

# Assignment 1

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1. a. The linear regression model can be written as,

$$P = \beta_0 + \beta_1 C + \epsilon$$

where,

- $P$  is the prime interest rate
- $C$  is the core inflation rate
- $\beta_0$  is the intercept of the regression line
- $\beta_1$  is the slope of the regression line
- $\epsilon$  is the error term

Assumptions of the linear regression model are,

- Linearity:** The relationship between  $P$  and  $C$  is linear
- Independence:** The error term  $\epsilon$  is independent of  $C$
- Normality:** The error term  $\epsilon$  is normally distributed
- Homoscedasticity:** The error term  $\epsilon$  has a constant variance across all levels of  $C$

2. **Remark:** from ch2A.pdf slide 9 and 13,

- $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$
- $\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$
- $S_{xy} = \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$
- $S_{xx} = \sum_{i=1}^n (x_i - \bar{x})^2$

Let  $e_i$  be the  $i$ th residual term. Then for each observation, we can see that the residual for each observation  $i$  is defined as:

$$e_i = y_i - \hat{y}_i$$

then we can say that the predicted value  $\hat{y}_i$  is equivalent to

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

$$(a) \sum_{i=1}^n e_i = 0$$

*Proof.* LHS:

$$\begin{aligned} \sum_{i=1}^n e_i &= \sum_{i=1}^n (y_i - \hat{y}_i) \\ &= \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) \\ &= \sum_{i=1}^n (y_i - \bar{y} + \hat{\beta}_1 \bar{x} - \hat{\beta}_1 x_i) \\ &= \sum_{i=1}^n y_i - n\bar{y} + \hat{\beta}_1 n\bar{x} - \hat{\beta}_1 \sum_{i=1}^n x_i \\ &= n\bar{y} - n\bar{y} + \hat{\beta}_1 n\bar{x} - \hat{\beta}_1 n\bar{x} \\ &= 0 \end{aligned}$$

$$LHS = 0 = RHS$$

$\therefore$  It is true that the sum of residuals  $e_i$  is zero. □

$$(b) \sum_{i=1}^n x_i e_i = 0$$

*Proof.* LHS:

$$\begin{aligned}
\sum_{i=1}^n x_i e_i &= \sum_{i=1}^n x_i (y_i - \hat{y}_i) \\
&= \sum_{i=1}^n x_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) \\
&= \sum_{i=1}^n (x_i y_i - x_i (\bar{y} - \hat{\beta}_1 \bar{x}) - \hat{\beta}_1 x_i^2) \\
&= \sum_{i=1}^n (x_i y_i - x_i \bar{y} + x_i \hat{\beta}_1 \bar{x} - \hat{\beta}_1 x_i^2) \\
&= \sum_{i=1}^n (x_i (y_i - \bar{y}) + \hat{\beta}_1 (\bar{x} x_i - x_i^2)) \\
&= \sum_{i=1}^n x_i y_i - \bar{y} \sum_{i=1}^n x_i - \hat{\beta}_1 \left( \sum_{i=1}^n x_i^2 - \bar{x} \sum_{i=1}^n x_i \right) \\
&= \left( \sum_{i=1}^n x_i y_i - n \bar{x} \bar{y} \right) - \hat{\beta}_1 \left( \sum_{i=1}^n x_i^2 - n \bar{x}^2 \right) \\
&= \left( \sum_{i=1}^n x_i y_i - n \bar{x} \bar{y} - n \bar{x} \bar{y} + n \bar{x} \bar{y} \right) - \hat{\beta}_1 \left( \sum_{i=1}^n x_i^2 - n \bar{x} \bar{x} - n \bar{x} \bar{x} + n \bar{x} \bar{x} \right) \\
&= \left( \sum_{i=1}^n x_i y_i - n \bar{x} \bar{y} - n \bar{x} \bar{y} + n \bar{x} \bar{y} \right) - \hat{\beta}_1 \left( \sum_{i=1}^n x_i^2 - 2n \bar{x} \bar{x} + n \bar{x} \bar{x} \right) \\
&= \left( \sum_{i=1}^n x_i y_i - \bar{y} \sum_{i=1}^n x_i - \bar{x} \sum_{i=1}^n y_i + n \bar{x} \bar{y} \right) - \hat{\beta}_1 \left( \sum_{i=1}^n x_i^2 - 2\bar{x} \sum_{i=1}^n x_i + n \bar{x}^2 \right) \\
&= \sum_{i=1}^n (x_i y_i - \bar{y} x_i - \bar{x} y_i + \bar{x} \bar{y}) - \hat{\beta}_1 \sum_{i=1}^n (x_i^2 - 2\bar{x} x_i + \bar{x}^2) \\
&= \sum_{i=1}^n (x_i (y_i - \bar{y}) + \bar{x} (y_i - \bar{y})) - \hat{\beta}_1 \sum_{i=1}^n (x_i - \bar{x})^2 \\
&= \sum_{i=1}^n (x_i + \bar{x}) (y_i - \bar{y}) - \hat{\beta}_1 \sum_{i=1}^n (x_i - \bar{x})^2 \\
&= S_{xy} - \hat{\beta}_1 S_{xx} \\
&= S_{xy} - \frac{S_{xy}}{S_{xx}} S_{xx} \\
&= S_{xy} - S_{xy} \\
&= 0
\end{aligned}$$

$$LHS = 0 = RHS$$

□

∴ It is true that the independent variables  $x_i$  is completely uncorrelated to the residuals  $e_i$ .

$$(c) \sum_{i=1}^n \hat{y}_i e_i = 0$$

*Proof.* LHS:

$$\begin{aligned}
\sum_{i=1}^n \hat{y}_i e_i &= \sum_{i=1}^n \hat{y}_i (y_i - \hat{y}_i) \\
&= \sum_{i=1}^n (y_i \hat{y}_i - \hat{y}_i^2) \\
&= \sum_{i=1}^n (y_i (\hat{\beta}_0 + \hat{\beta}_1 x_i) - (\hat{\beta}_0 + \hat{\beta}_1 x_i)^2) \\
&= \sum_{i=1}^n (y_i (\bar{y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 x_i) - (\bar{y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 x_i)^2) \\
&= \sum_{i=1}^n (y_i \bar{y} - y_i \hat{\beta}_1 \bar{x} + y_i \hat{\beta}_1 x_i \\
&\quad - (\bar{y}^2 - 2\hat{\beta}_1 \bar{y} \bar{x} + \hat{\beta}_1^2 \bar{x}^2 + 2\hat{\beta}_1 \bar{y} x_i - 2\hat{\beta}_1^2 \bar{x} x_i + \hat{\beta}_1^2 x_i^2)) \\
&= (\sum_{i=1}^n y_i \bar{y} - \sum_{i=1}^n y_i \hat{\beta}_1 \bar{x} + \sum_{i=1}^n y_i \hat{\beta}_1 x_i \\
&\quad - \sum_{i=1}^n (\bar{y}^2 - 2\hat{\beta}_1 \bar{y} \bar{x} + \hat{\beta}_1^2 \bar{x}^2 + 2\hat{\beta}_1 \bar{y} x_i - 2\hat{\beta}_1^2 \bar{x} x_i + \hat{\beta}_1^2 x_i^2)) \\
&= (n\bar{y}^2 - \hat{\beta}_1 \bar{y} \bar{x} + \hat{\beta}_1 \bar{y} \sum_{i=1}^n x_i \\
&\quad - (\sum_{i=1}^n \bar{y}^2 - \sum_{i=1}^n 2\hat{\beta}_1 \bar{y} \bar{x} + \sum_{i=1}^n \hat{\beta}_1^2 \bar{x}^2 + \sum_{i=1}^n 2\hat{\beta}_1 \bar{y} x_i - \sum_{i=1}^n 2\hat{\beta}_1^2 \bar{x} x_i + \sum_{i=1}^n \hat{\beta}_1^2 x_i^2)) \\
&= (n\bar{y}^2 - (n\bar{y}^2 - 2\hat{\beta}_1 n\bar{y} \bar{x} + \hat{\beta}_1^2 n\bar{x}^2 + 2\hat{\beta}_1 n\bar{y} \bar{x} - 2\hat{\beta}_1^2 n\bar{x} \bar{x} + \hat{\beta}_1^2 \sum_{i=1}^n x_i^2))
\end{aligned}$$

proof continued.

$$\begin{aligned}\sum_{i=1}^n \hat{y}_i e_i &= (n\bar{y}^2 - (n\bar{y}^2 - \hat{\beta}_1^2 n\bar{x}^2 + \hat{\beta}_1^2 n\bar{x}^2)) \\ &= (n\bar{y}^2 - n\bar{y}^2) \\ &= 0\end{aligned}$$

$$LHS = 0 = RHS$$

□

∴ It is true that the predicted values  $\hat{y}_i$  is completely orthogonal to the residuals  $e_i$ .