Advanced Monetary Policy

Technical note

2-period deterministic

consumption-savings problem

rhys.m.bidder@kcl.ac.uk

Spring 2024

Abstract

This note lays out a core economic problem that features in almost every macro model these days - a consumer's decision of how much to consume today, versus save for future consumption. The main points can be got across in a two period model without any randomness.

1 Introduction

At various points in this course we will need to make use of results derived from the assumption that people are choosing their consumption, savings and investments in a forward looking way. In our applications, we will be allowing for randomness and perhaps some frictions that complicate the problem of the consumer (or 'household'). But the many of the main points can be conveyed in a simple model where we eliminate randomness and introduce a minimalist structure. Indeed, we need only imagine the person making decisions two periods - we can consider them making a decision 'today' on how much to consume or save, with implications for 'tomorrow' - the last period of their life.

From this starkly simple setup it is quite amazing how much intuition we can achieve that will translate almost immediately to much richer situations later in the course.

There is some mathematical formalism that might be unfamiliar to some (or many) of you, so I want to start you (as we will with programming also) on this stuff early, and at a slow pace. I will try to introduce some mathematical concepts but typically you can just 'blindly accept' many of the results - I don't expect you to have a deep mathematical toolkit.

The end point of this note is for you to understand better the origin of one of the equations from the three equation macro model that Prof. Yates showed you. He may have called it the IS curve and in a simple model it is derived from (plus a few assumptions) an equation we will obtain in this note - the Euler Equation. This equation relates consumption growth and returns on assets (in our case a riskless 'bank account' or 'bond'). In its various guises, the Euler Equation (EE) could be said to be at the heart of dynamic macro and asset pricing.

2 Consumption-Savings problem

Here we lay out, solve and analyze the consumption savings problem. Something like it will appear in almost every macro model (and implicitly asset pricing model) you will ever see.

2.1 Statement of problem

We consider an agent who lives for two periods: 1 and 2 ('today' and 'tomorrow'). She receives an endowment of income in each period, call them m_1 and m_2. We could easily endogenize her income, by imagining her working for a wage, and choosing her optimal labor supply, but since we are focused on consumption-savings, we abstract from that. Perhaps she's a 'trust fund baby'.

She can save in a bank account or borrow from the bank at gross (real) interest rate R_1 in period 1 and R_2 in period 2. Denote her consumption in the two periods as c_1 and c_2 and her savings (which could be negative - indicating borrowing) as s_1 and s_2. We assume there is no risk in the economy. In particular, she knows today not only what rate she will get (pay) on her savings (borrowing) but also what her income will be next period.

She is choosing a plan for c_1 and c_t, which, will imply a plan for savings. A formal statement of the problem is

$$\max_{c_1, c_2, s_1, s_2} U(c_1, c_2)$$

where

$$\begin{aligned} U(c_1, c_2) &\equiv u(c_1) + \beta u(c_2) \\ u(x) &\equiv \frac{x^\gamma - 1}{1 - \gamma} \end{aligned}$$

and subject to the 'period budget constraints'

$$\begin{aligned} c_1 + s_1 &= m_1 \\ c_2 + s_2 &= m_2 + R_1 s_1 \end{aligned}$$

The period budget constraints say that income in any period must go towards either consumption or saving in that period. Income in the first period comes only from the endowment. In the second period, income may be supplemented by savings (or reduced by needing to pay back borrowings, if $s_1 < 0$). These represent the **'what can I get'** or 'feasibility' part of the problem.

The **'what do I want'** or 'preferences' part of the problem relates to the utility function, U. The agent has preferences over sequences of consumption (in this case two element vector - consumption today and tomorrow) which can be represented with a utility function, U that takes today and tomorrow's consumption as inputs and returns a number. For preferences to be represented by a utility function we mean that if a vector of consumption $\vec{c}^A$ is strictly preferred to another vector $\vec{c}^B$, then $U(c_1^A, c_2^A) > U(c_1^B, c_2^B)$. If she is indifferent between the two, then U must be such that $U(c_1^A, c_2^A) = U(c_1^B, c_2^B)$. This must apply for the whole space of consumption sequences considered.

We assume that preferences are such that U takes a particular form - one that is time separable (i.e. has separate bits that relate to different periods) and involves time discounting. In addition, it is assumed that we can appeal to a fixed 'period utility function' u to express how consumption in either period contributes to the overall value of U. u represents the agent's preferences over a one shot amount of consumption (rather than a vector). U acts on two numbers and u acts on a single number. We will come back to what γ represents but you should note that it affects 'curvature' of the u function. More simply, you should notice that u is increasing in its input. This will imply that the agent (very reasonably) prefers more to less.

Time discounting is captured by $\beta < 1$ which means that when taking decisions in t the agent's preferences are such that a given amount of consumption yields less utility (contributes less to U) than the same amount received today. Apart from being empirically useful, this makes sense - humans are evolved to prefer 'jam today, rather than jam tomorrow' and it may also reflect some latent fear of dying at any point!

In summary, the agent wants to pick the combination of consumption that yields her highest utility (as captured by U) as that means she will be consuming the most preferred consumption sequence she can afford, given the two period budget constraints.

2.2 Solving the problem - Using math

Note first that we can immediately a lot about the solution, assuming the agent has sensible (prefers more consumption to less) preferences. Clearly, she will spend all her resources and, in particular, she will allocate nothing to savings in the last period.[1]

As such, we can restate the problem, eliminating s_2

$$\max_{c_1,c_2,s_1} U(c_1, c_2)$$

subject to

$$\begin{aligned} c_1 + s_1 &= m_1 \\ c_2 &= m_2 + R_1 s_1 \end{aligned}$$

In fact, we can simplify further as a choice of c_1 and c_2 implies s_1 so we don't need to carry it around as an extra variable. So we can substitute it out of the problem. Note we have (from the first period BC) $s_1 = m_1 - c_1$ (obviously - that's the definition of saving) which we can then put into the second period

[1] She's dead after that, so it can't be optimal to set $s_2 > 0$ and we assume no one will allow her to die in debt. Remember there is no risk, so she knows from the start that there's no 'day after tomorrow'.

BC, to obtain what we call the **intertemporal budget constraint**:

$$c_1 + \frac{1}{R_1}c_2 = m_1 + \frac{1}{R_1}m_2 \tag{1}$$

That is, we can replace two 'period' budget constraints involving savings, with one 'intertemporal' budget constraint involving only consumption and income. It has a very intuitive form. The left hand side is the **present value of lifetime consumption** (from the perspective of the first period) and the right hand side is the **present value of lifetime income**. You'd better be spending all of your lifetime income and no more if you are maximizing your utility and not getting in trouble with creditors!

Does this mean she must cover period 1's consumption with period 1's income, or period 2's consumption with period 2's income? **No!** She can consume more in one period than income in that period, provided this is offset by 'savings' out of the other period's income. It can be useful to rearrange Equation 1 to see this more clearly:

$$c_1 - m_1 = \frac{1}{R_1}(m_2 - c_2)$$

You can see that the agent has to 'pay the piper'. Sure she can consume loads today, but that means she is going to have to consume hardly anything tomorrow. Clearly, the interest rate on borrowing/saving is key in this financial trade-off. The RHS (present value - or PV - of lifetime income) is fixed and any consumption sequence that satisfies Equation 1 is **feasible**, though it may not be **optimal**.

We are thus solving

$$\max_{c_1, c_2} U(c_1, c_2)$$

subject to Equation 1. Amazingly, we can actually make this problem even more simplified, notationally, by noting that we can use Equation 1 to eliminate c_2 because

$$c_2 = m_2 + R_1(m_1 - c_1)$$

Again this is just a rearrangement of Equation 1 that makes explicit that in period 2 the agent will consume her income, plus any savings (including the return) she brings into the period. Of course, if $c_1 > m_1$ then those 'savings' are negative and actually represent debt. In that case we will have $c_2 < m_2$ as she has to use some of her period 2 consumption to pay off debt, rather than for consumption. So now we are solving

$$\max_{c_1} u(c_1) + \beta u(m_2 + R_1(m_1 - c_1))$$

which is a nice univariate problem. All the constraints have been mashed together to leave us with a choice in c_1. Now we can use our differentiation tricks (see the technical math note in Keats) to find the derivative with respect to c_1 and set it equal to zero - the condition for optimality.

Note that here we will need to use the 'chain rule' discussed in the uploaded math note. It arises because of the second term and in particular because of $u(m_2 + R_1(m_1 - c_1))$. In the language of the math note, this takes the form $g(h(x))$ where g is u, here, and $h(x) = m_2 + R_1(m_1 - x)$. The optimality condition (remember - setting derivative equal to 0) is

$$u'(c_1) + \beta u'(m_2 + R_1(m_1 - c_1)) \times (-R_1) = 0$$

where the second term on the right features the application of the chain rule (the derivative of the inner function, h, is $-R_1$). At this point, it's more convenient to restore c_2 explicitly in how we write the expression

$$u'(c_1) = \beta R_1 u'(c_2)$$

or, given our assumed form of u

$$c_1^{-\gamma} = \beta R_1 c_2^{-\gamma}$$

What's the point? The point is that if the household is making optimal choices, this imposes a tight connection between marginal utilities (thus consumption) in the two periods **and** the interest rate. This is the **Euler Equation** for optimal saving. It encodes both preferences (in the form of marginal utility and time discounting) and constraints (in the form of the 'relative price' of consumption in the two periods - i.e. the interest rate) as it has arisen from maximizing utility subject to those constraints.

2.3 Solving the problem - Using intuition

In fact, we could derive this 'optimality condition' (which is what the EE is) without any of the above rigmarole and with barely any math.

Let us ask: What must be the case at the optimum? It must be the case that there is no *feasible* (i.e. affordable) tiny change in consumption - shifting from period 1 to 2 or vice versa - that can improve the agent's utility. If such a change *is* possible then that contradicts the assertion that they have picked an optimal plan. That means that any marginal cost of reducing (increasing) consumption today must be balanced exactly by the marginal benefit of increasing (decreasing) consumption tomorrow.

What's the cost of reducing consumption today slightly? That would be U_{c_1}, which turns out to be $u'(c_1)$. What's the benefit? That would be U_{c_2}, which in this case is $R_1 u'(c_2)$. Why is the latter true? Because the tiny bit of reduced consumption today implies a tiny bit of increased saving, which earns

R_1 and the utility gained is $\beta R_1 u'(c_2)$. notice that time preference moderates the contribution of the second period. As such marginal cost = marginal benefit implies $u'(c_1) = R_1 u'(c_2)$, which is just the Euler equation. So we can derive it without really going through all the setup we previously did (in this simple case).

In fact, even if you had infinite periods (as opposed to 2-periods in our simple example) the EE would still hold between consecutive periods:

$$1 = \beta R_t \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}$$

Why? Remember our basic intuition. This still holds for any pairs of periods. If the gain from reducing (increasing) consumption in a given period is more than the loss from increasing (reducing) it in other periods, then the household can't be optimizing.

2.4 Digression: Extending the intuition beyond 2-periods

This goes for two non-consecutive periods too in a world of more than 2 periods. What is the calculation then? If I reduce my consumption today by a little bit, I lose $u'(c_1)$ but if I use those savings to increase by consumption the **day after tomorrow** I would gain $\beta^2 R_1 R_2 u'(c_{t+2})$. Implicitly I would earn the 1 period return from period 1 to period 2, reinvest the gross return, and earn the 1 period return from period 2 to period 3. So we must have

$$u'(c_1) = \beta^2 R_1 R_2 u'(c_3) \tag{2}$$

at an optimum. Again we are assuming no risk and that the agent knows what one period interest rate will be prevailing in period 2, which she will earn on savings from then into period 3. Note that the time discounting effect is bigger ($\beta^2 < \beta$ since $\beta < 1$) as we are comparing with consumption further in the future.

Another way of seeing this is to iterate on two Euler equations since we assume that in the second period, the person will still be making optimal decisions, so an EE will hold between period 2 and period 3 also. That is, we know that an optimum will imply

$$\begin{aligned} u'(c_1) &= \beta R_1 u'(c_2) \\ u'(c_2) &= \beta R_2 u'(c_3) \end{aligned}$$

but we can put the second equation into the first (replace $u'(c_2)$ in the first equation with what the second equation has on the RHS) and then, again, we get Equation 2.

We will come back to this when we discuss bond pricing and the riskless term structure (or 'yield curve'). There are important implications for studying conventional and unconventional monetary policy. For now, think about what a 2-period riskless asset must offer as a return (from period 1 to 3) given the above EE was constructed imagining the household investing consecutively in two, 1-period assets. You might think that the two-period asset's return must be equal to the (in our case known) product of the two one-period returns - and you would be right!

2.5 Interpreting the Euler equation

Let's dig into the EE and see what it can tell us.

2.5.1 The effect of rate changes on consumption growth

If we rearrange the Euler equation we can get an insight into how interest rates might affect a person's consumption decisions.

$$\frac{C_2}{C_1} = (\beta R_1)^{\frac{1}{\gamma}}$$

where we used the result (see the math note) that $x^b = y$ implies $x = y^{\frac{1}{b}}$. If we then define $\sigma \equiv \gamma^{-1}$ and $G_2 \equiv \frac{C_2}{C_1}$ we can rewrite this as

$$G_2 = (\beta R_1)^{\sigma}$$

Using the result from the math note that the log of products equals the sum of logs, we get

$$\log G_2 = \log \beta^{\sigma} + \log R_1^{\sigma}$$

Using the fact (again see the math note) that 'exponents become coefficients' when one applies logs, we can rewrite this as

$$\log G_2 = \sigma \log \beta + \sigma \log R_1 \tag{3}$$

The gross growth rate of consumption, G_2, is related to the net growth rate, g_2 by $G_2 \equiv 1 + g_2$. The gross interest rate, similarly, is related to the net interest rate, r_1 by $R_1 \equiv 1 + r_1$. We recall that logs imply that, approximately, $\log 1 + x \approx x$ for small x. Thus for plausible net growth in consumption and for plausible real interest rates it is approximately the case that

$$
\begin{aligned}
g_2 &= \phi + \sigma \times r_1 \\
\phi &\equiv \sigma \log \beta
\end{aligned}
$$

So we have that variations in interest rates will induce a person to vary the growth rate of their consumption. The higher is R, all else equal, the faster is consumption growth. The lower is R, the slower is consumption growth. Why? Well the interest rate pins down the **relative price of consumption today vs consumption tomorrow**. If R is high the price of consumption today, relative to consumption tomorrow is high, so you optimally consume less today **relative to tomorrow**. Note the implication for monetary policy. If a monetary policymaker can influence the real rate of interest then she can influence the shape of the path of consumption (among other things).

What controls how strongly the rate changes influence g_2? The parameter σ (or γ^{-1}) determines the sensitivity. If σ is large then a small change in rates can cause a big change in consumption growth. If σ is small, then a large change in rates would be required to tilt consumption growth. With the preference structure we have chosen in this example, the 'intertemporal elasticity of substitution' is equal to σ. The intertemporal elasticity of substitution (IES) tells us how willing is a person to tolerate differences in consumption across different time periods.

Think about your own preferences. Suppose the interest rate is zero ($r_1 = 0$ or $R_1 = 1$), then the price of consumption today is the same as the price of consumption tomorrow. Now, we all prefer to have things now, rather than later ($\beta < 1$), so perhaps we will want a bit more consumption today than tomorrow, since we are deciding things today. However, it is natural to think that we don't want to consume *everything* today (exhausting our IBC) and then nothing tomorrow! That would be very unpleasant.[2]

In fact, the preferences we are working with - and the preferences that anyone would sensibly use to model people's behavior - encode a preference for smooth consumption paths. People with these preferences, once we allow for time preference (β), will prefer a path with relatively similar consumption (a 'flat' consumption profile) over time, rather than a path with a strong slope. However, suppose the interest rate increases? In that case, you are being paid, in a sense, to deviate from the 'flat' consumption profile and instead to raise c_2 relative to c_1. You like smooth consumption, yes, but if you're offered a killer return on savings, then that will induce you to respond. The better return is worth it. The more willing you are to substitute consumption across time (the higher is σ, in this case) the more you will

[2] Recall marginal utility is u' and $u'(c) = c^{-\sigma}$ so as $c \to 0$ marginal utility goes to ∞. That is, you are absolutely desperate to have *some* consumption.

respond to the price movements.

2.5.2 The effect of time preference on consumption growth

All else equal, the higher is β, the steeper will be the consumption profile - i.e. the higher will be consumption growth, or the higher will be c_2 relative to c_1. All else equal, a person will choose to save more in the first period, the more she values utility coming from the second period.

We should think of β as being between 0 and 1, but typically close to 1. Why do we think this? Well, we can see what interest rates are - perhaps $R \approx 1.01$ for an annual interest rate. And people seem to have pretty flat consumption, perhaps growing in real terms around 1% on average, but let's say $G_2 \approx 1$. Some people have done research on consumption responses to interest rate changes and maybe find $\sigma \approx 1$ (or possibly lower). Then the Euler equation will imply

$$1 \approx (\beta 1.01)^1$$

which suggests $\beta \approx 0.99$. Note that if $\sigma = 1$, then for a person to choose a flat consumption profile, the interest rate should be $R_1 = \beta^{-1}$. All of these insights are coming from the Euler equation.

What might lead to β increasing? Well, in our simple setup, we don't have a way to think about that as it's a fixed parameter. But in some sense β is capturing a deep desire to save. Anything that means that at a given interest rate, there is more desired saving, will have a similar impact (in some dimensions) as a higher β. We see that a higher β implies a lower interest rate is consistent with a given growth rate of consumption. Remember the debate (Tony mentioned it) about declining r^*. Well, some people point to increased desire for precautionary saving by Asian countries in response to the various financial and currency crises they had in the late 90s as one reason why global interest rates fell. Basically a lot of Asian countries wanted to build up reserves to defend their currencies.

Another interesting interpretation is that a drop in β might capture the fact that although the policymaker might be setting R, the rate that people face is actually R plus a spread demanded by banks. If the spread increases (perhaps because banks are scared of risk, or are weak themselves) then if our model does not explicitly allow for this, it will look a bit like β has increased - that people are becoming more patient, and not spending as much (at least relative to the future) as before. In fact, they're just facing a higher cost of today's consumption relative to tomorrow's. They aren't actually facing R and if our model isn't rich enough to have a banking sector, estimating our model might attribute a change in behavior to a change in β.

2.5.3 Consumption growth vs consumption

Does the Euler equation tell us what optimal consumption is? **No!** It tells us what optimal consumption *growth* is. We can't say, using the EE alone, what is happening to consumption in either period.

We know that when R_1 goes up, c_1 will decline relative to c_2 but we don't know if c_1 will actually decline. It might actually increase. Why? Well, suppose m_1 is very high and m_2 is very low. That is, the person receives lots of income today but hardly any tomorrow. Then she likely will be saving a lot in period 1 (remember, people like smooth consumption paths). This means that higher rates are 'good' for her - she is a saver - so she effectively gets a wealth boost and this leads her to increase consumption today and tomorrow, though she will increase her consumption tomorrow by proportionally more. The wealth effect strengthens the substitution effect of c_2 becoming relatively cheap, so we can be sure c_2 will go up. The wealth effect fights against the 'substitution' effect of c_1 becoming relatively expensive, which would otherwise push c_1 down, so that c_1 may actually go up. We can't make these predictions without knowing something about the agent's budget - the EE is not enough.

A way to see that the EE doesn't pin down consumption is to note that if a particular vector of consumption (not necessarily optimal) solves the EE, say (c_1^A, c_2^A) then so will any other profile that satisfies the same growth rate. For $\alpha > 0$ we (obviously) have

$$\frac{c_2^A}{c_1^A} = \frac{\alpha c_2^A}{\alpha c_1^A}$$

If α is close to zero then it implies a consumption plan with really low consumption. That's not optimal. If α is super high then it implies a crazy massive amount of consumption, which is not affordable. You can probably see that the 'right' α, consistent with the optimum, is the one that ensures the EE holds **and the person spends all of her lifetime income**. The extra condition that pins down the solution is provided by that requirement - spend all your income, but no more.

So let's combine the information from the EE with the information from the IBC.

$$\begin{aligned} c_2 &= (\beta R_1)^\sigma c_1 \\ c_2 &= m_2 + R_1(m_1 - c_1) \end{aligned}$$

so we have

$$(\beta R_1)^\sigma c_1 = m_2 + R_1(m_1 - c_1)$$

which we can rearrange to get

$$c_1 = \frac{(\beta R)^{-\sigma}(m_2 + R_1 m_1)}{1 + (\beta R_1)^{-\sigma} R_1} \tag{4}$$

This actually gives us the answer of 'what will the person consumer in period 1' (provided we know what her income is in the two periods and what the interest rate and preference parameters are).

2.5.4 Special case of log period utility

Let us examine a special case where we let $\sigma \to 1$. As discussed in an uploaded video, if someone with our u has a γ that is really close to 1 they effectively become someone whose $u(x) = \log x$ - which is known as having 'log utility'. Here we set $\sigma = 1$ and examine the optimal consumption.[3]

$$\begin{aligned} c_1 &= \frac{(\beta R_1)^{-1}(m_2 + R_1 m_1)}{1 + (\beta R_1)^{-1} R_1} \\ &= \frac{1}{1+\beta}(m_1 + \frac{1}{R_1} m_2) \end{aligned}$$

And what does that say? It says that the person will always consume in period 1, a fixed fraction of the PV of their lifetime income. The fraction is determined by β - the more patient you are (higher β) the lower is the fraction (and the higher will be the fraction devoted to c_2). Note that the second term on the RHS is the PV of lifetime income (discounted flow of income). That will change with R_1 and thus so will c_1 but consumption will change to maintain the same fraction. You can also see intuitively what would happen if $\beta = 1$ (you don't have any time preference). Then you would split your PV of lifetime income equally (the fraction becomes 1/2), which makes sense. Remember, though, we are dealing with a log consumer. These results do not hold for σ away from 1.

3 Linking with previous macro course

In your previous Monetary/Financial course you encountered (something like) the following equations:

$$\pi = \beta E[\pi] + \kappa(y - y^*) + \epsilon_{PC} \tag{5}$$

$$y - y^* = E[y - y^*] - \sigma(i - E[\pi]) + \epsilon_{IS} \tag{6}$$

$$i = a(\pi - \pi^*) + b(y - y^*) + \epsilon_{Pol} \tag{7}$$

[3]If you try to set $\gamma = 1$ in u something confusing will happen, though if you know L'Hopital's rule, you will see why log is the limiting case (and why our numerator in u has a -1.

with Equation 5 being the New Keynesian Phillips Curve, Equation 6 being an aggregate demand, or 'dynamic IS' curve (sometimes I might say 'schedule' instead of curve) and Equation 7 being a Taylor-type policy rule for the short rate.[4]

You should now begin to see where the dynamic IS curve comes from. The details aren't quite the same but you can see that it is in some sense derived from an Euler equation. In the DIS there are expectations (we can add those without changing really how the EE looks), and shocks (we could add a random β) and we have an output gap rather than consumption (this comes from approximation and from the idea that everyone's consumption added together must be connected with aggregate output - as they are consuming that output). But you can see that essentially it comes from the sort of stuff we have been discussing in this note for a single agent - some measure of activity (output or consumption) is related to its future value or expectation, and is connected to the real interest rate.

[4] The equations are simplified (note the absence of time subscripts) but gets across the points I want to make.