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6.042J Problem Set - 2

1.(a) $a_1 < a_2 \geq a_3 \leq a_4 \quad a_5$

If $a_2 < a_3$, then $a_1 < a_2 < a_3$ (3-chain) is created. Therefore, $a_2 > a_3$

If $a_3 > a_4$, then $a_2 > a_3 > a_4$ (3-chain) is created. Therefore, $a_3 < a_4$

If $a_3 > a_1$, then $a_1 < a_3 < a_4$ (3-chain) is created. Therefore, $a_3 < a_1$ $\square$

(b) If $a_4 > a_2$, then $a_1 < a_2 < a_4$ (3-chain) is created. Therefore, $a_4 < a_2$

So, $a_3 < a_4$ and $a_4 < a_2 \Rightarrow a_3 < a_4 < a_2$ $\square$

(c) (by case analysis)

Case I: If $a_4 < a_5$, then $a_3 < a_4 < a_5$ makes a 3-chain

Case II: If $a_5 < a_4$, then $a_2 > a_4 > a_5$ makes a 3-chain

$\therefore$ Any value of a_5 makes a 3-chain

(d) If we repeat steps (a), (b), (c) for $a_1 > a_2$, we get the same result (in (c)), with just the sign reverse. The previous steps assumed no 3-chains existed, but we ended up in a 3-chain anyways. $\therefore$ All sequences of 5 distinct integers must contain a 3-chain $\square$

2. $$P(n) := \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 \quad \forall n \in \mathbb{N}$$

Base Case: $P(0) = 0 = \left(\frac{0(0+1)}{2}\right)^2 = 0$ ✓

Inductive Step: Assume $P(n)$, prove $P(n+1)$

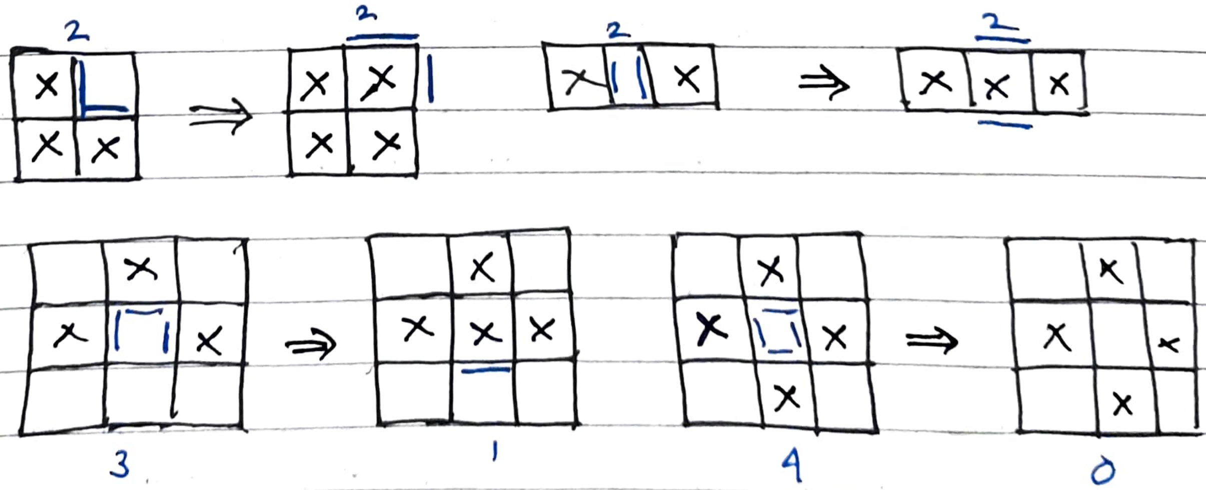
$$\sum_{i=1}^{n+1} i^3 = \underbrace{1^3 + 2^3 + \dots + n^3}_{P(n)} + (n+1)^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 = (n+1)^2\left(\frac{n^2}{4} + n + 1\right)$$

$$= \frac{(n+1)^2(n^2+4n+4)}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2 = P(n+1)$$

$\therefore$ $P(n)$ is true $\forall n \in \mathbb{N}$ $\square$

3. Thm: If fewer than n students in class are initially infected, the class will never be completed infected.



NOTE - The 'boundary' (perimeter) never increases.

Invariant: The number of edges in the perimeter of infected students do not exceed the starting state. (I)

Let p be the # edges that have an infected person on only one side

For the starting state with ~~k~~ less than n infected, ~~$p > 0$~~. $0 < p < 4(n-1)$

In the final state, $p = 4n$, because the perimeter is the boundary of the grid

Using induction to prove the invariant,

Let $P(k)$ be the proposition that the perimeter does not exceed I after k time steps.

Base Case: $P(0)$, The starting state has perimeter I ✓

Inductive Step: Assume $P(n)$, prove $P(n+1)$

If after n timesteps $\# p \leq I$, then after the next timestep, $\# p$ won't exceed because in the best case, no new edges are created (see previous page). $\therefore$ $P(n+1)$ is true.

Thus, $P(n)$ is true $\forall n \in \mathbb{N}$ $\square$

Since, it is impossible to exceed the perimeter in the starting state which is ~~<~~ less than $4n$, the final state with $p = 4n$ can never be reached. $\square$

4. $P(n) := \forall k \leq n,\ a^k = 1$

Base Case: $P(0) := a^0 = 1$ ✓

The flaw is in the proving $P(0) \Rightarrow P(1)$, i.e. in the inductive step for $n=0$

$$a^{0+1} = \frac{a^0 \cdot a^0}{a^{-1}}$$

a^{-1} is not proved, $-1 < 0$ (below the base case)

$\therefore$ The proof is bogus.

5. Base Case: $G_0 = 0$, $G_1 = 1$
Constructor Case: $G_n = 5G_{n-1} - 6G_{n-2}$

$$P(n) := G_n = 3^n - 2^n$$

Base Case:

$P(0) := G_0 = 3^0 - 2^0 = 1 - 1 = 0$ ✓
$P(1) := G_1 = 3^1 - 2^1 = 3 - 2 = 1$ ✓

Constructor Case: Assume $P(k)$ is true $\forall k < n$

$$G_n = 5G_{n-1} - 6G_{n-2} = 5(3^{n-1} - 2^{n-1}) - 6(3^{n-2} - 2^{n-2})$$

$$= \frac{5}{6}(2 \times 3^n - 3 \times 2^n) - \frac{\cancel{6}^{1}}{\cancel{9 \times 4}_{3 \times 2 = 6}}(4 \times 3^n - 9 \times 2^n)$$

$$= \frac{5}{6}(2\times 3^n - 3\times 2^n) - \frac{1}{6}(4\times 3^n - 9\times 2^n)$$

$$\cancel{= \frac{4}{6}(6\times 2^n - 2\times 3^n}$$

$$= \frac{1}{6}(10\times 3^n - 15\times 2^n - 4\times 3^n + 9\times 2^n)$$

$$= \frac{1}{6}(6\times 3^n - 6\times 2^n) = 3^n - 2^n \checkmark$$

$$\therefore \forall n \in \mathbb{N} \quad G_n = 3^n - 2^n \qquad \square$$

6(a) A row move moves a tile from cell i to cell $i+1$ or ~~i+~~ $i-1$. This does not change the relative order with tiles ~~b~~ before or after it. Therefore a row move does not change the order of the tiles. $\square$

(b) Column Move:

A column moves changes the order of 3 tiles between the blank and concerned tile. Thus ~~if~~ a column move on cell i changes the relative order of cells $(i-1, i-2, i-3)$ and $(i+1, i+2, i+3)$. The ~~remaining~~ order of remaining tiles remain unchanged

(c) From (a), a row move cannot change the order of any tiles. So, after a row move, the parity remains unchanged.

(d) For a column move, 3 tiles are affected (order reverse)

Pf. by case analysis.

The initial possibilities for the 3 tiles are (I, I, C); (I, C, C); (I, I, I); (C, C, C)

$$\begin{aligned}
III &\rightarrow CCC \Rightarrow -3 \\
CCC &\rightarrow III \Rightarrow +3 \\
IIC &\rightarrow CCI \Rightarrow -2+1 = -1 \\
ICC &\rightarrow CII \Rightarrow +2-1 = +1
\end{aligned}$$

Since the # inversion ~~switche~~ changes by an odd number, the parity switches everytime a column move is made.

(e) (by induction)

P(n) := After n moves, the parity of the number of inversions is different from the parity of the row containing the blank square

Base Case: P(0) → parity = odd (#inversion = 1)
row 4 = even ✓

Inductive step: Assume $P(n)$, prove $P(n+1)$.

Two possible moves:

(i) Row move: From ~~(a)~~ (c), ~~#~~ a row move doesn't affect parity of # inversions and the blank is in the same row ~~of~~ as after n steps.

(ii) Column Move: After a column move, the parity switches but so does the row number.

Thus $P(n) \Rightarrow P(n+1)$

$\therefore P(n)$ is true $\forall n$

(f) In the start state, the parity of # inversions is ~~even~~ odd and and parity of row no. is even.
In the target state, the parity of #inv is even (0) and the parity of row no. is also even.
But, from (e), the parities after n step can never be the same. So we cannot reach the target state from the starting configuration. $\square$

7. ~~NOTE~~ Let n_z be the no. of Z-lings and n_b be the no. of B-link.

Observe that if ~~the~~ $n_z > n_b$, then the no. of ~~Z~~ ~~n~~ Z-ling in further generations keep on increasing.

Also, the excess of Z/B-lings are responsible for increasing the no. of offsprings. Since Z×B ~~one paren~~ gives an offspring of each type, not changing the count.

So, a stronger hypothesis is that $n_z \leq n_b$.

Let $P(n) :=$ $n_z \leq n_b$ after n generations. ~~of~~

Base Case: $P(1)$. In the first generation, $n_z = 200$, and $n_b = 800$. $200 < 800$ ✓

Inductive Step: Assume $P(n)$, Prove $P(n+1)$

Assume $n_z \leq n_b$ after the n^{th} generation. An excess of B-lings $(n_b - n_z)$ will produce $\lfloor n_b - n_z \rfloor$ B-lings and $\lfloor n_b - n_z \rfloor / 2$ Z-lings

Where $\lfloor x \rfloor$ rounds to the smallest even no.

So, the no. of Z-lings and B-lings in the $(n+1)^{th}$ generation are:

$$n'_z = n_z + \lfloor n_b - n_z \rfloor / 2 \implies n'_z < n'_b$$
$$n'_b = n_z + \lfloor n_b - n_z \rfloor$$

$\therefore$ $P(n+1)$ is true.

Thus $P(n) := n_z \leq n_b$ is true $\forall n$ generation

As a corollary, $Q(n) := n_z \leq 2n_b$ is also true since $P(n)$ is true.

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