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On the Optimality of Equilibrium when the Market Structure is Incomplete*

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1. INTRODUCTION

In the Arrow–Debreu theory of general equilibrium, it is assumed that markets exist for current goods and also for goods to be delivered at future dates and in uncertain events. Under this assumption, and if economic agents face no transaction costs, all economic decisions will be made at one time and markets will open only once.

In the real world, however, we observe that few markets for contingent futures goods exist at any one time, and that markets reopen many times. In recent years, several authors have constructed general equilibrium models that reflect this incomplete and sequential aspect of real world trading. Two main approaches have been taken: the temporary equilibrium approach and the rational expectations approach. The temporary equilibrium approach assumes that economic agents have given expectations of future prices and investigates whether there exist prices which clear current markets (for a discussion of this approach, see Arrow and Hahn [2] and Grandmont [5]). The rational expectations approach, on the other hand, regards expectations as variables and investigates whether there exists a set of current prices *and* expected prices such that all markets, both current and future, are cleared (for a discussion of this approach, see Radner [11]).

Most of the work that has been done on incomplete and sequential markets has been concerned with establishing the existence of equilibrium. The subject of the optimality of equilibrium has received considerably less attention. Now it is clear that a temporary equilibrium will not generally enjoy any optimality properties since the actions of economic agents are

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based on price expectations that may turn out to be quite incorrect. What is perhaps more surprising is that, as the present paper will demonstrate, rational expectations equilibria are also not generally optimal in economies where the market structure is incomplete.

In order to establish the suboptimality of rational expectations equilibria, it is necessary first to provide a satisfactory definition of an optimal allocation. Clearly, it would be unreasonable to expect an equilibrium allocation to be fully Pareto optimal in an economy where certain markets are missing. (See [7] for sufficient conditions for Pareto optimality.) The most that we can hope for is that an equilibrium will be constrained Pareto optimal, that is, Pareto optimal relative to the set of allocations that can be achieved through the existing market structure. A result of this type has been obtained by Diamond [3] when there is one good and the economy lasts for only two periods. The main conclusion of the present paper is that Diamond's result does not hold in more general cases. In fact, we will show that if the economy contains more than one good or lasts for more than two periods, then an equilibrium generally will not even be Pareto optimal relative to the set of competitive equilibria; that is, given one rational expectations equilibrium, it may be possible to find another rational expectations equilibrium that Pareto-dominates it.

As well as establishing that a rational expectations equilibrium is not optimal, we will show that such an equilibrium may not exist under the usual continuity and convexity assumptions. The reason for nonexistence is that consumers' budget correspondences are not generally upper semicontinuous when markets are incomplete. (This problem arises even when bankruptcy is prohibited.) Radner [11] guarantees the upper semicontinuity of budget correspondences by imposing a bound on forward transactions. Unfortunately, any such bound is quite arbitrary, and the equilibrium of the economy will in general depend on the bound chosen.

Finally, we will consider the effect on a rational expectations equilibrium of opening new markets, that is, of permitting trades that were previously prohibited. Our intuition tells us that the introduction of additional markets ought to make people better off in some sense. We will present an example, however, where the result of opening new markets is in fact to make everybody worse off. (Of course, this cannot happen if enough new markets are opened to make the market structure complete in the Arrow–Debreu sense.)

Throughout the paper, we will confine our attention to a pure exchange economy. There are two reasons for this. First, no general theory of producer behavior has been developed for the case where markets are incomplete. Secondly, we wish to show that an equilibrium is suboptimal even when there is no production. If we include production, of course, the suboptimality described here will still be important. (See [9] for a discussion of a model with production.) In fact, the recent work on models of the stock market [4, 12] suggests that, if production is included, there will be a second type of inefficiency resulting from the suboptimal production decisions of firms.

In Section 2, we present a simple example of the suboptimality of equilibrium. In Section 3, we formulate a general model of an economy with incomplete markets. We discuss the existence of equilibrium in Section 4 and the optimality of equilibrium in Section 5. Finally, in Section 6, we consider the effect on the economy of opening new markets.

2. An Example

In this section we will present a simple example of the suboptimality of equilibrium in an economy with incomplete markets. Consider an exchange economy that lasts for two periods and where there is no uncertainty. Assume that there are two consumers and that two goods are available in each period. Consumer i receives an endowment $\omega_1^i \in R_+^2$ at date 1 and $\omega_2^i \in R_+^2$ at date 2 $(i = 1, 2)$. (We define $R_+^l = \{x \in R^l \mid x_j \geqslant 0$ for $j = 1,..., l\}$. We also will use the following notation. If $x \in R^l$, $x \geqslant 0$ means $x_j \geqslant 0$ for $j = 1,..., l$; $x > 0$ means $x \geqslant 0$ and $x \neq 0$; $x \gg 0$ means $x_j > 0$ for $j = 1,..., l$.) We will assume that Consumer i's utility function may be written as

$$U^i(x_1^i, x_2^i) = V^i(x_1^i) + \beta^i W^i(x_2^i) \tag{2.1}$$

for some functions V^i, W^i and some $\beta^i > 0$; $x_1^i \in R_+^2$ is Consumer i's consumption at date 1 and $x_2^i \in R_+^2$ is Consumer i's consumption at date 2 $(i = 1, 2)$.

Suppose that spot markets open at dates 1 and 2, but that, for some reason, perhaps because futures markets are relatively costly to organize, futures contracts are not permitted. Suppose also that there are no borrowing or lending possibilities and that goods cannot be stored.

Assume that consumers have point expectations of date 2 prices, that both consumers have the same expectations, and that these expectations turn out to be correct. Let p_1 be the price vector at date 1 and p_2 be the expected price vector at date 2. Then Consumer i $(i = 1, 2)$ chooses (x_1^i, x_2^i) to maximize $U^i(x_1^i, x_2^i)$ subject to

$$\begin{aligned} p_1 x_1^i &\leqslant p_1 \omega_1^i, \\ p_2 x_2^i &\leqslant p_2 \omega_2^i, \\ x_1^i, x_2^i &\geqslant 0. \end{aligned}$$

(If $a, b \in R^l$, ab denotes the inner product $\sum_{j=1}^l a_j b_j$.)

Since U^i is separable, we may represent equilibrium by means of two Edgeworth boxes, one for each period.

In Fig. 1, 0, 0′ are the endowment points at dates 1 and 2, respectively. At date 1, indifference curves are given by V^i = constant and, at date 2, by W^i = constant.

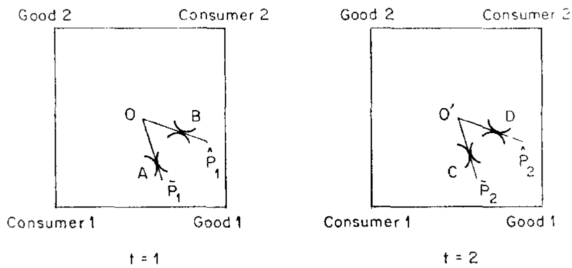

FIGURE 1

Suppose that V^i and W^i are such that there are two equilibria in each Edgeworth box; A and B at date 1, and C and D at date 2. (The same argument can be used when there are more than two equilibria at each date.) Then there are four equilibria for the two period economy, (A, C), (A, D), (B, C), (B, D), and the corresponding equilibrium prices are $(\bar{p}_1, \bar{p}_2)$, $(\bar{p}_1, \hat{p}_2)$, $(\hat{p}_1, \bar{p}_2)$, $(\hat{p}_1, \hat{p}_2)$. It should be noted that consumers' price expectations are not the same in all these equilibria, but in every case the expectations are self-fulfilling.

Consider now whether these equilibria are optimal in any sense. It is clear that, except by chance, none of them will be Pareto optimal, since, in the absence of futures markets, there is no reason why the marginal rate of substitution between consumption at date 1 and consumption at date 2 should be the same for both consumers. However, we might hope that an equilibrium is optimal in a weaker sense, that is, Pareto optimal relative to all allocations that can be achieved by using the existing markets. We will now establish that equilibria are not generally optimal in this weaker sense by showing that one equilibrium may be Pareto-dominated by another.

Consider the two equilibria (A, D) and (B, C). Consumer 1 is better off at date 1 (that is, V^1 is higher), and worse off at date 2 (that is, W^1 is lower), in (B, C) than in (A, D). The opposite is the case for Consumer 2. Since the equilibria of the economy are independent of β^1 and β^2, we can choose these parameters arbitrarily. It is clear from (2.1), however, that if we choose β^1 small enough and β^2 large enough, both consumers will prefer the (B, C) equilibrium to the (A, D) equilibrium, and so the (B, C) equilibrium will Pareto-dominate the (A, D) equilibrium.

In this case the (A, D) equilibrium is clearly not optimal in any sense. The (B, C) equilibrium is Pareto-superior and can be achieved without opening any new markets; all that is required is a change in date 1 prices and a change in consumers' expectation of date 2 prices. There are, however, no forces in the market that will tend to lead the economy to (B, C) rather than to (A, D); if the economy starts off at (A, D), it will stay there.

This simple example shows that when markets are incomplete, an equilibrium may be Pareto-dominated by another allocation which can be achieved using the same markets. It might be argued that the example is rather restrictive since it assumes that no opportunities exist for transferring wealth from one period to another. The purpose of this paper is to show that the same phenomenon occurs in much more general circumstances. In fact, it occurs however complicated the market structure is, except in two cases: one is an extension of a case analyzed by Diamond [3], and the other is the case where markets are complete.

3. A Formal Model

In this section we will present a formal model of an exchange economy with incomplete markets. Such a model has been analyzed by Radner [11]. Our development will follow Radner's closely, but we will find it convenient to use a somewhat different notation.

We will consider an economy which extends through three periods in an uncertain environment. (Radner considers the more general case where the economy lasts for T periods. All our results generalize directly to the case $T > 3$.) We will assume that there is a finite set S of alternative states. Each state in S is to be interpreted as a particular history of the environment from date 1 to date 3. The set of events observable at date t will be represented by a partition, $\mathscr{P}_t$, of S. It is assumed that the sequence of partitions, $\mathscr{P}_t$, is monotone nondecreasing in fineness, that is, $\mathscr{P}_2$ is as fine as $\mathscr{P}_1$ and $\mathscr{P}_3$ is as fine as $\mathscr{P}_2$. (A partition $\mathscr{P}$ is said to be as fine as a partition $\mathscr{P}'$ if, for every A' in $\mathscr{P}'$ and A in $\mathscr{P}$, either $A \subset A'$ or $A \cap A' = \varnothing$.) $\mathscr{P}_1$ is taken to be $\{S\}$. The observable events thus form a "tree."

We will assume without loss of generality that the true history of the environment is known at date 3, so that each event at date 3 contains a single element of S.

It should be noted that in contrast to another model of Radner [10], everybody has the same information about which state has occurred.

The Market Structure

In the Arrow–Debreu model, it is assumed that all contingent futures commodity markets are open at date 1. There will then be no need for markets to reopen at subsequent dates. In contrast to this, we will assume that contingent futures commodity markets may not exist at date 1, and that markets reopen at each subsequent date. The set of markets that is open at each date will be taken as given, and we will not inquire into why some markets are open and some are closed. For a discussion of a model in which transaction costs determine the market structure, see Hahn [8].

We assume that at each date there is a finite set of goods, numbered $1,..., H$. At each date, the market structure will be represented by means of securities. At date 1, markets for current goods open and there are also markets for F securities (F is finite). For each $f = 1,..., F$, security f is represented by an ordered pair (a_2^f, a_3^f), where a_2^f and a_3^f are functions from S to R_+^H. The interpretation of this ordered pair is that if a consumer holds z units of security f from date 1 onwards and if s is the true history of the environment, then the consumer will receive the vector of goods $za_2^f(s)$ at date 2 and the vector of goods $za_3^f(s)$ at date 3 (z may be positive or negative). We will think of $a_2^f(s)$ and $a_3^f(s)$ as being the "dividends" which security f pays.

The advantage of introducing securities into the analysis is that we can use them to represent a wide variety of market structures. For example, if $a_2^f(s) = a_3^f(s) = 0$ for all f and s, then only trading in current goods is possible. At the other extreme, if securities $1,..., F$ span the set of all possible ordered pairs (a_2, a_3), then the situation is equivalent to one in which there are complete Arrow–Debreu markets. Moreover, by picking the dividends of the securities appropriately, we can represent all the market structures lying between these two extremes.

Since the true history of the environment is not generally known to consumers until date 3, we must constrain the functions a_2^f, a_3^f to be consistent with the information that consumers possess. Hence, we impose the following condition:

$$\text{For } t = 2, 3, \text{ every set } E \text{ in } \mathscr{P}_t, \text{ and all } f = 1,..., F,$$
$$a_t^f(s) = a_t^f(s') \text{ if } s \text{ and } s' \text{ are both in } E. \tag{3.1}$$

Mathematically, condition (3.1) simply says that the function a_t^f is $\mathscr{P}_t$-measurable; that is, measurable with respect to the partition $\mathscr{P}_t$.

Consider now date 2. We assume that markets for current goods reopen and that the F securities that were available at date 1 can be retraded. We also assume that G new securities become available (G is finite).

Without loss of generality, we may take G to be independent of the history of the environment up to date 2 since we can always ensure that this is the case by creating artificial securities that yield nothing in every state.

The G new securities will be indexed by $f = F + 1, \ldots, F + G$. For each $f = F + 1, \ldots, F + G$, security f is represented by a function a_3^f which maps S into R_+^H. If a consumer holds z units of security f and if state s occurs, the consumer will receive the vector of goods $za_3^f(s)$ at date 3. Since s is known at date 3, a condition like (3.1) is not required.

Finally, at date 3, since the economy terminates, futures markets are irrelevant, and only markets for current goods open.

The market structure described here is more general than Radner's [11], since Radner does not allow for the possibility of joint contracts; that is, Radner does not consider the case where a particular contingent futures commodity can be purchased only if other contingent futures commodities are purchased simultaneously. For an approach which is equivalent to ours, see Wilson [13].

Prices

At date 1, prices for current goods and for securities $1, \ldots, F$ are announced. It is assumed that consumers also have expectations of the prices which will rule in markets at dates 2 and 3.

We will represent goods prices by an ordered triple $p = (p_1, p_2, p_3)$, where, for each t, p_t is a $\mathscr{P}_t$-measurable function from S to R_+^H. We will call such a triple a *goods price system*. For $t > 1$, $p_t(s)$ is the vector of goods prices which are expected to rule at date t in state s. The vector $p_1(s)$ represents the announced goods prices at date 1. (As a consequence of the measurability condition on p_t, $p_1(s)$ is in fact independent of s.)

Similarly, we will represent security prices by an ordered pair $\pi = (\pi_1, \pi_2)$, where π_1 is a $\mathscr{P}_1$-measurable function from S to R_+^F and π_2 is a $\mathscr{P}_2$-measurable function from S to R_+^{F+G}. We will call such a pair a *security price system*. The vector $\pi_1(s)$ refers to security prices at date 1 and the vector $\pi_2(s)$ to expected security prices at date 2.

Finally, a *price system* is an ordered pair (p, π), where p is a goods price system and π is a security price system.

It should be noted that in representing prices in this way we are making the strong assumption that each consumer associates a unique set of prices with each event. We are also assuming that all consumers have the same price expectations, that is, that $p_t(s)$ is the same for all consumers. Of course, this does not mean that consumers agree on the joint probability distribution of future prices, since different consumers might assign different probabilities to the same event.

Consumers

We will assume that there is a finite set of consumers, numbered 1,..., I. Each consumer receives and consumes a nonnegative endowment of goods at each date and in each event. The consumers' initial endowments of securities are zero. There are no production activities and goods cannot be stored.

An *endowment stream* for Consumer i is an ordered triple $\omega^i = (\omega_1{}^i, \omega_2{}^i, \omega_3{}^i)$, where, for each t, $\omega_t{}^i$ is a $\mathscr{P}_t$-measurable function from S to $R_+{}^H$. The vector $\omega_t{}^i(s)$ represents Consumer i's endowment at date t in state s.

Similarly, a *consumption plan* for Consumer i is an ordered triple $x^i = (x_1{}^i, x_2{}^i, x_3{}^i)$, where, for each t, $x_t{}^i$ is a $\mathscr{P}_t$-measurable function from S to $R_+{}^H$. The vector $x_t{}^i(s)$ represents Consumer i's consumption bundle at date t in state s.

Finally, a *security trading plan* for Consumer i is an ordered pair $z^i = (z_1{}^i, z_2{}^i)$, where $z_1{}^i$ is a $\mathscr{P}_1$-measurable function from S to R^F and $z_2{}^i$ is a $\mathscr{P}_2$-measurable function from S to R^{F+G}. The vector $z_t{}^i(s)$ represents Consumer i's security holdings at date t in state s.

Budget Constraints

Let us turn now to consumers' preference and budget sets. Each Consumer i is assumed to have an intertemporal utility function U^i, defined on the set of all consumption plans. (The utility function could, of course, be replaced by a suitable preference preordering.) The utility function may be sufficiently regular to permit scaling in terms of subjective probabilities and von Neumann–Morgenstern utilities, but we will not need to assume this.

Each consumer faces a sequence of budget constraints instead of a single budget constraint, since budgets must be balanced each time markets open. At date 1, Consumer i's budget constraint is

$$p_1(s)\, x_1{}^i(s) + \pi_1(s)\, z_1{}^i(s) \leqslant p_1(s)\, \omega_1{}^i(s). \tag{3.2}$$

Condition (3.2) says that the cost of current consumption plus the cost of security acquisitions cannot exceed the value of Consumer i's endowment. Notice that (3.2) is just one constraint since $p_1(s)$, $x_1{}^i(s)$, $\pi_1(s)$, $z_1{}^i(s)$, and $\omega_1{}^i(s)$ are the same for all s.

At date 2, if event $E \in P_2$ has occurred, Consumer i's budget constraint is

$$\begin{aligned} &p_2(s)\, x_2{}^i(s) + \pi_2(s)\, z_2{}^i(s) \\ &\quad \leqslant p_2(s)\, \omega_2{}^i(s) + p_2(s)\left(\sum_{f=1}^{F} z_{1f}^i(s)\, a_2{}^f(s)\right) + \sum_{f=1}^{F} \pi_{2f}(s)\, z_{1f}^i(s), \end{aligned} \tag{3.3}$$

where $s \in E$. Condition (3.3) is similar to (3.2) except that there are two new terms on the right-hand side. The first of these is the market value at date 2 of the dividends from the securities purchased at date 1. The second new term is the market value at date 2 of the date 1 securities themselves.

Finally, at date 3, if state s has occurred, Consumer i's budget constraint is

$$p_3(s)\, x_3^{\,i}(s) \leqslant p_3(s)\, \omega_3^{\,i}(s) + p_3(s)\left(\sum_{f=1}^{F+G} z_{2f}^{i}(s)\, a_3^{\,f}(s)\right). \tag{3.4}$$

Given a price system (p, π), let $B^i(p, \pi, \omega^i)$ be the set of pairs of consumption plans and security trading plans, (x^i, z^i), for Consumer i that satisfy (3.2), (3.3), and (3.4) for all s. Define

$$\bar{B}^i(p, \pi, \omega^i) = \{x^i \mid (x^i, z^i) \in B^i(p, \pi, \omega^i) \text{ for some } z^i\}.$$

Then, Consumer i is assumed to choose a consumption plan in $\bar{B}^i(p, \pi, \omega^i)$, which maximizes U^i.

It should be noted that a security trading plan might involve a consumer agreeing to supply in the future more of a commodity than the consumer will ever possess. However, we do insist that each consumer be solvent in every eventuality (even in eventualities that the consumer believes occur with probability zero), so that consumers will always be able to honor their commitments by buying back commodities of which they do not posses enough.

Equilibrium

We define an equilibrium for the economy described above under the assumption that consumers' price expectations are self-fulfilling.

A *Radner equilibrium* for the economy is an array $(x^1,..., x^I, z^1,..., z^I)$ of consumption and security trading plans, and a price system (p, π), such that

(3.5) $(x^i, z^i) \in B^i(p, \pi, \omega^i)$ for each i;

(3.6) for each i, $U^i(x^i) \geqslant U^i(x'^i)$ for all $x'^i \in \bar{B}^i(p, \pi, \omega^i)$;

(3.7) $\sum_i x_t^{\,i}(s) \leqslant \sum_i \omega_t^{\,i}(s)$ for all t and s;

(3.8) $\sum_i z_t^{\,i}(s) \leqslant 0$ for all t and s.

Conditions (3.7) and (3.8) state that the plans of consumers are consistent and hence the prices which consumers expect to occur will be market clearing prices. It follows that consumers' price expectations actually are fulfilled in equilibrium. (We are assuming implicitly that, as the future unfolds, consumers do not change their tastes.)

Pareto-Optimum

Finally, we define a Pareto optimum. A *consumption allocation* is an array $(x^1, \ldots, x^I)$ of consumption plans, satisfying (3.7). The consumption allocation $(x'^1, \ldots, x'^I)$ is said to *Pareto-dominate* the consumption allocation $(x^1, \ldots, x^I)$ if

$$U^i(x'^i) > U^i(x^i) \qquad \text{for all } i. \tag{3.9}$$

A consumption allocation is said to be a *Pareto optimum* if it is not Pareto-dominated by another consumption allocation. (This definition is slightly weaker than the usual one, which says that a consumption allocation is Pareto optimal if it is impossible to make some people better off and nobody worse off.)

In defining a Pareto optimum, we are assuming that the information structure of consumers is fixed; this assumption will be made throughout the paper.

4. The Existence of Equilibrium

Let us consider now whether an equilibrium exists for the economy of Section 3. We will present an example that shows that even under all the standard convexity and continuity assumptions, an equilibrium may fail to exist.

Example 1. There are two consumers, two goods, and two states, and consumers know which state has occurred at date 2. Each consumer receives a zero endowment of goods at dates 1 and 3 and each consumer's utility depends only on consumption at date 2. Consumer 1's preferences at date 2 are represented by the von Neumann–Morgenstern utility function $2^{5/2}x_1^{1/2} + 2x_2^{1/2}$ and Consumer 2's preferences at date 2 are represented by the von Neumann–Morgenstern utility function $2x_1^{1/2} + 2^{5/2}x_2^{1/2}$; x_1 is consumption of the first good and x_2 is consumption of the second good. Each consumer believes that each state occurs with probability 1/2, so that

$$U^1 = 2^{3/2}\{x_1^1(1)\}^{1/2} + \{x_2^1(1)\}^{1/2} + 2^{3/2}\{x_1^1(2)\}^{1/2} + \{x_2^1(2)\}^{1/2},$$
$$U^2 = \{x_1^2(1)\}^{1/2} + 2^{3/2}\{x_2^2(1)\}^{1/2} + \{x_1^2(2)\}^{1/2} + 2^{3/2}\{x_2^2(2)\}^{1/2},$$

where $x_h^i(s)$ is Consumer i's consumption of good h in state s. (Time subscripts are left out since all consumption takes place at date 2.) Consumer 1's endowment vector at date 2 is (5/2, 50/21) if state 1 occurs and (13/21, 1/2) if state 2 occurs; Consumer 2's endowment vector at date 2 is (1/2, 13/21) if state 1 occurs and (50/21, 5/2) if state 2 occurs.

There are two securities at date 1. One unit of the first security yields one unit of the first good at date 2, whichever state occurs; one unit of the second security yields one unit of the second good at date 2, whichever state occurs.

We will show that no equilibrium exists for this economy. Since consumers are interested only in consumption at date 2, the only economically relevant markets are the security markets at date 1 and the spot markets at date 2. Let $p(s) \in R_+^2$ be the equilibrium expected price vector at date 2 if state s occurs. (We again leave out the time subscript.) Then the monetary return of security 1 at date 2 is $p_1(1)$ if state 1 occurs and $p_1(2)$ if state 2 occurs, and hence the vector of monetary returns of security 1 is $(p_1(1), p_1(2))$. Similarly, the vector of monetary returns of the second security is $(p_2(1), p_2(2))$. We consider two cases.

Case 1. *$p(1)$ and $p(2)$ are linearly independent.* In this case the vectors of monetary returns of the two securities are linearly independent and span the set of all vectors of monetary returns. Hence, the situation is equivalent to one in which contingent futures markets for date 2 goods exist at date 1 [1]. Let $q_h(s)$ be the implicit price at date 1 of one unit of good h to be delivered at date 2 in state s. Then, Consumer 1 maximizes U^1 subject to

$$q(1)\,x^1(1)+q(2)\,x^1(2) \leqslant (5/2)q_1(1)+(50/21)q_2(1)+(13/21)q_1(2)+(1/2)q_2(2),$$
$$x^1(1),\, x^1(2) \geqslant 0,$$

and Consumer 2 maximizes U^2 subject to

$$q(1)\,x^2(1)+q(2)x^2(2) \leqslant (1/2)q_1(1)+(13/21)q_2(1)+(50/21)q_1(2)+(5/2)q_2(2),$$
$$x^2(1),\, x^2(2) \geqslant 0.$$

The equilibrium condition is

$$x_h^1(s) + x_h^2(s) = 3 \qquad \text{for} \quad h = 1, 2 \text{ and } s = 1, 2.$$

It is easy to show that an equilibrium is given by $q_1(1) = q_2(1) = q_1(2) = q_2(2) = 1/4$; $x^1(1) = x^1(2) = (8/3, 1/3)$, $x^2(1) = x^2(2) = (1/3, 8/3)$. Since consumers' utility functions satisfy the gross substitutes assumption [2], this is the only equilibrium (if we normalize the q_h to sum to unity). (The gross substitutes assumption is actually a restriction on demand functions rather than utility functions. When we say that a utility function satisfies the gross substitutes assumption, we will mean that, given the usual complete market situation, and under the assumption that endowments are nonnegative, the utility function gives rise to a demand function satisfying the gross substitutes assumption.) However, by the definition of the $q_h(s)$,

$$\frac{q_1(s)}{q_2(s)} = \frac{p_1(s)}{p_2(s)} \quad \text{for} \quad s = 1, 2,$$

and therefore, in equilibrium,

$$\frac{p_1(1)}{p_2(1)} = \frac{p_1(2)}{p_2(2)} = 1,$$

which contradicts the assumption that $p(1)$ and $p(2)$ are linearly independent. Hence, Case 1 is ruled out.

Case 2. *$p(1)$ and $p(2)$ are linearly dependent.* In this case, the vectors of monetary returns of the two securities are linearly dependent, and hence there is effectively only one security at date 1. Therefore, there is no way of transferring wealth from state 1 to state 2, and we may assume that in equilibrium there is no trading at date 1.

Consider date 2 and suppose that state 1 has occurred. Then, Consumer 1 maximizes $2^{3/2}\{x_1^1(1)\}^{1/2} + \{x_2^1(1)\}^{1/2}$ subject to

$$p(1)\, x^1(1) \leqslant (5/2)p_1(1) + (50/21)p_2(1), \qquad x^1(1) \geqslant 0,$$

and Consumer 2 maximizes $\{x_1^2(1)\}^{1/2} + 2^{3/2}\{x_2^2(1)\}^{1/2}$ subject to

$$p(1)\, x^2(1) \leqslant (1/2)p_1(1) + (13/21)p_2(1), \qquad x^2(1) \geqslant 0.$$

The equilibrium condition is

$$x_h^1(1) + x_h^2(1) = 3$$

for $h = 1, 2$, and it is easy to show that the unique equilibrium is given by $x^1(1) = (62/21, 31/21)$, $x^2(1) = (1/21, 32/21)$ and $p(1) = (2/3, 1/3)$. (We normalize prices to sum to unity.)

Suppose now that state 2 occurs. Then, Consumer 1 maximizes $2^{3/2}\{x_1^1(2)\}^{1/2} + \{x_2^1(2)\}^{1/2}$ subject to

$$p(2)\, x^1(2) \leqslant \frac{13}{21} p_1(2) + \frac{1}{2} p_2(2), \qquad x^1(2) \geqslant 0,$$

and Consumer 2 maximizes $\{x_1^2(2)\}^{1/2} + 2^{3/2}\{x_2^2(2)\}^{1/2}$ subject to

$$p(2)\, x^2(2) \leqslant \frac{50}{21} p_1(2) + \frac{5}{2} p_2(2), \qquad x^2(2) \geqslant 0.$$

The equilibrium condition is

$$x_h^1(2) + x_h^2(2) = 3$$

for $h = 1, 2$, and it is easy to show that the unique equilibrium is given by $x^1(2) = (32/21, 1/21)$, $x^2(2) = (31/21, 62/21)$ and $p(2) = (1/3, 2/3)$.

It follows that $p(1) = (2/3, 1/3)$ and $p(2) = (1/3, 2/3)$ are linearly independent vectors, which contradicts the assumption of Case 2. Hence, Case 2 is also ruled out. Therefore, since neither Case 1 nor Case 2 is possible, we have shown that an equilibrium does not exist for the economy of Example 1.

It may not be clear from this example why the usual proof of the existence of equilibrium breaks down. It is in fact because consumers' budget correspondences are not upper semicontinuous. To see this, assume that at date 1 the price of the first security is $(121 + 158r)/(42 + 158r)$ and the price of the second security is 1. Consider a sequence of date 2 price vectors, $p^r(1) = (1 + (1/r), 1), p^r(2) = (1, 1)$, where $r = 1, 2,\ldots$. Then, the consumption plan represented by $x^1(1) = x^1(2) = (3/2, 3/2)$ lies in Consumer 1's budget set $\bar{B}^1$ for all r, since the budget constraints are satisfied if Consumer 1 sells $(1 + (79r/21))$ units of security 1 and purchases $[(121/42) + (79r/21)]$ units of security 2. In the limit, however, when the prices of both securities are one and $p(1) = p(2) = (1, 1)$, the consumption plan $x^1(1) = x^1(2) = (3/2, 3/2)$ does not lie in Consumer 1's budget set. This discontinuity in budget correspondences leads, of course, to a discontinuity in consumers' demand correspondences, and hence the usual techniques cannot be applied to prove the existence of equilibrium.

Example 1 does not seem in any way pathological, and there does not appear to be any reasonable assumption which rules out such cases. Radner gets round the problem by imposing an upper bound on forward transactions. That is, he constrains security trading plans z^i to satisfy

$$z_{tf}^i(s) \leqslant L \tag{4.1}$$

for all t, s and f, where L is a fixed positive number. Under this assumption, budget and demand correspondences are upper semicontinuous, and the following theorem may be proved. (A proof is given in Radner [11].)

THEOREM 1. *Assume the following for all i.*

(4.2) *If* $({}^rx^i)$ *is a sequence of consumption plans converging to the consumption plan* x^i, *then* $U^i({}^rx^i) \to U^i(x^i)$.

(4.3) *If* $\hat{x}^i$ *and* x^i *are consumption plans, then* $U^i(\hat{x}^i) > U^i(x^i) \Rightarrow U^i(\lambda\hat{x}^i + (1 - \lambda)x^i) > U^i(x^i)$ *for* $0 < \lambda < 1$.

(4.4) $\omega_{th}^i(s) \gg 0$ *for all s, t and h.*

Then, if security trading plans are constrained to satisfy (4.1) *for all i, a Radner equilibrium exists.*

(We define convergence and addition of consumption plans in the obvious way, regarding the set of all consumption plans as a subset of Euclidean space.

The main objection to Radner's approach is that the upper bound L is not determined endogenously, but instead is fixed a priori. It might seem that a natural upper bound on forward contracts exists since supplies of goods are limited. However, this is not the case, since a consumer can always agree to deliver a large amount of a commodity forward, while at the same time planning to buy the commodity back at a later date.

Hence, the choice of L is arbitrary. This is particularly disturbing since the equilibrium of the economy in general will depend on L. To see this, consider again Example 1. If the equilibrium is independent of L, then we can choose L large enough so that (4.1) is not binding. We will thus obtain an equilibrium for the economy without the constraint (4.1), which contradicts the fact that, as we have already shown, no such equilibrium exists.

For these reasons, we will not impose a constraint like (4.1) in the remainder of this paper, preferring instead to accept the fact that the equilibrium which we have defined may not exist. However, it should be emphasized that the results in Section 5 about the suboptimality of equilibrium also hold if (4.1) is imposed. In fact, one of our reasons for not imposing (4.1) is that we wish to show that the suboptimality of equilibrium is not a consequence of the presence of artificial constraints, but instead arises directly from the incompleteness of the market structure.

5. The Optimality of Equilibrium

We turn now to an analysis of the optimality of a Radner equilibrium. As was noted in Section 2, there is no reason to expect a Radner equilibrium to be fully Pareto optimal when markets are incomplete. Guesnerie and Jaffray [7] have found sufficient conditions for Pareto optimality, but these conditions are quite restrictive. We are led therefore to consider whether a Radner equilibrium is optimal in a somewhat weaker sense. One approach is to attempt to characterize the set of all consumption allocations which can be achieved using existing markets—we will call such allocations *feasible*—and to investigate whether a Radner equilibrium is Pareto optimal relative to this set.

The main difficulty with this approach is finding a suitable definition of a feasible allocation. Diamond [3] has considered this problem when there is only one good and the economy lasts two periods. (Diamond also allows for production activities.) In this case, markets open only at date 1, and it is natural to say that a consumption allocation is feasible if it can be realized by some reallocation of date 1 endowments and some allocation of date 1 securities. In the general case, however, when there is more than one good or there are more than two periods, there appears to be no satisfactory generalization of Diamond's notion of feasibility. The problem is that now markets reopen after date 1, and therefore any sensible definition of a feasible allocation must allow for some exchanges of goods after date 1. If we permit arbitrary exchanges, however, all consumption allocations can be achieved. Therefore exchanges must be restricted in some way, but it is not clear what an appropriate restriction is.

Because of this difficulty, we will not attempt to characterize the set of all feasible consumption allocations. Instead, we will confine our attention to a subset of this set. If we can show that in general a Radner equilibrium is Pareto-dominated by an allocation in this subset, then we will certainly have shown that the equilibrium is not Pareto optimal relative to the whole set.

The subset that we will choose is the set of competitive equilibria. Any reasonable definition of a feasible allocation would certainly include all the competitive equilibria, since equilibrium allocations clearly can be achieved using existing markets.

Given the endowment streams $\omega^1,..., \omega^I$, let $E(\omega^1,..., \omega^I)$ be the set of consumption allocations which can be achieved as competitive equilibria; that is, the consumption allocation $(x^1,..., x^I) \in E(\omega^1,..., \omega^I)$ if there exist security trading plans $z^1,..., z^I$ and a price system (p, π) such that (3.5), (3.6), (3.7), and (3.8) are satisfied.

Definition of Optimality.

(5.1) Given the endowment streams $\omega^1,..., \omega^I$, we will say that the consumption allocation $(x^1,..., x^I) \in E(\omega^1,..., \omega^I)$ is *weakly optimal* if $(x^1,..., x^I)$ is not Pareto-dominated by any $(x'^1,..., x'^I) \in E(\omega^1,..., \omega^I)$.

(5.2) Given the endowment streams $\omega^1,..., \omega^I$, we will say that the consumption allocation $(x^1,..., x^I) \in E(\omega^1,..., \omega^I)$ is *strongly optimal* if $(x^1,..., x^I)$ is not Pareto-dominated by any $(x'^1,..., x'^I) \in E(\omega'^1,..., \omega'^I)$, where $\omega'^1,..., \omega'^I$ are endowment streams such that, for some $u^1,..., u^I \in R^F$, with $\sum_i u^i = 0$,

$$\sum_i \omega_1'^i(s) = \sum_i \omega_1{}^i(s) \text{ for all } s,$$

and

$$\omega_t'^i(s) = \omega_t{}^i(s) + \sum_{f=1}^{F} u_f{}^i a_t{}^f(s) \text{ for all } i,\ t = 2, 3 \text{ and all } s.$$

(Of course, there is no guarantee that strong and weak optima will exist since the set of competitive equilibria may be empty.)

Condition (5.1) says that an equilibrium is optimal if it is not Pareto-dominated by another equilibrium achieved with the same endowment streams. Condition (5.2) says that an equilibrium is optimal if it is not Pareto-dominated by an equilibrium achieved with possibly different endowment streams; however, these endowment streams are restricted so that it is possible to move from the original endowment streams to the new endowment streams by redistributing date 1 endowments and reallocating date 1 securities. If no such restrictions were placed on the new endowment streams, any Pareto optimal consumption allocation could be achieved by redistribution, and hence, strong optimality would be equivalent to full Pareto optimality.

It may be shown that if there is one good in the economy and if consumers' utilities depend only on consumption at dates 1 and 2, then, under weak assumptions, strong optimality is equivalent to constrained Pareto optimality in Diamond's sense [3]. Hence, Diamond's results show that an equilibrium is strongly optimal in this case.

Since we will show that in general an equilibrium is not even weakly optimal, the reader may wonder why we bother to concern ourselves with strong optimality. The reason is that weak optimality is sometimes too restrictive a concept. For example, if there is a unique equilibrium, then it is automatically weakly optimal. In Section 6, we will see that even if an equilibrium is weakly optimal, it may not be strongly optimal.

For the rest of this section, unless otherwise stated, we will confine our attention to weak optimality. We have already given an example in Section 2 where an equilibrium is not weakly optimal. In that example, however, we assumed that there was no way of transferring wealth from one period to another. We also assumed that there were multiple equilibria in each period. Such multiple equilibria would be ruled out, for example, by the gross substitutes assumption [2]. We now give an example where, even though the utility functions satisfy the gross substitutes assumption and there are some opportunities for transferring wealth from one period to another, there is an equilibrium which is not weakly optimal. (Although the gross substitutes assumption rules out multiple equilibria in an Arrow-Debreu economy, it does not rule out multiple equilibria in an economy with incomplete markets. The reason for this is the existence of budget constraints at each date.)

EXAMPLE 2. This example is identical to Example 1 of Section 4 except that we change the dividends of the securities. Assume that, at date 2, one unit of the first security yields one unit of the first good if state 1 occurs and two units of the first good if state 2 occurs; one unit of the second security yields two units of the second good if state 1 occurs and one unit of the second good if state 2 occurs. Then, if we repeat the argument of Example 1, the equilibria of Cases 1 and 2 now both exist. In the Case 1 equilibrium,

$$U^1 = U^2 = 2^{5/2}\left(\frac{8}{3}\right)^{1/2} + 2\left(\frac{1}{3}\right)^{1/2},$$

and, in the Case 2 equilibrium,

$$U^1 = U^2 = 2^{3/2}\left(\frac{62}{21}\right)^{1/2} + \left(\frac{31}{21}\right)^{1/2} + 2^{3/2}\left(\frac{32}{21}\right)^{1/2} + \left(\frac{1}{21}\right)^{1/2}.$$

Since it is easy to show that

$$2^{5/2}\left(\frac{8}{3}\right)^{1/2} + 2\left(\frac{1}{3}\right)^{1/2} > 2^{3/2}\left(\frac{62}{21}\right)^{1/2} + \left(\frac{31}{21}\right)^{1/2} + 2^{3/2}\left(\frac{32}{21}\right)^{1/2} + \left(\frac{1}{21}\right)^{1/2},$$

it follows that the Case 2 equilibrium is Pareto-dominated by the Case 1 equilibrium, and hence the Case 2 equilibrium is not weakly optimal.

In this example, there are two goods and the economy last for essentially only two periods (there is no economic activity at date 3). We now show that in a three period economy, an equilibrium may not be weakly optimal even if there is only one good.

EXAMPLE 3. The mathematical structure of Example 3 is identical to that of Example 2. There are two consumers, one good and two states, and consumers know which state has occurred at date 2. Each consumer receives a zero endowment of the good at date 1 and each consumer's utility depends only on consumption at dates 2 and 3. We assume that

$$U^1 = 2^{3/2}\{x_1{}^1(1)\}^{1/2} + \{x_2{}^1(1)\}^{1/2} + 2^{3/2}\{x_1{}^1(2)\}^{1/2} + \{x_2{}^1(2)\}^{1/2},$$
$$U^2 = \{x_1{}^2(1)\}^{1/2} + 2^{3/2}\{x_2{}^2(1)\}^{1/2} + \{x_1{}^2(2)\}^{1/2} + 2^{3/2}\{x_2{}^2(2)\}^{1/2}.$$

These are the same utility functions as in Example 1, but the interpretation is different. Now it is the *h* subscript, not the *t* subscript, that is left out, so that $x_t{}^i(s)$ refers to Consumer *i*'s consumption in state *s* at *date t*. Consumer 1's endowment is 5/2 units of the good at date 2 and 50/21 units of the good at date 3 if state 1 occurs, and 13/21 units of the good at date 2 and 1/2 unit of the good at date 3 if state 2 occurs. Consumer 2's endowment is 1/2 unit of the good at date 2 and 13/21 units of the good at date 3 if state 1 occurs, and 50/21 units of the good at date 2 and 5/2 units of the good at date 3 if state 2 occurs.

There are two securities at date 1. One unit of the first security yields one unit of the good at date 2 if state 1 occurs and two units of the good at date 2 if state 2 occurs. One unit of the second security yields two units of the good at date 3 if state 1 occurs and one unit of the good at date 3 if state 2 occurs. There are no new securities at date 2, but security 2 is retraded.

Let us normalize prices so that the spot price of the good is one at date 2 in both states. Let $\pi_1 \in R_+^2$ be the vector of equilibrium security prices at date 1 and let $\pi_{22}(s) \in R_+$ be the equilibrium price of security 2 at date 2 in state s. Then, at date 2, the vector of monetary returns of security 1 is $(1, 2)$ and the vector of monetary returns of security 2 is $(\pi_{22}(1), \pi_{22}(2))$. As in Examples 1 and 2, we may distinguish between two cases: the case where these vectors are linearly independent and the case where they are linearly dependent. Furthermore, as in Example 2, there is an equilibrium in each case, and the equilibrium in which the vectors are linearly independent Pareto-dominates the equilibrium in which they are linearly dependent. Hence, the latter equilibrium is not weakly optimal.

Examples 2 and 3, and also the example of Section 2, show that if markets are incomplete and there are at least two goods or at least three dates, one equilibrium may be Pareto-dominated by another equilibrium. The reason for this may be given intuitively as follows. When markets are incomplete, trading opportunities depend to some extent on expected prices. One equilibrium may be better for all consumers than another because greater gains from trade can be realized at the equilibrium expected prices. In Example 2, for instance, when $p(1)$ and $p(2)$ are linearly independent, the trading opportunities are the same as they would be if a full set of contingent futures commodity markets existed at date 1. When $p(1)$ and $p(2)$ are linearly dependent, however, the situation is equivalent to one in which there are no opportunities for futures trading at all. Moreover, since consumers are perfectly competitive, expected prices are taken as given, and hence there are no forces within the market which prevent $p(1)$ and $p(2)$ from being linearly dependent.

Examples 2 and 3 and the example of Section 2 share an interesting property which throws further light on the reasons for the suboptimality of equilibrium. If we compare the Pareto-inferior equilibrium and the Pareto-superior equilibrium in each of these examples, we see that, at the Pareto-inferior equilibrium prices, there do not exist security trading plans which provide consumers with *exactly* the monetary returns they need to purchase the consumption allocations corresponding to the Pareto-superior equilibrium. This is most easily seen in the example of Section 2. At the (A, D) equilibrium prices, Consumer 1 can purchase the consumption bundle (B, C) only by exchanging income in state 2 for income in state 1. However, by assumption, such an exchange is not feasible since no markets for securities exist at date 1.

Theorem 2 shows that this property of Pareto-inferior equilibria generalizes.

THEOREM 2. *Suppose that* $(x^1,..., x^I) \in E(\omega^1,..., \omega^I)$ *is an equilibrium consumption allocation and that the corresponding equilibrium price system is* (p, π). *Let* $(\hat{x}^1,..., \hat{x}^I)$ *be a consumption allocation and assume that for all* i, *except possibly for some* $i = i_0$, *there exists a security trading plan* $\hat{z}^i$ *such that*

$$p_2(s)\, \hat{x}_2{}^i(s) + \pi_2(s)\, \hat{z}_2{}^i(s) = p_2(s)\, \omega_2{}^i(s) + p_2(s)\left(\sum_{f=1}^{F} \hat{z}_{1f}^i(s)\, a_2{}^f(s)\right) + \sum_{f=1}^{F} \pi_{2f}(s)\, \hat{z}_{1f}^i(s) \tag{5.3}$$

and

$$p_3(s)\, \hat{x}_3{}^i(s) = p_3(s)\, \omega_3{}^i(s) + p_3(s)\left(\sum_{f=1}^{F+G} \hat{z}_{2f}^i(s)\, a_3{}^f(s)\right) \tag{5.4}$$

for all s. *Then* $(\hat{x}^1,..., \hat{x}^I)$ *does not Pareto-dominate* $(x^1,..., x^I)$.

Proof. Suppose that $(\hat{x}^1,..., \hat{x}^I)$ Pareto-dominates $(x^1,..., x^I)$. Consider any $i \neq i_0$. Since (5.3) and (5.4) imply that Consumer i could have achieved the consumption plan $\hat{x}^i$ by purchasing the security trading plan $\hat{z}^i$, it must be the case that Consumer i's budget constraint at date 1 would have been violated. That is,

$$p_1(s)\, \hat{x}_1{}^i(s) + \pi_1(s)\, \hat{z}_1{}^i(s) > p_1(s)\, \omega_1{}^i(s). \tag{5.5}$$

Consider now Consumer i_0. It is easy to show that since $(\hat{x}^1,..., \hat{x}^I)$ satisfies (3.7), (5.3) and (5.4) imply that

$$p_2(s)\, \hat{x}_2^{i_0}(s) + \pi_2(s)\, \hat{z}_2^{i_0}(s) \leqslant p_2(s)\, \omega_2^{i_0}(s) + p_2(s)\left(\sum_{f=1}^{F} \hat{z}_{1f}^{i_0}(s)\, a_2{}^f(s)\right) + \sum_{f=1}^{F} \pi_{2f}(s)\, \hat{z}_{1f}^{i_0}(s) \tag{5.6}$$

and

$$p_3(s)\, \hat{x}_3^{i_0}(s) \leqslant p_3(s)\, \omega_3^{i_0}(s) + p_3(s)\left(\sum_{f=1}^{F+G} \hat{z}_{2f}^{i_0}(s)\, a_3{}^f(s)\right)$$

$$\text{for all } s, \text{ where } \hat{z}^{i_0} = -\sum_{i \neq i_0} \hat{z}^i. \tag{5.7}$$

Therefore, the security trading plan $\hat{z}^{i_0}$ must have been too expensive for Consumer i_0 at date 1, that is,

$$p_1(s)\,\hat{x}_1^{i_0}(s) + \pi_1(s)\,\hat{z}_1^{i_0}(s) > p_1(s)\,\omega_1^{i_0}(s). \tag{5.8}$$

Summing (5.5) over $i \neq i_0$ and adding (5.8), we obtain

$$p_1(s)\left(\sum_i \hat{x}_1{}^i(s)\right) > p_1(s)\left(\sum_i \omega_1{}^i(s)\right),$$

which contradicts the fact that $(\hat{x}^1,\ldots,\hat{x}^I)$ satisfies (3.7). Q.E.D.

Theorem 2 suggests a natural sufficient condition for Pareto optimality. Let us define the market structure to be complete if, given any vector of monetary returns, there exists a security trading plan which yields exactly those monetary returns at each date and in each event. More formally:

DEFINITION OF COMPLETE MARKET STRUCTURE. Let Λ be the set of ordered triples $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, where, for each t, λ_t is a $\mathscr{P}_t$-measurable function from S to R. We will say that the market structure is *complete* if, given any price system (p, π) and any $\lambda \in \Lambda$, there exists a security trading plan z such that, for all s,

$$\lambda_2(s) + \pi_2(s)\,z_2(s) = p_2(s)\left(\sum_{f=1}^{F} z_{1f}(s)\,a_2{}^f(s)\right) + \sum_{f=1}^{F} \pi_{2f}(s)\,z_{1f}(s), \tag{5.9}$$

$$\lambda_3(s) = p_3(s)\left(\sum_{f=1}^{F+G} z_{2f}(s)\,a_3{}^f(s)\right). \tag{5.10}$$

It should be noted that the market structure is necessarily complete if a full set of contingent futures commodity markets or Arrow–Debreu securities exists at date 1. (Strictly speaking, this is only true if the price of a security which yields nothing at every date and in every event is set equal to zero. In equilibrium, of course, the price of such a security must be zero as long as some consumer is nonsatiated at each date and in each event.) It will also be complete in other cases, however.

THEOREM 3. *Suppose that the market structure is complete. Then, if $(x^1,\ldots,x^I) \in E(\omega^1,\ldots,\omega^I)$, $(x^1,\ldots,x^I)$ is a Pareto optimum.*

Proof. Apply Theorem 2, putting $\lambda_t(s) = p_t(s)\,\hat{x}_t{}^i(s) - p_t(s)\,\omega_t{}^i(s)$. Q.E.D.

Theorem 3 gives us a sufficient condition for Pareto optimality which generalizes that of Guesnerie and Jaffray [7]. We turn next to a sufficient condition for strong optimality.

DEFINITION. We will say that the market structure is *complete up to date* 2 if, given any price system (p, π) and any $\lambda \in \Lambda$, there exists a security trading plan such that

$$\lambda_2(s) = p_2(s)\left(\sum_{f=1}^{F} z_{1f}(s)\, a_2^{f}(s)\right) + \sum_{f=1}^{F} \pi_{2f}(s)\, z_{1f}(s) \tag{5.11}$$

for all s.

In other words, the market structure is complete up to date 2 if, given any vector of monetary returns, there exists a security trading plan which yields exactly those monetary returns at each date and in each event, *excluding date* 3.

THEOREM 4. *Suppose that the market structure is complete up to date* 2, *and there is only one good in the economy* ($H = 1$). *Then, if* $(x^1,..., x^I) \in E(\omega^1,..., \omega^I)$, $(x^1,..., x^I)$ *is strongly optimal.*

Proof. We will actually be able to prove a stronger result. We will show that $(x^1,..., x^I)$ cannot be Pareto-dominated by any allocation which is achieved by an arbitrary redistribution of endowments at dates 1 and 2 and an allocation of securities at date 2.

Let $(\hat{x}^1,..., \hat{x}^I)$ be such an allocation. Then, for each i and s, there exists $u^i(s) \in R^{F+G}$ such that

$$\hat{x}_3^{i}(s) = \omega_3^{i}(s) + \sum_{f=1}^{F+G} u_f^{i}(s)\, a_3^{f}(s).$$

Therefore, since the market structure is complete up to date 2, the conditions of Theorem 2 are satisfied (put $\hat{z}_2^{i}(s) = u^i(s)$ and $\lambda_2(s) = p_2(s)\, \hat{x}_2^{i}(s) - p_2(s)\, \omega_2^{i}(s) + \pi_2(s)\, \hat{z}_2^{i}(s)$). Hence $(\hat{x}^1,..., \hat{x}^I)$ cannot Pareto-dominate $(x^1,..., x^I)$. Q.E.D.

Theorem 4 generalizes Diamond's result on the constrained optimality of equilibrium in a one good world to the case where there are three periods. (More generally, if the economy lasts for T periods, Theorem 4 holds under the assumption that markets are complete up to date $T - 1$.)

In an important sense, Theorems 3 and 4 provide us with the most general possible sufficient conditions for optimality. In fact, it can be shown that if the conditions of Theorems 3 and 4 do not hold, and if we rule out some minor exceptional cases, then it is always possible to choose utility functions and endowment streams for consumers such that there exists an equilibrium which is not weakly optimal. Of course, if we are prepared to restrict the utility functions and endowment streams, other sufficient conditions may be obtained. For example, if all consumers are identical and possess quasiconcave utility functions, an equilibrium is fully Pareto optimal, whatever the market structure is, since there are no gains from trade!

In conclusion, we have shown that an equilibrium is Pareto-optimal if markets are complete, and strongly optimal if markets are complete up to date 2 and there is only one good in the economy. In other cases, however, there is no reason to expect an equilibrium to be even weakly optimal, unless restrictions are imposed on consumers' utility functions and endowments.

6. The Consequences of Opening New Markets

So far, we have assumed that the market structure of the economy is fixed. We will now investigate the consequences of opening a new market, that is, we will consider what happens if a new security becomes available at date 1 or date 2. Our intuition tells us that this ought to improve the situation in some sense. We will give an example, however, where this is not the case. In this example, there is a unique equilibrium in the initial situation and a unique equilibrium after the new market has been opened, and each consumer is worse off after the new market has been opened.

The consequence of opening new markets has been investigated in a different context by Green and Sheshinski [6]. They also find that opening new markets may make everybody worse off, but their results apply to a production situation where there are transaction costs. In contrast, we will show that everybody may be made worse off even in an exchange economy where there are no transaction costs.

Example 4. There are two consumers, two goods and two states, and consumers do not know which state has occurred until date 3. Consumers' utility functions are given by

$$U^1 = \{x_{11}^1\} + \{(x_{11}^1 + 1)^{1/4}(x_{31}^1(1))^{1/4} + (x_{32}^1(1))^{1/4}\} + \beta^1\{2^{1/4}(x_{31}^1(2))^{1/4} + (x_{32}^1(2))^{1/4}\}, \tag{6.1}$$

$$U^2 = \{x_{21}^2\} + \beta^2\{2^{1/4}(x_{31}^2(1))^{1/4} + (x_{32}^2(1))^{1/4}\} + \{(x_{21}^2 + 1)^{1/4}(x_{31}^2(2))^{1/4} + (x_{32}^2(2))^{1/4}\}, \tag{6.2}$$

where β^1, β^2 are positive numbers. ($x^i_{th}(s)$ is Consumer i's consumption of good h at date t in state s. We leave out s if $t = 1$ or 2, since state s is not known then, and hence $x^i_{th}(s)$ must be independent of s.)

Consumer 1 receives nothing at date 1, one unit of the first good at date 2, and one unit of each good at date 3, whichever state occurs. Consumer 2 receives one unit of the first good at date 1, nothing at date 2, and one unit of each good at date 3, whichever state occurs.

Initially, there are no securities at dates 1 and 2. Then, a single security becomes available at date 1; one unit of this security yields one unit of the first good at date 2, whichever state occurs.

INITIAL EQUILIBRIUM. Consider the equilibrium when there are no securities. At dates 1 and 2, there will be no trading since in each case only one good is in positive supply. Suppose that state 1 occurs. Then, at date 3, Consumer 1's tastes are represented by the second term in (6.1); that is, since $x^1_{11} = 0$, by the function $(x^1_{31}(1))^{1/4} + (x^1_{32}(1))^{1/4}$. Similarly, Consumer 2's tastes are represented by the second term in (6.2), that is, by the function $\beta^2(2^{1/4}(x^2_{31}(1))^{1/4} + (x^2_{32}(1))^{1/4})$. Since these two functions satisfy the gross substitutes assumption, there is a unique equilibrium at date 3 in state 1. Furthermore, since the consumers' marginal rates of substitution between goods one and two differ at the endowment point, the equilibrium must involve trade and each consumer must be better off after trade has taken place. Therefore,

$$(x^1_{31}(1))^{1/4} + (x^1_{32}(1))^{1/4} > 1 + 1, \tag{6.3}$$

$$2^{1/4}(x^2_{31}(1))^{1/4} + (x^2_{32}(1))^{1/4} > 2^{1/4} + 1. \tag{6.4}$$

Suppose that state 2 occurs. Then at date 3, Consumer 1's tastes are represented by the third term in (6.1), that is, by the function $\beta^1\{2^{1/4}(x^1_{31}(2)^{1/4} + (x^1_{32}(2))^{1/4}\}$. Consumer 2's tastes are represented by the third term in (6.2); that is, since $x^2_{21} = 0$, by the function $\{(x^2_{31}(2))^{1/4} + (x^2_{32}(2))^{1/4}\}$. Again there is a unique equilibrium which involves trade and hence,

$$2^{1/4}(x^1_{31}(2))^{1/4} + (x^1_{32}(2))^{1/4} > 2^{1/4} + 1, \tag{6.5}$$

$$(x^2_{31}(2))^{1/4} + (x^2_{32}(2))^{1/4} > 1 + 1. \tag{6.6}$$

In the overall equilibrium at date 1, consumers' utilities are given by

$$U^1 = (x^1_{31}(1))^{1/4} + (x^1_{32}(1))^{1/4} + \beta^1\{2^{1/4}(x^1_{31}(2))^{1/4} + (x^1_{32}(2))^{1/4}\}, \tag{6.7}$$

$$U^2 = \beta^2\{2^{1/4}(x^2_{31}(1))^{1/4} + (x^2_{32}(1))^{1/4}\} + \{(x^2_{31}(2))^{1/4} + (x^2_{32}(2))^{1/4}\}. \tag{6.8}$$

EQUILIBRIUM WITH THE SECURITY. If the security is introduced at date 1, then, since Consumer 1's utility is independent of consumption at date 2 and Consumer 2's utility is independent of consumption at date 1, it is clear that, in equilibrium, the consumers will simply swap their endowments at dates 1 and 2. This means that $x_{11}^1 = 1$ and hence, if state 1 occurs, Consumer 1's tastes at date 3 are now represented by the function $2^{1/4}(x_{31}^1(1))^{1/4} + (x_{32}^1(1))^{1/4}$. Similarly, if state 2 occurs, Consumer 2's tastes at date 3 are now represented by the function $2^{1/4}(x_{31}^2(2))^{1/4} + (x_{32}^2(2))^{1/4}$. Since Consumer 1's tastes at date 3 are unchanged if state 2 occurs, and Consumer 2's tastes at date 3 are unchanged if state 1 occurs, it follows that there will no longer be any gains from trade at date 3. Hence, consumers' equilibrium utilities are

$$U^1 = 1 + (2^{1/4} + 1) + \beta^1(2^{1/4} + 1), \tag{6.9}$$

$$U^2 = 1 + \beta^2(2^{1/4} + 1) + (2^{1/4} + 1). \tag{6.10}$$

Let us compare (6.7) and (6.9), and (6.8) and (6.10). Since the equilibria with and without the security are independent of β^1 and β^2, we can choose β^1 and β^2 arbitrarily. But it is clear that, in view of (6.5) and (6.4), for sufficiently large β^1 and β^2, the expression in (6.7) is greater than the expression in (6.9) and the expression in (6.8) is greater than the expression in (6.10). Therefore, for sufficiently large β^1 and β^2, the equilibrium with the security market is worse for both consumers than the equilibrium without the security market. Hence, opening the security market makes both consumers worse off.

It may seem puzzling that expanding the market opportunities available to the consumers can make them worse off. Intuitively, the reason is that, if β^1 and β^2 are large, the utilities of Consumers 1 and 2 are determined mainly by consumption at date 3. If consumers are given the opportunity to trade at date 1, they will in fact eliminate all the gains from trade at date 3. However, since the consumers are pricetakers, the interdependence between the gains from trade at dates 1 and 3 will not be recognized and hence individual maximizing behavior will lead to an outcome which is socially undesirable.

We can modify the above example to fulfill our earlier promise to construct an equilibrium which is weakly optimal but not strongly optimal. Replace U^2 by

$$U^2 = \{x_{21}^2\} + \beta^2\{2^{1/4}(x_{31}^2(1))^{1/4} + (x_{32}^2(1))^{1/4}\}$$
$$+ \{(x_{11}^2 + 1)^{1/4}(x_{21}^2 + 1)^{1/4}(x_{31}^2(2))^{1/4} + (x_{32}^2(2))^{1/4}\},$$

and assume that Consumer 1 has one unit of the first good at date 1 and Consumer 2 has one unit of the first good at date 2. (Date 3 endowments remain the same.) Then, a unique equilibrium exists. Now suppose that Consumer 1 gives all his endowment of the first good at date 1 to Consumer 2. Then, it is easy to show that a unique equilibrium again exists and that, if β^1 is large enough, both consumers are made better off.

Conclusion

We have shown that an economy with incomplete markets differs in many important respects from an economy with complete markets. First, the usual continuity and convexity assumptions are not sufficient to ensure the existence of equilibrium. Secondly, an equilibrium may be Pareto-dominated by another allocation which can be achieved using the same markets.

We have also seen that if we start off in a situation where markets are incomplete, opening new markets may make things worse rather than better. In this respect, an economy with incomplete markets is like a typical second best situation. Only if all imperfections are removed, that is, in this case all markets are opened, can we be sure that there will be any overall improvement.

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