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18.01 Exam – II

1\. a) $\sin(\pi + 1/100) = \overset{0}{\sin\pi}\cdot\cos(1/100) + \cos(\pi)\sin(1/100)$

$= -\sin(1/100)$

$\sin x \approx x \quad (x \approx 0)$

$\sin(\pi + 1/100) = -\sin(1/100) = -\frac{1}{100} = \underline{\underline{-0.01}}$

b) $\sqrt{101} = \sqrt{100+1} = \sqrt{100\left(1+\frac{1}{100}\right)} = 10\sqrt{1+\frac{1}{100}}$

$= 10\left(1+\frac{1}{100}\right)^{1/2}$

$(1+x)^r \approx 1 + rx \quad (x \approx 0)$

$\Rightarrow 10\left(1+\frac{1}{200}\right) = 10 + 0.05 = \underline{\underline{10.05}}$

2\. $f(x) = \frac{4}{x} + x + 1$

discontinuities

$f(0^+) = \frac{4}{0^+} + 0^+ + 1 \to \infty$

$f(0^-) = \frac{4}{0^-} + 0^- + 1 \to -\infty$

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endpoints

$f(\infty) = \frac{4}{\infty} + \infty + 1 \Rightarrow \infty$

$f(-\infty) = \frac{4}{-\infty} - \infty + 1 \rightarrow -\infty$

At the end points, the 4/x term almost disappears and the graph can be approximated to the straight line $y = x + 1$

$f'(x) = \frac{-4}{x^2} + 1$

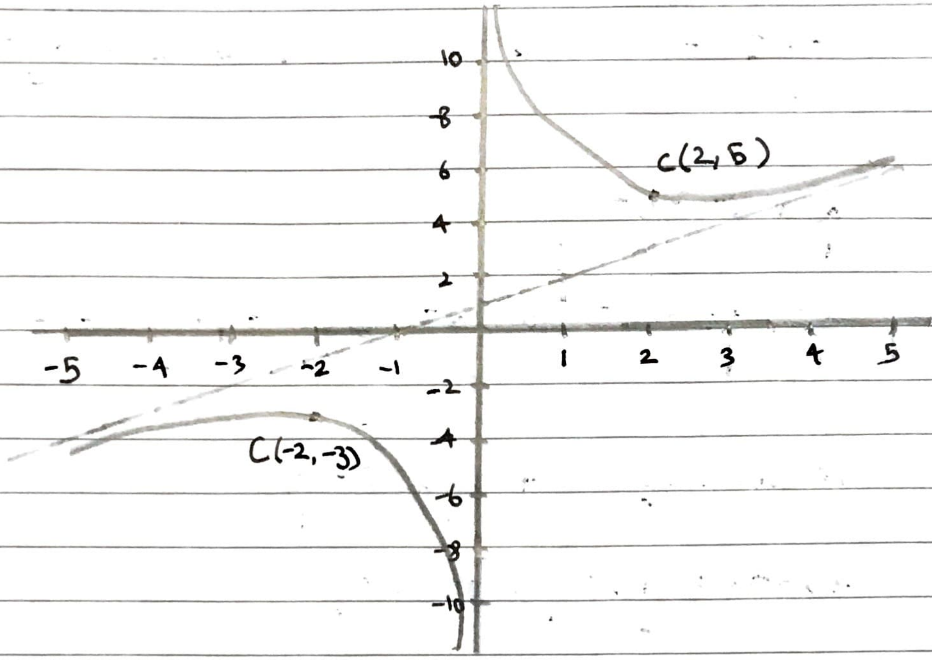
critical points

$f'(x) = \frac{-4}{x^2} + 1 = 0 \Rightarrow x = \pm 2$

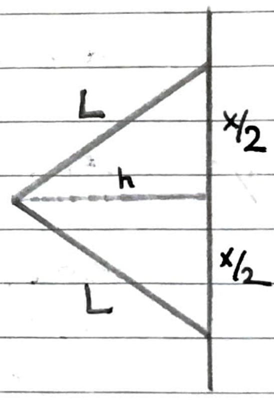
$f(2) = 5$; $f(-2) = -3$

inflection points

$f''(x) = \frac{d}{dx}\left(-4x^{-2}+1\right) = \frac{+8}{x^3} \Rightarrow$ no inflection points



3. $\frac{x^2}{4} + h^2 = L^2$



differentiating w.r.t x

$$\frac{2x}{4} + 2h\frac{dh}{dx} = 0$$

$$\frac{dh}{dx} = \frac{-x}{4h}$$ //

$$A = \frac{1}{2}xh$$

$$\frac{dA}{dx} = \frac{1}{2}\left(x\frac{dh}{dx} + h\right) = 0$$

$$\frac{-x^2}{4h} + h = 0 \Rightarrow \frac{+x^2}{4h} = h \Rightarrow \underline{\underline{x^2 = 4h^2}}$$

$$\frac{x^2}{4} + h^2 = L^2$$

$$\frac{x^2}{4} + \frac{x^2}{4} = L^2 \Rightarrow x^2 = 2L^2 \Rightarrow \boxed{x = \sqrt{2}\,L}$$

at extremes $\quad A = \frac{1}{2} \times \sqrt{L^2 - \frac{x^2}{4}}$

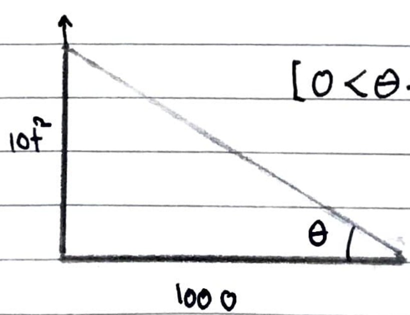
$$A(x=0) = \frac{1}{2} \times 0 \times \sqrt{L^2 - 0^2} = 0 //$$

$$A(x=2L) = \frac{1}{2} \times 2L \times \sqrt{L^2 - \frac{4L^2}{4}} = 0 //$$

So, the critical point is a maximum.

Hence, the region enclosed has the largest area for $\underline{\underline{x = \sqrt{2}\,L}}$

4.



$[0 < \theta < \pi/2]$

$$\tan\theta = \frac{10t^2}{1000}$$

differentiating w.r.t t

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{2t}{\cancel{100}_{50}}$$

$$\boxed{\frac{d\theta}{dt} = \frac{t\cos^2\theta}{50}}$$

At $t = 10$, $h = 1000$

$$\tan\theta = \frac{1000}{1000} = 1 \Rightarrow \theta = \pi/4$$

$$\frac{d\theta}{dt} = \frac{10 \times 1}{50 \cdot 2} = \frac{1}{10} \text{ rad/sec}$$

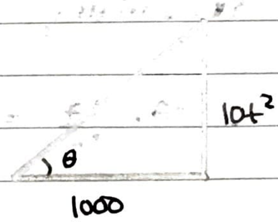
$$\frac{d\theta}{dt} = \frac{t\cos^2\theta}{50} > 0 \quad \text{for } 0 < \theta < \pi/2 \text{ and } t > 0$$

So the function is always increasing : $[0, \infty)$

$$\frac{d^2\theta}{dt^2} = \frac{d}{dt}\left(\frac{t\cos^2\theta}{50}\right) = \frac{\cos^2\theta + t(-2\sin\theta\cos\theta)\,d\theta/dt}{50}$$

$$= \frac{50\cos^2\theta - 2t^2\sin\theta\cos^3\theta}{2500} = 0$$

$$50\cos^2\theta = 2t^2 \sin\theta \cos^3\theta$$



$$t^2 \sin\theta \cos\theta = 25$$

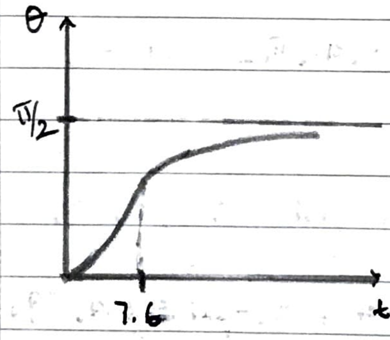
$$25 = t^2 \times \left(\frac{10t^2}{(10^6 + 10^2 t^4)^{1/2}}\right)\left(\frac{10^3}{(10^6 + 10^2 t^4)^{1/2}}\right)$$

$$10^4 t^4 = 25\left(10^6 + 10^2 t^4\right)$$

$$(10^4 - 2500)\, t^4 = 25 \times 10^6$$

$$t^4 = \frac{25 \times 1\cancel{0}0 \times 10^4}{750\cancel{0}} \Rightarrow t^4 = \frac{10000}{3}$$

$t \simeq 7.6$ sec $\leftarrow$ inflection point.



The function is concave down in the interval: $\underline{\underline{[7.6, \infty)}}$

5. a) i) $\int \cos(3x)\,dx$

guess: $\sin(3x)$. $\frac{d}{dx}\sin(3x) = 3\cos(3x)$

$$\int \cos(3x)\,dx = \frac{1}{3}\sin(3x) + c \quad //$$

ii) $\int x e^{-x^2}\,dx$

guess: e^{-x^2}. $\frac{d}{dx}e^{-x^2} = -2x e^{-x^2}$

$$\int x e^{-x^2}\,dx = -\frac{1}{2}e^{-x^2} + c \quad //$$

b) $\frac{dy}{dx} = \frac{1}{y^3}$, $y(0) = 1$

$y^3\,dy = dx \Rightarrow \int y^3\,dy = \int dx$

$\frac{1}{4}y^4 = x + c \Rightarrow y = (4x + c)^{1/4}$

$y(0) = 1$

$1 = c^4 \Rightarrow c = 1$ $\quad \boxed{y = (4x+1)^{1/4}}$

6. $f'(x) = e^{x^2}$

From MVT,

$f(x) = f(a) + f'(c)(x-a)$, $a<c<x$

Let $\cancel{\alpha}(a, x) = (0, 1)$

$f(1) = f(0) + e^{c^2}(1-0)$

$f(1) = 10 + e^{c^2}$

Range of e^{c^2} , $0<c<1$

$1 \leq e^{c^2} < e$

$10+1 < 10 + e^{c^2} < 10+e$

$\Rightarrow$ $11 < f(1) < 10+e$

$$\boxed{\begin{array}{l} A = 11 \\ B = 10+e \end{array}}$$