[\left\{-x\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)\right\}_0^{l/2} - \int_0^{l/2} -1\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)dx\right]$$

$$+\frac{2}{l}\left[\left\{-(l-x)\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)\right\}_{l/2}^{l} - \int_{l/2}^{l}(+1)\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)dx\right]$$

$$= \frac{2}{l}\left[\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\left\{\sin\frac{n\pi x}{l}\right\}_0^{l/2}\right] + \frac{2}{l}\left[\frac{+l^2}{2n\pi}\cos\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\left\{\sin\frac{n\pi x}{l}\right\}_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] + \frac{2}{l}\left[\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

and $$B_n = \frac{l}{an\pi}\cdot\frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}dx = \frac{2}{an\pi}\int_0^l x(l-x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{an\pi}\left[\left\{-x(l-x)\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)\right\}_0^l - \int_0^l (2x-l)\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)dx\right]$$

$$= \frac{2}{an\pi}\left[0 - \left\{(2x-l)\sin\frac{n\pi x}{l}\left(\frac{l^2}{n^2\pi^2}\right)\right\}_0^l + \int_0^l 2\sin\frac{n\pi x}{l}\cdot\left(\frac{l^2}{n^2\pi^2}\right)dx\right]$$

$$= \frac{2}{an\pi}\left[-\frac{l^3}{n^2\pi^2}(\sin n\pi - 0) - 2\frac{l^3}{n^3\pi^3}\left(\cos\frac{n\pi x}{l}\right)_0^l\right]$$

$$= \frac{-4l^3}{ax^4\pi^4}(\cos n\pi - 1) = \begin{cases} \dfrac{8l^3}{an^4\pi^4}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$\therefore$ The desired solution is

$$u(x, t) = \frac{4l}{\pi^2}\sum_1^{\infty}\left[\frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{an\pi t}{l}\sin\frac{n\pi x}{l}\right] + \frac{8l^3}{a\pi^4}\sum_1^{\infty}\left[\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\,a\pi t}{l}\sin\frac{(2n-1)\pi x}{l}\right]$$ **Ans.**

EXAMPLE 29.20. (*a*) A thin uniform tightly stretched vibrating string, fixed at the points $x = 0$ and $x = l$, satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{subject to the initial condition} \quad y(x, 0) = y_0 \sin^3 \frac{\pi x}{l}$$

Find the displacement $y(x, t)$ at any x and any time t.

[*GGSIPU III Sem End Term 2009*]

(*b*) The vibrations of an elastic string is governed by the partial differential equation $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$. The length of the string is π and the ends are fixed. The initial velocity is zero and the initial deflection is $u(x, 0) = 2(\sin x + \sin 3x)$. Find the deflection $u(x, t)$ of the vibrating string for $t > 0$.

[*GGSIPU III Sem II Term 2010*]

SOLUTION: (*a*) Here the boundary conditions and the initial conditions can be specified as

(*i*) $y(0, t) = 0$, (*ii*) $y(l, t) = 0$

(*iii*) $\dfrac{\partial y}{\partial t} = 0$ at $t = 0$ and (*iv*) $y(x, 0) = y_0 \sin^3 (\pi x/l)$.

The solution of the given equation which satisfies the conditions (*i*), (*ii*) and (*iii*) as obtained in Example 29.19 above, is

$$y = c'_3 \sin \frac{n\pi x}{l} \cos \frac{cn\pi t}{l}, \quad n = 1, 2, 3, \ldots \qquad \ldots(1)$$

Since c'_3 depends on n, the more general solution is

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{cn\pi t}{l} \qquad \ldots(2)$$

Now applying the condition (*iv*) in (2), gives

$$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

Using $\sin^3 \theta = \dfrac{1}{4}(3 \sin \theta - \sin 3\theta)$ here, we get

$$\frac{y_0}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right) = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{2\pi x}{l} + a_3 \sin \frac{3\pi x}{l} + a_4 \sin \frac{4\pi x}{l} + \ldots$$

This will be satisfied if $a_1 = \dfrac{3y_0}{4}$, $a_2 = 0$, $a_3 = -\dfrac{y_0}{4}$, $a_4 = a_5 = \ldots = 0$.

Thus, the final solution of the given equation, is

$$y = \frac{y_0}{4}\left(3 \sin \frac{\pi x}{l} \cos \frac{c\pi t}{l} - \sin \frac{3\pi x}{l} \cos \frac{3\pi t c}{l}\right). \qquad \textbf{Ans.}$$

(*b*) The suitable solution is $u = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mt + c_4 \sin mt)$ ...(1)

The initial and boundary conditions are

$$u(0, t) = 0, \; u(\pi, t) = 0, \; u(x, 0) = 2(\sin x + \sin 3x) \text{ and } \frac{\partial u}{\partial t}(x, 0) = 0.$$

Since $u(0, t) = 0$ we get $c_1 = 0$; and since $u(\pi, t) = 0$, we have

$$c_2 \sin m\pi\,(c_3 \cos mt + c_4 \sin mt) = 0 \Rightarrow \sin m\pi = 0 \quad \therefore \quad m = n \text{ where } n \in I.$$

Thus,

$$u = \sin nx\,(c_3 \cos nt + c_4 \sin nt)$$

$\therefore$

$$\frac{\partial u}{\partial t} = \sin nx\,(-c_3 \cdot n \sin nt + c_4 \cdot n \cos nt)$$

and since $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$, we have $c_4 = 0$.

Therefore, $u = c_3 \sin nx \cos nt = b_n \sin nx \cos nt$.

Most general solution is $u = \sum_{n=1}^{\infty} b_n \sin nx \cos nt$.

Since $u(x, 0) = 2 \sin x + \sin 3x$, we have

$$2 \sin x + 2 \sin 3x = \sum b_n \sin nx \Rightarrow b_1 = 2,\ b_2 = 0,\ b_3 = 2,\ b_4 = b_5 = \ldots = 0.$$

Therefore, the solution is $y = 2 \sin x \cos t + 2 \sin 3x \cos 3t$. **Ans.**

EXAMPLE 29.21. (a) Solve the boundary value problem $\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}$, given that $y(0, t) = 0$, $y(5, t) = 0$, $y(x, 0) = 0$ and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 5 \sin \pi x$. [*GGSIPU II Sem II Term 2007*]

(b) Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, $0 < x < l$, $0 < t < 4$ with boundary conditions and initial condition $u(x, 0) = f(x)$ and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x)$, $0 \le x \le l$.

[*GGSIPU III Sem End Term 2011*]

SOLUTION: (a) Applying the method of separation of variables to the wave equation $\frac{\partial^2 y}{\partial t^2} = 2^2 \frac{\partial^2 y}{\partial x^2}$.

The suitable solution is $y = (c_1 \cos px + c_2 \sin px)\ (c_3 \cos 2pt + c_4 \sin 2pt)$

Applying the initial condition $y(x, 0) = 0$, we have $c_3 (c_1 \cos px + c_2 \sin px) \Rightarrow c_3 = 0$

$$\therefore \quad y = c_4 (c_1 \cos px + c_2 \sin px) \sin 2pt = (c_1' \cos px + c_2' \sin px) \sin 2pt$$

where c_1' and c_2' are arbitrary constants.

Now using $y(0, t) = 0$ we get $0 = c_1' \sin 2pt \Rightarrow c_1' = 0$

$$\therefore \quad y = c_2' \sin 2pt \sin px.$$

Further, since $y(5, t) = 0$ we have $c_2' \sin 2pt \sin 5p = 0 \Rightarrow \sin 5p = 0 = \sin n\pi$

thus, we have $p = n\pi/5$, $n = 1, 2, 3, \ldots$

Therefore

$$y = c_2' \sin (n\pi 2t/5) \sin (n\pi x/5).$$

Also the boundary condition $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 5 \sin \pi x$, gives

$$c_2' \frac{n\pi 2}{5} \cos\left(\frac{n\pi 2t}{5}\right)_{t=0} \sin\left(\frac{n\pi x}{5}\right) = 5 \sin \pi x$$

$$\Rightarrow \quad n = 5 \quad \text{and} \quad c_2' = \frac{5}{2\pi}$$

Therefore, we have $y = \dfrac{5}{2\pi} \sin \pi x \sin 2\pi t.$ **Ans.**

(b) The suitable solution is $u = (C_1 \cos px + C_2 \sin px)\,(C_3 \cos c\,pt + C_4 \sin c\,pt)$

Since $u(0, t) = 0$ we have $C_1 = 0.$

and since $u(l, t) = 0,$ we have $0 = (C_2 \sin pl)\,(C_3 \cos cpt + C_4 \sin cpt)$

$$\Rightarrow \sin pl = 0 \quad \therefore \quad p = \frac{n\pi}{l},\ n \in I.$$

$$\therefore \quad u = C_2 \sin\frac{n\pi x}{l}\left[C_3 \cos\frac{cn\pi t}{l} + C_4 \sin\frac{n\pi ct}{l}\right] = \sin\frac{n\pi x}{l}\left(c_3' \cos\frac{cn\pi t}{l} + c_4' \sin\frac{cn\pi t}{l}\right).$$

Next $u(x, 0) = f(x)$ gives $c_3' \sin\dfrac{n\pi x}{l} = f(x),$ hence $c_3' = \dfrac{f(x)}{\sin\frac{n\pi x}{l}}.$

and $$u = f(x)\cos\frac{n\pi ct}{l} + c_4' \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}.$$

$$\therefore \quad \frac{\partial u}{\partial t} = -f(x)\sin\frac{n\pi ct}{l}\cdot\left(\frac{n\pi c}{l}\right) + c_4' \sin\frac{n\pi x}{l}\cdot\cos\left(\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right).$$

Since $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = g(x)$ we have

$$g(x) = 0 + c_4' \sin\frac{n\pi x}{l}\cdot\left(\frac{n\pi c}{l}\right) \qquad \therefore \quad c_4' = \frac{l}{n\pi c}\,\frac{g(x)}{\sin\frac{n\pi x}{l}}.$$

$$\therefore \quad u = f(x)\cos\frac{n\pi ct}{l} + \frac{l}{n\pi c}\,g(x)\sin\frac{n\pi ct}{l}.$$ **Ans.**

EXAMPLE 29.22. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in equilibrium state. If it is set vibrating by giving to each of its points a velocity $\mu x(l - x),$ find the displacement of the string at any point x from one end at any point of time $t.$

SOLUTION: The partial differential equation for the vibrating string is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

As per the boundary conditions provided, the form of solution of (1), is

$$y = (c_1 \cos px + c_2 \sin px)\,(c_3 \cos apt + c_4 \sin apt) \qquad ...(2)$$

Here $y(0, t) = 0,$ gives $0 = (c_1 + 0)\,(c_3 \cos apt + c_4 \sin apt) \quad \Rightarrow \quad c_1 = 0.$

Then (2) becomes $y = c_3 \sin px \,(c_1 \cos apt + c_2 \sin apt)$

or $y = \sin px \,(c'_1 \cos apt + c'_2 \sin apt)$ where c'_1 and c'_2 are arbitrary constants ...(3)

Further, since the string is initially at rest, $y(x, 0) = 0$

$\therefore$ (3) gives $0 = \sin px \,(c'_1 + 0) \Rightarrow c'_1 = 0$.

Therefore, (3) becomes $y = c'_2 \sin px \sin apt$. ...(4)

Also, from the condition $y(l, t) = 0$, we get $0 = c'_2 \sin pl \sin apt$

which gives $\sin pl = 0 \Rightarrow pl = n\pi$ or $p = \frac{n\pi}{l}$, $n = 1, 2, 3, \ldots$

$\therefore$ (4) becomes $y = c'_2 \sin \frac{n\pi x}{l} \sin \frac{an\pi t}{l}$, $n = 1, 2, 3, \ldots$

Thus, the most general solution can be written as

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sin \frac{an\pi t}{l}. \qquad \text{...(5)}$$

$$\therefore \quad \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{an\pi t}{l} \left(\frac{an\pi}{l}\right).$$

Using the boundary condition $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \mu x(l - x)$, we get

$$\mu x(l - x) = \sum_{n=1}^{\infty} b_n \left(\frac{an\pi}{l}\right) \sin \frac{n\pi x}{l}. \qquad \text{...(6)}$$

To determine b_n we expand $\mu x(l - x)$ as a half range Fourier sine series in $(0, l)$, to get

$$\mu x(l - x) = \sum_{n=1}^{\infty} b'_n \sin \frac{n\pi x}{l} \qquad \text{...(7)}$$

where $b'_n = \frac{2}{l} \int_0^l \mu x(l - x) \sin \frac{n\pi x}{l}\, dx = \frac{4\mu l^2}{\pi^3 n^3} [1 - (-1)^n]$ (on integrating by parts twice)

Comparing (6) and (7), yields $\frac{an\pi}{l} b_n = b'_n = \frac{4\mu l^2}{\pi^3 n^3} [1 - (-1)^n]$

$$\therefore \quad b_n = \frac{l}{an\pi} \cdot \frac{4\mu l^2}{\pi^3 n^3} [1 - (-1)^n] = \begin{cases} 0 & \text{when } n \text{ is even} \\ \frac{8\mu l^3}{a\pi^4 n^4} & \text{when } n \text{ is odd} \end{cases}.$$

Thus, the required solution is $y(x, t) = \sum_{n=1,3,5,\ldots} \frac{8\mu l^3}{a\pi^4 n^4} \sin \frac{n\pi x}{l} \sin \frac{n\pi a t}{l}$

or
$$y = \frac{8\mu l^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin \frac{(2n-1)\pi x}{l} \sin \frac{(2n-1)a\pi t}{l}. \qquad \textbf{Ans.}$$

EXAMPLE 29.23. Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ under the conditions $u = 0$ when $x = 0$ and when $x = \pi$, $\frac{\partial u}{\partial t} = 0$ when $t = 0$ and $u(x, 0) = x$, $0 < x < \pi$.

[GGSIPU III Sem End Term 201

SOLUTION: The possible solution suiting the boundary and initial conditions is

$$u = (C_1 \cos px + C_2 \sin px)(C_3 \cos apt + C_4 \sin apt).$$

Since $u = 0$ at $x = 0$ we have $C_1(C_3 \cos apt + C_4 \sin apt) = 0$ hence $C_1 = 0$.

Therefore $u = C_2 \sin px\,(C_3 \cos apt + C_4 \sin apt) = \sin px(C_3' \cos apt + C_4' \sin apt)$.

Further since $u = 0$ at $x = \pi$ we get $\sin \pi p = 0$ hence $p\pi = n\pi$, $n \in N$

$$p = n, \quad n \in I, \text{ hence } u = \sin nx(C_3' \cos ant + C_4' \sin ant).$$

Also, $\frac{\partial u}{\partial t} = 0$ when $t = 0$, hence $C_4' = 0$

$\therefore$ $u = \sin nx\,(C_3' \cos ant)$ where $n = 0, 1, 2, \ldots$

In general we can write $u(x, t) = \sum_{n=0}^{\infty} C_n \sin nx \cos ant$.

$$\therefore \quad u(x, 0) = \sum_{0}^{\infty} C_n \sin nx$$

Given that $u(x, 0) = x$, $0 < x < \pi$, we can write half-range Fourier series for x, $0 < x < \pi$, as

$$u(x, 0) = \sum b_n \sin nx$$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx = \frac{2}{\pi}\left[-\frac{x \cos nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} \frac{1(-\cos nx)}{n}\,dx$$

$$= -\frac{2}{n}\cos n\pi + \frac{2}{n\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = -\frac{2}{n}(-1)^n + 0$$

$$\therefore \quad u(x, 0) = \sum_{1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

But $u(x, 0) = \sum_{0}^{\infty} C_n \sin nx$ hence $C_n = \frac{2}{n}(-1)^{n+1}$.

Thus, $u(x, t) = \sum_{n=0}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx \cos ant$ **Ans.**

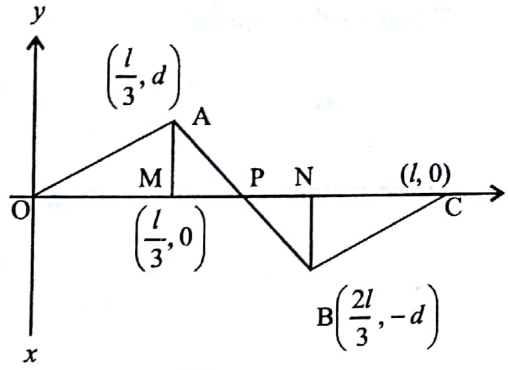
EXAMPLE 29.24. The points of trisection of a string are pulled aside through the same distance on opposite position of equilibrium and the string is released from rest. Find the expression for the displacement of the string at subsequent time and show that the mid point of the string always remains at rest.

[GGSIPU III Sem End Term 2009, II Term 201.

SOLUTION: The adjoining figure depicts the initial position of the string in parts OA, AB and BC.

Here OM = MN = NC = $l/3$

and AM = NB = d, OP = PC = $l/2$.



Equation of the part OA of the string, is $y = \left(\frac{3d}{l}\right)x$,

equation of the part AB of the string, is $y = \frac{3d}{l}(l-2x)$

and the equation of the part BC of the string, is $y = \frac{3d}{l}(x - l)$.

The vibrations of the string are given by the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{...(1)}$$

and the boundary conditions can be listed as

$$(i)\ \ y(0,t)=0 \quad (ii)\ \ \frac{\partial y}{\partial t}=0 \ \text{ at } \ t=0, \quad (iii)\ \ y(l,t)=0$$

and

$$(iv)\ \ y(x,0)=\begin{cases} \frac{3dx}{l}, & 0<x<l/3 \\ \frac{3d}{l}(l-2x), & l/3<x<2l/3. \\ \frac{3d}{l}(x-l), & 2l/3<x<l \end{cases}$$

As in Example 29.20(a) the solution of (1) satisfying the boundary conditions (*i*), (*ii*) and (*iii*), is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{an\pi t}{l} \quad \text{...(2)}$$

so that

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}. \quad \text{...(3)}$$

This is also the Fourier half range sine series of $y(x, 0)$ where

$$b_n = \frac{2}{l}\int_0^l y\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^{l/3}\left(\frac{3d}{l}\right)x\sin\frac{n\pi x}{l}\,dx + \frac{2}{l}\int_{l/3}^{2l/3}\frac{3d}{l}(l-2x)\sin\frac{n\pi x}{l}\,dx + \frac{2}{l}\int_{2l/3}^{l}\frac{3d}{l}(x-l)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{6d}{l^2}\left[-\left\{x\cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)\right\}_0^{l/3} - \int_0^{l/3} -\cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)dx + \left\{(l-2x)\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)\right\}_{l/3}^{2l/3}\right.$$

$$\left. -\int_{l/3}^{2l/3}(-2)\left(-\cos\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right)dx + \left\{-(x-l)\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)\right\}_{2l/3}^{l} - \int_{2l/3}^{l} -\cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)dx\right]$$

$$= \frac{6d}{l^2}\cdot\frac{l}{n\pi}\left[-\frac{l}{3}\cos\frac{n\pi}{3} + \left\{\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right\}_0^{l/3} - \left(\frac{-l}{3}\cos\frac{2n\pi}{3} - \frac{l}{3}\cos\frac{n\pi}{3}\right)\right.$$

$$\left. -2\left\{\sin\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right\}_{l/3}^{2l/3} - \frac{l}{3}\cos\frac{2n\pi}{3} + \frac{l}{n\pi}\left\{\sin\frac{n\pi x}{l}\right\}_{2l/3}^{l}\right]$$

$$= \frac{6d}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right) = -\frac{12d}{n^2\pi^2}\cos\frac{n\pi}{2}\sin\frac{n\pi}{6}.$$

EXAMPLE 29.25. (a) Find the temperature in a this metal root of length l with both ends insulated and with initial temperature in the rod $\sin(\pi x/l)$.

[GGSIPU III Sem End Term 2013]

(b) The equation for heat conduction along a bar of length l is $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ neglecting the radiation. Find an expression for $u(x, t)$ if the ends of the bar are maintained at zero temperature and if initially the temperature is T at the centre of the bar and falls uniformly to zero with time.

[GGSIPU IIIrd Sem. End Term 2004]

SOLUTION: (a) Temperature function $u(x, t)$ in the rod follows $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$.

Its suitable solution is $u = (c_1 \cos px + c_2 \sin px)\, e^{-c^2 p^2 t}$.

Both the ends of the rod are insulated hence $u(0, t) = 0$ and $u(l, t) = 0$

it implies that $c_1 = 0$ and $\sin pl = 0$ then $pl = n\pi$ or $p = \frac{n\pi}{l}$, $n = 1, 2, 3, \ldots$

Thus, we have $\quad u = c_2 \sin \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$, $n = 1, 2, 3, \ldots$

Since c_2 depends on the value of n, the general solution can be written as

$$u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}.$$

Therefore, $u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot 1$

But we are given that $u(x, 0) = \sin \frac{\pi x}{l}$, hence $\sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l} = \sin \frac{\pi x}{l}$.

Thus, we have $b_1 = 1$ and $b_2 = b_3 = \ldots = 0$.

$\therefore \quad u = b_1 \sin \frac{\pi x}{l} e^{-c^2\pi^2 t/l^2} \quad \Rightarrow \quad u = \sin \frac{\pi x}{l} e^{-c^2 \pi^2 t/l^2}$. **Ans.**

(b) The boundary conditions are $u(0, t) = 0$, $u(l, t) = 0$,

and the initial condition are

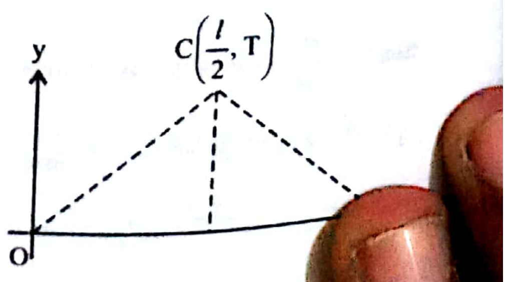
$$\left.\begin{aligned} u(x, 0) &= \frac{2T}{l} x && \text{for } 0 \le x \le l/2 \\ &= \frac{2T}{l}(l - x) && \text{for } l/2 \le x \le l. \end{aligned}\right\} \qquad \ldots(1)$$

Using the method of separation of variables, the choice of solution under the given boundary conditions, is

$$u = (c_1 \cos mx + c_2 \sin mx) e^{-a^2 m^2 t} \qquad \ldots(2)$$

The condition $u(0, t) = 0$ gives, $c_1 = 0$, hence

$$u = c_2 \sin mx \; e^{-a^2 m^2 t} \qquad \ldots(3)$$



The condition $u(l, t) = 0$ gives $\sin ml = 0 \Rightarrow ml = n\pi$ or $m = \frac{n\pi}{l}$ where $n \in N$.

hence $u = c_2 \sin \frac{n\pi x}{l} \, e^{-a^2 n^2 \pi^2 t/l^2}$ ...(4)

$\therefore$ the most general solution is $u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-a^2 n^2 \pi^2 t/l^2}$. ...(5)

Applying the initial condition (1) here, we have $u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$.

where
$$b_n = \frac{2}{l}\int_0^l u(x, 0) \sin \frac{n\pi x}{l} dx$$
$$= \frac{2}{l}\int_0^{l/2} \frac{2T}{l} x \sin \frac{n\pi x}{l} dx + \frac{2}{l}\int_{l/2}^{l} \frac{2T}{l}(l - x) \sin \frac{n\pi x}{l} dx$$
$$= \frac{4T}{l^2}\left[\left\{-x \cos \frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)\right\}_0^{l/2} - \int_0^{l/2} (-1) \cos \frac{n\pi x}{l}\cdot\frac{l}{n\pi} dx\right.$$
$$\left. + \left\{-1(l - x) \cos \frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)\right\}_{l/2}^{l} - \int_{l/2}^{l} \cos \frac{m\pi l}{l}\cdot\frac{l\,dx}{n\pi}\right]$$
$$= \frac{4T}{l^2}\left[-\frac{l^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\left\{\sin \frac{n\pi x}{l}\right\}_0^{l/2} + \frac{l^2}{2n\pi} \cos \frac{n\pi}{2} - \frac{l^2}{n\pi^2}\left\{\sin \frac{n\pi x}{l}\right\}_{l/2}^{l}\right]$$
$$= \frac{4T}{l^2}\left(\frac{l^2}{n^2\pi^2}\right)\left(\sin \frac{n\pi}{2} + \sin \frac{n\pi}{2}\right) = \frac{8T}{n^2\pi^2} \sin \frac{n\pi}{2}$$

Substituting this value of b_n in (4), we get

$$u(x, t) = \frac{8T}{\pi^2} \sum_{1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{l} e^{-a^2 n^2 \pi^2 t/l^2}$$ which is the required solution. **Ans.**

EXAMPLE 29.26. A bar of length l with insulated sides is initially at 0°C temperature throughout. The end $x = 0$ is kept at 0°C for all time and the heat is suddenly applied such that $\frac{\partial u}{\partial x} = 10$ at $x = l$ for all time. Find the temperature function $u(x, t)$.

SOLUTION: The temperature function $u(x, t)$ satisfies the partial differential equation.

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad ...(1)$$

in the present case. The method of separation of variables is to be used here.
The possible solutions of (1) are

$$u = (c_1 \cos nx + c_2 \sin nx) \, e^{-a^2 n^2 t} \quad ...(2)$$
$$u = (c_1 e^{nx} + c_2 e^{-nx}) \, e^{-a^2 n^2 t} \quad ...(3)$$
$$u = (c_1 x + c_2) \quad ...(4)$$

...(5)

The given boundary conditions are $u(0, t) = 0$ ...(6)

and $\left(\frac{\partial u}{\partial x}\right)_{x=l} = 10$ for all t ...(7)

and $u(x, 0) = 0.$

The boundary condition (6) is such that none of the above mentioned solutions (2), (3) and (4) satisfies it, therefore we shall use a combination of (2) and (4) to satisfy all the given conditions of the problem.

Let the solution be $u = c_1 x + c_2 + (c_3 \cos nx + c_4 \sin nx)\, e^{-a^2 n^2 t}$ ...(8)

The condition $u(0, t) = 0$ when applied to (8), gives

$$0 = c_2 + c_3\, e^{-a^2 n^2 t} \quad \text{for all } t. \quad \therefore \quad c_2 = 0 \text{ and } c_3 = 0.$$

Therefore, (8) becomes $u = c_1 x + c_4 \sin nx\ e^{-a^2 n^2 t}$ ...(9)

$$\therefore \quad \frac{\partial u}{\partial x} = c_1 + n \cdot c_4 \cos nx\ e^{-a^2 n^2 t}$$

Using the condition (6), gives $10 = c_1 + n \cdot c_4 \cos nl e^{-a^2 n^2 t}$

$\Rightarrow \quad c_1 = 10$ and $\cos nl = 0$ as $c_4 \neq 0.$

$\therefore \quad nl = m\pi + \pi/2$ or $n = (2m + 1)\,\pi/(2l)$ where $m \in I$

Therefore, (9) becomes

$$u = 10x + c_4 \sin \frac{(2m+1)\,\pi\, x}{2l}\, e^{-a^2 \frac{(2m+1)^2 \pi^2 t}{4l^2}}, \quad m = 0,1,2,3,...$$

The most general solution can be written as

$$u = 10x + \sum_{m=0}^{\infty} b_m \sin \frac{(2m+1)\,\pi x}{2l}\, e^{-a^2 \frac{(2m+1)^2 \pi^2 t}{4l^2}}$$

Using here the condition (7), we get

$$\sum_{m=0}^{\infty} b_m \sin\left(\frac{(2m+1)\,\pi x}{2l}\right) \cdot 1 = -\,10x \qquad ...(10)$$

The left hand side in (10) should be the half range Fourier sine series for $-\,10x$ for which

$$b_m = \frac{2}{l} \int_0^l -10x \sin\left(\frac{(2m+1)\pi\, x}{2l}\right) dx$$

$$= \frac{20}{l}\left[x \cos\left(\frac{(2m+1)\,\pi\, x}{2l}\right) \cdot \frac{2l}{(2m+1)\,\pi}\right]_0^l - \frac{20}{l} \int_0^l 1 \cdot \cos\left(\frac{(2m+1)\,\pi\, x}{2l}\right) \cdot \frac{2l}{(2m+1)\,\pi}\, dx$$

$$= 0 - \frac{40}{(2m+1)\,\pi}\left[\sin\left(\frac{(2m+1)\,\pi\, x}{2 \cdot l}\right)\left\{\frac{2l}{(2m+1)\,\pi}\right\}\right]_0^l = -\,\frac{80\, l}{(2m+1)^2\, \pi^2} \sin\left(\frac{(2m+1)\,\pi}{2}\right)$$

Thus, the final solution is

$$u = 10x - \frac{80\, l}{\pi^2} \sum_{m=0}^{\infty} \frac{\sin\,(2m+1)\,\pi x/2}{(2m+1)^2} \cdot e^{\frac{-a^2 (2m+1)^2 \pi^2 t}{4l^2}}.$$

Ans.

EXAMPLE 29.27. A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to $0°C$ and are kept at that temperature. Prove that the temperature function $u(x, t)$ is given by $u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{\frac{-c^2\pi^2 n^2 t}{l^2}}$ where b_n is determined from the equation $u(x, 0) = u_0$.

[GGSIPU III Sem End Term 2005]

SOLUTION: The temperature function $u(x, t)$ satisfies $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ whose suitable solution is

$$u(x, t) = (c_1 \cos px + c_2 \sin px)\, e^{-c^2p^2t} \qquad \dots(1)$$

As per boundary conditions $u(0, t) = 0 = u(l, t)$ we have $c_1 e^{-c^2p^2t} = 0 \Rightarrow c_1 = 0$

and $\quad c_2 \sin pl\, e^{-c^2p^2t} = 0 \Rightarrow \sin pl = 0$ or $pl = n\pi$ or $p = n\pi/l$.

$$\therefore \quad u = c_2 \sin \frac{n\pi x}{l} e^{\frac{-c^2n^2\pi^2t}{l^2}} \quad \text{where } n = 1, 2, 3, \dots$$

hence C_2 has to be replaced by b_n then adding all such solutions, we have

$$u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{\frac{-c^2\pi^2n^2t}{l^2}}$$

Since $u(x, 0) = u_0$ we have $u_0 = \Sigma b_n \sin \frac{n\pi x}{l}$.

which is a half range Fourier sine series of u_0, hence

$$b_n = \frac{2}{l}\int_0^l u_0 \sin \frac{n\pi x}{l} dx = \frac{-2u_0}{l}\left[\cos \frac{n\pi x}{l}\right]_0^l \cdot \frac{l}{n\pi} = \frac{2u_0}{n\pi}(1-\cos n\pi) = \begin{cases} 0 & \text{when } n \text{ is even} \\ \dfrac{4u_0}{n\pi} & \text{when } n \text{ is odd.} \end{cases}$$

Therefore $$u(x, t) = \frac{4u_0}{\pi} \Sigma \frac{1}{n} \sin \frac{n\pi x}{l} e^{\frac{-c^2n^2\pi^2t}{l^2}}, \quad n = 1, 3, 5, \dots$$

or $$u(x, t) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{l} e^{-c^2(2n-1)^2\pi^2t/l^2}.$$

EXERCISE 29C

1. Obtain the form for Y as a function of y alone if $u = Y \sin mx$ is a solution of the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and then find the solution of the equation such that $u = 0$ when $y = -1$ and $u = \sin x$ when $y = 1$.

2. Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions

   (*i*) $u(0, y) = 0$, (*ii*) $u(1, y) = 0$,

   (*iii*) $u(x, \infty) = 0$ and (*iv*) $u(x, 0) = \sin^3 \pi x$ for $0 < x < 1$.

3. A long rectangular plate of width a cm with insulated surfaces has its temperature equal to zero on both long sides and on one of the short sides so that $u(0, y) = 0$, $u(a, y) = 0$, $u(x, \infty) = 0$ and $u(x, 0) = kx$. Find the expression for the temperature at any point of the plate in the steady state.

4. A square plate of width 'a' is bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = a$. The edges $x = 0$, $y = a$ are kept at zero temperature, the edge $y = 0$ is insulated and the edge $x = a$ is kept at the temperature u_0. Find the steady state temperature u at any point of the plate.

5. A tightly stretched string of length l is attached at $x = 0$ and at $x = l$. Find the expression for y the displacement of the string at a distance x, given that $y = A \sin \frac{2\pi x}{l}$ at $t = 0$.

   [*GGSIPU III Sem II Term 2009*]

6. An elastic string of length l is stretched. One end is taken as the origin and at a distance $2l/3$ from this end the string is displaced through a distance 'a' transversely and the released from rest. Obtain $y(x, t)$ the vertical displacement at any x if y satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

7. A string is stretched tightly between $x = 0$ and $x = l$ and both the ends are given the displacement $y = a \sin pt$ perpendicular to the string. The displacement y in the string satisfies the equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 y}{\partial t^2}$. Obtain the expression for the displacement in the string at any time t.

8. Solve the differential equation $\frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u = \sin t$ at $x = 0$ and $\frac{\partial u}{\partial x} = \sin t$ at $x = 0$.

   [*GGSIPU III Sem II Term 2011*]

9. A rod l cm long with insulated lateral surface is initially at zero temperature throughout. If one end is kept at $0^\circ C$ and the other at 100° C find the temperature function $u(x, t)$ where x is the distance measured along the rod from the end which is at 0° C.

10. Solve the equation $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$ for the conduction of heat along a thin rod without radiation subject to the following boundary and initial conductions

(i) u remains finite for any lapse of time

(ii) $\frac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$

(iii) $u = x(l - x)$ at $t = 0$ between $x = 0$ and $x = l$.

11. Using the method of separation of variables, solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{where} \quad u(x, 0) = 6e^{-3x} \quad x > 0, \quad t > 0.$$

[GGSIPU III Sem End Term 2010]

12. Solve the equation $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ by the method of separation of variables.

13. An insulated rod of length l has its ends A and B maintained at 0 °C and 100 °C respectively until steady state conditions prevail. If B is suddenly reduced to 0 °C and maintained at 0 °C, find the temperature at a distance x from A at time t.

❑❑❑