18.01 Practice Questions for Exam - I

1. a) $\frac{d}{dt}\left(\frac{3t}{\ln t}\right)\Big|_{e^2}$

$$\frac{d}{dt}\left(\frac{3t}{\ln t}\right) = \frac{3\ln t - 3t/t}{(\ln t)^2} = \frac{3(\ln t - 1)}{(\ln t)^2}$$

at $t = e^2$, $\ln t = 2$

$$\frac{d}{dt}\left(\frac{3t}{\ln t}\right)\Big|_{e^2} = \frac{3(2-1)}{2^2} = \frac{3}{4}$$

b) $\lim_{u\to 0} \frac{3u}{\tan 2u}$

$$\frac{3u}{\tan 2u} = \frac{3u}{\sin 2u} \cdot \cos 2u = \frac{2u}{\sin 2u} \cdot \frac{3\cos 2u}{2}$$

$$\lim_{u\to 0} \frac{2u}{\sin 2u} \cdot \frac{3\cos 2u}{2} = \frac{3}{2}$$

c) $\frac{d^3}{dx^3} \sin kx$

$f(x) = \sin kx$
$f'(x) = \cos(kx) \cdot k$
$f''(x) = -k\sin(kx) \cdot k$
$f'''(x) = \cancel{k^2\cos} - k^3\cos(kx)$

d) $\frac{d}{d\theta}(a+k\sin^2\theta)^{1/3} = \frac{1}{3}(a+k\sin^2\theta)^{-2/3} \cdot (k\cdot 2\sin\theta.\cos\theta)$

$$= \frac{2k\sin\theta\cos\theta}{3(a+k\sin^2\theta)^{2/3}} //$$

2. $f'(x_0) = f' \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$

$$\left.\frac{d}{dx} x^3\right|_{x_0} = \lim_{\Delta x \to 0} \frac{(x_0+\Delta x)^3 - x_0^3}{\Delta x}$$

$$= \frac{\cancel{x_0^3} + 3x_0^2\Delta x + 3x_0\Delta x^2 + \Delta x^3 - \cancel{x_0^3}}{\Delta x}$$

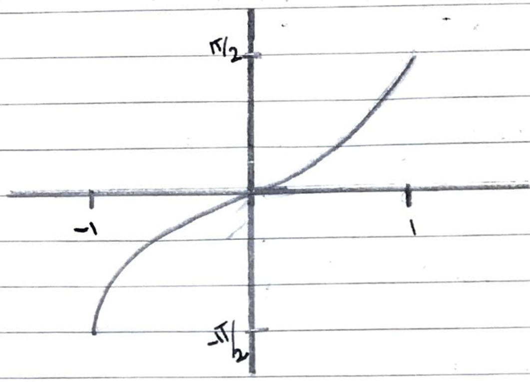
$$= \frac{\cancel{\Delta x}(3x_0^2 + 3x_0\Delta x + \Delta x^2)}{\cancel{\Delta x}} \xrightarrow[\Delta x \to 0]{} \underline{\underline{3x_0^2}}$$

3. $\lim_{h\to 0} \frac{1-\sqrt[3]{1+h}}{h} = -\lim_{h\to 0} \frac{\sqrt[3]{1+h} - \sqrt[3]{1}}{h}$

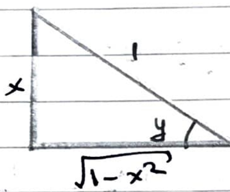
$$= -\left.\frac{d}{dx} x^{1/3}\right|_1 = \left.-\frac{1}{3}x^{-2/3}\right|_{x=1} = \frac{-1}{3} //$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

4. $y = \sin^{-1} x$



$x = \sin y$



$\frac{d}{dx}(x = \sin y)$

$1 = \cos y \cdot y' \Rightarrow y' = \frac{1}{\cos y}$

$$\boxed{\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}}$$

5. $f(x) = \begin{cases} ax + b, & x > 0 \\ 1 - x + x^2, & x \leq 0 \end{cases}$

a) continuous.

$f(0) = \lim_{x \to 0} f(x)$

(i) $f(0) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0} (1 - x + x^2) = 1$ //

(ii) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} ax + b = b$ //

$\boxed{b = 1}$ $\boxed{a \to \text{anything}}$

b) differentiable

$\lim_{\Delta x \to 0^+} \frac{\Delta f(x)}{\Delta x} = \frac{d}{dx}(ax + b) = a$ //

$\lim_{\Delta x \to 0^-} \frac{\Delta f}{\Delta x} = \frac{d}{dx}(1 - x + x^2)\Big|_{x=0} = 2x - 1 = -1$ //

$\therefore \boxed{a = -1}$

from diff $\Rightarrow$ continuous

$\boxed{b = 1}$

6. $x^2y + y^3 + x^2 = 8$

implicit differentiation

$2xy + x^2y' + 3y^2y' + 2x = 0$

$(x^2 + 3y^2)y' = -2x - 2xy$

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$$y' = \frac{-2x(1+y)}{x^2+3y^2} = 0$$

$\underline{x=0}$:

$y^3 = 8 \Rightarrow y = \pm 2$; $(0,2)$, $(0,-2)$

$\underline{y=-1}$:

$-x^2 - 1 + x^2 = 8$

no x exists.

So the points with horizontal slope are $(0,2)$ and $(0,-2)$

7. $y = f(x)$

$y - y_0 = m(x - x_0)$

$y - y_0 = f'(x_0)(x - x_0)$

intersecting x-axis , $y = 0$

$-y_0 = f'(x_0)(x - x_0) \Rightarrow \boxed{x = x_0 - \frac{y_0}{f'(x_0)}}$

8. $V = \frac{4}{3}\pi r^3$

$\left.\frac{dV}{dt}\right|_{r=20} = -10 \text{ cm}^3/s$ to find $\left.\frac{dr}{dt}\right|_{20}$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3}r^2 = 4\pi r^2 \cdot \frac{dr}{dt}$$

at $r = 20$

$$-10 = 4\pi(20)^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{-1}{4\pi} \cdot \frac{10}{(20)^2}$$

$$\frac{dr}{dt} = \frac{-1}{4\pi} \cdot \frac{1}{20 \cdot 2} = \frac{-1}{160\pi} \text{ cm/s}$$

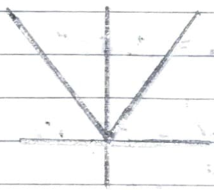
9. a) $\sec x = \dfrac{1}{\cos x}$

undefined at $x = \pm\frac{\pi}{2} + 2n\pi$, $n =$ Integers.
~~(jump)~~ ∞

b) $\dfrac{1+x^2}{1-x^2}$

undefined at $x = \pm 1$ //
~~(jump)~~ ∞

c) $\dfrac{d}{dx}|x|$



$f(x) = |x|$

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ \text{undefined}, & x = 0 \end{cases}$$

~~Fr~~ $f'(x)$ is discontinuous at $x = 0$

10. $A = A_0 e^{-rt}$ $(r > 0)$

a) $A = A_0 e^{-rt}$

$A = A_0 / 4$

$e^{-rt} = \frac{1}{4} \Rightarrow -rt = \ln(1/4)$

$rt = \ln(4) \Rightarrow \boxed{t = \frac{\ln(4)}{r}}$

b) $\frac{dA}{dt} = A_0 e^{-rt} \cdot -r = -A_0 r e^{-rt}$

$\frac{dA}{dt}$ at $t = \frac{\ln 4}{r}$, $= \frac{-A_0 r}{4}$ // g/s