COLUMBIA UNIVERSITY
Department of Economics

Mathematical Methods for Economists (Ph.D.)

ECON G6410 | Fall 2016
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Problem Set 10

1. For each of the following dynamical systems (X, f), describe all the sets $S \subseteq X$ which are invariant under f, list all its fixed points, and describe the stable set of each fixed point.

   (a) $X = \mathbb{R}, f(x) = x$

   (b) $X = \mathbb{R}, f(x) = -x$

   (c) $X = \mathbb{R}, f(x) = x + 1$

   (d) $X = \mathbb{R}, f(x) = x/2$

   (e) $X = \mathbb{R}_+, f(x) = 1.2x$

   (f) $X = \mathbb{R}_+, f(x) = 0.2\sqrt{x} + 0.8x$

2. Consider the dynamic system of $x = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$ in the space $\mathbb{R}^3$, which is implicitly defined by the following system of equations:

$$\begin{aligned} p_t + q_t &= -2p_{t-1} - q_{t-1} + 3.5r_{t-1} \\ q_t &= -p_{t-1} + 1.5r_{t-1} \\ q_t + r_t &= -4p_{t-1} - q_{t-1} + 5.5r_{t-1} \end{aligned}$$

   for any $t \in \mathbb{N}$.

   (a) Express the system of equations as $Bx_t = Cx_{t-1}$, where B and C are 3×3 matrices. Write down B and C explicitly.

   (b) Find the transition matrix A of this linear dynamic system, i.e. find A s.t. $x_t = Ax_{t-1}$.

(c) Find the three eigenvalues of A.

(d) Diagonalize the matrix A (in the general sense), i.e. find an invertible **real** matrix P s.t.

$$P^{-1}AP = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{bmatrix}$$

Be specific about the matrix P you find, as well as the value of λ, α, and β.

(e) Prove that the zero vector $x^* = 0$ is the unique fixed point of this dynamic system.

(f) Prove that the fixed point $x^* = 0$ is not globally asymptotically stable, by finding one $x_0 \in \mathbb{R}^3$ s.t. $x_t := A^t x_0 \not\to 0$.

(g) Find one $\hat{x}_0 \neq 0$ s.t. $\hat{x}_t := A^t \hat{x}_0 \to 0$.

3. Prove that $f(x) = 2x^2 - 1$ on $[-1, 1]$ is conjugate to $g(x) = x^2 - 2$ on $[-2, 2]$

4. Consider the Hénon map defined by

$$x_n = a - x_{n-1}^2 + by_{n-1}$$

$$y_n = x_{n-1}$$

Let $a = 1.4$ and $b = 0.3$.

(a) Using a program, calculate the dynamics over 5000 periods, and plot x_t against t. Use a seed of $(x_0, y_0) = (0, 0)$. Plot y_n against x_n.

(b) Plot a histogram of the trajectory of x_t with 40 bins.

(c) Redo (a) and (b) with a perturbed seed of $(x_0', y_0') = (0 + 10^{-8}, 0 + 10^{-8})$. How different is the trajectory of the perturbed seed from the initial seed?

(d) Plot the the difference in trajectories $\Delta_n = ln|x_n - x_n'|$ against t. What would you say this plot tells you about the dependence of future behaviour of the system on initial conditions?