

ECON G6410 Problem Set 10

1. (a) Every subset of X is invariant under f .

Every point in X is a fixed point. For all $x \in X$, the corresponding stable set is $W(x) = \{x\}$.

- (b) The subsets of X that are invariant under f take the form of

$$S = T \cup (-T),$$

where $T \subset \mathbb{R}$.

The system has one fixed point $x = 0$; its stable set is $W(0) = \{0\}$.

- (c) The subsets of X that are invariant under f take the form of

$$S = T \cup \left(\bigcup_{x \in \mathbb{N}} \{t + x : t \in T\} \right),$$

where $T \subset \mathbb{R}$.

The system has no stable point.

- (d) The subsets of X that are invariant under f take the form of

$$S = T \cup \left(\bigcup_{x \in \mathbb{N}} \left\{ \frac{t}{2^x} : t \in T \right\} \right),$$

where $T \subset \mathbb{R}$.

The system has one fixed point $x = 0$; its stable set is $W(0) = \mathbb{R}$. This is because for any $x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} f^n(x) = \lim_{n \rightarrow \infty} \frac{x}{2^n} = 0.$$

- (e) The subsets of X that are invariant under f take the form of

$$S = T \cup \left(\bigcup_{x \in \mathbb{N}} \{1.2^x t : t \in T\} \right),$$

where $T \subset \mathbb{R}_+$.

The system has one fixed point $x = 0$; its stable set is $W(0) = \{0\}$.

- (f) The subsets of X that are invariant under f take the form of

$$S = T \cup \left(\bigcup_{x \in \mathbb{N}} \{f^x(t) : t \in T\} \right),$$

where $T \subset \mathbb{R}_+$.

Setting $f(x) = x$ gives two solutions $x = 0$ and $x = 1$, which are the fixed points of the system.

To find their stable sets, note that $f(x) > x$ for $x \in (0, 1)$ and $f(x) < x$ for $x \in (1, \infty)$, so

$$\lim_{n \rightarrow \infty} f^n(x) = 1$$

for any $x \in (0, \infty)$. Therefore, $W(1) = (0, \infty)$ and $W(0) = \{0\}$.

2. (a)

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$
$$C = \begin{pmatrix} -2 & -1 & 3.5 \\ -1 & 0 & 1.5 \\ -4 & -1 & 5.5 \end{pmatrix}.$$

(b) Since B is invertible,

$$A = B^{-1}C$$
$$= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} C$$
$$= \begin{pmatrix} -1 & -1 & 2 \\ -1 & 0 & 1.5 \\ -3 & -1 & 4 \end{pmatrix}.$$

(c) Set $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} -1 - \lambda & -1 & 2 \\ -1 & -\lambda & 1.5 \\ -3 & -1 & 4 - \lambda \end{vmatrix} = 0$$
$$2\lambda^3 - 6\lambda^2 + 5\lambda - 2 = 0$$
$$(\lambda - 2)(2\lambda^2 - 2\lambda + 1) = 0$$
$$\lambda = 2 \text{ or } \frac{1+i}{2} \text{ or } \frac{1-i}{2}.$$

(d) First find the eigenvectors.

For $\lambda_1 = 2$:

$$(A - 2I)z = 0$$
$$\begin{pmatrix} -3 & -1 & 2 \\ -1 & -2 & 1.5 \\ -3 & -1 & 2 \end{pmatrix} z = 0,$$

which implies

$$\begin{aligned} -3z_1 - z_2 + 2z_3 &= 0 \\ -z_1 - 2z_2 + 1.5z_3 &= 0. \end{aligned}$$

Solve the system and get an eigenvector

$$z = (1, 1, 2)'.$$

For $\lambda_2 = \frac{1+i}{2}$: for the real part,

$$\begin{aligned} &\left(A - \left(\frac{1+i}{2}\right)I\right)(u + iv) = 0 \\ &\begin{pmatrix} -1.5 - 0.5i & -1 & 2 \\ -1 & -0.5 - 0.5i & 1.5 \\ -3 & -1 & 3.5 - 0.5i \end{pmatrix} (u + iv) = 0, \end{aligned}$$

which implies

$$\begin{aligned} (-1.5 - 0.5i)u_1 - u_2 + 2u_3 + (0.5 - 1.5i)v_1 - iv_2 + 2iv_3 &= 0 \\ -u_1 - (0.5 + 0.5i)u_2 + 1.5u_3 - iv_1 + (0.5 - 0.5i)v_2 + 1.5iv_3 &= 0, \end{aligned}$$

or

$$\begin{aligned} -1.5u_1 - u_2 + 2u_3 + 0.5v_1 &= 0 \\ -0.5u_1 - 1.5v_1 - v_2 + 2v_3 &= 0 \\ -u_1 - 0.5u_2 + 1.5u_3 + 0.5v_2 &= 0 \\ -0.5u_2 - v_1 - 0.5v_2 + 1.5v_3 &= 0. \end{aligned}$$

Solve the system and get an eigenvector

$$u + iv = (-1, 0, -1)' + i(1, 1, 1)'.$$

For $\lambda_3 = \frac{1-i}{2}$, an eigenvector is

$$u - iv = (-1, 0, -1)' + i(1, 1, 1)'.$$

Let

$$\begin{aligned} P &= \begin{pmatrix} z & u & v \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \end{pmatrix}. \end{aligned}$$

Then

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & -0.5 & 0.5 \end{pmatrix},$$

i.e. $\lambda = 2, \alpha = \beta = 0.5$.

(e) Let $Ax = x$; then $(A - I)x = 0$. Because

$$\det(A - I) = \begin{vmatrix} -2 & -1 & 2 \\ -1 & -1 & 1.5 \\ -3 & -1 & 3 \end{vmatrix} = 0.5,$$

$A - I$ is invertible and the system has a unique solution

$$\begin{aligned} x^* &= (A - I)^{-1} 0 \\ &= 0. \end{aligned}$$

(f) Let $P^{-1}AP = \Lambda$; then $A = P\Lambda P^{-1}$ and thus

$$A^t = P\Lambda^t P^{-1}.$$

Note that Λ^t takes the form

$$\Lambda^t = \begin{pmatrix} 2^t & 0 & 0 \\ 0 & a_t & b_t \\ 0 & c_t & d_t \end{pmatrix},$$

where

$$D_t \equiv \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix}^t.$$

We want to first show $\lim_{t \rightarrow \infty} D_t = 0$. For a matrix C , denote the entry with the largest absolute value as k_C . Because

$$D_2 = \begin{pmatrix} 0 & 0.5 \\ -0.5 & 0 \end{pmatrix},$$

for an arbitrary matrix M ,

$$D_2 M = \begin{pmatrix} 0.5m_{21} & 0.5m_{22} \\ -0.5m_{11} & -0.5m_{12} \end{pmatrix}.$$

Therefore, we have

$$k_{D_2M} = \frac{1}{2}k_M.$$

So

$$\begin{aligned} k_{D_t} &= \left(\frac{1}{2}\right)^{\frac{t}{2}} k_I \\ &= \left(\frac{1}{2}\right)^{\frac{t}{2}} \end{aligned}$$

for any even number integer t , and

$$\begin{aligned} k_{D_t} &= \left(\frac{1}{2}\right)^{\frac{t-1}{2}} k_{D_1} \\ &= \left(\frac{1}{2}\right)^{\frac{t+1}{2}} \end{aligned}$$

for any odd integer t . Apparently $\lim_{t \rightarrow \infty} k_{D_t} \rightarrow 0$; so $\lim_{t \rightarrow \infty} D_t = 0$.

It follows that

$$\begin{aligned} \lim_{t \rightarrow \infty} A^t &= P \lim_{t \rightarrow \infty} \Lambda^t P^{-1} \\ &= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2^t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2^t & 0 & 0 \\ 2^t & 0 & 0 \\ 2^{t+1} & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -2^t & 0 & 2^t \\ -2^t & 0 & 2^t \\ -2^{t+1} & 0 & 2^{t+1} \end{pmatrix}. \end{aligned}$$

Let $x_0 = (1, 0, 0)'$; then

$$\lim_{t \rightarrow \infty} A^t x_0 = \lim_{t \rightarrow \infty} (-2^t, -2^t, -2^{t+1}) \neq 0.$$

(g) Let $\hat{x}_0 = (1, 0, 1)'$. It is easy to verify that $\lim_{t \rightarrow \infty} A^t \hat{x}_0 = 0$.

3. We want to show there exists a homeomorphism $h : [-1, 1] \rightarrow [-2, 2]$ such that $h \circ f = g \circ h$. We guess that h is of the polynomial form $h(x) = a + bx + cx^2$. Then

$$\begin{aligned} h \circ f(x) &= a + b(2x^2 - 1) + c(2x^2 - 1)^2 \\ &= 4cx^4 + (2b - 4c)x^2 + a - b + c, \\ g \circ h(x) &= (a + bx + cx^2)^2 - 2 \\ &= c^2x^4 + 2bcx^3 + (b^2 + 2ac)x^2 + 2abx + a^2 - 2. \end{aligned}$$

The equivalence between the two implies

$$\begin{aligned}4c &= c^2 \\ 0 &= 2bc \\ 2b - 4c &= b^2 + 2ac \\ 0 &= 2ab \\ a - b + c &= a^2 - 2.\end{aligned}$$

Solve the system and get

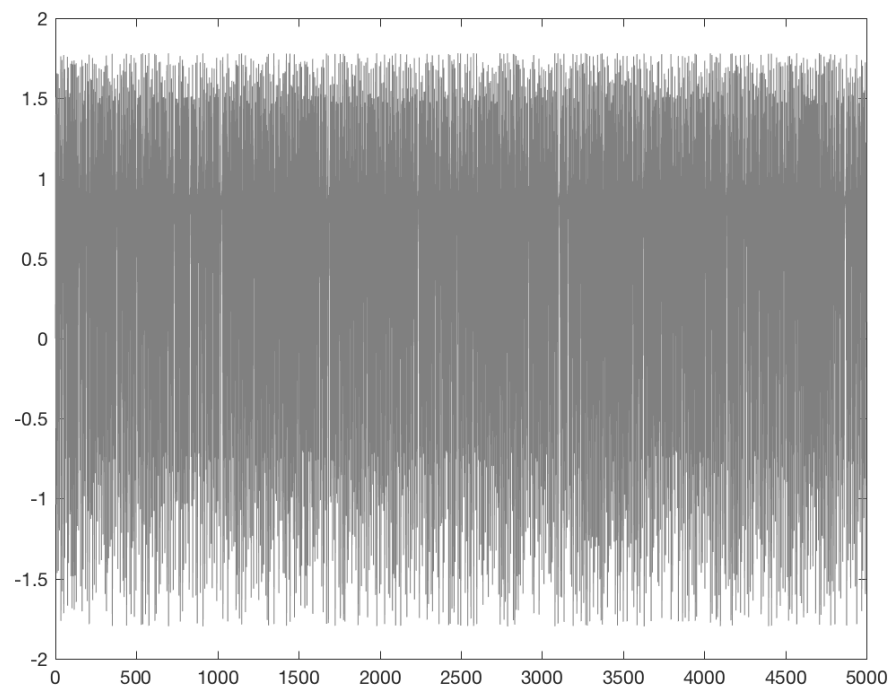
$$\begin{cases} a = -1 \\ b = 0 \\ c = 0 \end{cases} \quad \text{or} \quad \begin{cases} a = 2 \\ b = 0 \\ c = 0 \end{cases} \quad \text{or} \quad \begin{cases} a = 0 \\ b = 2 \\ c = 0 \end{cases}.$$

The third set of solutions corresponds to $h(x) = 2x$, which is a homeomorphism from $[-1, 1]$ to $[-2, 2]$. Therefore, f and g are conjugate.

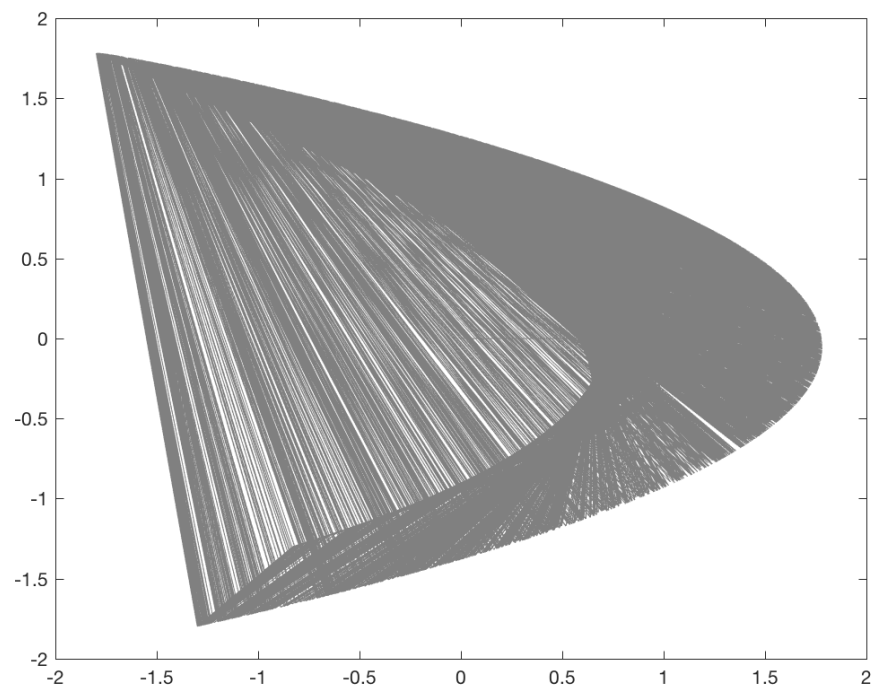
4. (a)

```
1 clear all;
2 close all;
3
4 % Part (a)
5 n=5000;
6 a=1.4;
7 b=0.3;
8 x=zeros(n,1);
9 y=zeros(n,1);
10
11 for i=2:n
12     x(i) = a - (x(i-1))^2 + b*y(i-1);
13     y(i) = x(i-1);
14 end
15
16 plot(1:5000,x,'Color',[0.5,0.5,0.5])
17 print('Fig_a1','-dpng')
18
19 plot(x,y,'Color',[0.5,0.5,0.5])
20 print('Fig_a2','-dpng')
```

Fig_a1.png:



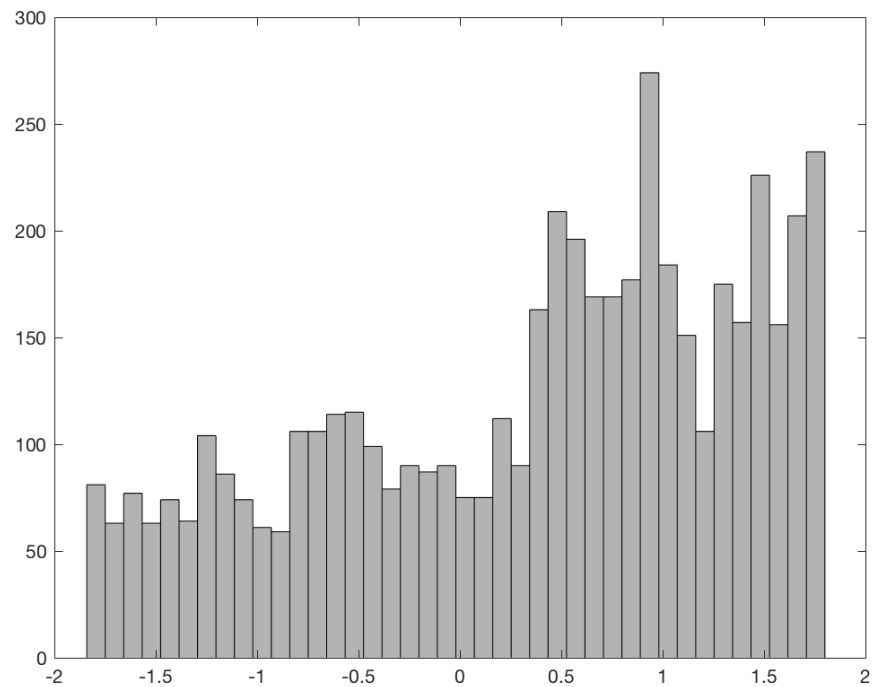
Fig_a2.png:



(b)

```
1 % Part (b)
2 histogram(x,40,'FaceColor',[0.5,0.5,0.5])
3 print('Fig_b','-dpng')
```

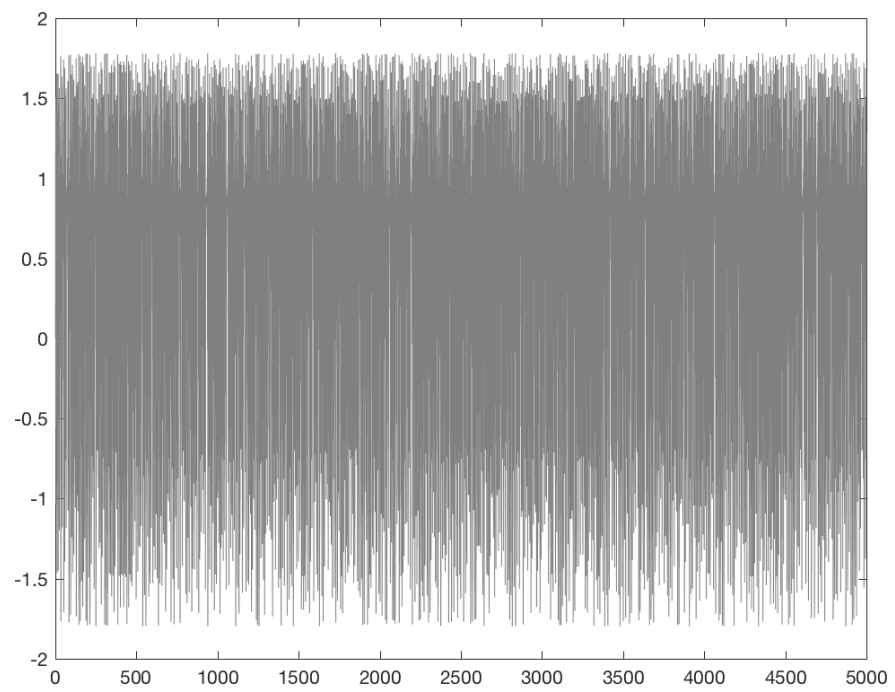
Fig_b.png:



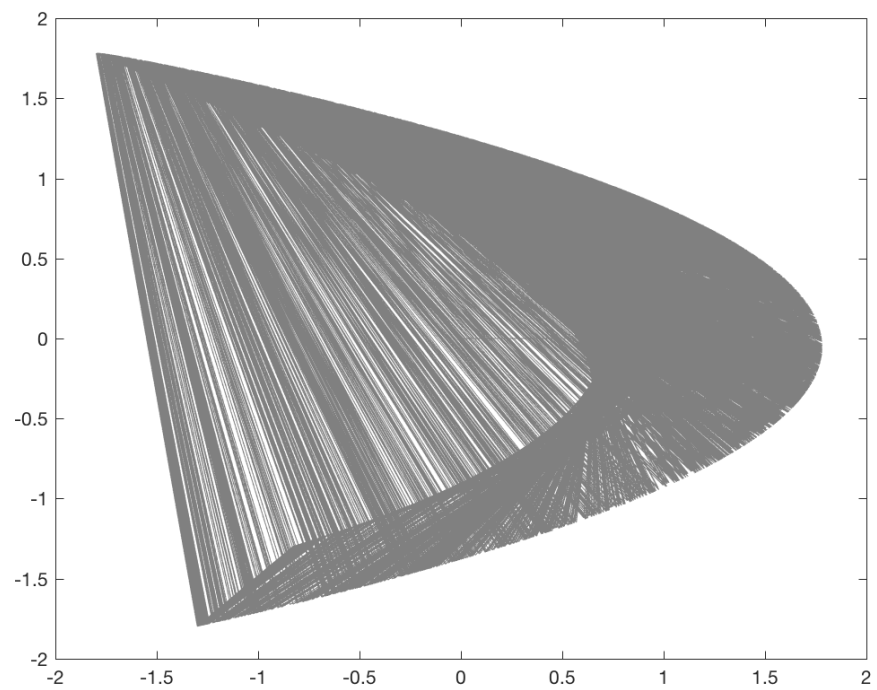
(c)

```
1 % Part (c)
2 xx=zeros(n,1);
3 yy=zeros(n,1);
4 xx(1)=10^(-8);
5 yy(1)=10^(-8);
6
7 for i=2:n
8     xx(i) = a - (xx(i-1))^2 + b*yy(i-1);
9     yy(i) = xx(i-1);
10 end
11
12 plot(1:5000,xx,'Color',[0.5,0.5,0.5])
13 print('Fig_c1','-dpng')
14
15 plot(xx,yy,'Color',[0.5,0.5,0.5])
16 print('Fig_c2','-dpng')
17
18 histogram(xx,40,'FaceColor',[0.5,0.5,0.5])
19 print('Fig_c3','-dpng')
```

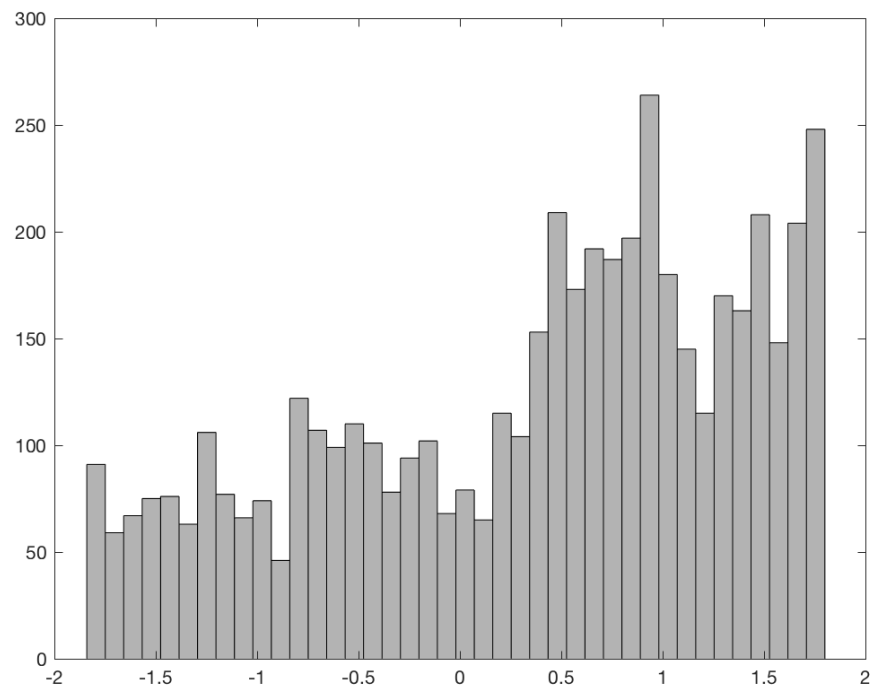

Fig_c1.png:



Fig_c2.png:



Fig_c3.png:

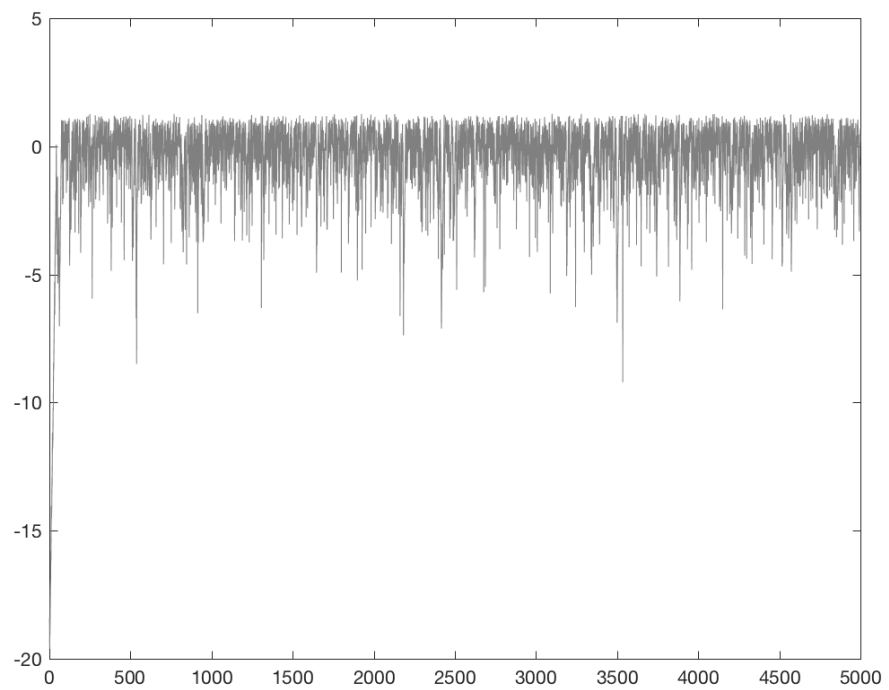


The trajectories are notably different from each other, but overtime they pass through approximately the same set of points.

(d)

```
1 % Part (d)
2 diff=log(abs(x-xx));
3 plot(1:5000,diff,'Color',[0.5,0.5,0.5])
4 print('Fig_d','-dpng')
```

Fig_d.png:



The trajectory depends heavily on the initial conditions in the sense that a small initial deviation leads to persistent differences later on.