Simulations for Problem 1

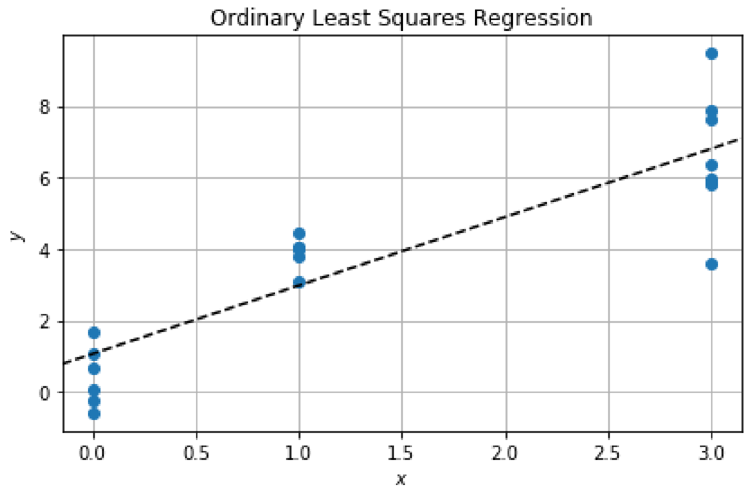
In [1]:
```
import collections
import itertools

import matplotlib.pyplot as plt
import multiprocessing
import numpy as np
import pandas as pd
from scipy import linalg
from scipy import stats

X_VALUES = [0, 1, 3]
MU = np.array([0.6, 3.6, np.nan, 6.8])


def simulate_data(n, random_seed=None):
    np.random.seed(random_seed)
    dist = stats.rv_discrete(values=list(zip(
        *[(x, 1/len(X_VALUES)) for x in X_VALUES])))
    x = dist.rvs(size=(n,))
    while len(np.unique(x)) == 1:
        x = dist.rvs(size=(n,))
    mu = MU[x]
    return x, stats.norm.rvs(loc=mu, scale=1)

def solve_beta_hat(x, y):
    X = np.column_stack((np.ones_like(x), x))
    return linalg.cho_solve(linalg.cho_factor(X.T.dot(X)), np.eye(X.shape[1])).dot
(X.T).dot(y)
```

In [2]:
```
x, y = simulate_data(20, random_seed=2019)
```

In [3]:
```
beta_hat = solve_beta_hat(x, y)
beta_hat
```

Out[3]:
```
array([1.07327318, 1.9097713 ])
```

In [4]:
```
fig = plt.figure(figsize=(6,4))
ax = fig.gca()
ax.grid(True)
ax.plot(x, y, 'o')
xlim_left, xlim_right = ax.get_xlim()
ax.plot([xlim_left, xlim_right],
        np.array([xlim_left, xlim_right])*beta_hat[1] + beta_hat[0], '--k')
ax.set_xlim(xlim_left, xlim_right);
ax.set_xlabel('$x$')
ax.set_ylabel('$y$')
ax.set_title('Ordinary Least Squares Regression')
fig.tight_layout()
fig.savefig('p1_ols.pdf', bbox_inches='tight')
```



In [5]:
```
def _estimate_beta1(n):
    x, y = simulate_data(n)
    return solve_beta_hat(x, y)[1]

def estimate_beta1(num_estimates, n):
    pool = multiprocessing.Pool(4)
    estimates = pool.map(_estimate_beta1, [n]*num_estimates)
    pool.close()
    return np.array(estimates)

def estimate_bias(num_trials):
    n_list = list(range(2, 31, 2)) + list(range(30, 101, 5))
    mean = []
    standard_errors = []
    for n in n_list:
        estimates = estimate_beta1(num_trials, n)
        mean.append(np.mean(estimates))
        standard_errors.append(np.sqrt(np.var(estimates, ddof=1)))
    return pd.DataFrame(collections.OrderedDict([
        ('$n$', n_list),
        ('$\\hat{\\beta}_1$', mean),
        ('$\\hat{\\sigma}$', standard_errors),
    ]))

#bias_simulations = estimate_bias(1000000)
#bias_simulations.to_pickle('p1_simulations.pkl')
bias_simulations = pd.read_pickle('p1_simulations.pkl')
bias_simulations
```

Out[5]:

	n	$\hat{\beta}_1$	$\hat{\sigma}$
0	2	2.222531	1.115256
1	4	2.119261	0.752701
2	6	2.058301	0.536735
3	8	2.028134	0.408784
4	10	2.014462	0.332531
5	12	2.008274	0.284690
6	14	2.005376	0.253067
7	16	2.003600	0.231102
8	18	2.002070	0.213511
9	20	2.001587	0.200356
10	22	2.001091	0.189500
11	24	2.001169	0.180668
12	26	2.000937	0.172259
13	28	2.000748	0.165419
14	30	2.000182	0.159203
15	30	2.000505	0.159288
16	35	2.000618	0.146462
17	40	2.000300	0.136427
18	45	2.000245	0.127994
19	50	2.000259	0.121297
20	55	2.000184	0.115340
21	60	2.000254	0.110308
22	65	2.000014	0.105664
23	70	2.000246	0.101802
24	75	2.000068	0.098277
25	80	2.000033	0.094885
26	85	2.000091	0.092020
27	90	2.000172	0.089272
28	95	1.999983	0.087027
29	100	1.999924	0.084749

In [6]:
```
def partition(n, k = None, m = None):
    k = n if k is None else k
    m = n if m is None else m
    if m is None or m >= n:
        yield [n]

    for f in range(n-1 if m >= n else m, 0, -1):
        if f*(k - 1) < n - f:
            break
        for p in partition(n-f, k - 1, f):
            yield [f] + p


def multiset_permuation(xs):
    current_permutation = len(xs)*[None]
    counts = collections.Counter(xs)

    def yield_permuation(j):
        if j == len(xs):
            yield tuple(current_permutation)
            return

        for x in counts:
            if counts[x] <= 0:
                continue
            counts[x] -= 1
            current_permutation[j] = x
            for permuation in yield_permuation(j + 1):
                yield permuation
            counts[x] += 1

    return yield_permuation(0)

def enumerate_multinomial_support(n, k):
    for p in partition(n, k):
        unordered_draw = [0]*(k - len(p)) + p
        for x in multiset_permuation(unordered_draw):
            yield x
```

In [7]:
```
def solve_expected_beta(n):
    support = enumerate_multinomial_support(n, 3)
    weighted_sum = np.zeros(2)
    total_prob = 0
    for sample in support:
        if np.isin(n, sample):
            continue
        x = np.hstack([count*[X_VALUES[i]] for i, count in enumerate(sample)])
        y = MU[x.astype(np.int)]
        expected_beta = solve_beta_hat(x, y)
        prob = stats.multinomial.pmf(sample, n, p = np.ones(len(X_VALUES))/len(X_V
ALUES))
        weighted_sum += expected_beta*prob
        total_prob += prob
    return weighted_sum/total_prob

expected_beta = np.array([solve_expected_beta(n) for n in bias_simulations['$n$'
]])
expected_beta
```

Out[7]:
```
array([[1.06666667, 2.22222222],
       [1.04168541, 2.11935822],
       [1.0336912 , 2.05765274],
       [1.02878485, 2.02807713],
       [1.02477543, 2.01456074],
       [1.02144138, 2.00820044],
       [1.01872191, 2.00502319],
       [1.01652181, 2.00331956],
       [1.01473662, 2.00233772],
       [1.01327484, 2.00173226],
       [1.01206379, 2.0013359 ],
       [1.01104808, 2.00106296],
       [1.01018608, 2.00086695],
       [1.00944647, 2.00072128],
       [1.00880554, 2.00060992],
       [1.00880554, 2.00060992],
       [1.00752558, 2.0004236 ],
       [1.00656859, 2.0003117 ],
       [1.00582669, 2.0002391 ],
       [1.00523495, 2.00018928],
       [1.0047521 , 2.00015359],
       [1.00435066, 2.00012714],
       [1.00401168, 2.00010698],
       [1.00372166, 2.00009128],
       [1.0034707 , 2.00007879],
       [1.00325143, 2.00006871],
       [1.0030582 , 2.00006045],
       [1.00288664, 2.00005359],
       [1.00273329, 2.00004784],
       [1.00259541, 2.00004297]])
```

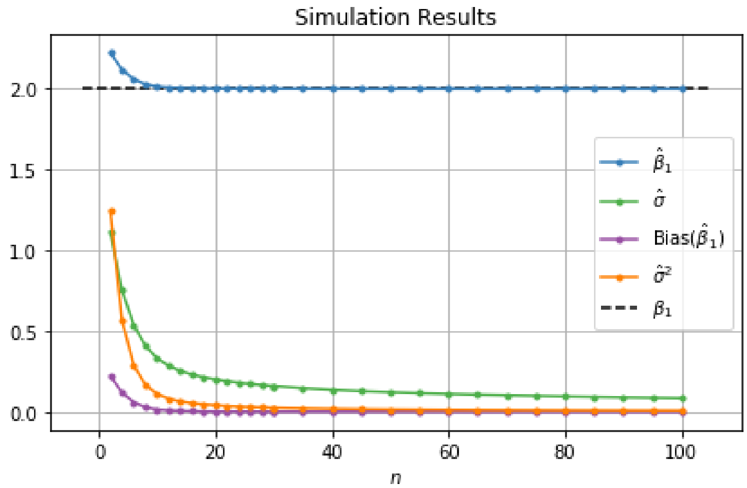
In [8]:
```
bias_simulations['$\\mathbb{E}[\\hat{\\beta}_1]$'] = expected_beta[:,1]
bias_simulations
```

Out[8]:

	n	$\hat{\beta}_1$	$\hat{\sigma}$	$\mathbb{E}[\hat{\beta}_1]$
0	2	2.222531	1.115256	2.222222
1	4	2.119261	0.752701	2.119358
2	6	2.058301	0.536735	2.057653
3	8	2.028134	0.408784	2.028077
4	10	2.014462	0.332531	2.014561
5	12	2.008274	0.284690	2.008200
6	14	2.005376	0.253067	2.005023
7	16	2.003600	0.231102	2.003320
8	18	2.002070	0.213511	2.002338
9	20	2.001587	0.200356	2.001732
10	22	2.001091	0.189500	2.001336
11	24	2.001169	0.180668	2.001063
12	26	2.000937	0.172259	2.000867
13	28	2.000748	0.165419	2.000721
14	30	2.000182	0.159203	2.000610
15	30	2.000505	0.159288	2.000610
16	35	2.000618	0.146462	2.000424
17	40	2.000300	0.136427	2.000312
18	45	2.000245	0.127994	2.000239
19	50	2.000259	0.121297	2.000189
20	55	2.000184	0.115340	2.000154
21	60	2.000254	0.110308	2.000127
22	65	2.000014	0.105664	2.000107
23	70	2.000246	0.101802	2.000091
24	75	2.000068	0.098277	2.000079
25	80	2.000033	0.094885	2.000069
26	85	2.000091	0.092020	2.000060
27	90	2.000172	0.089272	2.000054
28	95	1.999983	0.087027	2.000048
29	100	1.999924	0.084749	2.000043

In [9]:
```
with open('p1_simulation_results.tex', 'w') as f:
    f.write(bias_simulations.to_latex(escape=False, index=False))
```

In [10]:
```
fig = plt.figure(figsize=(6, 4))
ax = fig.gca()
ax.grid(True)

ax.plot(bias_simulations['$n$'], bias_simulations['$\\hat{\\beta}_1$'],
        '.-', label='$\\hat{\\beta}_1$', c=plt.cm.Set1(1))
ax.plot(bias_simulations['$n$'], bias_simulations['$\\hat{\\sigma}$'],
        '.-', label='$\\hat{\\sigma}$', c=plt.cm.Set1(2))
ax.plot(bias_simulations['$n$'], bias_simulations['$\\hat{\\beta}_1$'] - 2,
        '.-', label='Bias($\\hat{\\beta}_1$)', c=plt.cm.Set1(3))
ax.plot(bias_simulations['$n$'], bias_simulations['$\\hat{\\sigma}$']**2,
        '.-', label='$\\hat{\\sigma}^2$', c=plt.cm.Set1(4))
ax.hlines(y=2, xmin=ax.get_xlim()[0], xmax=ax.get_xlim()[1],
          linestyles='--', label='$\\beta_1$')
ax.legend()
ax.set_xlabel('$n$')
ax.set_title('Simulation Results')
fig.tight_layout()
fig.savefig('p1_simulation_results.pdf', bbox_inches='tight')
```



In [11]:
```
fig = plt.figure(figsize=(6, 4))
ax = fig.gca()
ax.grid(True)

ax.plot(bias_simulations['$n$'], bias_simulations['$\\hat{\\beta}_1$'] - 2,
        '.-', label='Simulated Bias($\\hat{\\beta}_1$)', c=plt.cm.Set1(0), alpha=
0.4)
ax.plot(bias_simulations['$n$'], bias_simulations['$\mathbb{E}[\\hat{\\beta}_1]$']
 - 2,
        '.-', label='Bias($\\hat{\\beta}_1$)', c=plt.cm.Set1(1), alpha=0.4)
ax.legend()
ax.set_title('Bias Comparison')
ax.set_xlabel('$n$')
ax.set_ylabel('Bias')
ax.set_yscale('symlog', linthreshy=1e-5)
fig.tight_layout()
fig.savefig('p1_bias_comparison.pdf', bbox_inches='tight')
```