1 The Problem

$$\max_{\{k_{t+J},n_t,z_{t+1}\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^t U\left(c_t^h,l_t^h\right)+E\sum_{t=0}^{\infty}\beta^t U\left(c_t^f,l_t^f\right)=E\sum_{t=0}^{\infty}\beta^t\frac{1}{\gamma}\left(\left(c_t^h\right)^{\mu}\left(1-n_t^h\right)^{1-\mu}\right)^{\gamma}+E\sum_{t=0}^{\infty}\beta^t\frac{1}{\gamma}\left(\left(c_t^f\right)^{\mu}\left(1-n_t^f\right)^{1-\mu}\right)^{\gamma},$$

where

$$\begin{aligned}
c_t^h+c_t^f &= \left[\left(\lambda_t^h\left(k_t^h\right)^{\theta}\left(n_t^h\right)^{1-\theta}\right)^{-\nu}+\sigma\left(z_t^h\right)^{-\nu}\right]^{-1/\nu}+\left[\left(\lambda_t^f\left(k_t^f\right)^{\theta}\left(n_t^f\right)^{1-\theta}\right)^{-\nu}+\sigma\left(z_t^f\right)^{-\nu}\right]^{-1/\nu}-\sum_{j=1}^{J}\phi_j s_{jt}^h-\sum_{j=1}^{J}\phi_j s_{jt}^f-z_{t+1}^h-z_{t+1}^f+z_t^h+z_t^f\\
&= \left[\left(\lambda_t^h\left(k_t^h\right)^{\theta}\left(n_t^h\right)^{1-\theta}\right)^{-\nu}+\sigma\left(z_t^h\right)^{-\nu}\right]^{-1/\nu}+\left[\left(\lambda_t^f\left(k_t^f\right)^{\theta}\left(n_t^f\right)^{1-\theta}\right)^{-\nu}+\sigma\left(z_t^f\right)^{-\nu}\right]^{-1/\nu}-\frac{1}{J}\sum_{j=1}^{J}s_{1,t+j-1}^h-\frac{1}{J}\sum_{j=1}^{J}s_{1,t+j-1}^f-z_{t+1}^h-z_{t+1}^f+z_t^h+z_t^f\\
&= \left[\left(\lambda_t^h\left(k_t^h\right)^{\theta}\left(n_t^h\right)^{1-\theta}\right)^{-\nu}+\sigma\left(z_t^h\right)^{-\nu}\right]^{-1/\nu}+\left[\left(\lambda_t^f\left(k_t^f\right)^{\theta}\left(n_t^f\right)^{1-\theta}\right)^{-\nu}+\sigma\left(z_t^f\right)^{-\nu}\right]^{-1/\nu}\\
&\quad -\frac{1}{J}\sum_{j=1}^{J}\left(k_{t+j}^h-(1-\delta)\,k_{t+j-1}^h\right)-\frac{1}{J}\sum_{j=1}^{J}\left(k_{t+j}^f-(1-\delta)\,k_{t+j-1}^f\right)-z_{t+1}^h-z_{t+1}^f+z_t^h+z_t^f\\
&\equiv y_t^h+y_t^f-\frac{1}{J}\sum_{j=1}^{J}\left(k_{t+j}^h-(1-\delta)\,k_{t+j-1}^h\right)-\frac{1}{J}\sum_{j=1}^{J}\left(k_{t+j}^f-(1-\delta)\,k_{t+j-1}^f\right)-z_{t+1}^h-z_{t+1}^f+z_t^h+z_t^f
\end{aligned} \tag{1}$$

Lagrangian multiplier: ξ_t

FOCs (for h variables; f variables are symmetric):

c_t^h :

$$\beta^t u_{tc}^h-\xi_t=0 \tag{2}$$

n_t^h:

$$0=\beta^t u_{tn}^h+\xi_t y_{tn}^h \tag{3}$$

k_{t+J}^h:

$$\begin{aligned}
0 &= \xi_{t+J}y_{t+J,k}^h-\frac{1}{J}\left(\sum_{t}^{t+J-1}\xi_t-\sum_{t+1}^{t+J}\xi_t\,(1-\delta)\right)\\
&= \xi_{t+J}y_{t+J,k}^h-\frac{1}{J}\left(\xi_t+\delta\xi_{t+1}+\ldots+\delta\xi_{t+J-1}+(\delta-1)\,\xi_{t+J}\right)
\end{aligned}$$

$$[= \xi_{t+4} y^h_{t+4,k} - \frac{1}{4}\left(\xi_t + \delta\xi_{t+1} + \delta\xi_{t+2} + \delta\xi_{t+3} + (\delta - 1)\,\xi_{t+4}\right)] \tag{4}$$

z^h_{t+1}:

$$0 = -\xi_t + \xi_{t+1}\left(y^h_{t+1,z} + 1\right) \tag{5}$$

2 First-Order Conditions

From (2),

$$\xi_t = \beta^t u^h_{tc} \tag{6}$$

Substitute (6) back into (4):

$$\begin{aligned} 0 &= \beta^{t+4} u^h_{t+4,c} y^h_{t+4,k} - \frac{1}{4}\left(\beta^t u^h_{tc} + \delta\beta^{t+1} u^h_{t+1,c} + \delta\beta^{t+2} u^h_{t+2,c} + \delta\beta^{t+3} u^h_{t+3,c} + (\delta - 1)\,\beta^{t+4} u^h_{t+4,c}\right) \\ 0 &= \beta^4 u^h_{t+4,c} y^h_{t+4,k} - \frac{1}{4}\left(u^h_{tc} + \delta\beta u^h_{t+1,c} + \delta\beta^2 u^h_{t+2,c} + \delta\beta^3 u^h_{t+3,c} + (\delta - 1)\,\beta^4 u^h_{t+4,c}\right) \end{aligned} \tag{7}$$

Substitute (6) back into (5):

$$\begin{aligned} 0 &= \beta^t u^h_{tc} - \beta^{t+1} u^h_{t+1,c}\left(y^h_{t+1,z} + 1\right) \\ 0 &= u^h_{tc} - \beta u^h_{t+1,c}\left(y^h_{t+1,z} + 1\right) \end{aligned} \tag{8}$$

Substitute (6) back into (3):

$$0 = u^h_{tn} + u^h_{tc} y^h_{tn} \tag{9}$$

The home- and foreign-country versions of (6) further gives

$$u^h_{tc} = u^f_{tc} \tag{10}$$

3 Steady State

At the steady state, (7) implies

$$\begin{aligned} 0 &= \beta^4 u^h_c y^h_k - \frac{1}{4}\left(1 + \delta\beta + \delta\beta^2 + \delta\beta^3 + (\delta - 1)\,\beta^4\right) u^h_c \\ 0 &= \beta^4 y^h_k\left(\bar{\lambda}^h, \bar{k}^h, \bar{n}^h, \bar{z}^h\right) - \frac{1}{4}\left(1 + \delta\beta + \delta\beta^2 + \delta\beta^3 + (\delta - 1)\,\beta^4\right) \end{aligned} \tag{11}$$

(8) implies

$$\begin{aligned} 0 &= u_c^h - \beta u_c^h \left(y_z^h + 1\right) \\ 0 &= 1 - \beta \left(y_z^h \left(\bar{\lambda}^h, \bar{k}^h, \bar{n}^h, \bar{z}^h\right) + 1\right) \end{aligned} \tag{12}$$

(9) implies

$$0 = u_n^h \left(\bar{c}^h, \bar{n}^h\right) + u_c^h \left(\bar{c}^h, \bar{n}^h\right) y_n^h \left(\bar{\lambda}^h, \bar{k}^h, \bar{n}^h, \bar{z}^h\right) \tag{13}$$

and (1) implies

$$\bar{c}^h = \bar{y}^h - \delta \bar{k}^h \tag{14}$$

4 Log-Linearization

Now log-linearize the system of (7), (8), (9), (10) and (1). Let π_{ab} be the elasticity of marginal utility of a with respect to b evaluated at the steady state, τ_{ab} be the elasticity of marginal product of a with respect to b evaluated at the steady state, and ζ_a be the elasticity of output with respect to a evaluated at the steady state.

(7):

$$\begin{aligned} \pi_{cc}\hat{c}_{t+4}^h + \pi_{cn}\hat{n}_{t+4}^h + \tau_{k\lambda}\hat{\lambda}_{t+4}^h + \tau_{kk}\hat{k}_{t+4}^h + \tau_{kn}\hat{n}_{t+4}^h + \tau_{kz}\hat{z}_{t+4}^h = \frac{1}{1 + \delta\beta + \delta\beta^2 + \delta\beta^3 + (\delta - 1)\beta^4} \left[\left(\pi_{cc}\hat{c}_t^h + \pi_{cn}\hat{n}_t^h\right) + \delta\beta\left(\pi_{cc}\hat{c}_{t+1}^h + \pi_{cn}\hat{n}_{t+1}^h\right)\right. \\ \left. + \delta\beta^2\left(\pi_{cc}\hat{c}_{t+2}^h + \pi_{cn}\hat{n}_{t+2}^h\right) + \delta\beta^3\left(\pi_{cc}\hat{c}_{t+3}^h + \pi_{cn}\hat{n}_{t+3}^h\right) + (\delta - 1)\beta^4\left(\pi_{cc}\hat{c}_{t+4}^h + \pi_{cn}\hat{n}_{t+4}^h\right)\right] \end{aligned} \tag{15}$$

(8):

$$\pi_{cc}\hat{c}_t^h + \pi_{cn}\hat{n}_t^h = \pi_{cc}\hat{c}_{t+1}^h + \pi_{cn}\hat{n}_{t+1}^h + (1 - \beta)\left(\tau_{z\lambda}\hat{\lambda}_{t+1}^h + \tau_{zk}\hat{k}_{t+1}^h + \tau_{zn}\hat{n}_{t+1}^h + \tau_{zz}\hat{z}_{t+1}^h\right) \tag{16}$$

(The equation above utilizes (12).)

(9):

$$\pi_{nc}\hat{c}_t^h + \pi_{nn}\hat{n}_t^h = \pi_{cc}\hat{c}_t^h + \pi_{cn}\hat{n}_t^h + \tau_{n\lambda}\hat{\lambda}_t^h + \tau_{nk}\hat{k}_t^h + \tau_{nn}\hat{n}_t^h + \tau_{nz}\hat{z}_t^h \tag{17}$$

(10):

$$\pi_{cc}\hat{c}_t^h + \pi_{cn}\hat{n}_t^h = \pi_{cc}\hat{c}_t^f + \pi_{cn}\hat{n}_t^f \tag{18}$$

(1):

$$\bar{c}^h \hat{c}_t^h + \bar{c}^f \hat{c}_t^f = \bar{y}^h \left(\zeta_\lambda \hat{\lambda}_t^h + \zeta_k \hat{k}_t^h + \zeta_n \hat{n}_t^h + \zeta_z \hat{z}_t^h\right) + \bar{y}^f \left(\zeta_\lambda \hat{\lambda}_t^f + \zeta_k \hat{k}_t^f + \zeta_n \hat{n}_t^f + \zeta_z \hat{z}_t^f\right) - \delta \bar{k}^h \sum_{j=1}^{4} \frac{1}{4}\left(\frac{1}{\delta}\hat{k}_{t+j}^h - \frac{1-\delta}{\delta}\hat{k}_{t+j-1}^h\right) - \delta \bar{k}^f \sum_{j=1}^{4} \frac{1}{4}\left(\frac{1}{\delta}\hat{k}_{t+j}^f - \frac{1-\delta}{\delta}\hat{k}_{t+j-1}^f\right) \quad (19)$$

(The equation above utilizes (14).)

5 Derivatives and Elasticities

$$\begin{aligned}
u_{tc}^h &= \mu \left(c_t^h\right)^{\gamma\mu-1} \left(1-n_t^h\right)^{\gamma(1-\mu)} \\
u_{tn}^h &= (\mu-1) \left(c_t^h\right)^{\gamma\mu} \left(1-n_t^h\right)^{\gamma(1-\mu)-1} \\
y_{tk}^h &= \theta \left[\left(\lambda_t^h \left(k_t^h\right)^\theta \left(n_t^h\right)^{1-\theta}\right)^{-\nu} + \sigma \left(z_t^h\right)^{-\nu}\right]^{-\frac{1+\nu}{\nu}} \left(\lambda_t^h \left(n_t^h\right)^{1-\theta}\right)^{-\nu} \left(k_t^h\right)^{-\nu\theta-1} \\
&= \theta \left(y_t^h\right)^{\nu+1} \left(\lambda_t^h \left(n_t^h\right)^{1-\theta}\right)^{-\nu} \left(k_t^h\right)^{-\nu\theta-1} \\
&= \theta \left(y_t^h\right)^{\nu+1} \left[\left(y_t^h\right)^{-\nu} - \sigma \left(z_t^h\right)^{-\nu}\right] \left(k_t^h\right)^{-1} \\
y_{tn}^h &= (1-\theta) \left[\left(\lambda_t^h \left(k_t^h\right)^\theta \left(n_t^h\right)^{1-\theta}\right)^{-\nu} + \sigma \left(z_t^h\right)^{-\nu}\right]^{-\frac{1+\nu}{\nu}} \left(\lambda_t^h \left(k_t^h\right)^\theta\right)^{-\nu} \left(n_t^h\right)^{\nu(\theta-1)-1} \\
&= (1-\theta) \left(y_t^h\right)^{\nu+1} \left(\lambda_t^h \left(k_t^h\right)^\theta\right)^{-\nu} \left(n_t^h\right)^{\nu(\theta-1)-1} \\
&= (1-\theta) \left(y_t^h\right)^{\nu+1} \left[\left(y_t^h\right)^{-\nu} - \sigma \left(z_t^h\right)^{-\nu}\right] \left(n_t^h\right)^{-1} \\
y_{tz}^h &= \sigma \left[\left(\lambda_t^h \left(k_t^h\right)^\theta \left(n_t^h\right)^{1-\theta}\right)^{-\nu} + \sigma \left(z_t^h\right)^{-\nu}\right]^{-\frac{1+\nu}{\nu}} \left(z_t^h\right)^{-\nu-1} \\
&= \sigma \left(y_t^h\right)^{\nu+1} \left(z_t^h\right)^{-\nu-1} \\
\pi_{cc} &= \gamma\mu - 1 \\
\pi_{cn} &= \gamma (1-\mu) \frac{\bar{n}^h}{\bar{n}^h - 1} \\
\pi_{nc} &= \gamma\mu \\
\pi_{nn} &= (\gamma (1-\mu) - 1) \frac{\bar{n}^h}{\bar{n}^h - 1} \\
\zeta_\lambda &= \left[\left(\bar{\lambda}^h \left(\bar{k}^h\right)^\theta \left(\bar{n}^h\right)^{1-\theta}\right)^{-\nu} + \sigma \left(\bar{z}^h\right)^{-\nu}\right]^{-1} \left(\left(\bar{k}^h\right)^\theta \left(\bar{n}^h\right)^{1-\theta}\right)^{-\nu} \left(\bar{\lambda}^h\right)^{-\nu}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\bar{y}^h\right)^{\nu} \left(\left(\bar{k}^h\right)^{\theta} \left(\bar{n}^h\right)^{1-\theta}\right)^{-\nu} \left(\bar{\lambda}^h\right)^{-\nu} \\
&= 1 - \sigma \left(\bar{y}^h\right)^{\nu} \left(\bar{z}^h\right)^{-\nu} \\
\zeta_k &= \theta \left[\left(\bar{\lambda}^h \left(\bar{k}^h\right)^{\theta} \left(\bar{n}^h\right)^{1-\theta}\right)^{-\nu} + \sigma \left(\bar{z}^h\right)^{-\nu}\right]^{-1} \left(\bar{\lambda}^h \left(\bar{n}^h\right)^{1-\theta}\right)^{-\nu} \left(\bar{k}^h\right)^{-\nu\theta} \\
&= \theta \left(\bar{y}^h\right)^{\nu} \left(\bar{\lambda}^h \left(\bar{n}^h\right)^{1-\theta}\right)^{-\nu} \left(\bar{k}^h\right)^{-\nu\theta} \\
&= \theta \left[1 - \sigma \left(\bar{y}^h\right)^{\nu} \left(\bar{z}^h\right)^{-\nu}\right] \\
\zeta_n &= (1-\theta) \left[\left(\bar{\lambda}^h \left(\bar{k}^h\right)^{\theta} \left(\bar{n}^h\right)^{1-\theta}\right)^{-\nu} + \sigma \left(\bar{z}^h\right)^{-\nu}\right]^{-1} \left(\bar{\lambda}^h \left(\bar{k}^h\right)^{\theta}\right)^{-\nu} \left(\bar{n}^h\right)^{\nu(\theta-1)} \\
&= (1-\theta) \left(\bar{y}^h\right)^{\nu} \left(\bar{\lambda}^h \left(\bar{k}^h\right)^{\theta}\right)^{-\nu} \left(\bar{n}^h\right)^{\nu(\theta-1)} \\
&= (1-\theta) \left[1 - \sigma \left(\bar{y}^h\right)^{\nu} \left(\bar{z}^h\right)^{-\nu}\right] \\
\zeta_z &= \sigma \left[\left(\bar{\lambda}^h \left(\bar{k}^h\right)^{\theta} \left(\bar{n}^h\right)^{1-\theta}\right)^{-\nu} + \sigma \left(\bar{z}^h\right)^{-\nu}\right]^{-1} \left(\bar{z}^h\right)^{-\nu} \\
&= \sigma \left(\bar{y}^h\right)^{\nu} \left(\bar{z}^h\right)^{-\nu} \\
\tau_{k\lambda} = \tau_{n\lambda} &= (\nu+1)\,\zeta_\lambda - \nu \\
&= (\nu+1) \left[1 - \sigma \left(\bar{y}^h\right)^{\nu} \left(\bar{z}^h\right)^{-\nu}\right] - \nu \\
\tau_{kk} &= (\nu+1)\,\zeta_k - \nu\theta - 1 \\
&= (\nu+1)\,\theta \left[1 - \sigma \left(\bar{y}^h\right)^{\nu} \left(\bar{z}^h\right)^{-\nu}\right] - \nu\theta - 1 \\
\tau_{kn} &= (\nu+1)\,\zeta_n + \nu\,(\theta-1) \\
&= (\nu+1)\,(1-\theta) \left[1 - \sigma \left(\bar{y}^h\right)^{\nu} \left(\bar{z}^h\right)^{-\nu}\right] + \nu\,(\theta-1) \\
\tau_{kz} = \tau_{nz} &= (\nu+1)\,\zeta_z \\
&= (\nu+1)\,\sigma \left(\bar{y}^h\right)^{\nu} \left(\bar{z}^h\right)^{-\nu} \\
\tau_{nk} &= (\nu+1)\,\zeta_k - \nu\theta \\
&= (\nu+1)\,\theta \left[1 - \sigma \left(\bar{y}^h\right)^{\nu} \left(\bar{z}^h\right)^{-\nu}\right] - \nu\theta \\
\tau_{nn} &= (\nu+1)\,\zeta_n + \nu\,(\theta-1) - 1 \\
&= (\nu+1)\,(1-\theta) \left[1 - \sigma \left(\bar{y}^h\right)^{\nu} \left(\bar{z}^h\right)^{-\nu}\right] + \nu\,(\theta-1) - 1 \\
\tau_{z\lambda} &= (\nu+1)\,\zeta_\lambda
\end{aligned}
$$

$$
\begin{aligned}
&= (\nu+1)\left[1-\sigma\left(\bar{y}^h\right)^{\nu}\left(\bar{z}^h\right)^{-\nu}\right] \\
\tau_{zk} &= (\nu+1)\,\zeta_k \\
&= (\nu+1)\,\theta\left[1-\sigma\left(\bar{y}^h\right)^{\nu}\left(\bar{z}^h\right)^{-\nu}\right] \\
\tau_{zn} &= (\nu+1)\,\zeta_n \\
&= (\nu+1)\,(1-\theta)\left[1-\sigma\left(\bar{y}^h\right)^{\nu}\left(\bar{z}^h\right)^{-\nu}\right] \\
\tau_{zz} &= (\nu+1)\,\zeta_z-\nu-1 \\
&= (\nu+1)\,\sigma\left(\bar{y}^h\right)^{\nu}\left(\bar{z}^h\right)^{-\nu}-\nu-1
\end{aligned}
$$

6 Steady State, Cont'd

Substitute the derivatives back to the steady-state equations (11), (12), (13) and (14):

$$
\begin{aligned}
0 &= \beta^4\theta\left(\bar{y}^h\right)^{\nu+1}\left(\left(\bar{y}^h\right)^{-\nu}-\sigma\left(\bar{z}^h\right)^{-\nu}\right)\left(\bar{k}^h\right)^{-1}-\frac{1}{4}\left(1+\delta\beta+\delta\beta^2+\delta\beta^3+(\delta-1)\,\beta^4\right) && (20)\\
0 &= 1-\beta\left(\sigma\left(\bar{y}^h\right)^{\nu+1}\left(\bar{z}^h\right)^{-\nu-1}+1\right) && (21)\\
0 &= (1-\mu)\,\bar{c}^h+\mu\left(1-\bar{n}^h\right)(1-\theta)\left(\bar{y}^h\right)^{\nu+1}\left[\left(\bar{y}^h\right)^{-\nu}-\sigma\left(\bar{z}^h\right)^{-\nu}\right]\left(\bar{n}^h\right)^{-1} && (22)\\
\bar{c}^h &= \bar{y}^h-\delta\bar{k}^h && (23)
\end{aligned}
$$

From (21),

$$
\bar{z}^h = \left(\sigma\frac{\beta}{1-\beta}\right)^{\frac{1}{\nu+1}}\bar{y}^h
$$

From (20):

$$
\bar{k}^h = \frac{4\beta^4\theta}{1+\delta\beta+\delta\beta^2+\delta\beta^3+(\delta-1)\,\beta^4}\left(\bar{y}^h\right)^{\nu+1}\left(\left(\bar{y}^h\right)^{-\nu}-\sigma\left(\bar{z}^h\right)^{-\nu}\right)
$$

Given (23), from (22):

$$
\bar{n}^h = \left(\left[(1-\theta)\left(\bar{y}^h\right)^{\nu+1}\left(\left(\bar{y}^h\right)^{-\nu}-\sigma\left(\bar{z}^h\right)^{-\nu}\right)\right]^{-1}\left(\frac{1}{\mu}-1\right)\left(\bar{y}^h-\delta\bar{k}^h\right)+1\right)^{-1}
$$

7 Log-Linearization, Cont'd

From (17), we can write $\hat{n}_t^h$ as

$$\hat{n}_t^h = \frac{1}{\pi_{nn} - \pi_{cn} - \tau_{nn}} \left[(\pi_{cc} - \pi_{nc})\, \hat{c}_t^h + \tau_{n\lambda} \hat{\lambda}_t^h + \tau_{nk} \hat{k}_t^h + \tau_{nz} \hat{z}_t^h \right]$$

Substitute this back to (15), (16), (18) and (19).

(15): Let

$$\begin{aligned}
m &= 1 + \delta\beta + \delta\beta^2 + \delta\beta^3 \\
s &= \pi_{nn} - \pi_{cn} - \tau_{nn} \\
p_c &= \pi_{cc} + \pi_{cn} (\pi_{cc} - \pi_{nc}) \\
q_c &= \pi_{cc} (\pi_{nn} - \tau_{nn}) - \pi_{cn}\pi_{nc} \\
p_\lambda &= \pi_{cn}\tau_{n\lambda} \\
p_k &= \pi_{cn}\tau_{nk} \\
p_z &= \pi_{cn}\tau_{nz};
\end{aligned}$$

then

$$\begin{aligned}
& \left[m q_c + \left(m + (\delta - 1)\beta^4 \right) \tau_{kn} (\pi_{cc} - \pi_{nc}) \right] \hat{c}_{t+4}^h \\
& + \left[m \pi_{cn} \tau_{n\lambda} + \left(m + (\delta - 1)\beta^4 \right) (\tau_{kn}\tau_{n\lambda} + s\tau_{k\lambda}) \right] \hat{\lambda}_{t+4}^h \\
& + \left[m \pi_{cn} \tau_{nk} + \left(m + (\delta - 1)\beta^4 \right) (\tau_{kn}\tau_{nk} + s\tau_{kk}) \right] \hat{k}_{t+4}^h \\
& + \left[m \pi_{cn} \tau_{nz} + \left(m + (\delta - 1)\beta^4 \right) (\tau_{kn}\tau_{nz} + s\tau_{kz}) \right] \hat{z}_{t+4}^h \\
= \;& p_c \hat{c}_t^h + p_\lambda \hat{\lambda}_t^h + p_k \hat{k}_t^h + p_z \hat{z}_t^h \\
& + \delta\beta p_c \hat{c}_{t+1}^h + \delta\beta p_\lambda \hat{\lambda}_{t+1}^h + \delta\beta p_k \hat{k}_{t+1}^h + \delta\beta p_z \hat{z}_{t+1}^h \\
& + \delta\beta^2 p_c \hat{c}_{t+2}^h + \delta\beta^2 p_\lambda \hat{\lambda}_{t+2}^h + \delta\beta^2 p_k \hat{k}_{t+2}^h + \delta\beta^2 p_z \hat{z}_{t+2}^h \\
& + \delta\beta^3 p_c \hat{c}_{t+3}^h + \delta\beta^3 p_\lambda \hat{\lambda}_{t+3}^h + \delta\beta^3 p_k \hat{k}_{t+3}^h + \delta\beta^3 p_z \hat{z}_{t+3}^h
\end{aligned} \tag{24}$$

(16):

$$\begin{aligned}
& [q_c + (1 - \beta)\, \tau_{zn} (\pi_{cc} - \pi_{nc})]\, \hat{c}_{t+1}^h \\
& + [\pi_{cn}\tau_{n\lambda} + (1 - \beta)(\tau_{zn}\tau_{n\lambda} + s\tau_{z\lambda})]\, \hat{\lambda}_{t+1}^h \\
& + [\pi_{cn}\tau_{nk} + (1 - \beta)(\tau_{zn}\tau_{nk} + s\tau_{zk})]\, \hat{k}_{t+1}^h
\end{aligned} \tag{25}$$

$$\begin{aligned} & + \left[\pi_{cn}\tau_{nz} + (1-\beta)\left(\tau_{zn}\tau_{nz} + s\tau_{zz}\right)\right]\hat{z}_{t+1}^{h} \\ = \quad & q_c\hat{c}_t^h + p_\lambda\hat{\lambda}_t^h + p_k\hat{k}_t^h + p_z\hat{z}_t^h \end{aligned}$$

(18):

$$q_c\hat{c}_t^h + p_\lambda\hat{\lambda}_t^h + p_k\hat{k}_t^h + p_z\hat{z}_t^h = q_c\hat{c}_t^f + p_\lambda\hat{\lambda}_t^f + p_k\hat{k}_t^f + p_z\hat{z}_t^f \tag{26}$$

(19):

$$\begin{aligned} & \delta s\bar{k}^h\hat{k}_{t+1}^h + \delta s\bar{k}^h\hat{k}_{t+2}^h + \delta s\bar{k}^h\hat{k}_{t+3}^h + s\bar{k}^h\hat{k}_{t+4}^h \\ & + \delta s\bar{k}^f\hat{k}_{t+1}^f + \delta s\bar{k}^f\hat{k}_{t+2}^f + \delta s\bar{k}^f\hat{k}_{t+3}^f + s\bar{k}^f\hat{k}_{t+4}^f \\ = \quad & 4\left[\zeta_n\left(\pi_{cc} - \pi_{nc}\right)\bar{y}^h - s\bar{c}^h\right]\hat{c}_t^h \\ & + 4\left[\zeta_n\left(\pi_{cc} - \pi_{nc}\right)\bar{y}^f - s\bar{c}^f\right]\hat{c}_t^f \\ & + 4\bar{y}^h\left(\zeta_\lambda s + \zeta_n\tau_{n\lambda}\right)\hat{\lambda}_t^h \\ & + 4\bar{y}^f\left(\zeta_\lambda s + \zeta_n\tau_{n\lambda}\right)\hat{\lambda}_t^f \\ & + \left[4\bar{y}^h\left(\zeta_k s + \zeta_n\tau_{nk}\right) + (1-\delta)\,s\bar{k}^h\right]\hat{k}_t^h \\ & + \left[4\bar{y}^f\left(\zeta_k s + \zeta_n\tau_{nk}\right) + (1-\delta)\,s\bar{k}^f\right]\hat{k}_t^f \\ & + 4\bar{y}^h\left(\zeta_z s + \zeta_n\tau_{nz}\right)\hat{z}_t^h \\ & + 4\bar{y}^f\left(\zeta_z s + \zeta_n\tau_{nz}\right)\hat{z}_t^f \end{aligned} \tag{27}$$

8 A Detour: $J = 1$

Change in steady-state values:

$$\bar{k}^h = \frac{\beta\theta}{1 + (\delta - 1)\beta}\left(\bar{y}^h\right)^{\nu+1}\left(\left(\bar{y}^h\right)^{-\nu} - \sigma\left(\bar{z}^h\right)^{-\nu}\right)$$

Change in log-linearized system (pre-substitution):

$$\pi_{cc}\hat{c}_{t+1}^h + \pi_{cn}\hat{n}_{t+1}^h + \tau_{k\lambda}\hat{\lambda}_{t+1}^h + \tau_{kk}\hat{k}_{t+1}^h + \tau_{kn}\hat{n}_{t+1}^h + \tau_{kz}\hat{z}_{t+1}^h = \frac{1}{1 + (\delta - 1)\beta}\left[\left(\pi_{cc}\hat{c}_t^h + \pi_{cn}\hat{n}_t^h\right) + (\delta - 1)\beta\left(\pi_{cc}\hat{c}_{t+1}^h + \pi_{cn}\hat{n}_{t+1}^h\right)\right]$$

$$\bar{c}^h\hat{c}_t^h + \bar{c}^f\hat{c}_t^f = \bar{y}^h\left(\zeta_\lambda\hat{\lambda}_t^h + \zeta_k\hat{k}_t^h + \zeta_n\hat{n}_t^h + \zeta_z\hat{z}_t^h\right) + \bar{y}^f\left(\zeta_\lambda\hat{\lambda}_t^f + \zeta_k\hat{k}_t^f + \zeta_n\hat{n}_t^f + \zeta_z\hat{z}_t^f\right) - \delta\bar{k}^h\left(\frac{1}{\delta}\hat{k}_{t+1}^h - \frac{1-\delta}{\delta}\hat{k}_t^h\right) - \delta\bar{k}^f\left(\frac{1}{\delta}\hat{k}_{t+1}^f - \frac{1-\delta}{\delta}\hat{k}_t^f\right)$$

New final system of equations (only the first and fourth equations change):

$$
\begin{aligned}
& \left[q_c + (1 + (\delta - 1)\,\beta)\,\tau_{kn}\,(\pi_{cc} - \pi_{nc})\right] \hat{c}^h_{t+1} \\
& + \left[\pi_{cn}\tau_{n\lambda} + (1 + (\delta - 1)\,\beta)\,(\tau_{kn}\tau_{n\lambda} + s\tau_{k\lambda})\right] \hat{\lambda}^h_{t+1} \\
& + \left[\pi_{cn}\tau_{nk} + (1 + (\delta - 1)\,\beta)\,(\tau_{kn}\tau_{nk} + s\tau_{kk})\right] \hat{k}^h_{t+1} \\
& + \left[\pi_{cn}\tau_{nz} + (1 + (\delta - 1)\,\beta)\,(\tau_{kn}\tau_{nz} + s\tau_{kz})\right] \hat{z}^h_{t+1} \\
= \quad & p_c\hat{c}^h_t + p_\lambda\hat{\lambda}^h_t + p_k\hat{k}^h_t + p_z\hat{z}^h_t
\end{aligned}
$$

$$
\begin{aligned}
& \left[q_c + (1 - \beta)\,\tau_{zn}\,(\pi_{cc} - \pi_{nc})\right] \hat{c}^h_{t+1} \\
& + \left[\pi_{cn}\tau_{n\lambda} + (1 - \beta)\,(\tau_{zn}\tau_{n\lambda} + s\tau_{z\lambda})\right] \hat{\lambda}^h_{t+1} \\
& + \left[\pi_{cn}\tau_{nk} + (1 - \beta)\,(\tau_{zn}\tau_{nk} + s\tau_{zk})\right] \hat{k}^h_{t+1} \\
& + \left[\pi_{cn}\tau_{nz} + (1 - \beta)\,(\tau_{zn}\tau_{nz} + s\tau_{zz})\right] \hat{z}^h_{t+1} \\
= \quad & q_c\hat{c}^h_t + p_\lambda\hat{\lambda}^h_t + p_k\hat{k}^h_t + p_z\hat{z}^h_t
\end{aligned}
$$

$$
\begin{aligned}
& q_c\hat{c}^h_t + p_\lambda\hat{\lambda}^h_t + p_k\hat{k}^h_t + p_z\hat{z}^h_t \\
& = q_c\hat{c}^f_t + p_\lambda\hat{\lambda}^f_t + p_k\hat{k}^f_t + p_z\hat{z}^f_t
\end{aligned}
$$

$$
\begin{aligned}
s\bar{k}^h\hat{k}^h_{t+1} + s\bar{k}^f\hat{k}^f_{t+1} = & \left[\zeta_n\,(\pi_{cc} - \pi_{nc})\,\bar{y}^h - s\bar{c}^h\right] \hat{c}^h_t \\
& + \left[\zeta_n\,(\pi_{cc} - \pi_{nc})\,\bar{y}^f - s\bar{c}^f\right] \hat{c}^f_t \\
& + \bar{y}^h\,(\zeta_\lambda s + \zeta_n\tau_{n\lambda})\,\hat{\lambda}^h_t \\
& + \bar{y}^f\,(\zeta_\lambda s + \zeta_n\tau_{n\lambda})\,\hat{\lambda}^f_t \\
& + \left[\bar{y}^h\,(\zeta_k s + \zeta_n\tau_{nk}) + (1 - \delta)\,s\bar{k}^h\right] \hat{k}^h_t \\
& + \left[\bar{y}^f\,(\zeta_k s + \zeta_n\tau_{nk}) + (1 - \delta)\,s\bar{k}^f\right] \hat{k}^f_t \\
& + \bar{y}^h\,(\zeta_z s + \zeta_n\tau_{nz})\,\hat{z}^h_t \\
& + \bar{y}^f\,(\zeta_z s + \zeta_n\tau_{nz})\,\hat{z}^f_t
\end{aligned}
$$