Biased Random Compass for Online Routing

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1 Introduction

In recent years, motivated primarily by the proliferation of wireless networks and GPS devices, much research has been done on routing algorithms for geometric networks [4]. In this research a network is modelled as a geometric graph $G = (V, E)$ whose vertex set V is a set of points in $\mathbb{R}^2$. We say that a routing algorithm $\mathcal{A}$ *works* for G if, for any pair of vertices $s, t \in V$, the algorithm always find a path from s to t in a finite number of steps.

The research on geometric routing algorithms largely focuses on utilizing geometric properties of a class of geometric graphs to reduce the complexity of, and information required by, routing algorithms. For example, when G is the unit disk graph of the points in V, then an algorithm, called FACE-1, of Bose *et al* [3] works and requires no preprocessing of G or additional state information at the vertices of G and requires only a constant size header associated with each packet.

A particularly interesting and restricted class of routing algorithms are so-called oblivious routing algorithms. An *oblivious* routing algorithm is one in which the decision about the next edge on the route to t for a packet currently located at node v is made based only on v, t, and the neighbourhood, $N(v)$, of v.[1] In particular, an oblivious algorithm does does not make use of information obtained in previous routing steps and can not use any global information about G.

Bose and Morin [2] showed that if G is Delaunay triangulation or a regular triangulation then deterministic oblivious routing algorithms named GREEDY and COMPASS, respectively, guarantee delivery of a packet between any source-destination pair. Bose *et al* [1] later proved a stronger result showing that a deterministic oblivious routing algorithm named GREEDY-COMPASS works for any convex subdivision G.

Oblivious routing algorithms are so simple, elegant, and practical that researchers have spent considerable effort designing geometric embeddings of graphs so that oblivious routing algorithms can be applied to the resulting embeddings. A famous example in this vein is due to Leighton and Moitra [5] who proved that every 3-connected planar graph $\tilde{G}$ admits an embedding G in $\mathbb{R}^2$ such that GREEDY works on G. The combination of the embedding and routing algorithm represents a form of *compact routing* [7].

Bose *et al* also showed that deterministic oblivious routing algorithm are, however, inherently limited. There exists 17 convex subdivisions $G_1, \ldots, G_{17}$ each with 17 vertices such that any deterministic oblivious routing algorithm does not work for at least one of these subdivisions. Thus, convex subdivisions form a class of geometric graphs that are too rich for deterministic oblivious routing algorithms [1].

[1]More precisely, a deterministic oblivious routing algorithm is a function $f : \mathbb{R}^2 \times \mathbb{R}^2 \times (\mathbb{R}^2)^+ \to \mathbb{R}^2$ that satisfies $f(v, t, N) \in N$ and $f(t, t, N) = t$ for all inputs.

The authors [1, 2] did, however, observe that, if randomization is allowed, then an oblivious algorithm, named RandomCompass, that uses one random bit per step works for any convex subdivision. They did not analyze the efficiency of Random-Compass except to note that, for some convex subdivisions G, and some pairs $s, t \in V$, the expected number of steps taken by Random-Compass when routing from s to t is $\Omega(|V|^2)$. Note that this is in contrast to deterministic oblivious routing algorithms where any route has length at most $|V|-1$ (otherwise the route contains a cycle that must repeat forever since the algorithm is oblivious and deterministic).

In the current paper, we show that a simple variant of RandomCompass called Biased-Random-Compass not only works for any convex subdivision G, but finds a path between any pair of vertices $s, t \in V$ of expected length $O(n)$. Thus, Biased-Random-Compass is a practical randomized oblivious routing algorithm that works efficiently for any convex subdivision.

The remainder of the paper is organized as follows: In 2 we describe Random-Compass and Biased-Random-Compass algorithms. In Section 3, we prove some properties of a directed graph G' induced by the Section BiasedRandomCompass algorithm. In Section 4 we prove our main theorem, namely that BiasedRandomCompass works and that the expected number of steps required to reach the destination is $O(n)$.

2 RandomCompass and BiasedRandomCompass

Let $G = (V, E)$ be a convex subdivision and fix a destination vertex $t \in V$. For a vertex $v \in V$, $v \neq t$, consider the line segment vt. If vt does not contain some edge $vw \in E$ then vt intersects the interior of some face f of G that is incident on v. Define $\mathrm{ccw}(v)$ and $\mathrm{cw}(v)$ be the neighbours of v on f in the clockwise and counterclockwise directions, respectively (see Figure 1.) In the degenerate case where vt contain some edge vw we use the convention that $\mathrm{cw}(v) = \mathrm{ccw}(v) = w$.

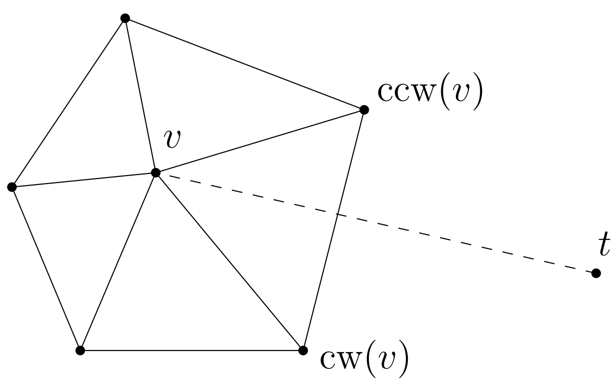

Figure 1: Definition of $\mathrm{cw}(v)$ and $\mathrm{ccw}(v)$

The Random-Compass algorithm [2, 1] works as follows: When a packet is situated at a node v and its final destination is t, the packet is forwarded to exactly one of $\mathrm{cw}(v)$ or $\mathrm{ccw}(v)$ with equal probability. Bose *et al* show that, with probability 1, Random-Compass reaches t in a finite number of steps. However, in case where G is a regular convex n-gon and the source vertex s and the destination vertex t are antipodal, Random-Compass takes a simple random walk on the vertices of G and will not reach t until its distance from its starting location s exceeds $n/2$. A well-known result on random walks is that the expected time for this to happen is $\Theta(n^2)$.

Motivated by the above, we consider the following BIASED-RANDOM-COMPASS algorithm: When a packet is situated at a node v and its final destination is t, the packet is forwarded to $\mathrm{cw}(v)$ with probability 1/3 otherwise (with probability 2/3) it is forwarded to $\mathrm{ccw}(v)$. The remainder of this paper is dedicated to proving that the expected number of steps taken by BIASED-RANDOM-COMPASS before it reaches its target is $O(n)$.

3 The Directed Graph G'

Let $G' = (V, E')$ be the directed graph derived from G by setting

$$E' = \{vw : v \in V \text{ and } w \in \{\mathrm{cw}(v), \mathrm{ccw}(v)\}\} \ .$$

Note that any route taken by BIASED-RANDOM-COMPASS is restricted to travelling on the directed edges of G'. In this section we consider some properties of G' that will allow us to prove our main result.

Claim 1. *Any simple directed cycle in G' contains t in its interior.*

Proof. The destination vertex t can not be on any cycle, because all the edges adjacent to t are directed towards t. Suppose by way of contradiction that t is exterior to some directed cycle C. Assume, without loss of generality that C is directed clockwise. Label the vertices of C $x_1, \ldots, x_k$ in order and select a vertex x_i such that (see Figure 2)

1. the line ℓ through x_i and t is locally tangent to C at x_i, and
2. the angles, measured counterclockwise, at x_i satisfy $\angle t x_i x_{i+1} > \angle t x_i x_{i-1}$.

The existence of such a vertex is easily established using the assumption that t is outside of C. Since G is a convex subdivision, there must be an edge $x_i y \in G$ not contained in the halfspace bounded by ℓ and containing C. But then $x_{i+1} \notin \{\mathrm{cw}(x_i), \mathrm{ccw}(x_i)\}$ contradicting the assumption that $x_i x_{i+1}$ is an edge of G'. □

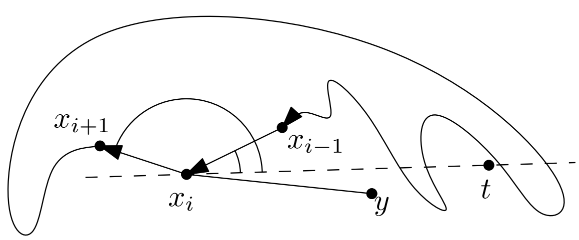

Figure 2: C does not contain t

Claim 2. *Any simple cycle in G' is convex.*

Proof. BROKEN - fix □

Lemma 1. *Any two simple cycles in G' can not share any vertices.*

Proof. Suppose two simple cycles A and B in G' share one or more vertices (see Figure 3). Without loss of generality, assume that cycle A is directed clockwise and label its vertices $x_1, \ldots, x_k$ in order. Let x_i be a vertex that is part of A and part of B but such that x_{i+1} is not in B.Let y be the successor of x_i in B. Without loss of generality, we may assume that y is in the interior of A (otherwise reverse the roles of A and B). Since A is convex (Claim 2) and contains t (Claim 1), $x_{i+1} = \mathrm{ccw}(x_i)$. But since B is also convex and contains t the existence of the edge $x_i y$ in G implies that $\mathrm{ccw}(x_i) \neq x_{i+1}$, a contradiction that completes the proof. □

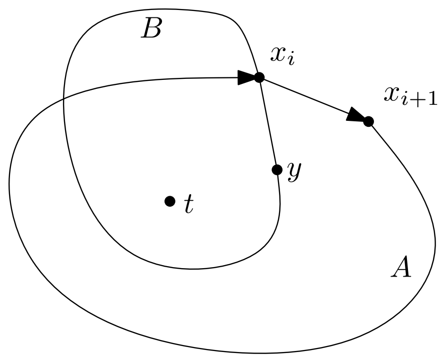

Figure 3: Two cycles share vertices

4 The Time Complexity of the Biased Algorithm

In this section, we prove an upper bound on the expected number of steps required for Biased-Random-Compass to reach its destination. Our argument makes use of a result from probability theory which we now review.

Let T be a positive, integral, finite random variable and let $X = \langle X_1, X_2, \ldots \rangle$ be an infinite sequence of random variables. We call T a *stopping time* for X if the event that $T = n$ is determined by the outcome of the random variable $X_1, \ldots, X_n$. We then have the following result for stopping times [6, Chapter 6]:

Theorem 1 (Wald's Equation for Stopping Times)**.** *Let $X = \langle X_1, X_2, \ldots \rangle$ be independent and identically distributed random variables with $\mathrm{E}[|X_i|] < \infty$, and let T be a stopping time for X with $\mathrm{E}[T] < \infty$. Then*

$$\mathrm{E}\left[\sum_{i=1}^{T} X_i\right] = \mathrm{E}[T] \cdot \mathrm{E}[X].$$

Next we define a labelling of the vertices of G' that allows us to quantify the progress of Biased-Random-Compass. Consider the strongly connected components $V_0(= \{t\}), V_1, \ldots, V_k$ of G' and order these components so that if G' contains an edge vw with $v \in C_i$ and $w \in C_j$ then $i > j$. That this ordering exists follows from the fact that the ordering condition defines a directed acyclic graph over $V_0, \ldots, V_k$ with one sink V_0.

This numbering partitions the vertices of G' into j classes $V_0 = \{t\}, V_1, \ldots, V_j$ according to their pseudo-distances to t.

Note that any path in G' from some vertex s to t visits the strongly connected components in reverse order. That is the path visits a sequence of components $V_{i_1}, \ldots, V_{i_r} = V_0$ and $i_j > i_{j+1}$ for all $j \in \{1, \ldots, r-1\}$. Our strategy is to show that the expected time BIASED-RANDOM-COMPASS spends at the vertices V_i is bounded by $O(|V_i|)$. Therefore, the total expected time spent by BIASED-RANDOM-COMPASS before reaching t is $O(n)$.

Each strongly connectec component V_i is one of three types:

1. V_i is a (doubly-directed) path,

2. V_i contains a single simple cycle, or

3. V_i is a singleton set.

That these three possibilities are exhaustive follows from Lemma 1. In particular, no V_i contains more than a single simple cycle. Next we consider each of the three cases, starting with the easiest first.

Case 3: If $V_i = \{v\}$ is a singleton set then, by definition, both $\mathrm{cw}(v)$ and $\mathrm{ccw}(v)$ are not in V_i so BIASED-RANDOM-COMPASS spends at most one step in V_i.

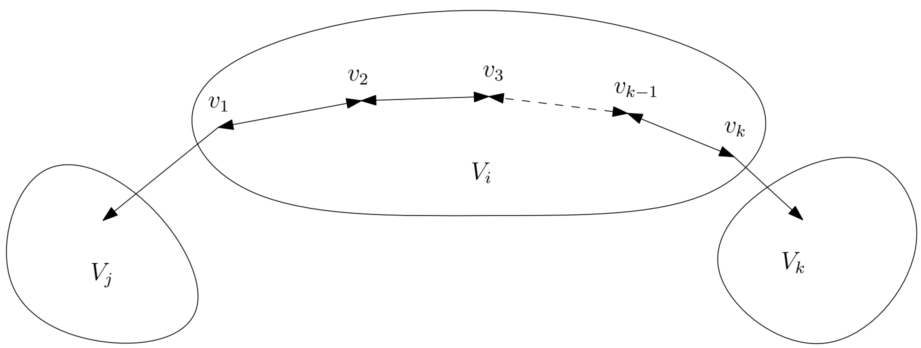

Figure 4: A chain of doubly directed edges

Case 1: Refer to Figure 4. Suppose there are k vertices in the chain of doubly directed edges, and $V_i = \{v_1, \ldots, v_k\}$. If the packet goes clockwise k more times than counterclockwise, then the packet will leave the chain and never come back.

We will first prove that the expected time to leave V_i is finite and then use this property to obtain a tighter bound. Suppose the packet goes clockwise (to $\mathrm{ccw}(v)$) with probability $p = \frac{2}{3}$ and counterclockwise (to $\mathrm{cw}(v)$) with probability $1 - p = \frac{1}{3}$. Let r and l be the number of steps to the clockwise and counterclockwise directions respectively after $c3k$ steps. After $c3k$ steps,

$$\begin{aligned}
\Pr[r-l \le k-1] &= \Pr\left[l \ge \frac{c3k-k+1}{2}\right] \\
&= \Pr\left[l \ge \frac{c3k-k+1}{c2k}ck\right] \\
&= \Pr\left[l \ge \frac{c3k-k+1}{c2k}\mathrm{E}[l]\right] \\
&\le \frac{c2k}{c3k-k+1}\,\text{(Markov's inequality)} \\
&= \frac{2}{3}-\epsilon\,\text{(where ϵ is a positive number)}
\end{aligned}$$

Define a round as $c3k$ steps. The probability that the packet leaves the chain in each round is then greater than $(\frac{1}{3}+\epsilon)$. The expected number of rounds required to leave the chain is therefore less than

$$\sum_{i=1}^{\infty} i\frac{1}{3}\left(\frac{2}{3}\right)^{i-1} = \frac{1}{3}\frac{1}{(1-\frac{2}{3})^2} = 3.$$

Hence, the total expected number of steps to leave the doubly directed chain is less than $c9k$.

Now that we know that the expected time to leave the chain is finite. We can use the Wald's Equation to get a tighter bound on the expected time. Let

$$X_i = \begin{cases} 1 & \text{if the packet goes clockwise at the i-th step} \\ -1 & \text{if the packet goes counterclockwise at the i-th step} \end{cases}$$

and

$$T = \min\left\{i : \sum_{j=1}^{i} X_j = k\right\}.$$

We know that $\mathrm{E}[T] < c9k$, so by Wald's Equation,

$$\begin{aligned}
\mathrm{E}\left[\sum_{j=1}^{T} X_j\right] &= \mathrm{E}[T]\cdot\mathrm{E}[X_i] \\
k &= \mathrm{E}[T]\cdot\left(\frac{2}{3}\cdot 1+\frac{1}{3}\cdot -1\right) \\
\mathrm{E}[T] &= 3k
\end{aligned}$$

In other words, the expected time to leave the chain is $3k$, i.e. $3|V_i|$.

Case 2: if the cycle consists of only singly directed edges, the probability of leaving the cycle at any node is at least $\frac{1}{3}$. Therefore, the expected number of steps in the cycle is at most

$$\sum_{i=1}^{\infty} i\frac{1}{3}\left(\frac{2}{3}\right)^{i-1} = 3 \le |V_i|.$$

If some parts of the cycle are doubly directed edges, we have the following claim.

Claim 3. *Any simple cycle in G' contains at least 3 singly directed edges.*

Proof. This claim is true when all edges are singly directed since any simple cycle contain at least 3 edges. Suppose there is only one singly directed edge and the remaining are doubly directed edges in a cycle C (see Figure 5). Let x_n be the head of the singly directed edge. There must be an edge

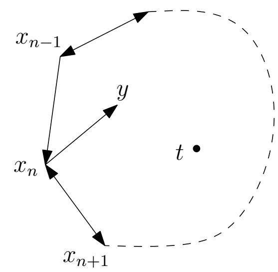

Figure 5: A cycle with one singly directed edge

$x_n y \in G$ contained in C. There can not be any other edge with one end on C and the other end in C, because all other edges are doubly connected. If $x_n y$ exists G can not be a convex subdivision. Therefore, there are at least two singly directed edges in C.

If there are two singly directed edges and the remaining are doubly directed in cycle C, they are either adjacent or separated. If they are adjacent (see Figure 6(a)), let x_n and x_{n+1} be the heads

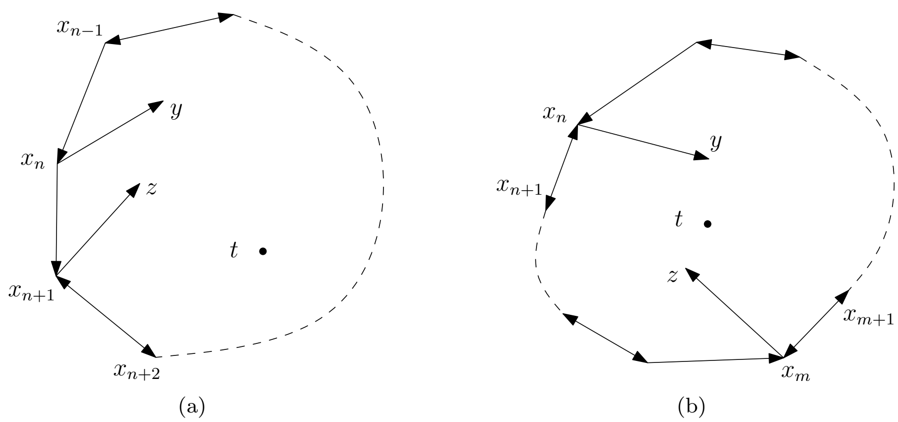

Figure 6: Cycles with two singly directed edges

of the two singly directed edges. There must be two edges $x_n y, x_{n+1} z \in G$ contained in C. There can not be any other edge with one end on C and the other end in C for the same reason as before. Hence, G can not be a convex subdivision.

If the two singly directed edges are separated (see Figure 6(b)), let x_n and x_m be the heads of the two edges. Similarly, there must be two edges $x_n y, x_m z \in G$ contained in C, and there can not be any other edge with one end on C and the other end in C. Vertices x_n, y, z, and x_m can not be collinear, otherwise, edges $x_n x_{n+1}$ and/or $x_m x_{m+1}$ can not be doubly directed. Hence, G can not be a convex subdivision.

Therefore, there are at least 3 singly directed edges in a cycle. $\square$

Suppose cycle C contains ℓ doubly directed edge chains $h_1, \ldots, h_\ell$ ($\ell \geq 3$), then it contains at least ℓ singly directed edges. The probability that a packet goes back to h_1 after leaving it is at most $p = \left(\frac{2}{3}\right)^\ell$. Suppose v is in h_1, the expected number of times that the packet visits h_1 is no more than

$$\sum_{i=0}^{\infty}(i+1)(1-p)p^i = \frac{1}{1-p}.$$

Let T be the number of times the packet visits h_1, and N_i be the number of steps taken in h_1 before leaving it at the i-th visit. According to Wald's equation and the discussion above,

$$\mathrm{E}\left[\sum_{i=1}^{T} N_i\right] = \mathrm{E}[T] \cdot \mathrm{E}[N] \leq \frac{1}{1-p} \cdot 3|h_1| \leq \left(1 - \left(\frac{2}{3}\right)^3\right)^{-1} \cdot 3|h_1| < 4.264|h_1|.$$

Since v is in h_1, the packet will not visit any other chain more often, then the expected number of times that the packet visits h_i for $1 \leq i \leq \ell$ is no more than $(1-p)^{-1}$. The expected total time spent in h_i is then no more than $4.264|h_i|$. Therefore, the expected total time spent in C is no more than

$$4.264|C| \leq 4.264|V_i|.$$

Theorem 2. *Let G be a convex subdivision of n vertices. The expected execute time of the* BIASED-RANDOM-COMPASS *algorithm on G is less than $4.264n$.*

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