

$$Q_{ff} = K_{fu} K_{uu}^{-1} K_{uf} \quad (1)$$

$$\frac{\partial Q_{ff}}{\partial \theta_i} = K_{fu} \frac{\partial K_{uu}^{-1} K_{uf}}{\partial \theta_i} + \frac{\partial K_{fu}}{\partial \theta_i} K_{uu}^{-1} K_{uf} \quad (2)$$

$$\frac{\partial Q_{ff}}{\partial \theta_i} = K_{fu} \left[K_{uu}^{-1} \frac{\partial K_{uf}}{\partial \theta_i} - (K_{uu}^{-1} \frac{\partial K_{uu}}{\partial \theta_i} K_{uu}^{-1}) K_{uf} \right] + \frac{\partial K_{fu}}{\partial \theta_i} K_{uu}^{-1} K_{uf} \quad (3)$$

$$\frac{\partial Q_{ff}}{\partial \theta_i} = K_{fu} K_{uu}^{-1} \left[\frac{\partial K_{uf}}{\partial \theta_i} - \left(\frac{\partial K_{uu}}{\partial \theta_i} K_{uu}^{-1} \right) K_{uf} \right] + \frac{\partial K_{fu}}{\partial \theta_i} K_{uu}^{-1} K_{uf} \quad (4)$$

$$R = Q_{ff} + \text{diag}[K_{ff} - Q_{ff}] + \sigma_{noise}^2 I \quad (5)$$

$$\frac{\partial R}{\partial \theta_i} = \frac{\partial Q_{ff}}{\partial \theta_i} + \text{diag} \left[\frac{\partial K_{ff}}{\partial \theta_i} \right] - \text{diag} \left[\frac{\partial Q_{ff}}{\partial \theta_i} \right] + \frac{\partial \sigma_{noise}^2}{\partial \theta_i} I \quad (6)$$

$$\log(y|X) = -\frac{1}{2} \log|R| - \frac{1}{2} \mathbf{y}^\top R^{-1} \mathbf{y} - \frac{n}{2} \log(2\pi) \quad (7)$$

$$\frac{\partial \log(y|X)}{\partial \theta_i} = -\frac{1}{2} \text{tr} \left(R^{-1} \frac{\partial R}{\partial \theta_i} \right) + \frac{1}{2} \mathbf{y}^\top R^{-1} \frac{\partial R}{\partial \theta_i} R^{-1} \mathbf{y} \quad (8)$$