ECON G6410 Problem Set 10

1. (a) Every subset of X is invariant under f.

   Every point in X is a fixed point. For all $x \in X$, the corresponding stable set is $W(x) = \{x\}$.

   (b) The subsets of X that are invariant under f take the form of

   $$S = T \cup (-T),$$

   where $T \subset \mathbb{R}$.

   The system has one fixed point $x = 0$; its stable set is $W(0) = \{0\}$.

   (c) The subsets of X that are invariant under f take the form of

   $$S = T \cup \left(\bigcup_{x \in \mathbb{N}} \{t + x : t \in T\} \right),$$

   where $T \subset \mathbb{R}$.

   The system has no stable point.

   (d) The subsets of X that are invariant under f take the form of

   $$S = T \cup \left(\bigcup_{x \in \mathbb{N}} \left\{ \frac{t}{2^x} : t \in T \right\} \right),$$

   where $T \subset \mathbb{R}$.

   The system has one fixed point $x = 0$; its stable set is $W(0) = \mathbb{R}$. This is because for any $x \in \mathbb{R}$,

   $$\lim_{n \to \infty} f^n(x) = \lim_{n \to \infty} \frac{x}{2^n} = 0.$$

   (e) The subsets of X that are invariant under f take the form of

   $$S = T \cup \left(\bigcup_{x \in \mathbb{N}} \{1.2^x t : t \in T\} \right),$$

   where $T \subset \mathbb{R}_+$.

   The system has one fixed point $x = 0$; its stable set is $W(0) = \{0\}$.

   (f) The subsets of X that are invariant under f take the form of

   $$S = T \cup \left(\bigcup_{x \in \mathbb{N}} \{f^x(t) : t \in T\} \right),$$

where $T \subset \mathbb{R}_+$.

Setting $f(x) = x$ gives two solutions $x = 0$ and $x = 1$, which are the fixed points of the system. To find their stable sets, note that $f(x) > x$ for $x \in (0,1)$ and $f(x) < x$ for $x \in (1, \infty)$, so

$$\lim_{n \to \infty} f^n(x) = 1$$

for any $x \in (0, \infty)$. Therefore, $W(1) = (0, \infty)$ and $W(0) = \{0\}$.

2. (a)

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} -2 & -1 & 3.5 \\ -1 & 0 & 1.5 \\ -4 & -1 & 5.5 \end{pmatrix}.$$

(b) Since B is invertible,

$$\begin{aligned} A &= B^{-1}C \\ &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} C \\ &= \begin{pmatrix} -1 & -1 & 2 \\ -1 & 0 & 1.5 \\ -3 & -1 & 4 \end{pmatrix}. \end{aligned}$$

(c) Set $\det(A - \lambda I) = 0$:

$$\begin{aligned} \begin{vmatrix} -1-\lambda & -1 & 2 \\ -1 & -\lambda & 1.5 \\ -3 & -1 & 4-\lambda \end{vmatrix} &= 0 \\ 2\lambda^3 - 6\lambda^2 + 5\lambda - 2 &= 0 \\ (\lambda - 2)\left(2\lambda^2 - 2\lambda + 1\right) &= 0 \\ \lambda &= 2 \text{ or } \frac{1+i}{2} \text{ or } \frac{1-i}{2}. \end{aligned}$$

(d) First find the eigenvectors.

For $\lambda_1 = 2$:

$$(A - 2I)\, z = 0$$

$$\begin{pmatrix} -3 & -1 & 2 \\ -1 & -2 & 1.5 \\ -3 & -1 & 2 \end{pmatrix} z = 0,$$

which implies

$$-3z_1 - z_2 + 2z_3 = 0$$
$$-z_1 - 2z_2 + 1.5z_3 = 0.$$

Solve the system and get an eigenvector

$$z = (1, 1, 2)'.$$

For $\lambda_2 = \frac{1+i}{2}$: for the real part,

$$\left(A - \left(\frac{1+i}{2}\right) I\right)(u + iv) = 0$$

$$\begin{pmatrix} -1.5 - 0.5i & -1 & 2 \\ -1 & -0.5 - 0.5i & 1.5 \\ -3 & -1 & 3.5 - 0.5i \end{pmatrix} (u + iv) = 0,$$

which implies

$$(-1.5 - 0.5i)\, u_1 - u_2 + 2u_3 + (0.5 - 1.5i)\, v_1 - iv_2 + 2iv_3 = 0$$
$$-u_1 - (0.5 + 0.5i)\, u_2 + 1.5u_3 - iv_1 + (0.5 - 0.5i)\, v_2 + 1.5iv_3 = 0,$$

or

$$-1.5u_1 - u_2 + 2u_3 + 0.5v_1 = 0$$
$$-0.5u_1 - 1.5v_1 - v_2 + 2v_3 = 0$$
$$-u_1 - 0.5u_2 + 1.5u_3 + 0.5v_2 = 0$$
$$-0.5u_2 - v_1 - 0.5v_2 + 1.5v_3 = 0.$$

Solve the system and get an eigenvector

$$u + iv = (-1, 0, -1)' + i\,(1, 1, 1)'.$$

For $\lambda_3 = \frac{1+i}{2}$, an eigenvector is

$$u - iv = (-1, 0, -1)' + i\,(1, 1, 1)'.$$

Let

$$P = \begin{pmatrix} z & u & v \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \end{pmatrix}.$$

Then

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & -0.5 & 0.5 \end{pmatrix},$$

i.e. $\lambda = 2, \alpha = \beta = 0.5$.

(e) Let $Ax = x$; then $(A - I)\,x = 0$. Because

$$\det(A - I) = \begin{vmatrix} -2 & -1 & 2 \\ -1 & -1 & 1.5 \\ -3 & -1 & 3 \end{vmatrix} = 0.5,$$

$A - I$ is invertible and the system has a unique solution

$$x^* = (A - I)^{-1}\,0$$
$$= 0.$$

(f) Let $P^{-1}AP = \Lambda$; then $A = P\Lambda P^{-1}$ and thus

$$A^t = P\Lambda^t P^{-1}.$$

Note that Λ^t takes the form

$$\Lambda^t = \begin{pmatrix} 2^t & 0 & 0 \\ 0 & a_t & b_t \\ 0 & c_t & d_t \end{pmatrix},$$

where

$$D_t \equiv \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix}^t.$$

We want to first show $\lim_{t\to\infty} D_t = 0$. For a matrix C, denote the entry with the largest absolute value as k_C. Because

$$D_2 = \begin{pmatrix} 0 & 0.5 \\ -0.5 & 0 \end{pmatrix},$$

for an arbitrary matrix M,

$$D_2 M = \begin{pmatrix} 0.5m_{21} & 0.5m_{22} \\ -0.5m_{11} & -0.5m_{12} \end{pmatrix}.$$

Therefore, we have

$$k_{D_2 M} = \frac{1}{2} k_M.$$

So

$$\begin{aligned} k_{D_t} &= \left(\frac{1}{2}\right)^{\frac{t}{2}} k_I \\ &= \left(\frac{1}{2}\right)^{\frac{t}{2}} \end{aligned}$$

for any even number integer t, and

$$\begin{aligned} k_{D_t} &= \left(\frac{1}{2}\right)^{\frac{t-1}{2}} k_{D_1} \\ &= \left(\frac{1}{2}\right)^{\frac{t+1}{2}} \end{aligned}$$

for any odd integer t. Apparently $\lim_{t\to\infty} k_{D_t} \to 0$; so $\lim_{t\to\infty} D_t = 0$.

It follows that

$$\begin{aligned} \lim_{t\to\infty} A^t &= P \lim_{t\to\infty} \Lambda^t P^{-1} \\ &= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2^t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2^t & 0 & 0 \\ 2^t & 0 & 0 \\ 2^{t+1} & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -2^t & 0 & 2^t \\ -2^t & 0 & 2^t \\ -2^{t+1} & 0 & 2^{t+1} \end{pmatrix}. \end{aligned}$$

Let $x_0 = (1, 0, 0)'$; then

$$\lim_{t\to\infty} A^t x_0 = \lim_{t\to\infty} \left(-2^t, -2^t, -2^{t+1}\right) \neq 0.$$

(g) Let $\hat{x}_0 = (1, 0, 1)'$. It is easy to verify that $\lim_{t\to\infty} A^t \hat{x}_0 = 0$.

3. We want to show there exists a homeomorphism $h : [-1, 1] \to [-2, 2]$ such that $h \circ f = g \circ h$. We guess that h is of the polynomial form $h(x) = a + bx + cx^2$. Then

$$\begin{aligned} h \circ f(x) &= a + b\left(2x^2 - 1\right) + c\left(2x^2 - 1\right)^2 \\ &= 4cx^4 + (2b - 4c)x^2 + a - b + c, \\ g \circ h(x) &= \left(a + bx + cx^2\right)^2 - 2 \\ &= c^2x^4 + 2bcx^3 + \left(b^2 + 2ac\right)x^2 + 2abx + a^2 - 2. \end{aligned}$$

The equivalence between the two implies

$$\begin{aligned}
4c &= c^2 \\
0 &= 2bc \\
2b - 4c &= b^2 + 2ac \\
0 &= 2ab \\
a - b + c &= a^2 - 2.
\end{aligned}$$

Solve the system and get

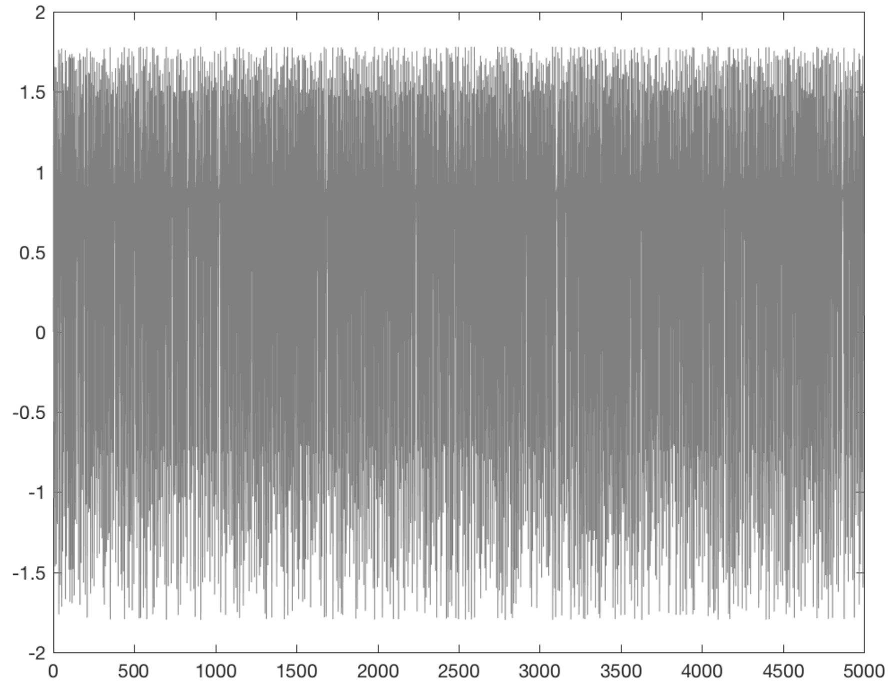
$$\begin{cases} a = -1 \\ b = 0 \\ c = 0 \end{cases} \quad \text{or} \quad \begin{cases} a = 2 \\ b = 0 \\ c = 0 \end{cases} \quad \text{or} \quad \begin{cases} a = 0 \\ b = 2 \\ c = 0 \end{cases}.$$

The third set of solutions corresponds to $h(x) = 2x$, which is a homeomorphism from $[-1, 1]$ to $[-2, 2]$. Therefore, f and g are conjugate.

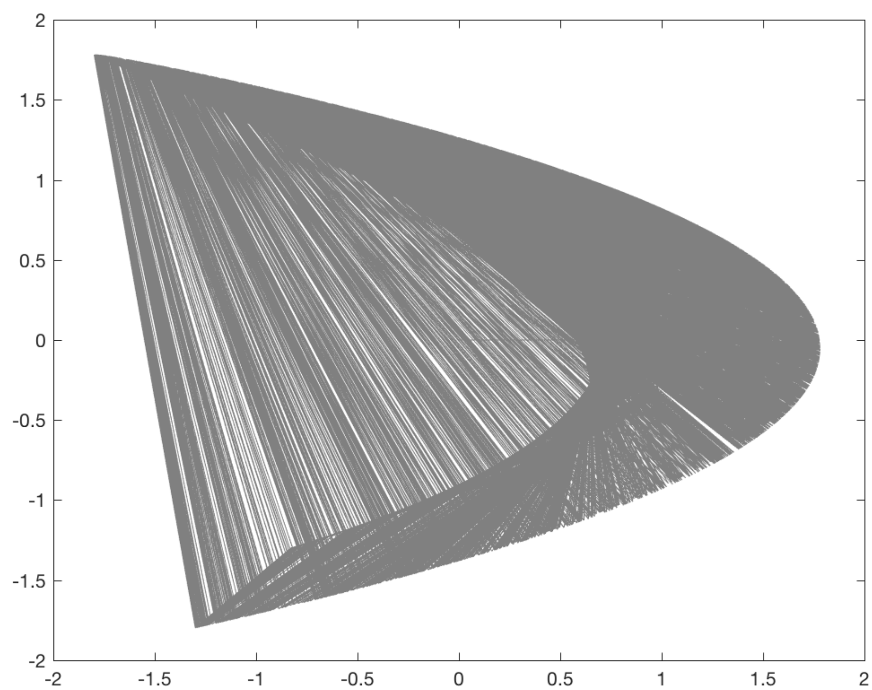
4. (a)

```
clear all;
close all;

% Part (a)
n=5000;
a=1.4;
b=0.3;
x=zeros(n,1);
y=zeros(n,1);

for i=2:n
   x(i) = a - (x(i-1))^2 + b*y(i-1);
   y(i) = x(i-1);
end

plot(1:5000,x,'Color',[0.5,0.5,0.5])
print('Fig_a1','-dpng')

plot(x,y,'Color',[0.5,0.5,0.5])
print('Fig_a2','-dpng')
```

Fig_a1.png:



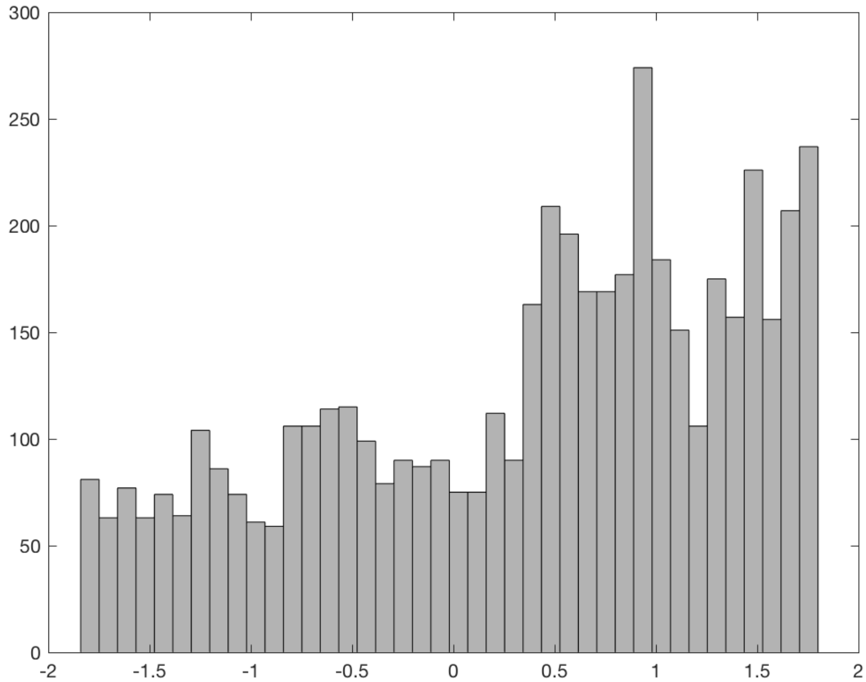
Fig_a2.png:



(b)

```
% Part (b)
histogram(x,40,'FaceColor',[0.5,0.5,0.5])
print('Fig_b','-dpng')
```

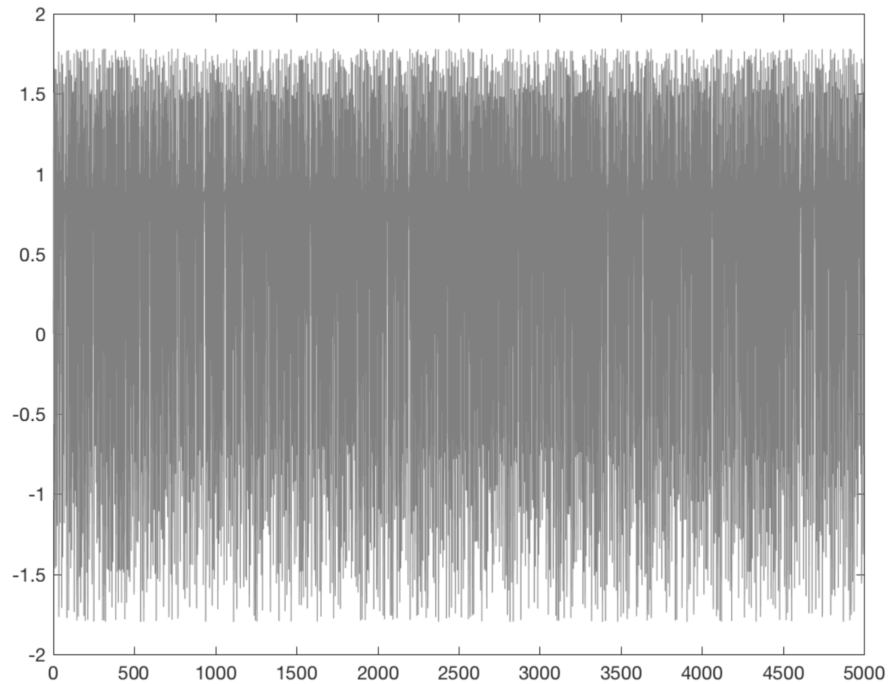
Fig_b.png:



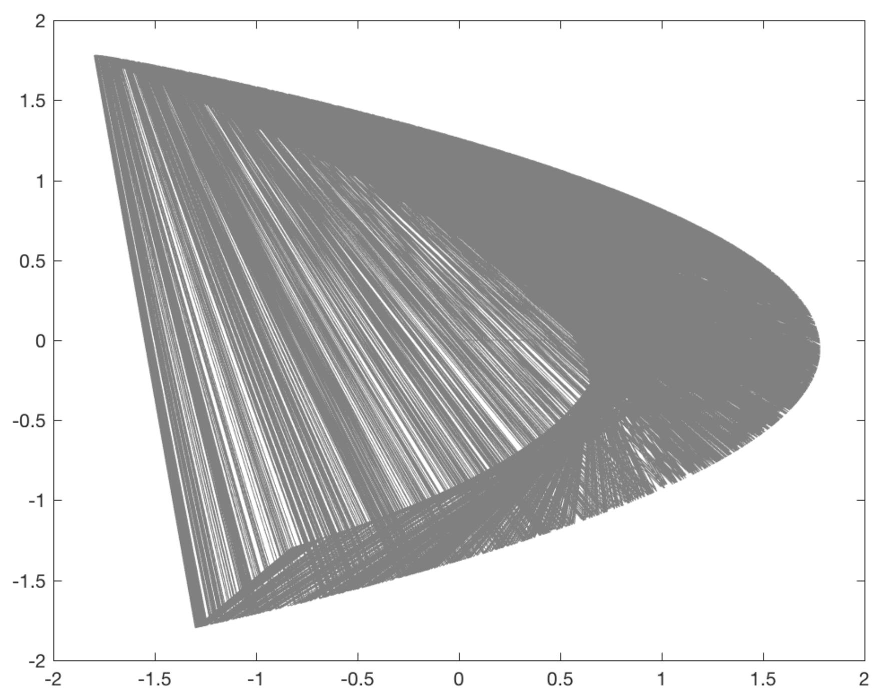
(c)

```
% Part (c)
xx=zeros(n,1);
yy=zeros(n,1);
xx(1)=10^(-8);
yy(1)=10^(-8);

for i=2:n
   xx(i) = a - (xx(i-1))^2 + b*yy(i-1);
   yy(i) = xx(i-1);
end

plot(1:5000,xx,'Color',[0.5,0.5,0.5])
print('Fig_c1','-dpng')

plot(xx,yy,'Color',[0.5,0.5,0.5])
print('Fig_c2','-dpng')

histogram(xx,40,'FaceColor',[0.5,0.5,0.5])
print('Fig_c3','-dpng')
```

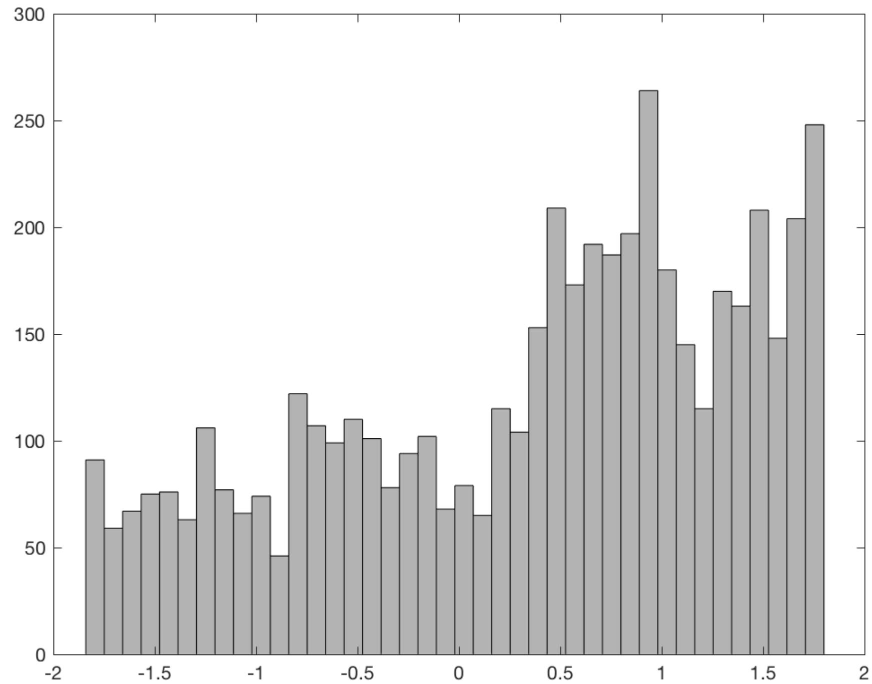
Fig_c1.png:



Fig_c2.png:



Fig_c3.png:

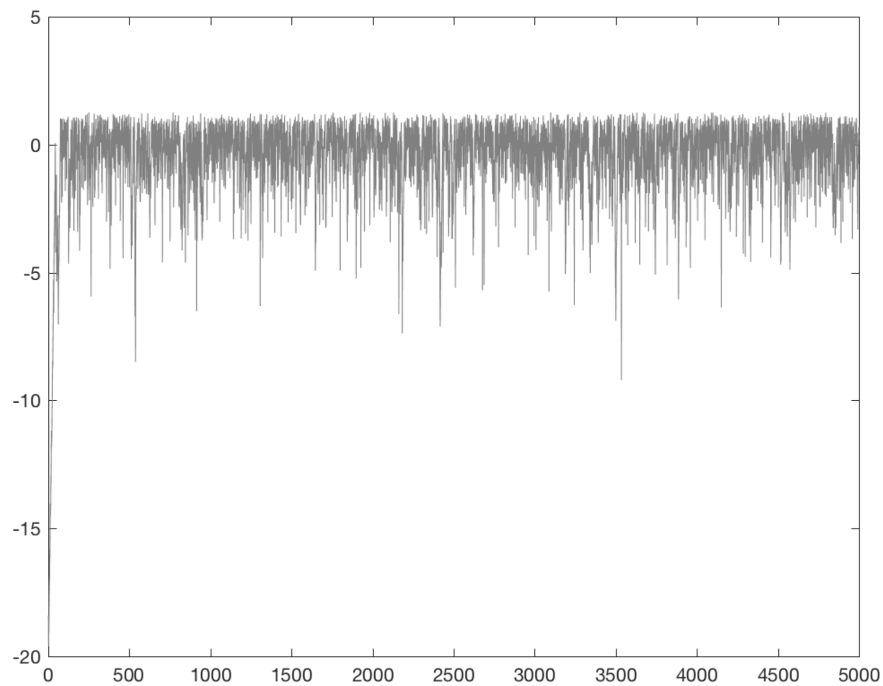


The trajectories are notably different from each other, but overtime they pass through approximately the same set of points, since the system is chaotic and the periodic points are dense.

(d)

```
% Part (d)
diff=log(abs(x-xx));
plot(1:5000,diff,'Color',[0.5,0.5,0.5])
print('Fig_d','-dpng')
```

Fig_d.png:



The trajectory depends heavily on the initial conditions in the sense that a small initial deviation leads to persistent differences later on.