CHAPTER 2

COMMODITIES AND PRICES

2.1. INTRODUCTION

The dual concepts of commodity and price are introduced in this chapter. The meanings of these terms, somewhat different from current usage, will be made precise in the next sections. Many examples will be given as illustrations.

It is possible to present in this introduction the essential features of the two concepts in a simplified and slightly imprecise manner. The economy is considered as of a given instant called the present instant. A commodity is characterized by its physical properties, the date at which it will be available, and the location at which it will be available. The price of a commodity is the amount which has to be paid *now* for the (future) availability of one unit of that commodity.

No theory of money is offered here, and it is assumed that the economy works without the help of a good serving as medium of exchange. Thus the role of prices is as follows. With each commodity is associated a real number, its price. When an economic agent commits himself to accept delivery of a certain quantity of a commodity, the product of that quantity and the price of the commodity is a real number written on the debit side of his account. This number will be called the amount paid by the agent. Similarly a commitment to make delivery results in a real number written on the credit side of his account, and called the amount paid to the agent. The balance of his account, i.e., the net value of all his commitments, guides his decisions in ways which will be specified in later chapters.

To link the preceding concept of price with the customary notion of an amount of money paid when and where the commodity is available, one must introduce the concept of price at a certain date, at a certain location. One obtains then, by comparing prices at the same location, at different dates, interest, and discount rates; by comparing prices at the same date, at different locations, exchange rates.

In the next chapters the theory will be developed in terms of the two general, abstract concepts of commodity and price. To have concrete translations of its results one must use the present chapter, which provides a great variety of interpretations for the two concepts, as a key.

2.2. Dates and Locations

The interval of time over which economic activity takes place is divided into a finite number of compact *elementary intervals* of equal length. These elementary intervals may be numbered in chronological order; the origin of the first one is called the present instant. Their common length, which may be a year, a minute, a week, . . . is chosen small enough for all the instants of an elementary interval to be indistinguishable from the point of view of the analysis. An elementary interval will be called a *date*, and the expression "at date t" will therefore be equivalent to "at some instant of the tth elementary interval."

Similarly the region of space over which economic activity takes place is divided into a finite number of compact *elementary regions*. These elementary regions, which may be arbitrarily numbered, are chosen small enough for all the points of one of them to be indistinguishable from the point of view of the analysis. An elementary region will be called a *location*, and the expression "at location s" will therefore be equivalent to "at some point of the sth elementary region."

2.3. Goods

The concept of a commodity can now be introduced by means of examples. The simplest is that of an economic *good* like wheat; it will be discussed in detail. There are indeed many kinds of wheat, and to have a well-defined good one must describe completely the wheat about which one is talking, and specify in particular its grade, e.g., No. 2 Red Winter Wheat. Furthermore wheat available now and wheat available in a week play entirely different economic roles for a flour mill which is to use them. Thus a good at a certain date and the same good at a later date are *different* economic objects, and the specification of the date at which it will be available is essential. Finally wheat available in Minneapolis and wheat available in Chicago play also entirely different economic roles for a flour mill which is to use them. Again, a good at a certain location and the same good at another location are *different* economic objects, and the specification of the location at which it will be available is essential. In the case now discussed a *commodity* is therefore defined by a specification of all its physical characteristics, of its availability date, and of its availability location. As soon as one of these three factors changes, a *different* commodity results.

The *quantity* of a certain kind of wheat is expressed by a number of bushels which can satisfactorily be assumed to be any (non-negative) real number. What is made available *to* an economic agent is called an *input* for him; what is made available *by* an economic agent is an *output* for him. For some agents inputs will be represented by non-negative numbers and outputs by non-positive numbers. For other agents the reverse convention will be made. A uniform convention might seem desirable, but a more flexible one will make interpretation easier. With one of the above conventions a quantity of wheat can be any real number.

Goods of the same type as wheat are cement, iron ore, crude rubber, wood pulp, cotton yarn, petroleum, water, gas, electricity (whose definition includes frequency and voltage, and whose quantity is expressed in kwhr), etc.

As the prototype of a second class of goods consider trucks. The complete description of this good includes model, mileage, . . . To define the corresponding commodity one must add its date and its location. A quantity of well-defined trucks is an integer; but it will be assumed instead that this quantity can be any real number. This assumption of perfect divisibility is imposed by the present stage of development of economics; it is quite acceptable for an economic agent producing or consuming a large number of trucks. Similar goods are machine tools, linotypes, cranes, Bessemer converters, houses, refrigerators, trees, sheep, shoes, turbines, etc.

Land requires special mention. Its condition is described by the nature of the soil and of the subsoil (the latter being of importance for construction work), the trees, growing crops and construction on it, etc. A quantity of land with specified condition, location, and date is expressed by a real number of acres.

Mineral deposits, oil fields, . . . are defined by a complete description of their content, their location, and, as always, their availability date. Their quantity is expressed by a real number of tons, barrels, . . .

2.4. Services

The first example of an economic *service* will be human labor. Its description is that of the task performed; thus one has the labor of a coal miner, of a truck driver, of a member of some category of teachers, of engineers, of draftsmen, of executives, etc. (all including any further specification necessary for a complete description). When one adds date and location one has again a well-defined *commodity*. The *quantity* of a specified type of labor is expressed by the time worked (a real number).

Another type of service is illustrated by the use of a truck. It will be assumed that a truck (and similar economic objects) can be in only a finite number of distinguishable conditions. The life of a truck is described by a succession of time-intervals during each of which it stays in the same condition. The lengths of those intervals depend on the intensity of use. Thus the description of the service "use of a truck" is that of the truck (therefore of its condition during the time the service is rendered) and of the conditions under which it is used (mileage per day for example). One adds, as usual, date and location. The quantity of such a service is expressed by the time during which it is rendered.

A more complex type of service is illustrated by the use of a hotel room. The description of this service includes a listing of everything which will be performed for the occupant. It must, of course, be dated and located. Its quantity is an integral number of days; but it will again be assumed instead that this quantity can be any real number. Of the same type is, for example, the use of an apartment.

For other services, time is not the expression of the quantity. Such is a storage service which is described, for example, by the type of warehouse (refrigerated or not . . .), the dates from which to which it is rendered, and the location. Its quantity is expressed, for example, by a real number of cubic feet. One observes that in this case the temporal specification requires not one but several dates. Many other services, whose purpose is no longer to change the date of a commodity, require similarly more than one date to be temporally specified (at least when the elementary time-intervals are short enough), e.g., services of a repair shop, of a laundry, of a beauty parlor, attendance at a show, at a course, etc. In every one of these cases a unit is easily recognized; it is as always supposed to be perfectly divisible.

Finally, transportation services are described by the conditions under which they are rendered (rail, road, air, water, pipelines, power lines, etc., and any further specification necessary for a complete description), the locations they involve, and (since again they require a time longer than an elementary time-interval) the dates they involve. Their quantities are expressed for goods, for example, by the weight or the volume transported. For passengers the unit of the service is obvious. Temporal and spatial specifications of such services require several dates and several locations. Their quantities can, by assumption, be any real numbers.

2.5. Commodities

Summing up, a commodity is a good or a service completely specified physically, temporally, and spatially. It is assumed that there is only a finite number l of distinguishable commodities; these are indicated by an index h running from 1 to l. It is also assumed that the quantity of any one of them can be any real number. From now on the full generality of the concept of commodity, as illustrated by all the examples above, should always be kept in mind. By focusing attention on changes of dates one obtains, *as a particular case* of the general theory of commodities which will be developed below, a theory of saving, investment, capital, and interest. Similarly by focusing attention on changes of locations one obtains, *as another particular case* of the same general theory, a theory of location, transportation, international trade and exchange. The interpretation of the results in those terms will be left to the reader, since it offers no difficulty once the definition of a commodity has been grasped.

The space R^l will be called the *commodity space*. For any economic *agent* a complete plan of action (made now for the whole future), or more briefly an *action*, is a specification for each commodity of the quantity that he will make available or that will be made available to him, i.e., a complete listing of the quantities of his inputs and of his outputs. With one of the sign conventions of 2.3 an action is therefore represented by a point a of R^l.

2.6. Prices

With each commodity, say the hth one, is associated a real number, its *price*, p_h. This price can be interpreted as the amount paid *now* by (resp. to) an agent for every unit of the hth commodity which will be made available to (resp. by) him.

The general term price covers a great variety of terms in current usage: prices proper, wages, salaries, rents, fares, fees, charges, royalties, . . .

Consider as an example the commodity No. 2 Red Winter Wheat available in Chicago a year from now. Its price is the amount which the buyer must pay *now* in order to have one bushel of that grade of wheat delivered to him at that location and at that date. Price as understood here is therefore very closely related to "price" as understood on a futures market. There a sale contract concerns a well-defined good to be delivered at a specified date, at a specified location. The "price" to be paid is also specified now (it is the "price" prevailing on the floor of the exchange), but it is understood that this "price" shall be paid *at the delivery date, at the delivery location.* This difference from the price concept which will be used here is inessential (see 2.7). A difference of another kind clearly exists. Organized futures markets concern only a small number of goods, locations, and dates (not too distant in the future), whereas it is implicitly assumed here that markets exist for *all* commodities.

The price p_h of a commodity may be positive (*scarce* commodity), null (*free* commodity), or negative (*noxious* commodity). In the last case an agent for whom that commodity is an output, i.e., who disposes of it, makes a payment to the agent for whom it is an input, i.e., receives from the latter a negative payment. The fact that the price of a commodity is positive, null, or negative is *not* an intrinsic property of that commodity; it depends on the technology, the tastes, the resources, . . . of the economy. For example, some industrial waste product may be a nuisance the disposal of which is costly; should an invention, i.e., a different technology, open uses for it, it might become a scarce commodity.

The *price system* is the l-tuple $p = (p_1, \cdots, p_h, \cdots, p_l)$; it can clearly be represented by a point of R^l. The *value* of an action a relative to the price system p is $\sum_{h=1}^{l} p_h a_h$, i.e., the inner product $p \cdot a$.

2.7. Interest, Discount, and Exchange

Imagine that a certain good circulates as money at location s, at date t, and let k be the index of the commodity thus defined. To obtain the price at s, at t of the hth commodity, $p_h^{s,t}$, i.e., the number of units of that money which must be paid at s, at t in order to have one unit of the hth commodity available, one would divide p_h by p_k. Doing this for all prices in p, one would obtain the price system at s, at t, $p^{s,t} = p(1/p_k)$. Instead of referring all prices to some money at s, at t, one might refer them, for example, to some good or service at s, at t. One is therefore led to the general concept of *price system at location s, at date t*, $p^{s,t}$, as derived from p by multiplication by a certain *positive* real number $\lambda^{s,t}$ (determined by the unit of value chosen at s, at t). In $p_h^{s,t}$ the location s and the date t correspond to payment, the location and date which are implicitly determined by h correspond to delivery; the first pair and the second are unrelated, in particular the payment date may be earlier than, simultaneous with, or later than the delivery date. p now appears as the price system at an unspecified location, at an unspecified instant (of which it is often convenient to think as the present instant).

Let t^1, t^2 be two dates such that $t^1 < t^2$. The number $\alpha^s_{t^1,t^2}$ defined by $p^{s,t^2} = p^{s,t^1}\alpha^s_{t^1,t^2}$ is called the *accumulation factor at s from t^1 to t^2*. In this section p is always assumed to be different from 0; therefore $\alpha^s_{t^1,t^2}$ is a uniquely defined positive number. Its meaning is simple: by giving one unit of value at s, at t^1, one receives $\alpha^s_{t^1,t^2}$ units of value at s, at t^2. When, in particular, $t^1 = t$ and $t^2 = t + 1$, one defines the *interest rate at s from t to t + 1* by $i^s_{t,t+1} = \alpha^s_{t,t+1} - 1$. It is the difference between the value at s, at $t + 1$, one receives and the unit of value at s, at t, one gives. The interest rates usually quoted, e.g., .02 or 2%, are rates per annum; here all interest (and discount) rates are *rates per elementary time-interval.* From $\alpha^s_{t,t+1} = 1 + i^s_{t,t+1}$ one derives

$$\alpha^s_{t^1,t^2} = (1 + i^s_{t^1,t^1+1})\cdots(1 + i^s_{t^2-1,t^2}),$$

a product of $t^2 - t^1$ terms. This prompts the definition of the *interest rate at s from t^1 to t^2*, $i^s_{t^1,t^2}$, as a certain average, by

$$\alpha^s_{t^1,t^2} = (1 + i^s_{t^1,t^2})^{t^2-t^1},$$

the positive root of $\alpha^s_{t^1,t^2}$ being taken.

Similarly the positive number $\beta^s_{t^2,t^1}$ defined by $p^{s,t^1} = p^{s,t^2}\beta^s_{t^2,t^1}$ is called the *discount factor at s from t^2 to t^1*. To receive one unit of value at s, at t^2, one gives $\beta^s_{t^2,t^1}$ units of value at s, at t^1. Clearly

$$\beta^s_{t^2,t^1} = \frac{1}{\alpha^s_{t^1,t^2}} = \frac{1}{(1 + i^s_{t^1,t^2})^{t^2-t^1}}.$$

One defines also the *discount rate at s from t^2 to t^1*, $d^s_{t^2,t^1}$, by

$$\beta^s_{t^2,t^1} = (1 - d^s_{t^2,t^1})^{t^2-t^1},$$

the positive root of $\beta^s_{t^2,t^1}$ being taken. For the hth commodity, $p^{s,t^1}_h = p^{s,t^2}_h\beta^s_{t^2,t^1}$ is called the price at s, at t^2 *discounted from t^2 to t^1*.

Let s^1, s^2 be two locations. The positive number ε^{s^2,s^1}_t defined by $p^{s^1,t} = p^{s^2,t}\varepsilon^{s^2,s^1}_t$ is called the *exchange rate at t, at s^1 on s^2*. One receives one unit of value at t, at s^2, by giving ε^{s^2,s^1}_t units of value at t, at s^1. For example, if the unit of value at New York (resp. London) is called dollar (resp. pound), the exchange rate at t at New York on London is the number of dollars at t (at New York) one pays for one pound at t (at London). One has

$$\varepsilon^{s^1,s^2}_t = \frac{1}{\varepsilon^{s^2,s^1}_t}.$$

In fact, the set of locations is partitioned into *nations*, and for all the locations s of a nation the price system at s, at a given date t, $p^{s,t}$, is the same (this statement is unrelated to the generally *false* statement that the same good or service available at t, at two different locations of a nation, has the same price). Then interest and discount rates, accumulation and discount factors are the same for all the locations of a nation, exchange rates are the same for all pairs of locations belonging respectively to a pair of nations; the nation only needs to be mentioned.

What has been said about the generality of the concept of commodity could be repeated now for the concept of price. It must always be remembered that when the price system p is known and the numbers $\lambda^{s,t}$ (p. 33) are given, all prices proper, wages, salaries, rents, fares, . . . , all accumulation and discount factors, interest and discount rates, all exchange rates are determined at every date, at every location.

2.8. Theory and Interpretations

To conclude this chapter it remains to sum up the formulation of all the above concepts in the language of the theory:

The number l of commodities is a given positive integer. An action a of an agent is a point of R^l, the commodity space. A price system p is a point of R^l. The value of an action a relative to a price system p is the inner product $p \cdot a$.

All that precedes this statement is irrelevant for the logical development of the theory. Its aim is to provide possible interpretations of the latter. Other interpretations will be presented in Chapter 7.

Notes

1. The idea that a good or a service available at a certain date (and a certain location) is a different commodity from the same good or service available at a different date (or a different location) is old. The first general mathematical study of an economy whose activity extends over a finite number of elementary time-intervals under conditions of perfect foresight was that of E. Lindahl [1]. A similar treatment of time recurs in J. R. Hicks [1] (see also G. Tintner [1], [2]).

The use of negative prices originated in K. J. Arrow [1] and T. C. Koopmans [1].

2. The assumption of a finite number of dates has the great mathematical convenience of enabling one to stay within a finite-dimensional commodity space. There are, however, conceptual difficulties in postulating a predetermined instant beyond which all

economic activity either ceases or is outside the scope of the analysis. It is therefore worth noticing that many results of the following chapters can be extended to infinite-dimensional commodity spaces. In general, the *commodity space* would be assumed to be a vector space L over the reals and, instead of a price vector p, one would consider a linear form v on L defining for every action $a \in L$ its *value* $v(a)$. In this framework could also be studied cases where the date, the location, the quality of commodities are treated as continuous variables.

3. Two important and difficult questions are not answered by the approach taken here: the integration of money in the theory of value (on this point see D. Patinkin [1] and his references), and the inclusion of indivisible commodities.