Advanced Monetary Policy

Technical note

The Euler equation and the real term structure

rhys.m.bidder@kcl.ac.uk

Spring 2024

Abstract

This note starts to consider asset pricing with *more than two periods* - with the aim of deriving and understanding the real term structure of riskless bonds.

1 Introduction

We will often discuss topics in which bonds at various horizons, and expectations of future monetary policy actions (and thus interest rates) must be considered. While our two-period models will continue (perhaps surprisingly) to be very useful - thanks to the power of recursion - we will want to consider assets whose payoffs are in the more distant future. Eventually we will derive the nominal term structure - the 'yield curve' that is frequently discussed in monetary policy circles. This note, however, will focus on purely real variables. However, as you should have noted in the other technical note on pricing nominal payoffs, there isn't really that much different *conceptually* when we introduce nominal variables. Much of the intuition we get from the real term structure will go through, but with nominal SDFs floating around in our equations, rather than real!

2 Zero coupon bonds of different maturities

Recall that there is a fundamental connection between prices and returns. It is probably easier to begin our analysis thinking in terms of the *prices* or riskless bonds, rather than their interest rates or returns. Again, it is useful to think in terms of 'zero coupon bonds' that pay nothing until maturity, at which point they pay off a single unit of consumption. By buying one of these today you are effectively buying future consumption (right?).

2.1 1 and 2 period bond maturities

Given our earlier notes and discussions, we know that the price of a one period riskless bond is

$$Q_1^{(1)} = E_1\left[\Lambda_{1,2}\right] \tag{1}$$

One way to see this in a slightly different light is to make the unit (i.e. '1') payoff of consumption explicit. . .

$$Q_1^{(1)} = E_1\left[\Lambda_{1,2} \times 1\right]$$

Note the time subscript on the price (when the price is prevailing, i.e. today, period 1), the bracketed maturity superscript (how far in the future the bond pays off), the information timing subscript on the expectations operator (I am forming expectations conditional on period 1 information), and the subscripts on the SDF (I am discounting period 2 payoffs back to period 1).

Ok - so what about a *two period* riskless zero coupon bond? That is, one that pays off a unit of consumption in period 3, as perceived from 'today' (period 1). What is its price today? Well, the price today should be (by an arbitrage assumption) the price of having a *one period* bond *tomorrow* - because that is what effectively I will have. Right? Think about it: I get no payoff tomorrow, other than my ownership of the bond, which in one more period will give me a payoff of 1. What is the price of a one period bond tomorrow? Well that would be pretty much as in Equation 1, but from the perspective of tomorrow (period 2):

$$Q_2^{(1)} = E_2 \left[\Lambda_{2,3}\right]$$

Note the time subscript on the price (when the price is prevailing, i.e. tomorrow, period 2), the bracketed maturity superscript (how far in the future the bond pays off), the information timing subscript on the expectations operator (I am forming expectations conditional on period 2 information), and the subscripts on the SDF (I am discounting period 3 realizations back to period 2). So we still have a '1-period ahead' SDF, but it's from the perspective of tomorrow, into the day after that. So the value today of buying a two period bond should be the value of pricing something that gives me $Q_2^{(1)}$ tomorrow:

$$Q_1^{(2)} = E_1 \left[\Lambda_{1,2} Q_2^{(1)}\right]$$

Again, be careful with notation - there are a lot of bits and pieces floating around. $Q_1^{(2)}$ is the value today of a two period bond. $Q_2^{(1)}$ is the value tomorrow of a one period bond.[1]

2.2 Longer bond maturities

We can get a long way by just stitching together a sequence of 'two-period' analyses. Let's expand on that by expressing $Q_2^{(1)}$ more explicitly.

$$Q_1^{(2)} = E_1 \left[\Lambda_{1,2} E_2 \left[\Lambda_{2,3}\right]\right]$$

What about that E_2 thing? No problem - it turns out that if you take an expectation with worse info of an expectation with better info, you just get back the expectation with worse info! This is the 'law of iterated expectations' and in our case, it just means that the expectation today of tomorrow's expectation of something is... my expectation today of that something. Yes I will have more info tomorrow on which basis to refine my expectation, but by definition I don't have that info yet - so unless I somehow *weirdly* am expecting to change my expectation (!) I'm just left with the expectations I can form today. **Long story short...** for a random variable X we have $E_1[E_2[X]] = E_1[X]$. As such, we have[2]

$$Q_1^{(2)} = E_1 \left[\Lambda_{1,2} \times \Lambda_{2,3}\right]$$

[1] We don't have the time or energy to get into arbitrage discussions but I am using 'value' and 'price' pretty much interchangeably. Ask yourself why prices should align with the SDF-discounted payoffs or the 'value'...

[2] Note that since $\Lambda_{1,2}$ will be known tomorrow, we can pull it in and out of the E_2 expectation, so $\Lambda_{1,2}E_2[\Lambda_{2,3}] = E_2[\Lambda_{1,2}\Lambda_{2,3}]$.

Now, suppose we want to price a zero coupon bond with a maturity of *three* periods, then you should be able to show, by the same logic

$$Q_1^{(3)} = E_1 \left[\Lambda_{1,2} \times \Lambda_{2,3} \times \Lambda_{3,4}\right]$$

In fact we can write, for maturity τ

$$\begin{aligned} Q_1^{(\tau)} &= E_1 \left[\Lambda_{1,2} \times \Lambda_{2,3} \times \ldots \times \Lambda_{\tau,\tau+1}\right] \\ &\equiv E_1 \left[\prod_{k=1}^{\tau} \Lambda_{k,k+1}\right] \end{aligned}$$

where the second line uses the definition of the 'product' symbol, $\prod$, which means my hand doesn't have to get tired writing out a sequence of $\times$ as in the first line.

There a few ways to look at this expression. First, we see that effectively we are defining a multi step SDF. So far we have worked with a one-step SDF, but effectively $\prod_{k=1}^{\tau} \Lambda_{k,k+1}$ is being used to discount (applying time preference and marginal utility tweaks) payoffs in $\tau + 1$ from the perspective of today, period 1. That is we could define a multi-step SDF, call it $\Lambda_1^{(\tau)}$ (as it is pricing τ periods ahead, from the perspective of 'today', time 1) by[3]

$$\Lambda_1^{(\tau)} \equiv \Lambda_1^{(\tau-1)} \Lambda_{1,2}$$

where the recursion begins with noting (as if $\Lambda_1^{(0)} = 1$)

$$\Lambda_1^{(1)} \equiv \Lambda_{1,2}$$

then

$$\Lambda_1^{(2)} \equiv \Lambda_1^{(1)} \Lambda_{1,2}$$

then

$$\Lambda_1^{(3)} \equiv \Lambda_1^{(2)} \Lambda_{1,2}$$

and so on...

This holds for whatever model for $\Lambda_{1,2}$ you may have, but under our particular preference assumptions (in particular, expected time-separable discounted utility - at this point we don't need to say what the form of u is though) we have

$$\Lambda_{t,t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

[3]We are using different notation as $\Lambda_1^{(\tau)}$ is a different object from the 1-step $\Lambda_{1,2}$ that we have worked with previously.

which means that[4]

$$
\begin{aligned}
Q_1^{(\tau)} &= E_1\left[\beta\frac{u'(c_2)}{u'(c_1)}\times\beta\frac{u'(c_3)}{u'(c_2)}\times\ldots\times\beta\frac{u'(c_{\tau+1})}{u'(c_\tau)}\right] \\
&\equiv \beta^\tau E_1\left[\prod_{k=1}^{\tau}\frac{u'(c_{k+1})}{u'(c_k)}\right] \\
&\equiv \beta^\tau E_1\left[\frac{u'(c_{\tau+1})}{u'(c_1)}\right]
\end{aligned}
$$

So we come back, **again**, to our nice marginal benefit = marginal benefit intuition (these are all just Euler equations). Why? Well rearrange...

$$Q_1^{(\tau)}u'(c_1) = \beta^\tau E_1[u'(c_{\tau+1})]$$

The marginal cost of buying (in real terms) a unit of consumption in the τ-distant future (the LHS) should be equal to the marginal benefit, which is the expected marginal utility in the future, discounted - not by β which would be the case for only one step in the future - but by β^τ (the RHS). We are talking about a unit of consumption τ periods into the future, so it is even more heavily discounted than in the 1-period case ($0 < \beta < 1$ implies $\beta^k < \beta$ for $k > 1$).

So, after all this, we have expressions for bond prices today (period 1) of arbitrary maturity. These show how future consumption is valued, from the perspective of today. Of course, we could make the notation more general and think about prices prevailing in an arbitrary t (rather than 'period 1'):

$$Q_t^{(\tau)} = E_t\left[\Lambda_{t,t+1}Q_t^{(\tau-1)}\right] = E_t\left[\Lambda_t^{(\tau)}\right] = \beta^\tau E_t\left[\prod_{k=1}^{\tau}\frac{u'(c_{t+\tau})}{u'(c_t)}\right] = \beta^\tau E_t\left[\frac{u'(c_{t+\tau})}{u'(c_t)}\right]$$

The 'yield to maturity', y_t^τ is defined by $(1+y_t^{(\tau)})^\tau = \frac{1}{Q_t^{(\tau)}}$ which implies

$$(Q_t^{(\tau)})^{-\frac{1}{\tau}} = 1 + y_t^{(\tau)}$$

and thus, approximately, we have

$$y_t^{(\tau)} = -\frac{1}{\tau}\log Q_t^{(\tau)}$$

Wait. Stop. What? The definition of the yield to maturity (YTM) was a bit funky, so let's take a step back. The return from today to τ periods in the future from holding the bond is $\frac{1}{Q_t^{(\tau)}}$. Now, imagine what 'one period return' *compounded over the τ horizon* would give that multi-period return. That's what this says (i.e. YTM is

[4]What happens in the last line here? It's called a telescopic product. Try writing this out explicitly for $\tau = 3$ and see how the numerators and denominators cancel with eachother, leaving only the 'first and last' marginal utilities.

that return):

$$(1 + y_t^{(\tau)})^\tau = \frac{1}{Q_t^{(\tau)}}$$

where the LHS is what you get by compounding the YTM as a constant gross return each period and the RHS is the return over the maturity horizon from buying the bond at $Q_t^{(\tau)}$ and getting a riskless payoff of 1 in τ periods' time. It's like re-expressing the longer horizon return in one-period equivalents. Why might that be useful? Well if you want a way to refer to long maturity rates in a way that is still in about the same scale as short maturity rates, then it's useful to convert to YTM form. The log stuff in the last few steps is again simply applying the log tricks from the math note. At issuance, the YTM gives us the implied interest rate on a long maturity bond, in a way that is comparable to rates on bonds of other maturities.

The relationship between $y_t^{(\tau)}$ and τ (or between $Q_t^{(\tau)}$ and τ) is known as the term structure of riskless real zero coupon bonds. You can move between speaking $y_t^{(\tau)}$ language or $Q_t^{(\tau)}$ language (each contains equivalent information, but represented differently).[5] If you plot $y_t^{(\tau)}$ against τ you have the 'yield curve'.

Notice the important difference between τ, which is used to index the cross section of **maturities**, and t, which is the time at which you are observing the yields, $y_t^{(\tau)}$ (or the prices $Q_t^{(\tau)}$). It *might* be interesting in some applications/debates to plot the evolution of yields at a particular τ over time, but that's not the yield curve. The yield curve comes from fixing the time period and considering how bonds of **multiple maturities** are priced **in that particular time period**.

2.3 Forwards/futures

You will see references to futures in a lot of monetary policy analysis (academic and policy). For example, people look at changes in futures prices either side of FOMC announcements to assess how much the market has been surprised and how much they are revised their 'view' of the future path of interest rates. They can in some sense (see below and also see here) be interpreted as expectations of future interest rates.[6]

The futures rate is defined as

$$f_t^{(\tau)} \equiv \log Q_t^{(\tau-1)} - \log Q_t^{(\tau)} \tag{2}$$

We see that $F_t^{(1)} = -\log Q_t^{(1)}$, because the price of a 'zero maturity bond' (i.e. consumption today), is 1 as a unit of consumption has real price 1, by definition. Put that into Equation 2 above and you get the desired result (remember $\log 1 = 0$ from the log tricks in the math note). But what is $-\log Q_t^{(1)}$? That is (again see note) $\log \frac{1}{Q_t^{(1)}}$ or $\log R_t$ (remember the definition of a riskless rate) and thus, $f_t^{(1)} \approx r_t$ (where $R_t \equiv 1 + r_t$ and again we use the log tricks/approximations).

[5] Are all bonds zero coupon? No, but you can imagine other riskless bonds, that do may coupons in periods before maturity as being made up of a particular set of zero coupon bonds, and - by arbitrage - infer their price also.

[6] See here for the relationship between forwards and futures - for our purposes the distinction is not important.

Being a little bit loose (this would be precise in continuous time), we have $f_t^{(1)} = y_t^{(1)} = r_t$. The short rate is equal to the 1-period yield to maturity, which is equal to the 1-period future's rate. The current rate pins down the short maturity part of the yield curve. We can think of the policymaker setting r_t, or at least influencing it by setting nominal short rates in a world with sticky prices.[7]

2.4 Taking stock

We have considered bonds that pay a unit of consumption for sure, τ periods in the future - what we call 'zero coupon bonds' of maturity τ. They have prices $Q_t^{(\tau)}$, which are tied to yields to maturity, $y_t^{(\tau)}$, which are in turn connected to futures rates, $f_t^{(\tau)}$. On the latter point, we see that $y_t^{(\tau)}$ is an average of futures rates relating to intermediate maturities. Specifically, we have

$$y_t^{(\tau)} = \frac{f_t^{(1)} + f_t^{(2)} + \ldots + f_t^{(\tau)}}{\tau}$$

Using the various definitions and expressions above, we can move between bond prices, yields and futures, depending on which is most relevant for our work. They are all connected - and ultimately - they all can be traced back to the behavior of the short rate, r_t and (suitably constructed) expectations of the short rate. The uploaded note from David Backus goes into this in more detail (concentrate on the early parts) and (much more difficult) a couple of nice references where these ideas are applied are here, here and here.

2.5 Expectations of future short rates

Why do people talk about the yield curve, bond prices or futures reflecting expectations of r_t in the future? To show this clearly, we would need to go beyond the scope of this course, technically, but for now, let us first eliminate risk. As such, we can drop all the E_t operators an manipulate equations in a simple way. We will also focus on the minimal case that makes the point - we will show that $f_t^{(2)}$ is equal to the future short rate prevailing in $t + 1$ (i.e. the riskless rate from $t + 1$ to $t + 2$).

[7] For the current short rate, r_t, we don't bother with a subscript relating to maturity, even though - as discussed repeatedly in previous notes - the riskless return is implicitly from today to tomorrow, as if there is maturity$= 1$. When we introduced yield to maturity we were thinking very explicitly in terms of returns over multiple periods, so there we bothered to include a maturity superscript. Same for futures. Warning: My futures notation deviates from the uploaded Backus note. My $f_t^{(\tau)}$ is his $f_t^{(\tau-1)}$.

If we had risk we would have

$$\begin{aligned}
f_t^{(2)} &\equiv \log Q_t^{(1)} - \log Q_t^{(2)} \\
&= \log E_t[\Lambda_{t,t+1}] - \log E_t[\Lambda_{t,t+1} Q_{t+1}^{(1)}] \\
&= \log E_t[\Lambda_{t,t+1}] - \log E_t[\Lambda_{t,t+1} E_{t+1}[\Lambda_{t+1,t+2}]] \qquad (3) \\
&= \log E_t[\Lambda_{t,t+1}] - \log E_t[\Lambda_{t,t+1} \Lambda_{t+1,t+2}]
\end{aligned}$$

Without risk we can write (since the SDFs are known in advance)

$$\begin{aligned}
f_t^{(2)} &= \log \Lambda_{t,t+1} - \log \Lambda_{t,t+1} \Lambda_{t+1,t+2} \\
&= \log \Lambda_{t,t+1} - \log \Lambda_{t,t+1} - \log \Lambda_{t+1,t+2} \\
&= -\log \Lambda_{t+1,t+2} \\
&= \log \frac{1}{\Lambda_{t+1,t+2}} \\
&= \log R_{t+1} \\
&\approx r_{t+1}
\end{aligned}$$

That is, the futures price with $\tau = 1$ in t gives us the short rate that will prevail in $t+1$. Trivially, since there is no risk, this is also the 'expectation' of the short rate in $t+1$. You should also see (convince yourself) that y_t^2 is the average of expected short rates (the average of r_t and r_{t+1}). Futures show you the rate starting at a point in the future, yields show you the average of rates up to a point in the future.

Now, let us re-introduce risk, and see how far we get.[8] We return to one of the intermediate steps above (Equation 3), which is

$$\begin{aligned}
f_t^{(2)} &= \log E_t[\Lambda_{t,t+1}] - \log E_t[\Lambda_{t,t+1} E_{t+1}[\Lambda_{t+1,t+2}]] \\
&= -\log R_t - \log E_t\left[\frac{\Lambda_{t,t+1}}{E_t[\Lambda_{t,t+1}]} E_t[\Lambda_{t,t+1}] E_{t+1}[\Lambda_{t+1,t+2}]\right] \\
&= -\log R_t - \log E_t\left[\frac{\Lambda_{t,t+1}}{E_t[\Lambda_{t,t+1}]} E_t[\Lambda_{t,t+1}] R_{t+1}^{-1}\right] \\
&= -\log E_t\left[\frac{\Lambda_{t,t+1}}{E_t[\Lambda_{t,t+1}]} R_{t+1}^{-1}\right]
\end{aligned}$$

As this can be very confusing the first time you see it, let me be explicit about each of the steps above (which are just manipulation in the end...):

[8] In fact, if you look at the expressions hard enough, and ignore the details, you can see the same basic pattern emerges but with adjustments for marginal utility (the stochastic discount factor messes with expectations) and Jensen's inequality.

- How did we get from the first to the second line?
  - We multiplied and divided inside the second brackets by $E_t[\Lambda_{t,t+1}]$
- How did we get from the second to the third line?
  - Definition of the riskless real rate (inverse of the expected SDF) applied to the rate prevailing in $t+1$
- How did we get from the third to the fourth line?
  - Definition of the riskless real rate (inverse of the expected SDF) applied to the rate prevailing in t

Ok, so now we reach something a bit deep. For a start, let's look at this slightly odd term, which I will call $\mathcal{T}_{t,t+1}$:

$$\mathcal{T}_{t,t+1} \equiv \frac{\Lambda_{t,t+1}}{E_t[\Lambda_{t,t+1}]}$$

2.6 Digression on risk neutral probabilities

What a strange thing $\mathcal{T}_{t,t+1}$ is! What are its properties?

- It is stochastic (random) as it depends on the SDF, $\Lambda_{t,t+1}$ (which depends on c_{t+1} not yet known in t)
- It is positive
- It's expected value is 1 (obviously - since $E_t\left[\frac{\Lambda_{t,t+1}}{E_t[\Lambda_{t,t+1}]}\right] = \frac{E_t[\Lambda_{t,t+1}]}{E_t[\Lambda_{t,t+1}]} = 1$)

Now, let's take a detour. Remember the example of a random payoff based on the outcome of a fair die, from an earlier technical note (after the numerical asset pricing example/problem I set). Each side of the die comes up with probability $\frac{1}{6}$. Imagine I win £10 if the number 1 comes up, £20 if the number 2 comes up, and so forth. The expected payoff is then

$$E[Payoff] = \frac{1}{6} \times 10 + \frac{1}{6} \times 20 + \frac{1}{6} \times 30 + \frac{1}{6} \times 40 + \frac{1}{6} \times 50 + \frac{1}{6} \times 60 = 35$$

Suppose we have a random variable, Z, that takes on positive values, depending on which side of the die comes up. Call the value z_1 in the case of rolling a 1, z_2 in the case of rolling a 2, and so on. Further, assume that its expected value is 1. That is

$$E[Z] = \overbrace{\frac{1}{6} \times z_1}^{p_1^Z} + \overbrace{\frac{1}{6} \times z_2}^{p_2^Z} + \overbrace{\frac{1}{6} \times z_3}^{p_3^Z} + \overbrace{\frac{1}{6} \times z_4}^{p_4^Z} + \overbrace{\frac{1}{6} \times z_5}^{p_5^Z} + \overbrace{\frac{1}{6} \times z_6}^{p_6^Z} = 1$$

where I have labeled the terms on the RHS p_i^Z for $i = 1 : 6$. That is

$$p_i^Z \equiv \frac{1}{6} z_i$$

If the p_i^Z are all positive and add to 1 then they can be interpreted as defining a discrete probability distribution. Very funky. So we started with a discrete probability distribution (each outcome had probability 1/6) and reweighted it to get another (the outcomes have probabilities p_i^Z). We obtained a legitimate set of probabilities because the thing doing the weighting was assumed to be a) positive and b) had expectation 1 *under the original probabilities*. This isn't dependent on the initial set of probabilities being uniformly 1/6. Generally, if the die (possibly unfair) implied probabilities, p_i for outcome i we would have had

$$p_i^Z \equiv p_i z_i$$

This should sound familiar. We've talked about the SDF weighting things, but the expected value of the SDF isn't 1 it's $Q_t^{(1)}$. But what if we divide it by its expected value? That's what $\mathcal{T}_{t,t+1}$ is. So if I am pricing a payoff I write

$$P_t = E_t\left[\Lambda_{t,t+1}\mathcal{P}_{t+1}\right]$$

but I can also write it as

$$\begin{aligned} P_t &= E_t[\Lambda_{t,t+1}]E_t\left[\mathcal{T}_{t,t+1}\mathcal{P}_{t+1}\right] \\ &= \frac{E_t\left[\mathcal{T}_{t,t+1}\mathcal{P}_{t+1}\right]}{R_t} \end{aligned}$$

where we (yet again) used the fact that the riskless rate is the inverse of $E_t[\Lambda_{t,t+1}]$. But what does this expression say? It says that the price of the random payoff is the numerator, discounted at the **riskless** rate. But it's a risky payoff, so why would we be discounting at the riskless rate? Because the adjustment for risk is now achieved by weighting with $\mathcal{T}_{t,t+1}$ in the expectation in the numerator. We aren't doing

$$P_t = \frac{E_t\left[\mathcal{P}_{t+1}\right]}{R_t}$$

which would be, maybe, how a naive non-economist would value the random payoff - i.e. not making any allowance for risk. Instead we are effectively using a different expectations operator, not using the probabilities underpinning E_t originally (the ones describing the randomness of $\mathcal{P}_{t+1}$) but using the probabilities re-weighted by $\mathcal{T}_{t,t+1}$, which imply a related distribution and one which is typically called the 'risk neutral distribution'. The original set of probabilities are called the 'physical distribution'. People often will write

$$P_t = \frac{E_t^Q\left[\mathcal{P}_{t+1}\right]}{R_t}$$

to emphasize that they are using the risk neutral distribution in expressing the asset price. Just to be clear, all of the following expressions are correct - it's just a matter of perspective:

$$
\begin{aligned}
P_t &= E_t^P\left[\Lambda_{t,t+1}\mathcal{P}_{t+1}\right] \\
P_t &= \frac{E_t^P\left[\mathcal{T}_{t,t+1}\mathcal{P}_{t+1}\right]}{R_t} \\
P_t &= \frac{E_t^Q\left[\mathcal{P}_{t+1}\right]}{R_t}
\end{aligned}
$$

Weighting with $\mathcal{T}_{t,t+1}$ gives you a 'risk neutral expectation' of a payoff that can then be discounted with a riskless rate (as the risk adjustments have been made to the probabilities). We have repeatedly been saying that - when pricing assets - people will weight random payoffs differently if they occur in contingencies when marginal utility is high (when the SDF is relatively high) or when it is low (when the SDF is relatively low). Now we just are taking the perspective that this weighting can be thought of as being like treating the 'bad' outcomes as more probable ($\mathcal{T}_{t,t+1} > 1$) and the 'good' outcomes as less probable ($\mathcal{T}_{t,t+1} < 1$). The risk neutral probabilities just make this literal. $\mathcal{T}_{t,t+1}$ is often called a 'change in measure' or 'Radon-Nikodym derivative' (the second link is quite an easy intro).

One final small point to make: since we divide the SDF by its expectation to obtain $\mathcal{T}_{t,t+1}$, the time discounting bit (β) cancels out. Hence to get the asset price, we need to discount by the riskless rate, which will still encode the time preference. Risk neutrality is all about the variation in marginal utility across contingencies, rather than time.

2.7 Returning to expectations of future short rates

Remember we had reached this point:

$$
f_t^{(2)} = -\log E_t\left[\frac{\Lambda_{t,t+1}}{E_t[\Lambda_{t,t+1}]}R_{t+1}^{-1}\right]
$$

which we can now write as

$$
f_t^{(2)} = -\log E_t^Q\left[R_{t+1}^{-1}\right] = \log\frac{1}{E_t^Q\left[R_{t+1}^{-1}\right]}
$$

Because of Jensen's inequality we can't say $\frac{1}{E_t^Q[R_{t+1}^{-1}]} = E_t^Q[R_{t+1}]$, though in continuous time with appropriate assumptions, we basically can. If we *could* do it though, we'd get something like

$$
f_t^{(2)} = \log E_t^Q\left[R_{t+1}\right]
$$

and, again, being loose (wrong) about Jensen's inequality, we would be tempted to write (taking the log inside the expectation - which is not ok!)

$$f_t^{(2)} = E_t^Q [\log R_{t+1}] = E_t^Q [r_{t+1}]$$

Bottom line is that futures rates essentially are informative about what expected short rates are, but under the risk neutral measure and with some minor technical caveats about Jensen's adjustments.

As we know, adjustments for risk are reflected in risk premia in prices/returns, and a lifetime can be spent trying to separate how much of $f_t^{(2)}$ reflects 'physical' expectations (what do I believe will happen future short rates?) from risk premia (the futures encode risk aversion and the degree to which a certain outcome for future rates is associated with a good or bad outcome for marginal utility). That is a **massive** topic and it comes up all them time when people look at the (nominal) yield curve and try to extrapolate inflation expectations or the expected path of Fed monetary policy. Often policymakers want to know the market's physical beliefs, but they can only observe prices that reflect (without any further assumptions) their risk neutral beliefs. Sometimes people combine futures prices with surveys/questionnaires that directly ask people for their physical measure beliefs. The idea is that comparing the two, you can learn something about risk and risk aversion, since the gap between them 'should' reflect that.

These issues come up all the time in unconventional monetary policy and so forth - where in addition to worrying about real rates and consumption growth, we add inflation and nominal rates into the mix...