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# 18.01 Problem Set - 4

## Part - I

26 1. b) $f(x) = \ln x$ , $[1, 2]$

$$f'(c) = \frac{f(b) - f(a)}{b-a} = \frac{\ln 2 - \ln 1}{2-1} = \underline{\underline{\ln 2}}$$

$$f'(x) = \frac{1}{x}$$

$$\frac{1}{c} = \ln 2 \Rightarrow \boxed{c = \frac{1}{\ln 2}} = 1.44$$

26 2. b) $\sqrt{1+x} < 1 + x/2$ , $x > 0$

$$f(x) = \sqrt{1+x} - (1 + x/2) \quad , \quad f(0) = 0$$

$$f'(x) = \frac{1}{2\sqrt{1+x}} - \frac{1}{2} = \frac{1}{2}\left(\frac{1}{\sqrt{1+x}} - 1\right) < 0 \; , \; x > 0$$

$$f(x) = f(0) + f'(c)(x)$$

$$f(x) < f(0) \Rightarrow \sqrt{1+x} - (1 + x/2) < 0$$

$$\therefore \underline{\underline{\sqrt{1+x} < 1 + x/2}}$$

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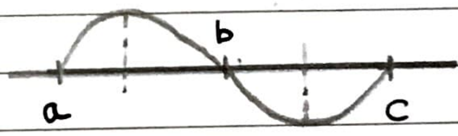
26 5. From MVT,

a)

$$f'(d) = \frac{f(b) - f(a)}{b - a}$$




$f'(d) = 0$ , $a < d < b$

$$f'(e) = \frac{f(c) - f(b)}{c - b} = 0 \quad , b < e < c$$

So $f'(x) = 0$ at $d, e$.

Now let the function be $f'(x)$.
Applying MVT across $(d, e)$

$$(f'(p))' = \frac{f'(d) - f'(e)}{d - e} = 0$$

Hence there is a point $p$ on $[a, c]$ where $f''(p) = 0$.

b) $f(x) = (x-a)(x-b)(x-c)$

$(uvw)' = u'vw + uv'w + uvw'$

$f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$

$f''(x) = (x-b) + (x-c) + (x-a) + (x-c) + (x-a) + (x-b)$

$= 6x - 2(a+b+c)$

$$f''(p) = 0$$

$$6p - 2(a+b+c) = 0 \Rightarrow \boxed{p = \frac{a+b+c}{3}}$$

266. $f(b) = f(a) + f'(x)(b-a) \quad , \quad a<b$

a) $f'(x) > 0.$

$\therefore f(b) = f(a) + (\text{'+ve'})$

$\Leftrightarrow \underline{\underline{a<b \Rightarrow f(a) < f(b)}}$ f increasing

b) $f'(x) = 0$

$\underline{\underline{f(b) = f(a)}}$ f constant.

3A 1. d) $d(e^{3x} \sin x) = (e^{3x} \cos x + 3e^{3x} \sin x)\, dx$

e) $\sqrt{x} + \sqrt{y} = 1.$

$$x^{1/2} + y^{1/2} = 1$$

$$d(x^{1/2} + y^{1/2}) = d(1)$$

$$\frac{1}{2} x^{-1/2} dx + \frac{1}{2} y^{-1/2} dy = 0$$

$$dy = -\frac{x^{-1/2} dx}{y^{-1/2}} = -dx\left(\frac{\sqrt{y}}{\sqrt{x}}\right) = \underline{\underline{-\left(\frac{1-\sqrt{x}}{\sqrt{x}}\right) dx}}$$

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3A 2.a) $\int (2x^4 + 3x^2 + x + 8)\,dx = \frac{2}{5}x^5 + x^3 + \frac{x^2}{2} + 8x + C$ /

c) $\int \sqrt{8+9x}\,dx \quad = \frac{1}{9}\int \sqrt{u}\,du$

$u = 8 + 9x$

$du = 9dx$

$= \frac{2}{3} \times \frac{1}{9} u^{3/2} + C = \frac{2}{27}(8+9x)^{3/2} + C$ //

e) $\int \frac{x}{\sqrt{8-2x^2}}\,dx$

guess : $\sqrt{8-2x^2}$

$\frac{d}{dx}\sqrt{8-2x^2} = \frac{1}{2} \times \frac{1}{\sqrt{8-2x^2}} \times -4x = -2\frac{x}{\sqrt{8-2x^2}}$

$\therefore \int \frac{x}{\sqrt{8-2x^2}} = -\frac{1}{2}\sqrt{8-2x^2} \cancel{\#} + c$ //

g) $\int 7x^4 e^{x^5}\,dx$

guess : $e^{x^5}$, $\frac{d}{dx} e^{x^5} = e^{x^5} \cdot 5x^4 = 5x^4 e^{x^5}$

$\therefore \int 7x^4 e^{x^5} = \frac{7}{5} e^{x^5} + C$ //

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(i) $\int \frac{dx}{3x+2}$

guess: $\ln(3x+2)$, $\frac{d}{dx}(\ln(3x+2)) = \frac{1}{3x+2} \times 3$

$\therefore \int \frac{dx}{3x+2} = \frac{\ln(3x+2)}{3} + c$ //

b) $\int \frac{x}{x+5} dx = \int \frac{x+5-5}{x+5} dx = \int 1 - \frac{5}{x+5} dx$

$\Rightarrow \int dx - 5\int \frac{dx}{x+5}$

$\therefore \int \frac{x\,dx}{x+5} = x - 5\ln|x+5| + c$ //

3A 3.a) $\int \sin(5x)\,dx = \frac{-\cos(5x)}{5} + c$ //

c) $\int \cos^2 x \sin x\,dx$

$u = \cos x$, $du = -\sin x\,dx$

$\int -u^2 du = -\frac{u^3}{3} + c = \frac{-\cos^3 x}{3} + c$ //

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e) $\int \sec^2(x/5)\, dx$

guess: $\tan(x/5)$ , $\frac{d}{dx}\tan(x/5) = \sec^2(x/5) \times \frac{1}{5}$

$\int \sec^2(x/5)\, dx = 5\tan x + c$ //

g) $\int \sec^9 x \tan x\, dx = \int \frac{1}{\cos^9 x} \cdot \frac{\sin x}{\cos x} dx$

$= \int \frac{\sin x\, dx}{\cos^{10} x} = \int \cos^{-10} x \sin x\, dx$

$u = \cos x$ , $du = -\sin x\, dx$

$\int \cos^{-10} x \sin x\, dx = -\int u^{-10} du = \frac{u^{-9}}{9} + c$

$= \int \sec^9 x \tan x\, dx = \frac{(\cos x)^{-9}}{9} + c = \frac{(\sec x)^9}{9} + c$ //

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BF 1. c) $\dfrac{dy}{dx} = \dfrac{3}{\sqrt{y}}$

$\sqrt{y}\,dy = 3\,dx \Rightarrow \int y^{1/2}dy = 3\int dx$

$\frac{2}{3}y^{3/2} = 3x + C$

$y^{3/2} = \frac{9x}{2} + C$ //

d) $\dfrac{dy}{dx} = xy^2$

$\dfrac{dy}{y^2} = x\,dx \Rightarrow \int y^{-2}dy = \int x\,dx$

$-y^{-1} = \frac{x^2}{2} + C \Rightarrow \frac{-1}{y} = \frac{x^2}{2} + C \Rightarrow y = \frac{-2}{x^2+C}$ //

2. a) $\dfrac{dy}{dx} = 4xy$ , $y(1) = 3$ . Find $y(3)$

$\dfrac{dy}{y} = 4\,x\,dx \Rightarrow \int \dfrac{dy}{y} = 4\int x\,dx$

$\ln y = \frac{4x^2}{2} + C \Rightarrow \ln y = 2x^2 + C$

At $x=1$, $y=3$, $\ln 3 = 2 + C \Rightarrow C = \ln(3) - 2$

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$\ln y = 2x^2 + \ln(3) - 2$

$\ln y/3 = 2(x^2-1) \Rightarrow |y| = 3e^{2(x^2-1)}$

$y = 3e^{2(9-1)} = \underline{\underline{3e^{16}}}$

e) $\frac{dy}{dx} = e^y$ , $y(3) = 0$ , Find $y(0)$

$e^{-y}dy = dx \Rightarrow \int e^{-y}dy = \int dx$

$-e^{-y} = x + c$

$e^{-y} = -x + c$

At $x = 3$, $y = 0$

$1 = -3 + c \Rightarrow c = 4$

$e^{-y} = -x + 4$ //

$e^{-y} = 4 \Rightarrow +y = \underline{\underline{- \ln 4}}$

$-x + 4 = e^{-y} > 0 \Rightarrow \boxed{x > -4}$

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3F 4 b) $\frac{dT}{dt} = k(T_e - T)$

$\frac{dT}{T_e - T} = k\,dt \Rightarrow \int \frac{dT}{T_e - T} = k\int dt$

$-\ln(T_e - T) = kt + C$

$\ln(T_e - T) = -kt + C$

$T_e - T = Ae^{-kt} \Rightarrow T = T_e - Ae^{-kt}$

At $t = 0$, $T = T_0$

$T_0 = T_e - A \Rightarrow A = T_e - T_0$

$T = T_e - (T_e - T_0)e^{-kt} = T_e(1 - e^{-kt}) + T_0 e^{-kt}$ //

c) As $t \to \infty$, $e^{-kt} \to 0$

$T = T_e(1 - 0) + T_0(0) = T_e$ //

d) $T_0 = 680°C$

$T_e = 40°C$

$T - T_e = Ae^{-kt} = (T_e - T_0)e^{-kt} = 640\,e^{-kt}$

$T(8) = 200$

$200 - 40 = 640\,e^{-k(8)}$

$$e^{-8k} = \frac{160}{640} = \frac{1}{4}$$

$$-8k = \ln(1/4)$$

$$\overset{4}{\cancel{-8}}k = -\cancel{2}\ln 2 \Rightarrow \boxed{k = \frac{\ln 2}{4}}$$

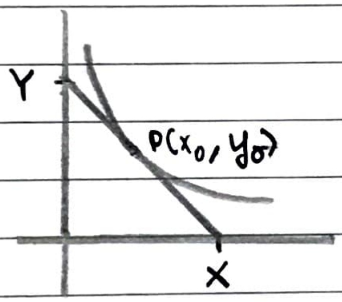
$$T = 40 + 640\, e^{-\frac{\ln 2}{4} t}$$

Find $t$ when $T = 50$

$$10 = 640 (e^{\ln 2})^{-t/4}$$

$$2^{-6} = 2^{-t/4} \Rightarrow \frac{t}{4} = 6 \Rightarrow \underline{\underline{t = 24 \text{ hrs}}} \ / 1 \text{day}$$

3F 8 b)




tangent line eqn : $(y - y_0) = f'(x)(x - x_0)$

Let X and Y be the intercepts

$$f'(x) = \frac{0 - Y}{X - 0} = \frac{-Y}{X} \longrightarrow (1)$$

Since P bisects the line ;

$$x_0 = \frac{X}{2}, \quad y_0 = \frac{Y}{2} \Rightarrow X = 2x_0, \ Y = 2y_0$$

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Substituting in ①

$$f'(x) = \frac{-\cancel{2}y_0}{2x_0}$$

$$\frac{dy}{dx} = \frac{-2y}{2x}$$ (general point) $x_0 \to x$, $y_0 \to y$

$$\frac{dy}{y} = -\frac{dx}{x} \Rightarrow \int\frac{dy}{y} = -\int\frac{dx}{x}$$

$$\ln y = -\ln x + C$$

$$y = e^{-\ln x + C} \Rightarrow y = \frac{A}{e^{\ln x}}$$ $(x > 0, y > 0)$

$$\boxed{y = \frac{A}{x}}$$

3B 2.a) $3-5+7-9+11-13$

$$a_i = (-1)^{i+1}(2i+1)$$

~~So~~ Summation: $\sum_{i=1}^{6} (-1)^{i+1}(2i+1)$

b) $\frac{1}{1} + \frac{1}{4} + \dots + \frac{1}{n^2} \Rightarrow \sum_{i=1}^{n} \frac{1}{i^2}$

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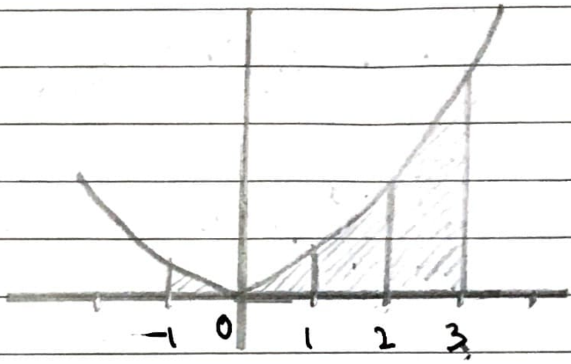
3B 3. b) $\int_{-1}^{3} x^2\,dx$




| $f(x)$ | 1 | 0 | 1 | 4 | 9 |
|---|---|---|---|---|---|
| $x$ | -1 | 0 | 1 | 2 | 3 |

$\Delta x = 1$

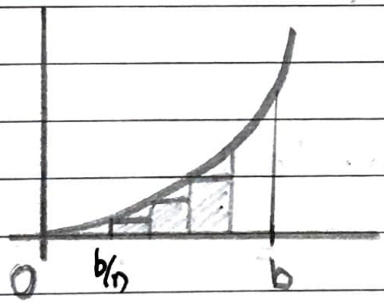
Upper: $1+1+4+9 = \underline{\underline{15}}$

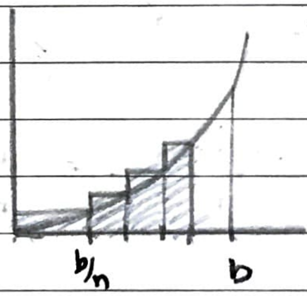
Lower: $0+0+1+4 = \underline{\underline{5}}$

Left: $1+0+1+4 = \underline{\underline{6}}$

Right: $0+1+4+9 = \underline{\underline{14}}$

4a)




$\Delta x = b/n$

$\int_{0}^{b} x^2\,dx$

Upper: $\left(\frac{b}{n}\right)\left(\frac{b}{n}\right)^2 + \left(\frac{b}{n}\right)\left(\frac{2b}{n}\right)^2 + \dots + \left(\frac{b}{n}\right)\left(\frac{nb}{n}\right)^2$

$= \left(\frac{b}{n}\right)^3\left(1^2 + 2^2 + \dots + n^2\right) = \left(\frac{b}{n}\right)^3 + 2^2\left(\frac{b}{n}\right)^3 + \dots + n^2\left(\frac{b}{n}\right)^3$

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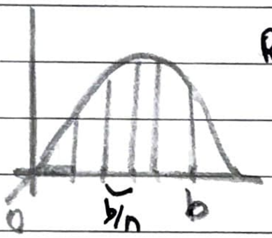
Lower: $\left(\frac{b}{n}\right)(0) + \left(\frac{b}{n}\right)\left(\frac{b}{n}\right)^2 + \dots + \left(\frac{b}{n}\right)\left(\frac{(n-1)b}{n}\right)^2$

$= \left(\frac{b}{n}\right)^3 \left(1^2 + \dots + (n-1)^2\right) = \left(\frac{b}{n}\right)^3 + 2^2\left(\frac{b}{n}\right)^3 + \dots + (n-1)^2\left(\frac{b}{n}\right)^3$

Upper − Lower $= n^2\left(\frac{b}{n}\right)^3 = \underline{\underline{\frac{b^3}{n}}} \xrightarrow[n\to\infty]{} 0$ ✓

3B 5. $f(x) = \sin x$ , $\Delta x = b/n$



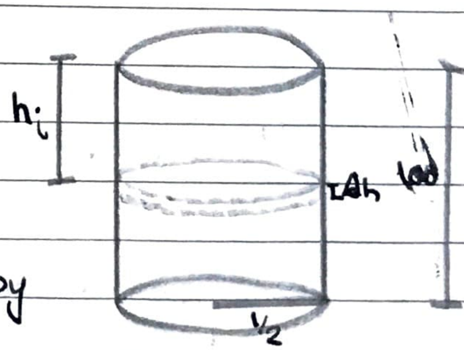

Right Sum: $\left(\frac{b}{n}\right)\sin(b/n) + \dots + \left(\frac{b}{n}\right)\sin\left(\frac{nb}{n}\right)$

$= \sum_{i=1}^{n} \left(\frac{b}{n}\right)\sin\left(\frac{ib}{n}\right)$

$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{b}{n}\right)\sin\left(\frac{ib}{n}\right) = \int_0^b \sin x\,dx$

$\therefore \quad \lim_{n\to 0} \sum_{i=1}^{n} \frac{\sin(ib/n)}{n} = \underline{\underline{\frac{1}{b}\int_0^b \sin x\,dx}}$

4J 1. Consider a thin disc of height $\Delta h$.




$\Delta V = \pi r^2 \Delta h$

Energy required to raise this volume by $h$:

$$E = k\Delta V \cdot h_i = k\pi r^2 h_i \Delta h$$

$$\text{Total Energy required} = \sum_{i=1}^{n} k\pi r^2 h_i \Delta h$$

$$\lim_{\Delta h \to 0} \sum_{i=1}^{n} k\pi r^2 h_i \Delta h = \frac{k\pi}{4} \int_0^{100} h\,dh = \frac{k\pi}{4} \frac{(100)^2}{2}$$

$$\boxed{E = \frac{k\pi (10)^4}{8}}$$

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18.01 Problem Set - 4

Part - II

1.a) $f(x) = f(a) + f'(c)(x-a)$

$f(x) = f(0) + f'(c)x \Rightarrow f(x) = f'(c)x \geq 0$

$f(x) \geq 0 \quad [f'(c) \geq 0,\ x \geq 0]$

b) $f(x) = x - \ln(1+x)$, $f(0) = 0 - \ln 1 = 0$

$f'(x) = 1 - \frac{1}{1+x} \geq 0 \quad [\because 1 > 1/_{1+x}]$

$f(0) = 0$

from (a), $f(x) \geq 0$

$x - \ln(1+x) \geq 0 \Rightarrow x \geq \ln(1+x)$

c) ~~f(x)=~~ $\ln(1+x) \geq x - \frac{x^2}{2}$

$f(x) = \ln(1+x) - x + \frac{x^2}{2}$, $f(0) = 0$

$f'(x) = \frac{1}{1+x} - 1 + x = \frac{1-1-x+x+x^2}{1+x} = \frac{x^2}{1+x} \geq 0$

So $f(x) \geq 0 \Rightarrow \ln(1+x) - x + \frac{x^2}{2} \geq 0 \Rightarrow \ln(1+x) \geq x - \frac{x^2}{2}$

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$$\ln(1+x) \leqslant x - \frac{x^2}{2} + \frac{x^3}{3} \quad , \quad x \geqslant 0$$

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ln(1+x)$$

$$f(0) = 0 - \ln 1 = 0$$

$$f'(x) = 1 - x + x^2 - \frac{1}{1+x} = \frac{1+x \cancel{-x} - \cancel{x^2} + \cancel{x^2} + x^3 - \cancel{1}}{1+x}$$

$$f'(x) = \frac{x^3}{1+x} \geqslant 0 \quad , \quad x \geqslant 0$$

from (a), $f(x) \geqslant 0$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \ln(1+x) \geqslant 0 \Rightarrow \underline{\underline{x - \frac{x^2}{2} + \frac{x^3}{3} \geqslant \ln(1+x)}}$$

d) $\ln(1+x) \leqslant \sum_{i=1}^{n} (-1)^{i+1} \frac{x^i}{i}$ if $n$ is odd

$\ln(1+x) \geqslant \sum_{i=1}^{n} (-1)^{i+1} \frac{x^i}{i}$ if $n$ is even

e) $\ln(1+x) \leqslant x \quad , \quad -1 < x \leqslant 0$

Let $u = -x$ ,

T.P $\ln(1-u) \leqslant -u \quad , \quad 0 \leqslant u < 1$

$f(x) = -u - \ln(1-u)$

$f(0) = 0 - \ln(1) = 0$

$f'(x) = -1 - \frac{1}{1-u}(-1) = -1 + \frac{1}{1-u}$

$f'(x) = -1 + \frac{1}{1-u} \geq 0$ for $0 \leq u < 1$

from (a)

$f(x) \geq 0 \Rightarrow -u - \ln(1-u) \geq 0$

$\ln(1-u) \leq -u \quad , \quad 0 \leq u < 1$

or.

$\underline{\underline{\ln(1+x) \leq x}} \quad , \quad -1 < x \leq 0$

2, a) $\displaystyle\int \frac{dx}{(1-x)^2}$

$u = 1-x \quad , \quad du = -dx$

$\displaystyle -\int \frac{du}{u^2} = -\int u^{-2}\,du = \frac{-1}{-1}u^{-1} = \frac{1}{u} + C = \frac{1}{1-x} + C$ //

$\displaystyle \frac{1}{1-x} - \underbrace{1}_{\text{constant}} = \frac{1-1+x}{1-x} = \underline{\underline{\frac{x}{1+x}}}$ , differ by constant.

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b) $\int \tan x \sec^2 x \, dx$

$u = \tan x$ , $du = \sec^2 x \, dx$

$\int u \, du = \frac{u^2}{2} + C = \frac{(\tan x)^2}{2} + C$ //

$\frac{\tan^2 x}{2} + \frac{1}{2} = \frac{\sec^2 x}{2}$ (differ by constant)

$\int \tan x \sec^2 x \, dx = \frac{\tan^2 x}{2} + C_1$

and

$\int \tan x \sec^2 x \, dx = \frac{\sec^2 x}{2} + C_2$

$\boxed{C_1 - C_2 = \frac{1}{2}}$

3 a) $m \frac{dv}{dt} = -C_1 v$

$\frac{dv}{v} = -\frac{C_1}{m} dt \Rightarrow \int \frac{dv}{v} = -\frac{C_1}{m} \int dt$

$\ln v = -\frac{C_1}{m} t + C_2 \Rightarrow v = e^{-\frac{C_1}{m} t} \cdot e^{C_2}$

$v = A e^{-ct}$ (where $A = e^{C_2}$, $c = C_1/m$)

At $t=0$, $v = v_0$

$v_0 = A \cdot 1 \Rightarrow A = v_0$

$$\boxed{v = v_0 e^{-ct}}$$

$$\frac{ds}{dt} = v_0 e^{-ct}$$

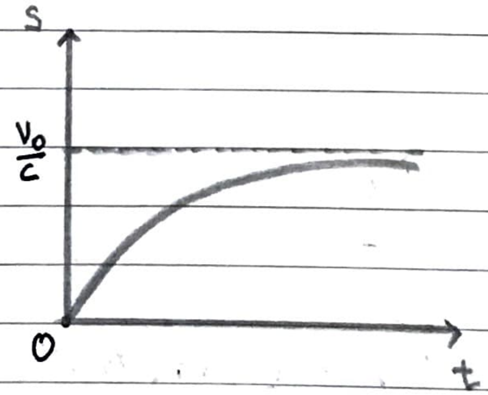
$$ds = v_0 e^{-ct} dt \Rightarrow \int ds = v_0 \int e^{-ct} dt$$

$$s = -\frac{v_0}{c} e^{-ct} + k$$

Assume at $t=0$, $s=0$.

$$0 = -\frac{v_0}{c} \times 1 + k \Rightarrow k = \frac{v_0}{c}.$$

$$s = \frac{v_0}{c}\left(1 - e^{-ct}\right)$$



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b) $m\frac{dv}{dt} = -c_1 v^2$

$\frac{dv}{v^2} = \frac{-c_1}{m} dt \Rightarrow \int \frac{dv}{v^2} = \frac{-c_1}{m} \int dt$

$-\frac{1}{v} = -\frac{c_1}{m} t + c_2$

At $t=0$, $v=v_0$

$-\frac{1}{v_0} = c_2$

$\frac{1}{v} - \frac{1}{v_0} = \frac{c_1 t}{m}$

$\frac{dt}{ds} = ct + \frac{1}{v_0} \Rightarrow ds = \frac{dt}{ct + 1/v_0} \Rightarrow \int ds = \int \frac{dt}{ct + 1/v_0}$

$s = \frac{\ln(ct + 1/v_0)}{c} + k$

At $t = 0$, $s = 0$ (assume)

$0 = \frac{-\ln v_0}{c} + k \Rightarrow k = \frac{\ln v_0}{c}$

$s = \frac{1}{c}\left( \ln(ct + 1/v_0) + \ln v_0 \right) = \frac{1}{c}\left( \ln(v_0 c t + 1) \right)$

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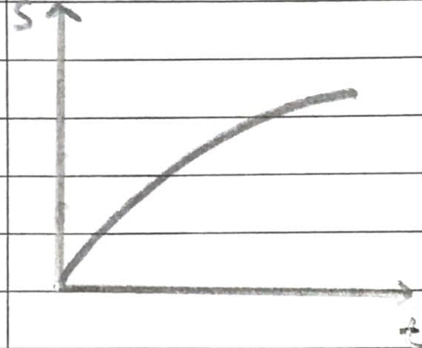


4. 3F 5. a) $\frac{dp}{dh} = -(0.13)p$

$\frac{dp}{p} = -0.13\,dh \Rightarrow \int \frac{dp}{p} = -0.13 \int dh$

$\ln p = -0.13h + C$

$p = Ae^{-0.13h} \quad (A = e^{C})$

When $h = 0$, $p = 1$, $1 = Ae^{0} \Rightarrow A = 1$

$\boxed{p = e^{-0.13h}}$

$p(10) = e^{-1.3} = \underline{\underline{0.273}}$ kg/cm$^2$

b) $\Delta p = p(0.1) - p(0) = e^{-0.013} - e^{0} = 0.987 - 1$

$\Delta p \approx -0.013 = -0.0129$ //

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using linear approx.

$e^x = 1 + x \quad (x \approx 0)$

$e^{-0.013} = 1 + (-0.013)$

$\Delta p = \cancel{1} - 0.013 - \cancel{1} = -0.013$ //

Percentage drop $= \dfrac{\Delta p}{p(0)} \times 100 = \underline{\underline{-1.3\ \%}}$

c) Green Building:

$\Delta p \approx p'(0)\,\Delta h$

$\dfrac{dp}{dh} = p' = -0.13\,p$

$p'(0) = -0.13\,p(0) = -0.13$ //

$\Delta h = +0.1$

$\Delta p \approx p'(0) \cdot \Delta h = -0.013$ //

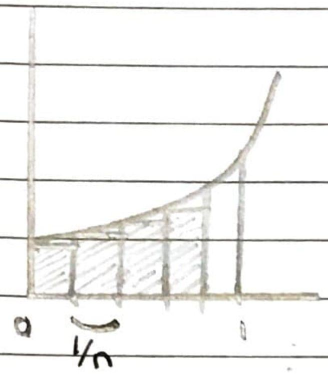
Mount Everest:

$\Delta p = p'(\cancel{1}0) \cdot \Delta h$

$\Delta h = 10$

$\Delta p = -0.13 \times 10 = -1.3 \leftarrow$ this doesn't make sense as $x \not\approx 0$. approximation not accurate

5. $\int_0^1 e^x \, dx$




$$\text{Area} = \frac{1}{n}\left(e^0 + e^{1/n} + \dots + e^{\frac{n-1}{n}}\right)$$

Geometric Series :

$$a,\ ar,\ ar^2,\ \dots$$

$$① \leftarrow S = a + ar + ar^2 + \dots + ar^n$$

$$② \leftarrow rS = \qquad ar + ar^2 + \dots + ar^n + ar^{n+1}$$

$$② - ①,\ (r-1)S = a(r^{n+1} - 1)$$

$$S = a\left(\frac{r^{n+1} - 1}{r-1}\right)$$

$$\text{Area} = \frac{1}{n}\sum_{i=1}^{n} e^{i/n} = \frac{1}{n}\left[\frac{\left(e^{1/n}\right)^{n-\cancel{1}+\cancel{1}} - 1}{e^{1/n} - 1}\right] \qquad \left(\begin{array}{l} r = e^{1/n} \\ a = 1 \\ n = n-1 \end{array}\right).$$

$$\text{Area} = \frac{1}{n}\left(\frac{e-1}{e^{1/n} - 1}\right)$$

$$\left(e^x \approx 1 + x,\ x \approx 0\right)$$

$$\lim_{n\to\infty} n\left(e^{1/n} - 1\right) \approx \lim_{n\to\infty} n\left(1 + \frac{1}{n} - 1\right) = \lim_{n\to\infty} \frac{n}{n} = \underline{\underline{1}}$$

$$\underset{(\text{exact})}{\text{Area}} = \lim_{n\to\infty} \frac{1}{n}\frac{(e-1)}{(e^{1/n} - 1)} = \underline{\underline{e - 1}}$$

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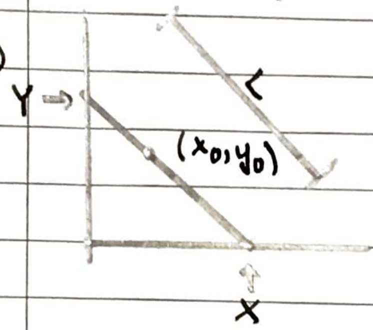
6. a)




Line Equation : $\dfrac{y-y_0}{x-x_0} = m_0$

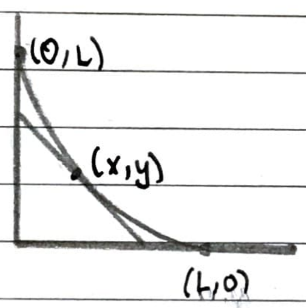
X: $\dfrac{0-y_0}{x-x_0} = m_0 \Rightarrow X = x_0 - \dfrac{y_0}{m_0}$

Y: $\dfrac{Y-y_0}{0-x_0} = m_0 \Rightarrow Y = y_0 - m_0 x_0$

$$X^2 + Y^2 = L^2$$

$$\boxed{L^2 = \left(x_0 - \frac{y_0}{m_0}\right)^2 + \left(y_0 - m_0 x_0\right)^2}$$

b)




$m_0 = y'$, $x_0 = x$, $y_0 = y$

$$L^2 = \left(x - \frac{y}{y'}\right)^2 + \left(y - y'x\right)^2$$

c) Differentiating w.r.t x

$$0 = 2\left(x - \frac{y}{y'}\right)\frac{d}{dx}\left(x - \frac{y}{y'}\right) + 2\left(y - y'x\right)\frac{d}{dx}\left(y - y'x\right)$$

$$\frac{d}{dx}\left(x - \frac{y}{y'}\right) = 1 - \left(\frac{y' \cdot y' - yy''}{(y')^2}\right) = \frac{yy''}{(y')^2}$$

$\frac{d}{dx}(y - y'x) = y' - y''x - y' = -y''x$

$0 = 2\left(x - \frac{y}{y'}\right)\frac{yy''}{(y')^2} + 2(y - y'x)(-y''x)$

$0 = (xy' - y)y''\left(\frac{y}{(y')^3}\right) + (xy' - y)y''(x)$

$0 = (xy' - y)(y'')\left(\frac{y}{(y')^3} + x\right)$

$\hookrightarrow$ something

d) I $\frac{y}{(y')^3} + x = 0$

$\frac{y}{(y')^3} = -x \Rightarrow (y')^3 = \frac{-y}{x} \Rightarrow y' = -\frac{y^{1/3}}{x^{1/3}}$

$\frac{dy}{dx} = \frac{-y^{1/3}}{x^{1/3}} \Rightarrow \frac{dy}{y^{1/3}} = -\frac{dx}{x^{1/3}} \Rightarrow \int \frac{dy}{y^{1/3}} = -\int \frac{dx}{x^{1/3}}$

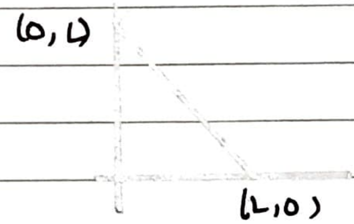
$\frac{3}{2}y^{2/3} = \frac{-3}{2}x^{2/3} + C \Rightarrow y^{2/3} + x^{2/3} = c \qquad (c = 2/3C)$

we know that $y(0) = L$,

$\therefore c = L^{2/3}$

Solution: $\boxed{x^{2/3} + y^{2/3} = L^{2/3}}$

II $y'' = 0$




$y = ax + b$

$y(0) = L, \quad y(L) = 0$

$\Rightarrow b = L \quad \Rightarrow 0 = aL + L \Rightarrow a = -1$

$\boxed{y = L - x}$ (trivial solution)

III $xy' - y = 0$

$$y' = \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$$

$\ln y = \ln x + c \quad (x, y > 0)$

$y = e^c x \Rightarrow y = Ax$ where $A > 0$

This solution is only meaningful when $L = 0$. The solution thus corresponds to a single point which is the origin.

$(x, y) = (0, 0)$ (very trivial solution)