Modernizing the Teaching of Confidence Intervals in Mathematical Statistics Courses

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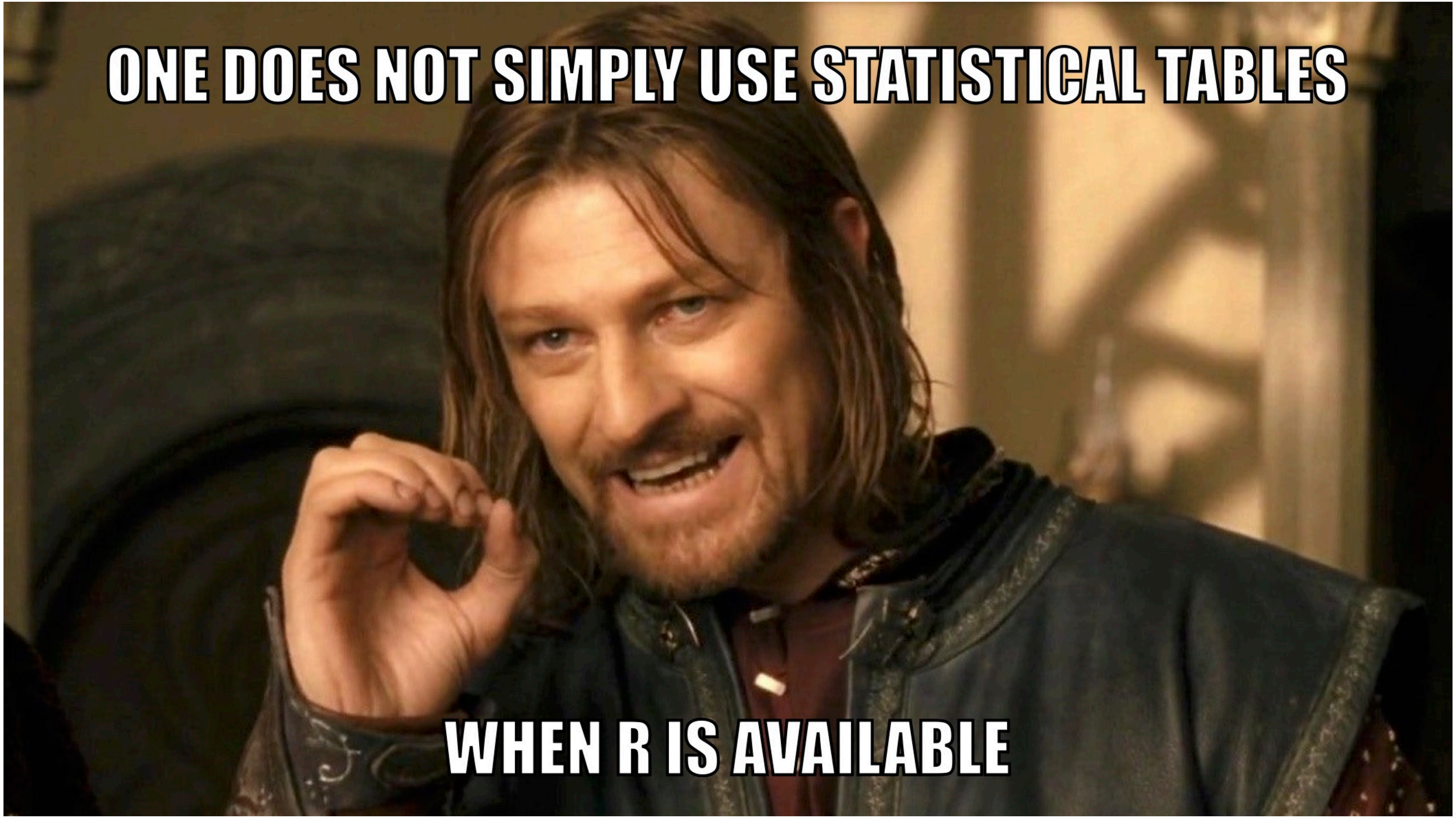
Motivation

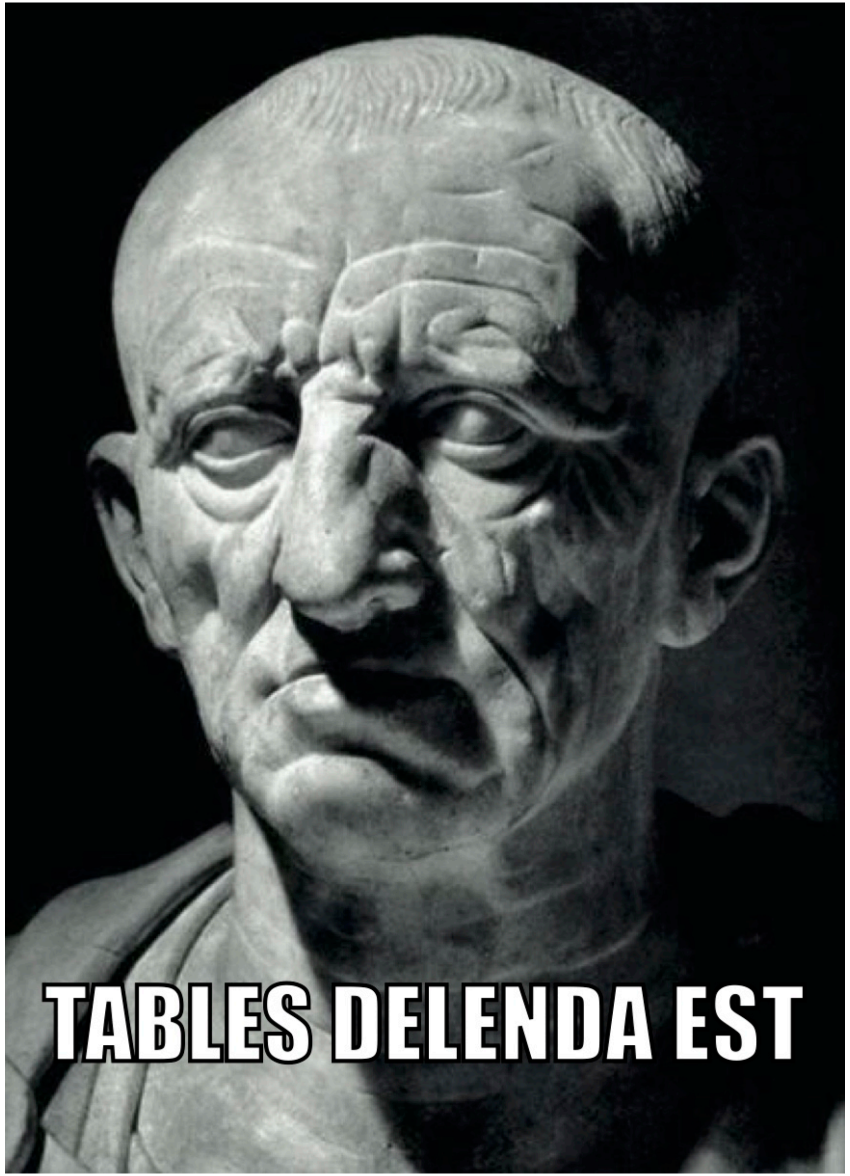
- A (two-sided) confidence interval $[\hat{\theta}_L, \hat{\theta}_U]$ for a distribution's parameter of interest θ satisfies the equation

$$P(\hat{\theta}_L \leq \theta \leq \hat{\theta}_U) = 1 - \alpha$$

  where $1 - \alpha$ is the *confidence coefficient*

- Typical textbook coverage of confidence interval construction is sub-optimal!
- A usual approach is to introduce the *pivotal method*, in which one defines a statistic that is a function of both the observed data and the unknown parameter of interest θ, and which has a distribution that does not depend on θ
- The pivotal method...
  - has limited utility
  - has the goal of "refactoring" an interval estimation problem such that it can be solved using statistical tables
- Regarding tables...


ONE DOES NOT SIMPLY USE STATISTICAL TABLES
WHEN R IS AVAILABLE


TABLES DELENDA EST

Confidence Interval Construction: Setup

- Let's assume that we sample n independent and identically distributed (iid) data from a distribution with parameter θ
- Let Y be a statistic formed from these data (e.g., $\bar{X}$)
- To determine an interval bound, we solve the following equation for θ:

$$F_Y(y_{\text{obs}}|\theta) - q = 0$$

  - $F_Y(\cdot)$ is the cumulative distribution function for Y
  - y_{obs} is the observed statistic value
  - q is an appropriate quantile...(we will return to this on the next slide)
- This is the approach dubbed *Pivoting the CDF* by Casella & Burger (2002; section 9.2.3)
  - Key: "realize that even if [the equation above] cannot be solved analytically, we really only need to solve [it] numerically since the proof that we have a $1 - \alpha$ confidence interval [does] not require an analytic solution"

    $\Rightarrow$ this is the only method a student needs to learn!

Confidence Interval Construction: Determining the Quantile

Interval Type	$E[Y]$ Increases With θ?	q for Lower Bound	q for Upper Bound
two-sided	yes	$1-\alpha/2$	$\alpha/2$
	no	$\alpha/2$	$1-\alpha/2$
one-sided lower	yes	$1-\alpha$	–
	no	α	–
one-sided upper	yes	–	α
	no	–	$1-\alpha$

- α is the assumed confidence coefficient
  - e.g., a 95% interval $\Rightarrow \alpha = 0.05$
- $E[Y]$ typically increases with θ (putting us on the "yes" line) but not always

Example: Analytic Solution

- We sample a single datum with value $X = x_{\text{obs}} = 0.6$ from the following distribution:

$$f_X(x) = \theta(1-x)^{\theta-1} \quad x \in [0,1]$$

- What is a 95% upper-bound on θ?
  - the statistic $Y = X$, trivially
  - students can determine that $F_Y(y|\theta) = 1 - (1-x)^{\theta}$
  - students can also determine that $E[Y] = 1/(\theta+1) \Rightarrow E[Y]$ *decreases* as θ increases
  - one-sided upper bound + "no" line $\Rightarrow q = 1 - \alpha = 0.95$
- Thus

$$F_Y(y_{\text{obs}}|\hat{\theta}_U) - q = 0 \;\;\Rightarrow\;\; 1 - (1-y_{\text{obs}})^{\hat{\theta}_U} - 0.95 = 0 \;\;\Rightarrow\;\; (1-y_{\text{obs}})^{\hat{\theta}_U} = 0.05 \;\;\Rightarrow\;\; \hat{\theta}_U \log(1-y_{\text{obs}}) = \log(0.05)$$

$$\Rightarrow \;\; \hat{\theta}_U = \frac{\log(0.05)}{\log(1-y_{\text{obs}})} = 3.269$$

Example: Numeric Solution

- Let's repeat the example on the last slide but derive the solution numerically
  - here we use the fact that $X \sim \text{Beta}(1, \theta)$

```
f <- function(theta,y.obs,q)
{
  pbeta(y.obs,shape1=1,shape2=theta) - q
}
uniroot(f,interval=c(0,100),y.obs=0.6,q=0.95)$root
```

```
## [1] 3.269429
```

- Note that...
  - the first argument to our user-defined function `f` is what we are solving for: θ
  - `interval` expresses the range of θ values to explore...as here, $\theta > 0$, we set the lower bound to 0
  - because $F_Y(\cdot)$ is strictly monotonic as a function of θ, there will only be one root within `interval`

For further details and more examples, see the `R Markdown` file included with this presentation

Thank You!