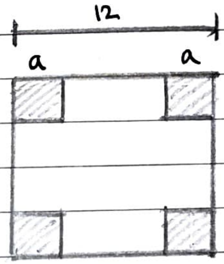
18.01 Problem Set - 3

Part - I



26 1. $V = lbh$

$= (12-2a)^2 \cdot a$,

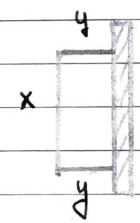
$(0 < a < 6)$ range.

$$\frac{dV}{da} = (12-2a)^2 + a\cdot(2(12-2a)-2)$$
$$= (12-2a)(12-2a-4a)$$
$$= (12-2a)(12-6a)$$

$\frac{dV}{da} = 0$, $a = 6, 2$

At end pts. $(a = 0, 6)$, the volume $\to 0$, so the volume is maximum at $\boxed{a=2}$.

2. $xy = 20000 \Rightarrow y = \frac{20000}{x}$



$$L = x + 2y = x + \frac{40000}{x}$$

$$\frac{dL}{dx} = 1 + 40000\left(\frac{-1}{x^2}\right) = 0$$

$$1 - \left(\frac{40000}{x^2}\right) = 0 \Rightarrow x^2 = 40000 \Longrightarrow \boxed{x = 200\ ft}$$

At extremes ($x = 0, \infty$) $L \to \infty$, so length minimum at $x = 200$

$$y = \frac{20000}{200} = \underline{\underline{100 \text{ ft}}}$$

Shortest length $= 200 + 2(100) = \underline{\underline{400 \text{ ft}}}$

5. $A = 2\pi r h + \pi r^2$

$V = \pi r^2 h$

$$\frac{dV}{dr} = \pi r^2 \cdot \frac{dh}{dr} + 2\pi r h \longrightarrow ①$$

$$\cancel{\frac{dA}{dr}}^{0} = 2\pi r \cdot \frac{dh}{dr} + 2\pi h + 2\pi r = 0$$

$$\Rightarrow \frac{dh}{dr} = \frac{-r-h}{r} \longrightarrow ②$$

Substituting ② in ①

$$\frac{\pi r^{\cancel{2}}(-r-h)}{\cancel{r}} + 2\pi r h = 0$$

$$-\pi r^2 - \pi r h + 2\pi r h = 0 \Rightarrow -\pi r^2 + \pi r h = 0$$

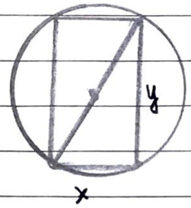
$$-r + h = 0 \Rightarrow \boxed{r = h}$$

At extremes ($r = 0, \infty$), $V \to 0$, so maximum at critical point.

Cylinder has max. volume when $\boxed{r = h}$.

__/__/__

11.



$x^2 + y^2 = (2r)^2$

differentiating on both sides.

$2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$

$S = cxy^3$

$S' = cx \cdot 3y^2 \cdot y' + cy^3 = cx \cdot 3y^2 \left(\frac{-x}{y}\right) + cy^3$

$S' = -cx^2 \cdot 3y + cy^3 = 0$

$3x^2 y = y^3 \Rightarrow 3x^2 = y^2 \Rightarrow \boxed{\frac{x}{y} = \frac{1}{\sqrt{3}}}$

At extremes ($x = 0, 2r$), $S = 0$, so maximum at critical point

The beam has max. strength when $\boxed{y/x = \sqrt{3}}$.

13. a) $f(x) = (200 + 5x)(100 - 2x)$ [increasing price]

$f'(x) = (200+5x)(-2) + (100-2x) \cdot 5$

$= -400 - 10x + 500 - 10x$

$= 100 - 20x = 0 \Rightarrow \boxed{x = 5}$

$g(x) = (200 - 5x)(100 + 2x) = f(-x)$ [decreasing price]

At $P = 200 + 25$ and $P = 200 - 25$, revenue is maximum.

$\boxed{P = 225, 175}$

b) $R = x \cdot p - x(10 - x/10^5)$, $x = 10^5(10 - P/2)$

$= 10^5 p(10 - P/2) - 10^5(10 - P/2)(P/2)$

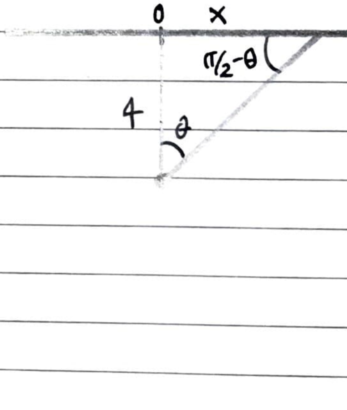
$R = 10^5(10 - P/2)(P/2) = \frac{10^6}{2} p - \frac{10^5}{4} p^2$

$\frac{dR}{dp} = \frac{10^6}{2} - \frac{10^5 \cdot p}{2} = 0 \Rightarrow \boxed{p = 10}$

The critical pt. is a maximum as $\frac{d^2R}{dp^2} = \frac{-10^5}{2} < 0$

$\therefore$ Max Profit at $\underline{\underline{p = 10 \text{ cents}}}$

2E2. $f = 3$ rev/min (frequency)



$\frac{d\theta}{dt} = 3 \times (2\pi \text{ rad})/\text{min} = 6\pi$ rad/min

$\tan\theta = \frac{x}{4} \Rightarrow x = 4\tan\theta$

$\frac{d}{dt}(x = 4\tan\theta)$

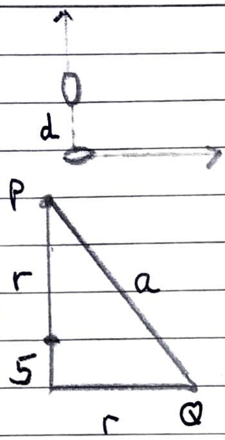
$\frac{dx}{dt} = 4\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{24\pi}{\cos^2\theta}$

$\left(\frac{\pi}{2} - \theta = \frac{\pi}{3} \right.$

$\left.\frac{dx}{dt}\right|_{\theta = 30^\circ} = \frac{24\pi}{(\sqrt{3})^2} \cdot (2)^2 = \underline{\underline{24\pi}}$ miles/min

$\left. \theta = \frac{\pi}{6} \right)$

2E3.



$$d = \text{speed} \times \text{time} = 30 \times \frac{1}{6} = 5 \text{ miles}$$

$$a = PQ = \sqrt{r^2 + (5+r)^2}, \quad \frac{dr}{dt} = 30 \text{ miles/hr}$$

$$a^2 = r^2 + (5+r)^2$$

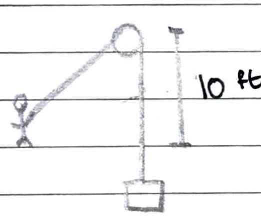
$$\cancel{2}a \cdot \frac{da}{dt} = 2r\frac{dr}{dt} + \cancel{2}(5+r)\frac{dr}{dt}$$

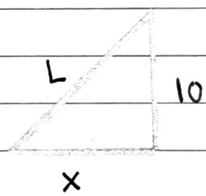
$$\frac{da}{dt} = \left(\frac{5+2r}{a}\right)\frac{dr}{dt}$$

when $r = 10$, $a = 5\sqrt{13}$

$$\frac{da}{dt} = \frac{\overset{5}{\cancel{25}}}{\cancel{5}\sqrt{13}} \cdot 30 = \frac{150}{\sqrt{13}} \text{ miles/hr.}$$

2E5.





$\frac{dL}{dt} = 4 \text{ ft/s}$, find $\frac{dx}{dt}$ at $x = 20$ ft

$$x^2 + 10^2 = L^2$$

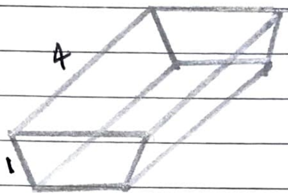
$$\frac{d}{dt}\left(x^2 + 10^2 = L^2\right)$$

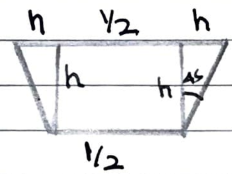
$$2x\frac{dx}{dt} = 2L\frac{dL}{dt} \Rightarrow \frac{dx}{dt} = \frac{4L}{x}$$

when $x = 20$, $L = \sqrt{(20)^2+(10)^2} = 10\sqrt{5}$ ft

$$\left.\frac{dx}{dt}\right|_{x=20} = \frac{4\times 10\sqrt{5}}{20} = 2\sqrt{5} \text{ ft/sec}$$

2E7.





$$V = \left[\left(\frac{1}{2}h^2\right)2 + \frac{h}{2}\right]4 = \left(h^2 + \frac{h}{2}\right)\cdot 4$$

$\frac{dV}{dt} = 1$, find $\frac{dh}{dt}$ at $h = 1/2$

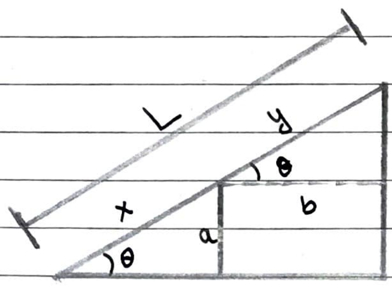
$$\frac{dV}{dt} = 8h\cdot\frac{dh}{dt} + \frac{4}{2}\frac{dh}{dt} = 1 \Leftrightarrow (8h+2)\frac{dh}{dt} = 1$$

$$\frac{dh}{dt} = \frac{1}{8h+2}, \quad \left.\frac{dh}{dt}\right|_{h=0.5} = \frac{1}{6} \text{ m/s}$$

18.01 Problem set -3

Part- II

1.a)



$$\sin\theta = \frac{a}{x} \Rightarrow x = \frac{a}{\sin\theta}$$

$$\cos\theta = \frac{b}{y} \Rightarrow y = \frac{b}{\cos\theta}$$

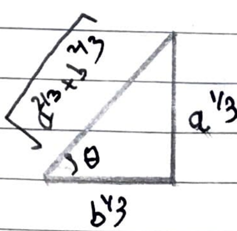
$$L = x + y = \frac{a}{\sin\theta} + \frac{b}{\cos\theta}$$

Find a θ which minimizes L.

$$\frac{dL}{d\theta} = a\left(\frac{-1}{(\sin\theta)^2}\cdot\cos\theta\right) + b\left(\frac{-1}{(\cos\theta)^2}\cdot(-\sin\theta)\right) = 0$$

$$\frac{-a\cos^3\theta + b\sin^3\theta}{(\sin\theta\cos\theta)^2} = 0 \Rightarrow a\cos^3\theta = b\sin^3\theta$$

$$\tan^3\theta = \frac{a}{b} \Rightarrow \boxed{\tan\theta = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}}$$



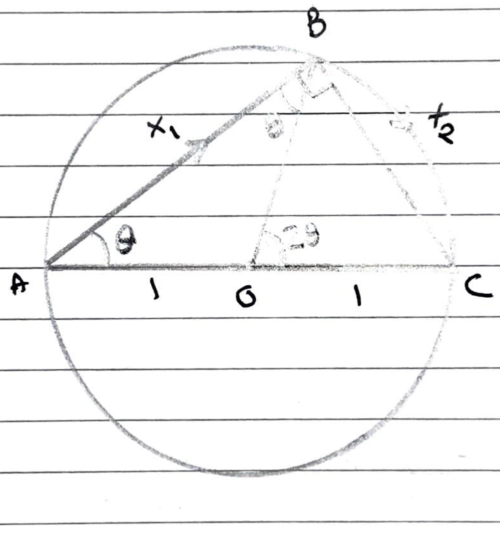
$$L = \frac{a}{\sin\theta} + \frac{b}{\cos\theta}$$

At extremes ($\theta = 0, \pi/2$), $L \to \infty$, so minimum at critical point.

$$\sin\theta = \frac{a^{1/3}}{(a^{2/3}+b^{2/3})^{1/2}}, \quad \cos\theta = \frac{b^{1/3}}{(a^{2/3}+b^{2/3})^{1/2}}$$

$$L = (a^{2/3}+b^{2/3})^{1/2}(a^{2/3}+b^{2/3}) = \underline{\underline{(a^{2/3}+b^{2/3})^{3/2}}}$$

b)



From ΔABC,

$$\cos\theta = \frac{x_1}{2} \Rightarrow \boxed{x_1 = 2\cos\theta} \quad (AB)$$

As $\angle AOB = 180° - 2\theta$, $\angle BOC = 2\theta$ //

$$x_2 = \frac{2\theta}{2\pi} \cdot 2\pi(1) = \underline{\underline{2\theta}} \Rightarrow \boxed{x_2 = 2\theta}$$

range $0 \leq \theta \leq \pi/2$

Time $= \dfrac{\text{Distance}}{\text{speed}}$

$$T = \frac{x_1}{3} + \frac{x_2}{6} = \frac{2\cos\theta}{3} + \frac{\theta}{3}$$

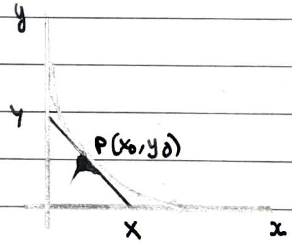
$$T' = \frac{-2\sin\theta + 1}{3} = 0 \Rightarrow -2\sin\theta + 1 = 0$$

$$\sin\theta = \frac{1}{2} \Rightarrow \boxed{\theta = \frac{\pi}{6}}$$

$$T'' = \frac{2\cos\theta}{3}$$ ← always positive in θ's range.

The man will reach the fastest when $\theta = \pi/6$ from the diameter AC.

2.



$$x^{2/3} + y^{2/3} = 1$$

Differentiating on both sides,

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} \cdot y' = 0 \Rightarrow y' = -\frac{x^{-1/3}}{y^{-1/3}} = \frac{-y^{1/3}}{x^{1/3}}$$

Tangent Line Equation:

$$y - y_0 = f'(x_0)(x - x_0)$$

$$y - y_0 = -\frac{y_0^{1/3}}{x_0^{1/3}}(x - x_0)$$

X-intercept $(y=0)$

$$+y_0 = \frac{+y_0^{1/3}}{x_0^{1/3}}(x - x_0) \Rightarrow y_0^{2/3} = \frac{x}{x_0^{1/3}} - x_0^{2/3}$$

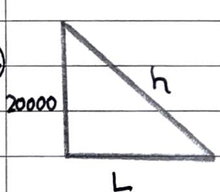
$$y_0^{2/3} + x_0^{2/3} = \frac{x}{x_0^{1/3}} \Rightarrow \boxed{X = x_0^{1/3}}$$

by symmetry $\boxed{Y = y_0^{1/3}}$

tangent length

$$X^2 + Y^2 = x_0^{2/3} + y_0^{2/3} = \underline{\underline{1}}$$

3.a)



$$L^2 + (20000)^2 = h^2$$

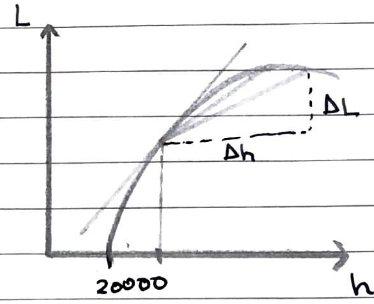
$$2L\frac{dL}{dh} = \cancel{2}h\frac{dh}{dh} \Rightarrow \boxed{\frac{dL}{dh} = \frac{h}{L}}$$

At $h_0 = 25000$, $L_0 = 15000$

$$\left.\frac{dL}{dh}\right|_{h=h_0} = \frac{25000}{15000} = \frac{5}{3} \approx 1.66\ldots7$$

Approximations were pretty close.

$L = \sqrt{h^2 - (20000)^2}$



Range of h : $[20000, \infty)$
Range of L : $[0, \infty)$

To prove $\frac{dL}{dh} \geq \frac{\Delta L}{\Delta h}$, we graph L w.r.t h

$$\underbrace{\frac{dL}{dh}}_{\text{slope of tangent}} \geq \underbrace{\frac{\Delta L}{\Delta h}}_{\text{slope of right secant.}} \quad (\Delta h > 0)$$

$L' = h/L$ ~~so~~ > 0, L is increasing,

$L'' = \left(L - \frac{h \times h}{L}\right) \times \frac{1}{L^2} = \frac{-(20000)^2}{L^3} < 0$, concave down

We can see visually that $\Delta L / \Delta h \leq \frac{dL}{dh}$ for $\Delta h > 0$.

For $|\Delta L| \leq 2|\Delta h|$,

$\frac{\Delta L}{\Delta h} \leq 2$, and $\frac{\Delta L}{\Delta h} \leq \frac{dL}{dh}$

So $\frac{dL}{dh} \geq 2$,

$\frac{dL}{dh} = \frac{h}{L} = 2$

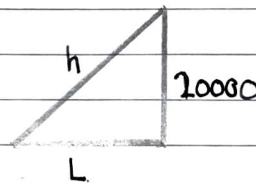
$L = h/2$

$$\frac{h^2}{4} + (20000)^2 = h^2 \Rightarrow \frac{3h^2}{4} = (20000)^2$$

$$h = \frac{40000}{\sqrt{3}} \simeq 23094 \text{ km}$$

$\therefore |\Delta L| \leq 2|\Delta h|$ is valid for h: $\underline{\underline{[20000, 23094]}}$

b) $L^2 = h^2 - (20000)^2$



$h_1 = 21000 \pm 10^{-2}$ km

$h_2 = 52000 \pm 10^{-2}$ km

$$L_1 = \sqrt{(21000)^2 - (20000)^2} = 6403.12 \text{ km}$$

$$L_2 = \sqrt{(52000)^2 - (20000)^2} = 48000 \text{ km}$$

errors : (satellite position accurate)

$L_{\text{with error}} - L_{\text{without error}}$ 1

$L_1 = 6403.12 \pm 0.03$ km

$L_2 = 48000 \pm 0.01$ km

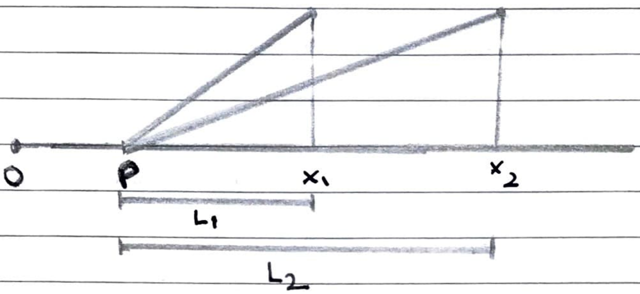
We have to specify position with respect to some unknown origin.

Now we consider different cases...

2 error $\boxed{\Delta L = \frac{dL}{dh} \cdot \underbrace{\Delta h}_{\text{error in } h}}$ linear approx.

Case-I: $x_1 < x_2$

Case-IA: $p < x_1 < x_2$



We can calculate OP using two ways -

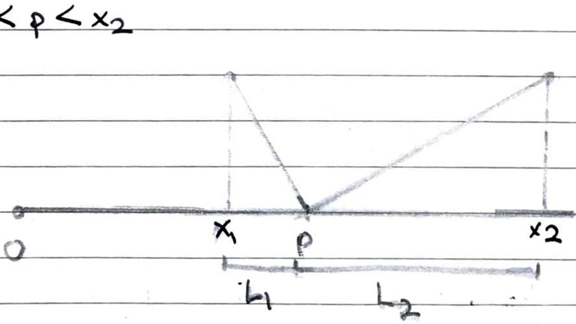
$\rightarrow OP = x_1 - L_1$

$= x_1 - 6403.12 \pm (10^{-10} + 0.03)$ //

$\rightarrow OP = x_2 - L_2$

$= x_2 - 48000 \pm (10^{-10} + 0.01)$ //

Case IB: $x_1 < p < x_2$

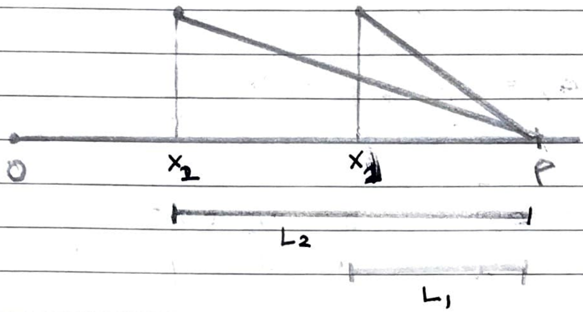


⇝ $OP = x_1 + 6403.12 \pm (10^{-10} + 0.03)$ //

⇝ $OP = x_2 - 48000 \pm (10^{-10} + 0.01)$ //

Case - II $x_1 > x_2$

Case II A



⇝ $OP = x_2 + 48000 \pm (10^{-10} + 0.01)$ km //

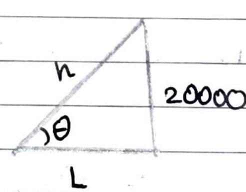
⇝ $OP = x_1 + 6403.12 \pm (10^{-10} + 0.03)$ km //

Case II B If P is between x_1 and x_2,

⇝ $OP = x_2 + 48000 \pm (10^{-10} + 0.01)$ //

⇝ $OP = x_2 - 6403.12 \pm (10^{-10} + 0.03)$ //

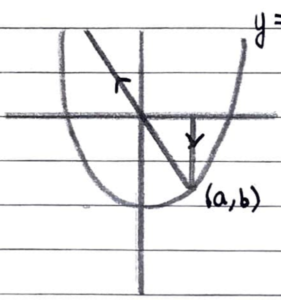
a) $\cos\theta = \dfrac{L}{h} \Rightarrow h = L\cos\theta$

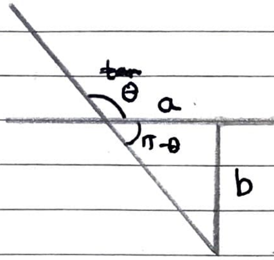


$$1 = L'\cos\theta + L(-\sin\theta)\frac{d\theta}{dh}$$

$$\boxed{L' = \sec\theta + L\tan\theta \frac{d\theta}{dh}}$$

4.





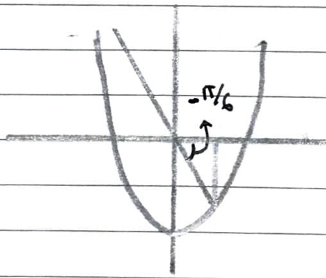
a) $\tan(\pi - \theta) = \frac{b}{a} \Rightarrow -\tan\theta = \frac{b}{a}$

$\boxed{\tan\theta = \frac{-b}{a}}$ $\qquad b = \frac{-1}{4} + a^2, \quad \frac{db}{da} = 2a$

$$\sec^2\theta \cdot \frac{d\theta}{da} = \frac{a(-db/da) - (1 \cdot (-b))}{a^2} =$$

$$= \frac{-2a^2 + a^2 - 1/4}{a^2} = -1 - \frac{1}{4a^2}$$

$$\boxed{\frac{d\theta}{da} = -\left(1 + \frac{1}{4a^2}\right)\cos^2\theta}$$



b) $\tan(\theta) = \frac{1/4 - a^2}{a}$

At $\theta = -\pi/6$,

$$\frac{-1}{\sqrt{3}} = \frac{1/4 - a2}{a} \Rightarrow a^2 - \frac{a}{\sqrt{3}} - \frac{1}{4} = 0$$

$$a = \frac{1/\sqrt{3} \pm \sqrt{1/3+1}}{2} = \frac{1/\sqrt{3} \pm 2/\sqrt{3}}{2}$$

$$a = \frac{1}{2\sqrt{3}}(1 \pm 2) = \frac{-1}{2\sqrt{3}}, \frac{\sqrt{3}}{2}$$

$$\boxed{a = \frac{-1}{2\sqrt{3}}}$$

using linear approximation to find Δa

$$\frac{d\theta}{da} = \frac{\Delta\theta}{\Delta a} \Rightarrow \Delta a = \frac{\Delta\theta}{(d\theta/da)}$$

$$\left.\frac{d\theta}{da}\right|_{\theta=-\pi/6} = -\left(1 + \frac{4 \times 3}{4}\right)\frac{3}{4} = -3$$

$$\Delta a = \frac{10^{-3}}{-3} = \pm 3.3 \times 10^{-4}$$

$$\boxed{a = \frac{-1}{2\sqrt{3}} \pm 3.3 \times 10^{-4}}$$

c) $a^2 + \tan\theta \cdot a - \frac{1}{4} = 0$

$$a = \frac{-\tan\theta \pm \sqrt{\tan^2\theta + 1}}{2} = \frac{-\tan\theta \pm \sec\theta}{2}$$

$$\boxed{a = \frac{-\sin\theta \pm 1}{2\cos\theta}}$$

—/—/—

5. a) $f(x) = x^3 - 9$; $x_0 = 2$

$f'(x) = 3x^2$

$$x_1 = x_0 - \frac{x_0^3 - 9}{3x_0^2} = x_0 - \frac{x_0}{3} + \frac{3}{x_0^2} = \frac{2}{3}x_0 + \frac{3}{x_0^2}$$

$$x_1 = \frac{2}{3}.2 + \frac{3}{4} = 2.08333$$

$$x_2 = \frac{2}{3}x_1 + \frac{3}{x_1^2} = 2.080088$$

$$x_3 = \frac{2}{3}x_2 + \frac{3}{x_2^2} = 2.080083$$

$$x_4 = \frac{2}{3}x_3 + \frac{3}{x_3^2} = 2.080083$$

n	x_n
0	2
1	2.083333
2	2.080088
3	2.080083
4	2.080083

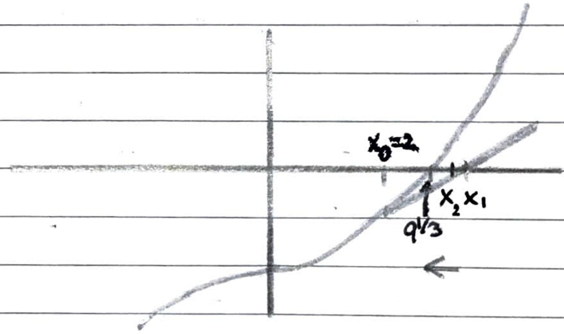
surpasses accuracy of calculator after $k = 4$

(b) (i) $k = 0$, $x_0 < 9^{1/3}$

$k > 0$, $x_k > 9^{1/3}$

(ii) $x_0 < x_1$, $k = 1$

$x_{k-1} > x_k$, $k > 1$



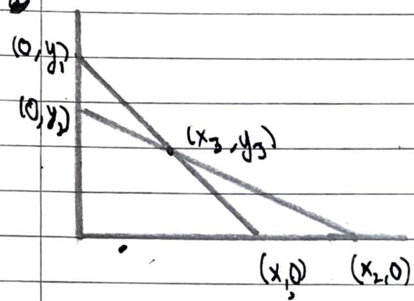
c) $$9^{1/3} \approx \left(8\left(1+\frac{1}{8}\right)\right)^{1/3} \approx 2\left(1+\frac{1}{8}\right)^{1/3}$$

$$\approx 2\left(1 + \frac{1}{24} + \frac{1}{3} \cdot \frac{-2}{3} \cdot \frac{1}{2} \cdot \frac{1}{8^2}\right) \approx 2.07986$$

$$\boxed{(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2}$$

$$\Delta e = 2.23 \times 10^{-4}$$

6-a)



$$y = mx + c$$

① $y = m_1 x_1 + c_1$

② $y = m_2 x_2 + c_2$

① $y = -\dfrac{y_1}{x_1} x + y_1$; ② $y = -\dfrac{y_2}{x_2} x + y_2$

$$-\frac{y_1}{x_1} x_3 + y_1 = -\frac{y_2}{x_2} x_3 + y_2$$

$$\left(\frac{y_2}{x_2} - \frac{y_1}{x_1}\right) x_3 = y_2 - y_1 \Rightarrow \boxed{x_3 = (y_2 - y_1)\left[\frac{1}{y_2/x_2 - y_1/x_1}\right]}$$

$$x_3 = \frac{c_2 - c_1}{m_1 - m_2}$$

$$y_3 = m_1 x_3 + c_1 = m_1\left(\frac{c_2 - c_1}{m_1 - m_2}\right) + c_1$$

$$= \frac{m_1 c_2 - \cancel{m_1 c_1} + \cancel{m_1 c_1} - m_2 c_1}{m_1 - m_2} = \frac{m_1 c_2 - m_2 c_1}{m_1 - m_2}$$

$$y_3 = \frac{-\frac{y_1}{x_1} y_2 + \frac{y_2}{x_2} y_1}{\frac{y_1}{x_1} - \frac{y_2}{x_2}} = \boxed{\frac{y_1 y_2 (x_1 - x_2)}{x_2 y_1 - x_1 y_2} = y_3}$$

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b) $x^2 + y^2 = 1$

$x_2 = x_1 + \Delta x \; ; \quad y_2 = y_1 + \Delta y$

$2x + 2yy' = 0 \Rightarrow y' = \dfrac{-x}{y}$

$y'\Big|_{x=x_1} = \dfrac{-x_1}{y_1}$ //

c) $x_3 = (y_2 - y_1)\left[\dfrac{1}{\dfrac{y_2}{x_2} - \dfrac{y_1}{x_1}}\right] = \Delta y \left[\dfrac{1}{\dfrac{y_1 + \Delta y}{x_1 + \Delta x} - \dfrac{y_1}{x_1}}\right]$

$x_3 = \dfrac{\Delta y\,(x_1(x_1 + \Delta x))}{\cancel{y_1 x_1} + \Delta y x_1 - \cancel{y_1 x_1} - y_1 \Delta x} = \underline{\underline{\dfrac{\Delta y\,(x_1(x_1 + \Delta x))}{x_1 \Delta y - y_1 \Delta x}}}$

$X = \lim\limits_{x_2 \to x_1} x_3 = \lim\limits_{\Delta x \to 0} x_3$

$= \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}\left(\dfrac{x_1^2 + x_1 \Delta x}{x_1 \dfrac{\Delta y}{\Delta x} - y_1 \dfrac{\Delta x}{\Delta x}}\right)\dfrac{\cancel{\Delta x}}{\cancel{\Delta x}} = +\dfrac{x_1}{y_1} \cdot \dfrac{x_1^2}{(x_1^2 + y_1^2)} \cdot y_1$

$\boxed{X = x_1^3} \qquad \because (x_1^2 + y_1^2 = 1)$

by symmetry

$\boxed{Y = y_1^3}$

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d) $X = x_1^3 \,, \quad Y = y_1^3$

$$X^{2/3} + Y^{2/3} = x_1^2 + y_1^2 = \underline{\underline{1}}$$

Thus the curve with the property is

$$\boxed{x^{2/3} + y^{2/3} = 1}$$