18.01 Practice Exam - I

1. a) $\dfrac{d}{dx} \dfrac{\sqrt{x}}{1+2x} \Big|_{x=1}$

$$\frac{d}{dx}\frac{\sqrt{x}}{1+2x} = \frac{(1+2x)(\sqrt{x})' - \sqrt{x}(1+2x)'}{(1+2x)^2}$$

$$= \left((1+2x)\cdot\frac{1}{2\sqrt{x}} - 2\sqrt{x} \right) \cdot \frac{1}{(2x+1)^2}$$

$$= \left(\frac{1+2x-4x}{2\sqrt{x}} \right) \cdot \left(\frac{1}{2x+1} \right)^2$$

$$= \left(\frac{1-2x}{2\sqrt{x}} \right) \cdot \frac{1}{(2x+1)^2} \Big|_{x=1} = \left(\frac{-1}{2} \cdot \frac{1}{9} \right) = \frac{-1}{18} //$$

b) $\dfrac{d}{du}(u \ln 2u)$

$$\frac{d}{dx}(uv)' = u'v + uv'$$

$$\frac{d}{du}(u \ln 2u) = 1 \cdot \ln 2u + u \cdot \frac{1}{2u} \cdot 2 = \underline{\underline{1 + \ln 2u}}$$

2. $\dfrac{d}{dt}\sqrt{1-k\cos^2 t} = \dfrac{d}{dt}\left(1-k\cos^2 t\right)^{1/2}$

chain rule

$$= \frac{1}{2}(1-k\cos^2 t)^{-1/2} \cdot (-k \cdot 2\cos t \cdot (-\sin t))$$

$$= \boxed{\frac{k\cos t \sin t}{\sqrt{1-k\cos^2 t}} = \frac{d}{dt}\sqrt{1-k\cos^2 t}}$$

b) $k=1$

$$f(t) = \sqrt{1-\cos^2 t} = \sin t$$

$$f'(t) = \underline{\underline{\cos(t)}} \quad . \quad ((\sin t)' = \cos t)$$

$$\left.\frac{k\cos t \sin t}{\sqrt{1-k\cos^2 t}}\right|_{k=1} = \frac{\cos t \cdot \sin t}{\sqrt{1-\cos^2 t}} = \frac{\cos t \cdot \cancel{\sin t}}{\cancel{\sin t}} = \underline{\underline{\cos t}}$$

3. $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \lim_{\Delta x \to 0}\left(\frac{1}{(x+\Delta x)^2} - \frac{1}{x^2}\right) \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} - \cancel{x^2} - \Delta x^2 - 2\Delta x(x)}{(x+\Delta x)^2 (x)^2} \cdot \frac{1}{\Delta x} = \lim_{\Delta x \to 0} \frac{-\Delta x - 2x}{((x+\Delta x)(x))^2}$$

$$\frac{d}{dx} x^{-2} = \frac{-2x}{x^4} = \frac{-2}{x^3} \,//$$

$$\boxed{\frac{d}{dx} x^{-2} = \frac{-2}{x^3}}$$

4. $y = \sin^{-1}x \Leftrightarrow \sin y = x$

implicit differentiation

$$(\cos y)\, y' = 1 \Rightarrow y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\boxed{y' = \frac{1}{\sqrt{1-x^2}}} \qquad (\because \sin y = x)$$

5. $f(x) = \begin{cases} ax+b &, x>1 \\ x^2-3x+2 &, x \le 1 \end{cases}$

(i) slope of tangents from boths must be equal.

$$\frac{d}{dx}(ax+b)\Big|_{x=1} = \frac{d}{dx}(x^2-3x+2)\Big|_{x=1}$$

$$a = 2x-3\Big|_{x=1} \Rightarrow \boxed{a=-1}$$

(ii) diff $\Rightarrow$ conti.

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} ax+b = b-1$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} (x^2-3x+2) = 0$$

$$b-1=0 \Rightarrow \boxed{b=1}$$

6. a) $$\lim_{u\to 0} \frac{\tan 2u}{u} = \frac{\sin 2u}{u} \cdot \frac{1}{\cos 2u} = \frac{\sin 2u}{2u} \cdot \frac{2}{\cos 2u}$$

$$\xrightarrow{u\to 0} (1) \cdot \frac{2}{1} = \underline{\underline{2}}$$

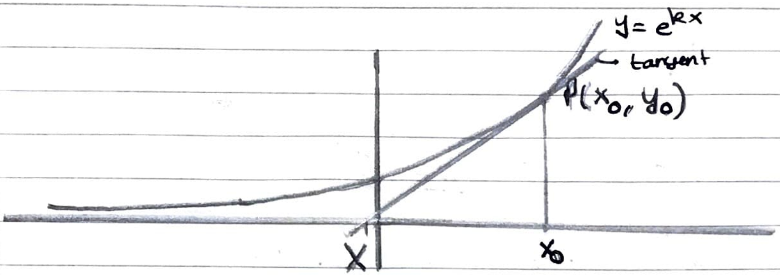
$$\boxed{\lim_{u\to 0} \frac{\tan 2u}{u} = 2}$$

b) $$\lim_{h\to 0} \frac{e^h-1}{h} = \lim_{h\to 0} \frac{e^{0+h}-e^0}{h} = \frac{d}{dx}e^x \Big|_{x=0}.$$

$$\frac{de^x}{dx} = e^x \Big|_{x=0} = e^0 = 1$$

$$\boxed{\lim_{h\to 0} \frac{e^h-1}{h} = 1}$$

7. $y = e^{kx}$, $k > 0$



eqn of tangent line

$$(y - y_0) = m(x - x_0)$$

$$m = \frac{d}{dx} e^{kx} \Big|_{x = x_0} = k e^{kx} \Big|_{x = x_0} = k e^{kx_0}$$

$$\Rightarrow (y - e^{kx_0}) = k e^{kx_0} (x - x_0)$$

to find x-intercept. $(y = 0)$

$$-\cancel{e^{kx_0}} = k \cancel{e^{kx_0}} (x - x_0)$$

$$\boxed{X = x_0 - \frac{1}{k}}$$