$$Q_{ff} = K_{fu} {K_{uu}}^{-1} K_{uf} \tag{1}$$

$$\frac{\partial Q_{ff}}{\partial \theta_i} = K_{fu} \frac{\partial {K_{uu}}^{-1} K_{uf}}{\partial \theta_i} + \frac{\partial K_{fu}}{\partial \theta_i} {K_{uu}}^{-1} K_{uf} \tag{2}$$

$$\frac{\partial Q_{ff}}{\partial \theta_i} = K_{fu} \Big[{K_{uu}}^{-1} \frac{\partial K_{uf}}{\partial \theta_i} - ({K_{uu}}^{-1} \frac{\partial K_{uu}}{\partial \theta_i} {K_{uu}}^{-1}) K_{uf} \Big] + \frac{\partial K_{fu}}{\partial \theta_i} {K_{uu}}^{-1} K_{uf} \tag{3}$$

$$\frac{\partial Q_{ff}}{\partial \theta_i} = K_{fu} {K_{uu}}^{-1} \big[\frac{\partial K_{uf}}{\partial \theta_i} - (\frac{\partial K_{uu}}{\partial \theta_i} {K_{uu}}^{-1}) K_{uf} \big] + \frac{\partial K_{fu}}{\partial \theta_i} {K_{uu}}^{-1} K_{uf} \tag{4}$$

$$R = Q_{ff} + diag[K_{ff} - Q_{ff}] + \sigma^2_{noise} I \tag{5}$$

$$\frac{\partial R}{\partial \theta_i} = \frac{\partial Q_{ff}}{\partial \theta_i} + diag[\frac{\partial K_{ff}}{\partial \theta_i}] - diag[\frac{\partial Q_{ff}}{\partial \theta_i}] + \frac{\partial \sigma^2_{noise} I}{\partial \theta_i} \tag{6}$$

$$log(y|X) = -\frac{1}{2} log|R| - \frac{1}{2} \mathbf{y}^\top R^{-1} \mathbf{y} - \frac{n}{2} log(2\pi) \tag{7}$$

$$\frac{\partial log(y|X)}{\partial \theta_i} = -\frac{1}{2} tr\Big(R^{-1} \frac{\partial R}{\partial \theta_i} \Big) - \frac{1}{2} \mathbf{y}^\top R^{-1} \frac{\partial R}{\partial \theta_i} R^{-1} \mathbf{y} \tag{8}$$