Flexible Shaft Model

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In this document the model of a generic flexible shaft is derived.

1 Equivalent mathematical model

The mass-spring-mass model is a valid approximation of the flexible shaft and it will be used to modelize it and its extrapolation to a generic simplified gear. Figure 1 shows a generic flexible shaft where τ_m is the torque actuated by a primer mover and τ_l is assumed to be a disturbance or a generic unknown load torque.

1.1 Model Derivation

The model presents two rotational inertia masses (J_m, J_l) connected by a torsional spring (k_ϑ) which include a friction term (b_ϑ). Both two masses are connected to the chassis by two virtual bearings modeled with two simple viscosity coefficients: b_m and b_l.

Mathematical model can be derived using Euler-Lagrange equations as follows

$$\frac{d}{dt}\left(\frac{\partial E_{kin}}{\partial \dot{q}_1}\right) - \frac{\partial E_{kin}}{\partial q_1} + \frac{\partial E_{pot}}{\partial q_1} = \underbrace{Q_{diss}^{(12)} + Q_1}_{\text{not conservative force}} \tag{1.1}$$

$$\frac{d}{dt}\left(\frac{\partial E_{kin}}{\partial \dot{q}_2}\right) - \frac{\partial E_{kin}}{\partial q_2} + \frac{\partial E_{pot}}{\partial q_2} = \underbrace{Q_{diss}^{(21)} + Q_2}_{\text{not conservative force}} \tag{1.2}$$

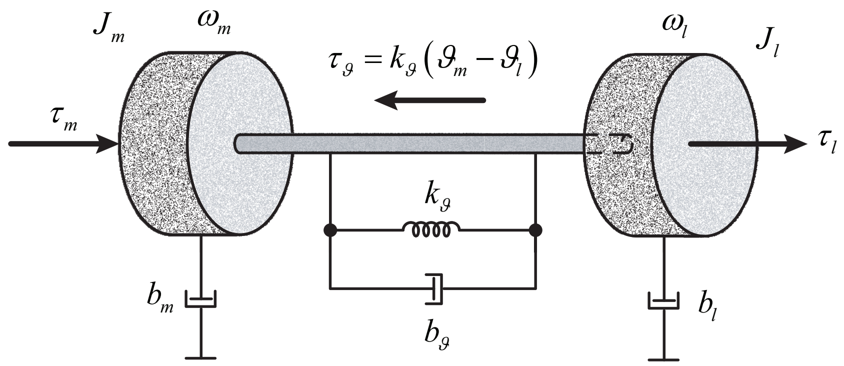

Figure 1: Generic flexible shaft model.

Where Q_1, Q_2, $Q_{diss}^{(12)}$ and $Q_{diss}^{(21)}$ are the generalized forces, like magnetic force, unknown load, disturbance and friction or in general not conservative force[1].

[1] A conservative force is a force which is derive by a potential and satisfies to the following conditions:

$$\vec{\nabla} \times \vec{f} = \vec{0} \tag{1.3}$$

$$\oint_{\mathscr{C}} \vec{f} \cdot d\vec{r} = 0 \tag{1.4}$$

$$\vec{f} = -\vec{\nabla}\phi \tag{1.5}$$

Applying Eqs. (1.1) and (1.2) for the case

$$\begin{aligned} q_1 &= \vartheta_m \\ q_2 &= \vartheta_l \end{aligned} \tag{1.6}$$

The kinematic and potential energy can be written as follows

$$E_{kin} = \frac{1}{2} J_m \dot{\vartheta}_m^2 + \frac{1}{2} J_l \dot{\vartheta}_l^2 \tag{1.7}$$

$$E_{pot} = \frac{1}{2} k_\vartheta \left(\vartheta_m - \vartheta_l\right)^2 \tag{1.8}$$

Applying the Lagrange equations we obtain the following motion equations

$$J_m \ddot{\vartheta}_m + k_\vartheta \left(\vartheta_m - \vartheta_l\right) = -b_m \dot{\vartheta}_m - b_\vartheta \left(\dot{\vartheta}_m - \dot{\vartheta}_l\right) + \tau_m \tag{1.9}$$

$$J_l \ddot{\vartheta}_l - k_\vartheta \left(\vartheta_m - \vartheta_l\right) = -b_l \dot{\vartheta}_l - b_\vartheta \left(\dot{\vartheta}_l - \dot{\vartheta}_m\right) + \tau_l \tag{1.10}$$

or

$$\begin{cases} \dot{\vartheta}_m = \omega_m \\ \dot{\omega}_m = -\dfrac{k_\vartheta}{J_m}\vartheta_m - \dfrac{b_m + b_\vartheta}{J_m}\omega_m + \dfrac{k_\vartheta}{J_m}\vartheta_l + \dfrac{b_\vartheta}{J_m}\omega_l + \dfrac{1}{J_m}\tau_m \\ \dot{\vartheta}_l = \omega_l \\ \dot{\omega}_l = \dfrac{k_\vartheta}{J_l}\vartheta_m + \dfrac{b_\vartheta}{J_l}\omega_m - \dfrac{k_\vartheta}{J_l}\vartheta_l - \dfrac{b_l + b_\vartheta}{J_l}\omega_l + \dfrac{1}{J_l}\tau_l \end{cases} \tag{1.11}$$

The state space representation become

$$\dot{\vec{z}} = \tilde{\mathbf{G}}\vec{z} + \tilde{\mathbf{H}}\ \tau_m + \tilde{\mathbf{M}}\ \tau_l \tag{1.12}$$

where $\vec{z} = \begin{bmatrix}\vartheta_m & \omega_m & \vartheta_l & \omega_l\end{bmatrix}^T$ and

$$\tilde{\mathbf{G}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_\vartheta}{J_m} & -\frac{b_m + b_\vartheta}{J_m} & \frac{k_\vartheta}{J_m} & \frac{b_\vartheta}{J_m} \\ 0 & 0 & 0 & 1 \\ \frac{k_\vartheta}{J_l} & \frac{b_\vartheta}{J_l} & -\frac{k_\vartheta}{J_l} & -\frac{b_l + b_\vartheta}{J_l} \end{bmatrix} \tag{1.13}$$

and $\tilde{\mathbf{H}} = \begin{bmatrix}0 & \frac{1}{J_m} & 0 & 0\end{bmatrix}^T$, $\tilde{\mathbf{M}} = \begin{bmatrix}0 & 0 & 0 & -\frac{1}{J_l}\end{bmatrix}^T$.

For our purpose, we want to change the state representation using a new state vector

$$\vec{x} = \mathbf{T}\vec{z} = \begin{bmatrix} \omega_m \\ \omega_l \\ \tau_\vartheta \end{bmatrix} \tag{1.14}$$

where[2] $\mathbf{T}$ is given by[3]

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ k_\vartheta & 0 & -k_\vartheta & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{T}^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{2k_\vartheta} \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2k_\vartheta} \\ 0 & 1 & 0 \end{bmatrix} \tag{1.17}$$

Basically, the linear transformation $\mathbf{T}$ consists in a model reduction. The minimum representation of a dynamical system can be reached applying some matrix analysis results, but also considering the total number of energy storage which are contained into the system. In the case of mass-spring-mass model, we can assume that three components can store energy: the two inertial masses J_m and J_l as kinetic energy, and the torsional spring as potential energy. That means the minimal representation of the mass-spring-mass model is made by three equations.

The new state space representation becomes

$$\dot{\vec{z}} = \tilde{\mathbf{G}}\vec{z} + \tilde{\mathbf{H}}\,\tau_1 + \tilde{\mathbf{M}}\,\tau_2 \tag{1.18}$$

$$\mathbf{T}^{-1}\dot{\vec{x}} = \tilde{\mathbf{G}}\mathbf{T}^{-1}\vec{x} + \tilde{\mathbf{H}}\,\tau_1 + \tilde{\mathbf{M}}\,\tau_2 \tag{1.19}$$

$$\dot{\vec{x}} = \mathbf{T}\tilde{\mathbf{G}}\mathbf{T}^{-1}\vec{x} + \mathbf{T}\tilde{\mathbf{H}}\,\tau_1 + \mathbf{T}\tilde{\mathbf{M}}\,\tau_2 \tag{1.20}$$

or

$$\dot{\vec{x}} = \tilde{\mathbf{A}}\vec{x} + \tilde{\mathbf{B}}\,\tau_1 + \tilde{\mathbf{E}}\,\tau_2 \tag{1.21}$$

or

$$\begin{bmatrix} \dot{\omega}_m \\ \dot{\omega}_l \\ \dot{\tau}_\vartheta \end{bmatrix} = \begin{bmatrix} -\frac{b_m+b_\vartheta}{J_m} & \frac{b_\vartheta}{J_m} & -\frac{1}{J_m} \\ \frac{b_\vartheta}{J_l} & -\frac{b_l+b_\vartheta}{J_l} & \frac{1}{J_l} \\ k_\vartheta & -k_\vartheta & 0 \end{bmatrix} \begin{bmatrix} \omega_m \\ \omega_l \\ \tau_\vartheta \end{bmatrix} + \begin{bmatrix} \frac{1}{J_m} \\ 0 \\ 0 \end{bmatrix} \tau_m + \begin{bmatrix} 0 \\ \frac{1}{J_l} \\ 0 \end{bmatrix} \tau_l \tag{1.22}$$

where

$$\vec{x} = \begin{bmatrix} \omega_m & \omega_l & \tau_\vartheta \end{bmatrix}^T$$

$$\tilde{\mathbf{A}} = \mathbf{T}\tilde{\mathbf{G}}\mathbf{T}^{-1} = \begin{bmatrix} -\frac{b_m+b_\vartheta}{J_m} & \frac{b_\vartheta}{J_m} & -\frac{1}{J_m} \\ \frac{b_\vartheta}{J_l} & -\frac{b_l+b_\vartheta}{J_l} & \frac{1}{J_l} \\ k_\vartheta & -k_\vartheta & 0 \end{bmatrix}$$

[2] where

$$\tau_\vartheta = k_\vartheta(\vartheta_m - \vartheta_l) \tag{1.15}$$

which can be written also as

$$\dot{\tau}_\vartheta = k_\vartheta(\omega_m - \omega_l) \tag{1.16}$$

which describes the dynamic of the shaft torsional torque.

[3] the matrix pseudo inverse has been used.

$$\tilde{\mathbf{B}} = \mathbf{T}\tilde{\mathbf{H}} = \begin{bmatrix} \frac{1}{J_m} \\ 0 \\ 0 \end{bmatrix} \quad \tilde{\mathbf{E}} = \mathbf{T}\tilde{\mathbf{M}} = \begin{bmatrix} 0 \\ \frac{1}{J_l} \\ 0 \end{bmatrix}$$

Consider the term $\tau_l(t)$ as a disturbance (or load) which perturbs the system.

For a more physical viewpoint the mass-spring-mass (or flexible shaft) dynamic equations can also be written as follows:

$$\begin{cases} \dot{\omega}_m = \dfrac{1}{J_m}\Big[-\big(b_m + b_\vartheta\big)\omega_m + b_\vartheta\omega_l - \tau_\vartheta + \tau_m\Big] \\ \dot{\omega}_l = \dfrac{1}{J_l}\Big[b_\vartheta\omega_m - \big(b_l + b_\vartheta\big)\omega_l + \tau_\vartheta + \tau_l\Big] \\ \dot{\tau}_\vartheta = k_\vartheta\big(\omega_m - \omega_l\big) \end{cases} \tag{1.23}$$

1.2 Natural oscillations

The frequency of the natural oscillation of the shaft can be evaluated imposing to zero the friction terms and calculating the eigenvalue of the transition matrix, as follows
setting $b_m = 0$, $b_l = 0$ and $b_\vartheta = 0$ we obtain the following characteristic polynomial

$$\Big|s\mathbf{I} - \tilde{\mathbf{A}}\Big| = \begin{vmatrix} s & 0 & 1/J_m \\ 0 & s & -1/J_l \\ -k_\vartheta & k_\vartheta & s \end{vmatrix} = s\Big[s^2 + k_\vartheta\frac{J_m + J_l}{J_m J_l}\Big] \tag{1.24}$$

setting $s = j\omega$ to the Eq. (1.24) and equating to zero

$$\omega^2 - k_\vartheta\frac{J_m + J_l}{J_m J_l} = 0 \tag{1.25}$$

we obtain

$$\omega_0 = \sqrt{k_\vartheta\frac{J_m + J_l}{J_m J_l}} \tag{1.26}$$

which represent the pulsation of the natural harmonic component of the system.

2 Extension to a generic gear model

The mathematical structure of the flexible shaft can be used also to develop a simplified gear.
Let

$$\omega_l = \omega_m/n_{tg} \tag{2.1}$$

be the relation between driver (m) and follower (l) speeds, where ω_l is the follower speed and ω_m is the driver speed. The term n_{tg} represents the ratio between the driver and the follower speeds of the gear.

The mathematical model of the simplified gear is here represented ($b_\vartheta = 0$ is assumed),

$$\begin{cases} \dot{\omega}_m = \dfrac{1}{J_m}\Big[\tau_m - \tau_\vartheta^m - b_m\omega_m\Big] \\ \dot{\omega}_l = \dfrac{1}{J_l}\Big[\tau_\vartheta^l + \tau_l - b_l\omega_l\Big] \\ \dot{\tau}_\vartheta^m = k_\vartheta\Big[\omega_m - \omega_l n_{tg}\Big] \\ n_{tg} = \dfrac{n_m}{n_l} \\ \tau_\vartheta^m = \dfrac{\tau_\vartheta^l}{n_{tg}} \end{cases} \tag{2.2}$$

where τ_ϑ^m is the torsional torque seen at the driver side as well as τ_ϑ^l is the torsional torque seen at the follower side, see also Figure 2.

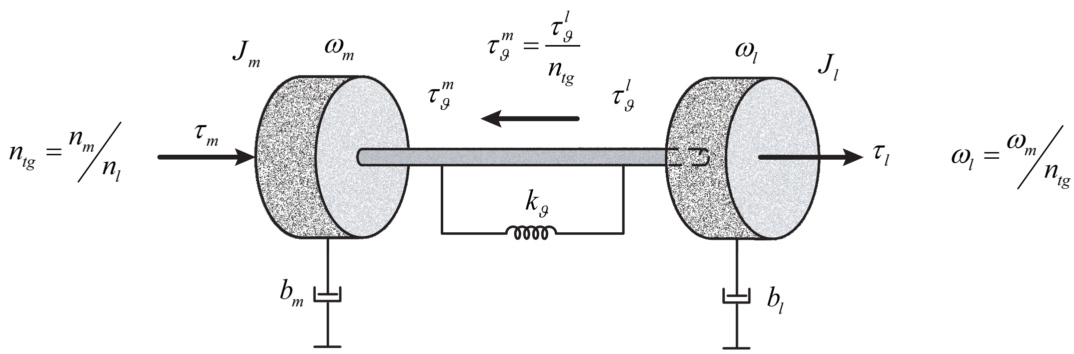

Figure 2: Generic flexible gear-shaft model.

3 Flexible Shaft - Model parameters settings

Parameters setting consist of

- J_m $\left[\mathrm{kg\,m^2}\right]$: Inertia of the m-side rotating mass.
- J_l $\left[\mathrm{kg\,m^2}\right]$: Inertia of the l-side rotating mass.
- b_m $\left[\mathrm{N\,s\,rad^{-1}}\right]$: Friction coefficient - m-side.
- b_l $\left[\mathrm{N\,s\,rad^{-1}}\right]$: Friction coefficient - l-side.
- k_ϑ $\left[\mathrm{N\,rad^{-1}}\right]$: Internal torsional spring.
- b_ϑ $\left[\mathrm{N\,s\,rad^{-1}}\right]$: Damping of the internal torsional spring.

4 Gear - Model parameters settings

Parameters setting consist of

- n_m $\left[\mathrm{ad}\right]$: with n_l forms the gear ratio $n_{tg} = \frac{n_m}{n_l}$.
- n_l $\left[\mathrm{ad}\right]$: with n_m forms the gear ratio $n_{tg} = \frac{n_m}{n_l}$.
- J_m $\left[\mathrm{kg\,m^2}\right]$: Inertia of the *driver*-side rotating mass.
- J_l $\left[\mathrm{kg\,m^2}\right]$: Inertia of the *follower*-side rotating mass.
- b_m $\left[\mathrm{N\,s\,rad^{-1}}\right]$: Friction coefficient - *driver*-side.
- b_l $\left[\mathrm{N\,s\,rad^{-1}}\right]$: Friction coefficient - *follower*-side.
- k_ϑ $\left[\mathrm{N\,rad^{-1}}\right]$: Internal torsional spring.