Pre-recording:
Tony's 3 equation NK model - in dynare

Rhys Bidder

KBS/QCGBF

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Disclaimer

The views expressed in this presentation, and all errors and omissions, should be regarded as those solely of the author, and are not necessarily those of the Bank of England or Qatar Central Bank.

Cast your mind back. . .

Cast your mind back...

- Cast your mind back to last term
- Tony introduced you to a three equation New Keynesian model

New Keynesian model: intuition in words

$\pi = \beta E(\pi) + \gamma(y - y^*) + shocks$

Inflation rises if expected inflation rises, or output rises above potential

$(y - y^*) = E(y - y^*) - \sigma(i - E(\pi)) + shocks$

Demand rises if demand if expected demand rises, or real interest rates fall

$i = a(\pi - \pi^*) + b(y - y^*)$

Policy maker raises interest rates if inflation rises above target, or output rises above potential. 'Taylor Rule'.

Refining Tony's 3-equation model

Minor changes. . .

- Small typo
- Changing notation
- Expressing all variables as deviations from steady state (or imagining SS is normalized to zero)

give us. . .

Reduced form 3-equation model

$$
\begin{aligned}
i_t &= \alpha_\pi \pi_t + \alpha_y y_t \\
y_t &= E_t[y_{t+1}] - \sigma(i_t - E_t[\pi_{t+1}]) + s_{IS,t} \\
\pi_t &= \beta E_t[\pi_{t+1}] + \kappa y_t + s_{PC,t}
\end{aligned}
$$

Taylor rule

$$i_t = \alpha_\pi \pi_t + \alpha_y y_t$$

Reduced form 3-equation model

Dynamic IS curve (Euler equation in disguise)

$$y_t = E_t[y_{t+1}] - \sigma(i_t - E_t[\pi_{t+1}]) + s_{IS,t}$$

Reduced form 3-equation model

New Keynesian Phillips Curve

$$\pi_t = \beta E_t[\pi_{t+1}] + \kappa y_t + s_{PC,t}$$

Reduced form 3-equation model

We will spend 3 weeks

- Explaining where the NKPC comes from
- Discussing (briefly) how we might end up with linear equations
- Solving this and related models in dynare
- Exploring how a NK model behaves, using the solution in Matlab

Ambitious, but doable - and it will transform your understanding of how modern macroeconomists work (or any economists, in many respects)

Equilibrium conditions

Equilibrium conditions

An equilibrium of a model (if it exists) is defined by a set of equations:

- Household optimality
- Firm optimality
- Market clearing
- Policy rules
- Technological and accounting constraints
- Shock processes

and some underlying distributional assumptions on innovations to shock processes

For an outstanding (though advanced) set of lecture notes on this, see Jesus F-V's teaching page (lectures 1-3 of 'Solving and Estimating Dynamic Equilibrium Models')

Equilibrium conditions

For now, we will assume (remember your econometrics class!) that $s_{PC,t}$ and $s_{IS,t}$ follow independent AR(1) processes with Gaussian (Normal) innovations

$$\begin{aligned} s_{k,t} &= \rho_k s_{k,t-1} + \epsilon_{k,t} \\ \epsilon_{k,t} &\sim N(0, \sigma_k^2) \end{aligned}$$

for $k \in \{PC, IS\}$

What are these shocks? We will discuss further but...

- IS: confidence, lower spreads offered by (unmodeled) banks on loans?
- PC: oil price shock, supply chain disruptions?
- $\epsilon_{k,t}$ are the 'news' or 'innovations' at t, with the 'shock' process $s_{k,t}$ exhibiting persistence (assume $\rho_k \in (0.1)$)

Equilibrium conditions vs Model solution

The three earlier equations (and the assumed exogenous shock processes) define the equilibrium

- In equilibrium, all those equations and assumptions must always be holdings, at all times and in all contingencies

But they do not constitute the *solution* of the model

- The solution is implicit in the equations
- The solution (in this case) is: **a set of functions that relate the endogenous variables to the state in such a way that the equilibrium conditions are (always and everywhere) satisfied**

WHAT?

Model solution

Model solution

Any of you who have studied dynamic systems in math (differential/difference equations) will already have the basic intuition

- You are given a differential equation describing how a variable, say, x_t, evolves over time
- But to **solve** it, you find an expression for x_t as a function of time (and other stuff, maybe. . .)
- If x_t given by that expression, then it will satisfy (solve) the original differential equation

Model solution - Simple example

Let's take an even simpler example and consider the following system of linear equations in two variables, y and π

$$\begin{aligned} y &= m_1 - 4\pi \\ y &= m_2 + 2\pi \end{aligned}$$

With basic matrix algebra we can express this as

$$Ax = b$$

where $x \equiv (y, \pi)'$, $b \equiv (m_1, m_2)'$ and

$$A = \begin{pmatrix} 1 & 4 \\ 1 & -1 \end{pmatrix}$$

See earlier notes and here for matrix-vector multiplication (to convince yourself the matrix equation is equivalent to the system of linear equations)

Model solution - Simple example

The solution then is

$$\begin{pmatrix} y \\ \pi \end{pmatrix} = A^{-1}b$$

Let's make this more concrete by assuming $m_1 = 5$ and $m_2 = -3$, then we have (**do this**)

$$\begin{pmatrix} y \\ \pi \end{pmatrix} = \begin{pmatrix} -1/3 \\ 4/3 \end{pmatrix}$$

If you're not convinced, put these values in the equations and see that they hold

- 5 - 4* $4/3 =$ -$1/3$
- -3 + 2*$4/3 =$ -$1/3$

Model solution - Simple example

Of course, geometrically (thinking in terms of pictures of two lines defined by the two equations), the solution is the values of y and π where the lines intersect (**draw the lines**)

Model solution - Less simple example

Now, here the solution was just a single vector (a single value (each) for y and π solved the system):

$$\begin{pmatrix} -1/3 \\ 4/3 \end{pmatrix}$$

But what if m_1 and m_2 took on different values depending on 'the weather' or the 'state of the world', s

- $\text{s} = 0$ (good weather) $\Rightarrow (m1, m2) = (5, -3)$
- $\text{s} = 1$ (bad weather) $\Rightarrow (m1, m2) = (6, -2)$

Then in good weather we have (as before) $(y, \pi) = (-1/3, 4/3)$ but in bad weather we have $(y, \pi) = (2/3, 4/3)$ (**show this**)

- **The solution is not a number, but it is a rule (indeed, a function) that maps from s to a number**

$$x = f(s)$$

Model solution - Transferring our intuition

Our system is a bit more complicated

$$
\begin{aligned}
i_t &= \alpha_\pi \pi_t + \alpha_y y_t \\
y_t &= E_t[y_{t+1}] - \sigma(i_t - E_t[\pi_{t+1}]) + s_{IS,t} \\
\pi_t &= \beta E_t[\pi_{t+1}] + \kappa y_t + s_{PC,t}
\end{aligned}
$$

But the basic thrust goes through: we are looking for **functions** (not just single numbers) that give us i, y and π that satisfy the equilibrium conditions, given a state of the world

- In this case the 'state of the world' is given by $s_t = (s_{PC,t}, s_{IS,t})$
  - Trust me?
- We could also say the state in t is $s_t = (s_{PC,t-1}, s_{IS,t-1}, \epsilon_{PC,t}, \epsilon_{IS,t})$
  - **Why?** (don't overthink it - remember the definition of an AR(1))
  - Side note: This is how dynare 'represents the state'
- **Dynare gives us those functions**

Model solution - Transferring our intuition

Dynare (**in this case**) gives us f_y, f_π and f_i such that

$$\begin{aligned}
f_i(s_t) &= \alpha_\pi f_\pi(s_t) + \alpha_y f_y(s_t) \\
f_y(s_t) &= E_t[f_y(s_{t+1})] - \sigma(f_i(s_t) - E_t[f_\pi(s_{t+1})]) + s_{IS,t} \\
f_\pi(s_t) &= \beta E_t[f_\pi(s_{t+1})] + \kappa f_y(s_t) + s_{PC,t}
\end{aligned}$$

More generally (if the system were nonlinear), it will only be able to give **approximations** to these functions

- Since we 'started' with a linear model (**where did that come from?**) it turns out that the functions have a simple (linear) form
- Thus, dynare gives them to us exactly
- Note that the function for i_t is trivial, given f_y and f_π (**why?**)

Model solution - Coefficients dynare gives us

In this case (I have uploaded code) dynare gives us coefficients $c_{j,k}$ for $j \in \{i, y, \pi\}$ and k corresponding to elements of the state, such that

$$
\begin{aligned}
i_t = f_i(s_t) &\equiv c_{i,1}s_{PC,t-1} + c_{i,2}s_{IS,t-1} + c_{i,3}\epsilon_{PC,t} + c_{i,4}\epsilon_{IS,t} \\
y_t = f_y(s_t) &\equiv c_{y,1}s_{PC,t-1} + c_{y,2}s_{IS,t-1} + c_{y,3}\epsilon_{PC,t} + c_{y,4}\epsilon_{IS,t} \\
\pi_t = f_\pi(s_t) &\equiv c_{\pi,1}s_{PC,t-1} + c_{\pi,2}s_{IS,t-1} + c_{\pi,3}\epsilon_{PC,t} + c_{\pi,4}\epsilon_{IS,t}
\end{aligned}
$$

which we can write as

$$v_t = Cs_t$$

where v_t is a vector containing the endogenous variables. That is, we get the equilibrium *solved* values by a simple matrix multiplication of C and a time varying state

- Essentially, C is the **solution of the model**
- So, in Matlab, build the coefficient matrix C and apply it to the values of the AR(1)s

Model solution - Coefficients obtained manually

As aforementioned, dynare likes to represent the state as

$$s_t = (s_{PC,t-1}, s_{IS,t-1}, \epsilon_{PC,t}, \epsilon_{IS,t})$$

But, I prefer to think of the state as

$$s_t = (s_{PC,t}, s_{IS,t})$$

Note that both contain the same information, but mine is more parsimonious (and simpler to handle if we are using 'pen and paper' math)

Model solution - Coefficients obtained manually

So,the 'solution' given my preferred representation of the state will be 6 coefficients (2 for each of y, π and i) such that

$$\begin{aligned} y_t &= c_{y,PC} s_{PC,t} + c_{y,IS} s_{IS,t} \\ \pi_t &= c_{\pi,PC} s_{PC,t} + c_{\pi,IS} s_{IS,t} \\ i_t &= c_{i,PC} s_{PC,t} + c_{i,IS} s_{IS,t} \end{aligned}$$

where we need to find what are the $c_{k,PC}$ and $c_{k,IS}$ (for $k \in \{y, \pi, i\}$)

Model solution - Coefficients obtained manually

So,the 'solution' given my preferred representation of the state will be 6 coefficients (2 for each of y, π and i) such that

$$
\begin{aligned}
y_t &= c_{y,PC} s_{PC,t} + c_{y,IS} s_{IS,t} \\
\pi_t &= c_{\pi,PC} s_{PC,t} + c_{\pi,IS} s_{IS,t} \\
i_t &= c_{i,PC} s_{PC,t} + c_{i,IS} s_{IS,t}
\end{aligned}
$$

where we need to find what are the $c_{k,PC}$ and $c_{k,IS}$ (for $k \in \{y, \pi, i\}$)

Chill. Out.

- This isn't that hard - it's just unfamiliar

All we are saying is:

- There are 'rules' to how variables in the economy vary over time
- The variables are functions of the state of the economy
- The state (here) is captured simply by the shock processes
- If you tell me the state, I can tell you what values all the variables take
- I do this by multiplying the states by the relevant coefficients
- The states and coeffs are just 'numbers'
- **You know how to multiply and add numbers!**

Model solution - Coefficients obtained manually

But how do we get the coefficients?

- We have equilibrium conditions
  - These conditions are satisfied by the policy functions (the f_k)
  - f_k simply imply multiplying states with coefficients
  - So we can stick these into the equilibrium conditions
- Sometimes, the equilibrium conditions involve y_{t+1}
  - **Panic?**
  - No. $y_{t+1} = f_y(s_{t+1})$ in the same way that $y_t = f_y(s_t)$
  - Remember, we are seeking constant *functions* (f_y etc.) not constant values of the variables (y)
  - **You know this, intuitively - the solution to a model with randomness is not going to be a world of constant variables!**
- We will end up with expectations of (linear functions of) s_{t+1}
  - **Panic?**
  - No. $E_t[s_{k,t+1}] = \rho_k s_{k,t}$ for $k \in \{PC, IS\}$
  - **Remember AR(1) properties from econometrics class**

Model solution - Coefficients obtained manually

After all that, we can write (not yet dealing with the $E_t[\cdot]$):

$$
\begin{aligned}
c_{i,PC} s_{PC,t} + c_{i,IS} s_{IS,t} &= \alpha_\pi \left(c_{\pi,PC} s_{PC,t} + c_{\pi,IS} s_{IS,t} \right) \\
&+ \alpha_y \left(c_{y,PC} s_{PC,t} + c_{y,IS} s_{IS,t} \right) \\
c_{y,PC} s_{PC,t} + c_{y,IS} s_{IS,t} &= E_t[c_{y,PC} s_{PC,t+1} + c_{y,IS} s_{IS,t+1}] \\
&- \sigma \big(c_{i,PC} s_{PC,t} + c_{i,IS} s_{IS,t} \\
&- E_t[c_{\pi,PC} s_{PC,t+1} + c_{\pi,IS} s_{IS,t+1}] \big) + s_{IS,t} \\
c_{\pi,PC} s_{PC,t} + c_{\pi,IS} s_{IS,t} &= \beta E_t[c_{\pi,PC} s_{PC,t+1} + c_{\pi,IS} s_{IS,t+1}] \\
&+ \kappa \left(c_{y,PC} s_{PC,t} + c_{y,IS} s_{IS,t} \right) + s_{PC,t}
\end{aligned}
$$

Model solution - Coefficients obtained manually

But remembering what AR(1) conditional expectations are, we thus have

$$
\begin{aligned}
c_{i,PC}s_{PC,t} + c_{i,IS}s_{IS,t} &= \alpha_\pi(c_{\pi,PC}s_{PC,t} + c_{\pi,IS}s_{IS,t}) \\
&+ \alpha_y(c_{y,PC}s_{PC,t} + c_{y,IS}s_{IS,t}) \\
c_{y,PC}s_{PC,t} + c_{y,IS}s_{IS,t} &= c_{y,PC}\rho_{PC}s_{PC,t} + c_{y,IS}\rho_{IS}s_{IS,t} \\
&- \sigma(c_{i,PC}s_{PC,t} + c_{i,IS}s_{IS,t} \\
&- c_{\pi,PC}\rho_{PC}s_{PC,t} - c_{\pi,IS}\rho_{IS}s_{IS,t}) + s_{IS,t} \\
c_{\pi,PC}s_{PC,t} + c_{\pi,IS}s_{IS,t} &= \beta(c_{\pi,PC}\rho_{PC}s_{PC,t} + c_{\pi,IS}\rho_{IS}s_{IS,t}) \\
&+ \kappa(c_{y,PC}s_{PC,t} + c_{y,IS}s_{IS,t}) + s_{PC,t}
\end{aligned}
$$

So all those nasty expectations are 'gone'

- Now we're back in high school math
- Conceptually, what remains is easy (though the expressions look ugly - that's why we use matrices!)

Model solution - Coefficients obtained manually

$$
\begin{aligned}
c_{i,PC} s_{PC,t} + c_{i,IS} s_{IS,t} &= \alpha_\pi \left(c_{\pi,PC} s_{PC,t} + c_{\pi,IS} s_{IS,t} \right) \\
&+ \alpha_y \left(c_{y,PC} s_{PC,t} + c_{y,IS} s_{IS,t} \right) \\
c_{y,PC} s_{PC,t} + c_{y,IS} s_{IS,t} &= c_{y,PC} \rho_{PC} s_{PC,t} + c_{y,IS} \rho_{IS} s_{IS,t} \\
&- \sigma \Big(c_{i,PC} s_{PC,t} + c_{i,IS} s_{IS,t} \\
&- c_{\pi,PC} \rho_{PC} s_{PC,t} - c_{\pi,IS} \rho_{IS} s_{IS,t} \Big) + s_{IS,t} \\
c_{\pi,PC} s_{PC,t} + c_{\pi,IS} s_{IS,t} &= \beta \left(c_{\pi,PC} \rho_{PC} s_{PC,t} + c_{\pi,IS} \rho_{IS} s_{IS,t} \right) \\
&+ \kappa \left(c_{y,PC} s_{PC,t} + c_{y,IS} s_{IS,t} \right) + s_{PC,t}
\end{aligned}
$$

This system involves

- Numbers I know (parameters)
- Numbers I want (solution coefficients)
- States in time t ($s_{PC,t}$ and $s_{IS,t}$)

Model solution - Coefficients obtained manually

If you look closely, you can see that on the LHS and RHS of each of the three equations, we have the sum of two terms:

- Something $\times$ $s_{PC,t}$ plus something $\times$ s_{IS}
- On the LHS, the somethings are the coefficients we want to solve for
- On the RHS, once we have collected terms multiplying the states, we have a mixture of the coefficients we want, combined with some model parameters

Since these equations must hold for *any* values of $s_{PC,t}$ and $s_{IS,t}$, they must hold one one is 1 and the other is 0 (and *vice versa*)

- **This means that the coefficient on $s_{PC,t}$ on the RHS must equal the coefficient on $s_{PC,t}$ on the RHS (and similarly for $s_{IS,t}$)**

Model solution - Coefficients obtained manually

This gives us a another system of equations, which is the final system we need to solve (thank heavens...)

$$
\begin{aligned}
c_{i,PC} &= \alpha_\pi c_{\pi,PC} + \alpha_y c_{y,PC} && (1)\\
c_{i,IS} &= \alpha_\pi c_{\pi,IS} + \alpha_y c_{y,IS} && (2)\\
c_{y,PC} &= c_{y,PC}\rho_{PC} - \sigma c_{i,PC} + c_{\pi,PC}\rho_{PC} && (3)\\
c_{y,IS} &= c_{y,IS}\rho_{IS} - \sigma c_{i,IS} + c_{\pi,IS}\rho_{IS} + 1 && (4)\\
c_{\pi,PC} &= c_{\pi,PC}\beta\rho_{PC} + \kappa c_{y,PC} + 1 && (5)\\
c_{\pi,IS} &= c_{\pi,IS}\beta\rho_{IS} + \kappa c_{y,IS} && (6)
\end{aligned}
$$

Equations (1)-(6) are a linear system of the form $Ax = b$ where

$$x = (c_{i,PC}, c_{i,IS}, c_{y,PC}, c_{y,IS}, c_{\pi,PC}, c_{\pi,IS})'$$

and the entries of A are a mixture of parameters and b is a few numbers

- **Can you derive this?**
- *Hint:* Rearrange the equations above

Model solution - Coefficients obtained manually

Thus $b = (0, 0, 0, -1, -1, 0)'$ and

$$A = \begin{pmatrix} -1 & 0 & \alpha_y & 0 & \alpha_\pi & 0 \\ 0 & -1 & 0 & \alpha_y & 0 & \alpha_\pi \\ -\sigma & 0 & \rho_{PC} - 1 & 0 & \rho_{PC} & 0 \\ 0 & -\sigma & 0 & \rho_{IS} - 1 & 0 & \rho_{IS} \\ 0 & 0 & \kappa & 0 & \beta\rho_{PC} - 1 & 0 \\ 0 & 0 & 0 & \kappa & 0 & \beta\rho_{IS} - 1 \end{pmatrix}$$

These entries are just numbers (once we assign numbers to the parameters)

Model solution - Coefficients obtained manually

Once you have A and b (and for a given set of parameter values, we know both) then we get coefficients as

$$x = A^{-1}b$$

and the model is solved (and that's why we use dynare - because this is **tedious and error-prone**)