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18.01 Problem Set 2B

Part - I

2A 2. $f(x) = \dfrac{1}{a+bx} = (a+bx)^{-1}$

$f'(x) = -1(a+bx)^{-2} \cdot b = \dfrac{-b}{(a+bx)^2}$

$f(x) \approx f(0) + f'(0) \cdot x$

$\approx \dfrac{1}{a} - \dfrac{bx}{a^2}$

$$\boxed{\dfrac{1}{a+bx} \approx \dfrac{1}{a} - \dfrac{b}{a^2}x} \quad x \approx 0$$

by formula, $(1+x)^r \approx 1+rx$

$(a+bx)^{-1} = (a(1+\frac{b}{a}x))^{-1} = \frac{1}{a}(1+\frac{b}{a}x)^{-1}$

$\approx \frac{1}{a}(1-\frac{b}{a}x) \approx \underline{\underline{\frac{1}{a} - \frac{b}{a^2}x}} \quad x \approx 0$

3. by formula

$f(x) = \dfrac{(1+x)^{3/2}}{1+2x} = (1+x)^{3/2}(1+2x)^{-1}$

$\Rightarrow (1+x)^r = 1+rx$

$$f(x) \approx \left(1+\frac{3}{2}x\right)(1-2x)$$

$$\approx 1 - 2x + \frac{3}{2}x - \cancel{3}x^2$$

$$\boxed{f(x) \approx 1 - \frac{1}{2}x}$$

by tangent line

$$f(x) = (1+x)^{3/2}(1+2x)^{-1}, \quad f(0) = 1$$

$$f'(x) = (1+x)^{3/2} \cdot \frac{-1}{(1+2x)^2} \cdot 2 + (1+2x)^{-1} \cdot \frac{3}{2}(1+x)^{1/2}$$

$$f'(0) = -2 + \frac{3}{2} = \frac{-1}{2}$$

$$f(x) \approx f(0) + f'(0)x \approx \underline{1 - \frac{1}{2}x}$$

7. $$f(x) = \frac{\sec x}{\sqrt{1-x^2}} = (\cos x)^{-1}(1-x^2)^{-1/2}$$

$$\approx \left(1-\frac{x^2}{2}\right)^{-1}\left(1+\frac{x^2}{2}\right) \approx \left(1+\frac{x^2}{2}\right)\left(1+\frac{x^2}{2}\right)$$

$$\approx \underline{1 + x^2}$$

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11. $pv^k = c \Rightarrow p = \dfrac{c}{v^k}$

Let $v = v_0 + \Delta v$

$$p = \frac{c}{(v_0+\Delta v)^k} = \frac{c}{v_0^k} \cdot \frac{1}{(1+\Delta v/v_0)^k} = \frac{c}{v_0^k}\left(1+\frac{\Delta v}{v_0}\right)^{-k}$$

$$\approx \frac{c}{v_0^k}\left(1 - \frac{k}{v_0}(\Delta v) + \frac{k(k+1)}{2v_0^2}(\Delta v)^2\right) //$$

12.a) $\dfrac{e^x}{1-x}$ (quadratic, $x \approx 0$)

$$\frac{e^x}{1-x} = e^x \cdot (1-x)^{-1} \approx \left(1+x+\frac{x^2}{2}\right)\left(1-1(-x)+\frac{(-1)(-2)(-x)^2}{2}\right)$$

$$\approx \left(1+x+\frac{x^2}{2}\right)\left(1+x+x^2\right)$$

$$\approx 1+x+x^2+x+x^2+\frac{x^2}{2}$$

$$\boxed{\frac{e^x}{1-x} \approx 1+2x+\frac{5}{2}x^2}$$

d) $\ln(\cos x)$ (quadratic, $x \approx 0$)

$$\ln(\cos x) \approx \ln\left(1-\frac{x^2}{2}\right) \approx \frac{-x^2}{2} //$$

e) $x \ln x$ (quadratic, $x \approx 1$)

$(x = 1 + h)$

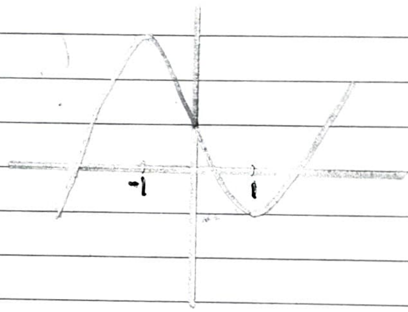
$$(1+h)\ln(1+h) \approx (1+h)\left(h - \frac{h^2}{2}\right)$$

$$\approx h - \frac{h^2}{2} + h^2 \approx h + \frac{h^2}{2}$$

Substituting $x - 1 = h$

$$x \ln x \approx (x-1)\left(1 + \frac{x-1}{2}\right) \approx \frac{(x-1)(x+1)}{2} \approx \underline{\underline{\frac{x^2-1}{2}}} \quad (x \approx 1)$$

2B 1.a) $y = x^3 - 3x + 1$



$y' = 3x^2 - 3$

$= 3(x-1)(x+1)$

$+ \quad - \quad +$

$-1 \quad 1$

→ $f(-1) = -1 + 3 + 1 = 3$, $f(1) = 1 - 3 + 1 = -1$, $f(0) = 1$

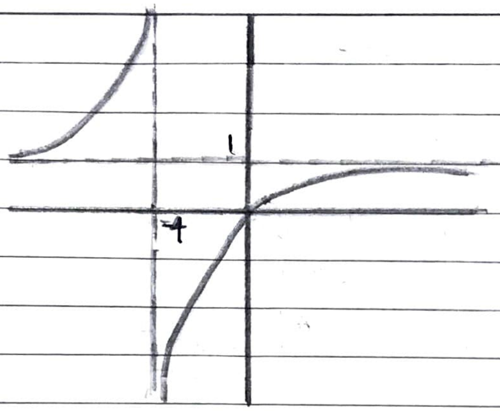
→ $f(x) \to \infty$, $x \to \infty$; $f(x) \to -\infty$, $x \to -\infty$

→ $y = 0$ has 3 solutions

e) $y = x/(x+4)$

$$y' = \frac{(x+4) - x}{(x+4)^2}$$

$$= 4/(x+4)^2$$



$+$

-4 everywhere else

→ $f(0) = 0$, $f(-4)$ undefined

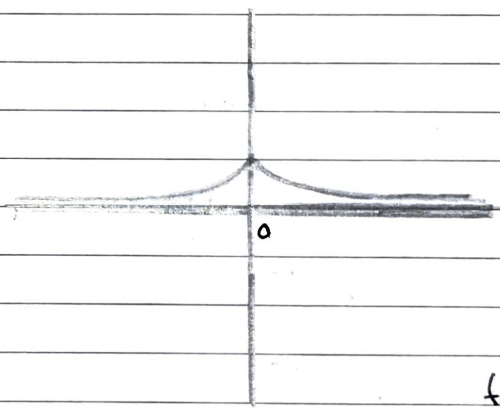
→ $f(x) \to 1,\ x \to \pm\infty$; $f(x) \to \infty,\ x \to -4^-$, $f(x) \to -\infty,\ x \to -4^+$

→ $y = 0$ has 1 solution.

b) $y = e^{-x^2}$

$y' = e^{-x^2} \cdot 2x$

$= -2x e^{-x^2}$



$+ \quad -$

0

f is even

→ $f(0) = e^0 = 1$,

→ $f(x) \to 0,\ x \to \pm\infty$

→ $y = 0$ has no solution (not counting $x = \pm\infty$)

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2B2. a) $y = x^3 - 3x + 1$

$y' = 3x^2 - 3$

$y'' = 6x$

inflection point : $\underline{\underline{x = 0}}$

e) $y = x/(x+4)$

$y' = 4/(x+4)^2 = 4 \cdot (x+4)^{-2}$

$y'' = 4 \cdot (-2) \cdot (x+4)^{-3}$

$= \dfrac{-08}{(x+4)^3} = -8/(x+4)^3$

<u>no</u> inflection points.

h) $y = e^{-x^2}$

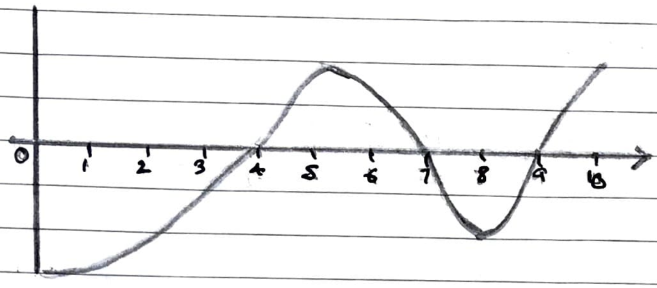
$y' = -2x\, e^{-x^2}$

$y'' = -2\left(e^{-x^2} + x \cdot (-2x e^{-x^2})\right)$

$= -2e^{-x^2}(1 - 2x^2)$

inflection points : $\underline{\underline{x = \pm 1/\sqrt{2}}}$

2B 4.



No. Cannot say for certain, the ~~ka~~ min. and max. value.
→ endpoints.

Minimum : $x = 8$ or $x = 0$
Maximum : $x = 5$ or $x = 10$.

2B 6 a) Let $y = ax^3 + bx^2 + cx + d$

$y' = 3ax^2 + 2bx + c$

one possibility : $y' = 3(x^2 - 1)$

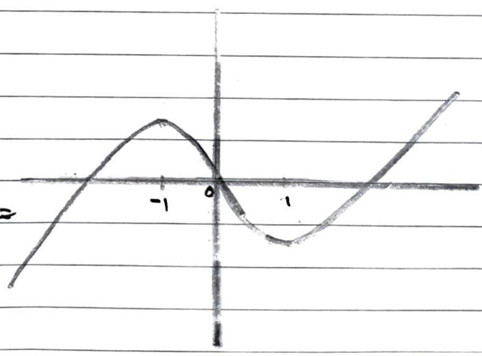
$a = \cancel{1/3}\ 1, \quad c = -3$

$\Rightarrow \boxed{y = x^3 - 3x}$

b) $y = x^3 - 3x$

$y' = 3(x^2 - 1)$

$f(0) = 0,\ f(1) = -2,\ \cancel{f(-1) =}$
$f(-1) = 2$



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28 7a) $f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

if f is increasing $\begin{cases} \Delta x > 0 \Rightarrow \Delta y > 0 \\ \Delta x < 0 \Rightarrow \Delta y < 0 \end{cases}$

in both cases $\Delta y / \Delta x > 0$.

$\therefore f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \geq 0$

b) $\frac{\Delta y}{\Delta x} > 0$ does not imply $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} > 0$

Ex. $u^2 > 0$, for $u \neq 0$ but $\lim_{u \to 0} u^2 = 0$.

Ex. $y = x^3$, $y' = 3x^2$ → increasing function for all x
but $y' = 0$ at $x = 0$.

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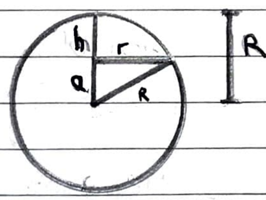
18.01 Problem Set - 2B

Part - II

1. a) $a = \sqrt{R^2 - r^2}$



$h = R - \sqrt{R^2 - r^2}$

$\boxed{\text{Area} = 2\pi R\left(R - \sqrt{R^2 - r^2}\right)}$

b) $A = 2\pi R\left(R - \sqrt{R^2 - r^2}\right)$

$= 2\pi R\left(R - R\sqrt{1 - r^2/R^2}\right)$

$= 2\pi R^2\left(1 - \sqrt{1 - \left(\frac{r}{R}\right)^2}\right)$

Linear: $x = -(r/R)^2$

$A = 2\pi R^2\left(1 - \sqrt{1+x}\right)$

$\approx 2\pi R^2\left(1 - \left(1 + \frac{1}{2}x\right)\right)$

$\approx 2\pi R^2 \cdot \frac{-1}{2}x = \pi R^2 \cdot \frac{r^2}{R^2} = \underline{\underline{\pi r^2}}$ (area of circle)

Quadratic:

$A \approx 2\pi R^2\left(1 - \left(1 + \frac{1}{2}x + \frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{1}{2}\right)x^2\right)\right)$

$\approx \pi r^2 + \frac{2\pi R^2}{8} \times \frac{r^4}{R^4} \approx \underline{\underline{\pi r^3 + \frac{\pi r^4}{4R^2}}}$

$R = \frac{3}{2}cm$, ~~$\pi = 3.14$, $r = 0.15$ cm~~

c) area $= 4\pi R^2 -$ (area removed) + (area added)

(i) area removed $= 100 \times \pi r^2$ (linear approximation)
area added $= 100 \times 2\pi r^2$

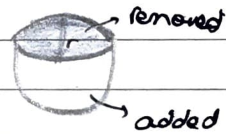
area of golf ball $= 4\pi R^2 - 100\pi r^2 + 200\pi r^2$
$= \underline{\underline{4\pi R^2 + 100\pi r^2}}$

(ii) area removed $= 100\left(\pi r^2 + \frac{\pi r^4}{4R^2}\right)$ [quadratic]

area of golf ball $= \underline{\underline{4\pi R^2 - 100\left(\pi r^2 + \frac{\pi r^4}{4R^2}\right) + 200\pi r^2}}$

(iii) area removed $= 100\left(2\pi R^2\left(1 - \sqrt{1 - \frac{r^2}{R^2}}\right)\right)$

area of golf ball $= 4\pi R^2 - 100\left(2\pi R^2\left(1 - \sqrt{1 - \frac{r^2}{R^2}}\right)\right)$
$+ 200\pi r^2$



~~Linear : A =~~

$R = 1.5$ cm, $r = 0.15$

linear : $A = 35.34$ cm^2 (99.96 % accuracy)
quadratic : $A = 35.33$ cm^2
exact : $A = 35.325$ cm^2

2. $f(x) = \dfrac{1}{(1+x^2)} = (1+x^2)^{-1}$

$f'(x) = \dfrac{-2x}{(1+x^2)^2}$

critical points

$f'(x) = 0 \cdot \boxed{x = 0}$ $\qquad$ $+ \quad \underset{0}{|} \quad -$ $\qquad$ $f(0) = 1$

$f''(x) = (f'(x))'$

$= \dfrac{(1+x^2)^2 \cdot (-2) - (-2x) \cdot (2(1+x^2) \cdot 2x)}{(1+x^2)^4}$

$= \dfrac{2(1+x^2)\left[-(1+x^2) + \overset{4}{\cancel{8}}x^2\right]}{(1+x^2)^4} = \dfrac{2\left[3x^2 - 1\right]}{(1+x^2)^3}$

inflection points

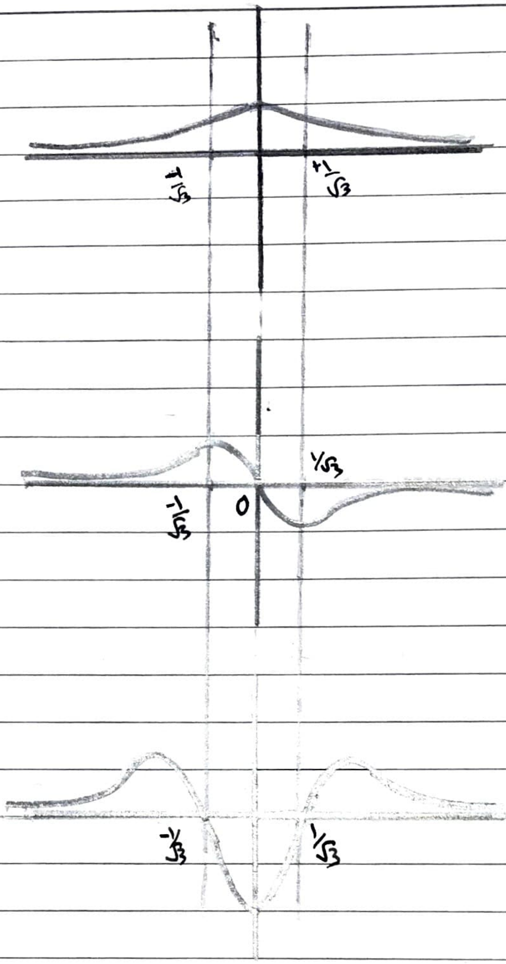
$f''(x) = 0$,

$3x^2 = 1 \Rightarrow \boxed{x = \pm 1/\sqrt{3}}$

endpoints

$x \to \pm\infty$, $f(x) \to 0$

- f is even and always positive.



$-1/\sqrt{3}$
$+1/\sqrt{3}$
$1/\sqrt{3}$
$-1/\sqrt{3}$
0
$-1/\sqrt{3}$
$1/\sqrt{3}$