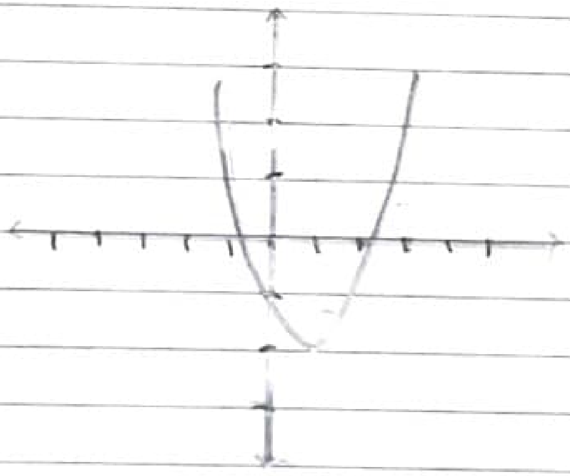
18.01 Problem Set - I

Part - I

1A 1a) $y = x^2 - 2x - 1$

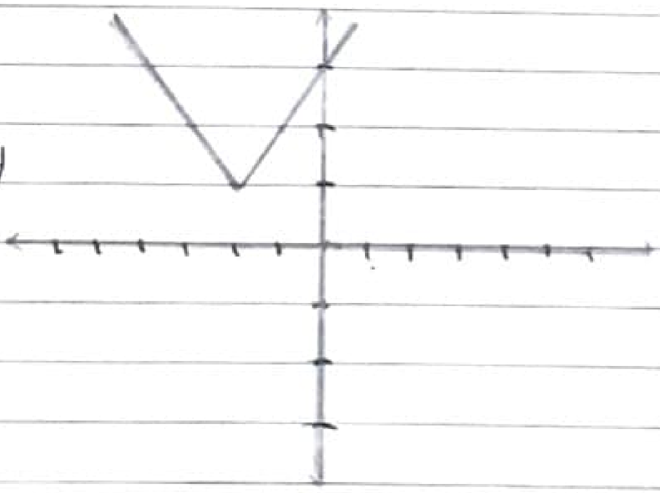
$= x^2 - 2x + 1 - 1 - 1$

$= (x-1)^2 - 2$



translate right by 1 and shift down by 2

2a) $y = 1 + |2 + x|$



translate left by 2 and shit up by 1

3a) $\dfrac{x^3 + 3x}{1 - x^4} \Rightarrow \dfrac{\text{odd}}{\text{even}} = \underline{\text{odd}}$

b) $\sin^2 x$

$\sin^2(-x) = (-\sin(x))^2 = \sin^2 x \Rightarrow \underline{\text{even}}$

e) $J_0(x^2)$

$J_0((-x)^2) = J_0(x^2) \Rightarrow \underline{\text{even}}$

6a) $\sin x + \sqrt{3}\cos x$

$= \left[\frac{1}{1+3}\sin x + \frac{\sqrt{3}}{1+3}\cos x\right] 4$ $\qquad \sin\phi = 1/4$, $\cos\phi = \sqrt{3}/4$

$= \left[\frac{1}{4}\sin x + \frac{\sqrt{3}}{4}\cos x\right] 4 = [\sin x\cos\phi + \cos x\sin\phi]\, 4$

$= 4\sin\left(x + \sin^{-1}\left(\frac{1}{4}\right)\right)$

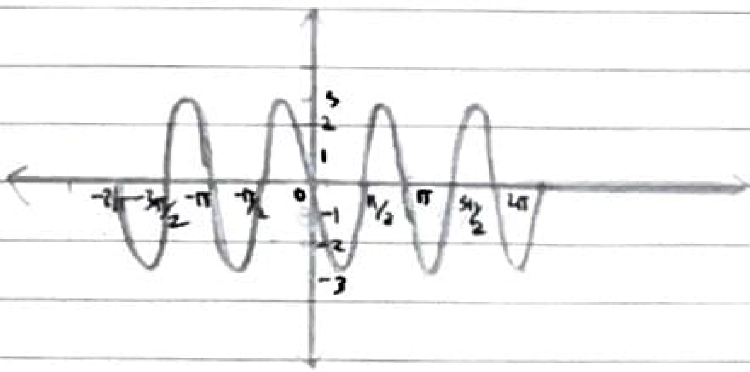
7a) $3\sin(2x-\pi)$

$3\sin(2(x-\pi/2))$

period $= \pi$

amplitude $= 3$

phase angle $= \pi/2$



1B 2a. $v = \frac{ds}{dt} = \frac{d}{dt}(bt - 16t^2) = b - 32t$

b. At maximum height $v = 0$,

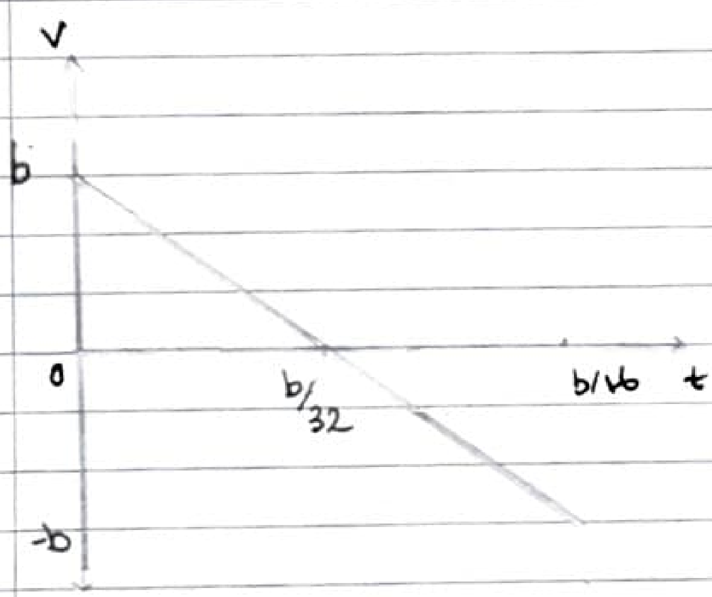
$b - 32t = 0$

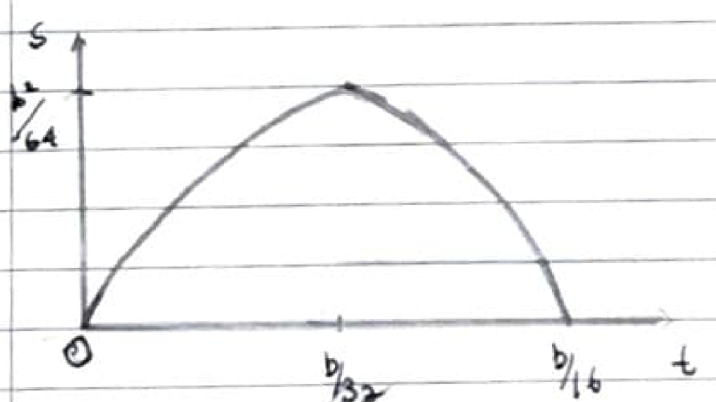
$t = \frac{b}{32}$ sec

c. $s_{max} = b \times \frac{b}{32} - 16 \times \frac{b}{32} \times \frac{b}{32} = \frac{b^2}{32} - \frac{b^2}{64}$

$s_{max} = \frac{b^2}{64}$ feet

d.





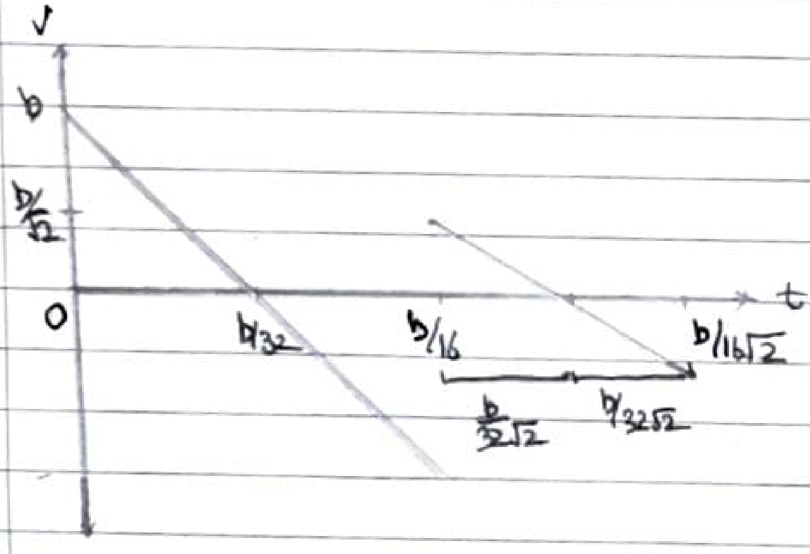
e. Let initial velocity of the second bounce be v_0

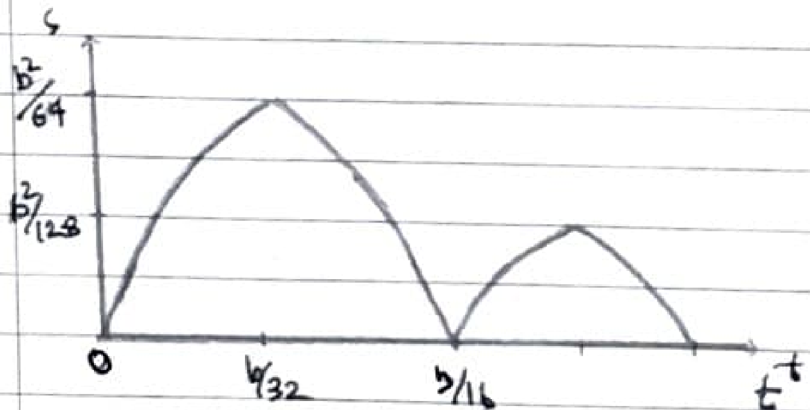
$$ {}_2S_{max} = \frac{v_0^2}{64} = \frac{1}{2}\,\frac{b^2}{64} \Rightarrow v_0 = \frac{b}{\sqrt{2}} $$

$$ s = \frac{bt}{\sqrt{2}} - 16t^2 $$

$$ v = \frac{b}{\sqrt{2}} - 32t $$

$v = 0$ when $t = \dfrac{b}{32\sqrt{2}}$ sec





b. $T = \frac{b}{16} + \frac{b}{16\sqrt{2}} + \frac{b}{32} \dots$

This ~~a~~ is a geometric progression

$u = \frac{b}{16}, \; r = \frac{1}{\sqrt{2}}$

$T = \frac{a}{1-r} = \frac{b\sqrt{2}}{16(\sqrt{2}-1)}$ sec

1C 1.a) To find - $\frac{dA}{dr}$

$A = \pi r^2 \qquad \frac{\Delta A}{\Delta r} = \frac{\pi(r+\Delta r)^2 - \pi r^2}{\Delta r}$

$= \frac{\pi(r^2 + 2r\Delta r + (\Delta r)^2) - \pi r^2}{\Delta r} = \frac{2\pi r\Delta r + (\Delta r)^2}{\Delta r}$

$= 2\pi r + \Delta r \xrightarrow[\Delta r \to 0]{} \underline{\underline{2\pi r}}$

3a. $f(x) = 1/(2x+1)$

$\frac{\Delta f}{\Delta x} = \frac{\frac{1}{2x+2\Delta x+1} - \frac{1}{2x+1}}{\Delta x} = \left[\frac{-2\Delta x}{(2x+1+2\Delta x)(2x+1)}\right]\frac{1}{\Delta x}$

$= \frac{-2}{(2x+1+2\Delta x)(2x+1)} \xrightarrow[\Delta x \to 0]{} \frac{-2}{(2x+1)^2}$

$$\boxed{f'(x) = \frac{-2}{(2x+1)^2}}$$

b. $f(x) = 2x^2 + 5x + 4$

$\frac{\Delta f}{\Delta x} = \frac{2(x+\Delta x)^2 + 5(x+\Delta x) + 4 - (2x^2 + 5x + 4)}{\Delta x}$

$= (\cancel{2x^2} + 4x\Delta x + 2(\Delta x)^2 + \cancel{5x} + 5\Delta x + 4 - (\cancel{2x^2} + \cancel{5x} + \cancel{4}))\frac{1}{\Delta x}$

$= 4x + 5 + 2\Delta x \xrightarrow[\Delta x \to 0]{} \underline{\underline{4x + 5}}$

$$\boxed{f'(x) = 4x + 5}$$

e. (a) $f'(x) = \dfrac{-2}{(2x+1)^2}$

$\underline{f'(x)=1}$: $1 = \dfrac{-2}{(2x+1)^2}$, no value of x satisfies.

$\underline{f'(x)=-1}$: $-1 = \dfrac{-2}{(2x+1)^2} \Rightarrow 2x+1 = \pm\sqrt{2}$

$\Rightarrow x = \dfrac{\pm\sqrt{2}-1}{2}$, $x_0 = \dfrac{\sqrt{2}-1}{2}$, $\dfrac{-(\sqrt{2}+1)}{2} = x_1$

$y_0 = \dfrac{1}{\sqrt{2}-1+1} = 1/\sqrt{2}$, $y_1 = \dfrac{-1}{\sqrt{2}}$ //

points are : $\left(\dfrac{\sqrt{2}-1}{2}, \dfrac{1}{\sqrt{2}}\right), \left(\dfrac{-(1+\sqrt{2})}{2}, \dfrac{-1}{\sqrt{2}}\right)$

$\underline{f'(x)=0}$: $0 = \dfrac{-2}{(2x+1)^2}$, no value of x satisfies

b) $f'(x) = 4x+5$

$\underline{f'(x)=1}$: $1 = 4x+5$, $x=-1$, $y=1$

points are $(-1, 1)$

$\underline{f(x) = -1}$: $-1 = 4x + 5$, $x = -3/2$

$$y = 2 \times \frac{9}{4} - \frac{5 \times 3}{2} + 4 = -1$$

points are $(-3/2, -1)$

$\underline{f'(x) = 0}$: $0 = 4x + 5$, $x = -5/4$

$$y = 2 \times \frac{25}{16} - \frac{25}{4} + 4 = 7/8$$

points are $(-5/4, 7/8)$

4.a $f(x) = 1/(2x+1)$ at $x = 1$

$$y - y_0 = m\,(x - x_0)$$

$$y - y_0 = \frac{-2}{(2x_0+1)^2}\,(x - x_0)$$

$(x_0, y_0) = (1, 1/3)$

$\Rightarrow$ $y - 1/3 = -2/9\,(x - 1)$ //

b. $f(x) = 2x^2 + 5x + 4$ at $x = a$

$$y - y_0 = m(x - x_0)$$

$$y - y_0 = (4x_0 + 5)(x - x_0)$$

$(x_0, y_0) = (a, 2a^2 + 5a + 4)$

$(y - 2a^2 - 5a - 4) = (4a+5)(x - a)$

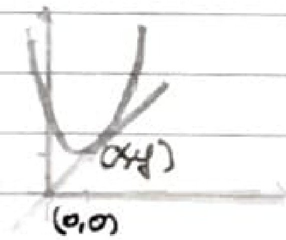
$y = (4a+5) - 4a^2 - 5a + 2a^2 + 5a + 4$

$y = (4a+5) - 2a^2 + 4$ //

5. $y = 1 + (x-1)^2$

$= 1 + x^2 - 2x + 1$

$y = x^2 - 2x + 2$



$\frac{dy}{dx} = 2x - 2$

$(y_0 - 0) = m(x_0 - 0)$

$y_0 = (2x_0 - 2)x_0$

$x_0^2 - \cancel{2x_0} + 2 = 2x_0^2 - \cancel{2x_0}$

$x_0^2 = 2$

$x_0 = \pm\sqrt{2}$ //

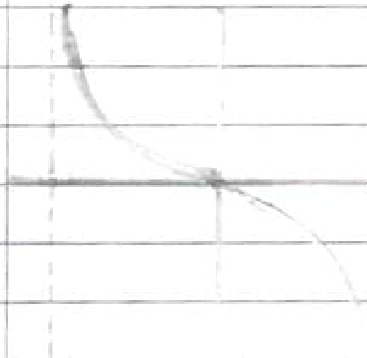
So $(x,y) \Rightarrow (\sqrt{2}, 4 - 2\sqrt{2}), (-\sqrt{2}, 4 + 2\sqrt{2})$

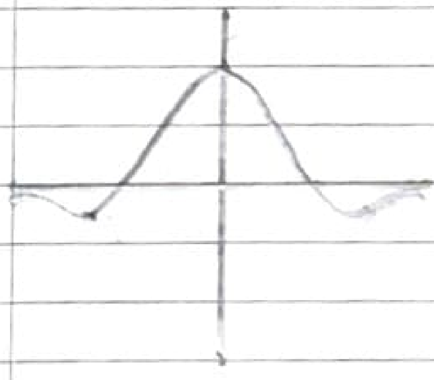
Tangent lines:

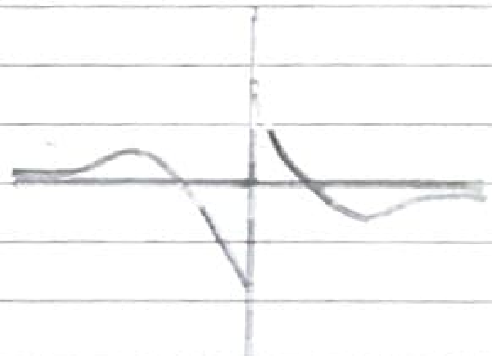
$y = (2\sqrt{2} - 2)x$

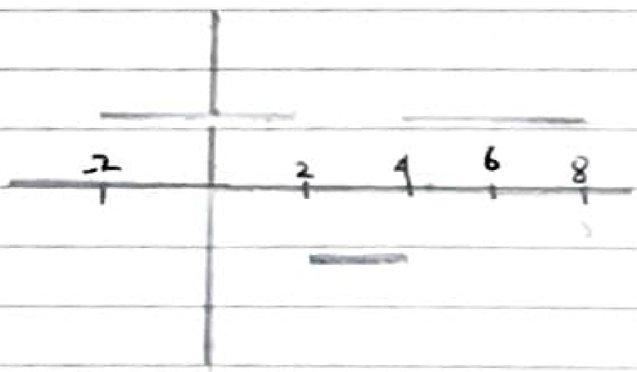
$y = -(2\sqrt{2} + 2)x$

6.a.





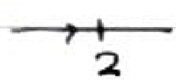




10.b) $\lim\limits_{x \to 2} \frac{4x}{x+1} = \frac{8}{3}$ //

d) $\lim\limits_{x \to -2} \frac{4x^2}{x+2}$ does not exist (undefined)

e) $\lim\limits_{x \to 2^-} \frac{4x^2}{2-x}$ does not exist $(+\infty)$



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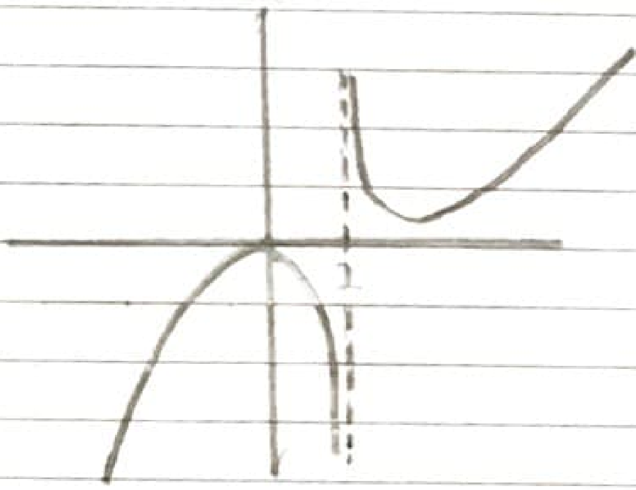
f) $\lim_{x\to\infty} \frac{4x^2}{x-2}$ does not exist $(+\infty)$

g) $\lim_{x\to\infty} \frac{4x^2}{x-2} - 4x = \cancel{\lim_{x\to\infty}} \frac{4x(x-\cancel{2})}{(x-2)}$ ~~does not exist (+∞)~~

$$= \frac{4x^2 - 4x^2 + 8x}{x-2} = \frac{8x}{x-2} = \frac{8}{1-\frac{2}{x}}$$

$$\lim_{x\to\infty} \frac{8}{1-\frac{2}{x}} = \underline{\underline{8}}$$

4a.



1c2 $f(x) = (x-a)g(x)$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\cancel{(x-a)}g(x) - 0}{\cancel{(x-a)}}$$

$$= g(a)$$

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1D3 a) $\frac{x-2}{x^2-4}$ discontinuity at $x = 2, -2$

$x=2$: $f(2) \to$ undefined

$$\lim_{x\to 2^+} \frac{x-2}{x^2-4} = \lim_{x\to 2^+} \frac{\cancel{x-2}}{\cancel{(x-2)}(x+2)} = \lim_{x\to 2^+} \frac{1}{x+2} = \frac{1}{4}//$$

$$\lim_{x\to 2^-} \frac{1}{x+2} = \frac{1}{4}// \Rightarrow \lim_{x\to 2} f(x) = 1/4$$

Removable discontinuity.

$x=-2$ $f(-2) \to$ undefined

$$\lim_{x\to -2^-} \frac{1}{x+2} = -\infty, \lim_{x\to -2^+} \frac{1}{x+2} = +\infty$$

Infinite discontinuity.

d) $f(x) = \begin{cases} x+a, & x>0 \\ a-x, & x<0 \end{cases}$

$x=0$: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x+a = a//$ limit exists

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} a-x = a//$$

$f(a) \to$ undefined

Removable discontinuity.

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e) $f'(x) = \begin{cases} 1 & , x > 0 \\ -1 & , x < 0 \end{cases}$

Jump discontinuity at $x = 0$.

6.a) $f(x) = \begin{cases} x^2 + 4x + 1 & , x \geqslant 0 \\ ax + b & , x < 0 \end{cases}$

for $f(x)$ to be ~~f~~ differentiable, (at $x=0$)

$$\lim_{\Delta x \to 0^-} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \begin{cases} 2x + 4 & , x \geqslant 0 \\ a & , x < 0 \end{cases}$$

$(x=0)$: $2 \cdot (0) + 4 = a \Rightarrow \boxed{a = 4}$

$f(x)$ must also be continuous

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 + 4x + 1 = 1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} 4x + b = b$$

$f(0) = 1$

$\Rightarrow \boxed{b = 1}$

8a) $f(x) = \begin{cases} ax+b, & x > 0 \\ \sin 2x, & x \leqslant 0 \end{cases}$

continuous

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$$

$$a(0) + b = \sin(2(0))$$

$$\boxed{b=0}$$

not differentiable

$\lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x}$ does not exist

$$\lim_{\Delta x\to 0^-} \frac{\Delta f}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{\sin(0+\Delta x) - 0}{\Delta x}$$

$$= \lim_{\Delta x\to 0^-} \frac{\sin(2\Delta x)}{2\Delta x}.2 \;\; > \left(\lim_{\Delta x\to 0^-} \frac{\sin(2\Delta x)}{2\Delta x}\right) 2 = 1\cdot 2 = 2 //$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{\Delta x\to 0^+} \frac{\Delta f}{\Delta x} = a //$$

for the f^n to be not differentiable $\boxed{a \neq 2}$

IE 1.a) $\frac{d}{dx}(x^{10} + 3x^5 + 2x^3 + 4) = 10x^9 + 15x^4 + 6x^2 /$

c) $\frac{d}{dx}(x/_2 + \pi^3) = 1/_2 //$

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2.b) $x^6 + 5x^5 + 4x^3$

$\frac{d}{dx} x^7 = 7x^6$

$F(x) = \frac{x^7}{7} + \frac{5}{6}x^6 + x^4$

$\frac{d}{dx} \frac{x^7}{7} = x^6$

3 $y = x^3 + x^2 - x + 2$

$y' = 3x^2 + 2x - 1$ → slope of tangent at x

$3x^2 + 2x - 1 = 0$
$3x^2 + 3x - x - 1 = 0$
$3x(x+1) - 1(x+1) = 0$
$(3x-1)(x+1) = 0 \Rightarrow x = 1/3, -1$

$x = -1, \ y = -1 + 1 + 1 + 2 = 3$
$x = 1/3, \ y = 49/27$

Slope is 0 (horizontal) at points $(-1, 3)$ and $(1/3, 49/27)$

4. a) $f(x) = \begin{cases} ax^2 + bx + 4, & x \leq 0 \\ 5x^5 + 3x^4 + 7x^2 + 8x + 4, & x < 0 \end{cases}$

for f(x) to be differentiable

$\frac{d}{dx} f(x) \Big|_{x \to 0^+} = \frac{d}{dx} f(x) \Big|_{x \to 0^-}$

& $8 = 2a(0) + b \Rightarrow \boxed{b = 8}$

'a' can be anything

5a) $f(x) = \frac{x}{1+x}$

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

$$\left(\frac{x}{1+x}\right)' = \frac{(1+x)x' - x(1+x)'}{(1+x)^2} = \frac{1}{(1+x)^2} //$$

1J 1.e) $f(x) = \frac{\sin x}{x}$

$$f'(x) = \frac{-\sin(x)(1) + x(\cos x)}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

2. $$\lim_{x \to \pi/2} \frac{\cos x}{x - \pi/2} = \lim_{x - \pi/2 \to 0} \frac{\cos(x) - \cos(\pi/2)}{x - \pi/2}$$

$$= \left(\cos\left(\frac{\pi}{2}\right)\right)' = -\sin(\pi/2) = -1 //$$

1F 1.a) I $(x^2+2)^2 = x^4 + 4 + 4x^2$

$$\frac{d}{dx}(x^4 + 4 + 4x^2) = 4x^3 + 8x //$$

II $$\frac{d}{dx}(x^2+2)^2 = 2(x^2+2) \cdot 2x = 4x^3 + 8x //$$

b) prefer II

$$\frac{d}{dx}(x^2+2)^{100} = 100(x^2+2)^{99} \cdot 2x = 200x(x^2+2)^{99} //$$

2. $f(x) = x^{10}(x^2+1)^{10}$

$(uv)' = uv' + u'v$

$f'(x) = x^{10}[10(x^2+1)^9\, 2x) + 10x^9(x^2+1)^{10}$

$= 20x^{11}(x^2+1)^9 + 10x^9(x^2+1)^{10}$

$= 10x^9(x^2+1)^9(2x^2+x^2+1)$

$= 10x^9(x^2+1)^9(3x^2+1)$

6. I Let $f(x)$ be even.

$f(-x) = f(x)$

differentiating on both sides,

$(f(-x))' = f'(x)$

$f'(-x) \cdot -1 = f'(x)$

$\Rightarrow f'(-x) = -f'(x) \Rightarrow f'(x)$ is odd.

II Let $f(x)$ be odd.

$f(-x) = -f(x)$

differentiating on both sides,

$(f(-x))' = -f'(x)$

$-f'(-x) = -f'(x)$

$\Rightarrow f'(-x) = f'(x) \Rightarrow f'(x)$ is even.

$$\frac{d}{dx}(uv)' = u'v + uv'$$

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7. b) $m = \dfrac{m_0}{\sqrt{1-v^2/c^2}}$, $\dfrac{dm}{dv} = ?$ $\qquad \left(\dfrac{u}{v}\right)' = \dfrac{vu' - uv'}{v^2}$

Let $y = \sqrt{x}$,

$$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} = \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{(\sqrt{x+\Delta x} + \sqrt{x})}{(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$= \frac{\cancel{\Delta x}}{\cancel{\Delta x}} \cdot \frac{1}{(\sqrt{x+\Delta x} + \sqrt{x})} \xrightarrow[\Delta x \to 0]{} \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$\frac{dm}{dv} = \frac{\cancel{\sqrt{1-v^2/c^2} \cdot 0}^{\,0} - m_0 \frac{d}{dv}(\sqrt{1-v^2/c^2})}{(1-v^2/c^2)}$$

$$= \frac{-m_0}{\cancel{2}\sqrt{1-v^2/c^2}} \cdot \frac{\cancel{2}v}{c^2} \cdot \frac{1}{(1-v^2/c^2)} = \frac{-m_0 v}{c^2(1-v^2/c^2)^{3/2}}$$ //

d) $Q = \dfrac{at}{(1+bt^2)^3}$, $\dfrac{dQ}{dt} = ?$

$$\frac{dQ}{dt} = \frac{(1+bt^2)^3 \cdot a - at \cdot 3(1+bt^2)^2 \cdot 2bt}{(1+bt^2)^6}$$

$$= \frac{a\cancel{(1+bt^2)^2}(1+bt^2 - 6bt^2)}{(1+bt^2)^{6-2=4}} = \frac{a(1-5bt^2)}{(1+bt^2)^4}$$ //

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1J 1. a) $\frac{d}{dx}(\sin(5x^2)) = \cos(5x^2) \cdot 10x$

$= 10x\cos(5x^2)$ //

b) $(\sin^2(3x))' = 2(\sin(3x)) \cdot \cos(3x) \cdot 3$

$= 6\sin(3x)\cos(3x)$ //

m) (i) $\frac{d}{dx}(\cos(2x)) = -\sin(2x) \cdot 2$

$= -2\sin(2x)$

$= -2(2\sin x\cos x)$

$= -4\sin x\cos x$ //

$\sin(a+b)$
$= \sin a\cos b +$
$\cos a\sin b$

(ii) $\frac{d}{dx}(\cos^2 x - \sin^2 x) = 2\cos x(-\sin x) - 2\sin x\cos x$

$= -4\sin x\cos x$ //

(iii) $\frac{d}{dx}(2\cos^2 x) = 4\cos x(-\sin x) = -4\sin x\cos x$ //

All the f^n are ~~equal~~ not equal ((iii) is different)

$\cos(a+b)$
$= \cos a\cos b$
$- \sin a\sin b$

$\cos(2x) = \cos x \cdot \cos x - \sin x \cdot \sin x$

$= \cos^2 x - \sin^2 x$ ✓

$= \cos^2 x - (1 - \cos^2 x)$

$= 2\cos^2 x - 1$ X

16 + b) $y = \frac{x}{x+5}$

$y' = \frac{(x+5)\cdot 1 - x \cdot 1}{(x+5)^2} = \frac{5}{(x+5)^2}$

$$y'' = \frac{(x+5)^2 \cdot 0 - 5(2(x+5))}{(x+5)^4}$$

$$= \frac{-10}{(x+5)^3} //$$

5\. a) $y = u \cdot v$

$y' = (uv)' = u'v + uv'$

$y'' = (u'v + uv')' = (u'v)' + (uv')'$

$= u''v + u'v' + u'v' + uv''$

$= u''v + 2u'v' + uv''$

$y''' = (u''v + 2u'v' + uv'')' = (u''v)' + 2(u'v')' + (uv'')'$

$= u'''v + u''v' + 2(u''v' + u'v'') + u'v''' + uv'''$

$= u'''v + 3u''v + 3u'v'' + uv''' //$

b) above answes match with Leibniz' formula.

$y = \underbrace{x^p}_{u}\underbrace{(1+x)^q}_{v}$

$y^{p+q} = u^{(p+q)}v + \binom{n}{1}u^{(p+q)-1}v^{(1)} + \ldots + uv^{(p+q)}$

$(x^p)^{(p+k)} = 0$ for all $k > 0$

and $((1+x)^q)^{q+k} = 0$ for all $k > 0$

So all the terms cancel except $\binom{p+q}{q} u^p q v^q$

$$y^{(p+q)} = 0 + 0 + \dots + \binom{p+q}{q}(x^p)^{(p)}\left((1+x)^q\right)^{(q)} + 0 + \dots + 0$$

$$= \binom{p+q}{q} p! \cdot q!$$

$$= \frac{(p+q)\cdot(p+q-1)\dots(p+1)\cdot p!}{\cancel{q!}} \cdot \cancel{q!}$$

$$= (p+q)(p+q-1)\dots(p+1)\cdot p\cdot(p-1)\dots 1 = (p+q)!$$

$$\Rightarrow \boxed{y^{(p+q)} = (p+q)!}$$

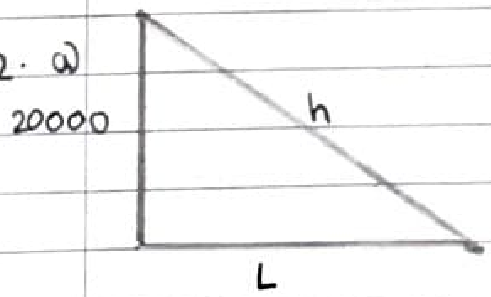
18.01 Problem Set - I

Part - II

1. $$\frac{x-1}{x+1} = \frac{(x-1)(x-1)}{(x+1)(x-1)} = \frac{(x-1)^2}{x^2-1}$$

$$= \frac{x^2-2x+1}{x^2-1} = \underbrace{\frac{x^2+1}{x^2-1}}_{\text{even}} + \underbrace{\frac{(-2x)}{x^2-1}}_{\text{odd}}$$

2. a)



$L = \sqrt{h^2-(20000)^2}$

$h_0 = 25000$ km

$L_0 = 15000$ km

$\Delta h = 1$ $\quad \Delta L = 1.67 \Rightarrow \frac{\Delta L}{\Delta h} = 1.67$

$\Delta h = 10^{-1}$ $\quad \Delta L = 0.167 \Rightarrow \frac{\Delta L}{\Delta h} = 1.67$

$\Delta h = 10^{-2}$ $\quad \Delta L = 0.0167 \Rightarrow \frac{\Delta L}{\Delta h} = 1.67$

$$|L - L_0| = |\Delta L| \leq \underline{2}\ |\Delta h|$$

b) $h_0 = 20001$ km

$L_0 = 200.0025$ km

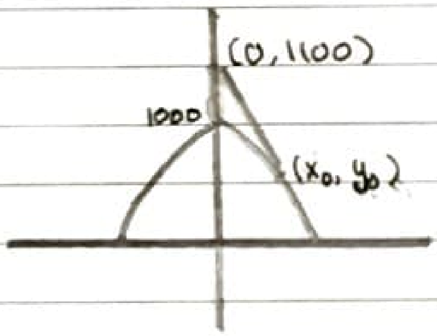
$\Delta h = 1$: $\Delta L = 282.85$ km $-L_0$ $\Rightarrow$ $\frac{\Delta L}{\Delta h} = \frac{282.85 - L_0}{1} = 82.85$ //

$\Delta h = 10^{-1}$: $\Delta L = 209.76$ km $-L_0$ $\Rightarrow$ $\frac{\Delta L}{\Delta h} = \frac{209.76 - L_0}{10^{-1}} = 97.6$ //

$\Delta h = 10^{-2}$: $\Delta L = 20.00$ km $-L_0$ $\Rightarrow$ $\frac{\Delta L}{\Delta h} = \frac{2000.0 - L_0}{10^{-2}} = 99.8$ //

estimated less acurately than part (a)

3.



tangent line

$(y - 1100) = m\,(x - 0)$

$m = f'(x) = \frac{d}{dx}(1000 - x^2) = -2x$

$y_0 - 1100 = -2x_0^2$

$1000 - x_0^2 - 1100 = -2x_0^2$

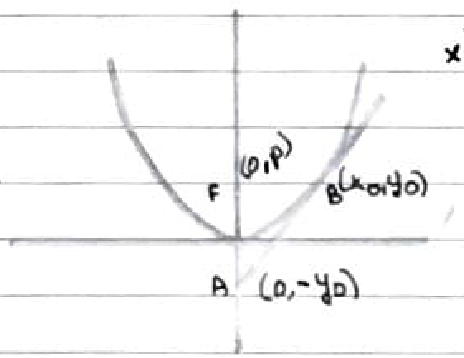
$x_0^2 = 100$

$x_0 = 10$

$y_0 = 1000 - 100 = \underline{\underline{900 \text{ feet}}}$

4.(a) tangent line:

$$y - y_0 = m(x - x_0)$$



$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{4p}\right) = \frac{2x}{4p} = \frac{x}{2p}$$

$$(y - y_0) = \frac{x_0}{2p}(x - x_0)$$

Let $(0, y)$ be the y-intercept

$$y - y_0 = -\frac{x_0^2}{2p} \Rightarrow y = \frac{x_0^2}{4p} - \frac{x_0^2}{2p} = -\frac{x_0^2}{4p} = -y_0$$

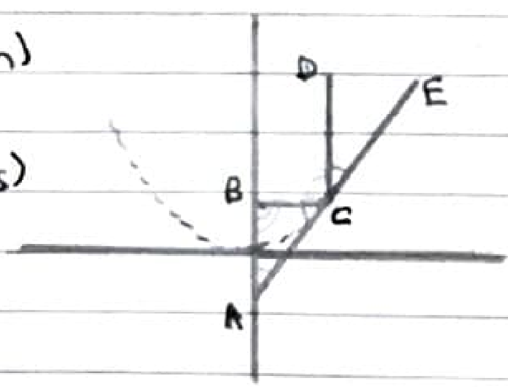
(b) $FA = p + y_0$

$$FB = \sqrt{x_0^2 + (y_0 - p)^2} = \sqrt{(4py_0) + y_0^2 - 2y_0p + p^2}$$

$$= \sqrt{y_0^2 + 2py_0 + p^2} = \sqrt{(y_0 + p)^2} = p + y_0$$

$\because FA = FB$, the Δ is isosceles.

(c) $\angle DCE = \angle ACB$ (reflection)
from (b)
$\angle ACB = \angle BAC$ (isosceles)



$\Rightarrow \angle BAC = \angle DCE$

$\Rightarrow$ AB ∥ CD (corresponding angles)
↳ y axis

So, the reflected ray points upward.

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5. a) $V = (10-t)^2/5 \text{ L}$

$$V_{avg} = \frac{V(5) - V(0)}{5 - 0} = \frac{5-20}{5} = \underline{\underline{-3 \text{ L/min}}}$$

b) $$V = \frac{100}{5} + \frac{t^2}{5} - \frac{20t}{5} = 20 + \frac{t^2}{5} - 4t$$

$$\frac{dV}{dt} = \frac{d}{dt}\left(20 + \frac{t^2}{5} - 4t\right) = 0 + \frac{2t}{5} - 4 = \underline{\frac{2t}{5} - 4}$$

$$\left(\frac{dV}{dt}\right)_{t=5} = 2-4 = \underline{\underline{-2 \text{ L/min}}}$$

6. 19d) $$\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right) = \lim_{1/x \to 0} \frac{\sin(1/x)}{1/x} = 1 //$$

f) $$\lim_{x \to 0} \frac{\sin^2 x}{3x^2} = \frac{1}{3} \lim_{x \to 0} \frac{(\sin x)}{x} \frac{(\sin x)}{x} = \frac{1}{3} //$$

g) $$\lim_{x \to 0} \frac{\sin(2x)}{\sin(3x)} \cdot \frac{2x}{2x} \cdot \frac{3x}{3x} = \lim_{x \to 0} \frac{\sin(2x)}{2x} \cdot \frac{2}{3} \cdot \lim_{x \to 0} \frac{1}{\frac{\sin(3x)}{(3x)}}$$

$$= \frac{2}{3} //$$

20. c) $$\lim_{x \to 0} \frac{x^2}{1-\cos^2 x} = \lim_{x \to 0} \frac{x^2}{\sin^2 x} = 1 //$$

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g) $\lim_{x\to 0} \frac{3x^2+4x}{\sin 2x} = \lim_{x\to 0} \frac{3x+4}{\left(\frac{\sin 2x}{2x}\right)2} = \frac{4}{2} = 2$ //

22.a)

θ	$f(\theta)$
0.1	0.4995
0.01	0.4999..
0.001	0.499999..
0.0001	~0.5

not calculating '-'ve values b/c function is even

limit approaches 0.5

22. b) $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} \cdot \frac{1+\cos\theta}{1+\cos\theta}$

$= \lim_{\theta\to 0} \frac{1-\cos^2\theta}{\theta^2} \cdot \frac{1}{1+\cos\theta}$

$= \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta^2} \cdot \frac{1}{1+\cos\theta} = 1 \cdot \frac{1}{2} = \frac{1}{2}$ //

7.a) u, v, w

$D(uvw) = D((uv)w) = (uv)'w + w'uv$
$= (u'v + uv')w + w'uv$
$= u'vw + uv'w + uvw'$
$= D(u)\cdot vw + u\cdot D(v)\cdot w + uv\cdot D(w)$

b) guess : $D(u_1 u_2 \cdots u_n) = u_1' u_2 \cdots u_n + u_1 u_2' \cdots u_n + \cdots + u_1 u_2 \cdots u_n'$

$P(n) : D(u_1 \cdots u_n) = u_1' u_2 \cdots u_n + \cdots + u_1 u_2 \cdots u_n'$

$P(n+1) : D(\underline{u_1 \cdots u_n} u_{n+1}) = (u_1 u_2 \cdots u_n)' u_{n+1} + (u_1 \cdots u_n) u_{n+1}'$

$= u_1' u_2 \cdots u_n u_{n+1} + \cdots + u_1 u_2 \cdots u_n' u_{n+1} + u_1 \cdots u_n u_{n+1}'$

So it is true for product of $n+1$ function.
Hence proved by induction