18.01 Practice Exam - II

1. $f(x) = 2x^3 + 3x^2 - 12x + 1$

$f'(x) = 6x^2 + 6x - 12$

critical points

$6x^2 + 6x - 12 = 0$

$x^2 + x - 2 = 0 \Rightarrow x^2 + 2x - x - 2 = 0$

$x(x+2) - 1(x+2) = 0$

$(x-1)(x+2) = 0 \Rightarrow x = \underline{\underline{1, -2}}$

Critical Points : $(1, -6)$; $(-2, 21)$

$+ \quad - \quad +$ $f'(x)$

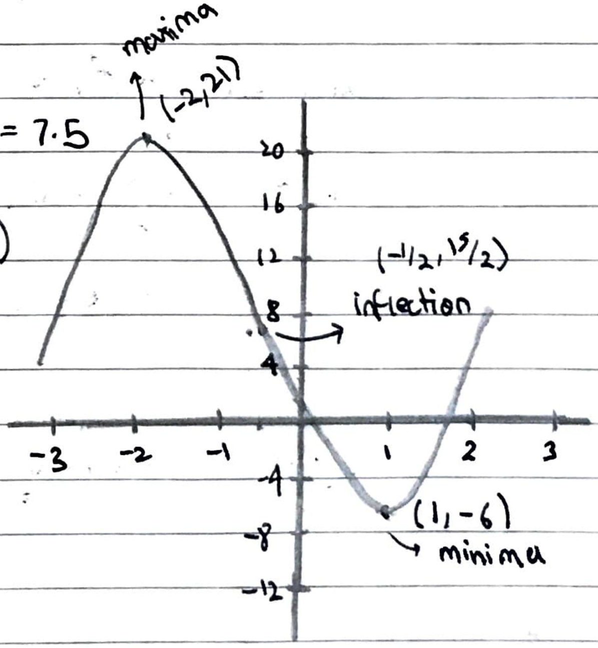
$-2 \quad 1$

inflection points

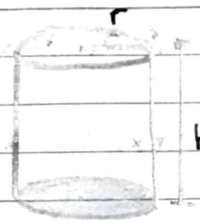
$f''(x) = 12x + 6 = 0$

$x = -1/2$, $f(-1/2) = 15/2 = 7.5$

Inflection Point : $(-1/2, 7.5)$



2. $V = \cancel{\pi} r^2 h = 64\cancel{\pi}$



$h = \dfrac{64}{r^2}$

$A = 2\pi r h + \pi r^2$

$A = 2\pi \cancel{r} \cdot \dfrac{64}{r^2} + \pi r^2 = \dfrac{2\pi \cdot 64}{r} + \pi r^2$

$\dfrac{dA}{dr} = \dfrac{-128\pi}{r^2} + 2\pi r = 0$

$\cancel{2}\cancel{\pi} r = \dfrac{\cancel{128}^{64} \cancel{\pi}}{r^2} \Rightarrow r^3 = 64 \Rightarrow \boxed{r = 4 \text{ inches}}$

is this a minimum?

extremes: $A(0^+) = \dfrac{128}{0^+} + \pi(0^+)^2 \Rightarrow \infty$

$A(\infty) = \dfrac{128}{\infty} + \pi(\infty)^2 \Rightarrow \infty$

$r = 4, \quad h = \dfrac{64}{r^2} = \dfrac{64}{16} = 4$

$\therefore$ For $r = h = 4$ inches, the can will require the least amount of metal.

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3.a) $\int e^{-3x}\,dx$

guess : e^{-3x} $\quad \frac{d}{dx} e^{-3x} = -3e^{-3x}$

$\int e^{-3x}\,dx = -\frac{1}{3} e^{-3x} + C$ //

b) $\int \cos^2 x \sin x\,dx$

$u = \cos x$, $du = -\sin x\,dx$

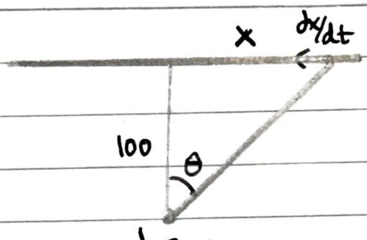
$-\int u^2\,du = -\frac{u^3}{3} + C = -\frac{\cos^3 x}{3} + C$ //

c) $\int \frac{x\,dx}{\sqrt{1-x^2}}$

$u = 1-x^2$, $du = -2x\,dx$

$\frac{-1}{2}\int u^{-1/2}\,du = -\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{1}} u^{1/2} + C = -\frac{\sqrt{1-x^2}}{\cancel{4}} + C$ //

4.



1rev in 8 min

2π rad in 8 min

thus $d\theta/dt = \underline{\underline{\dfrac{2\pi}{8} \text{ rad/min}}}$

$$\tan\theta = \frac{x}{100} \Rightarrow x = 100\tan\theta$$

$$\frac{dx}{dt} = 100\sec^2\theta \cdot \frac{d\theta}{dt} = 100\sec^2\theta \cdot \frac{\cancel{2}\cdot\pi}{4} = \underline{\underline{25\pi\sec^2\theta}}$$

At $\theta = \pi/3$

$$\frac{dx}{dt} = \frac{25\pi}{(1/2)^2} = 100\pi \text{ m/min} = 100\pi \frac{\cancel{m}\ \cancel{km}}{min} \times \frac{1km}{1000m} \times \frac{60 min}{1 hr}$$

$$\frac{dx}{dt} \simeq \underline{\underline{18.85}} \text{ km/h}$$

5. a) $e^{-x}(1+cx)^{1/2} \approx (1-x)\left(1+\dfrac{cx}{2}\right)$

$$\approx 1 + \frac{cx}{2} - x$$

For the function to be constant. $\boxed{c=2}$

b) $\frac{dx}{dt} = 2t\sqrt{1-x^2}$

$\frac{dx}{\sqrt{1-x^2}} = 2t\,dt \Rightarrow \int \frac{dx}{\sqrt{1-x^2}} = 2\int t\,dt$

$\sin^{-1}x = t^2 + c \Rightarrow x = \sin(t^2 + c)$

$x(0) = 1$

$1 = \sin(c) \Rightarrow c = \pi/2$

$\boxed{x = \sin(t^2 + \pi/2)}$

6. $\ln(1+x) < x \quad , \quad x > 0$

Let $f(x) = \ln(1+x) - x$

$f(0) = \ln 1 - 0 = 0$

$f'(x) = \frac{1}{1+x} - 1 < 0 \quad \because \left[\frac{1}{1+x} < 1\right]$

Hence f is decreasing,

$f(x) < f(0) \quad , \quad x > 0$

$\ln(1+x) - x < 0 \Rightarrow \boxed{\ln(1+x) < x}$, $x > 0$

b) I $f(x) = x^3 + x + c$

$f'(x) = 3x^2 + 1 > 0$ for any x

Thus f is alway increasing. Thus f can have atmost one zero. For the curve to have more than one zero, it should be wavy.

II Let 'a' and 'b' be zeroes of the function f(x). i.e
$f(a) = f(b) = 0$

Using MVT,

$$\frac{\overset{0}{f(b)} - \overset{0}{f(a)}}{b - a} = f'(c) \Rightarrow \underline{\underline{f'(c) = 0}}$$

but this is not possible as $f'(x) = 3x^2 + 1 > 0$ for all x.
Hence our assumption is wrong and the function cannot have two zeroes.