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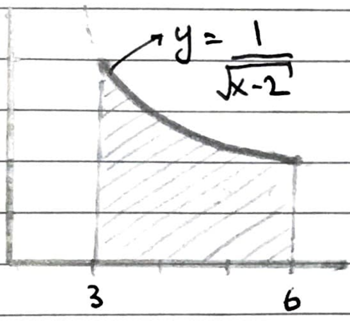
18.01 Problem Set - 5

Part - I

3C 1. Area $= \displaystyle\int_3^6 \frac{dx}{\sqrt{x-2}}$



guess: $\sqrt{x-2}$, $\frac{d}{dx}(x-2)^{1/2} = \frac{1}{2\sqrt{x-2}}$

$$\int_3^6 \frac{dx}{\sqrt{x-2}} = \left(2\sqrt{x-2}\right)\Big|_3^6$$

$$= 2(2) - 2(1) = 4 - 2 = \underline{\underline{2}}$$

2a) $\displaystyle\int_0^2 \sqrt{3x+5}\,dx = \int_0^2 (3x+5)^{1/2}\,dx$

~~guess~~ $= \dfrac{2(3x+5)^{3/2}}{3\times 3} = \dfrac{2}{9}(3x+5)^{3/2}\Big|_0^2$

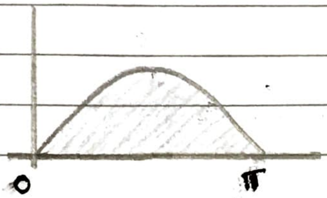
$$= \underline{\underline{\frac{2}{9}\left(11^{3/2} - 5^{3/2}\right)}}$$

3a) $\displaystyle\int_1^2 \frac{x\,dx}{x^2+1}$

$u = x^2+1$, $du = 2x\,dx \Rightarrow x\,dx = \dfrac{du}{2}$

$u_1 = 1^2+1 = 2$
$u_2 = 2^2+1 = 5$

$$\frac{1}{2}\int_{2}^{5} \frac{du}{u} = \frac{1}{2}\ln u \Big|_{2}^{5} = \frac{1}{2}(\ln 5 - \ln 2)$$

5a) 

$$\int_{0}^{\pi} \sin x\,dx = (-\cos x)\Big|_{0}^{\pi}$$

$$= (-\cos\pi) - (-\cos 0) = (1) - (-1) = 2$$

3E 6. b) $\sin^2 x < \sin x$ for $0 < x < \pi$

$$\int_{0}^{\pi} \sin^2 x\,dx < \int_{0}^{\pi} \sin x\,dx = 2$$

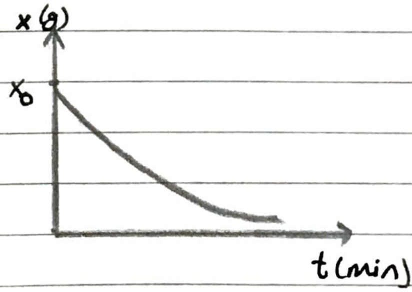
$$\boxed{\int_{0}^{\pi} \sin^2 x\,dx < 2}$$

c) $\sqrt{x^2+1} > x$, $10 < x < 20$

$$\int_{10}^{20} \sqrt{x^2+1}\,dx > \int_{10}^{20} x\,dx = \frac{x^2}{2}\Big|_{10}^{20} = 200 - 50 = 150$$

$$\boxed{\int_{10}^{20} \sqrt{x^2+1}\,dx > 150}$$

4T 2 $x = x_0 e^{-kt}$



Amount of Radiation ΔR produced in time Δt -

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$$\Delta R = xr\Delta t = rx_0 e^{-kt}\Delta t$$

The total radiation produced is the sum of all the intervals.

$$R = \sum_{i=0}^{60} rx_0 e^{-kt_i}\Delta t \longrightarrow \int_0^{60} rx_0 e^{-kt} dt$$

$$R = rx_0 \int_0^{60} e^{-kt} dt = -\frac{rx_0}{k} (e^{-kt}) \Big|_0^{60}$$

$$\boxed{R = \frac{rx_0}{k}(1 - e^{-60k})}$$

Ɛ1. $$L(x) = \int_1^x \frac{dt}{t}$$

$$L(1/a) = \int_1^{1/a} \frac{dt}{t}$$

$\boxed{u = \frac{1}{t}}$, $du = -\frac{1}{t^2} dt \Rightarrow \frac{dt}{t} = -t\,du = -\frac{du}{u}$

$u_1 = 1$
$u_2 = 1/1/a = a$

$$L(1/a) = -\int_1^a \frac{du}{u} = \underline{\underline{-L(a)}}$$

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3a) $\displaystyle\int_1^e \frac{\sqrt{\ln x}}{x}\,dx$

$u = \ln x$, $du = \frac{dx}{x}$

$u_1 = 0$

$u_2 = \ln e = 1$

$\displaystyle\int_0^1 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_0^1 = \underline{\underline{\frac{2}{3}}}$

3D1. a) To prove : $\displaystyle\int_0^x \frac{dt}{\sqrt{t^2+a^2}} = \ln(x+\sqrt{x^2+a^2}) - \ln a$

Let $\displaystyle G(x) = \int_0^x \frac{dt}{(t^2+a^2)^{1/2}}$

By FTC 2, $G'(x) = \dfrac{1}{\sqrt{x^2+a^2}}$

Let $H(x) = \ln(x+\sqrt{x^2+a^2}) - \ln a$

$H'(x) = \dfrac{1}{x+\sqrt{x^2+a^2}}\left(1 + \dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+a^2}}\right) = \left(\dfrac{\cancel{x+\sqrt{x^2+a^2}}}{\cancel{x+\sqrt{x^2+a^2}}}\right)\dfrac{1}{\sqrt{x^2+a^2}}$

$H'(x) = \dfrac{1}{\sqrt{x^2+a^2}}$

$G'(x) = H'(x) \Rightarrow G(x) = H(x) + C$

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Plugging $x=a$,

$\cancel{G(a)}^{0} = \cancel{H(a)}^{0} + c \Rightarrow c=0$

Thus $G(x) = H(x)$. Hence Proved

b) $\displaystyle\int_c^x \frac{dt}{\sqrt{t^2+a^2}} = \ln(x+\sqrt{x^2+a^2})$

$G(c) = \displaystyle\int_c^c \frac{dt}{\sqrt{t^2+a^2}} = 0 = \ln(c+\sqrt{c^2+a^2})$

$c+\sqrt{c^2+a^2} = 1 \Rightarrow \sqrt{c^2+a^2} = 1-c$

$c^2+a^2 = (1-c)^2 \Rightarrow \cancel{c^2}+a^2 = \cancel{c^2}-2c+1$

$$\boxed{c = \frac{1-a2}{2}}$$

4\. b) Let the function be $F(x)$,

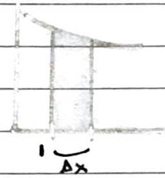
$F'(x) = \sin(x^3) = f(x)$, $F(0)=2$

$F(x) = \displaystyle\int_0^x \sin x^3 dx + 2$

c) $F(1) = -1$

$F(x) = \displaystyle\int_1^x \sin x^3 dx - 1$

5\. a) $$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_1^{1+\Delta x} \frac{t}{\sqrt{1+t^4}}\, dt$$



$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \cdot (\text{Area}) = \frac{1}{\cancel{\Delta x}} \cdot \underbrace{\cancel{\Delta x}}_{\text{base}} \cdot f(1)$$

$$\lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x}} \cdot \frac{1}{\sqrt{1+1^4}} = \frac{1}{\sqrt{2}} //$$

b) $$F(x) = \int_0^x \frac{t}{\sqrt{1+t^4}}\, dt$$

$$\frac{\Delta F}{\Delta x} = F'(x), \quad \Delta x \to 0$$

At $x = 1$,

$$\frac{\Delta F}{\Delta x} = F(1+\Delta x) - F(1) = \frac{1}{\Delta x}\left(\int_0^{1+\Delta x} f(t)\, dt - \int_0^{1} f(t)\, dt \right)$$

$$\frac{\Delta F}{\Delta x} = \frac{1}{\Delta x} \int_1^{1+\Delta x} \frac{t}{\sqrt{1+t^4}}\, dt$$

$$F'(x) = \frac{x}{\sqrt{1+x^4}} \quad (\text{by FTC2})$$

$$F'(1) = \frac{1}{\sqrt{2}}$$

$$\therefore \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_1^{1+\Delta x} \frac{t\, dt}{\sqrt{1+t^4}} = F'(1) = \frac{1}{\sqrt{2}} //$$

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8a) $F(x) = \int_0^x f(t)\,dt = 2x(\sin x + 1)$

by FTC 2, $F'(x) = f(x)$

$F'(x) = \frac{d}{dx}(2x(\sin x + 1))$

$= 2(\sin x + 1) + 2x(\cos x) = 2(x\cos x + \sin x + 1)$

$F'(\pi/2) = f(\pi/2) = 2(\frac{\pi}{2}\cancel{\cos \pi/2}^{0} + \cancel{\sin \pi/2}^{1} + 1) = \underline{\underline{4}}$

3E 2.a) $E(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-u^2/2}\,du$

$t = u/\sqrt{2}$, $dt = \frac{du}{\sqrt{2}}$

$t_1 = 0$

$t_2 = x/\sqrt{2}$

$E(x) = \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}\sqrt{\pi}} \int_0^{x/\sqrt{2}} e^{-t^2}\,dt = \underline{\underline{\frac{F(x/\sqrt{2})}{\sqrt{\pi}}}}$

$\lim_{x\to\infty} E(x) = \frac{1}{\sqrt{\pi}} \lim_{x\to\infty} F(x/\sqrt{2}) = \frac{1}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = \underline{\underline{\frac{1}{2}}}$

c) $\frac{1}{\sqrt{2\pi}} \int_a^b e^{-u^2/2}\,du = \frac{1}{\sqrt{2\pi}} \int_0^b e^{-u^2/2}\,du - \frac{1}{\sqrt{2\pi}} \int_0^a e^{-u^2/2}\,du$

$= \underline{\underline{E(b) - E(a)}}$

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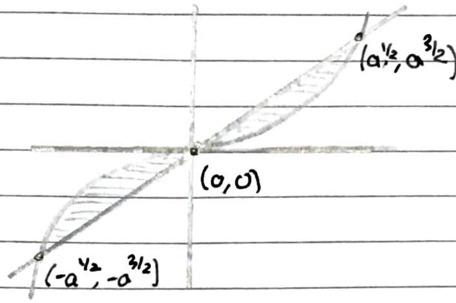
4A-1 b) $y = x^3, \ y = ax$

$x^3 = ax \Rightarrow x = 0, \pm\sqrt{a}$

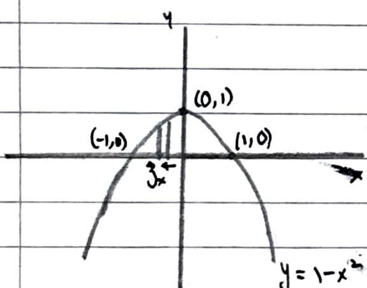


$$\text{Area} = 2\int_0^{\sqrt{a}} (ax - x^3)\,dx$$

$$= 2\left(\frac{ax^2}{2} - \frac{x^4}{4}\right)\Big|_0^{\sqrt{a}}$$

$$= 2\left(\frac{a^2}{2} - \frac{a^2}{4}\right) = \frac{2a^2}{4} = \frac{a^2}{2}$$

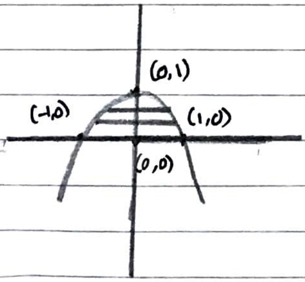
4A 2. Method I - vertical slices



$$\text{Area} = 2\int_0^1 (\overset{ht}{1-x^2})\,\overset{base}{dx} \quad \text{(symmetry)}$$

$$= 2\left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{4}{3}$$

Method II - horizontal slices



$y = 1-x^2 \Rightarrow x^2 = 1-y$

$x = \pm\sqrt{1-y}$

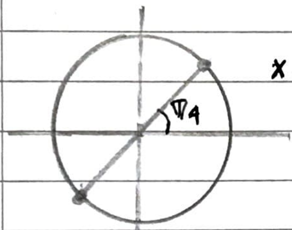
$$\text{Area} = \int_0^1 \left(\sqrt{1-y} - (-\sqrt{1-y})\right) dy$$

$$\text{Area} = 2\int_0^1 \sqrt{1-y}\,dy = 2\left(-\frac{2}{3}(1-y)^{3/2}\right)\Bigg|_0^1$$

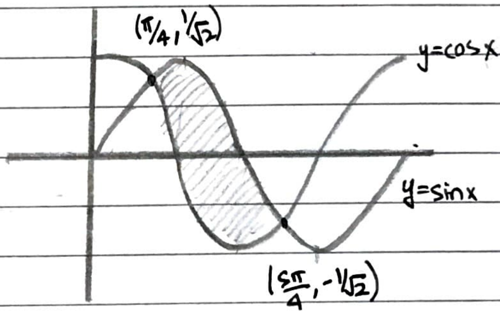
$$= 2\left(0-\left(-\frac{2}{3}\right)\right) = \frac{4}{3}$$

4A 4. $y = \cos x$, $y = \sin x$

$\cos x = \sin x$



$x = \frac{\pi}{4} + n\pi$

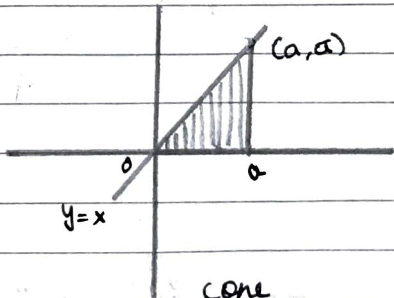


$$\text{Area} = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx = \left(-\cos x - \sin x\right)\Bigg|_{\pi/4}^{5\pi/4}$$

$$= \left(\left(+\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right) = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

4B 1. d) $dV = \cancel{2}\pi y^2 dx = \cancel{2}\pi x^2 dx$

$$\int dV = V = \int_0^a \cancel{2}\pi x^2 dx$$

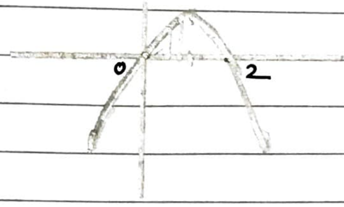


$$V = \cancel{2}\pi\left(\frac{x^3}{3}\right)\Bigg|_0^a = \frac{1}{3}\pi a^2$$

e) $y = 2x - x^2 \; , \; y = 0$

$= -(x^2 - 2x) + 1 - 1$

$= -(x^2 - 2x + 1) + 1 = -(x-1)^2 + 1$



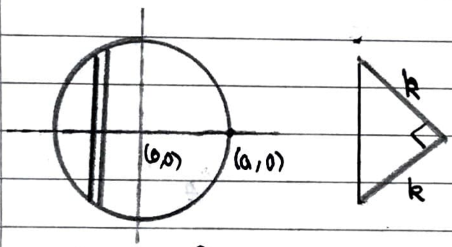
$2x - x^2 = 0 , \; x(2-x) = 0$

$$\text{volume} = \int_0^2 \pi (2x - x^2)^2 dx = \pi \int_0^2 (4x^2 + x^4 - 4x^3)\, dx$$

$$= \pi \left(\frac{4x^3}{3} + \frac{x^5}{5} - x^4 \right)_0^2 = \pi \left(\frac{2^5}{3} + \frac{2^5}{5} - 2^4 \right)$$

$$= 2^4 \pi \left(\frac{2}{3} + \frac{2}{5} - 1 \right) = \underline{\underline{\frac{16\pi}{15}}}$$

~~48~~ 6.



$x^2 + y^2 = a^2$

$y = \pm \sqrt{a^2 - x^2}$

hypotenuse length = segment length = $2\sqrt{a^2 - x^2}$

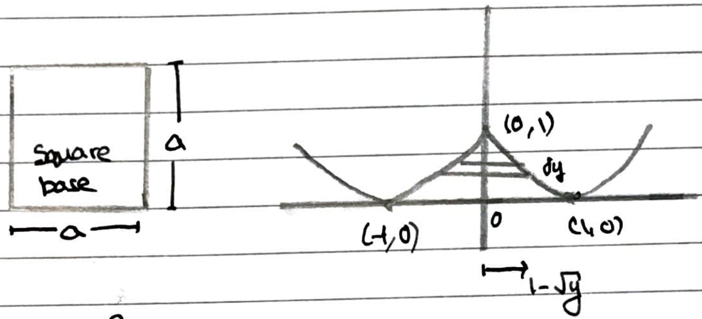
$2k^2 = 4a^2 - 4x^2 \Rightarrow k^2 = 2(a^2 - x^2)$

Area of $\Delta = \frac{1}{2} k^2 = (a^2 - x^2)$

$$\text{Volume} = 2\int_0^a (\text{Ar}(\Delta))\,dx = 2\int_0^a \frac{2(a^2-x^2)}{2}\,dx$$

$$= 2\left(a^2x - \frac{x^3}{3}\right)\Big|_0^a = 2\left(a^3 - \frac{a^3}{3}\right) = \frac{4a^3}{3}$$

4B 7.



$y = (x-1)^2, \; y = (x+1)^2, \; y = 0$

$\pm\sqrt{y} = x - 1 \Rightarrow x = 1 - \sqrt{y}$ (consider left part)

by symmetry,

$a = 2(1-\sqrt{y})$

$$\text{Volume} = \int \text{Area(square)}\cdot dy = \int_0^1 (2(1-\sqrt{y}))^2\,dy$$

$$= 4\int_0^1 (1 + y - 2\sqrt{y})\,dy = 4\left(y + \frac{y^2}{2} - \frac{4}{3}y^{3/2}\right)\Big|_0^1$$

$$= 4\left(1 + \frac{1}{2} - \frac{4}{3}\right) = \underline{\underline{\frac{2}{3}}}$$

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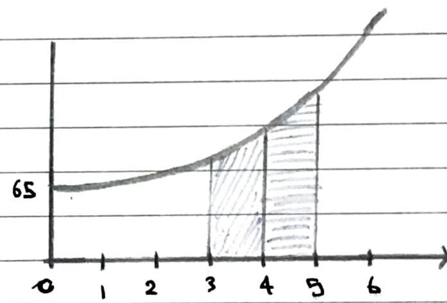
18.01 Problem Set - 5

Part - II

1-a) $y' = \frac{y}{100}$, $y(0) = 65°$

$\frac{dy}{dt} = \frac{y}{100} \Rightarrow \frac{dy}{y} = \frac{dt}{100} \Rightarrow \int \frac{dy}{y} = \int \frac{dt}{100}$

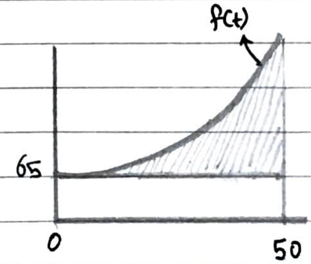
$\ln y = \frac{t}{100} + C \Rightarrow y = Ae^{t/100}$ $(A = e^C)$



when $t = 0$, $y = 65$

$65 = Ae^0 \Rightarrow A = 65°$

$\boxed{y = 65e^{t/100}}$



b) Cooling Degree Day

$$\sum_{i=0}^{49} \left(\int_i^{i+1} f(t)\,dt - 65 \right)$$

relating this to area

$$\int_0^{50} f(t)\,dt - 65 \cdot 50 = \int_0^{50} 65e^{t/100}\,dt - 65 \cdot 50$$

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$$\int_0^{50} 65e^{t/100}\,dt$$

$u = t/100$, $du = \dfrac{dt}{100}$

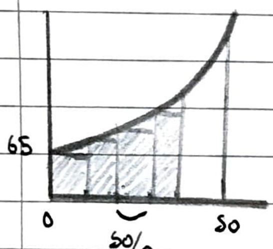
$u_1 = 0$
$u_2 = 1/2$

$$65\int_0^{1/2} \times 100\, e^u\,du = 6500\,(e^u)\Big|_0^{1/2} = \boxed{6500\left[e^{1/2}-1\right]}$$

The number of cooling degree days is $6500[e^{1/2}-1] - 65.50$

$$= 65 \times 50 \times [e^{1/2}-1] = \underline{\underline{3250[e^{1/2}-1]}}$$

c) $\int_0^{50} 65e^{t/100}\,dt$



$$\text{Area} = \frac{50}{n}\left\{65\left(e^{0/100} + e^{\frac{1\times 50}{100n}} + e^{\frac{2\times50}{100n}} + \dots + e^{\frac{(n-1)50}{100n}}\right)\right.$$

$$= \frac{50.65}{n}\left(e^0 + e^{1/2n} + e^{2/2n} + \dots + e^{n-1/2n}\right)$$

the above is a geometric series with
$a = 1$, $r = e^{1/2n}$, $n' = n-1$

$$S = a\left(\frac{r^{n'+1}-1}{r-1}\right)$$ (for geometric series

$$\text{Area} = \frac{50.65}{n}\left(\frac{(e^{1/2n})^n - 1}{e^{1/2n}-1}\right) = \frac{3250}{n}\left(\frac{e^{1/2}-1}{e^{1/2n}-1}\right)$$

$(e^x \approx 1+x,\ x \approx 0)$

$$\lim_{n\to\infty} n\left(e^{1/2n}-1\right) \approx \cancel{n}\left(\cancel{1}+\frac{1}{2\cancel{n}}\cancel{-1}\right) \approx \underline{\underline{\frac{1}{2}}}$$

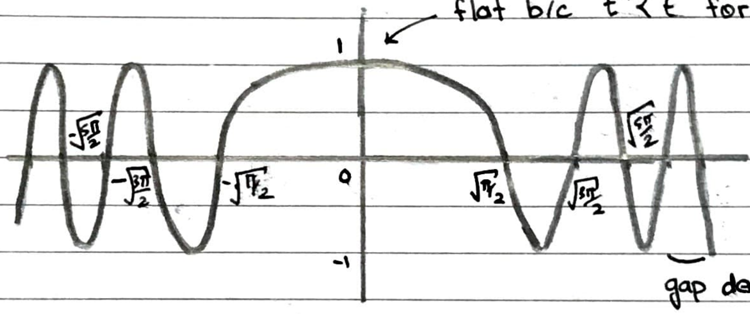
$$\underset{\text{(exact)}}{\text{Area}} = \lim_{n\to\infty} \frac{3250}{n}\,\frac{(e^{1/2}-1)}{(e^{1/2n}-1)} = \underline{\underline{6500\left(e^{1/2}-1\right)}}$$

2. a) $\cos(t^2)=0$

$t^2 = \frac{\pi}{2}+n\pi \Rightarrow t = \pm\sqrt{\pi/2+n\pi}$ $(n=0,1,\dots)$

$\cos((-t)^2) = \cos(t^2)$ [even f^n]

flat b/c $t^2 < t$ for $t<1$



gap decreases as n increase ($\sqrt{x}$)

b) $f(x) = \int_0^x \cos(t^2)\,dt$

$f'(x) = \cos(x^2)$ (from FTC 2)

$\cos(x^2)=0$ when $x = \pm\sqrt{\pi/2+n\pi}$ $(n=0,1,2,\dots)$

Critical Points $\rightarrow x = \pm\sqrt{\pi/2+n\pi}$ $(n=0,1,2,\dots)$

$f''(x) = -\sin(x^2)\cdot 2x = -2x\sin(x^2)$

I $f''(x) = 0$: inflection points

$-2x\sin(x^2) = 0 \Rightarrow x = \pm\sqrt{n\pi}$ $(n = 0, 1, 2, \ldots)$

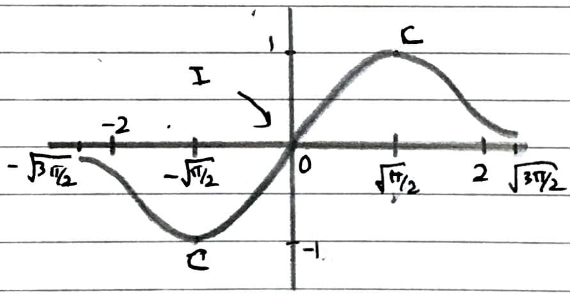
II $f''(x) > 0$ (for critical points): [Minimum] $\cup$

$x = -\sqrt{\pi/2 + n_e\pi}$ $(n_e = 0, 2, 4 \ldots)$
and
$x = \sqrt{\pi/2 + n_o\pi}$ $(n_o = 1, 3, 5 \ldots)$

III $f''(x) < 0$ (for critical points): [Maximum] $\cap$

$x = \sqrt{\pi/2 + n_o\pi}$ $(n_o = 1, 3, 5, \ldots)$
and
$x = -\sqrt{\pi/2 + n_e\pi}$ $(n_e = 0, 2, 4, \ldots)$

c) $\because f'(x) = \cos(x^2)$ is an even function, $f(x)$ is odd.



d) Let $\Delta f = f(0.1) - f(0)$ and $\Delta x = 0.1$

$$\Delta f = \int_0^{0.1} \cos(x^2)\,dx \Rightarrow \frac{\Delta f}{\Delta x} = \underbrace{\frac{1}{0.1}\int_0^{0.1} \cos(x^2)\,dx}_{}$$

Avg. value of $\cos(x^2)$ over $0 \leq x \leq 0.1$

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$$\frac{\Delta f}{\Delta x} = Avg(f') = 0.999975$$

$$Avg(f') = \frac{\cos((0.1)^2) + \cos(0^2)}{2} = 0.999975$$

$$\therefore \quad \Delta f = f(0.1) - \cancel{f(0)}^{0} = \Delta x(0.999975)$$

$$\boxed{f(0.1) = 0.099997}$$

e) i) $f(x) = \int_0^x \cos(t^2)\,dt$

$$t = \sqrt{\frac{\pi}{2}}\,u \Rightarrow dt = \sqrt{\pi/2}\,du$$

$u = \sqrt{2/\pi}\,t$, $\quad u_1 = \sqrt{2/\pi}\,x$

$u_2 = 0$

$$= \int_0^{\sqrt{2/\pi}\,x} \cos\left(\frac{\pi}{2}u^2\right) \cdot \sqrt{\frac{\pi}{2}}\,du = \underline{\underline{\sqrt{\frac{\pi}{2}}\,g\left(\sqrt{\frac{2}{\pi}}\,x\right)}}$$

- The factor '$\pi/2$' is chosen because all maximums, minimums and zeroes of the cosine function are multiples of $\pi/2$.

(ii) $f(x) = \int_0^x \cos(t^2)\,dt$

$$v = t^2 \Rightarrow dv = 2t\,dt \Rightarrow dt = \frac{dv}{2\sqrt{v}}$$

$v_1 = x^2$,

$v_2 = 0$

$$= \int_0^{x^2} \frac{\cos(v)}{2\sqrt{v}}\, dv = \frac{1}{2} h(x^2)$$

(iii) $k(x) = \sqrt{x} \int_0^1 \cos(xt^2)\, dt \ , \quad x > 0$

$z = xt^2, \quad dz = 2xt\, dt \Rightarrow dt = \frac{dz}{2xt} = \frac{dz\sqrt{x}}{2x\sqrt{z}}$

$\Downarrow$

$t = \sqrt{z/x} \ , \quad z_1 = x, \quad z_2 = 0$

$$= \sqrt{x} \int_0^x \cos(z) . \frac{dz}{2\sqrt{x}\sqrt{z}} = \frac{1}{2} \int_0^x \frac{\cos(z)}{\sqrt{z}}\, dz = \frac{1}{2} h(x)$$

$k(x) = \frac{1}{2} h(x) \Rightarrow k(x^2) = \frac{1}{2} h(x^2) = f(x)$

$$\boxed{k(x^2) = f(x)}$$

3. a) (i) $\frac{dV}{dt} = -cA(h)$

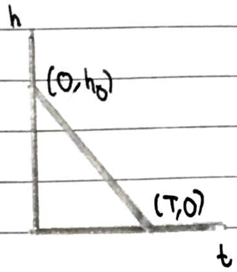
$V = \int_0^h A(x)dx$

$\frac{dV}{dx} = \frac{d}{dx} \int_0^h A(x)dx = A(h)$ (by FTC2)

by chain rule,

$$\cancel{\frac{dV}{dx}} = \cancel{dx} \quad \frac{dx}{dt} = \frac{dx}{dV} . \frac{dV}{dt} = \frac{1}{A(h)} (-c\, A(h)) = -c$$

(ii)

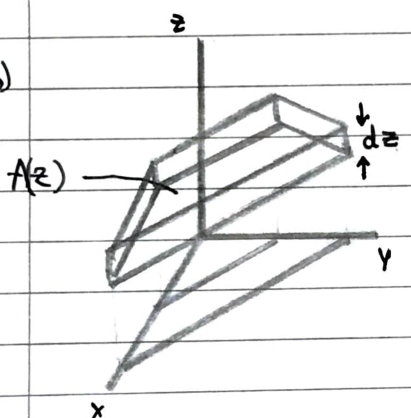


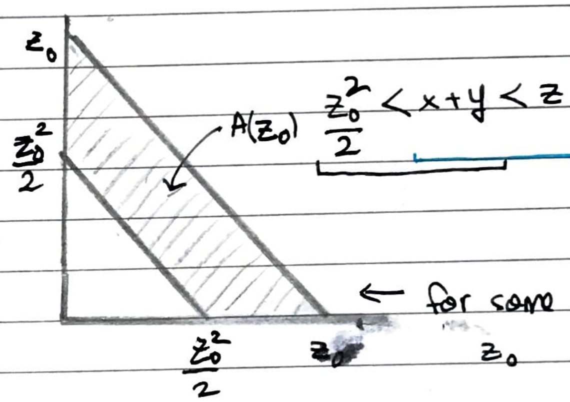
$$h = mt + b$$

$$\boxed{h = -ct + h_0}$$

$$h = 0, \quad 0 = -cT + h_0 \Rightarrow \boxed{T = \frac{h_0}{c}}$$

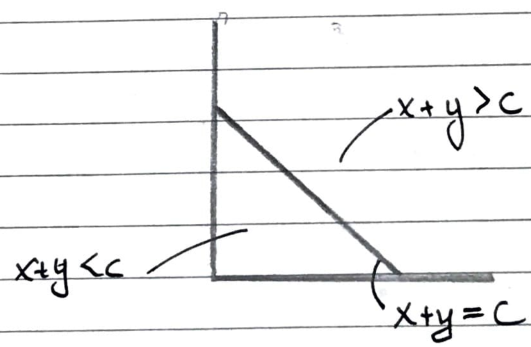
(b)





$$\frac{z^2}{2} < x+y < z$$

$$\frac{z^2}{2} < z \Rightarrow \boxed{z < 2}$$



$$\text{Volume} = \int_0^{2} A(z)\,dz$$

$$A(z) = \frac{1}{2}z^2 - \frac{1}{2}\left(\frac{z^2}{2}\right)^2 = \frac{z^2}{2} - \frac{z^4}{8}$$

$$V = \int_0^2 \frac{z^2}{2} - \frac{z^4}{8}\,dz = \left(\frac{z^3}{6} - \frac{z^5}{40}\right)\Big|_0^2 = \frac{8}{15}$$