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18.01 Problem Set 2A

Part - I

1F 3. $y = x^{1/n}$

$y^n = x$

$\frac{d}{dx}(y^n) = 1$

$ny^{n-1} \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{n} \cdot \frac{1}{(x^{1/n})^{n-1}}$

$$\boxed{\frac{dy}{dx} = \frac{1}{n} x^{1/n - 1}}$$

5. $\sin x + \sin y = 1/2$

find dy/dx:

$\cos x + \cos y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-\cos x}{\cos y} = 0$

$x = \frac{\pi}{2} + 2n\pi, \frac{-\pi}{2} + 2n\pi$

I $x = -\pi/2 + 2n\pi$

$\sin x + \sin y = 1/2$

$-1 + \sin y = 1/2$

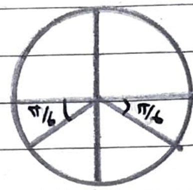
$\sin y = 3/2$

no possible y exists

II $x = \pi/2 + 2n\pi$

$\sin x + \sin y = 1/2$

$1 + \sin y = 1/2 \Rightarrow \sin y = -1/2$



$$y = \frac{-\pi}{6} + 2n\pi, \; \frac{7\pi}{6} + 2n\pi$$

The points are $\boxed{(\pi/2 + 2n\pi, -\pi/6 + 2n\pi), \; (\pi/2 + 2n\pi, 7\pi/6 + 2n\pi).}$

8.c) $c^2 = a^2 + b^2 - 2ab\cos\theta$, $\frac{da}{db} = ?$

$$\frac{d}{db}\left(c^2 = a^2 + b^2 - 2\cos\theta \, ab\right)$$

$$0 = 2a \cdot \frac{da}{db} + 2b - 2\cos\theta\left(\frac{da}{db} \cdot b + a \cdot 1\right)$$

$$2aa' + 2b - 2b\cos\theta \, a' - 2a\cos\theta = 0$$

$$(a - b\cos\theta)\, a' = a\cos\theta - b$$

$$\boxed{a' = \frac{a\cos\theta - b}{a - b\cos\theta}}$$

1A. 5.b) $y = x^2 + 2x \leftarrow f(x) \Rightarrow y = (x+1)^2 - 1$

exchange x and y

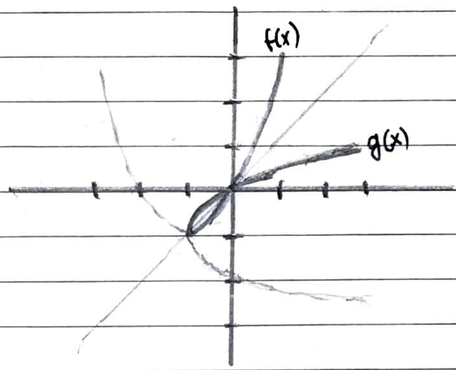
$x = y^2 + 2y \leftarrow g(x)$

$y^2 + 2y - x = 0$

$$y = \frac{-2 \pm \sqrt{4 + 4x}}{2} = \underline{\underline{-1 \pm \sqrt{1 + x}}}$$

Restricting domain of $f(x)$ to $[-1, \infty)$

$$y = -1 + \sqrt{1+x}$$



5A · 1. a) $\tan^{-1}\sqrt{3} = \theta$

$\sqrt{3} = \tan\theta$

$\theta = \underline{\underline{\pi/3}}$

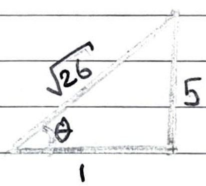
b) $\sin^{-1}(\sqrt{3}/2) = \theta$

$\sqrt{3}/2 = \sin\theta$

$\theta = \underline{\underline{\pi/3}}$

c) $\theta = \tan^{-1} 5$

$\tan\theta = 5$



$\sin\theta = 5/\sqrt{26}$

$\cos\theta = 1/\sqrt{26}$

$\sec\theta = \sqrt{26}$

3. f) $y = \sin^{-1}(9/x)$

using implicit diferentiation

$\sin y = 9/x$

$\frac{d}{dx}\left(\sin y = \frac{a}{x}\right)$

$(\cos y) y' = a \frac{d}{dx} x^{-1}$

$(\cos y) y' = -\frac{a}{x^2} \Rightarrow y' = \frac{-a}{x^2 \cos(\cancel{a/x}) \sin^{-1}(a/x))}$

$y' = \frac{-a}{x^2\sqrt{1-\sin^2(\sin^{-1}(a/x))}} = \frac{-a}{x^2\sqrt{1-\frac{a^2}{x^2}}} = \frac{-a}{x\sqrt{\cancel{a}x^2-a^2}}$

b) $y = \sin^{-1}\sqrt{1-x}$

Using ~~chaing~~ chain rule,

$y' = \frac{1}{\sqrt{1-(1-x)}} \cdot \frac{d}{dx}\sqrt{1-x} = \frac{1}{\sqrt{x}} \frac{d}{dx}(1-x)^{1/2}$

$= \frac{1}{\sqrt{x}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1-x}} \cdot -1 = \frac{-1}{2\sqrt{x(1-x)}}$

1H 1. a) $y = y_0 e^{kt}$

At $t = \lambda$, $e^{k\lambda} = 1/2$

$k\lambda = \ln(1/2) \Rightarrow \boxed{\lambda = \frac{-\ln 2}{k}}$

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b) $y_1 = y_0 e^{kt_1}$

$$y_2 = y_0 e^{k(t_1+\tau)}$$
$$= y_0 e^{kt_1} \cdot e^{k\tau}$$

$$= y_1 e^{-\ln 2} = \frac{y_1}{2}$$ //

2. $[H^+]_0$, $[H^+]_d$

$$[H^+]_d = \frac{[H^+]_0}{2}$$ (volume is doubled)

Take $-\log$ on both sides.

$$-\log[H^+]_d = -\log\left(\frac{[H^+]_0}{2}\right)$$

$$-\log[H^+]_d = -(\log[H^+]_0 - \log 2)$$

$$-\log[H^+]_d = -\log[H^+]_0 + \log 2$$

$$\Rightarrow \boxed{pH_{diluted} = pH_{original} + \log 2}$$

3. a) $\ln(y+1) + \ln(y-1) = 2x + \ln x$

$$\ln((y+1)(y-1)) = 2x + \ln x$$

$$(y+1)(y-1) = e^{2x} \cdot e^{\ln x}$$
$$= xe^{2x}$$

$$y^2 - 1 = xe^{2x}$$

$$y^2 = 1 + xe^{2x} \Rightarrow \boxed{y = \pm\sqrt{1+xe^{2x}}}$$

5. a) $\dfrac{e^x + e^{-x}}{e^x - e^{-x}} = y$

$u = e^x$

$$\frac{u + 1/u}{u - 1/u} = y$$

$$\frac{u^2+1}{u^2-1} = y \Rightarrow u^2+1 = yu^2 - y$$

$$y+1 = (y-1)u^2$$

$$u^2 = \frac{y+1}{y-1} \Rightarrow e^{2x} = \frac{y+1}{y-1}$$

Applying ln on both sides,

$$2x = \ln\left(\frac{y+1}{y-1}\right) \Rightarrow \boxed{x = \ln\sqrt{\frac{y+1}{y-1}}}$$

b) $y = e^x + e^{-x}$

$u = e^x$

$y = u + u^{-1}$

$$y = u + \frac{1}{u} \Rightarrow uy = u^2 + 1$$

$$u^2 - yu + 1 = 0$$

$$u = \frac{y \pm \sqrt{y^2 - 4}}{2}$$

$$e^x = \frac{y \pm \sqrt{y^2 - 4}}{2} \Rightarrow \boxed{x = \ln\left(\frac{y \pm \sqrt{y^2-4}}{2}\right)}$$

II 1.c) $\frac{d}{dx} e^{-x^2} = e^{-x^2}(-2x) = \underline{\underline{-2xe^{-x^2}}}$

d) $\frac{d}{dx}(\cancel{\ln x}\ x(\ln x - 1)) = (\ln x - 1) + 1 = \underline{\underline{\ln x}}$

e) $\frac{d}{dx} \ln(x^2) = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$ //

f) $\frac{d}{dx}(\ln x)^2 = 2\ln x \cdot \frac{1}{x} = \frac{2\ln x}{x}$ //

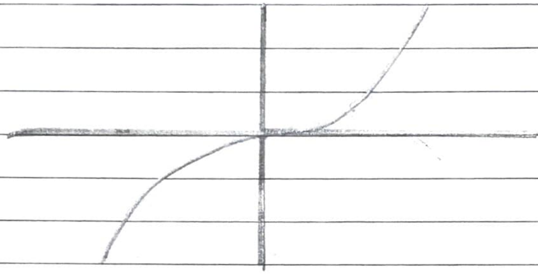
m) $\frac{d}{dx}\frac{(1-e^x)}{(1+e^x)} = \frac{(1+e^x)(-e^x) - (1-e^x)(e^x)}{(1+e^x)^2}$

$$= \frac{-e^x(1+e^x+1-e^x)}{(1+e^x)^2} = \underline{\underline{\frac{-2e^x}{(1+e^x)^2}}}$$

4a) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n} = \left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)^3 = e^3$ //

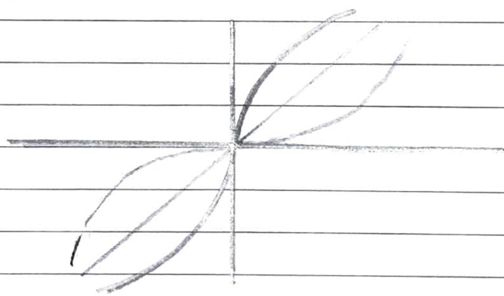
4.2(a) 5A 5.a) $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$, $y' = \dfrac{e^x + e^{-x}}{2} = \cosh x$

$\lim\limits_{x \to \infty} \dfrac{e^x - e^{-x}}{2} = \infty$, $\lim\limits_{x \to -\infty} \dfrac{e^x - e^{-x}}{2} = -\infty$



b) $y = \sinh x \Leftrightarrow x = \sinh^{-1} y$

graph of $y = \sinh^{-1} x$



defined for all r

c) $y = \sinh^{-1} x$, $x = \sinh y$,

implicit differentiation.

$\cosh^2 x - \sinh^2 x = 1$

$\cosh y \cdot y' = 1 \Rightarrow y' = \dfrac{1}{\cosh y} = \dfrac{1}{\sqrt{1 + \sinh^2 y}} = \dfrac{1}{\sqrt{1+x^2}}$ //

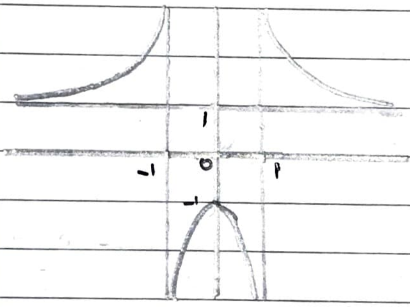
18.01 Problem set 2A

Part - II

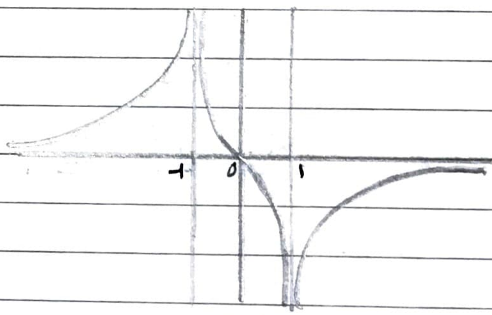
1. $f(x) = \dfrac{x^2+1}{x^2-1}$

$$f'(x) = \frac{(x^2-1)\cdot(2x) - (x^2+1)\cdot 2x}{(x^2-1)^2} = \frac{2x\,(\cancel{x^2}-1-\cancel{x^2}-1)}{(x^2-1)^2}$$

$$f'(x) = \frac{-4x}{(x^2-1)^2}$$



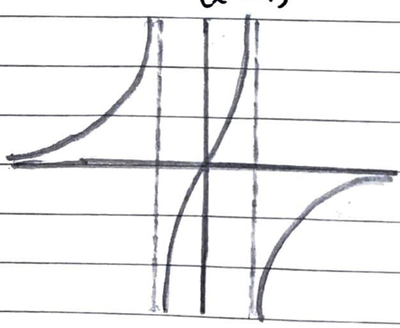
f(x)



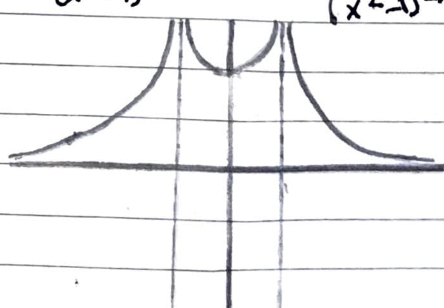
f'(x)

$$g(x) = \frac{-2x}{x^2-1}$$

$$g'(x) = \frac{(x^2-1)\cdot -2 - (-2x)\cdot 2x}{(x^2-1)^2} = \frac{(-2x^2+2+4x^2)}{(x^2-1)^2} = \frac{2(x^2+1)}{(x^2-1)^2}$$



g(x)



g'(x)

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2.a) $\frac{d}{dx}(\tan^3(x^4))$

$= 3(\tan^2(x^4)) \cdot \frac{d}{dx}(\tan(x^4))$

$= 3(\tan^2(x^4)) \sec^2(x^4) \cdot \frac{d}{dx} x^4$

$= 3(\tan^2(x^4)) \sec^2(x^4) \cdot 4x^3$

$= 12x^3(\tan^2(x^4)) \sec^2(x^4)$ //

b) I $\frac{d}{dy}(\sin^2 y \cdot \cos^2 y)$

$= 2\sin y \cdot \cos y \cdot \cos^2 y + 2\cos y(-\sin y)\sin^2 y$

$= 2\sin y\cos y(\cos^2 y - \sin^2 y)$

$= \underline{\underline{\sin(2y) \cdot \cos(2y)}}$

II $\frac{d}{dy}(\sin^2 y\cos^2 y) = \frac{d}{dy}((\sin y \cdot \cos y)^2) = \frac{d}{dy}\left(\left(\frac{2\sin y\cos y}{2}\right)^2\right)$

$= \frac{d}{dy}\left((\sin 2y)^2 \cdot \frac{1}{4}\right)$

$= \frac{1}{4} \cdot 2\sin(2y) \cdot \cos(2y) \cdot 2 = \underline{\underline{\sin(2y)\cos(2y)}}$

3. $y = uv$

$y' = u'v + uv'$ (product rule)

$y'' = (u'v + uv')'$

$$y'' = (u'v)' + (uv')'$$
$$= (u''v + u'v') + (u'v' + uv'')$$
$$= u''v + 2u'v' + uv''$$

b) $$y''' = (u''v + 2u'v' + uv'')'$$
$$= (u''v)' + 2(u'v')' + (uv'')'$$
$$= u'''v + u''v' + 2u''v' + 2u'v'' + u'v'' + uv'''$$
$$= u'''v + 3u''v' + 3u'v'' + uv'''$$

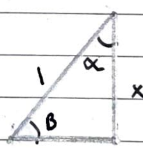
4. a) $$y = \cos^{-1}x$$

$$\cos y = x$$

$$\frac{d}{dx}(\cos y) = 1 \Rightarrow (-\sin y)\, y' = 1$$

$$y' = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-\cos^2(\cos^{-1}(x))}} = \underline{\underline{\frac{-1}{\sqrt{1-x^2}}}}$$

b) $$\frac{d}{dx}\underbrace{\cos^{-1}x}_{\alpha} + \frac{d}{dx}\underbrace{\sin^{-1}x}_{B} = 0$$



$\cos\alpha = x$, $\sin B = x$

As α and B are complementary angles.

$$\frac{d}{dx}(\alpha + B) = \frac{d}{dx}\left(\frac{\pi}{2}\right) = 0$$

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5. 8.28. a) $M = \frac{2}{3} \log_{10} \frac{E}{E_0}$

$M_1 = \frac{2}{3} \log_{10} \frac{E_1}{E_0}$, $M_2 = \frac{2}{3} \log \frac{E_2}{E_0}$, $M_1 = M_2 + 1$

$\frac{2}{3} \log_{10} \frac{E_1}{E_0} = \frac{2}{3} \log_{10} \frac{E_2}{E_0} + 1$

$\log \frac{E_1}{E_0} = \left(\log \frac{E_2}{E_0} + \frac{3}{2} \right)$

$10 \qquad 10$

$\frac{E_1}{\cancel{E_0}} = \frac{E_2}{\cancel{E_0}} \cdot 10^{3/2} \Rightarrow \boxed{\frac{E_1}{E_2} = 10^{3/2}}$

b) $6 = \frac{2}{3} \log_{10} \frac{E}{E_0}$

$9 = \log_{10} \frac{E}{E_0} \Rightarrow E = E_0 \cdot 10^9 = 7 \times 10^6$ kwh

c) no. of days $= \frac{7 \times 10^6}{3 \times 105} = \underline{\underline{23.33}}$ days

10. proof by contradiction

$\log_3 2 = p/q$

$3^{\log_3 2} = 3^{p/q}$

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$$2 = 3^{p/q}$$
$$2^{q} = 3^{p}$$

the above equation is not solvable.
So no p/q exists.

$\therefore \log_3 2$ is irrational

11. 1. $\log(1/2) < 2 \cdot \log(1/2)$ is the flaw

because $\log(1/2)$ is negative for all bases > 1.

8.4 18. $y = \sqrt[3]{(x+1)(x-2)(2x+7)}$

$$\ln y = \frac{1}{3}\left(\ln(x+1) + \ln(x-2) + \ln(2x+7)\right)$$

$$\frac{y'}{y} = \frac{1}{3}\left(\frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{2x+7}\right)$$

$$y' = \frac{1}{3}\left(\frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{2x+7}\right) \cdot \sqrt[3]{(x+1)(x-2)(2x+7)}$$

19(a) $\frac{d}{dx}\left(\frac{e^{x}(x^2-1)}{\sqrt{6x-2}}\right)$

$$\ln y = \ln e^{x} + \ln(x^2-1) - \frac{1}{2}\ln(6x-2)$$

$$\frac{y'}{y} = x + \ln(x^2-1) - \frac{1}{2}\ln(6x-2)$$

$$\frac{y'}{y} = 1 + \frac{2x}{x^2-1} - \frac{1}{2}\cdot\frac{1}{6x-2}\cdot 6$$

$$= 1 + \frac{2x}{x^2-1} - \frac{3}{6x-2}$$

$$y' = \left(1 + \frac{2x}{x^2-1} - \frac{3}{6x-2}\right)\frac{e^x(x^2-1)}{\sqrt{6x-2}}$$

$$= \frac{e^x(x^2-1)}{\sqrt{6x-2}} + \frac{2xe^x}{\sqrt{6x-2}} - \frac{3e^x(x^2-1)}{(6x-2)^{3/2}} \quad //$$

6. $y = u_1 u_2 \cdots u_n$

$$\ln y = \ln u_1 + \ln u_2 + \ln u_3 + \ldots + \ln u_n$$

$$\frac{d}{dx}(\ln y) = \frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \ldots + \frac{u_n'}{u_n}$$

$$\frac{y'}{y} = \frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \ldots + \frac{u_n'}{u_n}$$

$$\frac{y'}{y} = \frac{u_1'u_2\cdots u_n + u_1u_2'\cdots u_n + \ldots + u_1u_2\cdots u_n'}{u_1\cdots u_n}$$

$$y' = u_1'u_2\cdots u_n + u_1u_2'\cdots u_n + \ldots + u_1u_2\cdots u_n' \quad //$$