4 Game Theory

Chapter from “Beyond Price Theory” by W B MacLeod

4.1 Introduction

Savage’s theory allows one to use the full apparatus of statistical decision theory even in situations for which probabilities have not been estimated by an experiment. The viewpoint of statistics that Savage addressed was one in which Nature is viewed as providing a fixed, well defined environment that is considered *independent* of the statistical methods that one uses to uncover her secrets.

In contrast, contract theory explicitly addresses the question of designing a contract which takes into account how each party expects the other party to act in the future. Game theory addresses the question of how to extend the rational choice model to situations where one is interacting with other rational decision makers. Interactions with children are a good example of the problem one faces, in part because they are often quite rational in their behavior (even though it may not seem so to adults).[1] They often enter into explicit agreements with their parents regarding each party’s behavior. Consider going to a store with a child, who then decides that she would like a candy bar, something that the parent would not normally wish to buy. In advance the child might know that asking for such a bar would elicit the typical response “No!”. However, once in the store some children anticipate the fact that their parents are very uncomfortable when they start to scream and roll on the floor. They use the threat of a temper tantrum to extract candy from the helpless parent.

Notice, that for the child this is entirely rational behavior because she knows that her parents are embarrassed by such a display and would succumb. This game can be formally modeled as follows:

1 See Harbaugh et al. (2007) for some nice evidence.

	Buy Candy (BC)	No Candy (NC)
Tantrum if No Candy (TNC)	$(2, -2)$	$(-1, -10)$
No Tantrum (NT)	$(2, -2)$	$(0, 2)$

Table 4.1
Candy Game

This matrix is a game in *normal form.* When represented in a table form like this it always has the same interpretation - the first column lists the set of strategies for player 1, while the first row lists the strategies for player 2. The table entries are in the form (u_1, u_2), and represent the Bernoulli utilities for each pair of actions for players 1 and 2 respectively. Unless otherwise stated, it is assumed that the players satisfy the axioms of subjective expected utility theory.

This simple game illustrates a number of features of strategic decision making that are important for understanding social interactions. Notice that if the child chooses TNC (Tantrum if No Candy), the parent is better off choosing BC (Buy Candy) to avoid the negative consequences of a tantrum, represented by the -10. If the threat is not actually carried out, then the child is indifferent between tantrum and no tantrum. Thus the pair $\{TNC, BC\}$ forms a *Nash Equilibrium.* Each person's strategy is optimal given his or her expectations regarding the other person's strategy. Notice that this is not the only Nash equilibrium. The payoffs are written down with the hypothesis that having a tantrum is costly for the child, in which case the outcome $\{NT, NC\}$ is also a Nash equilibrium.

The point is that if the child knows that she will not receive a candy in any case, then it is better not to have a tantrum even when there is no candy; but the child also understands that if she backs down too easily, then the threat will never work. Thus children, will often have tantrums to prove to their parents that their threat is *credible.* If the threat is to carry out a costly action when individuals do not behave as one wishes (in this case the parent does not provide the candy), then enforcement of behavior may require the development of a reputation for having tantrums. Parents can of course fight back with their own strategies. One common strategy is to enter in a *contract* with the child. For example, if there is no temper tantrum while shopping the parent may offer say ice cream at the end of the outing, under the assumption that the combination of ice cream and peace while shopping is preferred to the alternative.

In this case, there is also an enforcement problem. Once the child has behaved well, and is in the car out of the public eye, the threat of a tantrum is much less costly for the parent. At that point the parent may attempt to renege on the original agreement. In particular, the extent to which a child would agree to such a contract depends upon the extent to which the child *believes* that the parent will honor the agreement. This illustrates not only the potentially complex inter-temporal behavior that is possible, but also the extent to which behavior is determined by one's beliefs regarding future behavior. In particular, one's beliefs are affected by the extent to which one believes that the person one is interacting with is rational. In the case of children, there is evidence that they do behave in a rational fashion, though the extent to which they can understand and respond to inter-temporal incentives can depend on age.[2] Before introducing more formally the concept of Nash equilibrium and its various refinements, we need to introduce the notion of a strategy into Savage's model.

2 Harbaugh et al. (2001) explicitly look at the extent to which children have rational preferences when making choices. They find that college students are no more rational than 7 year olds!

The Concept of a Strategy

Formally, Savage's concept of an act is sufficiently rich to provide a complete model of a strategy. A difficulty with his approach is that one's payoff depends upon the other person's decision; different choices by the *other* players technically result in different states. In fact, one can view nature as a player as well, in which case the model simply entails mapping all possible strategies by the other players into states, and then the decision maker forms beliefs over this state space before making a choice.[3] In practice, game theory supposes that strategies are distinct from the notion of a state. Let us introduce the basic model with the well known "Monty Hall" based upon the television game show *Let's Make A Deal,* for which Monty Hall was the host. The problem, as posed by Marilyn vos Savant in her *Parade* Magazine column, September 1991:

> "Suppose you're on a game show, and you're given a choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the other doors, opens another door, say No. 3, which has a goat. He then says to you, 'Do you want to pick door No. 2?' Is it to your advantage to take the switch?"

This problem has produced a great deal of controversy concerning what is the "correct" solution. Selvin (1975b) provides one of the first solutions to this problem. He views this as a "problem in probability" - yet really the issue is not probability per se but detailing the actions of each party, and linking those actions to outcomes. More precisely, this is a problemthat can be easily solved using the game theoretic approach.[4] The difficulty is that this is not simply a problem in probability because one needs to understand and model how the actions of Monty affect the gains from switching. In particular the fact that Monty knows which door hides the prize is crucial to the solution to the problem.

3 Mertens and Zamir (1985) show that mathematically it is possible to create a *universal type space* that captures all characteristics of all the players in the game.

4 Wikipedia has a full article on the problem describing the various "solutions". See http://en.wikipedia.org/wiki/Monty_Hall_problem .

We begin by outlining a general method for describing any game played with a discrete number of moves, and then we shall apply the technique to the Monty Hall problem. The first step is to enumerate the set of players, given by $N = \{0, 1, 2, \cdots, \bar{n}\}$. Uncertainty is introduced by observing that whenever there is an uncertain event this can be modeled as a choice by Nature, who as a matter of convention is always player 0. Second, one constructs a timeline which orders the time at which players make their moves. This does not have to be real time, but the sequence is important to allow us to properly model information flows.

Games in practice are not typically defined by their extensive forms, but by the "rules of the game". For example, in the case of chess, players take turns choosing moves that are consistent with the rules of the game. Rather than defining an abstract extensive form, let us suppose that the rules of the game are available, and that one *constructs* the extensive form through repeated application of the rules of the game, beginning with a node known at the root.

Definition Given players $N = \{0, 1, ..., \bar{n}\}$, a rule book R, and a play starting at date $t = 1$ at the root r, an *extensive form game* Ξ is defined recursively as follows:

1. The game begins with the root node r, and set $\Sigma(0) = \{r\}$. The set $\Sigma(t), t \geq 1$ represents the nodes created at date t as a result of the moves made at date $t-1$, and for which there will be further moves. Let Z represent the final nodes of the game. We suppose that $r \notin Z$ and we suppose that the rule book R has clear criteria that determine whether a node n is in Z. In the case of chess, Z consists of the set of board positions that correspond to checkmate or stalemate.
2. The set $\Sigma(t)$ is partitioned into information sets, $E_{tk} \in \Phi_t$, $k = 0, ..., k_t$, where k_t is the number of information sets at time t, and Φ_t is a partition of $\Sigma(t)$. Each information set is associated with a

player $i_{tk} \in N$, with the interpretation that this player does not know which node in E_{tk} has occurred. At this information set player i_{tk} must choose an action $a_{ik} \in A_{tk}$. Player i's *strategy* at information set E_{tk} is given by the function $\gamma^i(E_{tk}) = a_{tk} \in A_{tk}$. Player $i's$ *behavioral strategy* at information set E_{tk} is defined by the function $\hat{\gamma}^i(E_{tk}) = \hat{a}_{tk} \in \Delta(A_{tk})$, that is, the player randomly selects an action from A_{tk}. If $i_{tk} = 0$, then this is a play by Nature. It is assumed that Nature always plays a behavioral strategy $\hat{a}^0_{tk} \in \Delta(A_{tk})$ that is known by all players, and corresponds to the probability that nature chooses an action in A_{tk}.

3. New nodes in $\Sigma(t+1)$ for the next date are created as follows. For each $n \in \Sigma(t)$, and for each $a_{tk} \in A_{tk}$, where E_{tk} is the information set containing n, a new node $n' = a_{tk}$ is created, with the following additional properties. If a_{tk} leads to a final move of the game, then n' is added to the list Z, otherwise it is added to the list $\Sigma(t+1)$. Define the predecessor function by $n = p(n')$, and the successor correspondence by $s(n) = p^{-1}(n) = \{n' | p(n') = n\}$. Let $S(E_{tk}) = \{n' | p^m(n') \in E_{tk}, \, for\, some\, m \geq 1\}$ be the set of all successor nodes, namely all the nodes that follow the information set E_{tk}, where $p^m(n')$ is the predecessor function for the choice made m periods ago.
4. Repeat steps 2 and 3 until $\Sigma(t) = \emptyset$ is empty, or equivalently $s(n) \subset Z$ for all $n \in \Sigma(t-1)$.
5. Let $\Phi^i = \{E_{tk} | i_{tk} = i\}$ be the information sets at which player i moves. Let Φ denote the set of all information sets.
6. For each $z \in Z$, the Bernoulli utility for player i of arriving at that node is specified by $u^i(z)$.

Whenever we use the symbol Ξ we mean a game in extensive form as defined above. In general there are many ways to describe a game. The rules of any finite game can be mapped into the extensive form structure described above. Notice that each element in Z corresponds to a *unique* description of the actions that lead to that node, and hence the elements in Z correspond precisely to the notion of a *state* - a complete description of the outcome of the game.

The standard approach to defining an extensive form game begins with a set of nodes $\Sigma(t)$ and endpoints, Z.[5] This approach is convenient to address purely theoretical issues, such as the existence of an equilibrium. Moreover, this approach suggests that the notion of a state or node in a game seems to be something that is exogenous and well defined. The definition here highlights the idea that states in a model are *constructed* and that in practice when playing a game individuals are typically aware of only a small subset of the full state space. For example, in the game of chess one never thinks of it in terms of all possible endpoints, but rather in terms of a small number of board positions that can be reached from the current position in the game. This perspective will become relevant when we turn to the issue of contractual incompleteness.

We complete the definition of a game with the introduction of a strategy. Let Γ^i and $\hat{\Gamma}^i$ denote respectively the set of strategies and behavioral strategies by player i defined in step 2. A strategy *profile* for the game is given by:

$$\hat{\gamma}^0 \times \gamma = \hat{\gamma}^0 \times \left\{\gamma^1, ..., \gamma^{\bar{n}}\right\} \in \hat{\Gamma}^0 \times \Gamma^1 \times ... \times \Gamma^{\bar{n}} = \hat{\Gamma}^0 \times \Gamma,$$

where Γ is the set of strategies for the players. This generates a probability distribution $\mu(\gamma) \in \Delta(Z)$. Given that $\hat{\gamma}^0$ is usually specified as part of the description of the game, it is not an explicit argument of μ. Hence the payoff of the individuals as a function of the *strategy profile* γ is defined by:

$$U^i(\gamma) = \sum_{z \in Z} u^i(z)\, \mu(z|\gamma),$$

5 See page 277 Mas-Colell et al. (1995).

where $\mu(z|\gamma)$ is the probability of z given the strategy profile γ. Notice that all the information necessary to describe the game is given by the finite strategy sets Γ^i, and the payoffs, $U^i(\cdot)$. When the strategies and payoffs are given in this manner without reference to the underlying information structure then this is called a game in *strategic* or *normal* form. In this case we would write $G(\Xi) = \left\{U^i, \Gamma^i\right\}_{i\in N}$. We shall also use the notation $U^i\left(\gamma^i, \gamma^{-i}\right)$ to indicate the expected utility of player i, given her strategy $\gamma^i \in \Gamma^i$, and the strategies of the other players, $\gamma^{-i} \in \Gamma^{-i}$.

Until we address the issue of repeated games, it is assumed that the game ends in a finite number of rounds, and hence the set of strategy profiles for player i, Γ^i is a finite set. A *mixed strategy* for player i is one for which the player randomizes over the elements of Γ^i, and thus corresponds to the set $\Delta(\Gamma^i)$. Given mixed strategies $\hat{\gamma}^i \in \Delta(\Gamma^i)$, the utility of agent i is:

$$U^i(\hat{\gamma}) = \sum_{\gamma\in\Gamma} U(\gamma)\hat{\gamma}^1(\gamma^1) \times ... \times \hat{\gamma}^{\bar{n}}(\gamma^{\bar{n}}), \tag{4.1}$$

where $\hat{\gamma}^i(\gamma^i)$ is the probability that pure strategy γ^i is chosen under mixed strategy $\hat{\gamma}$.

In practice it is convenient to work with the subset of mixed strategies known as behavioral strategies, $\hat{\Gamma}^i \subset \Delta(\Gamma^i)$, as defined in step 2 above. A behavioral strategy is much less complex than a more general mixed strategy. Mixed strategies require the player to have complete plans of play, and then randomize over these plans. In contrast, behavioral strategies do not require randomization to occur until the player is called upon to make a choice - in that case the randomization occurs over the set of choices A_{tk}, a much smaller set than the set of all possible strategies.

When each player has perfect recall then it is the case that any outcome achievable with a mixed strategy can also be achieved with a behavior strategy. A player has perfect recall if she can recall all the strategies she has played previously:

Definition: Formally, player i has perfect recall if for every information set $E_{tk} \in \Phi^i$, and for any choice a_{tk} at information set E_{tk}, then we have that for any other information set $E_{t'k'} \in \Phi^i$ either:

1. $E_{t'k'} \cap S(a_{tk}) = \emptyset$, that is the player at information set $E_{t'k'}$ knows that a_{tk}did not lead to the current information set; or
2. $E_{t'k'} \subset S(a_{tk})$ - player i knows for sure that a_{tk} has occurred before he arrives at information set $E_{t'k'}$.

Given this assumption we have the following result.

Proposition: Suppose that player i has perfect recall, then for any mixed strategy, there exists a behavioral strategy that results in the same payoff for player i.

The proof of this proposition is left as an exercise for the interested reader. We shall suppose that all players have perfect recall, and hence without loss of generality players can be assumed to use behavioral strategies.

The Monty Hall Problem

Let us now illustrate the construction of an extensive form game using the Monty Hall problem. This is illustrated in figure 4.1. At time $t = 1$ Nature moves and places the key in one of the boxes, A, B or C. When the player, denoted by P, chooses her strategy, Nature's choice is not observed. Hence there is only one information set, denoted E_{21}, containing all the nodes generated by Nature. Given E_{21}, the Player then selects A, B or C. Notice that this implies that the same action must be applied to each node in the same information set.

For the purposes of the example, the Player has chosen B. At that point Monty Hall (MH) plays. In this case he knows not only the location of the car, but also the choice made by the Player. Thus given P's choice, Monty Hall has three information sets, E_{31}, E_{32}, and E_{33}, corresponding to the three choices made by Nature (in total at $t = 3$, there are $3 \times 3 = 9$ information sets, 3 for each choice made by P).

The question then is exactly what Monty's strategy is at this point. This is not always clear, Monty could for example show the location of the car, but then that would not be very interesting. In this example, consistent with the analysis of the problem, it is assumed that Monty opens a door with the following two properties: it was not chosen by P (and hence he opens A or C), and the car is not behind the door.

What many analyses of the problem miss is that we are not really told the probability Monty chooses box C or A (See Selvin (1975b) and the later correspondence in Selvin (1975a)).

Now Savage's model teaches us to assign subjective probabilities to Monty's choice, say p and $1 - p$ respectively. Thus, this problem *cannot be* a problem in probability since we are not given information on how Monty will choose. The Savage model simply implies that we should assign some probability to Monty's choices.

First consider information set E_{41}, corresponding to the event that P has chosen B and Monty has chosen C (with probability p). Given this information the individual must choose A, B or C. Notice that a quick calculation shows that the probability of arriving at node P_1 is $1/3$, while the probability of arriving at node P_2 is $p/3$. Hence we can conclude using the definition of conditional probability that $\Pr(P_1|E_{41}) = \frac{1/3}{1/3+p/3} = \frac{1}{1+p}$. Letting u be the utility from winning, and 0 the utility for not winning, then conditional upon E_{41}, the utility of the Player for each of the three strategies is given by:

$$
\begin{aligned}
E(U|E_{41}, A) &= \frac{1}{1+p}u, \\
E(U|E_{41}, B) &= \frac{p}{1+p}u, \\
E(U|E_{41}, C) &= 0.
\end{aligned}
$$

Since $p \in [0,1]$, then playing A is always an optimal strategy, though the Player is indifferent between A and B if $p = 1$. If one does a similar calculation, one will find that choosing C when event E_{42} occurs is optimal, and yields a payoff $E(U|E_{42}, C) = \frac{1}{2-p}u$. Notice that we can compare the strategy of choosing B and staying with one choice (denoted by BB) to the strategy of choosing B and then switching (denoted by BS) as follows. The probability of arriving at E_{41} is $\frac{1+p}{3}$, and at E_{42} is $\frac{2-p}{3}$. Therefore the payoff from each strategy is:

$$E(U|BB) = E(U|E_{41}, B) \times \Pr(E_{41}) + E(U|E_{42}, B) \times \Pr(E_{42}) = u/3,$$
$$E(U|BS) = E(U|E_{41}, A) \times \Pr(E_{41}) + E(U|E_{42}, C) \times \Pr(E_{42}) = u/2.$$

Thus, we find that regardless of the strategy followed by Monty Hall, the strategy of switching will result in a 50% chance of winning, while the strategy of sticking to one's choice results in a 33% chance of winning.

While writing out the extensive form seems complex, a web search on the Monty Hall problem will show that it is very controversial. The extensive form allows one to think about all the possible variants of the game.

4.2 Rational Choice in a Game

We solved the Monty Hall problem by viewing Monty's strategies as a play by Nature that required the assignment of subjective probabilities. Game theory is concerned with modeling choice when it is common knowledge that all agents are rational (or more precisely decision makers satisfy the Aumann-Anscombe axioms of choice). Let us now consider the problem faced by rational decision makers in a game defined in normal form by $G = \{U^i, \Gamma^i\}_{i \in N}$. In game theory the standard hypothesis is that the rules of the game are common knowledge, and that all players are rational in the sense that they satisfy the axioms of subjective

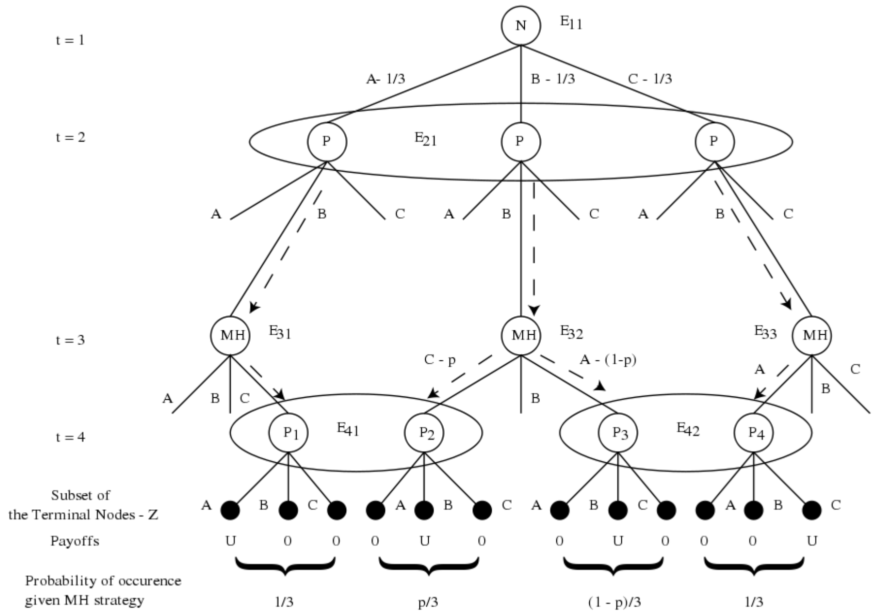

Figure 4.1
Extensive Form for Monty Hall Problem

expected utility theory. It is well known that these assumptions by themselves are not sufficient to uniquely determine the outcome of the game in all cases.[6] For example, consider the famous *battle of the sexes* game.[7]

6 See Schelling (1980) for an excellent discussion of this issue. At the time he wrote the first version of his book there was cold war between the US and the Soviet Union and the threat of nuclear force. It was a serious issue trying to determine the conditions under which the Soviet Union might use their weapons.

7 See section 5.3 of Luce and Raiffa (1989). Many of the well known examples in game theory come from this classic text.

George	Lucy	
	Don't Buy (DB)	Buy Opera Tickets (B)
Buy Hockey Tickets (B)	(1,2)	(-1,-1)
Don't Buy (DB)	(-1,-1)	(2,1)

Table 4.2
"Battle of the Sexes"

In this game, the two parties, say George and Lucy, discuss the possibility of going out the next evening. George has a co-worker who will sell him Hockey tickets, while Lucy has the chance to get Opera tickets. Unfortunately, they did not have a chance to finish their discussion on where to go, but while at work they each must independently make a decision whether to buy the tickets or not.

The problem is that George prefers the Opera, while Lucy prefers Hockey, and both hate to waste money. Thus the question is whether or not they should buy the tickets. If they both buy, then they will be upset, while if neither buys they won't go out. The question is, what should they do? The interesting point here is that even though both parties are rational, if they cannot communicate with each other, there is no way for them to agree upon a single rational decision.

We will come back to this game in the next section. For the moment, let us consider situations where rational choice theory *can* guide us. Consider now the the prisoner's dilemma problem:[8]

8 See section 5.4 of Luce and Raiffa (1989). See also Axelrod (1981) for an extended discussion of how the prisoner's dilemma problem can be applied to politics.

	Keep Quiet/Cooperate	Confess/Cheat
Keep Quiet/Cooperate	(2,2)	(-1,3)
Confess/Cheat	(3,-1)	(0,0)

Table 4.3
"Prisoner's Dilemma"

In this game two white collar criminals have agreed not to confess to the authorities. They are in separate cells where they must decide whether to keep quiet (cooperate with each other) or confess (cheat upon the other). If neither confesses, they are free to keep their ill gotten gains of 2 each. If one confesses and the other does not, then the cheater gains 3, while the cooperator loses 1. Finally, if they both confess, the payoffs are normalized at 0 each.

In this case rational choice theory can make a unique prediction. Notice that regardless of the other player's decision, the strategy of cheating is *always* a better choice than the strategy of cooperation. The strategy of Cheat is called a *dominant strategy,* and would always be used by a rational player. Notice that when a dominant strategy exists, decision making by the player is very easy because their optimal choice does not vary with the actions of the other player. Given that this is a desirable feature of any economic mechanism, there is a great deal of research into the question of how to design institutions with the feature that players can restrict their attention to dominant strategies (see exercise 6).

Let us now consider the problem of determining the implications when it is common knowledge that all players are rational - each player knows that other players make decisions consistent with their preferences, and they all know they share this belief.[9] The most straightforward way to think about this problem is to suppose that each player i chooses an optimal action given her beliefs over possible choices by the other players.

9 See Lewis (1967) for more formal definition and discussion.

This implies that if there is a strategy $\gamma^i \in \Gamma^i$ that a rational player i would never choose, then other players must assign zero probability to this strategy in their own beliefs. Bernheim (1984) and Pearce (1984) provide a formal model of this idea that proceeds as follows.

In the context of Savage's model, players would have some probability distribution over the actions of the other players. Suppose that player i believes that the other players will select strategies from the set $\Gamma'^{-i} \subset \Gamma^{-i}$, then a rational player's strategies will come from the set $H^i(\Gamma^{'-i})$, where:

$$H^i(\Gamma'^{-i}) = \bigcup_{\hat{\gamma}^j \in \Delta(\Gamma''^j), j \in N^{-i}} \left\{ \arg\max_{\gamma^i \in \Gamma^i} U^i\left(\gamma^i, \hat{\gamma}^{-i}\right) \right\},$$

$N^{-i} = \{1, ..., i-1, i+1, ..., \bar{n}\}$. These are the set of strategies for player i that maximize her payoff when she believes that the other players choose their strategies from the set Γ'^{-i}. Notice that we suppose that the other players are choosing mixed strategies. These strategies really represent player i's beliefs. Since we have no way to restrict these beliefs other than saying that they lie in Γ'^{-i} then we take the union over all possible beliefs. Since the set of strategies is finite, this mapping is always well defined.

We now define the set of strategies that are *rationalizable* - these are the strategies consistent with the assumption that rational choice is common knowledge. Begin with the full set of strategies, $\Gamma_0^i = \Gamma^i$, then set $\Gamma_1^i = H^i(\Gamma_0^{-i})$. Namely, we know that rational individuals would only choose strategies from the set $\Gamma_1 = \left\{\Gamma_1^i\right\}_{i \in N}$. If this is the case, the rational person i would only select strategies from $\Gamma_2^i = H^i(\Gamma_1^{-i})$. More generally, if one lets

$$\Gamma_t^i = H^i(\Gamma_{t-1}^{-i}),$$

then it follows that $\Gamma_t^i \subset \Gamma_{t-1}^i$. When combined with the fact that the number of strategies is finite, this allows one to conclude that these sets converge to limit sets Γ_*^i with the feature that $\Gamma_*^i \subset H^i(\Gamma_*^{-i})$. These strategies are called *rationalizable*, because player i's strategy has the feature that it is a best reply to some beliefs that place positive weights only upon the other players' *rationalizable* strategies.

As an example, consider the battle of the sexes game 4.2 in which George and Lucy are considering going to the Opera or to a Hockey game. Consider first the problem from George's perspective. Since George cannot communicate with Lucy, then he may view her decision as a *state of the world.* In that case the choice to buy or not are two acts, the consequence of which is determined by Lucy's choice. If George is rational in the sense of satisfying our theory of subjective expected utility, then we may suppose he assigns a subjective probability p_L^e to Lucy's choosing to buy the hockey tickets. In this case George's payoffs from Don't Buy (DB) and Buy (B) as a function of his belief regarding Lucy's choice are:

$$U^G(DB, p_L^e) = 3p_L^e - 1 \tag{4.2}$$

$$U^G(B, p_L^e) = 1 - 2p_L^e. \tag{4.3}$$

These payoffs are illustrated in figure 4.2. Notice that for belief $p_L^e < p^*$ the optimal strategy is to choose B, while for $p_L^e > p^*$ don't buy is an optimal strategy. When $p_L^e = p^* = 0.4$ either strategy is optimal, and hence any lottery $\{(1-\rho_G), DB; \rho_G, B\}$ is optimal for George.

Let $\rho_G = 0$ denote DB, and $\rho_G = 1$ denote B, then George's behavior as a function of his beliefs can be described by the *best reply correspondence:*

$$r^G(p_L^e) = \arg\max_{\rho\in[0,1]} U^G(\rho, p_L^e) = \begin{cases} 0, & \text{if } p_L^e > p^* \\ [0,1], & \text{if } p_L^e = p^* \\ 1, & \text{if } p_L^e < p^* \end{cases}. \tag{4.4}$$

Rational choice theory, as modeled by subjective expected utility theory, simply predicts that George chooses a strategy to maximize his utility

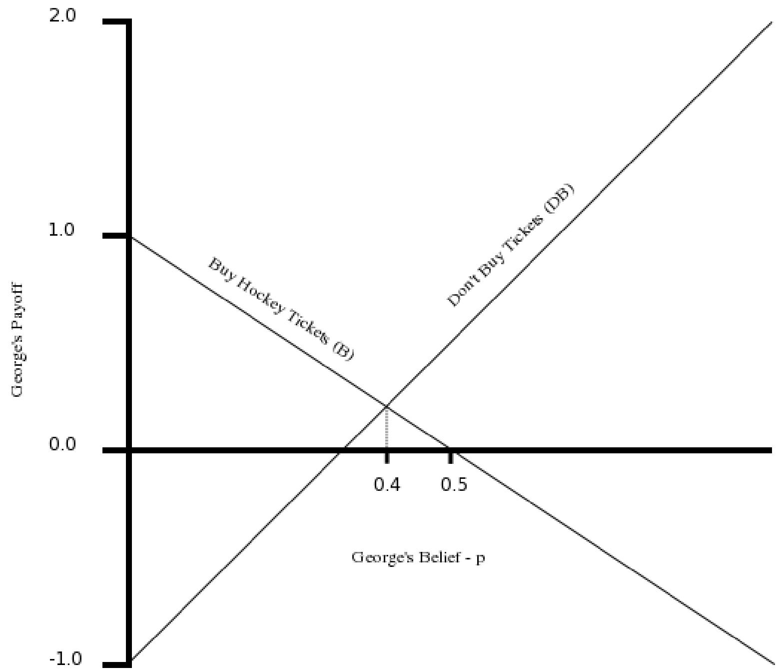

Figure 4.2
George's Payoffs given Beliefs

given his beliefs. The central question of game theory is how to model the formation of these beliefs. Lucy carries out the same calculus, and hence forms a belief p_G^e regarding the probability that George chooses B. In that case her best reply correspondence, where ρ_L is the probability that Lucy chooses B is given by:

$$r^L\left(p_G^e\right) = \arg\max_{\rho\in[0,1]} U^L\left(p_G^e, \rho\right) = \begin{cases} 0, & \text{if } p_G^e > p^{**} \\ [0,1], & \text{if } p_G^e = p^{**} \\ 1, & \text{if } p_G^e < p^{**} \end{cases}, \tag{4.5}$$

where $p^{**} = 0.4$ is the strategy by George that makes Lucy indifferent between buying and not buying the opera ticket.[10]

In this case all strategies are rationalizable - even though preferences are well defined and it is common knowledge that both parties are rational, there is no way to make a prediction in this game. This illustrates the important point that rational choice theory alone is not sufficient to make a unique prediction regarding individual behavior in strategic situations. At one level this may seem unsatisfactory, but then again it does appear to be consistent with the casual observation that making predictions about economic events seems to be a difficult, if not impossible, task!

In this book we are concerned with incentive contracts - the kind of agreements that individuals would make in the shadow of rational choice. As a matter of law, a necessary condition for a contract to be enforceable is that parties have achieved a "meeting of the minds". This idea is modeled by supposing that individuals correctly anticipate how

10 The careful reader might observe that we have not strictly applied the Savage's model to this situation. That would require expanding the state space to include Lucy's beliefs as explicit states. But if the hypothesis that they are rational is common knowledge, then Lucy would have to form states that would include George's beliefs regarding Lucy's beliefs and so on. Thus we would get a hierarchy of beliefs, greatly complicating the analysis. See Myerson (1991) chapter 2 for a further discussion. Mertens and Zamir (1985) provide a formal analysis of this procedure.

the other person will act. This is captured by the concept of a Nash equilibrium (Nash (1950)).

4.3 Nash Equilibrium

The Nash equilibrium concept was first introduced by Cournot (1838) in his famous formulation of the duopoly problem. He envisioned two firms adjusting their output each period to maximize their profits given the other firm's output last period. He showed that this process leads to an outcome at which each firm's output is optimal given the other firm's strategy. John Nash (1950) extended this idea to allow for an arbitrary number of players, and provided a general existence proof. In the context of our model of rational choice, the Nash equilibrium concept adds to the criteria of rationalizability the requirement that each player correctly anticipates how the others will play.

Formally, consider a game (in normal form) given by $G = \left\{\Gamma^i, U^i\right\}_{i \in N}$, where $N = \{1, ..., n\}$ is the set of players, Γ^i is the set of strategies available to player i, and $U^i(\gamma), \gamma \in \Gamma = \Gamma^1 \times ... \times \Gamma^n$ is the utility for player i. We write $U^i(\hat{\gamma})$ when referring to mixed strategies, as defined in 4.1. We assume throughout that agents have perfect recall, hence we will not in general distinguish between mixed strategies and behavioral strategies unless it is necessary.

Definition: A strategy vector $\gamma^* \in \Gamma$ is a *Pure Strategy Nash Equilibrium* for the game $G = \left\{\Gamma^i, U^i\right\}_{i \in N}$ if for every $i \in N$:

$$U^i(\gamma^*) \geq U^i\left(\gamma^i, \gamma^{-i*}\right), \text{ for every } \gamma^i \in \Gamma^i. \tag{4.6}$$

The vector $\left(\gamma^i, \gamma^{-i*}\right)$ represents γ^*,with the $i'th$ coordinate replaced by γ^i. This definition does not depend upon the nature of the strategy space Γ, and applies equally to the case where Γ is an infinite set, as in the case of mixed strategies. If the mixed strategies are defined over a finite set of pure strategies then Nash (1950) proves the existence of an equilibrium. See exercise 8 for the proof.

Consider again the case of Lucy and George. Suppose they can agree beforehand that George will buy the hockey tickets when he goes to work. Notice that at the time that George buys the tickets Lucy cannot observe this action, nor will she know the outcome until arriving home in the evening. Hence, George buys the tickets because he *believes* Lucy will not buy the opera tickets and hence knows that he is better off with this action. This agreement is *self-enforcing:* both individuals have an incentive to follow through upon the agreement.

For this game there also exists a mixed strategy Nash equilibrium. We can find this equilibrium by observing that Nash equilibria are best replies to the other player's strategy, and hence a strategy pair (ρ_G, ρ_L) forms a Nash equilibrium if:

$$\rho_G \in r^L(\rho_L), \tag{4.7}$$

$$\rho_L \in r^G(p_G), \tag{4.8}$$

where these reaction functions are defined in 4.4 and 4.5. The reaction functions for this game are illustrated in figure 4.3. In the case of George, it is optimal to buy the hockey tickets when the probability of Lucy buying the opera tickets is between 0 and 0.4. If Lucy chooses buy (B) with a probability of 0.4 then George is indifferent between buying and not buying. The situation is similar for Lucy. Thus, in this game there are three Nash equilibria: $(1,0)$, $(0,1)$ and $(.4,.4)$. The first two equilibria are called pure strategy equilibria, while $(.4,.4)$ is called a mixed strategy equilibrium because it entails randomization by at least one of the players.

The concept of a Nash equilibrium has proven to be very durable in economics. It is almost universally accepted as *the* definition of rational play. Strictly speaking, it is a purely static concept that is much more restrictive that the hypothesis of rational choice alone. One possible

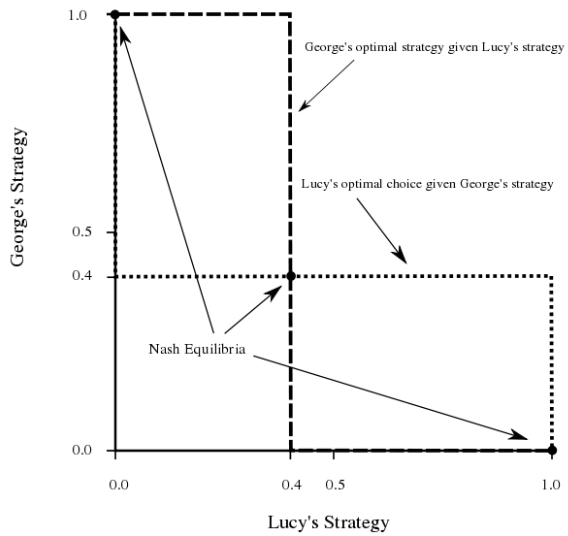

Figure 4.3
Reaction Functions for the Battle of the Sexes Game

reason for its durability is that it is the plausible outcome of a number of different strategic situations. The first of these, as we discussed above, is Cournot's dynamic duopoly model in which players are assumed to choose the optimal strategy with beliefs determined by the previous period's play. There is now a vast literature exploring various behavioral foundations for the concept of a Nash equilibrium.[11]

For the purposes of contract theory, the Nash equilibrium concepts is a natural *necessary* condition.[12] If it is common knowledge that two parties are rational then, for any agreement, they would expect the other party to take actions consistent with their self-interest (as represented by their preferences).[13] A key feature of a legally binding contract is that parties have a *meeting of the minds*, a condition that is formally captured by the requirement that one's beliefs regarding the other person's play are correct. The rest of this book very much relies upon this interpretation of a Nash Equilibrium. However, a Nash equilibrium is not always a *sufficient* condition to capture the *meeting of the minds* interpretation of the concept.

4.4 The Problem of Commitment

In this section we show that the Nash equilibrium concept is not a *sufficient* condition for rational choice in contract-like situations. This can be illustrated with the following modification of the battle of the sexes game. Suppose that George (the one who prefers Opera) has the first chance to buy the hockey tickets, and then leaves a message for Lucy saying whether or not he was able to buy the tickets. Lucy then decides

11 See **?** and Binmore (1998) for an entertaining wide range of discussion of the foundations of game theory.

12 See the discussion in chapter 5 of Schelling (1980) explicitly linking the notion of a Nash equilibrium to the concept of an agreement.

13 Self interest does not mean that parties do not behave altruistically, only that if they are altruistic, then this should be reflected in their preferences.

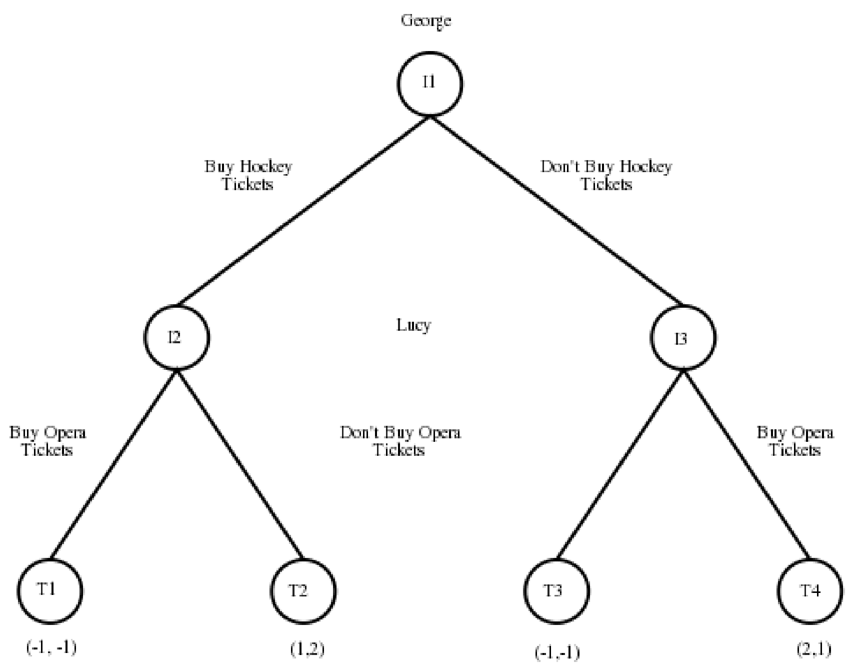

Figure 4.4
Extensive Form for Battle of the Sexes Game

whether or not to buy Opera tickets *after* George has made his move. The strategies and payoffs can be illustrated using the *extensive form*, as shown in figure 4.4.

In this figure I1, I2 and I3 denote *information sets (nodes).* In this case George has no information when deciding what to do, but Lucy knows George's choice at the time that she decides, and hence is able to make a different decision depending upon whether she has information I2 or I3. Here the strategy is a pair that specifies how she will play in each information set. For example $\{B, DB\}$ represents B if I2 occurs (George buys the hockey tickets) and DB if I3 occurs (George does not buy the tickets). In total Lucy has four possible strategies. Given the payoffs at the terminal nodes T1-T4, the normal form for this game is:

George	Lucy			
	{B,B}	{B,DB}	{DB,B}	{DB,DB}
DB	$(2,1)^*$	$(-1,-1)$	$(2,1)^*$	$(-1,-1)$
B	$(-1,-1)$	$(-1,-1)$	$(1,2)$	$(1,2)^*$

Table 4.4
Normal Form of the Battle of the Sexes Game

For this game there are three pure strategy Nash equilibria: $\{DB,\{B,B\}\}$ and $\{DB,\{DB,B\}\}$ resulting in an outcome of $(2,1)$, and $\{B,\{DB,DB\}\}$ resulting in $(1,2)$. An interesting feature of the third equilibrium is that it requires Lucy not to buy opera tickets if George does not buy hockey tickets. This equilibrium has a very familiar interpretation - it corresponds to Lucy saying to George:

"You better buy the hockey tickets, because under no condition will I buy the opera tickets."

This action appears to be an *incredible threat.* It supposes that if for whatever reason George does not buy Hockey tickets, then Lucy will, against her own self-interest, not buy the Opera tickets. However, it *is* a Nash equilibrium. If Lucy believes that George will indeed buy the tickets, this threat of not buying the opera tickets is optimal. The root of the problem is that rational learning theory has no satisfactory way to deal with zero probability events. Namely, at the equilibrium, Lucy assigns zero probability to George not buying the tickets. This implies that even if Lucy observes that George has not bought the hockey tickets, the strict application of Bayes' rule implies that she ignores this information.[14]

Selten (1975) provides the most widely accepted solution to the problem. He proceeds in two steps. The first is to recognize that it is most

14 See Diaconis and Zabell (1982) for a discussion and formal solution to the problem.

appropriate to suppose that each time a person plays, he can be viewed as a new player, whose preferences happen to be the same as other players in the game with the same index. The attentive reader will notice that this assumption ensures that the set of behavioral strategies and mixed strategies are always the same.

In essence this ensures that each time a choice is made, it is done with respect to only the information available at the time the decision is made. This transforms the original game into one that Selten calls the *agent normal form.* Referring back to the construction of the extensive form, this requires that for each information set, E_{tk}, one creates a new player with the same preferences as player i_{tk}, the player who will choose an action at stage t.

Second, in order to deal with zero probability events, Selten requires the equilibrium to be stable against a perturbation of the game that results in all information sets being reached with strictly positive probability. This ensures that players can always use Bayes' rule to update their beliefs when an information set is reached. The result is Selten (1975)'s concept of a *trembling hand perfect equilibrium, also known as a perfect equilibrium.* In our example, this would have the effect of creating two Lucys, Lucy-2 who plays when event $I2$ occurs, and Lucy-3 who plays when $I3$ occurs. In that case the normal form of the game would be:

George	Lucy 2		Lucy 3	
	B	DB	B	DB
DB			$(2, 1, 1)$	$(-1, -1, -1)$
B	$(-1, -1, -1)$	$(1, 2, 2)$		

Table 4.5
Agent Normal Form

In this game there are now three pure strategy Nash equilibria, $\{B, DB, DB\}$, $\{DB, DB, B\}$, and $\{DB, B, B\}$. Consider the equilibrium $\{B, DB, DB\}$, and suppose there is a small chance that George plays DB. In that case Lucy-2 would optimally choose B rather than DB. That is, a small perturbation in B's strategy leads to a large change in Lucy's strategy, and hence this equilibrium is *not stable.* More formally, suppose then that each player i chooses strategy $\gamma \in \Gamma^i$ with probability of at least $\varepsilon_\gamma^i > 0$, and let $\hat{\epsilon} = \left\{\varepsilon_\gamma^i\right\}_{i \in N, \gamma \in \Gamma^i}$ be the vector of $\varepsilon' s$. Denote this new strategy space by $\hat{\Gamma}_{\hat{\epsilon}}^i$.

Definition: Given any game represented by its agent Normal Form: $\left\{U^i, \hat{\Gamma^i}\right\}_{i \in N}$, then the mixed strategy γ^* is a *trembling hand perfect Nash equilibrium* if there exists a sequence of Nash equilibria $\gamma_{\hat{\epsilon}_n}^*$ for the perturbed game $\left\{U^i, \hat{\Gamma}_{\hat{\epsilon}_n}^i\right\}_{i \in N}$, such that as$\hat{\epsilon}_n \rightarrow \hat{0}$, $\gamma_{\hat{\epsilon}_n}^* \rightarrow \gamma^*$.

By convergence one means that the probability of each action in $\gamma_{\hat{\epsilon}_n}^*$ converges to the probability for the corresponding action in γ^*. The proof of existence follows from the existence of a Nash equilibrium, and the fact that the set of mixed strategies is closed. See exercise 8.

Perfect equilibria have the property that they are "close" to some equilibrium in the perturbed game. Even though this concept is defined in terms of local stability, it has some rather remarkable decision theoretic properties, including the elimination of "incredible" threats. The connection between perfect equilibria and decision theory is worked out in Kreps and Wilson (1982). They introduce a number of desirable properties that an equilibrium should satisfy, and show that perfect equilibria satisfy these properties.

To fix these ideas, consider the game illustrated in figure 4.5. This game has two Nash equilibria that depend upon what George believes Lucy will do, which in turn depends upon Lucy's beliefs regarding

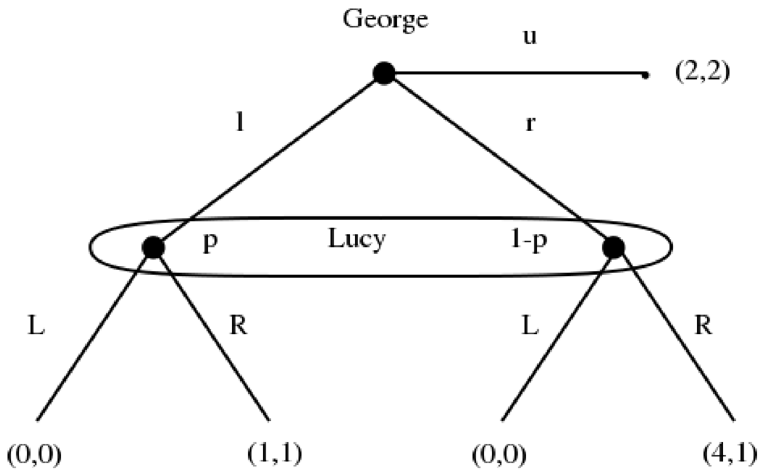

Figure 4.5
Beliefs in Extensive Form

George. Suppose Lucy thinks George will play up, then the *ex ante* probability of Lucy making a choice is zero. Hence it is a Nash equilibrium for George to choose u and for Lucy to choose L.

Notice that Bayesian decision theory is of no use here because it provides no way to update beliefs. However, Lucy's decision is *inconsistent* with the hypothesis that she is a rational decision maker in the sense of Savage. If George believes that Lucy is rational in the sense of Savage, then he must suppose that conditional upon being given a choice, Lucy must form a belief over George's play, given by p, the probability that George chose l. We call this belief Lucy's *beliefs* at this information set. Given these beliefs, she would always choose R. Hence, if George believes Lucy is rational in the sense of Savage, he will always play r, and never u or l. Given this, Lucy would rationally set $p = 0$, resulting in a unique equilibrium for this game. It fact it is easy to see that this is the unique perfect equilibrium.

More formally, let G be an extensive form game with a perfect equilibrium $\hat{\gamma}^* \in \Gamma$, and suppose that $\hat{\gamma}_{\varepsilon_n}$is the sequence of strategies in the perturbed game that is converging to $\hat{\gamma}^*$. Let us suppose that it is in agent form, in this sense we index players by tk, where t is date of decision, k indexes the information sets at date t, and the utility function of agent tk is given by $U^{i_{tk}}$. In addition, we now keep track of the beliefs of the players. Let μ_{tk} be a probability distribution over the nodes in E_{tk}, with the interpretation that these represent the beliefs of player tk at this information set. These are called *beliefs.* Consistent with the Nash equilibrium concept, it is assumed that the strategies of the players are common knowledge, which in turn places some structure upon beliefs.

Since the strategies $\hat{\gamma}_{\varepsilon_n}$ are completely mixed, then each node in the game occurs with positive probability. Thus for each information set, E_{tk}, we can define the probability of a node $n \in E_{tk}$ occurring, conditional upon reaching this information set, as given by:

$$\mu_{tk}^{\epsilon_n}(n) = \frac{\Pr(n|\hat{\gamma}_{\varepsilon_n})}{\Pr(E_{tk}|\hat{\gamma}_{\varepsilon_n})} > 0. \tag{4.9}$$

The probability distribution, $\mu_{tk}^{\epsilon_n}$ is called player $i_{tk}'s$ beliefs at information set E_{tk}. Given that it is in a compact set, and there is a finite number of possible information sets, then we can take a subsequence that ensures there are limit points: $\mu_{tk} = \lim_{n\to\infty} \mu_{tk}^{\epsilon_n}$. Note that in general, not all information sets are reached in equilibrium, nevertheless these limit beliefs are defined for *all* information sets. Together, the strategy profile γ and the beliefs, $\{\mu_{tk}\}_{E_{tk}\in\Phi}$ are called an *assessment.*

This structure allows us to formally model how a rational person would play if there is a deviation from the equilibrium strategy. In particular, for each information set E_{tk},we can define a "sub-game" that corresponds to the occurrence of E_{tk}. Let r_{tk} define the root at which Nature plays, and selects nodes in E_{tk} with probabilities given by μ_{tk}. The set of nodes for the game is given by $S(E_{tk}) \cap \Sigma$,with final nodes $S(E_{tk}) \cap Z$, where the payoffs are defined as before. We can now consider strategies that are restricted to these nodes. Let $\mu_{tk}(z|\hat{\gamma})$ be the probability distribution over z in this restricted game, then for each information set E_{tk} we can let $G_{tk} = \{\Gamma^i, U^i_{tk}\}_{i\in N}$ denote the game in strategic form where:

$$U^i_{tk}(\hat{\gamma}) = \sum_{z\in Z} u^i(z)\mu_{tk}(z|\hat{\gamma}).$$

Given this definition, we have the following proposition.

Proposition 4.1 . *Let $\hat{\gamma}^*$ be a perfect equilibrium strategy profile of the game G with finite payoffs, and let μ_{tk} be a set of beliefs associated with this equilibrium. Then the* assessment $\left\{\hat{\gamma}^*, \{\mu_{tk}\}_{E_{tk}\in\Phi}\right\}$*is* sequentially rational, *namely for every $i = i_{tk}$* and game G_{tk}:

$$U^i_{tk}(\hat{\gamma}^*) \geq U^i_{tk}(\gamma'^i, \hat{\gamma}^{-i*}), \forall \gamma'^i \in \Gamma^i.$$

Proof: Suppose that the strategy is not sequentially rational, then there is an information set E_{tk}, player i, and associated action, a'_{tk} such that $U^i_{tk}(\gamma'^i, \hat{\gamma}^{-i*}) > U^i_{tk}(\hat{\gamma}^*)$, where γ'^i is the same as $\hat{\gamma}^{i*}$, except at information set E_{tk} where a'_{tk} is played. Let $\delta = U^i_{tk}(\gamma'^i, \hat{\gamma}^{-i*}) - U^i_{tk}(\hat{\gamma}^*) > 0$. Let $\hat{\gamma}^n$ be the completely mixed Nash equilibrium converging to $\hat{\gamma}^*$ used to define the perfect equilibrium, and let ϵ^n be the associated minimum probability with which each strategy in the Agent normal form is played. Let $\tilde{\gamma}^{in}$ be a completely mixed strategy that is identical to $\hat{\gamma}^{in}$ except it converges to choosing a_{tk} at information set A_{tk}. Then there is a n' that $\left|U^i_{tk}(\gamma'^i, \hat{\gamma}^{-i*}) - U^i_{tk}(\tilde{\gamma}^{in'}, \hat{\gamma}^{-in'})\right| < \delta/3$ and $\left|U^i_{tk}(\hat{\gamma}^{n'}) - U^i_{tk}(\hat{\gamma}^*)\right| < \delta/3$. Thus we have:

$$\begin{aligned}
&U^i_{tk}(\tilde{\gamma}^{in'}, \hat{\gamma}^{-in'}) - U^i_{tk}(\hat{\gamma}^{n'}) \\
&= U^i_{tk}(\gamma'^i, \hat{\gamma}^{-i*}) - U^i_{tk}(\hat{\gamma}^*) - U^i_{tk}(\gamma'^i, \hat{\gamma}^{-i*}) + U^i_{tk}(\tilde{\gamma}^{in'}, \hat{\gamma}^{-in'}) - U^i_{tk}(\hat{\gamma}^{n'}) + U^i_{tk}(\hat{\gamma}^*) \\
&\geq U^i_{tk}(\gamma'^i, \hat{\gamma}^{-i*}) - U^i_{tk}(\hat{\gamma}^*) - \left|U^i_{tk}(\gamma'^i, \hat{\gamma}^{-i*}) - U^i_{tk}(\tilde{\gamma}^{in'}, \hat{\gamma}^{-in'})\right| - \left|U^i_{tk}(\hat{\gamma}^{n'}) - U^i_{tk}(\hat{\gamma}^*)\right| \\
&> \delta - \delta/3 - \delta/3 \\
&> 0.
\end{aligned}$$

Hence $\hat{\gamma}^{in'}$ is not a best response at E_{tk}. But since the $\hat{\gamma}^n$ is completely mixed, this implies that E_{ik} is reached with strictly positive probability. The payoffs for $z \notin S(E_{tk})$ have the same probabilities of occurring under $\hat{\gamma}^{in'}$ and $\hat{\gamma}^{in}$, from which we conclude that $\hat{\gamma}^n$ is not a Nash equilibrium in the perturbed game, contradicting the hypothesis that γ^* is a perfect equilibrium.

From this result we obtain immediately the following corollary:

Corollary: Suppose that the game G_{tk} defines a proper subgame, in the sense that for all information sets $E \in \Phi$, $E \cap S(E_{tk}) \in \{E, \emptyset\}$. Then if $\hat{\gamma}^*$ is a perfect equilibrium strategy profile of the game G, it is also a Nash equilibrium strategy profile of the game G_{tk}.

The final result can be used to eliminate some of the equilibria in the game between Lucy and George without resorting to the construction of a perturbed game. We can solve the game illustrated in figure 4.4 by backward induction - namely we go to the last period, and then consider a game at which each of the final period information sets are reached with probability one. It is clear that in this case Lucy has a unique optimal strategy, namely buy the opera tickets if and only if George has not bought the hockey tickets. Given these choices, George's unique best choice is to not buy the hockey tickets. Since this is the only outcome, then this is the unique perfect equilibrium of this game.

We have emphasized the decision theoretic foundations of the perfect equilibrium concept. For complicated extensive form games, that approach provides a convenient method to find the equilibria of a game. We discuss this further in the next section. In simple normal form games one can use simple dominance arguments to find the perfect equilibria. Since perfect equilibria are constructed from games that place positive probability weights on all strategies, then any strategy with the feature that it is dominated by another strategy would not be used. Here dominance is defined as follows.

Definition: Given a game $G = \{U^i, \Gamma^i\}_{i \in N}$, then a strategy γ^i is *weakly dominated* by $\tilde{\gamma}^i$ if for every $\gamma^{-i} \in \Gamma^{-i}$ it is the case that $U^i\left(\tilde{\gamma}^i, \gamma^{-i}\right) \geq U^i\left(\gamma^i, \gamma^{-i}\right)$, with strict inequality for at least one strategy profile $\gamma^{-i} \in \Gamma^{-i}$. A strategy profile γ is *weakly undominated* if for each player i there exists no $\tilde{\gamma}^i$ that weakly dominates player i's strategy.

It is straightforward to show the following proposition.

Proposition 4.2 . *Consider the game* $G = \left\{U^i, \hat{\Gamma}^i\right\}_{i \in N}$ *where* $\hat{\Gamma}^i$ *is the set of mixed strategies over a finite set* Γ^i*, then if* γ^* *is a perfect equilibrium it is weakly undominated.*

The converse is not true, namely, in the case of 3 or more players if a strategy is weakly undominated, then this does not necessarily imply that it is perfect. Rather, the converse is true only in the case of two player games:

Proposition 4.3 . *Consider the two person game* $G = \left\{U^i, \hat{\Gamma}^i\right\}_{i \in \{1,2\}}$ *where* $\hat{\Gamma}^i$ *is the set of mixed strategies over a finite set* Γ^i*, then* γ^* *is a perfect equilibrium if and only if it is weakly undominated.*

See section 3.2 of van Damme (1991) for a proof of this result.

4.5 Sequential Equilibrium

Though the perfect equilibrium concept provides an elegant solution to the problem of how to model rational play in a game, it suffers from two problems. The first of these is that it is well defined only in the case of a finite game. Second, it is very difficult to compute explicitly. Hence in practice, one typically uses a weaker equilibrium concept based upon some subset of the properties of a perfect equilibrium.

The first of these is the concept of a *subgame perfect Nash equilibrium,* introduced in Selten (1965), (See van Damme (1991) for an excellent and comprehensive review of solution concepts in game theory).

Definition: A strategy profile $\hat{\gamma}$ is a *subgame perfect Nash equilibrium* if for every information set E_{tk} that is a singleton (consists of a single node) and that the corresponding subgame G_{tk} is a proper subgame, then $\hat{\gamma}$ is a Nash equilibrium of the game G_{tk}.

From corollary 4.4 it follows that all perfect equilibria are subgame perfect, but there are situations where the converse does not hold. For example, the game in 4.5 that we discuss above has no proper subgames beginning at a singleton.

The notion of a subgame perfect equilibrium is very useful for the study of repeated games that is studied in the next section. A repeated game is one in which the same normal form game is played for multiple periods, and all past plays can be observed. Subgame perfect is the property that in each period, regardless of past play, players are assumed to play a Nash equilibrium from that point on. It is also the case that for games with perfect information (all information sets are singletons), the set of perfect equilibria and subgame perfect equilibria coincide and can be easily computed using backward induction.

Subgame perfection does not provide a good way to model credible play over time when there is asymmetric information. In that case parties may not know the characteristics of the other players, and hence there are typically no proper subgames, and thus subgame perfect would not refine the set of equilibria. In these cases one uses the notion of a perfect Bayesian equilibrium:

Definition: An assessment $\{\gamma, \{\mu_{tk}\}_{E_{tk}\in\Phi}\}$ forms a *perfect Bayesian equilibrium* in an extensive form game G if the beliefs μ_{tk} are such that:

1. The strategies are sequentially rational, as defined in proposition 4.1.
2. For every game G_{tk} corresponding to information set E_{tk}, for any information set $E \subset S(E_{tk})$, if $\Pr\{E|\gamma, E_{tk}\} > 0$ then for each $n \in E$ it must be the case that:

$$\mu_E(n) = \frac{\Pr(n|\gamma, E_{tk})}{\Pr(E|\gamma, E_{tk})}, \tag{4.10}$$

where μ_E is the belief at information set E, and $\Pr(\bullet|\gamma, E_{tk})$ is the probability measure over the nodes in $S(E_{tk})$ when nature is assumed to select nodes in E_{tk} with probability given by μ_{tk}.[15]

The point here is that beliefs should be consistent with equilibrium play whenever possible. In practice there is not a single accepted definition of a perfect Bayesian Nash Equilibrium. Rather, authors tend to define the concept in a way that is convenient for the problem at hand. The term "perfect Bayesian Nash equilibrium" is appropriate whenever the solution satisfies sequential rationality, and Bayes' rule where "possible". See part III of Fudenberg and Tirole (1991) for further discussion.

Finally, there is the sequential equilibrium concept introduced by Kreps and Wilson (1982). This paper is important because it was the first to explicitly link notions of an equilibrium to decision theory. Kreps and Wilson explicitly introduced the idea of an assessment. They say that beliefs are *consistent* with a strategy profile $\hat{\gamma}$ if they are the limit points of beliefs derived via equation 4.9 from any sequence of completely mixed strategies that converges to $\hat{\gamma}$. Thus we have:

15 More precisely, these probabilities form what Myerson (1986) calls a *conditional probability system*. Namely, one has a well defined condition probability for all events, even zero probability events. Myerson (1986) shows that beliefs constructed via small perturbations, as we have done for the perfect equilibrium concept, defines a conditional probability system.

Definition: An assessment consisting of a strategy profile $\hat{\gamma}$ and associated beliefs μ_{tk} is a *sequential* equilibrium if:

1. The beliefs are consistent with $\hat{\gamma}$.
2. The strategies are sequentially rational, as defined in proposition 4.1.

The difference between a sequential equilibrium and a perfect equilibrium is that the former does not require the test sequence of completely mixed strategies used to generate the beliefs to form a Nash equilibrium in the perturbed game. Like the perfect equilibrium concept, ensuring that beliefs are consistent is difficult, so in practice one typically uses some form of perfect Bayesian Nash equilibrium for games with asymmetric information, and subgame perfect for games with complete information and for repeated games.

Proposition 4.4 . *Every perfect equilibrium of a finite game* $G = \left\{U^i, \Gamma^i\right\}_{i\in N}$ *is also a sequential equilibrium.*

See exercise 8 for the proof. This ensures that for finite games at least a sequential equilibrium exists. The benefit of a sequential equilibrium is that it is well defined and exists for some infinite games where the notion of a perfect equilibrium is not even well defined. We return to this issue in the context of some explicit models in the subsequent chapters.

4.6 Repeated Games and Payoff Possibilities Set

The extensive form model for game theory illustrates that the theory is very general - in principle it can be applied to any decision making situation. However, it is a very complex model that contains many elements that are unknowable to an outside observer. Moreover, even if one did have complete information regarding the strategic game being played by individuals, the notion of a sequential equilibrium does not produce a unique solution. As a consequence, there is a large literature in game theory on how best to select a Nash equilibrium from the set of possible equilibria.[16] Despite many years of research into the issue there is no accepted solution to the problem of equilibrium selection.

In this section we provide a very brief introduction to repeated game theory, a theory that is widely used to model the interaction of individuals over time. One of the most important applications has been the use of the prisoner's dilemma game to understand the problem of cooperation. Axelrod (1984)'s book argued that the prisoner's dilemma is a parable on life that can be used to find ways for self-interested individuals to enter into cooperative relationships.

It is worthwhile to keep in mind the distinction between a repeated game and a relational contract. The goal of game theory is to study how rational individuals choose strategies for *any* strategic game. In contrast, a contract is the *design* of a game that rational individuals would play, including the payoffs they would receive after different choices. The goal of contract theory is to achieve a particular allocation of resources, and thus it is less concerned with understanding all the behavior that is possible in a game, except to the extent that the contract needs to anticipate future behavior.

Thus, the analysis in this section is restricted to the study of the payoffs that are possible in a repeated game. We shall show that with the following simple structure, any payoff in a repeated game with symmetric information in each period can be achieved. First, individuals agree, either implicitly or explicitly, upon payoffs and the associated choices that will generate these payoffs each period. Then, individuals are expected to follow these actions each period. Should an individual be observed to deviate from this agreement, in the next period the subgame perfect

16 van Damme (1991) has an excellent review of the early literature on refinements of Nash equilibria.

equilibrium that gives the lowest payoff to this deviator is played.[17] This strategy provides necessary and sufficient conditions for the characterization of all payoffs that can be supported by some subgame perfect Nash equilibrium.

This result follows from the theory of dynamic programming, which shows that optimal decision making over time can be reduced to a trade-off between current payoffs, and a reduced form representation of the future defined by a *value function.* Beginning with the work of Abreu (1988), we can apply this same kind of reasoning to games played over time where it is assumed the game played each period is the same. This allows us to characterize behaviors that can be shown to be equilibria in the much more complex game that entails a full analysis of all possible strategies.

The approach here follows the work of Abreu et al. (1990) who show that one can represent the future payoffs in terms of a payoff possibilities set. The notation and technical development follows Bergin and MacLeod (1993), who, building upon Telser (1980), view strategies in a repeated game as a self-enforcing agreement between two parties. The term *agreement* refers to the idea that parties have correct expectations regarding how each party will play. The term *self-enforcing* means that parties play a subgame perfect Nash equilibrium. If we are in a case with asymmetric information then *a self-enforcing agreement* would refer to outcomes that are the consequence of playing a perfect Bayesian Nash equilibrium.

A Basic Repeated Game Model

Consider a two person normal form game, $G = \left\{u^i, A^i\right\}_{i=1,2}$, where the space A^i has n possible actions denoted by $a^i \in A^i = \left\{a^i_1, ..., a^i_n\right\}, i =$

17 If more that one player deviates in any period then there is no punishment. The notion of a Nash equilibrium only considers deviations by single players. We come back to this issue below when we discuss renegotiation.

$1, 2$. The results of this characterization extend easily to more than two players (see Myerson (1991) for details). The game is played repeatedly in periods $t = 0, 1, 2..$, with the following sequence of steps each period:

1. The past sequence of plays, denoted by $h^t \in H^t$, is observed by both players, $h^0 = \emptyset$.
2. The players observe $p^t \in [0, 1]$, where it is assumed that p^t is uniformly distributed over $[0, 1]$ and independently drawn each period.
3. Each player chooses their action $a^{it} \in A^i$.
4. Each player realizes a payoffs $u^{it} = u^i \left(a^{1t}, a^{2t}\right)$.

The introduction of a common shock in step 2 is a formal way to implement public randomization in the repeated game. It allows us to have a convex payoff space while avoiding the complexity that comes with the use of mixed strategies. If mixed strategies are used, then if players only observe actions, then they do not know the probability that an action is chosen, and hence there are no proper subgames each period. With public randomization we gain the convenience of a convex payoff space, while ensuring that agents have symmetric information at the beginning of each period.[18]

We will exploit the public signal to provide a complete characterization of all SPNE payoffs. Specifically, with a common shock the strategy specifies which observable actions are to be chosen when $p^t \in [0, 1]$ occurs, and thus deviations from equilibrium play can be immediately detected and punished.

18 One could make the mixed strategy observable via a process known as *strategy purification*. One can suppose that players observe a private shock p^{it} and then choose their action, and that p^{it} becomes commonly observable after actions are chosen. Requiring agents to coordinate upon a commonly observed event is rather natural in contract theory (such as rate of inflation or adverse events), but is not a natural assumption for mixed strategies. For example, mixed strategies are natural in sporting events, such as when defending against a penalty kick in soccer. In this case the goalie has little time to react to a kick, and so typically dives randomly to one side, just as the attacker randomly chooses to go left, right or middle.

The choice in period t is given by a (Borel measurable) function:

$s^{it} : [0,1] \to A^i.$

Given an choice pair $s^t = \left\{s^{1t}, s^{2t}\right\} \in S = S^1 \times S^2$, where S^i is the set of possible choices in period t for player i, we can define the *event* $E\left(a, s^t\right)$ that generates an action $a = \left\{a^1, a^2\right\} \in A = A^1 \times A^2$ by:

$$E\left(a, s^t\right) = \left[p^t | s^t\left(p^t\right) = a\right],$$

with the corresponding probability that action a is chosen given by:

$$\lambda\left(a, s^t\right) = Prob\left\{E\left(a, s^t\right)\right\}.$$

This is the probability that action pair a is observed. Without loss of generality we suppose that whenever $\lambda\left(a, s^t\right) = 0$ then $E\left(a, s^t\right) = \emptyset$.[19] Thus we can define the payoff for player i in period t by:

$$u^i\left(s^t\right) = \int_0^1 u^i\left(s^t\left(p\right)\right) dp = \sum_{a \in A} u^i\left(a\right) \lambda\left(a, s^t\right). \tag{4.11}$$

Thus the set of possible payoffs is given by the convex, compact set:

$$F = conv\left\{\left\{u^1\left(a\right), u^2\left(a\right)\right\} | a \in A\right\} \subset \Re^2,$$

where *conv* denotes convexification of the set. This defines all the payoff possibilities within the stage game. Since players are strategic this implies that when faced with a player that wishes to lower their payoff they can always mitigate by choosing the action that maximizes their payoff after each opponent's action. Namely, player i can always ensure that she gets at least:

$$\bar{u}^i = min_{a^j \in A^j} max_{a^i} u^i\left(a^1, a^2\right), j \neq i.$$

19 Without this assumption we would have to worry about zero probability events where players might choose non-equilibrium play. This would lead to unnecessary complications.

This is the utility that player i can guarantee, regardless of player j's action. This is called the individually rational payoff or the minimax payoff. If we allowed mixed strategies then we would have that this level is also equal to the $max - min$. With pure strategies this is not necessarily the case (see exercise 9). Accordingly, we define the set of individually rational payoffs for the normal form game G by:

$$\bar{F} = \left\{ u \in F | u^i \geq \bar{u}^i, i = 1, 2 \right\}.$$

Let A^{NE} be the set of Nash equilibria when the payoffs are given by $\{u^1(a), u^2(a)\}$, then we have the following result:

Proposition 4.5 . *The set of Nash equilibrium payoffs for the game G is given by:*

$$F^{NE} = conv \left\{ u^1(a), u^2(a) | a \in A^{NE} \right\} = \left\{ u^1\left(s(p^t)\right), u^2\left(s(p^t)\right) | s(p^t) \in S^{NE} \right\} \subset \bar{F},$$

where S^{NE} is the set of Nash equilibrium choices for game G.

The proof is straightforward and left to the reader. This set is sometimes call the set of *correlated equilibria* because parties can correlate their choices via the public signal p^t. Rather than introduce a new equilibrium concept we have included the correlation signal as part of the definition of the game. Since we are not using mixed strategies, then we cannot be sure that a Nash equilibrium exists, and we need the assumption:

Assumption(Existence) There exists at least one Nash equilibrium for the game G, namely $F^{NE} \neq \emptyset$.

We are now in a position to define the payoffs for the repeated game and the associated strategies. We suppose that play occurs in continuous time and that the length of a period is Δ units (seconds or days) long. At the beginning of period t the sequence of moves described above are carried out immediately, and then the chosen actions a^t are held fixed for the period after p^t is observed. It is assumed that the discount rate is given by r and thus the payoff in period t given action a^t is given by:

$$\int_0^{\Delta} u^i\left(a^t\right)e^{-r\tau}d\tau = \frac{\left(1-e^{-r\Delta}\right)}{r}u^i\left(a^t\right).$$

When the game is repeated each period we shall suppose that players agree, either tacitly or explicitly, upon the choices s^t to play each period. Let $s = \left\{s^0, s^1, ..., s^{T-1}\right\} \in S^T$ denote this agreement, where T is the number of periods, which may be infinite. The payoff under this sequence in the repeated game starting in period τ is defined by:

$$U^i\left(s,\tau\right) = \frac{(1-\delta)}{\left(1-\delta^{T-\tau}\right)}\sum_{t=\tau}^{T-1}\delta^{t-\tau}u^i\left(s^t\right).$$

Let $U^i\left(s\right) = U^i\left(s,0\right)$ be the payoff at the beginning of the game. The factor $(1-\delta)/\left(1-\delta^{T-\tau}\right)$ renormalizes the payoff stream so that regardless of the values for the discount rate δ, the duration of a period, Δ and the number of periods T, we have that $U\left(s\right) = \left\{U^1\left(s\right), U^2\left(s\right)\right\} \in F$.

Strategies and Subgame Perfect Equilibria

Let us now define the set of possible strategies in this model. The history of the game in period t is the sequence of realizations of our random shock and the action chosen by each person. It is given by:

$$h^t = \left\{\left(p^0, a^0\right), \left(p^1, a^1\right), ..., \left(p^{t-1}, a^{t-1}\right)\right\} \in H^t,$$

where H^t is the set of possible histories in period t, and as a matter of convention $H^0 = \emptyset$. A strategy for player i is denoted by $\sigma^i = \left\{\sigma^{i0}, \sigma^{i1}, ...\right\} \in \Sigma^i$, and is a (Borel) measurable function:

$$\sigma^{it} : H^t \rightarrow S^i.$$

The consequence of playing a strategy pair is a sequence of choices that are determined recursively. Consider the recursive determination of the payoffs associated with a strategy σ:

1. In period 0 there is no history dependence, and the strategy σ^0 determines s^0, which in turn determines $u^{i0} = u^i\left(s^0\right)$.
2. To work out the payoff in period 1 we need to compute the probability of $a \in A$ in period 1 under the strategy σ. This is given by finding the pairs $\left\{p^0, p^1\right\}$ that give rise to choice a:

   $$E\left[a, \sigma, t=1\right] = \left\{\left\{p^0, p^1\right\} | \sigma^1\left(p^0, s^0\left(p^0\right)\right)\left(p^1\right) = a\right\}.$$

   Since the strategies and choices are measurable functions, then this is a measurable set for which the probability $\lambda\left(a, \sigma, 1\right)$ is well defined and we can compute the probability that a is chosen in period 1. This procedure can be applied recursively to determine the probability that action a is chosen in any period t under strategy σ. Hence, the choice in period t is well defined for each t and we can define $\lambda\left(a, \sigma, t\right) = Prob\left[E\left[a, \sigma, t\right]\right] \in \left[0, 1\right]$ as the probability that action a is chosen in period t under strategy σ.

Subgame perfect requires that an individual's strategy be an optimal response after any subgame. Accordingly, let $\sigma\left(h^{t^0}\right)$ denote the strategy from period t^0 onwards given a history h^{t^0}, and let $\lambda\left(a, \sigma\left(h^{t^0}\right), t\right)$ be the associated probability of choosing action a in period $t \geq t^0$. We now define the payoff from a strategy σ after any subgame h^{t^0} by:

$$U^i\left(\sigma, h^{t^0}\right) = \frac{(1-\delta)}{\left(1-\delta^{(T-t^0)}\right)} \sum_{t=t^0}^{T-1} \delta^{t-t^0} \sum_{a \in A} u^i\left(a\right) \lambda\left(a, t, \sigma\left(h^{t^0}\right)\right).$$

As a matter of convention we let $U^i\left(\sigma\right) = U^i\left(\sigma, h^0\right)$ be the payoff at the beginning of the game. We are now in position (finally!) to define the notion of a subgame perfect Nash equilibrium for this game:

Definition 4.6 . A strategy $\sigma \in \Sigma$ is a subgame perfect Nash equilibrium (SPNE) if:

$$U^i\left((\sigma^i, \sigma^{-i}), h^{t^0}\right) \geq U^i\left((\tilde{\sigma}^i, \sigma^{-i}), h^{t^0}\right),$$

for every $t^0 = 0, ...T$, every $h^{t^0} \in H^T$, $i = 1, 2$ and every $\tilde{\sigma}^i \in \Sigma^i$. Let $\Sigma^{SPNE}(T)$ be the set of SPNE for the game that is repeated T times. When we write Σ^{SPNE} this refers to the case $T = \infty$.

The set of possible SPNE is very large, with many strategies that are essentially equivalent. For example, it is only the probability of an action a being chosen that determines the payoff, and hence parties can coordinate upon the public signal p^t in many different ways.

Observe that playing a Nash equilibrium each period - namely setting $\sigma^t(h^t) = s$ for all $t \geq 0, h^t \in H^t$ for some $s \in S^{NE}$ is a SPNE. Hence the existence of a Nash equilibrium in the stage game G ensures the existence of a SPNE. In the end, players are interested only in payoffs, and not the structure of the equilibrium per se. The goal of repeated game theory is to show that payoffs that are superior to the payoffs from playing a Nash equilibrium to the stage game can be sustained. Our goal is to characterize the set of such payoffs defined by:

$$F^{SPNE}(T) = \left\{u \in F | u = \left(U^1(\sigma), U^2(\sigma)\right) \ \mathit{such\,that}\ \sigma \in \Sigma^{SPNE}(T)\right\}.$$

Given that NE to the stage game can be sustained in the repeated game we immediately have:

Proposition 4.7 . *For all $T \geq 1$ the set of subgame perfect Nash equilibrium payoffs satisfies:*

$$F^{NE} \subset F^{SPNE}(T) \subset \bar{F}.$$

In the next section we work out the equilibria in the repeated prisoner's dilemma game to illustrate how one can construct equilibria that are superior to the static Nash equilibrium. This is followed by a general characterization of the set of equilibrium payoffs.

Cooperation in the Repeated Prisoner's Dilemma Game

The repeated prisoner's dilemma game has received a great deal of attention as a parable on life (see in particular Axelrod (1984)). It captures the idea that relationships may not be efficient because the temptation to cheat is stronger than the gains from sticking to an efficient agreement. We shall show that if the prisoner's dilemma game is played repeatedly then there are SPNE equilibria that give payoffs that are superior to the Nash equilibrium in the one-shot game. These equilibria can be interpreted as *self-enforcing agreements.* In chapter 9 we use these ideas to build a theory of *relational contracts.*

Thus we view the payoff stream in equilibrium as an agreement or *promise* between the two parties on how they will play each period. For example, suppose that player 1 has more bargaining power, and they agree to play (C, C) 50% of the time and (D, C) the rest of the time. This is implemented by supposing that (C, C) is played whenever $p^t \geq 0.5$ and (D, C) is played when $p^t < 0.5$. The expected payoff to players in the prisoners dilemma are given in table 4.3, and hence this agreement leads to payoffs:

$$\begin{aligned} u^{1*} &= 0.5 \times u^1(C, C) + 0.5 \times u^1(D, C) = 1 + 1.5 = 2.5, \\ u^{2*} &= 0.5 \times u^2(C, C) + 0.5 \times u^2(D, C) = 1 - 0.5 = 0.5. \end{aligned}$$

Observe that in the prisoner's dilemma the minmax payoffs and the Nash equilibrium payoffs are the same and equal to zero. Thus the agreement to play this way each period is better than the worst that could happen to either player in the game. These points are illustrated in Figure 4.6. Here the dotted area are the set of payoffs that are individually rational.

Next, we work out conditions for the agreed upon payoffs to be a SPNE. It must be the case that any deviation from the agreement will be punished. In the prisoner's dilemma each player can guarantee getting at least zero (by playing D each period) and thus a necessary condition for an equilibrium is that no player is better off deviating and then getting zero for the rest of the game - the worst possible payoff. This implies that the following conditions must be satisfied:

1. $p^t < 0.5$:
   a. $(1-\delta)\, u^1 (C, D) + \delta u^{1*} \geq (1-\delta)\, u^1 (D, D) + \delta \times 0.$
   b. $(1-\delta)\, u^2 (C, D) + \delta u^{2*} \geq (1-\delta)\, u^2 (C, D) + \delta \times 0$
2. $p^t \geq 0.5$:
   a. $(1-\delta)\, u^1 (C, C) + \delta u^{1*} \geq (1-\delta)\, u^1 (D, C) + \delta \times 0$
   b. $(1-\delta)\, u^2 (C, C) + \delta u^{2*} \geq (1-\delta)\, u^2 (C, D) + \delta \times 0$

In the case of 1(a) this implies that player 1 cannot gain by non-cooperating even though the other player is not cooperating. The necessary condition is:

$$(1-\delta)\left(u^1 (C, D) - u^1 (D, D)\right) \geq -\delta u^{1*}.$$

which implies $\delta \geq \frac{1}{4.5}$. More generally, it can be easily checked that in order to have all the inequalities satisfied we must have:

$$\delta \geq max\left\{\frac{1}{4.5}, \frac{1}{1.5}\right\} = 2/3. \tag{4.12}$$

This condition is also *sufficient.* That is, suppose that this condition is satisfied, and players agree to play (C, C) 50% of the time and (C, D) 50% of the time. They then use the strategy of playing the agreement until there is a deviation, at which point players agree to play the static Nash equilibrium. Now in any period condition (4.12) ensures that they cannot gain from deviating from the agreement.

In order to be a SPNE it must form a NE for all histories. Consider any history in which there has been a deviation, then this strategy calls for player to choose the static Nash equilibrium, which is itself a SPNE. Hence, when condition (4.12) then the agreement to play (C, C) 50% of the time and (D, C) the rest of the time can be supported by a SPNE.

The discussion thus far seems to imply that cooperation is possible because of the existence of a credible threat if there is deviation. It should be highlighted that what makes cooperation possible is the *difference* in payoffs between continued cooperation and movement to a worst equilibrium. In particular, if the game is finitely repeated then cooperation in the prisoner's dilemma is not a SPNE. In the last period of play there is no future, and hence the unique outcome is to play (D, D) regardless of past play. Now, in the next to last period, since there is not chance to cooperate in the next period, then (D, D) is again the unique equilibrium. Applying this reasoning backwards we see that in a finitely repeated game there can be no cooperation.

This reasoning breaks down if there are at least two equilibria in the last period (see exercise 13). In that case one can use the choice of equilibrium in the last period to incentivize behavior in the next to last period. The reason infinitely repeated games are different is that in any period there is always a chance to meet again, which in turn can be used to provide an incentive to cooperate. It is possible to provide a unified approach to the characterization of all the payoffs using the idea that it is *future* payoffs that provide incentives of performance today.

Characterization of the Payoffs in Repeated Game

In this section we characterize the set of SPNE payoffs, F^{SPNE}, and show how this set can be used to construct a self-enforcing agreement. The basic idea builds upon extending Bellman's principal of optimality to repeated games (see Abreu (1988) and Abreu et al. (1990)). The idea is simple - in order to implement an action profile a^t today all one needs to know are the payoffs that are potential SPNE equilibria tomorrow.

We follow Abreu and Sannikov (2013) and suppose that $W^{t+1} \subset F$ is a compact set representing the payoffs that are possible beginning in period $t+1$. Under the assumption that any payoff in the set W^{t+1} is the outcome of a SPNE beginning in period $t+1$ we can construct the set of SPNE payoffs that are possible in period t, denoted by W^t.

First, we define two functions. If player i deviates in period t the worst payoff that she might receive is given by:

$$P^i\left(W^{t+1}\right) = min\left\{u^i \middle| \left(u^1, u^2\right) \in W^{t+1}\right\},$$

which always exists given that W^{t+1} is compact. Let $P\left(W^{t+1}\right) = \left\{P^1\left(W^{t+1}\right), P^2\left(W^{t+1}\right)\right\}$Next, we define the most a player can gain by defecting upon an agreed upon action pair $a \in A$:

$$h^i(a) = max_{\tilde{a}^i \in A^i} u^i\left(\tilde{a}^i, a^{-i}\right) - u^i(a) \geq 0.$$

Let $h(a) = \left\{h^1(a), h^2(a)\right\}$. Notice that $a \in A^{NE}$ if and only if $h(a) = \hat{0} = \{0, 0\}$.

Definition 4.8 . A point $v \in F$ is *generated by the set* W^{t+1} in a repeated game of length T if there is a payoff vector $w \in W^{t+1}$ and an action $a \in A$ such that the *adding up constraint* is satisfied:

$$v = (1-\delta)\frac{1}{1-\delta^{T-t}}u(a) + \delta\frac{1-\delta^{T-t-1}}{1-\delta^{T-t}}w, \tag{4.13}$$

and the *incentive constraint* is satisfied:

$$h(a) \leq \frac{\delta}{(1-\delta)}\left(1-\delta^{T-t-1}\right)\left(w - P\left(W^{t+1}\right)\right) \tag{4.14}$$

Let $V\left(W^{t+1}, t, T\right)$ be the set of points generated by W^{t+1}.

The set $V\left(W^{t+1}, t, T\right)$ is not empty given that a Nash equilibrium exists. In period $t = T-1$ then $V\left(W^{t+1}, T-1, T\right)$ is exactly the set of Nash equilibrium payoffs to the game without the correlation device. The following result can be readily established:

Lemma 4.9 . *Given that a Nash Equilibrium to the static games exists ($F^{NE} \neq \emptyset$) then for every compact subset $W \subset F$ we have $V(W, T-1, T)$ is a non-empty, compact subset of F. Moreover, if $W \subset W'$ then $V(W, t, T) \subset V(W', t, T)$.*

Proof. Let

$$\hat{W}(a, W) = \left\{ w \in W | h(a) \leq \frac{\delta}{(1-\delta)} \left(1 - \delta^{T-t-1}\right)(w - P(W)) \right\}$$

be the set of future payoffs that can implement action a. This set might be empty, but if not, it is compact. It is not empty for $a \in A^{NE}$. Next let

$$V(a) = \left\{ v = (1-\delta) \frac{1}{1-\delta^{T-t}} u(a) + \delta \frac{1-\delta^{T-t-1}}{1-\delta^{T-t}} w | w \in \hat{W}(a, W) \right\}$$

be the set of points generated by the action a. This is compact whenever $W(a)$ is not empty. It follows that:

$$V(W, t, T) = \cup_{a \in A} V(a)$$

is the finite union of compact sets and thus is also compact. Clearly $\hat{W}(a, W) \subset \hat{W}(a, W')$ from which we get the final condition □

The set of points generated by W^{t+1} are payoffs associated with actions in period t that can be enforced with continuation payoffs taken from W^{t+1}. The next issue is the relationship between these payoffs and the set of SPNE.

Lemma 4.10 . *Suppose that each payoff in W^{t+1} can be supported by some SPNE then $v \in V\left(W^{t+1}, t, T\right)$ if and only if there is a SPNE from period t with payoff v and continuation payoffs taken from W^{t+1}.*

Proof. For each $w \in W^{t+1}$ there is an associated strategy σ_w that forms a SPNE starting in period $t+1$ that ignores history before period $t+1$. We now show that a point v generated by W^{t+1}can also be supported by the following SPNE. Let a_v be the strategy that defines v via the adding up constraint and satisfies the incentive constraint, and let $w(v)$ be the associated payoff in W^{t+1}. Then v is supported by the following strategy σ_v defined from period t onwards:

1. In period t: $\sigma_v(h^t) = a(v)$ for all $h^t \in H^t$.

2. If both parties choose a_v or both deviate from a_v play the SPNE strategy associated with $w(v)$.

3. If agent i deviates (and $-i$ does not) play the SPNE strategy associated with payoff $P^i\left(W^{t+1}\right)$.

Under this strategy after every history, play in periods $\tau \geq t+1$ is some SPNE and hence we know for all histories there is no incentive to deviate from the specified strategy. If player i deviates from a_v in period t the incentive constraint ensures that she is worst off and hence choosing a_v is optimal for both players. This shows that v is supported by this SPNE.

Next, consider the converse. Suppose there is a SPNE strategy σ that entails the choice of a in period t ($\sigma(h^t) = a$ for all h^t), and all continuation payoffs lie in W^{t+1}. Let $v = U(\sigma, h^t)$ - by construction v satisfies the adding up constraint. Let $w \in W^{t+1}$ be the equilibrium continuation payoff. Since σ is a SPNE then it defines a NE in period t, so it must be the case that for any action $\tilde{a}^i$ there is an associated $w\left(\tilde{a}^i\right) \in W^{t+1}$ such that:

$$\begin{aligned} v^i &= (1-\delta)\frac{1}{1-\delta^{T-t}}u(a) + \delta\frac{1-\delta^{T-t-1}}{1-\delta^{T-t}}w \geq (1-\delta)\frac{1}{1-\delta^{T-t}}u\left(\tilde{a}^i, a^{-i}\right) + \delta\frac{1-\delta^{T-t-1}}{1-\delta^{T-t}}w^i\left(\tilde{a}^i\right), \\ &\geq (1-\delta)\frac{1}{1-\delta^{T-t}}u\left(\tilde{a}^i, a^{-i}\right) + \delta\frac{1-\delta^{T-t-1}}{1-\delta^{T-t}}P^i\left(W^{t+1}\right). \end{aligned} \tag{4.15}$$

Hence, this strategy must also satisfy the incentive constraint, from which we conclude that v is generated by W^{t+1}. □

The last inequality was first pointed out by Abreu (1988) - namely, one can fully characterize the set of SPNE by finding the worst SPNE that is possible for each play and using it as a threat when there is deviation from agreed upon play. The public randomizing device at the beginning of each period implies that the set of payoffs in period t that correspond to SPNE with continuation payoffs in $W^{t+1}$is exactly defined by:

$$B\left(W^{t+1}, t, T\right) = conv\left\{V\left(W^{t+1}, t, T\right)\right\}. \tag{4.16}$$

The convexificaton of V preserves both compactness and monotonicity properties, hence if W is compact, then so is $B(W, t, T)$, and $B(W, t, T) \subset B(W', t, T)$ whenever $W \subset W'$. We now use this operator to characterize the set of SPNE payoffs possible for the repeated game. We begin with the case of finitely repeated games.

Proposition 4.11 . *Suppose that a game is repeated $T < \infty$ times, then the set of SPNE payoffs in period t is defined recursively by:*

$$F^{SPNE}(T-1) = F^{NE} = B(\{\emptyset\}, T-1, T)$$
$$F^{SPNE}(t) = B\left(F^{SPNE}(t+1), t, T\right), t \leq T-2.$$

Lemma 4.10 implies that if $F^{SPNE}(t+1)$ consists of all SPNE payoffs in period $t+1$ then $F^{SPNE}(t)$ consists of all SPNE in period t. In the last period there is no future, and hence the only possible SPNE are the Nash equilibria of the one shot game. Thus we can recursively define all the SPNE recursively beginning in period $T-1$. Notice that if there is a unique Nash equilibrium, then F^{NE} is a single point w^{NE}. This implies that $w^{iNE} = P^i\left(F^{NE}\right)$, and hence the incentive constraint is $h(a) = 0$ every period, which in turn implies that the unique SPNE is the Nash equilibrium of the one-shot game.

Corollary 4.12 . *If the Nash equilibrium of the one shot game is unique then repeating this action is the unique SPNE in the finitely repeated game.*

Infinitely repeated games have no last period and hence this procedure does not work. Moreover, when $T = \infty$ then the δ^Tterms are zero and the B operator is time invariant - we denote it simply by B. This implies that the set of SPNE payoffs is time invariant, denoted by F^{SPNE}. Moreover, the argument regarding the necessary and sufficient conditions for a point to correspond to a SPNE implies that:

$$F^{SPNE} = B\left(F^{SPNE}\right).$$

The next proposition provides an explicit way to compute F^{SPNE}.

Proposition 4.13 . *The set of payoffs supported by some SPNE satisfies:-*

$$F^{SPNE} = B\left(F^{SPNE}\right) = lim_{n\to\infty} B^n\left(F\right)$$

Proof. Notice that since $F^{SPNE} \subset \bar{F}$ and $F^{SPNE} \subset B\left(F^{SPNE}\right)$ then the monotonicity of the B operator implies that $F^{SPNE} \subset B^n\left(\bar{F}\right)$ for all n. Since the operator preserves compactness, then we have

$$lim_{n\to\infty} B^n\left(\bar{F}\right) = \bigcap_{n=1,2,\dots} B^n\left(\bar{F}\right)$$

which converges to some compact set F' with the feature $F' = B\left(F'\right)$ and $F^{SPNE} \subset F'$. For any $w \in F'$ one can use the algorithm above to construct an equilibrium sequence of actions that are incentive compatible each period, and hence forms a SPNE. This implies $F' \subset F^{SPNE}$. Thus $F' = F^{SPNE}$. □

This provides a way to construct F^{SPNE}. Recently, Abreu and Sannikov (2013), building upon the work of Cronshaw and Luenberger (1994) and Judd et al. (2003), show that with a finite strategy space the set of payoffs is a polytope that can be computed very quickly.[20] For the purposes of relational contract theory, the most important result is the characterization of the set of possible equilibrium payoffs in terms of the worst SPNE.

20 Their web sites have a link to the algorithm that can be used to compute the set of equilibrium payoffs.

Corollary 4.14 . *(Abreu (1988))Let $u_w^i = P^i\left(F^{SPNE}\right)$ then an agreement to play $s \in S$ every period in an infinitely repeated game can be supported by some SPNE if and only if for any action a played with positive probability:*

$$one - shot - gain = h^i\left(a\right) \leq \frac{\delta}{\left(1-\delta\right)}\left(u^i\left(s\right) - u_w^i\right) = future - lost. \tag{4.17}$$

In many contractual settings the worst payoff is generated by the threat of separation, which can often be easily computed independently of the discount rate and other features of the environment. This determines $u^i(s) - u_w^i$ in 4.17, which then allows for an elementary characterization of the set of self-enforcing agreements. Observe that in this case when the time between moves is small ($\Delta \to 0$) then $\delta \to 1$, and the right hand side is not binding when $\left(u^i\left(s\right) - u_w^i\right) > 0$. The folk theorem for repeated games establishes the fact that when $\delta \to 1$ then $u_w^i \to \bar{u}^i$, the min-max payoff, and hence $F^{SPNE} \to \bar{F}$.

4.7 Summary

In this chapter we have provided a very brief overview of game theory. The chapter reviewed the main themes that are relevant to the contracting institutions to be discussed later. Game theory illustrates the importance of carefully defining the information and actions available to all participants. Risk can be introduced by viewing Nature as a player that chooses states.

The Monty Hall problem illustrates the importance of explicitly modeling the actions of agents in the environment. The reason it seems hard is because it is often approached as a problem in probability theory, yet as we saw, it cannot be solved with making assumptions regarding the actions of Monty Hall. Game theory provides a way to systematically model and study the actions of individuals.

Second, though Savage's theory of rational choice includes games as a special case, it provides no guidance on how to model the probability assessments of individuals. Game theory begins with the hypothesis that individuals know that they are playing other individuals with well defined goals. This knowledge can be used to constrain the beliefs regarding how one expects others to play. The fundamental solution concept in Game theory is the notion of a Nash equilibrium. It requires that these beliefs regarding the evolution of play be correct - namely at a Nash equilibrium each person is making an optimal decision correctly anticipating how others will play.

Game theory differs from rational choice theory in a second fundamental way. When an individual is choosing option A over B, she must contemplate the counterfactual - what would happen if she chooses B? The difficulty is that if at a Nash equilibrium action B is not expected, then it is assigned a probability of zero by the other players. If they observe B, Bayes' rule cannot be used, and hence there is no method in decision theory to guide the updating of beliefs. Depending upon the information structure of the game, game theory has a number of solutions, these include the concept of a subgame perfect Nash equilibrium (a concept that is widely used in repeated game theory), and the notion of a sequential equilibrium, which extends the notion of subgame perfection to games with asymmetric information.

In general, the theory does not produce a unique prediction for an arbitrary game. There is a long history of attempts to find a solution concept that yields a unique prediction for any game - with Kohlberg and Mertens (1986) possibly providing the most satisfactory solution. However, the project has not proven to be successful. Even if they had found a solution, we would be faced with the fundamental fact that given a particular game the literature on experimental game theory has documented a large body of evidence players vary greatly in how they act in practice (see for example Colin Camerer (2003)'s review of experimental game theory).

For example, Brandts and MacLeod (1995) help players deal with the complexity of analyzing a given game, and provide recommendations on how to play. Even then, players often deviate from recommended play. What we do know is that when individuals are aware that they are playing against another individual this is used to inform their choices. In later chapters, we will use game theory as a way to formally model strategic interaction, particularly contract formation. It goes beyond price theory in that individuals' beliefs on how others will play is an essential ingredient of the model.

Finally, repeated game theory solves the open question of what the limits are of informal cooperation when parties meet repeatedly. The key insight is that the set of possible informal agreements can be characterized by first finding the *worst* possible outcome for each individual. Given this information, the set of possible agreements are those that use these worst outcomes as punishments. This result is central to the theory of relational contracts that we study in chapter 9. These relational contracts rely upon the existence of a *rent* in the relationship. The next natural question is: how do parties agree to divide this rent? We address this issue in the next chapter.

4.8 Exercises

1. Suppose that we observe a contestant refusing to change her decision in the Monty Hall problem. Can we conclude that the contestant is acting irrationally? Why or why not?
2. Suppose now that Monty Hall is free to choose whether to open a second door or not, how does that change the analysis?
3. Provide a simple example showing a mixed strategy that cannot be replaced by a behavioral strategy.

4. Prove that when players have perfect recall then for any mixed strategy by player i, there exists a corresponding behavioral strategy that yields the same expected payoff.
5. Prove that in a finite game of perfect information (every information set consists of a single node), there exists an equilibrium that can be found by backwards induction, and that this equilibrium is perfect.
6. Prove proposition 4.2.
7. Consider the second price auction game. A single good is to be sold to one of $n \geq 2$ possible buyers at a price p. Buyer i has a value $v^i \in [0,1]$ for the good, that is drawn from the distribution $f^i(v^i)$. The distributions are common value, but each buyer knows only her valuation. After the buyers observe their valuations, they submit bids b^i for the object. The buyer with the highest bid wins, and pays a price p equal to the second highest bid, and earns utility $v^i - p$. The losers earn zero.

   a. Describe the normal form of this game, along with the payoffs of the players.
   b. Prove that it is a dominant strategy for play i to set $b^i = v^i$.
   c. Suppose $f^i(v) = 1$ for all v and i. Work out the expected revenues for the seller in this game as a function of the number of players n.
   d. For the case in (c) work out the Nash equilibrium when the price is equal to the winner's bid. What is the expected revenue in this case?

8. For any finite game $G = \{U^i, \Gamma^i\}_{i \in N}$ prove there exists a perfect equilibrium and that every perfect equilibrium is also a sequential equilibrium. This result follows from Kakutani's theorem. Let $A \subset \Re^n$ then $f : A \to A$ is a correspondence if $f(x)$ is a subset of A. In the context of games in mixed strategies, $\{U_i, \Gamma_i\}_{i \in N}$, the correspondence

we have in mind is:

$$f_i(\gamma) = argmax_{\gamma_i \in \hat{\Gamma}_i} U_i\left(\gamma^i, \gamma^{-i}\right). \tag{4.18}$$

One can readily show that this correspondence is *upper-hemicontinuous.* Namely, the graph of f, $\{(x,y)\,|y \in f(x), x \in A\}$ is a closed set and for any compact set $B \subset A$ then $f(B) = \{y|y \in f(x), x \in B\}$ is compact. Then Kakutani's theorem states that if A is a compact, convex subset of $\Re^n$ and $f : A \to A$ is a upper-hemicontinuous correspondence, such that $f(x)$ is convex for all $x \in A$ then there is an x^* such that $x^* \in f(x^*)$. To show the existence of a Nash equilibrium one begins with the observation that the set of mixed strategies is a compact ,convex set and that f defined in 4.18 is upper-hemicontinuous and convex-valued.

9. Provide an example of a finite game for which

$$min_{a^2} max_{a^1} u^1\left(a^1, a^2\right) > max_{a^1} min_{a^2} u^1\left(a^1, a^2\right).$$

10. Prove that we have equality in the previous exercise if we allow for mixed strategies. Hint: Consider the zero sum game where $u^2\left(a^1, a^2\right) = -u^1\left(a^1, a^2\right)$, and use the fact that a Nash equilibrium in mixed strategies exists.
11. Consider the game illustrated in figure 4, find the set of sequential equilibria and perfect equilibria for the game. Now change the payoff corresponding to (l, R) to $(1, -1)$, what is the set of sequential equilibria and perfect equilibria? Specify the entire set of equilibrium assessments.
12. Consider the following Final Offer Game. Suppose that two players have to divide a dollar. Player 1 offers Player 2 an amount x. If accepted, then Player 1 gets $1 - x$ and Player 2 gets x. If rejected they both get zero. What are the pure strategy Nash equilibria for this

game? What is the unique perfect equilibrium? Suppose the game is played n times, where the players just add up their winnings (so the maximum gain is n dollars). What is the perfect equilibrium? Now suppose that the game is played with pennies, so that the $x \in \{0, 1, 2..., 100\}$. Work out the perfect equilibria in the one period game, and then work out the maximum equilibrium offer possible in the first round in an n period game as a function of n.

13. Suppose that in addition to the payoffs for the PD, players have the option of quitting the relationship, denoted by Q. If either player quits the the payoff is -1. Suppose that each period they simultaneously choose $A \in \{C, D, Q\}$. Show that there are two Nash equilibria to this game. Next characterize the payoffs when the game is repeated T times. Next suppose the game is played over one period, and that the time between moves is Δ. Characterize the set of payoffs as $\Delta \to 0$.

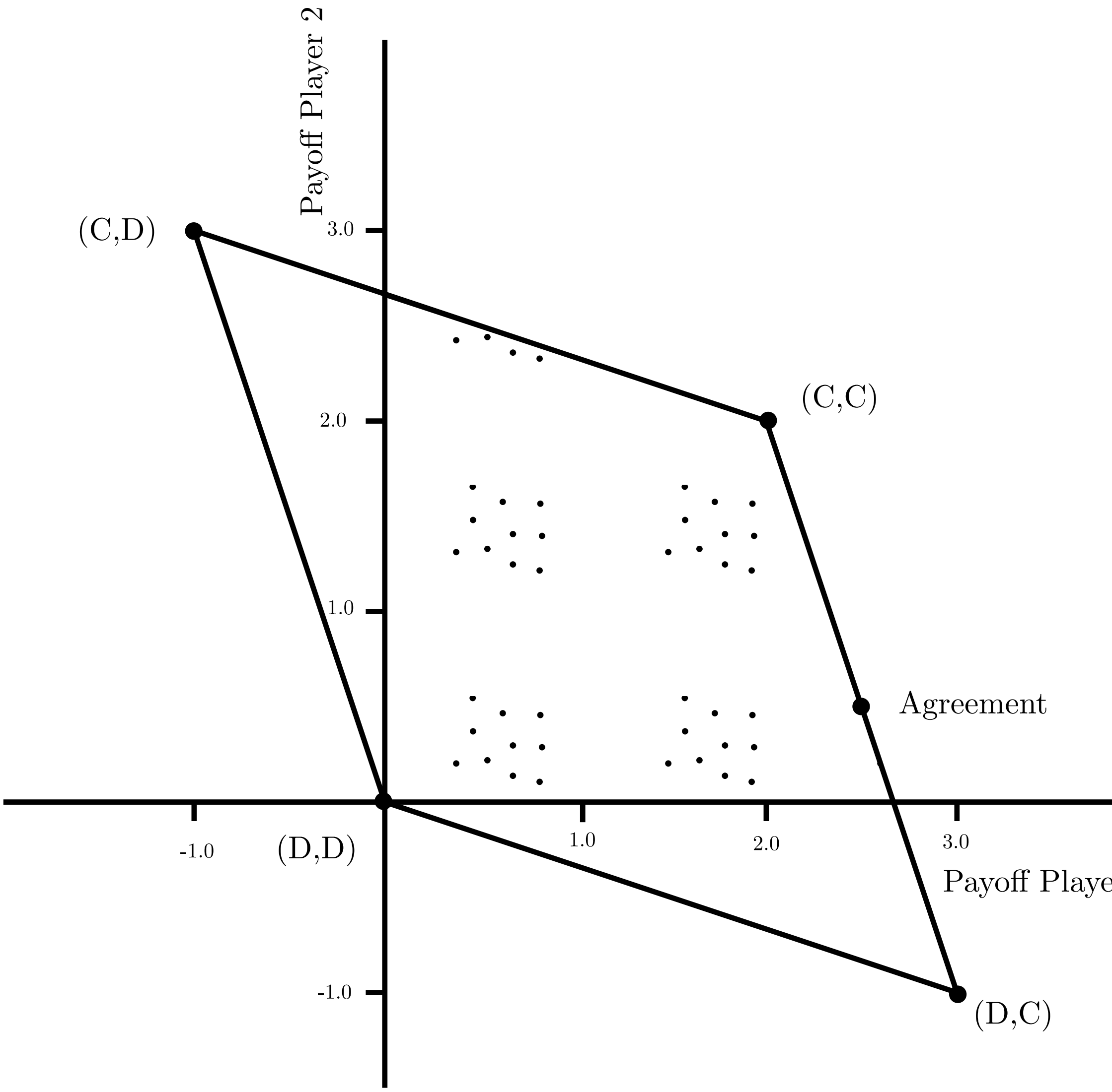

Figure 4.6
Payoffs in Prisoner's Dilemma Game

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