ECON G6410 Problem Set 10

1. (a) Every subset of X is invariant under f.

   Every point in X is a fixed point. For all $x \in X$, the corresponding stable set is $W(x) = \{x\}$.

   (b) The subsets of X that are invariant under f take the form of

   $$S = T \cup (-T),$$

   where $T \subset \mathbb{R}$.

   The system has one fixed point $x = 0$; its stable set is $W(0) = \{0\}$.

   (c) The subsets of X that are invariant under f take the form of

   $$S = T \cup \{t + 1 : t \in T\},$$

   where $T \subset \mathbb{R}$.

   The system has no stable point.

   (d) The subsets of X that are invariant under f take the form of

   $$S = T \cup \left\{ \frac{t}{2} : t \in T \right\},$$

   where $T \subset \mathbb{R}$.

   The system has one fixed point $x = 0$; its stable set is $W(0) = \mathbb{R}$. This is because for any $x \in \mathbb{R}$,

   $$\lim_{n \to \infty} f^n(x) = \lim_{n \to \infty} \frac{x}{2^n} = 0.$$

   (e) The subsets of X that are invariant under f take the form of

   $$S = T \cup \{1.2t : t \in T\},$$

   where $T \subset \mathbb{R}_+$.

   The system has one fixed point $x = 0$; its stable set is $W(0) = \{0\}$.

   (f) The subsets of X that are invariant under f take the form of

   $$S = T \cup \left\{ 0.2\sqrt{t} + 0.8t : t \in T \right\},$$

   where $T \subset \mathbb{R}_+$.

Setting $f(x) = x$ gives two solutions $x = 0$ and $x = 1$, which are the fixed points of the system. To find their stable sets, note that $f(x) > x$ for $x \in (0,1)$ and $f(x) < x$ for $x \in (1, \infty)$, so

$$\lim_{n \to \infty} f^n(x) = 1$$

for any $x \in (0, \infty)$. Therefore, $W(1) = (0, \infty)$ and $W(0) = \{0\}$.

2. (a)

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} -2 & -1 & 3.5 \\ -1 & 0 & 1.5 \\ -4 & -1 & 5.5 \end{pmatrix}.$$

(b) Since B is invertible,

$$A = B^{-1}C$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} C$$

$$= \begin{pmatrix} -1 & -1 & 2 \\ -1 & 0 & 1.5 \\ -3 & -1 & 4 \end{pmatrix}.$$

(c) Set $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} -1 - \lambda & -1 & 2 \\ -1 & -\lambda & 1.5 \\ -3 & -1 & 4 - \lambda \end{vmatrix} = 0$$

$$2\lambda^3 - 6\lambda^2 + 5\lambda - 2 = 0$$

$$(\lambda - 2)\left(2\lambda^2 - 2\lambda + 1\right) = 0$$

$$\lambda = 2 \text{ or } \frac{1+i}{2} \text{ or } \frac{1-i}{2}.$$

(d) First find the eigenvectors.
For $\lambda_1 = 2$:

$$(A - 2I) z = 0$$

$$\begin{pmatrix} -3 & -1 & 2 \\ -1 & -2 & 1.5 \\ -3 & -1 & 2 \end{pmatrix} z = 0,$$

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which implies

$$-3z_1 - z_2 + 2z_3 = 0$$
$$-z_1 - 2z_2 + 1.5z_3 = 0.$$

Solve the system and get an eigenvector

$$z = (1, 1, 2)'.$$

For $\lambda_2 = \frac{1+i}{2}$: for the real part,

$$\left(A - \left(\frac{1+i}{2} \right) I \right) (u + iv) = 0$$

$$\begin{pmatrix} -1.5 - 0.5i & -1 & 2 \\ -1 & -0.5 - 0.5i & 1.5 \\ -3 & -1 & 3.5 - 0.5i \end{pmatrix} (u + iv) = 0,$$

which implies

$$(-1.5 - 0.5i) u_1 - u_2 + 2u_3 + (0.5 - 1.5i) v_1 - iv_2 + 2iv_3 = 0$$
$$-u_1 - (0.5 + 0.5i) u_2 + 1.5u_3 - iv_1 + (0.5 - 0.5i) v_2 + 1.5iv_3 = 0,$$

or

$$-1.5u_1 - u_2 + 2u_3 + 0.5v_1 = 0$$
$$-0.5u_1 - 1.5v_1 - v_2 + 2v_3 = 0$$
$$-u_1 - 0.5u_2 + 1.5u_3 + 0.5v_2 = 0$$
$$-0.5u_2 - v_1 - 0.5v_2 + 1.5v_3 = 0.$$

Solve the system and get an eigenvector

$$u + iv = (-1, 0, -1)' + i (1, 1, 1)'.$$

For $\lambda_3 = \frac{1+i}{2}$, an eigenvector is

$$u - iv = (-1, 0, -1)' + i (1, 1, 1)'.$$

Let

$$P = \begin{pmatrix} z & u & v \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \end{pmatrix}.$$

Then

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & -0.5 & 0.5 \end{pmatrix},$$

i.e. $\lambda = 2, \alpha = \beta = 0.5$.

(e) Let $Ax = x$; then $(A - I)x = 0$. Because

$$\det (A - I) = \begin{vmatrix} -2 & -1 & 2 \\ -1 & -1 & 1.5 \\ -3 & -1 & 3 \end{vmatrix} = 0.5,$$

$A - I$ is invertible and the system has a unique solution

$$x^* = (A - I)^{-1} 0$$
$$= 0.$$

(f) Let $P^{-1}AP = \Lambda$; then $A = P\Lambda P^{-1}$ and thus

$$A^t = P\Lambda^t P^{-1}.$$

Note that Λ^t takes the form

$$\Lambda^t = \begin{pmatrix} 2^t & 0 & 0 \\ 0 & a_t & b_t \\ 0 & c_t & d_t \end{pmatrix},$$

where

$$D_t \equiv \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix}^t.$$

We want to first show $\lim_{t \to \infty} D_t = 0$. For a matrix C, denote the entry with the largest absolute value as k_C. Because

$$D_2 = \begin{pmatrix} 0 & 0.5 \\ -0.5 & 0 \end{pmatrix},$$

for an arbitrary matrix M,

$$D_2 M = \begin{pmatrix} 0.5m_{21} & 0.5m_{22} \\ -0.5m_{11} & -0.5m_{12} \end{pmatrix}.$$

Therefore, we have

$$k_{D_2 M} = \frac{1}{2} k_M.$$

So

$$k_{D_t} = \left(\frac{1}{2}\right)^{\frac{t}{2}} k_I$$
$$= \left(\frac{1}{2}\right)^{\frac{t}{2}}$$

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for any even number integer t, and

$$k_{D_t} = \left(\frac{1}{2}\right)^{\frac{t-1}{2}} k_{D_1}$$

$$= \left(\frac{1}{2}\right)^{\frac{t+1}{2}}$$

for any odd integer t. Apparently $\lim_{t\to\infty} k_{D_t} \to 0$; so $\lim_{t\to\infty} D_t = 0$.

It follows that

$$\lim_{t\to\infty} A^t = P \lim_{t\to\infty} \Lambda^t P^{-1}$$

$$= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2^t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^t & 0 & 0 \\ 2^t & 0 & 0 \\ 2^{t+1} & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2^t & 0 & 2^t \\ -2^t & 0 & 2^t \\ -2^{t+1} & 0 & 2^{t+1} \end{pmatrix}.$$

Let $x_0 = (1,0,0)'$; then

$$\lim_{t\to\infty} A^t x_0 = \lim_{t\to\infty} \left(-2^t, -2^t, -2^{t+1}\right) \neq 0.$$

(g) Let $\hat{x}_0 = (1,0,1)'$. It is easy to verify that $\lim_{t\to\infty} A^t \hat{x}_0 = 0$.

3. We want to show there exists a homeomorphism $h : [-1,1] \to [-2,2]$ such that $h \circ f = g \circ h$. We guess that h is of the polynomial form $h(x) = a + bx + cx^2$. Then

$$h \circ f(x) = a + b\left(2x^2 - 1\right) + c\left(2x^2 - 1\right)^2$$
$$= 4cx^4 + (2b - 4c)x^2 + a - b + c,$$
$$g \circ h(x) = \left(a + bx + cx^2\right)^2 - 2$$
$$= c^2 x^4 + 2bcx^3 + \left(b^2 + 2ac\right)x^2 + 2abx + a^2 - 2.$$

The equivalence between the two implies

$$4c = c^2$$
$$0 = 2bc$$
$$2b - 4c = b^2 + 2ac$$
$$0 = 2ab$$
$$a - b + c = a^2 - 2.$$

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Solve the system and get

$$\begin{cases} a = -1 \\ b = 0 \\ c = 0 \end{cases} \quad \text{or} \quad \begin{cases} a = 2 \\ b = 0 \\ c = 0 \end{cases} \quad \text{or} \quad \begin{cases} a = 0 \\ b = 2 \\ c = 0 \end{cases}.$$
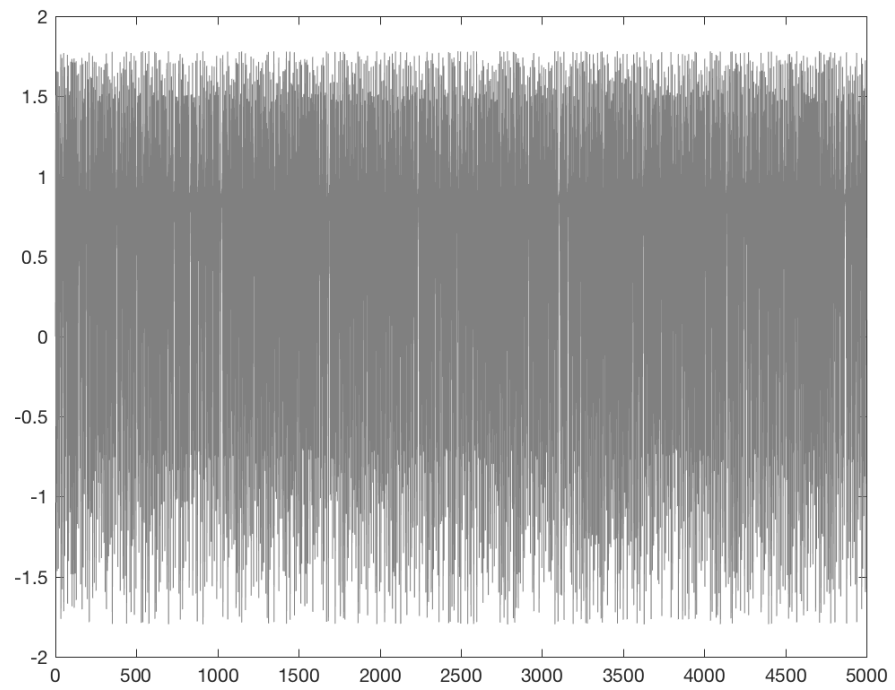
The third set of solutions corresponds to $h(x) = 2x$, which is a homeomorphism from $[-1, 1]$ to $[-2, 2]$. Therefore, f and g are conjugate.
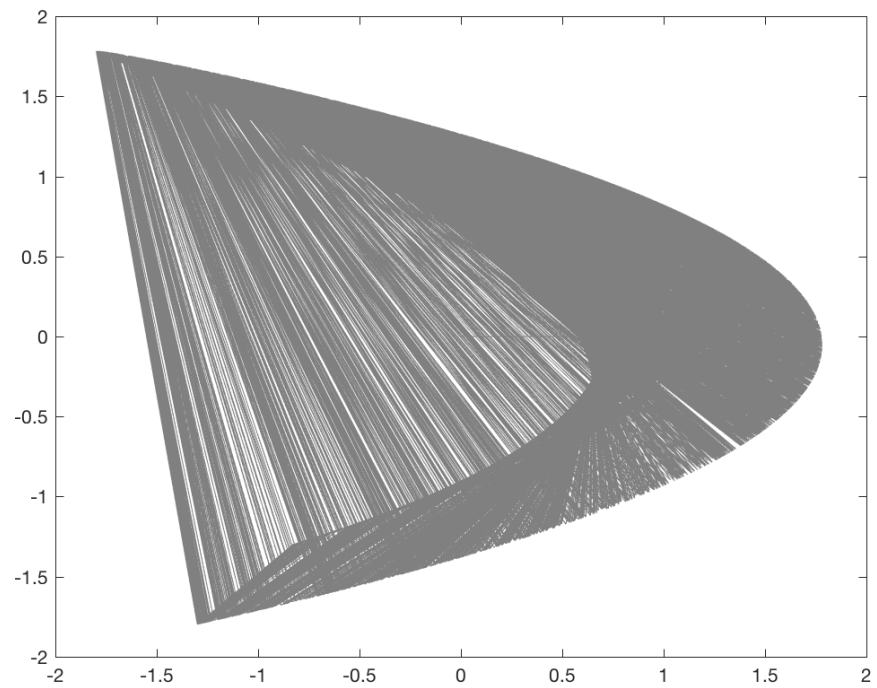
4. (a)

```matlab
1  clear all;
2  close all;
3
4  % Part (a)
5  n=5000;
6  a=1.4;
7  b=0.3;
8  x=zeros(n,1);
9  y=zeros(n,1);
10
11 for i=2:n
12    x(i) = a - (x(i-1))^2 + b*y(i-1);
13    y(i) = x(i-1);
14 end
15
16 plot(1:5000,x,'Color',[0.5,0.5,0.5])
17 print('Fig_a1','-dpng')
18
19 plot(x,y,'Color',[0.5,0.5,0.5])
20 print('Fig_a2','-dpng')
```
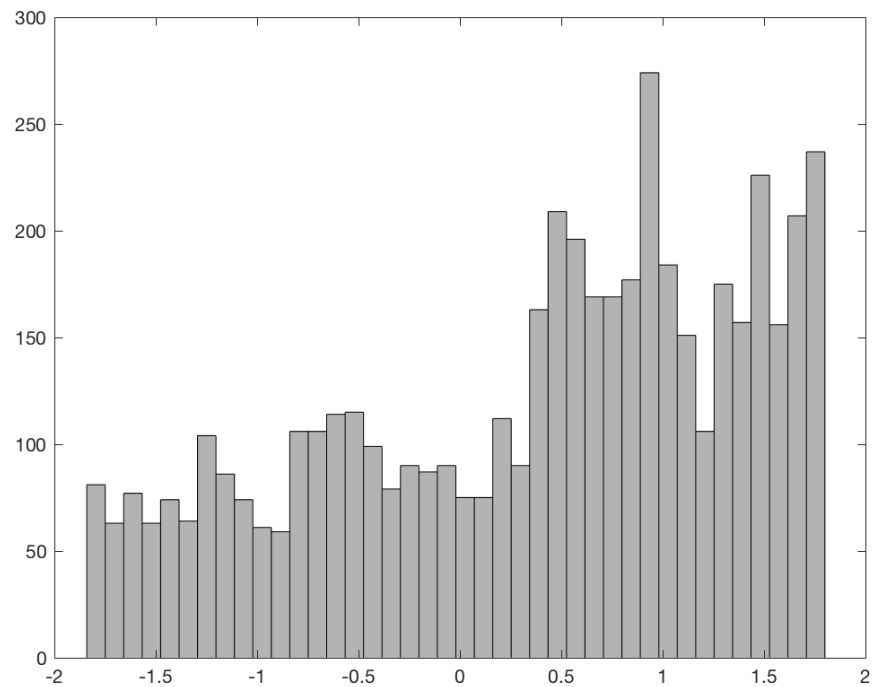
Fig_a1.png:

Fig_a2.png:

(b)

```
1  % Part (b)
2  histogram(x,40,'FaceColor',[0.5,0.5,0.5])
3  print('Fig_b','-dpng')
```
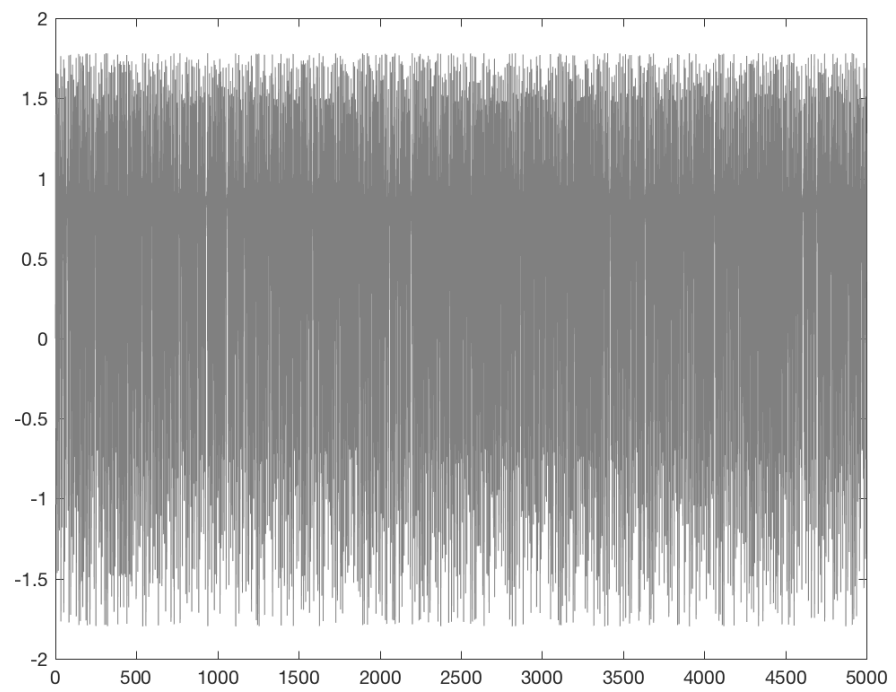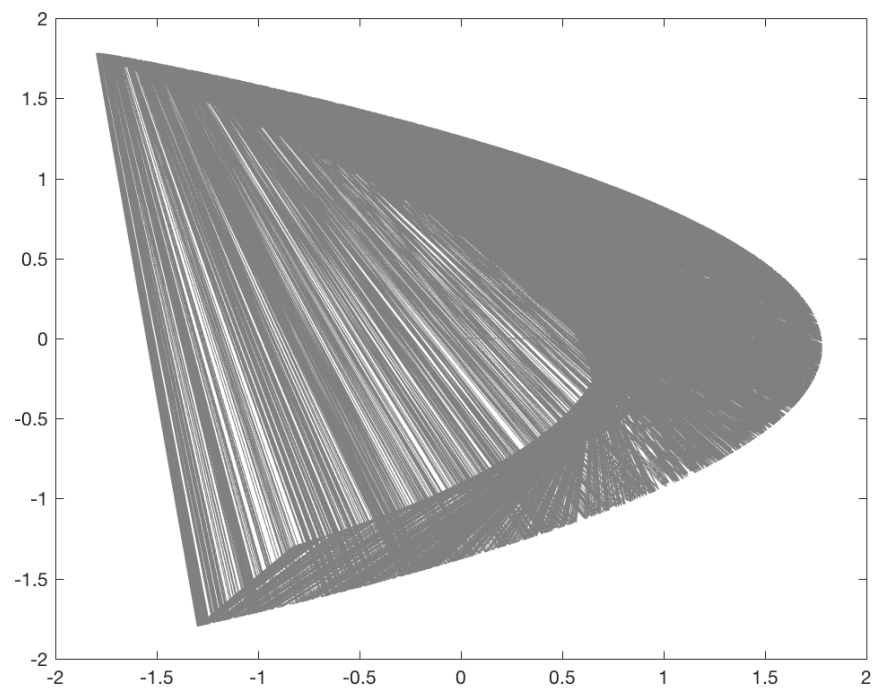
Fig_b.png:

(c)

```matlab
% Part (c)
xx=zeros(n,1);
yy=zeros(n,1);
xx(1)=10^(-8);
yy(1)=10^(-8);

for i=2:n
    xx(i) = a - (xx(i-1))^2 + b*yy(i-1);
    yy(i) = xx(i-1);
end

plot(1:5000,xx,'Color',[0.5,0.5,0.5])
print('Fig_c1','-dpng')

plot(xx,yy,'Color',[0.5,0.5,0.5])
print('Fig_c2','-dpng')

histogram(xx,40,'FaceColor',[0.5,0.5,0.5])
print('Fig_c3','-dpng')
```
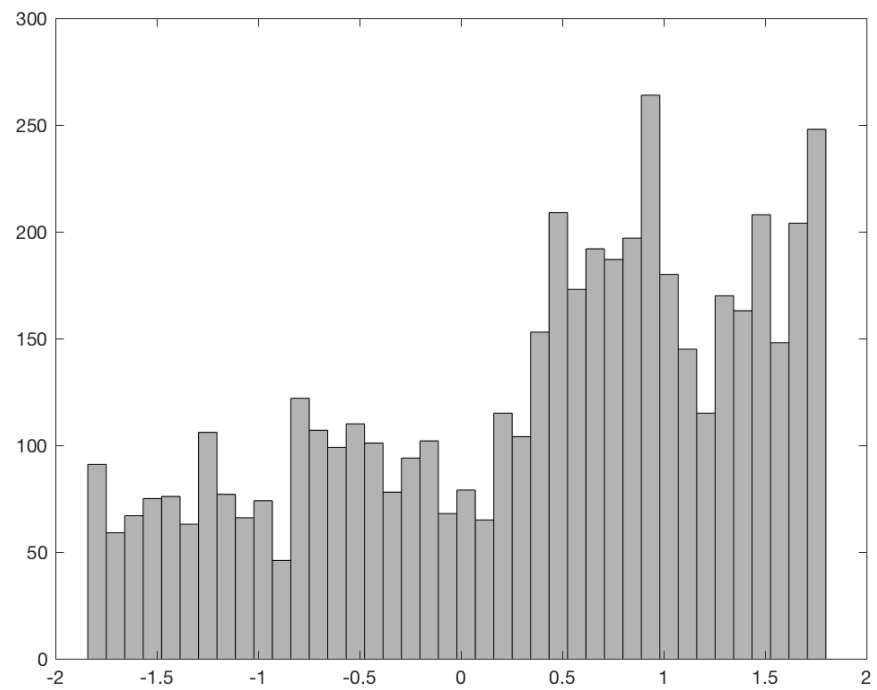
Fig_c1.png:

Fig_c2.png:

Fig_c3.png:

Both trajectories are roughly the same.

(d)

```
1  % Part (d)
2  diff=log(abs(x-xx));
3  plot(1:5000,diff,'Color',[0.5,0.5,0.5])
4  print('Fig_d','-dpng')
```

Fig_d.png: