Advanced Monetary Policy

Technical note

The Euler equation and asset pricing

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Abstract

This note expands on our understanding of the Euler equation, building on the previous 2-period deterministic consumption problem. We will take an asset pricing perspective and show some numerical examples (associated with Matlab code that generates the numerical results).

1 Introduction

The Euler equation emerges from the optimal decisions of an agent allocating her savings/investments with the aim of spreading her consumption over time and over contingencies. In the previous technical note, we emphasized the former perspective (smoothing consumption over time). In this note we begin to think about risk. By risk, we mean that for any given time period in the future, there may be uncertainty over what contingency, or 'state of the world', will prevail in that period. For example, we do not typically know for sure if the next period will feature a boom or a recession, or whether our assets will pay off well, or not.

Previously we introduced the EE by emphasizing the idea of a 'small' consumer taking prices as given. They were choosing their consumption growth, faced with a return on a bank account. This will remain true in this note. However, with a few assumptions - beyond the scope of this course - the consumption growth in the EE can be taken to be the consumption growth of the 'average' person within the economy. Indeed (and typically this is done in a lot of consumption-based empirical finance) we may even be able to insert *aggregate* consumption growth into the EE and imagine it still holding.

This then opens the door to an interesting perspective. Aggregate consumption growth is determined alongside other aggregate variables, such as asset prices, interest rates and returns. At this level (rather than at the level of the individual) it is less useful to think of prices as influencing aggregate quantities (consumption growth) than thinking of them all as jointly determined as part of a big macroeconomic system where 'everything depends on everything'. In this context, it becomes much more natural to think of the EE as being an 'asset pricing' equation. From this perspective, we say - ok, consumption growth behaves like 'this' so the EE then tells us that asset prices should behave like 'that'.

Why might the asset pricing perspective be useful? Well, aggregate consumption growth is likely very closely tied to aggregate output growth, at least in terms of trend. In that case, if we have a theory of *trend output growth* (e.g. the great stagnation or a drop in g^*) then we can put that into the EE, restricting the consumption growth part. This may allow us to make predictions about the average level of interest rates, all else equal (this is part of the questions set at the end of this note). Similarly, if we have information - or a model - for the *volatility* of consumption, or how it *correlates* with asset payoffs, that will also yield implications for asset prices.

2 Notation

In the two period consumption problem (see the other note), we imagined a person who could deposit (save) and borrow from a 'bank' at a riskless real rate. We wrote the riskless rate as R_1. Although the investor receives the rate in $t = 2$, I wanted to make clear that it was known 'today' (period 1), hence the subscript '1'. In this note, I will change the notation somewhat and subscript *returns* with the period in which the return is *realized*, so as to be consistent with what follows. As such, I will write R_2^S, using the superscript **S** to make clear that the return tomorrow is **S**afe (known today).

3 Euler equation - No risk (banks and bonds)

What is a return? It is the ratio of the payoff you get 'tomorrow' to the outlay you paid 'today' (obviously). Let us denote the payoff in period t by $\mathcal{P}_t$.

Let us take a stock as an example. For a stock, what is the payoff 'tomorrow'? You get ownership of the stock tomorrow, which is worth the price of the stock tomorrow, P_2^{Stk}. You also may get a dividend - denote that by D_2. So we have $\mathcal{P}_2^{Stk} = P_2^{Stk} + D_2$. The gross return from today to tomorrow is therefore

$$R_2^{Stk} = \frac{\mathcal{P}_2^{Stk}}{P_1^{Stk}} = \frac{P_2^{Stk} + D_2}{P_1^{Stk}}$$

which, as stated, is simply the payoff of the stock, relative to its price today.

You can think of 'investing' in a riskless bank deposit in this way. You pay a 'dollar' into the riskless asset (the deposit) and get back a payoff of R_2^S next period. The price of a *gross* return is 1. Right? To get a gross return from today to tomorrow, you can invest 1 unit, so that means the price is '1'.[1] Of course, I could invest more than 1 dollar in my deposit account. If I invest K dollars today, I would then get $R_2^S \times K$ tomorrow. I'm buying the return on every dollar in the K dollars. Similarly, if I buy K stocks, I would end up getting $R_2^{Stk} \times K$.

Now, imagine there is another riskless asset available to invest in - in addition to the riskless bank account. By basic arbitrage you should be able to guess that it must be offering today the same return as the bank account. One such asset is a 'riskless bond'. Most people regard UK or US government debt, for example, as 'riskless', so let us think about the agent being able to 'buy' one of these bonds (effectively, 'lending' to the government).[2]

[1] It looks a bit weird to have the payoff notated as a return, but if the denominator (the outlay or 'price') is 1, then it is what it is!

[2] Continue to ignore, for a moment, the fact that they are effectively only riskless in nominal terms - as in these notes we have not even spoken about inflation. Some governments do issue inflation linked bonds that are arguably free of credit risk and can

Bonds have various properties - such as exactly how and when they pay interest (or 'coupons') in addition to any repayment of 'principal'. Economists often find it useful to think about a particular type of bonds with a very simple structure - those that promise a payment of one unit of consumption (for sure) at maturity and that pay no coupons. They are often called 'zeros' or 'zero coupon bonds'. Such bonds, of maturity $k > 0$, are imagined as being sold at a price, $P_t^{Safe,k}$ in period t. In our simple case, we are just considering a bond maturing tomorrow, sold today for $P_1^{Safe,1}$ (as $k = 1$).

Let us consider such a bond that promises one unit of consumption tomorrow. Given time preference ($\beta < 1$), you should expect its price to be somewhat less than 1 (you will show this later). What is the riskless return on this bond? Well, the payoff is 1 and the price is $P_1^{Safe,1}$ so we must have the return is $\frac{1}{P_1^{Safe,1}}$. But by no arbitrage we must then have

$$R_2^S = \frac{1}{P_1^{Safe,1}}$$

Thus, you can think about investing in a riskless opportunity in terms of a riskless bank account offering (today) R_2^S as a gross return on your unit of consumption foregone (saved) today. But you can *also* think in terms of a bond offering a riskless unit of consumption tomorrow in return for paying (today) $P_1^{Safe,1}$ units of consumption. There really isn't any economic difference in our simple model.

So, going back to our two period consumption problem from the previous note, I could just as easily have written the budget constraints as[3]

$$\begin{aligned} c_1 + P_1^{S,1} b_1 &= m_1 \\ c_1 &= m_2 + b_1 \end{aligned}$$

rather than

$$\begin{aligned} c_1 + s_1 &= m_1 \\ c_1 &= m_2 + R_2^S s_1 \end{aligned}$$

The rewritten budget constraint for period 1 above says that the agent can use income today (m_1) to consume or to save, where she saves by buying the b_1 amounts of the riskless bonds. For each unit of bonds she buys, she must pay $P_1^{S,1}$ today, and will receive a unit of consumption next period. The return

perhaps be regarded as real riskless bonds.

[3] Again, we are assuming savings in period 2 are zero, since the person will die in $t = 3$. Also, no creditor will let her die in debt.

she gets on her savings will be the same as if we had imagined her using a bank account and the amount she chooses to save or consume will be unchanged. That is, $P_1^{S,1} b_1$ in the 'bonds world' will be equal to s_1 in the 'bank account' world.[4]

In the end, the Euler equation in the bank case will be

$$u'(c_1) = \beta R_2^S u'(c_2)$$

and in the bond case it will be

$$u'(c_1) P_1^{S,1} = \beta u'(c_2)$$

You should be able to give a marginal benefit = marginal cost intuition for the second representation.

4 Euler equation - Adding some risk

In the previously released 2-period consumption note, we had

$$1 = \beta R_1 \frac{u'(c_2)}{u'(c_1)}$$

whereas - recalling our change in notation - that would translate here to

$$1 = \beta R_2^S \frac{u'(c_2)}{u'(c_1)}$$

Most (all?) assets offer returns that are *random*. In addition, people are also typically unsure about what their consumption is going to be next period. This means that typically we will see Euler equations featuring expectations. For a risky return, R_2^{Risk}, and with risky consumption in $t = 2$ (given information available at $t = 1$), we might encounter

$$u'(c_1) = \beta E_1 \left[R_2^{Risk} u'(c_2) \right]$$

This is also an Euler equation and our previous intuition goes through in a similar way. If I sacrifice a little bit of consumption today (losing $u'(c_1)$ of utility on the margin, a condition of optimality is that this 'marginal cost' is balanced by the 'marginal benefit' of what I get from consuming the marginal additional

[4] In fact, we could give her access to both! We could write the budget constraint with both savings technologies available. We would get Euler equations for bonds and for bank account and they would look pretty much the same. At that point, in our model, we can't predict the split of a given amount of savings between the two assets, there is an indeterminacy. There's nothing that will pin down the split between the two riskless investments (and that's fine).

consumption I get from my investments. The difference in this case is, first, I don't know what the return from the asset is going to be. Second, many contingencies could happen tomorrow, each with a particular probability. These contingencies may imply different payoffs - and thus returns - on the asset. They may also imply high or low realizations for my consumption. If consumption next period turns out to be high, I may not value *more* consumption much on the margin ($u'(c_2)$ may be low). However, if consumption tomorrow turns out to be low, I may value very highly the payoffs from an asset because it gives me more consumption when $u'(c_2)$ is high.

There is randomness in $u'(c_2)$ (coming from randomness over c_2) and there is randomness from R_2^{Risk}. Acknowledging this randomness, and the fact that marginal utility and the return may be correlated, all I can do in $t = 1$ is take an investment decision to align my marginal cost today, with the *expected* marginal benefit tomorrow.

A familiar example comes from investing in the stock market. Stock returns are random and there is a tendency for them to perform badly (well) in a general economic downturn (boom). Consumption tends to be relatively low (high), therefore, when returns are low (high). This *comovement* is what makes stocks risky - not their variability *per se*. Rather than insuring you, they expose you to exaggerated movements in the same direction as your consumption is already going. As such, stocks are lower priced than they might otherwise we.

5 The stochastic discount factor

Note that people sometimes rearrange/rewrite the EE to obtain

$$\begin{aligned} 1 &= \beta E_1 \left[R_2^{Risk} \frac{u'(c_2)}{u'(c_1)} \right] \\ &= E_1 \left[\beta R_2^{Risk} \frac{u'(c_2)}{u'(c_1)} \right] \\ &\equiv E_1 \left[\Lambda_{1,2} R_2^{Risk} \right] \end{aligned}$$

where $\Lambda_{1,2} \equiv \beta \frac{u'(c_2)}{u'(c_1)}$. There is a lot we could say about $\Lambda_{1,2}$, which is the 'stochastic discount factor' (SDF) for real payoffs in $t = 2$, discounting from the perspective of $t = 1$. It's a beautiful thing, but going into it in full detail is beyond the scope of this course. Nevertheless, we will get across some of the intuition for its role and importance.

Why is it called 'stochastic'? Well, 'stochastic' is another word for 'random'. **Why is it random?** It is random (from the perspective of $t = 1$) because it depends on c_2, which is (generally) not yet known in $t = 1$.

In what way(s) is $\Lambda_{1,2}$ 'discounting'? Well, for a start, it involves β, which captures the idea that payoffs in the future are somewhat less valuable than payoffs today. This reflects time preference, which we previously discussed in the riskless case. So, β captures *time* discounting. However, $\Lambda_{1,2}$ *also* applies discounting by comparing marginal utility tomorrow, in different contingencies, to marginal utility today.

Marginal utility tomorrow ($u'(c_2)$) depends on how the world's randomness turns out next period, because that determines how c_2 turns out. In contingencies where $u'(c_2)$ is low (think of this like a boom tomorrow), then the return in that contingency gets down-weighted in the expectation, relative to simply $\beta E_1[R_2^{Risk}]$. In contingencies where $u'(c_2)$ is high (think of this like a downturn tomorrow when you are really desperate), then the return in that contingency gets up-weighted in the expectation, relative to simply $\beta E_1[R_2^{Risk}]$.

To see this, remember the shape of $u(x) = \frac{x^{1-\gamma}-1}{1-\gamma}$. Or remember that $u'(x) = x^{-\gamma}$. Notice that $u'(x) > 0$ for $x > 0$ i.e. marginal utility from consumption is always positive - so you always prefer more consumption to less. But note, also, that $u'(x)$ is declining as x gets bigger (or note $u''(x) < 0$) so marginal utility is 'high' when consumption is low, and low when consumption is high. How rapidly this occurs - how much curvature there is in utility, in a sense, depends on γ. Previously we spoke about $\sigma \equiv \gamma^{-1}$ being the elasticity of intertemporal consumption. It turns out that γ captures the agent's risk aversion also - how willing they are to substitute consumption across *contingencies*, rather than *time*.[5]

Think of expectations of a random variable as being an average - a weighted average where the outcomes for the variable are weighted by probabilities. You imagine all the possible outcomes, what the payoffs are in those outcomes, and you'll weigh them up. All else equal, you might weight the most likely outcomes a bit more heavily and the less likely outcomes a bit less heavily. That's what an expected value *is*. If we were just calculating the expected return, that would be $E_1[R_2^{Risk}]$. But there is something else that most people will consider - in addition to the payoff and the probability: you will also think to yourself whether the asset will insure you or exaggerate the risk you face. That's the $\frac{u'(c_2)}{u'(c_1)}$ bit. Add in the fact that you prefer to have consumption today rather than tomorrow ($\beta < 1$) and you can see that $\Lambda_{1,2}$ is tweaking your expectations of the asset's returns to reflect your preferences and your consumption. So you end up with $E_1[\Lambda_{1,2} R_2^{Risk}]$, rather than $E_1[R_2^{Risk}]$.

[5] That's a bit weird, and presumably not a good description of peoples' preferences. Why should your preference of how variable your consumption is over time, in a deterministic world be pinned down by your preferences over how variable it is across random contingencies? In fact, this very strong restriction seems empirically very problematic - if you pick γ to match facts about risk aversion, you'll have a tough time matching facts about substitution over time, and *vice versa*. This is why people often use Epstein-Zin preferences in empirical finance, which adds an extra parameter to preferences and allows the two types of substitution attitudes to be separately adjusted.

6 Numerical example

Let's really take this weighted average intuition seriously and try an example with actual numbers. Suppose there are 3 economies and each features 3 assets (it's not important that there are as many economies as assets). Let there be two possible outcomes in $t = 2$ - boom and bust, or 'high growth' and 'low growth'. Without it being hugely important for our application, let us assume that 'high' and 'low' growth are equally likely.

6.1 Assets

Let the first asset be a riskless asset that pays 1 unit of consumption in $t = 2$, for sure, regardless of the state of the economy ($\mathcal{P}^1_{2,H} = \mathcal{P}^1_{2,L} = 1$). Let the second asset, pay 1.07 units in the boom and 0.93 in the bust. Finally, let the third asset pay the reverse configuration from asset two, 0.93 in the boom and 1.07 in the bust. You likely have already noticed that with 50:50 chance of boom and bust, all of these assets promise the same expected *payoff*, of 1 unit.[6]

Now let us specify three economies. For simplicity, let us think of all the economies having agents with the same 'log' preferences (the limiting case as $\gamma \to 1$, which we discussed in the previous note). You will be able to play with the code and see what happens to the results when you set risk aversion to a higher level.

6.2 Economies

Within the first economy, let us imagine that optimal behavior, other (unspecified) aspects of the economy, and the requirements of equilibrium, have generated a world where there is high growth and high volatility in consumption. So the expected gross rate of consumption growth is 1.04, which reflects the gross rate of consumption growth being 1.06 (net rate of 6%) in the boom and 1.02 (net rate of 2%) in the 'bust'. The second economy has the same expected gross rate of consumption growth as the first economy (1.04), but has gross rates of consumption growth of 1.045 and 1.035 in boom and bust respectively. That is, the average growth is the same, but the volatility is lower. Finally, imagine a low growth and low volatility economy, with average (expected) gross rate of consumption growth of 1.02, with gross growth of 1.025 and 1.015 in boom and bust.

[6] Asset 1: $0.5 * 1 + 0.5 * 1 = 1$. Asset 2: $0.5 * 1.07 + 0.5 * 0.93 = 1$. Asset 3: $0.5 * 0.93 + 0.5 * 1.07 = 1$.

6.3 Question

Find the prices and expected returns of the three assets, priced in each of the three economies. You can (and should) do this with a calculator and maybe some pen/paper. You may also run the code provided. After you have done the latter to check your work, try increasing the risk aversion and recalculate. We will discuss the results in the lab. One particular thing to ask yourself - is asset 3 risky? In what sense?

Hint: For each economy, make a table with a row for each asset. Create columns for:

- the asset payoffs in booms and busts (two columns)
- consumption growths in booms and busts (two columns)
- the value of the SDF in booms and busts (two columns)
- $E_1[\Lambda_{1,2}\mathcal{P}_2]$ where $\mathcal{P}_2$ is the period 2 payoff (one column containing the expected discounted payoff - i.e. the price of the asset)
- the expected return on the asset (one column)

Hint: As a simple example of calculating expectations. Suppose I am rolling a fair dice. There are six sides. Each side is equally likely to come up. So each side will come up with probability $\frac{1}{6}$. Imagine I win £10 if the number 1 comes up, £20 if the number 2 comes up, and so forth. What is the expected payoff?

$$E[Payoff] = \frac{1}{6} \times 10 + \frac{1}{6} \times 20 + \frac{1}{6} \times 30 + \frac{1}{6} \times 40 + \frac{1}{6} \times 50 + \frac{1}{6} \times 60 = 35$$

So here the weighted average was equally weighted (all the weights were $\frac{1}{6}$), and we averaged the payoffs.