18.01 Practice Questions for Exam - II

1. ~~f(∞)~~ $f(x) = 3x^5 - 5x^3 + 1$

$f(\infty) \longrightarrow \infty$

$f(-\infty) \longrightarrow -\infty$

$f'(x) = 15x^4 - 15x^2 = 0$

$15x^2(x^2 - 1) = 0 \Rightarrow x = 0, 1, -1$

Critical Points: $(0,1)$; $(1,-1)$; $(-1,3)$

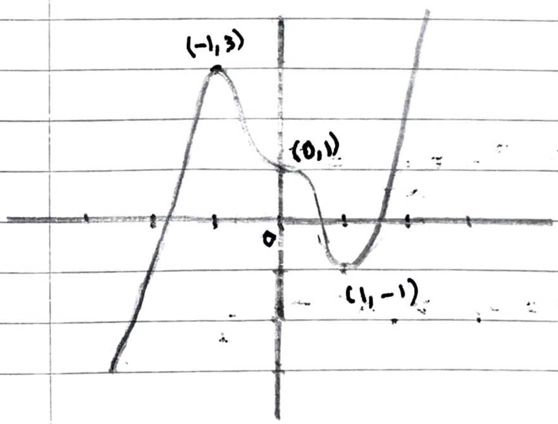
+	−	−	+

$-1 \quad 0 \quad 1$

$f''(x) = 60x^3 - 30x = 0$

$30x(2x^2 - 1) = 0 \Rightarrow x = 0, \pm 1/\sqrt{2}$

Inflection Points: $(0,1)$; $(1/\sqrt{2}, -0.237)$; $(-1/\sqrt{2}, 2.237)$



$f(1) = -1$; $f(2) = 57$
$f(-1) = 3$; $f(-2) = -55$

Appox. location of zeroes:
$[-2,-1]$; $[1,2]$

2. $f(x) = 4x^2 - \frac{1}{x}$

discontinuities

$f(0^+) = 4(0^+)^2 - \frac{1}{0^+} \rightarrow -\infty$

asymtotes

$f(0^-) = 4(0^-)^2 - \frac{1}{0^-} \longrightarrow +\infty$

ends

$f(+\infty) \Rightarrow \infty$

$f(-\infty) \rightarrow \infty$

zeroes

$f(x) = 0 \Rightarrow 4x^2 = \frac{1}{x} \Rightarrow 4x^3 = 1$

$x = 1/4^{1/3}$

critical points

$f'(x) = 8x - \frac{d}{dx} x^{-1} = 8x + \frac{1}{x^2} = 0$

$8x = -\frac{1}{x^2} \Rightarrow 8x^3 = -1 \Rightarrow \boxed{x = -1/2}$

Critical Point : $(-1/2, 2)$

$$f''(x) = \frac{d}{dx}(8x + x^{-2}) = 8 - \frac{2}{x^3}$$

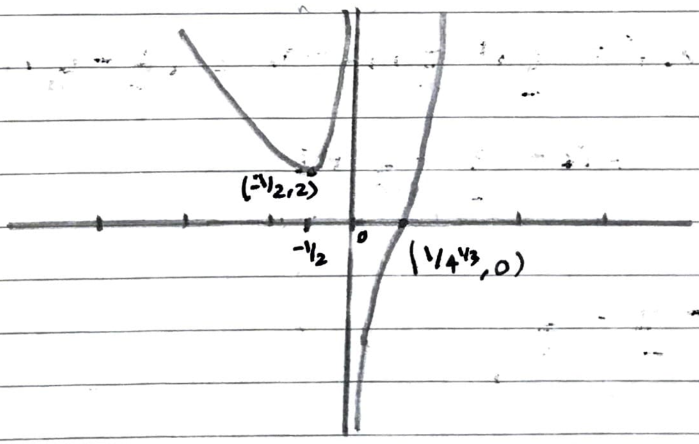
$f''(1/2) = 8 + 16 = 24 \Rightarrow$ local minimum

inflection points

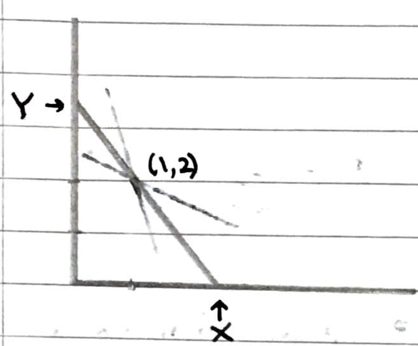
$f''(x) = 0 \Rightarrow \frac{2}{x^3} = 8 \Rightarrow x^3 = 1/4$

$x = 1/4^{1/3}$

Inflection Point : $(1/4^{1/3}, 0)$



3.



eqn. of line :

$$\frac{y-2}{x-1} = m$$

intercepts:

$$\frac{0-2}{X-1} = m \Rightarrow X = 1 - 2/m$$

$$\frac{Y-2}{0-1} = m \Rightarrow Y = 2 - m$$

$$A = \frac{1}{2} X Y = \frac{1}{2}\left(1 - \frac{2}{m}\right)(2-m) = \left(\frac{1}{2} - \frac{1}{m}\right)(2-m)$$

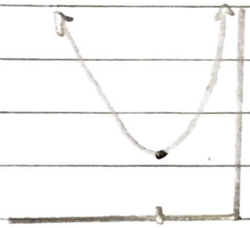
$$= 1\cancel{2} - \frac{m}{2} - \frac{2}{m} + \cancel{1}$$

$$\boxed{A = 2 - \frac{m}{2} - \frac{2}{m}}$$

$$A' = -\frac{1}{2} + \frac{2}{m^2} = 0 \Rightarrow \frac{2}{m^2} = \frac{1}{2} \Rightarrow m = \pm 2$$

$$\boxed{m = -2}$$

_/_/_



ends

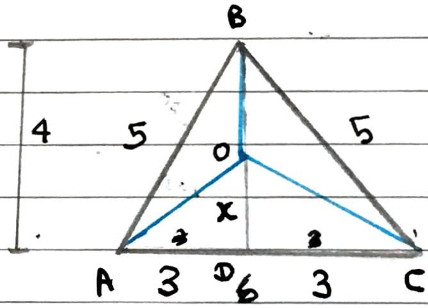
$A(0^-) \to \infty$
$A(-\infty) \to \infty$

So critical point is a minimum.

$$A(-2) = 2 + \frac{2}{2} + \frac{2}{2} = 4$$

Line : $\boxed{y-2 = -2(x-1)}$

4. By symmetry, we can say that the center of the Y must lie along BD.



$$BD = \sqrt{5^2 - 3^2} = 4 \,//$$

length of wire $= L = (4-x) + 2\sqrt{x^2+9}$

$$L' = -1 + \frac{\cancel{2}}{\cancel{2}\sqrt{x^2+9}} \cdot 2x = -1 + \frac{2x}{(x^2+9)^{1/2}}$$

$$L' = 0 \Rightarrow 2x = \sqrt{x^2+9}$$

$$4x^2 = x^2 + 9 \rightarrow 3x^2 = 9 \Rightarrow x = \pm\sqrt{3}$$

$\boxed{x = \sqrt{3}}$

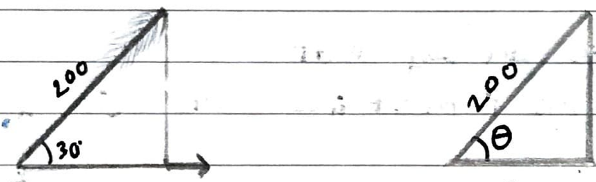
$L(\sqrt{3}) = 4 + 3\sqrt{3} \approx 9.20$

At the extremes,

$L(0) = 3 + 3 + 4 = 10 > 9.20$
$L(4) = 5 + 5 + 0 = 10 > 9.20$

Thus $\underline{\underline{4 + 3\sqrt{3}}}$ is the minimum length of wire needed

5.



$\frac{ds}{dt} = 50$ ft/s

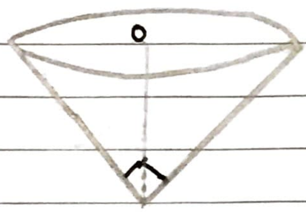
$\cos\theta = \frac{s}{200} \Rightarrow s = 200\cos\theta$

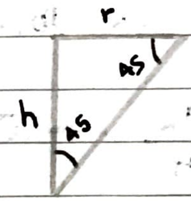
$\frac{ds}{dt} = -200\sin\theta \cdot \frac{d\theta}{dt}$

At $\theta = \pi/6$, $ds/dt = 50$

$50 = -\cancel{200}^{100} \times \frac{1}{\cancel{2}} \frac{d\theta}{dt} \Rightarrow \boxed{\frac{d\theta}{dt} = \frac{-1}{2} \text{ rad/sec}}$

6.a)





$$\underline{\underline{r = h}}$$

$$\frac{dV}{dt} = 3 \text{ cm}^3/\text{s}$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3 \qquad [r = h]$$

$$\frac{dV}{dt} = \frac{1}{\cancel{3}}\pi \cancel{3} h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{\pi h^2}\frac{dV}{dt}$$

At $h = 2$ cm,

$$\frac{dh}{dt} = \frac{1}{4\pi} \times 3 = \underline{\underline{\frac{3}{4\pi}}} \text{ cm/s}$$

b) Assume the rate of evaporation is proportional to the surface area on top. i.e.

$$\frac{dV}{dt} = c\pi r^2 = c\pi h^2 \quad (c < 0) \quad \rightarrow \text{①}$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3$$

differentiating on both sides.

$$\frac{dv}{dt} = \pi h^2 \frac{dh}{dt} \longrightarrow (2)$$

Equating (1) and (2)

$$\cancel{\pi h^2} \frac{dh}{dt} = c\cancel{\pi h^2} \Rightarrow \boxed{\frac{dh}{dt} = c = [\text{constant}]}$$

7. $$f(x) = \frac{e^{-\lambda x}}{1+2\sin x}$$

$$\frac{e^{-\lambda x}}{1+2\sin x} \approx \frac{1-\lambda x}{1+2x} = 1 \Rightarrow \boxed{\lambda = -2}$$ linear approximation

To find $f(0.1)$, use quadratic approximation

$$f(x) = e^{2x}(1+2\sin x)^{-1}$$

$$\approx e^{2x}(1+2x)^{-1} = \left(1 + 2x + \frac{4x^2}{2}\right)\left(1 - 2x + \frac{(-1)(-2)}{2} 4x^2\right)$$

$$\approx \left(1+2x+2x^2\right)\left(1-2x+4x^2\right) \approx 1 - 2x + 4x^2 + 2x - 4x^2 + 2x^2$$

$$f(x) \approx 1 + 2x^2 = 1 + \frac{2}{100} = \underline{\underline{1.02}}$$

8. **Mean-Value Theorem**:

$\rightarrow$ $\dfrac{f(b)-f(a)}{b-a} = f'(c)$, for some c, $a<c<b$

$\rightarrow$ $f(b) = f(a) + f'(c)(b-a)$

a) $f'(x) > 0$, f is differentiable.

Consider an interval (a,b) where $a<b$, then there exists some 'c' such that

$f(b) = f(a) + \underbrace{f'(c)(b-a)}$, $a<c<b$

$\because f'(c) > 0$ (given), thus $f(b) = f(a) +$ [+'ve quantity]

$f(b) > f(a)$ // , $a<b \Rightarrow f(a) < f(b)$

The function $f(x)$ is increasing.

b) $e^x > 1+x$ $(x>0)$

Let $f(x) = e^x - (1+x)$

$f(0) = e^0 - 1 = 0$

$f'(x) = e^x - 1 > 0$ $(x>0)$

Hence the function is increasing,

$f(x) > f(0)$

$e^x - (1+x) > 0 \Rightarrow \boxed{e^x > 1+x}$

9.a) $\displaystyle\int \frac{dx}{(3x+2)^2}$

guess: $\dfrac{1}{3x+2}$. $\dfrac{d}{dx}(3x+2)^{-1} = \dfrac{-3}{(3x+2)^2}$

$$\int \frac{dx}{(3x+2)^2} = \frac{-1}{3}(3x+2)^{-1} + C$$

b) $\displaystyle\int \sin 2x \sin x \, dx$

$\sin 2x = 2\sin x \cos x$

$2\displaystyle\int \sin^2 x \cos x \, dx$

$u = \sin x$, $du = \cos x \, dx$

$$2\int u^2 du = \frac{2}{3}u^3 + C = \frac{2}{3}\sin^3 x + C$$

__/__/__

c) $\int \frac{\ln^2 x}{x}\, dx$

$u = \ln x$, $du = \frac{dx}{x}$

$\int u^2 du = \frac{1}{3}u^3 + C = \frac{\ln^3 x}{3} + C$ //

10. $\frac{dy}{dx} = xy^2 + x$, $y(0) = 1$

$\frac{dy}{dx} = x(y^2+1) \Rightarrow \frac{dy}{y^2+1} = x\,dx \Rightarrow \int \frac{dy}{y^2+1} = \int x\,dx$

$\tan^{-1} y = \frac{x2}{2} + C$

$y(0) = 1$

$\tan^{-1} 1 = C \Rightarrow C = \pi/4$

$\boxed{y = \tan\left(\frac{x2}{2} + \frac{\pi}{4}\right)}$

11. a) $\frac{dT}{dt} = k(T - T_e)$

b) $\frac{dT}{(T-T_e)} = k\,dt \Rightarrow \int \frac{dT}{(T-T_e)} = k \int dt$

$\ln|T - T_e| = kt + c$

$T - T_e = Ae^{kt} \quad (A = e^c)$

$\boxed{T = T_e + Ae^{kt}}$

$T(0) = 20$ Find time when $T = 60$

$T_e = 100$

$T(5) = 30$

$20 = 100 + Ae^0 \Rightarrow \boxed{A = -80}$

$30 = 100 - 80e^{5k}$

$e^{5k} = \frac{7}{8} \Rightarrow 5k = \ln(7/8) \Rightarrow \boxed{k = \frac{1}{5}\ln(7/8)}$

Find t.

$60 = 100 - 80e^{kt} \Rightarrow e^{kt} = 1/2$

$kt = \ln(1/2) \Rightarrow t = \frac{5\ln(1/2)}{\ln(7/8)} \simeq$ 26 mins