Extending Flexible Boolean Semantics for Plural Definites in Mathematical Language

Investigating the Semantics of Mathematical Language

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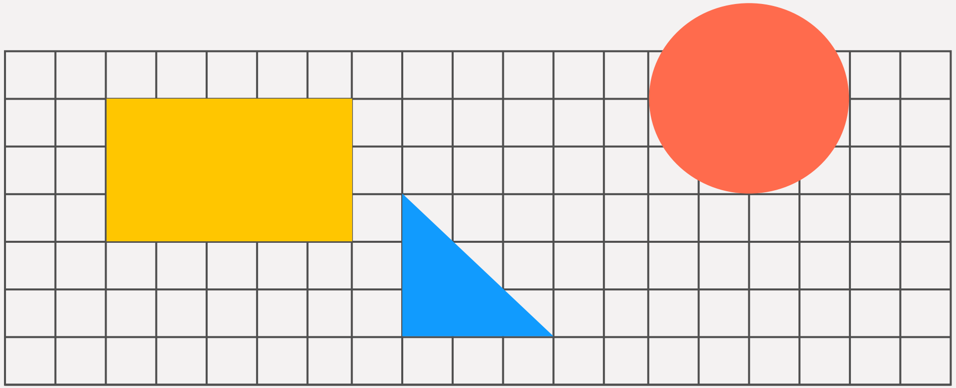

A Mathematico-Linguistic Puzzle

(1) A. x and y are prime.
 B. x is prime.

(2) A. x and y are coprime.
 B. # x is coprime.

(3) A. P and Q are countable.
 B. P is countable.

(4) A. P and Q are equinumerous.
 B. # P is equinumerous.

Why does A entail B in (1) and (3), but not in (2) and (4)?

Why study the Language of Mathematics?

LoM: Statements found in mathematics papers and textbooks.

Underexplored in formal semantics:
Test-bed for semantic theories

Key to understanding how
mathematical meaning is constructed

Foundation for auto-formalisation and
math-aware NLP

A Mathematico-Linguistic Puzzle

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    B. x is prime.

(2) A. x and y are coprime.
    B. # x is coprime.

(3) A. P and Q are countable.
    B. P is countable.

(4) A. P and Q are equinumerous.
    B. #P is equinumerous.

Why does A entail B in (1) and (3), but not in (2) and (4)?

Coordination and Distributivity

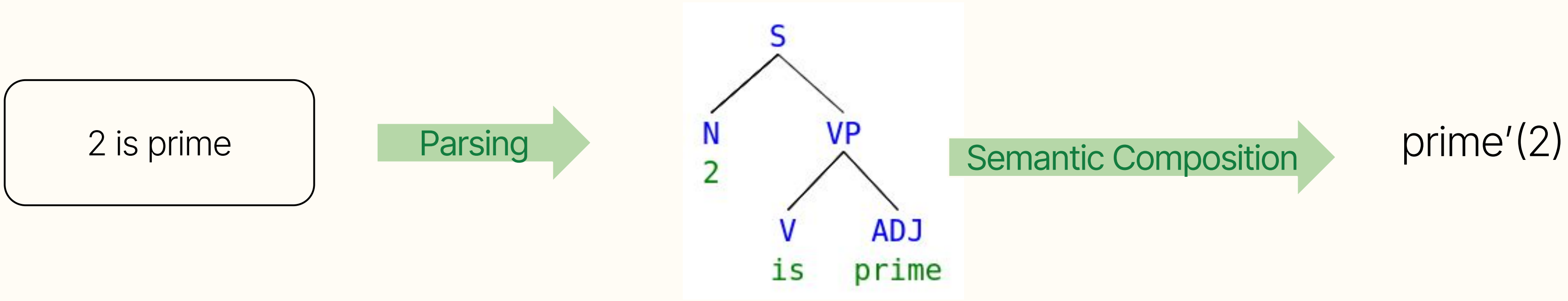
X and Y are P

Coordination
2 and 3,
The integers and the rationals,
The set X and the set Y

Distributivity
Prime, countable

Collectivity
Coprime, equinumerous

Montague Semantics: Building Meaning from Parts

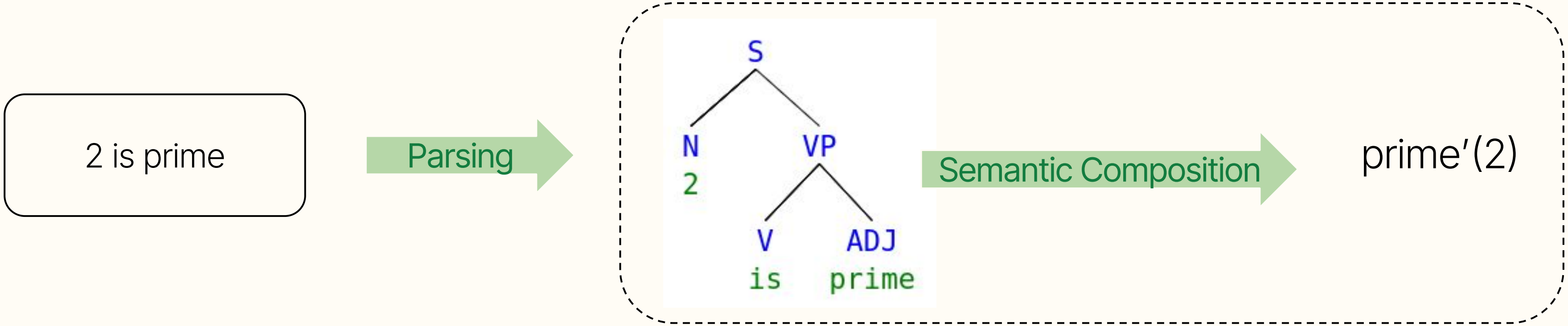


Montague Semantics in Action: Types

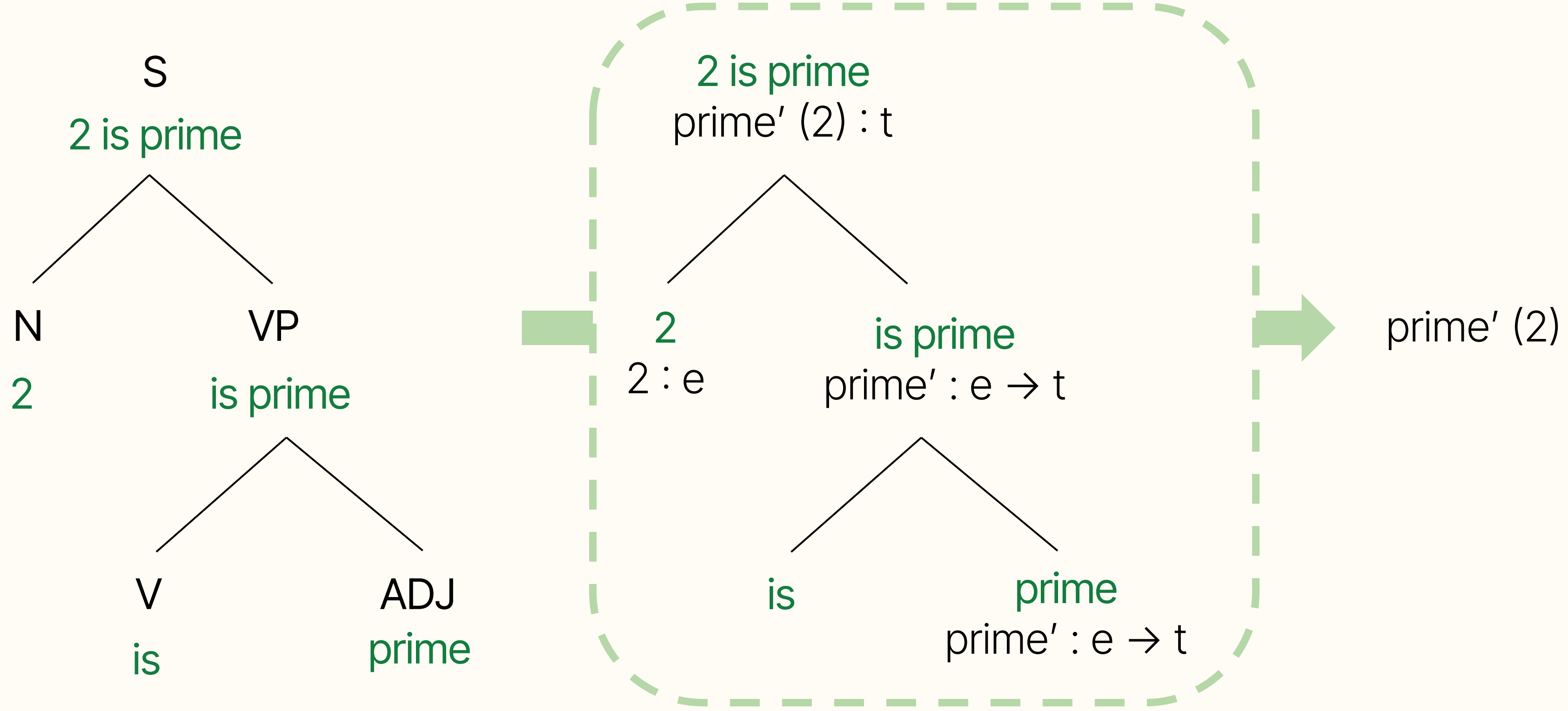
$e \quad t \quad e \to t \quad t \to t \quad e \to (e \to t) \quad (e \to t) \to t \quad \ldots$

Linguistic Expression	Type
2 is prime	t
2	e
prime	e → t

Semantic Composition: Zooming In



Montague Semantics in Action: Composition



Boolean Extensions of Montague Semantics

Sentence	Syntactic Category Coordinated	Types
2 is prime and **3 is prime**	Sentence	t
2 and **3** are prime	Noun Phrase	e
2 is **prime** and **even**	Adjective	$e \to t$

Boolean Semantics:

Domains = Boolean algebras

and = meet

$\wedge$ for t

$\cap$ for $e \to t$

$[\![\text{2 is prime and 3 is prime}]\!]$
$= [\![\text{2 is prime}]\!] \wedge [\![\text{3 is prime}]\!]$
$= \text{prime}'(2) \wedge \text{prime}'(3)$

$[\![\text{2 is prime and even}]\!]$
$= 2 \in [\![\text{prime and even}]\!]$
$= 2 \in \text{prime}' \cap \text{even}'$

Boolean Operation on Entities?

Sentence	Syntactic Category Coordinated	Type
2 and **3** are prime	Noun Phrase	e

Boolean Semantics:

Domains = Boolean algebras

And = meet

$\wedge$ for t

$\cap$ for $e \rightarrow t$

??? for e

$$\boxed{\mathbf{M}(x) = \{ P \mid P(x) \}}$$

$$[\![2 \text{ and } 3]\!] = \mathbf{M}(2) \cap \mathbf{M}(3)$$
$$= \{ P \mid P(2) \} \cap \{ Q \mid Q(3) \}$$

Winter, Y.: Flexibility Principles in Boolean Semantics: The Interpretation of Coordination, Plurality, and Scope in Natural Language, chap. Coordination and Collectivity. The MIT Press (02 2002).

Flexible Boolean Semantics

$$\boxed{\mathbf{M}(x) = \{\, P \mid P(x) \,\}}$$

$$\begin{aligned}[\![\text{2 and 3}]\!] &= \mathbf{M}(2) \cap \mathbf{M}(3) \\ &= \{\, P \mid P(2) \,\} \cap \{\, Q \mid Q(3) \,\}\end{aligned}$$

$$\begin{aligned}[\![\text{2 and 3 are prime}]\!] &= \text{prime}' \in [\![\text{2 and 3}]\!] \\ &= \text{prime}' \in \{\, P \mid P(2) \,\} \cap \{\, Q \mid Q(3) \,\} \\ &= \text{prime}'(2) \wedge \text{prime}'(3) \\ &= [\![\text{2 is prime}]\!] \wedge [\![\text{3 is prime}]\!] \\ &= [\![\text{2 is prime and 3 is prime}]\!]\end{aligned}$$

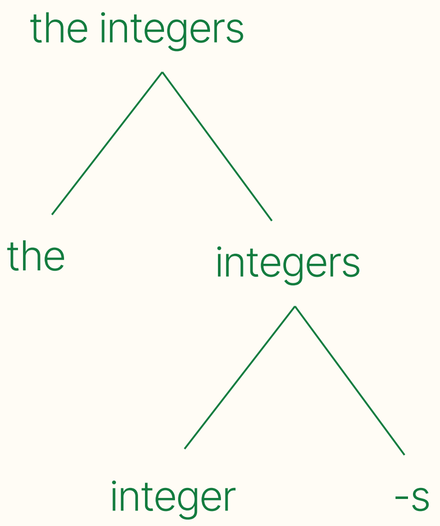
Flexible!

Partee, B., Rooth, M.: Generalized conjunction and type ambiguity. In: Bäuerle, R., Schwarze, C., von Stechow, A. (eds.) Meaning, Use, and Interpretation of Language, pp. 361–383. De Gruyter, Berlin, Boston (1983)

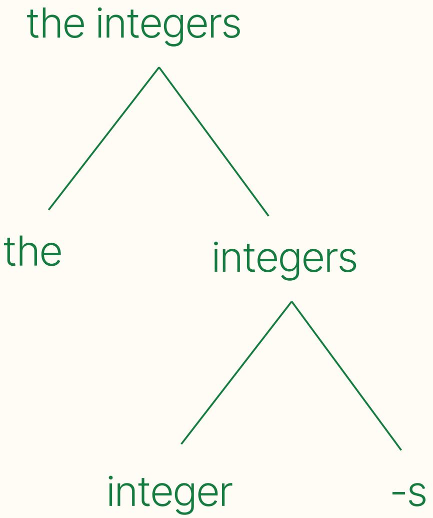
Sets as Subjects

The integers are countable

- [[-s]]
- [[the]]
- [[integer]]
- [[countable]]



Plural Forming *-s*



$$[\![\text{integer}]\!] : e \to t = Z$$

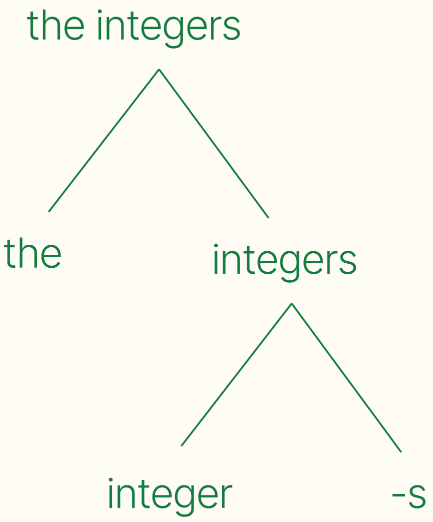
$$[\![\text{-s}]\!] : (e \to t) \to ((e \to t) \to t)$$
$$= \lambda P\, \lambda Q.(Q \neq \varnothing \wedge Q \subseteq P)$$

$$[\![\text{integers}]\!] : (e \to t) \to t$$
$$= [\![\text{-s}]\!]\, [\![\text{integer}]\!]$$
$$= \lambda Q.(Q \neq \varnothing \wedge Q \subseteq Z)$$

Link, G.: The logical analysis of plurals and mass terms: A lattice-theoretical approach. In: Bäuerle, R., Schwarze, C., von Stechow, A. (eds.) Meaning, Use, and Interpretation of Language, pp. 302–323. De Gruyter, Berlin, Boston (1983).

Sharvy, R.: A more general theory of definite descriptions. The Philosophical Review 89(4), 607–624 (1980),

Plural Definite Article



$[\![\text{the}]\!] : ((e \to t) \to t) \to (e \to t)$
$= \lambda P.\ (\text{unique largest } A \text{ s.t. } P(A))$

$[\![\text{integers}]\!] : (e \to t) \to t$
$= \lambda Q.(Q \neq \varnothing \wedge Q \subseteq Z)$

$[\![\text{the integers}]\!] : e \to t$
$= [\![\text{the}]\!]\ [\![\text{integers}]\!] = Z$

$[\![\text{the reals}]\!] : e \to t = R$

Sharvy, R.: A more general theory of definite descriptions. The Philosophical Review 89(4), 607–624 (1980)

Sets as Subjects : Revisited

countable : $(e \to t) \to t$

countable′(Z), countable′(Q)	True
countable′(R)	False

[[the integers are countable]]
[[countable]][[the integers]]
= countable′(Z)

[[the reals are countable]]
[[countable]][[the reals]]
= countable′(R)

Where Conventional Typing Goes Wrong

Both sets and properties have type $e \to t$

[[The integers and the reals]]
= [[the integers]] $\cap$ [[the reals]]
= $Z \cap R$
= Z

[[The integers and the reals are countable]]
= countable′(Z)
= [[The integers are countable]]

[[prime and even]]
= [[prime]] $\cap$ [[even]]

- x is prime and even
- x belongs to the intersection of prime integers and even integers
- x is prime and x is even
- prime′(x) $\wedge$ even′(x)

A Typing Fix

Core issue: Sets and properties behave differently when coordinated

Fix:

1. Change the type system by adding another constructor S

   $e \quad t \quad S(e) \quad e \to t \quad S(e) \to t \quad (e \to t) \to t \quad S(S(e)) \to t \quad \ldots$

2. Let "the integers" and "the reals" have type $S(e)$ instead of $e \to t$.

   $[\![\text{-s}]\!] : (e \to t) \to (S(e) \to t)$ and **not** $[\![\text{-s}]\!] : (e \to t) \to ((e \to t) \to t)$
   $[\![\text{the}]\!] : (S(e) \to t) \to S(e)$ and **not** $[\![\text{the}]\!] : ((e \to t) \to t) \to (e \to t)$

3. Do **not** define "and" as $\cap$ for $S(e)$

Typing Fix and Flexibility for Correct Derivations

Fix: 4. Extend **M**

$$\mathbf{M} : e \to (e \to t) \to t = \lambda x .\{ P \mid P(x) \}$$
$$\mathbf{M'} : S(e) \to (S(e) \to t) \to t = \lambda A.\ \{ P \mid P(A) \}$$

$$\begin{aligned}[\![\text{The integers and the reals}]\!] &= \mathbf{M'}([\![\text{the integers}]\!]) \cap \mathbf{M'}([\![\text{the reals}]\!]) \\ &= \mathbf{M'}(Z) \cap \mathbf{M'}(R) \\ &= \{ P \mid P(Z) \} \cap \{ Q \mid Q(R) \}\end{aligned}$$

$$\begin{aligned}&[\![\text{The integers and the reals are countable}]\!] \\ &= \text{countable}' \in [\![\text{The integers and the reals}]\!] \\ &= \text{countable}' \in \{ P \mid P(Z) \} \cap \{ Q \mid Q(R) \} \\ &= \text{countable}'(Z) \wedge \text{countable}'(R) \\ &= [\![\text{The integers are countable and the reals are countable}]\!]\end{aligned}$$

Conclusion, Limitations and Future Work

- Flexible Boolean Semantics

- Coordination of sets differs from coordination of properties

- Typing fix which distinguishes between sets and properties

"The integers and reals are countable" is true.

Quantificational vs predicative NPs
- The integers and reals are countable

Generalized quantifier theory
- All prime numbers greater than *x* are odd.
- Some numbers are irrational.
- Exactly five primes are less than *y*.

ON IT

Winter, Yoad. Flexibility principles in Boolean semantics: The interpretation of coordination, plurality, and scope in natural language. MIT press, 2002.