Metrics Pset 9

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1. Let the AR(1) process be $X_t = \phi X_{t-1} + \epsilon_t$ and $|\phi| < 1$. Therefore, $(1 - \phi L)X_t = \epsilon_t$. $X_t = \frac{1}{1-\phi L}\epsilon_t = \sum_{j=0}^{\infty} \phi^j \epsilon_{t-j}$, which is $MA(\infty)$. Let the MA(1) process be $X_t = \epsilon_t + \theta\epsilon_{t-1}$ and $|\theta| < 1$. $X_t = (1 + \theta L)\epsilon_t$. Then $\epsilon_t = \frac{1}{1+\theta L} X_t$. $\epsilon_t = \sum_{j=0}^{\infty} (-\theta)^j X_{t-j}$.

2. Autocovariance of MA(q)

   $MA(q) = \sum_{i=0}^{q} \theta_i \epsilon_i$. The ACF function of the process

   $ACF(X_t, X_{t-s}) = E[(\sum_{i=0}^{q} \theta_i \epsilon_{t-i})(\sum_{i=s}^{s+q} \theta_{i-s}\epsilon_{t-j})] = \sum_{i=s}^{q} \theta_i \theta_{i-s} \sigma^2$

3. Auto-covariance of AR(2)

   Let the AR(2) process be $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$. Multiplying the equation on both sides with y_{t-s} and taking expectations we have $\gamma(s) = E(y_t y_{t-s}) = \phi_1 E(y_{t-1}y_{t-s}) + \phi_2 E(y_{t-2}y_{t-s}) = \phi_1\gamma(s-1) + \phi_2\gamma(s-2)$. By definition, the auto-correlation function is $\rho(0) = 1$ and $\rho = \frac{\gamma(s)}{\gamma(0)}$ for $s \geq 1$. So for $s \geq 1$, we have $\rho(s) = \phi_1\rho(s-1) + \phi_2\rho(s-2)$. For $s = 1$, this becomes $\rho(1) = \phi_1\rho(0) + \phi_2\rho(1) = \phi_1 + \phi_2\rho(1) = \frac{\phi_1}{1-\phi_2}$ and recursively $\rho(2) = \phi_1\rho(1) + \phi_2\rho(0) = \frac{\phi_2+\phi_2(1-\phi_2)}{1-\phi_2}$. With this recursive relation, we can find $\gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + \sigma^2 = \phi_1\rho(1)\gamma(0) + \phi_2\rho(2)\gamma(0) + \sigma^2$. Thus, $\gamma(0) = \frac{1}{1-\phi_1\rho(1)-\phi_2\rho(2)}\sigma^2 = \frac{(1-\phi_2)}{\Phi}\sigma^2$. Using the recursive relation $\gamma(s) = \phi_1\gamma(s-1) + \phi_2\gamma(s-2)$ we can find $\gamma(2), \gamma(3)....$, which completely describes the evolution of auto-covariance.

4. aggregation

   (a) Let $X_t = \epsilon_t + \theta\epsilon_{t-1}$ and $Y_t = e_t + \phi e_{t-1}$ be two MA(1) independent stationary processes. Let $Z_t = X_t + Y_t$. Then $EZ_t = EX_t + EY_t = 0$. $Var(Z_t) = Var(X_t + Y_t) = Var(X_t) + Var(Y_t) = (1+\theta^2)\sigma_X^2 + (1+\phi^2)\sigma_Y^2$ because of independence. $E(Z_t Z_{t-k}) = E[(X_t + Y_t)(X_{t-k} + Y_{t-k})] = E(X_t X_{t-k} + X_t Y_{t-k} + Y_t X_{t-k} + Y_t Y_{t-k})$. $E(X_t Y_{t-k}) = 0$ and $E(Y_t X_{t-k}) = 0$ because of independence and $E(X_t X_{t-k}) = 0$ and $E(Y_t Y_{t-k}) = 0$ for $k > 1$. For $k = 1$, $E(X_t X_{t-k}) = \theta\sigma_X^2$ and $E(Y_t Y_{t-k}) = \phi\sigma_Y^2$. Therefore, $E(Z_t Z_{t-k}) = \begin{cases} 0 & k > 1 \\ \theta\sigma_X^2 + \phi\sigma_Y^2 & k = 1 \end{cases}$. Hence, Z_t is MA(1).

(b) We can rearrange and express our two AR(1) processes as $X_t = \frac{1}{1-\phi_1 L}\epsilon_t$ and $Y_t = \frac{1}{1-\phi_2 L}\eta_t$. Let $Z_t = X_t + Y_t$. Then if $\phi_1 = \phi_2$, we have $(1-\phi_1 L)Z_t = \epsilon_t + \eta_t$. Then we have $Z_t = \phi_1 Z_{t-1} + u_t$, where $u_t = \epsilon_t + \eta_t$. We need to show u_t is white noise. First, $E(u_t) = E(\epsilon_t + \eta_t) = 0$. Second, $E(u_t^2) = E(\epsilon_t^2 + \eta_t^2 + 2\epsilon_t\eta_t) = E(\epsilon_t^2) + E(\eta_t^2) = \sigma_\epsilon^2 + \sigma_\eta^2$ since $E(\epsilon_t\eta_t) = 0$ by independence. Third, $E(u_t u_{t-s}) = E(\epsilon_t\epsilon_{t-s} + \epsilon_t\eta_{t-s} + \eta_t\epsilon_{t-s} + \eta_t\eta_{t-s}) = 0$ because of independence and both η_t and ϵ_t are white noises. Therefore, u_t is white noise and Z_t is AR(1).

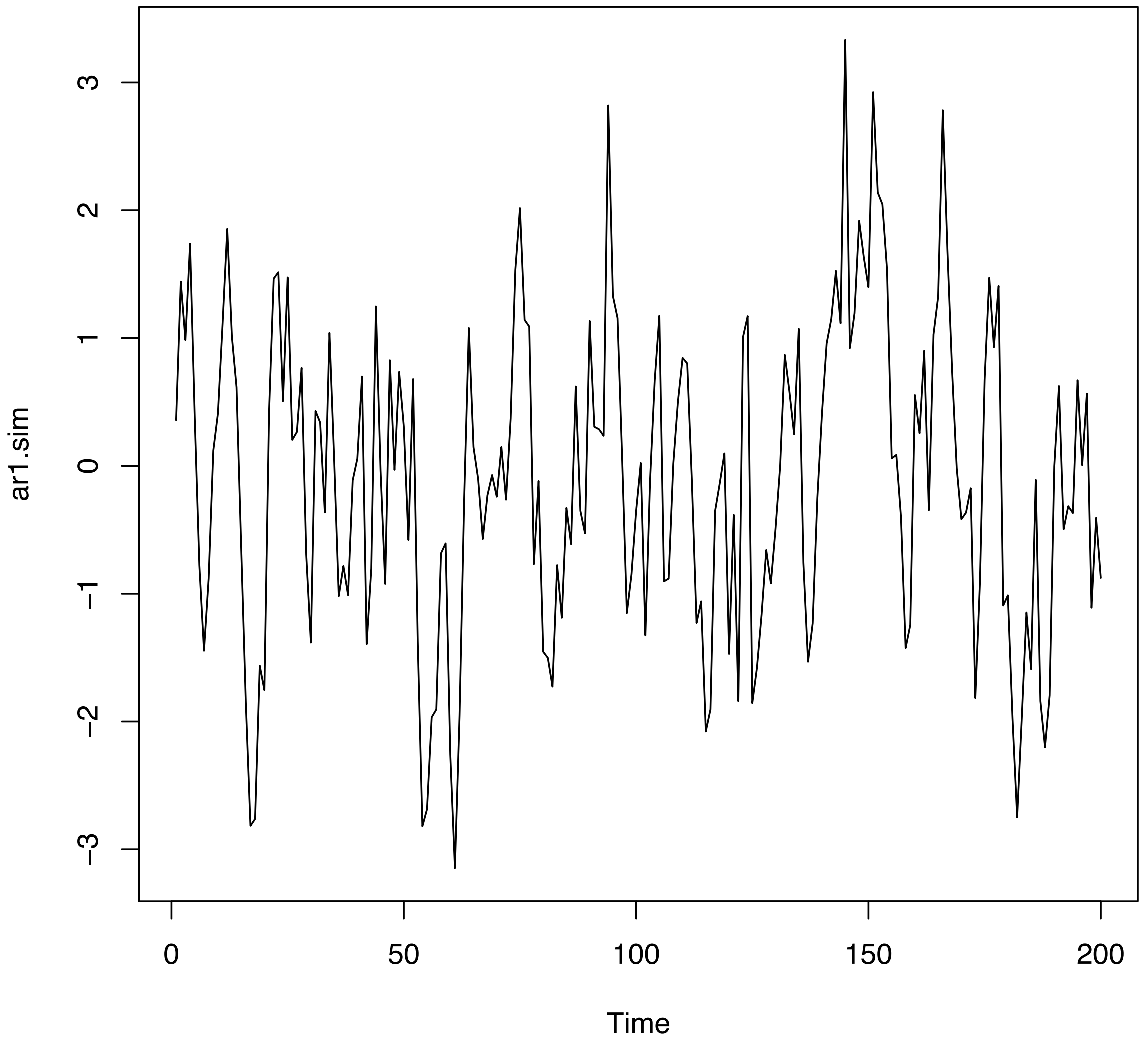
(c) Now, following the same definition of Z_t in (b), we have the following $(1-(\phi_1+\phi_2)L + \phi_1\phi_2 L^2)Z_t = (1-\phi_2 L)\epsilon_t + (1-\phi_1 L)\eta_t$. The LHS is AR(2) and the RHS is the sum of two independent stationary MA(1) processes which by (a) is an MA(1) process. Since $\phi_1, \phi_2 < 1$, we have a stationary $ARMA(2,1)$.

5. By the definition of long run variance, $lim_{T\to\infty} T \cdot var(\bar{X}_T) = \gamma_0 + 2\sum_{j=1}^{\infty}\gamma_j$, $\gamma_0 = \frac{\sigma^2}{1-\rho^2}$ and $\gamma_j = \rho^j \frac{\sigma^2}{1-\rho^2}$. Then $lim_{T\to\infty} T \cdot var(\bar{X}_T) = \frac{\sigma^2}{1-\rho^2} + \frac{2\sigma^2}{(1-\rho^2)(1-\rho)} = \frac{\sigma^2}{(1-\rho)^2}$.

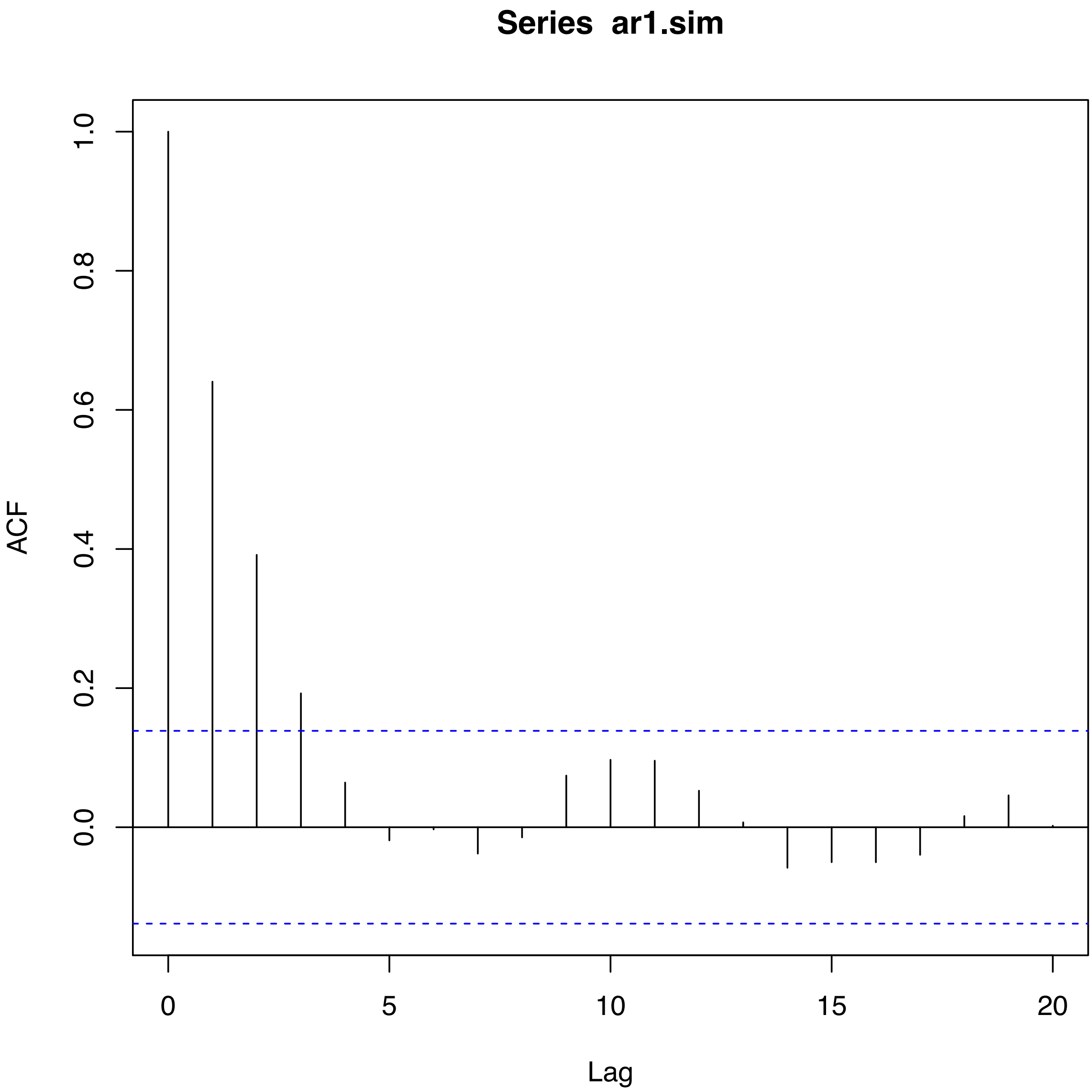
6. We have two equations about γ_0 and γ_1. $(1+\tau^2)\sigma_e^2 = 10$. $\tau\sigma_e^2 = 3$. So plug $\sigma_e^2 = \frac{3}{\tau}$ into the first equation. Then we have $3\tau^2 - 10\tau + 3 = 0$. And the two roots are $\tau_1 = \frac{1}{3}$ and $\tau_2 = 3$. We pick the τ_1 so that the Wold decomposition is invertible and $\sigma_e^2 = 9$ as result. So theWold decomposition is $X_t = e_t + \frac{1}{3}e_{t-1}$ with $\sigma_e^2 = 9$.

7. simulation

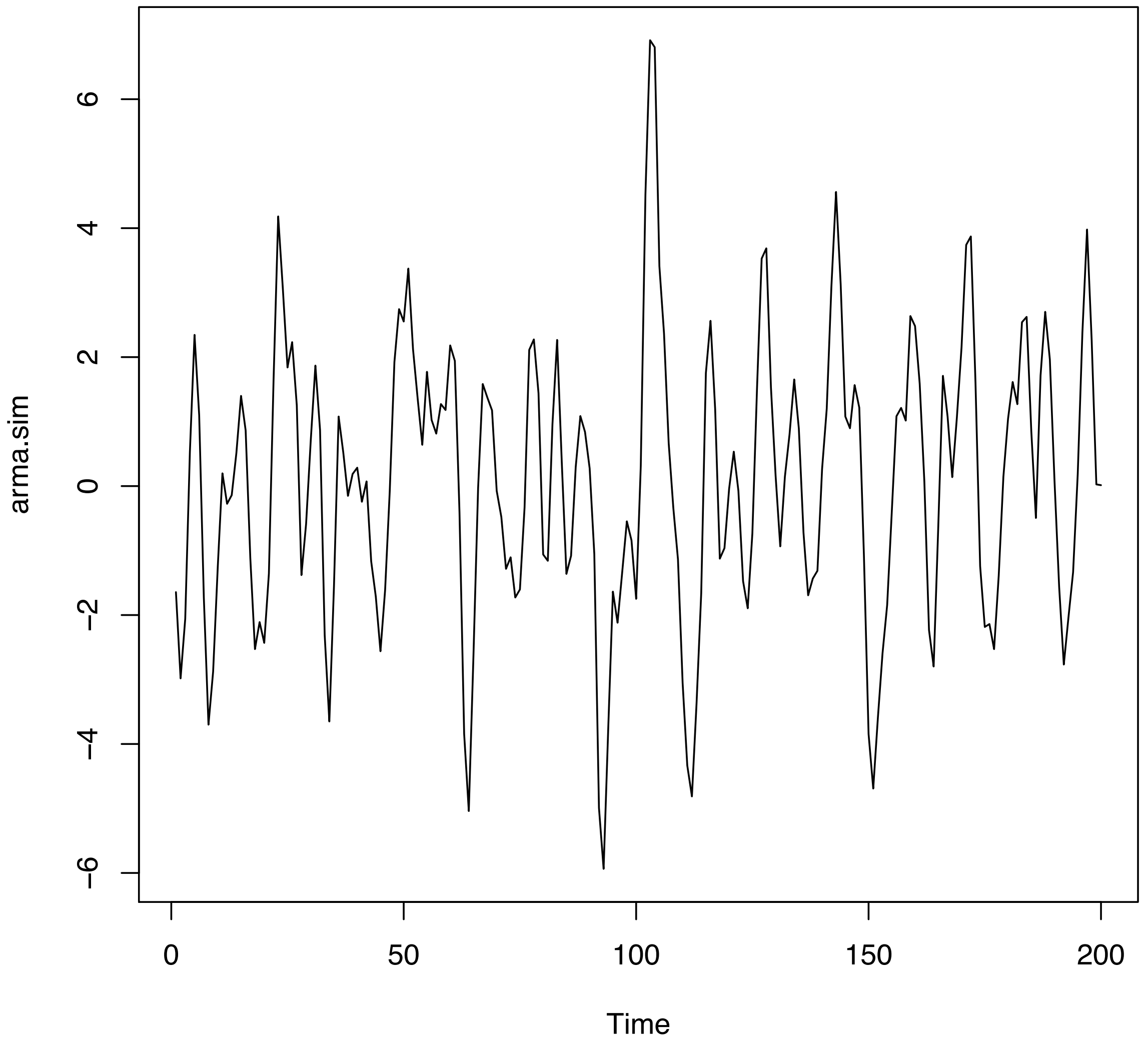
```
ar1.sim<-arima.sim(model=list(ar=c(.7)),n=200)
ts.plot(ar1.sim)
```



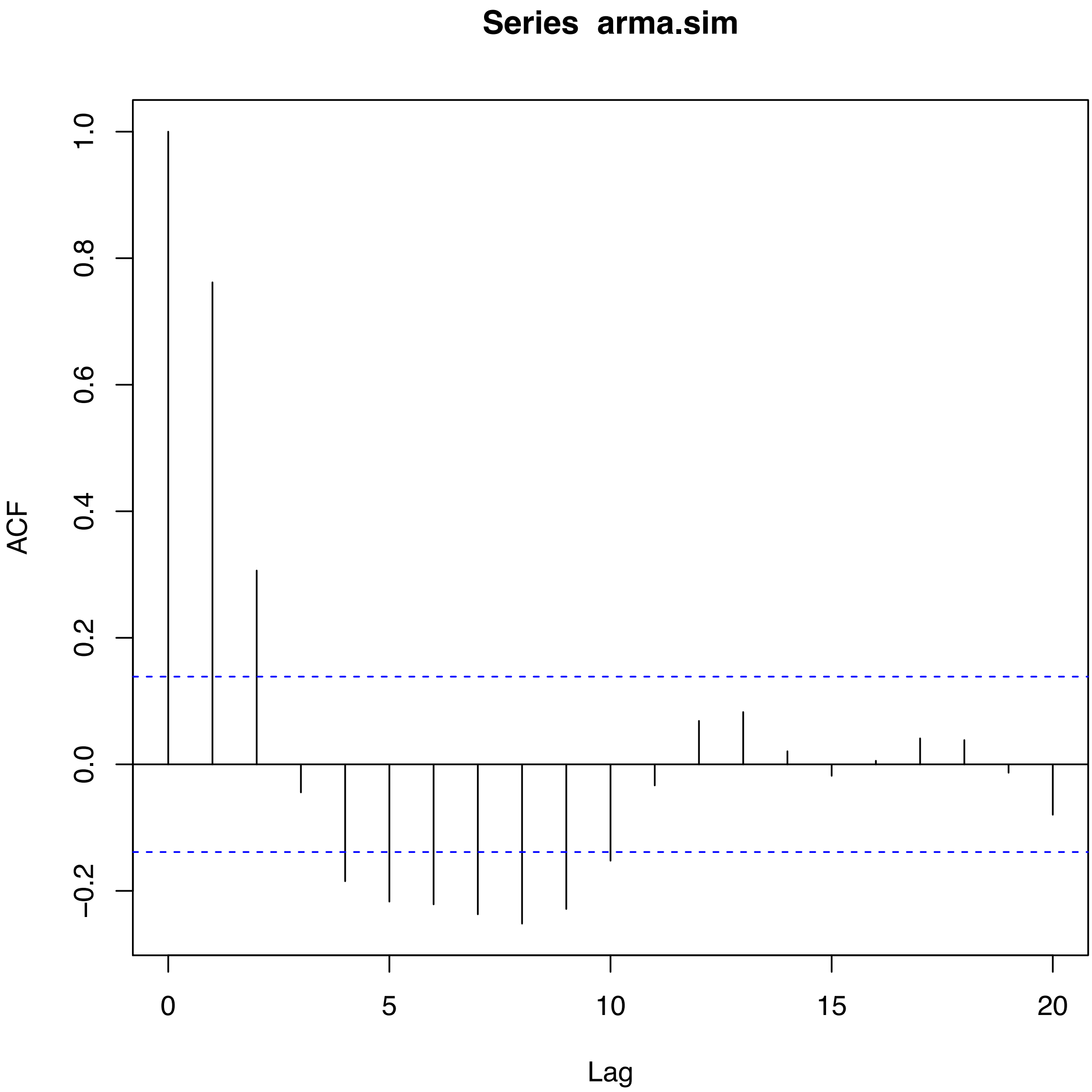

```
ar.acf<-acf(ar1.sim,20,type="correlation",plot=T)
```




```
arma.sim<-arima.sim(model=list(ar=c(.7,-.3),ma=c(.8,.4)),n=200)
ts.plot(arma.sim)
```




```
arma.acf<-acf(arma.sim,20,type="correlation",plot=T)
```


Series arma.sim
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
20
Lag