

April 23, 2024

## Lecture #1 :

- What is a problem? → Knowing a relationship between 2 sets
- What is an algorithm → mapping of input to outputs.
  - ↳ essentially a function, a procedure
- Birthday Problem
  - ↳ How to know if 2 people in class have identical birthdays?

• How do we prove correctness for any arbitrarily large input?

↳ induction

• How do we argue that an algorithm is efficient.

↳ easy answer is record time it takes

↳ problem: computer speed dependent!

↳ hence we count it abstractly

↳ if a computer takes a fixed amount of time to perform an operation or step then how many of those steps <sup>or ops</sup> does this algo take?

↳ Asymptotic Analysis

↳ we expect to be dependent on input size.

↳ i.e. "linear w.r.t. size of input"

↳  $O(\cdot)$  = upper bound

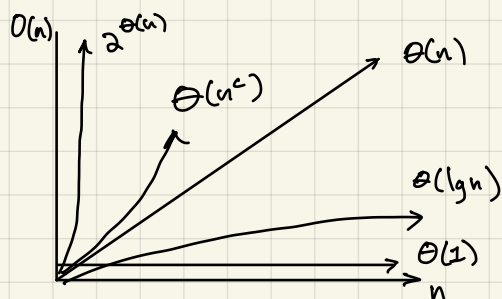
↳  $\Omega(\cdot)$  = lower bound

↳  $\Theta(\cdot)$  = up & low bound

• Types of asymptotic functions

- \*  $\Theta(1)$  = constant time
- \*  $\Theta(\lg n)$  = logarithmic time
- \*  $\Theta(n)$  = linear time
- \*  $\Theta(n \lg n)$  =
- \*  $\Theta(n^2)$  = quadratic time
- \*  $\Theta(n^c)$  = polynomial time → considered to be efficient enough
- \*  $2^{\Theta(n)}$  = exponential → Bad!

↑  
more efficient



• Model of Computation (our abstract model of a computer)

↳ "Word - RAM"

↓  
how big of a chunk can be retrieved from RAM and operated on CPU.

↳ random-access memory  
↳ can access any addr in memory at a fixed amt of time and bring it to the CPU to operate on it.

↳ Nowadays, this is 64 bit.

(used to be 32-bit) →  $2^{32}$   
↳ 4GB drive

## Studying Setup

↳ Lectures 1hr/sesh ; T, Th

↳ Recit 1hr/sesh ; W, F

↳ P.S. : Monday ; 2 hrs doing

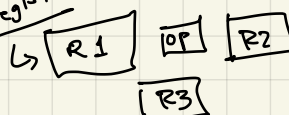
↳ P.S. Checking : Saturday Mornings  
1.5 hr for G.006

1.5 hr for G.042J

→ What ops can this model do @  $O(1)$  time "constant time"

- R/W on constant time.
- integer arith
- logical ops
- bitwise ops

essentially those that can be done on a fixed size of CPU registers



## Lecture 1: Introduction

The goal of this class is to teach you to **solve** computation problems, and to **communicate** that your solutions are **correct** and **efficient**.

### Problem

- Binary relation from **problem inputs** to **correct outputs**
- Usually don't specify every correct output for all inputs (too many!)
- Provide a verifiable **predicate** (a property) that correct outputs must satisfy
- 6.006 studies problems on large general input spaces
- Not general: small input instance
  - **Example:** In this room, is there a pair of students with same birthday?
- General: arbitrarily large inputs
  - **Example:** Given any set of  $n$  students, is there a pair of students with same birthday?
  - If birthday is just one of 365, for  $n > 365$ , answer always true by pigeon-hole
  - Assume resolution of possible birthdays exceeds  $n$  (include year, time, etc.)

### Algorithm

- Procedure mapping each input to a **single** output (deterministic)
- Algorithm **solves** a problem if it returns a correct output for every problem input
- **Example:** An algorithm to solve birthday matching
  - Maintain a **record** of names and birthdays (initially empty)
  - Interview each student in some order
    - \* If birthday exists in record, return found pair!
    - \* Else add name and birthday to record
  - Return None if last student interviewed without success

## Correctness

- Programs/algorithms have fixed size, so how to prove correct?
- For small inputs, can use case analysis
- For arbitrarily large inputs, algorithm must be **recursive** or loop in some way
- Must use **induction** (why recursion is such a key concept in computer science)
- **Example:** Proof of correctness of birthday matching algorithm
  - Induct on  $k$ : the number of students in record
  - **Hypothesis:** if first  $k$  contain match, returns match before interviewing student  $k + 1$
  - **Base case:**  $k = 0$ , first  $k$  contains no match
  - Assume for induction hypothesis holds for  $k = k'$ , and consider  $k = k' + 1$
  - If first  $k'$  contains a match, already returned a match by induction
  - Else first  $k'$  do not have match, so if first  $k' + 1$  has match, match contains  $k' + 1$
  - Then algorithm checks directly whether birthday of student  $k' + 1$  exists in first  $k'$   $\square$

## Efficiency

- How fast does an algorithm produce a correct output?
  - Could measure time, but want performance to be machine independent
  - **Idea!** Count number of fixed-time operations algorithm takes to return
  - Expect to depend on size of input: larger input suggests longer time
  - Size of input is often called ' $n$ ', but not always!
  - Efficient if returns in **polynomial time** with respect to input
  - Sometimes no efficient algorithm exists for a problem! (See L20)
- Asymptotic Notation: ignore constant factors and low order terms
  - Upper bounds ( $O$ ), lower bounds ( $\Omega$ ), tight bounds ( $\Theta$ )  $\in, =, \text{is, order}$
  - Time estimate below based on one operation per cycle on a 1 GHz single-core machine
  - Particles in universe estimated  $< 10^{100}$

input	constant	logarithmic	linear	log-linear	quadratic	polynomial	exponential
$n$	$\Theta(1)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(n \log n)$	$\Theta(n^2)$	$\Theta(n^c)$	$2^{\Theta(n^c)}$
1000	1	$\approx 10$	1000	$\approx 10,000$	1,000,000	$1000^c$	$2^{1000} \approx 10^{301}$
Time	1 ns	10 ns	1 $\mu s$	10 $\mu s$	1 ms	$10^{3c-9} s$	$10^{281}$ millenia

## Model of Computation

- Specification for what operations on the machine can be performed in  $O(1)$  time
- Model in this class is called the **Word-RAM**
- **Machine word**: block of  $w$  bits ( $w$  is word size of a  $w$ -bit Word-RAM)
- **Memory**: Addressable sequence of machine words
- **Processor** supports many **constant time** operations on a  $O(1)$  number of words (**integers**):
  - **integer** arithmetic:  $(+, -, *, //, \%)$
  - **logical** operators:  $(\&\&, ||, !, ==, <, >, <=, ==>)$
  - **(bitwise** arithmetic:  $(\&, |, <<, >>, \dots)$
  - Given word  $a$ , can **read** word at address  $a$ , **write** word to address  $a$
- Memory address must be able to access every place in memory
  - Requirement:  $w \geq \#$  bits to represent largest memory address, i.e.,  $\log_2 n$
  - 32-bit words  $\rightarrow$  max  $\sim 4$  GB memory, 64-bit words  $\rightarrow$  max  $\sim 16$  exabytes of memory
- **Python** is a more complicated model of computation, implemented on a Word-RAM

## Data Structure

- A **data structure** is a way to store non-constant data, that supports a set of operations
- A collection of operations is called an **interface**
  - **Sequence**: Extrinsic order to items (first, last,  $n$ th)
  - **Set**: Intrinsic order to items (queries based on item keys)
- Data structures may implement the same interface with different performance
- **Example: Static Array** - fixed width slots, fixed length, static sequence interface
  - `StaticArray(n)`: allocate static array of size  $n$  initialized to 0 in  $\Theta(n)$  time
  - `StaticArray.get_at(i)`: return word stored at array index  $i$  in  $\Theta(1)$  time
  - `StaticArray.set_at(i, x)`: write word  $x$  to array index  $i$  in  $\Theta(1)$  time
- Stored word can hold the address of a larger object
- Like Python `tuple` plus `set_at(i, x)`, Python `list` is a **dynamic array** (see L02)

```

1 def birthday_match(students):
2     '''
3     Find a pair of students with the same birthday
4     Input: tuple of student (name, bday) tuples
5     Output: tuple of student names or None
6     '''
7     n = len(students)                # O(1)
8     record = StaticArray(n)          # O(n)
9     for k in range(n):              # n
10        (name1, bday1) = students[k]  # O(1)
11        # Return pair if bday1 in record
12        for i in range(k):           # k
13            (name2, bday2) = record.get_at(i) # O(1)
14            if bday1 == bday2:        # O(1)
15                return (name1, name2)  # O(1)
16            record.set_at(k, (name1, bday1)) # O(1)
17    return None                       # O(1)

```

### Example: Running Time Analysis

- Two loops: outer  $k \in \{0, \dots, n-1\}$ , inner is  $i \in \{0, \dots, k\}$
- Running time is  $O(n) + \sum_{k=0}^{n-1} (O(1) + k \cdot O(1)) = O(n^2)$
- Quadratic in  $n$  is **polynomial**. Efficient? Use different data structure for record!

Handwritten diagram illustrating the summation formula for the running time analysis. The formula is  $S_n = n \frac{(a_1 + a_n)}{2}$ . Annotations include:
 

- $n$ : # of terms
- $a_1$ : 1st term
- $a_n$ : last term
- $n-1$ : (n-1)

 A blue arrow points from the  $O(n^2)$  result in the text to the formula.

## How to Solve an Algorithms Problem

1. Reduce to a problem you already know (use data structure or algorithm)

#### Search Problem (Data Structures)

Static Array (L01)  
 Linked List (L02)  
 Dynamic Array (L02)  
 Sorted Array (L03)  
 Direct-Access Array (L04)  
 Hash Table (L04)  
 Balanced Binary Tree (L06-L07)  
 Binary Heap (L08)

#### Sort Algorithms

Insertion Sort (L03)  
 Selection Sort (L03)  
 Merge Sort (L03)  
 Counting Sort (L05)  
 Radix Sort (L05)  
 AVL Sort (L07)  
 Heap Sort (L08)

#### Shortest Path Algorithms

Breadth First Search (L09)  
 DAG Relaxation (L11)  
 Depth First Search (L10)  
 Topological Sort (L10)  
 Bellman-Ford (L12)  
 Dijkstra (L13)  
 Johnson (L14)  
 Floyd-Warshall (L18)

2. Design your own (recursive) algorithm

- Brute Force
- Decrease and Conquer
- Divide and Conquer
- **Dynamic Programming** (L15-L19)
- Greedy / Incremental