MODULE - 3

TREE AND GRAPH ALGORITHMS

- **SYLLABUS**:
  - Trees
    - ★ Properties
    - ★ Pendant vertex.
  - Distance and centres in a tree.
    - ★ Rooted and binary trees.
    - ★ Counting trees.
    - ★ Spanning trees.
    - ★ Prim's algorithm and Kruskal's algorithm
    - ★ Djikstra's shortest path algorithm.
    - ★ Floyd-Warshall shortest path algorithm.
- **TextBook**:
  - ★ Narsingh Deo, "Graph Theory", PHI, 1979

- **TREE:**

A tree is a connected graph without any circuit.

- Tree is always a simple graph.

- **Theorem-1**:

"There is one and only one path between every pair of vertices in a tree T."

- **PROOF**:

Since, T is a connected graph, there must exist at least one path between every pair of vertices in T. Suppose that between two vertices A and B of T, there are two distinct paths. The union of these two paths will contain a circuit. Then T can't be a tree. ∴ there exists one and only one path between every pair of vertices.

- **Theorem-2**

"If in a graph G, there is one and only path between every pair of vertices, then G is a tree."

◆ PROOF:

Existence of a path between every pair of vertices assures that G is connected.

Then to prove G is a tree, it is enough to prove that G has no circuit.

A circuit in a graph implies that there is at least one pair of vertices a, b, such that there is a path b/w a and b. Since we have a single path b/w every pair of vertices, G can't have circuits. ∴ G is a tree.

■ Theorem-3 :

"A tree with 'n' vertices has 'n-1' edges".

◆ PROOF :

The theorem can be proved by induction on the number of vertices.

Case-I : ($n=1$).

- Since T has no circuits, self-loop isn't possible. So no edge.

Case-II : ($n=2$)

- Only one edge exists [No parallel edge or self-loop".]

Case-III : ($n=k+1$).

Suppose that the theorem is true for all $n \le k$. Now we have to prove that the result is true for $n=k+1$.

Consider a tree with '$k+1$' vertices. Let 'e' be an edge of 'T'. If we remove 'e' from 'T', the graph will become disconnected. The graph '$T-e$' has two components T_1 and T_2. Let the number of vertices in T_1 be 'n_1' and the number of vertices in T_2 be 'n_2'.

$$\boxed{n_1+n_2=k+1}$$

The values n_1 and n_2 are $\le k$.

∴ T_1 and T_2 are two trees having the number of vertices $\le k$. The result is hold. ∴ the number of edges in $T_1 = n_1-1$ and the number of edges in $T_2 = n_2-1$.

∴ The total number of edges in T

$$= (n_1-1)+(n_2-1)+1$$

$$= (n_1+n_2)-2+1$$

$$= (k+1)-1$$

$$= k$$

∴ It is true for 'k+1'. Hence the result is true for all integral values of k.

Hence proved.

$\Rightarrow \boxed{k=1}$.

∴ Number of components = 1. That is, G is a connected graph.

- $\boxed{\text{Theorem-4}}$:

"A connected graph with 'n' vertices and 'n-1' edges is a tree."

- $\boxed{\text{PROOF}}$:

Let G be a connected graph with n vertices and n-1 edges. Then prove that G has no circuit. We have to prove the converse that if G has no cycle and 'n-1' edges, then G is connected. We decompose G into k components $c_1, c_2, \ldots, c_k$. Each component is connected and it has no cycle since G has no cycle. Hence each G_i is a tree.

Let n_i denote the number of ~~edges~~ vertices in c_i, for $i=1,2,\ldots,k$. Then $\boxed{n=\sum_{i=1}^{k} n_i}$, and each c_i has 'n_i-1' edges. The total number of edges in G $=\boxed{\sum_{i=1}^{k}(n_i-1)=n-k}$.

Given that 'G' has 'n-1' edges,

$\therefore \boxed{n-k=n-1}$.

- $\boxed{\text{Theorem-5}}$:

"A graph is a tree iff it's minimally connected."

- $\boxed{\text{PROOF}}$:

A connected graph is said to be minimally connected if removal of anyone edge from it disconnects the graph. Suppose that the given graph is minimally connected. A minimally connected graph can't have a circuit. Thus a minimally connected graph is a tree.

Conversely suppose that G is a tree. Then it's connected and circuitless. ∴ the removal of an edge from tree disconnects the graph. ∴ the graph is minimally connected.

- $\boxed{\text{Theorem-6}}$:

"A graph G with 'n' vertices, 'n-1' edges, and no circuit is connected & is a tree."

◆ **PROOF**:

Suppose that there exists a circuitless graph G with 'n' vertices and 'n-1' edges which is disconnected. In that case, G will consist of two or more circuitless components without loss of generality, G consists of two components G_1 and G_2, Add an edge 'e' between a vertex v_1 in G_1, and v_2 in G_2. Since there was no path b/w v_1 and v_2 in G, adding 'e' doesn't create a circuit. Thus $G \cup e$ is a circuitless connected graph with 'n' vertices and 'n' edges. This is not possible. ∴ G is connected.

● ~~DISTANCE BETWEEN TWO VERTICES:~~

● PROPERTIES OF A TREE:

★ A tree is a connected graph without any circuit.

★ A tree is always a simple graph.

★ There is one and only one path between every pair of vertices in a tree.

★ A tree with 'n' vertices has 'n-1' edges.

★ A graph is a tree iff it is minimally connected.

● DISTANCE BETWEEN TWO VERTICES:

"The distance between two vertices v_i and v_j is denoted as '$d(v_i, v_j)$' and it is defined as length of the shortest path between v_i and v_j."

★ **Metric**:

"A function $f(x,y)$ is said to be a metric if it satisfies the following conditions

[a] Non-negativity : $\boxed{f(x,y) \geq 0}$

[b] Symmetry : $\boxed{f(x,y) = f(y,x)}$

[c] Triangle Inequality : $\boxed{f(x,y) \leq f(x,z) + f(z,y)}$

● **Theorem**:

"The distance between vertices of a connected graph is a metric."

◆ **PROOF**:

★ The distance between two vertices is always a non-negative value for them to exist. $\therefore d(v_i, v_j) \geq 0 \rightarrow$ Non-negativity

★ The distance from a vertex v_i to another vertex v_j in a connected graph G is equal to that from v_j to v_i. $\therefore d(v_i, v_j) = d(v_j, v_i)$

↳ Symmetry

★ For any three vertices v_i, v_j, v_k in a connected graph G, $d(v_i, v_j) \leq d(v_i, v_k) + d(v_k, v_j)$.

↳ Triangle Inequality

● **ECCENTRICITY OF A VERTEX:**

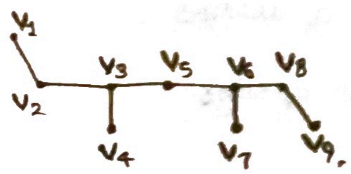
"The eccentricity of a vertex 'v' in a connected graph G is denoted by E(v) and it is defined as the distance from v to the vertex farthest from v in G," i.e.,

$$\boxed{E(v) = \max_{v_i \in V} d(v, v_i)}$$

● **CENTRE OF A TREE:**

"A vertex with minimum eccentricity in graph G is called the centre of G"

e.g.



T

★ Centre of the above tree = v_5.

● **RADIUS AND DIAMETER:**

"The eccentricity of a centre in a tree is its radius and the length of the longest path in a tree is its diameter"

■ **Theorem**:

"In any tree (with two or more vertices), there are at least two pendant vertices."

◆ **PROOF**:

A tree with n vertices has n-1 edges.

$\therefore$ Sum of the degrees of the vertices $= 2e$

$= \underline{2(n-1)}$.

$2(n-1)$ degree to be divided among 'n'
$= 2n-2$
vertices. $\because$ no vertex can be of zero degree, we must have at least two vertices of degree 1 in a tree.

■ **Theorem**:

"Every tree has either one or two centres."

◆ **PROOF**:

"Let T be a tree having more than two vertices. Then T has two or more pendant vertices. Deleting all the pendant vertices from T, the resulting graph T' is again a tree. The removal of all pendant vertices from T uniformly reduces the eccentricities of the remaining vertices. ∴ the centres of T are also the centres of T'. From T', remove all pendant vertices and get another tree T''. Continuing this process, we either get a vertex or an edge. If we get a vertex, then the tree has only one centre. Else two. 'Hence proved.'"

- ROOTED AND BINARY TREES:

★ "A tree in which one vertex is distinguished from all other vertices is called 'Rooted tree' and the distinguished vertex is called 'Root of the tree.'

★ A binary tree is defined as a rooted tree in which there is exactly one vertex of degree two and each of the remaining vertex having degree 1 or 3.

- The vertex of degree two serves as the root.

? Prove that the number of vertices in a binary tree is always odd.

An: "Consider a binary tree with 'n' vertices. One vertex is of degree 2 and all other vertices are of degree 1 or 3. ∴ $(n-1)$ degrees are odd degree vertices. In a graph, the number of odd degree vertices is always even.

$$\therefore n-1 = \text{Even}$$
$$\Rightarrow n = \text{Even}+1$$
$$= \underline{\underline{\text{Odd}}}$$

That is, the number of vertices in a binary tree is always odd."

? Prove that the number of pendant vertices in a Binary tree is $\frac{n+1}{2}$.

An: Let 'n' denote the number of vertices and 'p' denote the number of pendant vertices. Then the number of 3-degree vertices $= n-1-p$.

- Sum of degrees of vertices = 2 × Number of edges $= \underline{\underline{2(n-1)}}$.

$2+p+(n-1-p)3 = 2(n-1)$.

$\Rightarrow 2+p+3n-3-3p = 2n-2$. $\Rightarrow n = 2p-1$

$\Rightarrow p = \dfrac{n+1}{2}$

- **INTERNAL VERTEX:**

★ "A non-pendant vertex in a tree is called internal vertex."

? Prove that the number of internal vertices in a binary tree is one less than the number of pendant vertices.

An: Let 'n' denote the total number of vertices and 'p' denote the number of pendant vertices.

Then $\boxed{p = \dfrac{n+1}{2}}$

Number of internal vertices $= n - p$

$$= n - \frac{(n+1)}{2}$$

$$= \frac{2n-[n+1]}{2}$$

$$= \frac{n-1}{2} = \frac{n+1}{2} - 1$$

$$= p - 1$$

- $\dfrac{n-1}{2}$

$= \dfrac{(n+1)-2}{2}$

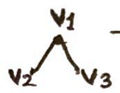
$= \dfrac{n+1}{2} - 1$

? Prove that the maximum number of vertices possible in the k^{th} level of a binary tree is 2^k.

An: We can prove this law by using the principle of mathematical Induction on levels.

★ when $\boxed{k=1}$,

maximum possible vertices $= 2^1 = 2$.

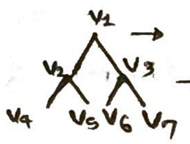


$\rightarrow \boxed{k=0}$

$\rightarrow \boxed{k=1}$

★ when $\boxed{k=2}$,

maximum possible vertices $= 2^2 = 4$



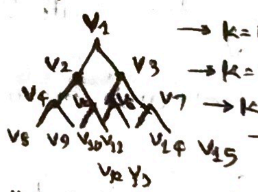
$\rightarrow k=0$

$\rightarrow k=1$

$\rightarrow k=2$

★ When $\boxed{k=3}$,

maximum possible vertices $= 2^3 = 8$.



$\rightarrow k=0$

$\rightarrow k=1$

$\rightarrow k=2$

$\rightarrow k=3$

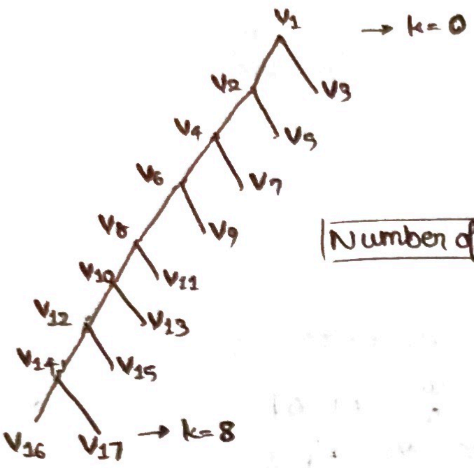
- Assume that the result is true for $(k-1)^{th}$ level. i.e., maximum possible vertices in the $(k-1)^{th}$ level $= 2^{k-1}$.

- $\therefore$ Maximum possible vertices in the k^{th} level $= 2 \cdot 2^{k-1} = 2^k$.

Hence, proved.

? Draw a binary tree with 17 vertices in maximum and minimum heights.

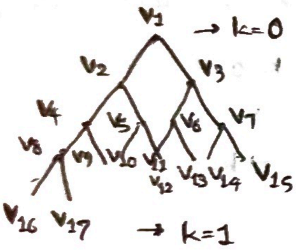
An: ▪ Maximum height : [n=17]



Number of levels = 8

[0 → 8]

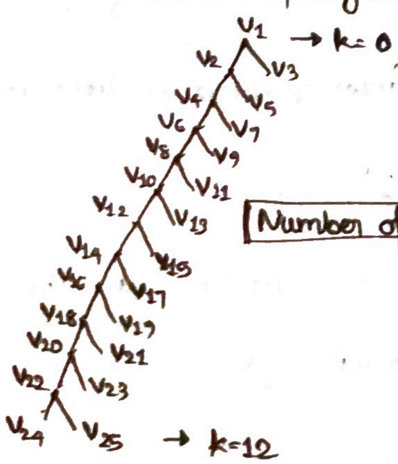
▪ Minimum height :



Number of levels = 4

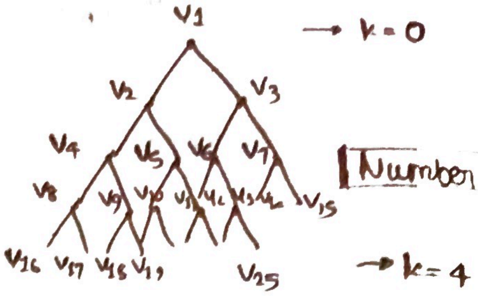
? Draw a binary tree with 25 vertices in maximum height and minimum height.

An: ▪ Maximum height : [n=25]



Number of levels = 12

▪ Minimum height :



Number of levels = 4

? Maximum level of a binary tree with 'n' vertices is '$\frac{n-1}{2}$'. Prove.

An. To construct a binary tree for a given 'n' such that the farthest vertex is as far as possible from the root, we must have exactly two vertices except at zero level. Hence the maximum level is '$\frac{n-1}{2}$'.

? Prove that the minimum possible height of a binary tree with 'n' vertices is $\lceil \log_2 (n+1) - 1 \rceil$. [Smallest integer function]

An. To draw a tree with minimum height we include maximum number of vertices in each level.

$\Rightarrow 2^0+2^1+2^2+\cdots+2^k \geq n$ $\left\{\begin{array}{l} k = \text{level} \\ n = \text{number of vertices} \end{array}\right\}$

$\Rightarrow \dfrac{1.[2^{k+1}-1]}{2-1} \geq n$

↳ Geometric Progression.

$\Rightarrow 2^{k+1}-1 \geq n$

$\Rightarrow 2^{k+1} \geq n+1$

$\Rightarrow \log_2[2^{k+1}] \geq \log_2(n+1)$

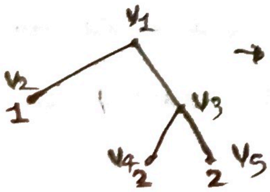
$\Rightarrow k+1 \geq \log_2(n+1)$

$\Rightarrow \underline{\underline{k = \lceil \log_2(n+1)-1 \rceil}}$ => Smallest Integer value

- **PATH LENGTH OF A BINARY TREE:**

"The path length of a binary tree can be defined as the sum of the path lengths from the root to all pendant vertices".

e.g.



→ Path length $= v_1v_2 + v_1v_4 + v_1v_5$

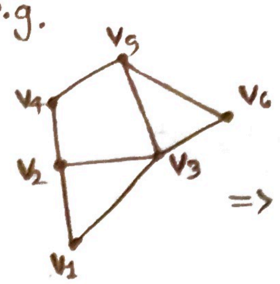
Path length $= 1+2+2 = \underline{\underline{5}}$

[Pendant vertices v_2, v_4, v_5]

- **SPANNING TREE:**

"A tree T is said to be a spanning tree of a connected graph G if T is a subgraph of G and T contains all vertices of G."

e.g.



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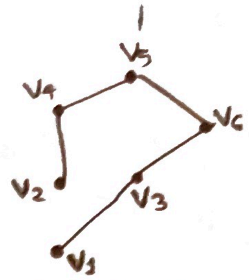


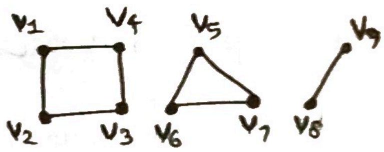
$\boxed{\text{Connected graph } G}$ $\boxed{\text{Spanning Tree, } T \text{ of } G}$

★ Since spanning trees are the largest tree in all trees of G, it is called "Maximal tree of G".

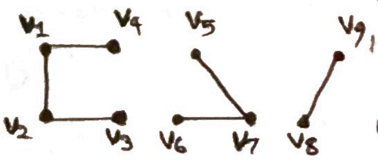
★ Each component of a disconnected graph has a spanning tree. Thus a disconnected graph with k' components has a spanning forest consisting of k' spanning trees.

★ A collection of trees are called forest.

- $\boxed{\text{Example}}$:



$\boxed{\text{A disconnected graph } G}$
[with 3 components]



$\boxed{\text{A spanning forest F of the disconnected graph G.}}$

[Consists of 3 spanning trees]

■ [Theorem 1]:

"Every connected graph has at least one spanning tree."

◆ [PROOF]:

If a graph has no circuit, then it is its own spanning tree. If the graph G has a circuit, then delete an edge from the circuit. This will still leave the graph connected. If there are more circuits, then ~~the~~ repeat the operation until there is no circuit left. It will result in a spanning tree. Hence proved.

■ [Theorem]:

"With respect to any of its spanning tree, a connected graph of 'n' vertices and 'e' edges has $(n-1)$ branches and $(e-n+1)$ chords."

■ [PROOF]:

★ Branch: Edges in the spanning tree.

★ Chord: The remaining edges.

If a connected graph has 'n' vertices, then its spanning tree has n vertices and $n-1$ edges. $\therefore$ $\boxed{\text{No. of branches} = n-1}$

● No. of chords $= e-(n-1)$

$\Rightarrow$ $\boxed{\text{No. of chords} = e-n+1}$

● RANK AND NULLITY:

Let G be a graph with 'e' edges, 'k' components, and 'n' vertices. Then

● $\boxed{\text{Rank} = n-k}$ [r]

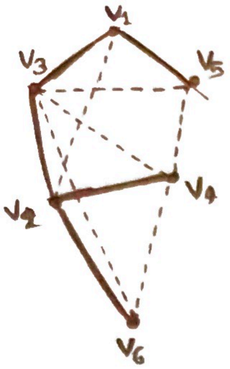
● $\boxed{\text{Nullity } [\mu] = e-n+k}$

For a connected graph, Rank $= n-1$ and Nullity $= e-n+1$. i.e., for a connected graph, rank equals no. of branches in ~~any~~ the spanning tree and nullity equals no. of chords in the tree.

★ Nullity is also referred to as its ~~phycomatic number~~ [cyclomatic number] or [First Betti Number].

● FUNDAMENTAL CIRCUIT:

Consider a spanning tree T in a connected graph G. Adding any one chord to T will create exactly one circuit. Such a circuit is called a Fundamental Circuit.



- ----- → Added chord
- ▬▬ → Branch.

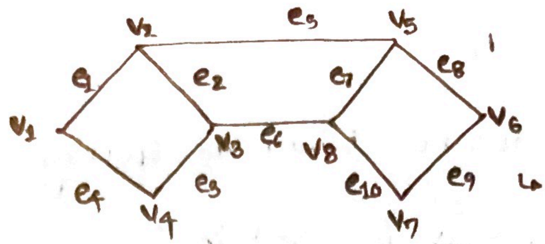
DISTANCE BETWEEN TWO SPANNING TREES:

"The distance between two spanning trees T_i and T_j of a graph G is defined as the number of edges of G present in T_i but not in T_j. It is denoted as $\boxed{d(T_i, T_j)}$."

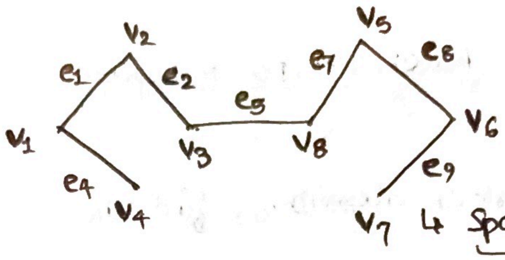
RINGSUM OF TWO SPANNING TREES:

"It is defined as a subgraph of G containing all edges that are either in T_i or in T_j, but not in both." It is denoted as $\boxed{T_i \oplus T_j}$."

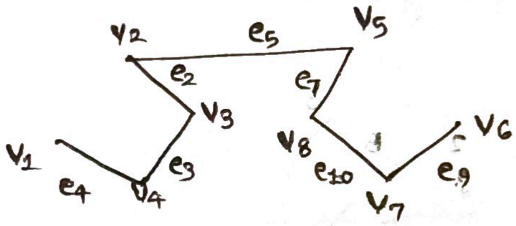
?



↳ Graph G



↳ Spanning Tree T_1



↳ Spanning Tree T_2

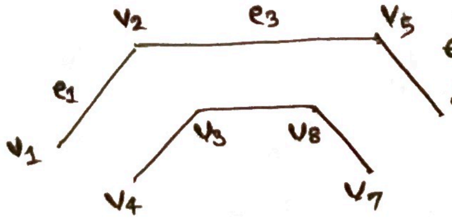
Find the distance and ringsum between the above two spanning trees T_1 and T_2.

An:
- Distance between T_1 and T_2

  = Total edges in T1/T2 − Number of common edges.

  $= 7 - 4 = \underline{\underline{3}} = d\,[T_1, T_2]$

- Ringsum of T_1 and T_2, $[T_1 \oplus T_2]$

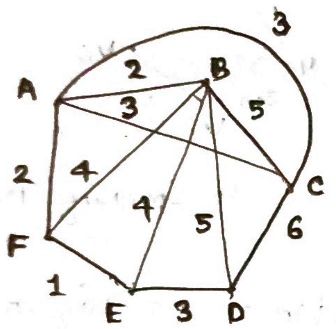


■ Theorem :

"The distance between the spanning trees of a graph is a metric."

● KRUSKAL'S ALGORITHM FOR MINIMUM SPANNING TREE:

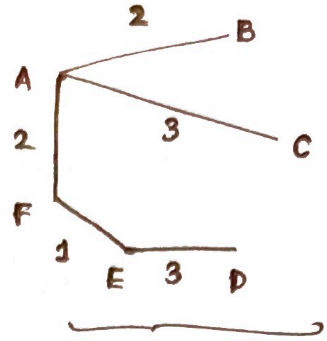
? Using Kruskal's algorithm, find the shortest spanning tree:



Ans: ★ Weights in the ascending order are:

- FE - 1
- AF - 2
- AB - 2
- AC - 3
- DE - 3
- BF - 4
- BE - 4
- BC - 5
- BD - 5
- CD - 6

■

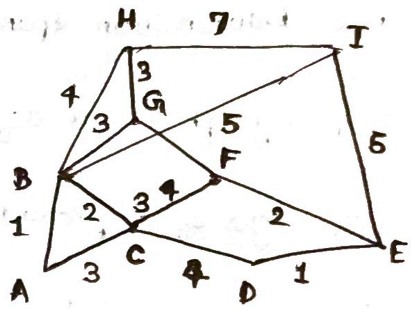


Minimum spanning tree

[Minimum weight]

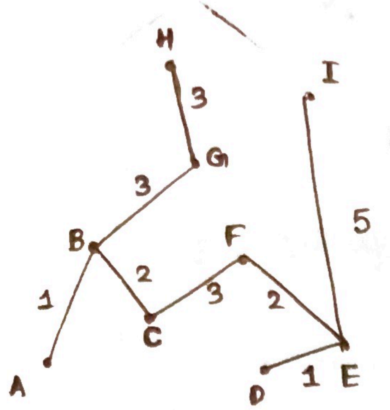
$= \underline{\underline{11}}$

?★ Find minimum spanning tree using Kruskal's algorithm:



Ans: • Weights arranged in ascending order

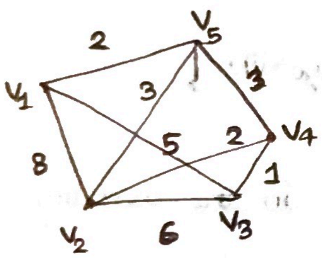
- AB - 1
- DE - 1
- BC - 2
- EF - 2
- AC - 3
- BG - 3
- CF - 3
- GH - 3
- CD - 4
- BI - 5



↳ Minimum weight = $\underline{20}$

PRIM'S ALGORITHM FOR MINIMUM SPANNING TREE:

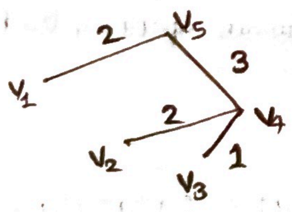
? Using Prim's algorithm find the shortest spanning tree.



=>

	V_1	V_2	V_3	V_4	V_5
V_1	-	8	5	∞	2
V_2	8	-	6	2	3
V_3	5	6	-	1	∞
V_4	∞	2	1	-	3
V_5	2	3	∞	3	-

[Matrix]



∴ Minimum weight = $\underline{8}$

■ **Kruskal's Algorithm**:

★ List all edges of the graph G in order of non-decreasing weight.

★ Select a smallest edge of G.

★ Then for each successive step select [from all remaining edges of G] another smallest edge that makes no circuit with the previously selected edges.

★ Continue until 'n-1' edges have been selected, and these edges will constitute the desired shortest spanning tree

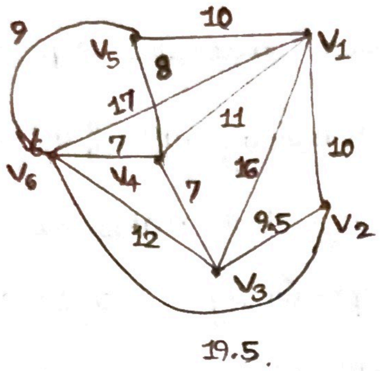
■ **Prim's Algorithm**:

★ Draw n isolated vertices and label them $v_1, v_2, \ldots, v_n$.

★ Tabulate the given weights of the edges of G in an 'n' by 'n' table. Set the weights of non-existent edges as very large [∞].

★ Start from vertex v_1 and connect it to its nearest neighbour [a vertex other than v_1 and v_k that has the smallest entry among all entries in rows 1 and k]. Let this vertex be v_i. Next regard the tree with vertices v_1, v_k, and v_i as one subgraph, and continue the process until all n vertices have been connected by n-1 edges. This will give the shortest spanning tree.

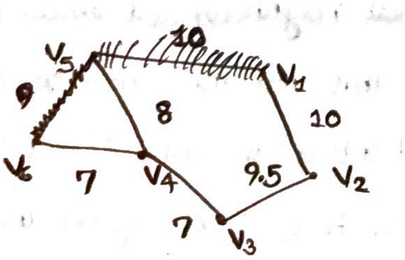
? Apply Prim's algorithm to find the minimum spanning tree.



An: Matrix →

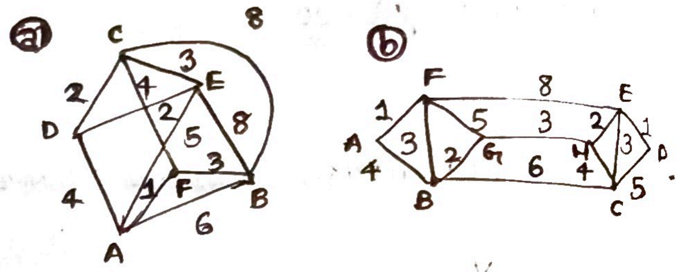
	V_1	V_2'	V_3	V_4	V_5'	V_6
V_1	–	10	16	11	10	17
V_2	10	–	9.5	∞	∞	19.5
V_3	16	9.5	–	7	∞	12
V_4	11	∞	7	–	~~17~~ 8	7
V_5	10	∞	∞	~~17~~ 8	–	9
V_6	17	19.5	12	7	9	–

∴ Minimum spanning tree →



∴ Minimum weight = **42.5**

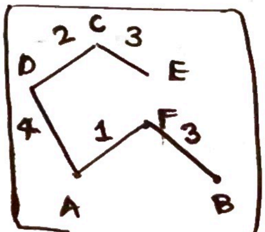
? Apply Prim's algorithm to find the minim. spanning tree to the given graph. Apply Kruskal's algorithm as well.



An: ⓐ $\boxed{\text{Kruskal's algorithm}}$

★ Weights arranged in the ascending order are:

- AF – 1 ✓
- CD – 2
- DE – 2
- BF – 3 ✓
- CE – 3
- AD – 4
- CF – 4
- AE – 5.
- AB – 6.
- BC – 8
- BE – 8



$\boxed{\text{Minimum spanning tree}}$

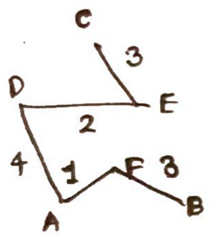
Minimum weight = 1 + 2 + 3 + 3 + 4

= **13**

(a) Prim's algorithm :

	A	B	C	D	E	F
A	-	6	∞	4	5	1
B	6	-	8	∞	8	3
C	∞	8	-	2	3	4
D	4	∞	2	-	2	∞
E	5	8	3	2	-	∞
F	1	3	4	∞	∞	-

6×6 table for Prim's Algorithm



Minimum spanning tree

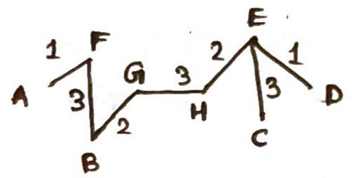
→ Minimum weight =

1+2+3+3+4 = **13**

Prim's Algorithm

	A	B	C	D	E	F	G	H
A	-	4	∞	∞	∞	1	∞	∞
B	4	-	6	∞	∞	3	2	∞
C	∞	6	-	5	3	∞	∞	4
D	∞	∞	5	-	1	∞	∞	∞
E	∞	∞	3	1	-	8	∞	2
F	1	3	∞	∞	8	-	5	∞
G	∞	2	∞	∞	∞	5	-	3
H	∞	∞	4	∞	2	∞	3	∞

8×8 table for Prim's algorithm



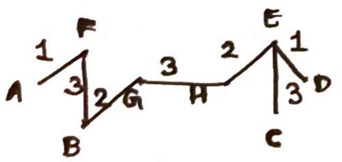
Minimum spanning tree

• Minimum weight = 1+1+2+2+3+3+3
= **15**

(b) Kruskal's algorithm

★ Weights arranged in the ascending order are:

- AF-1
- DE-1
- BG-2
- EH-2
- BF-3
- CE-3
- GH-3
- AB-4
- CH-4
- CD-5
- FG-5
- BC-6
- EF-8



Minimum spanning tree

• Minimum weight =

1+1+2+2+3+3+3 = **15**

? A tree has 5 vertices of degree 2, 3 vertices of degree 3, and 4 vertices of degree 4. How many pendant vertices does it have?

An: Let the number of pendant vertices be 'p'.

★ Total number of vertices $= 5+3+4+p$
$= \underline{12+p}$

★ Number of edges $= n-1 = [12+p]-1$
$= \underline{11+p}$

★ Sum of the degree of vertices $= 2e$
$= 2[11+p] = \underline{22+2p}$

$\Rightarrow (5\times2) + (3\times3) + (4\times4) + p = 22 + 2p$

$\Rightarrow p = 35 - 22 = \underline{13}$

★ Number of edges = $11 + p$ = $\underline{24}$

★ Number of vertices = $e+1 = \underline{25}$

? A tree has '2n' vertices of degree 1, '3n' vertices of degree 2, and 'n' vertices of degree 3. Determine the number of vertices and edges in the tree

An: Sum of the degrees of vertices =

2x Number of edges $= 2[1+$ No. of vertices$]$

$\Rightarrow [2n] + [2\times3n] + [3\times n] = [2n+3n+n-1]\,2$

$\Rightarrow 11n = [6n-1]\,2$

$\Rightarrow \frac{11}{2} n = 6n - 1$

$\Rightarrow 11n = 12n - 2$

$\Rightarrow \underline{n=2}$

? Show that if a tree has exactly two pendant vertices, then the degree of every other vertices is 2.

An: Let the 'n' vertices be $v_1, v_2, \ldots, v_n$.

Let $d(v_1) = d(v_2) = 1$.

- Sum of degrees $= 2e$

$\Rightarrow \sum_{i=1}^{n} d[v_i] = 2[n-1]$

$\Rightarrow 1+1+\sum_{i=3}^{n} d[v_i] = 2[n-1]$

$\Rightarrow \sum_{i=3}^{n} d[v_i] = 2[n-2]$

None of these vertices have either 0 or ...

Degree of each vertex is $\underline{2}$,

MODULE 4

CONNECTIVITY AND PLANAR GRAPHS

- **SYLLABUS**:
  - Vertex Connectivity
  - Edge Connectivity
  - Cut set and Cut Vertices
  - Fundamental circuits
  - Planar graphs
  - Kuratowski's Theorem [Proof Not Required]
  - Different representations of planar graphs.
  - Euler's Theorem
  - Geometric Dual.

- **TextBook**:
  - ★ Narsingh Deo, "Graph Theory", PHI, 1979

- **CUT VERTEX:**

"A cut vertex in a connected graph G is a vertex whose removal increase the number of components."

★ If V is a cut vertex of a connected graph G, then G-V is disconnected. A cut vertex is also called a "cut point".

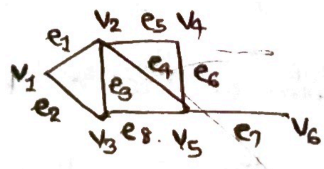
- **CUT EDGE:**

"An edge whose removal increases the number of components of a connected graph G." Cut edge is also called bridge.

- **CUT SET[S]:**

"The set of all minimum number of edges of G whose removal disconnects a graph G, is called its cut set [S]".

e.g.



★ Cut vertex - V_5

★ Cut edge - e_7

★ Cut set - $\{e_7\}$, $\{e_1, e_2\}$, $\{e_5, e_4, e_8\}$, $\{e_1, e_1, e_4, e_5$

NOTE:

* In a tree, every edge is a cut-edge as well as cut set.
* In a tree, every internal vertex is a cut vertex.

■ **Theorem**:

"Every cut set in a connected graph G must contain at least one branch of every spanning tree of G."

◆ **PROOF**:

Spanning tree contains all vertices of G. If the cut set doesn't contain branch of any spanning tree, then the removal of edge doesn't ~~contain~~ disconnect the graph. So, S contains at least one branch of any spanning tree of G.

■ **Theorem**:

"In a connected graph G, any minimal set of edges containing at least one branch of every spanning tree of G is a cut-set."

◆ **PROOF**:

In a given connected graph G, let Q be a minimal set of edges containing at least one branch of every spanning tree of G. Consider G-Q, the subgraph that remains after removing the edges of Q from G. ∵ G-Q contains no spanning tree of G, it is disconnected. Also ∵ Q is the minimal set of edges with this property, an edge E from Q returned to G-Q will create at least one spanning tree. Thus G-Q+e is a connected graph. ∴ Q is a minimal set of edges whose removal from Q disconnects G. ∴ Q is a cut set.

■ **Theorem**:

"Every circuit has an even number of edges in common with any cut set."

◆ **PROOF**:

Consider a cut set S in a graph G. Let the removal of S partition the vertices of G into two disjoint subsets V_1 and V_2. Consider a circuit γ in G. If all vertices

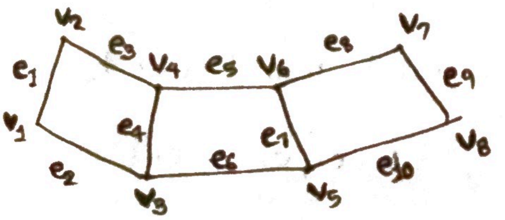
$\mathcal{H}$ are entirely within one of the vertex set V_1 or V_2, the number of edges common to S and $\mathcal{H}$ is 0, which is even.

If on the other hand, suppose that some vertices of $\mathcal{H}$ are in V_1 and some in V_2. Because of the closed nature of the circuit, the number of edges we traversed between V_1 and V_2 must be even. And every edge in S has one end in V_1 and other in V_2, no other edge in G has this property, the number of edges common to S and $\mathcal{H}$ is even.

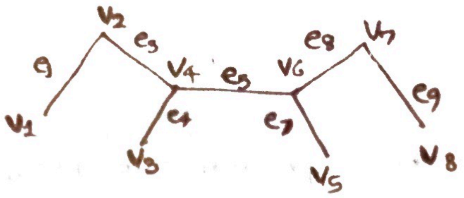
• FUNDAMENTAL CUTSET:

Consider a spanning tree T of a connected graph G. Cut sets of G that contain only one branch of the spanning tree are called fundamental cut sets of T with respect to G. 'Fundamental cut-sets' are also called Basic cutsets.'

e.g



Connected graph G



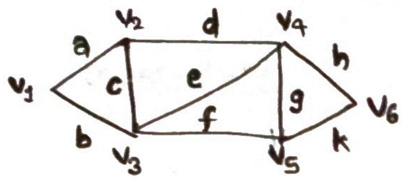
Spanning Tree T

- Here the fundamental cut-sets are $\{e_3\}$, $\{e_5\}$, and $\{e_8\}$. [Since they contain only one branch of the spanning tree T and their removal disconnects the graph].

■ **Theorem**:

"The Ringsum of any cutsets in a graph is either a third cutset or an edge-disjoint union of cutsets"

♦ **PROOF**: **Example**:



■ $S_1 = \{d, e, f\}$ ■ $S_2 = \{f, g, h\}$

The cutset S_1 partition the vertices into two vertex sets namely $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6\}$. S_2 partition the vertices into $V_3 = \{v_1, v_2, v_3, v_4\}$ and $V_4 = \{v_5, v_6\}$.

- $[V_1 \cap V_4] \cup [V_2 \cap V_3] = \varphi \cup \{v_4\} = \{v_4\} = V_5$
- $[V_1 \cap V_3] \cup [V_2 \cap V_4] = \{v_1, v_2, v_3\} \cup \{v_5, v_6\}$
$= \{v_1, v_2, v_3, v_5, v_6\} = V_6$

- $S_1 \oplus S_2 = \{d, e, g, h\}$ = ~~a~~ third cut set S_3

- S_3 partitions the graph into two vertex sets $\{v_1, v_2, v_3, v_5, v_6\}$ and $\{v_4\}$, which is equivalent to the vertex sets V_6 and V_5 respectively.

↳ Let cutset $S_3 = \{d, e, g, h\}$ and $S_4 = \{f, g, k\}$

★ Then S_3 partitions the graph into: V_5 and V_6.

★ S_4 also partitions the graph into ~~V_5 and V_6~~. ~~[Same as that of S_3]~~. $V_7 = \{v_5\}$ and $V_8 = \{v_1, v_2, v_3, v_4, v_6\}$.

- $(V_5 \cap V_8) \cup [V_6 \cap V_7] = \{v_5\} \cup \{v_4\} = \{v_4, v_5\} = V_9$

- $[V_5 \cap V_7] \cup [V_6 \cap V_8] = \{v_1, v_2, v_3, v_4\} \cup \phi$

$= \{v_1, v_2, v_3, v_4\}$

$= V_8$

- $S_3 \oplus S_4 = \{d, e, f, h, k\}$ = Union of two disjoint cut sets $\{d, e, f\}$ and $\{h, k\}$.

• **Statement**:

"Let S_1 and S_2 be two cut-sets in a connected graph G. Let (V_1, V_2) be a partition of the vertex set of G with respect to S_1 and (V_3, V_4) w.r.t. S_2. Now we consider $[V_1 \cap V_4] \cup [V_2 \cap V_3] = V_5$. The ringsum $S_1 \oplus S_2$ is a cutset, if the subgraph containing V_5 and V_6, each remain connected after $S_1 \oplus S_2$ is removed. Hence $S_1 \oplus S_2$ is a cut-set in the subgraph containing V_5 and V_6. (V_5, V_6) is a partition of vertex set V of G. Otherwise it's an edge-disjoint union of cut-sets.

■ **Theorem**:

"With respect to a given spanning tree T, a chord c_i, that determines a fundamental circuit Γ occurs in every fundamental cut-set associated with the branches in Γ and in no other."

◆ ~~EXAMPLE~~ : **PROOF**:

Consider a spanning tree T in a given connected graph G. Let c_i be a chord w.r.t. T. Let the fundamental circuit made by c_i be Γ. Every branch of any spanning tree has fundamental cut-set associated with it. Let S_1 be the fundamental cut-set associated with branch b_1.

$S_1 = \{b_1, c_1, c_2, \ldots, c_m\}$

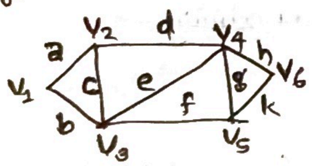
∵ there must be an even number of edges common to Γ and S_1, c_i is on of the chord $c_1, c_2, \dots, c_m$. Exactly same arguments hold for fundamental cut sets associated with $b_2, b_3, \dots, b_k$. ∴ The chord c_i contained in every fundamental cut-set associated with the branches of Γ.

If possible, suppose that S' is a fundamental cut-set contains the ~~chord~~ chord c_i and doesn't contain any branch $b_1, b_2, \dots, b_k$. Then only one edge common to S' and Γ is c_n. This is a contradiction. ∴ c_i doesn't contain no other fundamental cut-set. Hence proved.

$\Gamma = \{c_i, b_1, b_2, \dots, b_k\}$

• **EXAMPLE**:

e.g.



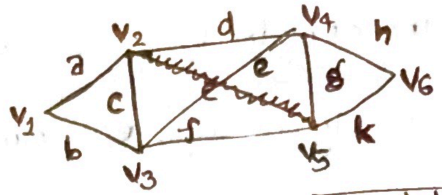
- $\Gamma = \{d, e, f, g\}$ } Fundamental circuit
- $d - \{d, e, f\}$; $g - \{f, g, k\}$
- $c - \{b, d, e, f\}$

} Fundamental cut-sets corresponding to the branches.

• **Theorem**:

"With respect to a given spanning tree T, a branch b_i that determines a fundamental cut-set S is contained in every fundamental circuit associated with the chords in S and in no others."

• **EXAMPLE**:



Connected graph G

- $S = \{b, c, e, f\}$ → Fundamental cut-set

↳ $b - \{a, b, c\}$
~~↳ c~~
↳ $e - \{c, d, e\}$
↳ $f - \{c, d, f, g\}$

} Fundamental circuits corresponding to the chords

• **PROOF**:

Consider a connected graph G and a spanning tree T of G. Let the fundamental cut-set S determined by a branch b_i be

$\boxed{S = \{b_i, c_1, c_2, \dots, c_k\}}$.

Let Γ_1 be the fundamental circuit determined by chord c_1.

i.e. $\boxed{\Gamma_1 = \{c_1, b_1, b_2, \dots, b_q\}}$

∴ the number of edges common to S and Γ_1 must be even, b_i is the one of the ~~chord~~ branch $b_1, b_2, \dots, b_q$. Exactly same arguments hold for a fundamental circuit made by chords $c_2, c_3, \dots, c_k$.

If possible, suppose that Γ' is a fundamental circuit made by a chord other than $c_1, c_2, \ldots, c_k$ in which b_i occurs.

$\because$ none of the chords $c_1, c_2, \ldots, c_k$ is in Γ', there is only one edge b_i common to Γ' and S. This is not possible. Hence proved.

- **CONNECTIVITY:**

★ Let G be a graph having k components. The minimum number of edges whose deletion from G increases the number of components of G is called the $\boxed{\text{EDGE CONNECTIVITY}}$. of G.

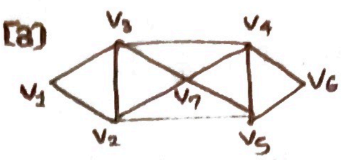
★ The number of edges in the smallest cut-set of a graph is its edge-connectivity

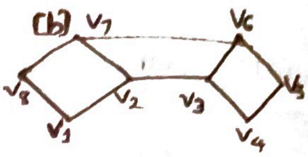
★ The Edge Connectivity of a tree is 1.

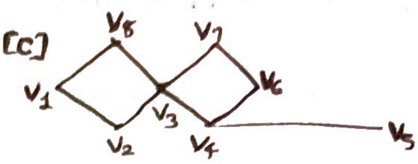
★ The minimum number of vertices whose deletion from graph G increases the number of components of G is called its $\boxed{\text{VERTEX CONNECTIVITY}}$.

★ The vertex connectivity of a tree is 1.

? Calculate the edge connectivity and vertex connectivity of the following graphs:

[a]


[b]


[c]


Ans: [a] Edge connectivity = 2

Vertex Connectivity = 2

[b] Edge Connectivity = 2

Vertex Connectivity = 2

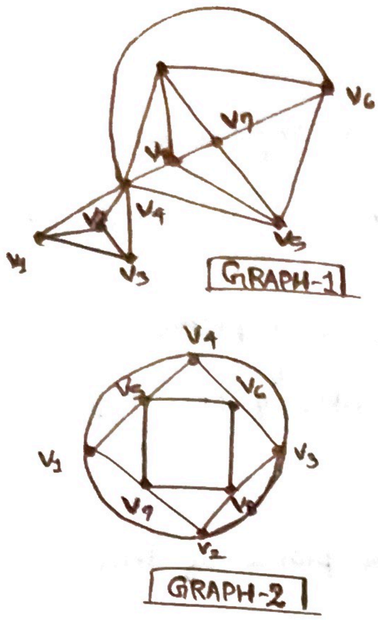
[c] Edge Connectivity = 1

Vertex Connectivity = 1

- APPLICATION OF CONNECTIVITY:

Suppose that we are given 'n' stations that are to be connected by means of e lines where $\boxed{e \geq n-1}$. Construct a graph with n vertices and e edges that has maximum possible connectivity.

For e.g., take n=8 and e=16.



Edge connectivity & Vertex Connectivity of Graph-2 is 4. But for Graph-1, Vertex Connectivity is 1 and Edge Connectivity is 3. In the second graph, even after any 3 stations are bombed, destroyed the remaining stations can still continue to communicate with each other.

- **Theorem**:

"The edge connectivity of a graph G can't exceed the degree of a vertex with the smallest degree in G."

- **PROOF**:

Let vertex v_i be the vertex with smallest degree in G. Let degree of $v_i = k$. Vertex v_i can be separated from G by removing 'k' edges incident on v_i. Hence proved.

- **Theorem 1**:

"The vertex connectivity of any graph G can never exceed the edge connectivity of G."

- **PROOF**:

Let α denote the edge connectivity of G.

∴ there exists a cut-set S in G with α edges. ~~Let~~ S partitions the vertices of G into two subsets V_1 and V_2. By removing atmost α vertices from V_1 [or V_2], on which the edges in S are incident, we can effect the removal of S together with all other edges incident on these vertices on G. Hence proved.

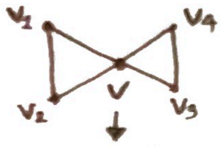
- **SEPARABLE GRAPH:**

"A connected graph is said to be separable if its vertex connectivity is one."

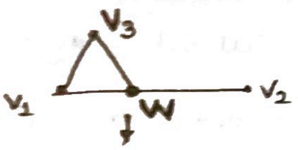
- Remaining connected graphs are non-separable.

- In a separable graph, a vertex whose removal disconnects the graph is called a cut vertex, a cut node, or an articulation node.

★ Separable graph Example:



Articulation vertex [point]



Articulation vertex.

- KURATOWSKI'S THEOREM:

- PLANAR GRAPH:

"A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect."

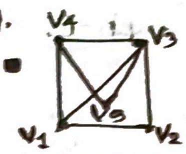
★ A graph that can't be drawn on a plane without a crossover of edges is called a non-planar graph.

★ The drawing of a geometric representation of a graph on any surface such that no edges intersect is called EMBEDDING.

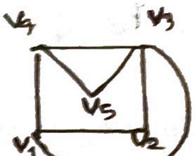
★ An embedding of a planar graph G on a plane is called "plane representation of G".

★ A graph G is planar if there exists a graph isomorphic to G that is embedded on a plane.

e.g.



Graph

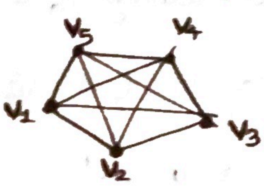


Plane embedding

- KURATOWSKI'S GRAPH:

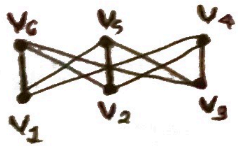
★ First Graph:

"A complete graph of 5 vertices [K_5] is non-planar"



★ Second Graph:

"Regular graph with 6 vertices and 9 edges."



- **PROPERTIES COMMON TO KURATOWSKI'S GRAPHS:**

[1] Both are regular graphs.

[2] Both are non-planar graphs.

[3] Removal of one edge or a vertex makes each a planar graph.

[4] Kuratowski's first graph is the non-planar graph with smallest number of vertices, second graph is the non-planar graph with smallest number of edges. Thus, both are simplest non-planar graphs.

- **EULER'S FORMULA:**

"If a connected planar graph G has n vertices, e edges, and r regions, then $\boxed{n-e+r=2}$."

◆ **PROOF**:

We prove the theorem by induction on e, where $\boxed{e = \text{Number of edges}}$

- **Case 1**: $e=0$

- If $e=0$, then G must have one vertex.

$\therefore n=1, e=0, r=1 \Rightarrow n-e+r = 1-0+1 = \underline{\underline{2}}$.

- **Case 2**: $e=1$.

- If $e=1$, then the number of vertices in G is either 1 or 2.

↳ If $e=1$, then $r=2$, $\boxed{n=1}$

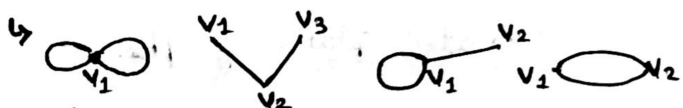
$\therefore n-e+r = 1-1+2 = \underline{\underline{2}}$

↳ If $\boxed{n=2}$, then,

$e=1, r=1$.

$\therefore n-e+r = 2-1+1 = \underline{\underline{2}}$

- **Case 3**: $e=2$



[a] [b] [c] [d]

- $n=1; r=3$ ▪ $n=3; r=1$ ▪ $n=2; r=2$ ▪ $n=2; r=2$

$\therefore n-e+r = 1-2+3 = \underline{\underline{2}}$

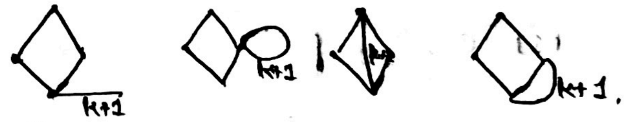
$\therefore n-e+r = 3-2+1 = \underline{\underline{2}}$

$\therefore n-e+r = 2-2+2 = \underline{\underline{2}}$

$\therefore n-e+r = 2-2+2 = \underline{\underline{2}}$

- Assume that the result is true for $e=k$.

$\therefore \boxed{n-k+r=2}$



[a] [b] [c] [d]

Add one new edge to G to form a $\boxed{[k+1]^{th} \text{ edge}}$ connected graph. Then there are 4 possibilities:

[a] The edge $k+1$ is incident with G only in one vertex.

Then $\boxed{r'=r}$; $\boxed{n'=n+1}$; $\boxed{e'=k+1}$

$\therefore n'-e'+r' = [n+1]-[k+1]+r$

$= n-k+r+1-1 = \underline{\underline{2}}$

[b] The edge $k+1'$ is a loop.

Then $\boxed{r' = r+1}$; $\boxed{n' = n}$; $\boxed{e' = k+1}$

$\therefore n' - e' + r' = 2$

[c] Let the edge $k+1'$ join two disjoint vertices of G.

Then, $\boxed{n' = n}$; $\boxed{r' = r+1}$; $\boxed{e = k+1}$

$\therefore n' - e' + r' = 2$

Thus the result is true for k+1 edges.

$\therefore$ by induction, the formula is true for all connected planar graphs.

■ **Corollary**:

"In any simple connected planar graph with f regions, e edges, and n vertices [e > 2], the following inequalities must hold:

[i] $\boxed{e \geq \frac{3}{2} f}$

[ii] $\boxed{e \leq 3n - 6}$ "

◆ **PROOF**:

[i] Since each region is bounded by at least 3 edges and each edge has exact 2 regions,

$\boxed{2e \geq 3f}$

$\Rightarrow \therefore \boxed{e \geq \frac{3}{2} f}$

[ii] By Euler's formula,

$\boxed{n - e + f = 2}$

From the first inequality,

$\boxed{n - e + \frac{2e}{3} \geq 2}$ $\because [e \geq \frac{3}{2} f]$

$\Rightarrow \boxed{e \leq 3n - 6}$

Hence proved.

? Using suitable inequality, prove that Kuratowski's graphs are non-planar.

An: ★ For planar graph, $\boxed{e \leq 3n - 6}$.

Here $e = 10$ and $n = 5$.

$\Rightarrow 3n - 6 = [3 \times 5] - 6 = 9$

$\boxed{10 \leq 9} \rightarrow$ NOT TRUE

$\therefore$ the graph is non-planar.

★ In Kuratowski's second graph, each region is bounded by at least 4 edges

$\therefore 2e \geq 4f$

$\Rightarrow e \geq 2f$

$\Rightarrow \boxed{f \leq \frac{e}{2}}$.

★ Consider Euler's formula.

Here $n = 6$; $e = 9$.

$9 \leq (2\times6)-4$

$\boxed{9 \leq 8} \rightarrow$ NOT TRUE

∴ Kuratowski's graphs are non-planar.

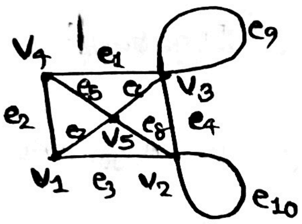
? How many edges must a planar graph have if it has 7 regions and 5 vertices. Draw one such graph.

An: By Euler's formula,

$$\boxed{n-e+f=2}$$

$$\Rightarrow \boxed{5-e+7=2}$$

$$\Rightarrow \boxed{e=10}$$



? Suppose G is a graph with 1000 vertices and 3000 edges. Is it a planar graph?

An: We have for a connected planar graph,

$$\boxed{e \leq 3n-6}$$

$$3000 \leq 3\times1000-6$$

$\boxed{3000 \leq 2994} \rightarrow$ FALSE

∴ It is a non-planar graph.

? A connected graph G has 9 vertices having degrees 2,2,2,3,3,3,4,4,5. Find out the number of edges and check whether it is planar or not.

An: Sum of degrees of vertices = 2 × no. of edges

$$\therefore e = \frac{2+2+2+3+3+3+4+4+5}{2} = \frac{28}{2} = 14$$

- For a planar graph, $e \leq 3n-6$

i.e., $14 \leq (3\times9)-6$

$\boxed{14 \leq 21} \rightarrow$ NOT TRUE

∴ It is not a planar graph.

? Find a graph G with degree sequence 4,4,3,3,3,3 such that G is planar.

An: • Number of vertices, $n = 6$.

- Number of edges, $e = \frac{(4\times2)+(3\times4)}{2} = 10$
- For a graph to be planar, $\boxed{e \leq 3n-6}$

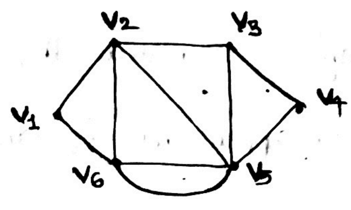
$\Rightarrow 10 \leq (3\times6)-6$

$\Rightarrow \boxed{10 \leq 12} \rightarrow$ NOT TRUE

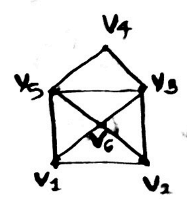
∴ We can't draw such a planar graph.

? Determine the number of regions connected by a planar graph of 6 vertices and 10 edges. Draw a simple and non-simple graph.

An:



NON-SIMPLE GRAPH
[Number of regions = 6]



SIMPLE GRAPH
[Regions = 6]

DETECTION OF PLANARITY AND KURATOWSKI'S THEOREM:

- Elementary Reduction in a Graph:

Step 1 › Disconnected planar graph is planar iff each of its components are planar. ∴ we need only consider one component at a time. Let G be a separable graph with blocks $G_1, G_2, \ldots, G_n$. Then we check the planarity of each block separately.

Step 2 › Since the addition or removal of self-loops don't affect planarity, remove them.

Step 3 › Since parallel edges doesn't affect planarity, eliminate edges in parallel by removing all, but one edge between every pair of vertices.

Step 4 › Elimination of vertex of degree 2 by merging two edges in series doesn't affect planarity. ∴ eliminate them.

★ After repeated application of elementary reduction, the given graph is reduced to:

[a] Any single edge K_2

OR

[b] A complete graph K_4

OR

[c] A non-separable simple graph

OR

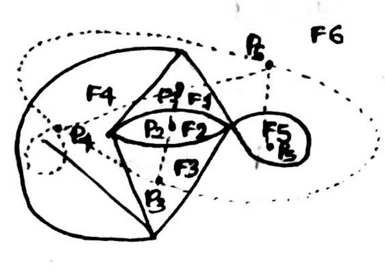
with $n \geq 5$, $e \geq 7$.

★ If H is a subgraph obtained from graph G by a series of elementary reduction, then G and H are said to be 'homeomorphic graphs'. H is also called "topological minor" of G.

- Kuratowski's Theorem:

"A necessary and sufficient condition for a graph G to be planar is that G doesn't contain either of Kuratowski's graphs or any graph homeomorphic to either of them."

GEOMETRIC DUAL OF A GRAPH:



.... → Dual — → Graph

★ [Procedure to obtain the geometric dual]

- Mark the regions as F_1, F_2, F_3, F_4, F_5 and F_6. Trace 6 points $P_1, P_2, \ldots, P_6$, one in each of the region. If two regions F_i and F_j are adjacent, draw a line joining points P_i and P_j, that intersect the common edge between F_i and F_j exactly once. If there are multiple edges common b/w F_i and F_j, draw one line b/w P_i and P_j for each of the common edges. For an edge E, lying entirely in one region, draw a self-loop at the point in that region, intersecting P exactly once. By this procedure we obtain a new graph G^* consisting of 6 vertices $V_1, V_2, \ldots, V_k$ and edges joining these vertices. Such a graph G^* is called the "Geometric Dual of G."

Pendant

★ "The relationship between a planar graph G and it's geometric dual G^*:

[i] An edge-forming self-loop in G yields a pendant edge in G^*.

[ii] A pendant edge in G yields a self-loop in G^*.

[iii] Edges that are in series in G produces parallel edges in G^*.

[iv] The number of edges constituting the boundary of a region F_i in G is equal to the degree of the corresponding vertex V_i in G^* and vice-versa.

[v] Graph G^* is also embedded in the plane. ∴ it is planar.

[vi] The dual of the graph is the dual of the dual itself. ∴ G and G^* are called dual graphs.

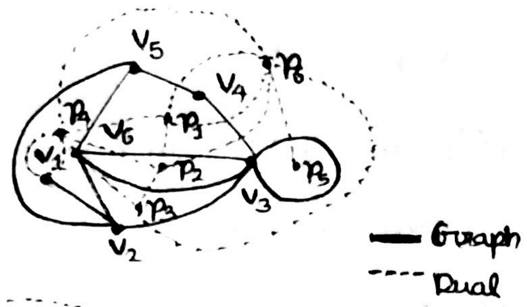
[vii] Let n, e, f, r, μ denotes the number of vertices, edges, faces, rank and nullity of the graph respectively [A connected graph G] and if $n^*, e^*, f^*, r^*, \mu^*$ are the corresponding numbers in dual graph G^*. Then,

$$\boxed{n^* = f;\ e^* = e;\ f^* = n;\ r^* = \mu;\ \mu^* = r}.$$

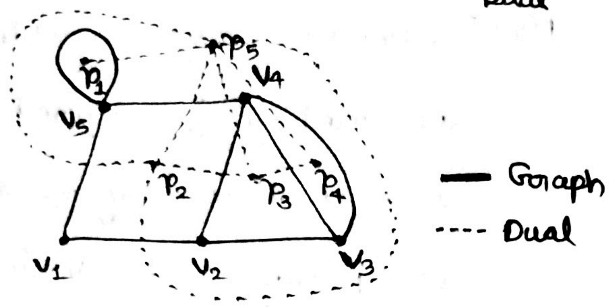
? Draw the geometric dual of the following graphs:

An:

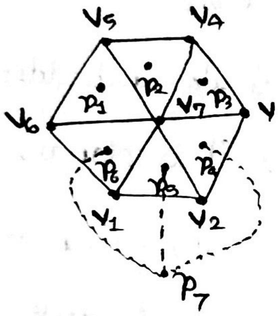
[a]



[b]



[c]



● DIJKSTRA'S SHORTEST PATH ALGORITHM:

[Module 3]

★ It is used to find the shortest distance between two vertices.

? Using Dijkstras algorithm, find the shortest path between the vertices s and t.