ECON G6410 Problem Set 10

1. (a) Every subset of X is invariant under f.

   Every point in X is a fixed point. For all $x \in X$, the corresponding stable set is $W(x) = \{x\}$.

   (b) The subsets of X that are invariant under f take the form of

$$S = T \cup (-T),$$

   where $T \subset \mathbb{R}$.

   The system has one fixed point $x = 0$; its stable set is $W(0) = \{0\}$.

   (c) The subsets of X that are invariant under f take the form of

$$S = T \cup \{t + 1 : t \in T\},$$

   where $T \subset \mathbb{R}$.

   The system has no stable point.

   (d) The subsets of X that are invariant under f take the form of

$$S = T \cup \left\{\frac{t}{2} : t \in T\right\},$$

   where $T \subset \mathbb{R}$.

   The system has one fixed point $x = 0$; its stable set is $W(0) = \mathbb{R}$. This is because for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} \frac{x}{2^n} = 0.$$

   (e) The subsets of X that are invariant under f take the form of

$$S = T \cup \{1.2t : t \in T\},$$

   where $T \subset \mathbb{R}_+$.

   The system has one fixed point $x = 0$; its stable set is $W(0) = \{0\}$.

   (f) The subsets of X that are invariant under f take the form of

$$S = T \cup \left\{0.2\sqrt{t} + 0.8t : t \in T\right\},$$

   where $T \subset \mathbb{R}_+$.

Setting $f(x) = x$ gives two solutions $x = 0$ and $x = 1$, which are the fixed points of the system. To find their stable sets, note that $f(x) > x$ for $x \in (0,1)$ and $f(x) < x$ for $x \in (1, \infty)$, so

$$\lim_{n\to\infty} f^n(x) = 1$$

for any $x \in (0, \infty)$. Therefore, $W(1) = (0, \infty)$ and $W(0) = \{0\}$.

2. (a)

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$
$$C = \begin{pmatrix} -2 & -1 & 3.5 \\ -1 & 0 & 1.5 \\ -4 & -1 & 5.5 \end{pmatrix}.$$

(b) Since B is invertible,

$$\begin{aligned} A &= B^{-1}C \\ &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} C \\ &= \begin{pmatrix} -1 & -1 & 2 \\ -1 & 0 & 1.5 \\ -3 & -1 & 4 \end{pmatrix}. \end{aligned}$$

(c) Set $\det(A - \lambda I) = 0$:

$$\begin{aligned} \begin{vmatrix} -1-\lambda & -1 & 2 \\ -1 & -\lambda & 1.5 \\ -3 & -1 & 4-\lambda \end{vmatrix} &= 0 \\ 2\lambda^3 - 6\lambda^2 + 5\lambda - 2 &= 0 \\ (\lambda - 2)\left(2\lambda^2 - 2\lambda + 1\right) &= 0 \\ \lambda &= 2 \text{ or } \frac{1+i}{2} \text{ or } \frac{1-i}{2}. \end{aligned}$$

(d) First find the eigenvectors.

For $\lambda_1 = 2$:

$$(A - 2I)z = 0$$
$$\begin{pmatrix} -3 & -1 & 2 \\ -1 & -2 & 1.5 \\ -3 & -1 & 2 \end{pmatrix} z = 0,$$

which implies

$$
\begin{aligned}
-3z_1 - z_2 + 2z_3 &= 0 \\
-z_1 - 2z_2 + 1.5z_3 &= 0.
\end{aligned}
$$

Solve the system and get an eigenvector

$$
z = (1, 1, 2)'.
$$

For $\lambda_2 = \frac{1+i}{2}$: for the real part,

$$
\begin{aligned}
\left(A - \left(\frac{1+i}{2}\right) I\right)(u + iv) &= 0 \\
\begin{pmatrix} -1.5 - 0.5i & -1 & 2 \\ -1 & -0.5 - 0.5i & 1.5 \\ -3 & -1 & 3.5 - 0.5i \end{pmatrix}(u + iv) &= 0,
\end{aligned}
$$

which implies

$$
\begin{aligned}
(-1.5 - 0.5i)\, u_1 - u_2 + 2u_3 + (0.5 - 1.5i)\, v_1 - iv_2 + 2iv_3 &= 0 \\
-u_1 - (0.5 + 0.5i)\, u_2 + 1.5u_3 - iv_1 + (0.5 - 0.5i)\, v_2 + 1.5iv_3 &= 0,
\end{aligned}
$$

or

$$
\begin{aligned}
-1.5u_1 - u_2 + 2u_3 + 0.5v_1 &= 0 \\
-0.5u_1 - 1.5v_1 - v_2 + 2v_3 &= 0 \\
-u_1 - 0.5u_2 + 1.5u_3 + 0.5v_2 &= 0 \\
-0.5u_2 - v_1 - 0.5v_2 + 1.5v_3 &= 0.
\end{aligned}
$$

Solve the system and get an eigenvector

$$
u + iv = (-1, 0, -1)' + i\,(1, 1, 1)'.
$$

For $\lambda_2 = \frac{1+i}{2}$, an eigenvector is

$$
u - iv = (-1, 0, -1)' + i\,(1, 1, 1)'.
$$

Let

$$
\begin{aligned}
P &= \begin{pmatrix} z & u & v \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \end{pmatrix}.
\end{aligned}
$$

Then

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & -0.5 & 0.5 \end{pmatrix},$$

i.e. $\lambda = 2, \alpha = \beta = 0.5$.

(e) Let $Ax = x$; then $(A - I)\,x = 0$. Because

$$\det(A - I) = \begin{vmatrix} -2 & -1 & 2 \\ -1 & -1 & 1.5 \\ -3 & -1 & 3 \end{vmatrix} = 0.5,$$

$A - I$ is invertible and the system has a unique solution

$$\begin{aligned} x^* &= (A - I)^{-1}\,0 \\ &= 0. \end{aligned}$$

(f) Let $P^{-1}AP = \Lambda$; then $A = P\Lambda P^{-1}$ and thus

$$A^t = P\Lambda^t P^{-1}.$$

Note that Λ^t takes the form

$$\Lambda^t = \begin{pmatrix} 2^t & 0 & 0 \\ 0 & a_t & b_t \\ 0 & c_t & d_t \end{pmatrix},$$

where

$$D_t \equiv \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix}^t.$$

We want to first show $\lim_{t\to\infty} D_t = 0$. For a matrix C, denote the entry with the largest absolute value as k_C. Because

$$D_2 = \begin{pmatrix} 0 & 0.5 \\ -0.5 & 0 \end{pmatrix},$$

for an arbitrary matrix M,

$$D_2 M = \begin{pmatrix} 0.5m_{21} & 0.5m_{22} \\ -0.5m_{11} & -0.5m_{12} \end{pmatrix}.$$

Therefore, we have

$$k_{D_2 M} = \frac{1}{2} k_M.$$

So

$$\begin{aligned} k_{D_t} &= \left(\frac{1}{2}\right)^{\frac{t}{2}} k_I \\ &= \left(\frac{1}{2}\right)^{\frac{t}{2}} \end{aligned}$$

for any even number integer t, and

$$
\begin{aligned}
k_{D_t} &= \left(\frac{1}{2}\right)^{\frac{t-1}{2}} k_{D_1} \\
&= \left(\frac{1}{2}\right)^{\frac{t+1}{2}}
\end{aligned}
$$

for any odd integer t. Apparently $\lim_{t\to\infty} k_{D_t} \to 0$; so $\lim_{t\to\infty} D_t = 0$.

It follows that

$$
\begin{aligned}
\lim_{t\to\infty} A^t &= P \lim_{t\to\infty} \Lambda^t P^{-1} \\
&= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2^t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2^t & 0 & 0 \\ 2^t & 0 & 0 \\ 2^{t+1} & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} -2^t & 0 & 2^t \\ -2^t & 0 & 2^t \\ -2^{t+1} & 0 & 2^{t+1} \end{pmatrix}.
\end{aligned}
$$

Let $x_0 = (1, 0, 0)'$; then

$$
\lim_{t\to\infty} A^t x_0 = \lim_{t\to\infty} \left(-2^t, -2^t, -2^{t+1}\right) \neq 0.
$$

(g) Let $\hat{x}_0 = (1, 0, 1)'$. It is easy to verify that $\lim_{t\to\infty} A^t \hat{x}_0 = 0$.

3. We want to show there exists a homeomorphism $h : [-1, 1] \to [-2, 2]$ such that $h \circ f = g \circ h$. We guess that h is of the polynomial form $h(x) = a + bx + cx^2$. Then

$$
\begin{aligned}
h \circ f(x) &= a + b\left(2x^2 - 1\right) + c\left(2x^2 - 1\right)^2 \\
&= 4cx^4 + (2b - 4c)x^2 + a - b + c, \\
g \circ h(x) &= \left(a + bx + cx^2\right)^2 - 2 \\
&= c^2x^4 + 2bcx^3 + \left(b^2 + 2ac\right)x^2 + 2abx + a^2 - 2.
\end{aligned}
$$

The equivalence between the two implies

$$
\begin{aligned}
4c &= c^2 \\
0 &= 2bc \\
2b - 4c &= b^2 + 2ac \\
0 &= 2ab \\
a - b + c &= a^2 - 2.
\end{aligned}
$$

Solve the system and get

$$\begin{cases} a = -1 \\ b = 0 \\ c = 0 \end{cases} \quad \text{or} \quad \begin{cases} a = 2 \\ b = 0 \\ c = 0 \end{cases} \quad \text{or} \quad \begin{cases} a = 0 \\ b = 2 \\ c = 0 \end{cases} .$$

The third set of solutions corresponds to $h(x) = 2x$, which is a homeomorphism from $[-1, 1]$ to $[-2, 2]$. Therefore, f and g are conjugate.