Differential Equations

Differential Equations

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This set of lecture notes covers topics typical of an introductory course in differential equations. The notes draw extensively from Phanuel Mariano's "Lecture notes in differential equations".

Contents

Chapter 1

Introduction

1.1 Modeling via differential equations, some solutions and definitions

We being by asking what a differential equation is. In your calculus career, you have seen quite a number of them already, just in disguise. For example, you might have been asked "what function is the antiderivative of x^2?" In other words, find a function $f(x)$ so that

$$\frac{d}{dx} f(x) = x^2.$$

In your past life, this sort of equation was solved by using integration, which will also be true in this class. Indeed, the reason that you spent so much time on techniques of integration was to give you the basic toolset needed to solve differential equations.

1.1.1 What are differential equations?

So let us consider the question "*What is a differential equation and what are its solutions?*" We can compare and contrast what we already know in algebra with what we know in differential equations.

- ***Algebraic equations:*** are equations formed from numerical variables.
  - Examples: $x^2 - 1 = 0$ or $x^2 + 1 = 0$
  - What are solutions? Are there even solutions? If so, how many?
    - Solutions to algebraic equations are numbers.
    - We can always check if a number is a solution: note that $x = 1$ is a solution of $x^2 - 1 = 0$ since $1^2 - 1 = 0$
    - But notice that there is another solution, $x = -1$ as well. So solutions are not *unique* in this case.
- ***Differential equations:*** are equations involving variables representing *functions and their derivatives.*
  - Examples: $\frac{dy}{dt} = 2y$ or $y' = 4y + e^t$.
  - What are solutions to differential equations?

  - ■ ***They are functions***! This is tricky because functions are more complicated than numbers. Functions have domains, ranges, etc.
  - ■ Solutions to differential equations are *not!* equations.
- ○ Is there even a solution? (**Existence**) If so, how many? (**Uniqueness**). These are the two main questions that one asks in differential equations.
- ○ ***Check:*** We can always check that a function is a solution to a differential equation.
- ○ Example: Show that $y(t) = 9e^{2t}$ is a solution to the differential equation $y' = 2y$.

  Solution: Plug $y(t)$ into the left hand side (LHS) of the equation, then plug into the right hand side (RHS) and check that they are equal.

$$LHS \stackrel{?}{=} RHS$$
$$\frac{d}{dt}(9e^{2t}) \stackrel{?}{=} 2 \cdot (9e^{2t})$$
$$18e^{2t} \stackrel{\checkmark}{=} 18e^{2t}$$

  What about $y = 9e^{2t} + 1$? Is this a solution? Take a pen and paper and try this yourself by hand. You will see that y is NOT actually a solution to the example equation.
- ○ Check that $y(t) = 1 + t$ is a solution to the differential equation

$$\frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t}.$$

  You should find that the answer is yes.

1.1.2 Studying first order equations

What are differential equations used for? The answer is pretty much everything in modern science. Derivatives measure rate of change. Since modern science, engineering, and applied mathematics study changing quantities in the physical world, *differential equations are the language of science.* Differential equations are models of the physical world that are used to make predictions.

- Meteorologists try to model the weather constantly (using very complicated differential equations), and they get it wrong all the time. *Modeling is hard.*
- Models frequently follow from qualitative observations about the world. A classical example is known as **Newton's Law of Cooling**, which states that "at a given time, the rate at which an object in an environment of constant ambient temperature is cooling is proportional to the difference between the object's temperature and the ambient temperature."

  Mathematically, we can translate this into a differential equation. Let T be the temperature of the object at time t, and A be the ambient temperature. Then
$$\frac{dT}{dt} = C(T - A),$$
  where C is the constant of proportionality.

- The solution to the equation

$$\frac{dP}{dt} = k \cdot P$$

models the population of a species at time t, assuming unrestricted growth. In words, the equation says "the rate at which a population grows at a given time is proportional to the current population at that time".

- A related equation called the **logistic growth model** describes population growth in an environment with restricted resources, and is given by

$$\frac{dP}{dt} = C \cdot P(M - P),$$

where the constant C is the constant of proportionality, and the constant M is called the **carrying capacity** of the model. How does this equation behave for as time progresses?

Definition 1.1.1 The standard form of a **first order differential equation** (that is, an equation with only first derivatives) is

$$\frac{dy}{dt} = f(t, y),$$

where $y = f(t)$ is a function and t is the independent variable.

An **initial value problem (IVP)** is a differential equation with an intial condition:

$$\frac{dy}{dt} = f(t, y), \text{ and } y(t_0) = y_0.$$

◊

- Consider the IVP

$$\frac{dy}{dt} = 2y, \quad y(0) = 9.$$

Is $y(t) = 9e^{2t}$ a solution to the IVP?

Yes. We've already checked that $y(t) = 9e^{2t}$ is a solution to the ODE, and furthermore we have that $y(0) = 9e^0 = 9$.

Definition 1.1.2 A **particular solution** to an ODE is simply one of the functions $y = y(t)$ that satisfy a differential equation $y' = f(t, y)$ for all t.

A **general solution** to an ODE is a parametrized collection of solutions that contains the solutions to every possible IVP built from that ODE. ◊

Example 1.1.3 Finding a general solution. To find the general solution to $\frac{dy}{dt} = 2y$, we can separate the y's and the t's to opposite sides and then integrate.

$$\begin{aligned}
\frac{dy}{dt} = 2y &\iff \frac{dy}{y} = 2dt \\
&\iff \int \frac{dy}{y} = \int 2dt \\
&\iff \ln|y| = 2t + C \\
&\iff |y| = e^{2t+C} = Ke^{2t}, \text{ where } K = e^C \\
&\iff y = ce^{2t}, \text{ where } c = \pm K.
\end{aligned}$$

Thus, the general solution must be of the form

$$y = ce^{2t}.$$

There will be a whole section on this technique (unsurprisingly named "separation of variables"). ☐

Definition 1.1.4 An **equilibrium solution** to an ODE is a *constant solution*$y(t) = y_0$. That is,

$$\frac{dy}{dt} = 0 \text{ for all } t.$$

◊

Example 1.1.5 Finding an equilibrium solution. Find the equilibrium solution of the following equation. Suppose that

$$y' = y^3 + y^2 - 6y.$$

For what values of $y(0) = y_0$ is $y(t)$ equilibrium, increasing, or decreasing? Factor to get

$$y' = y(y-2)(y+3),$$

and create a sign chart (like in calculus!). The equilibrium solutions are $y = -3, y = 0$, and $y = 2$. We can observe that solutions decrease for $y_0 \in (-\infty, -3) \cup (0, 2)$ and that they increase for $(-3, 0) \cup (2, \infty)$. ☐

1.1.3 Solutions to some differential Equations

- ***A Linear Differential Equation:*** Pick your favorite real numbers a, b, y_0 and consider the IVP

  $$\frac{dy}{dt} = ay - b, \quad y(0) = y_0.$$

  The ***general solution*** to this differential equation is

  $$y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}$$

  We will see how one can get this very soon!

- ***Example:*** Find the solution to

  $$\frac{dy}{dt} = -2y + 8, \quad y(0) = 5.$$

  Solution: The mysterious formula above says that

  $$a = -2, b = -8$$

  and

  $$y_0 = 9$$

  so the solution is

  $$y(t) = 4 + \left(5 - \frac{8}{2}\right) e^{-2t} = 4 + e^{-2t}.$$

1.1.4 Studying general differential equations

- In this class, we will only study **ordinary differential equations** (ODE): contains only ordinary derivatives:

  Example: $\frac{d^2y}{dt^2} + \frac{dy}{dt} = -1$

- There is a whole separate course where one can study **partial differential equations**(PDE):

  Ex: $\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = -1$

- System of equations:

$$\frac{dx}{dt} = x - xy$$
$$\frac{dv}{dt} = y - 3x$$

- The **order** of the equation speaks to the highest order derivative in the equation

$$y' + 3y = 0 \text{ 1st order}$$
$$y'' + 3y' = 2t \text{ 2nd order}$$
$$\frac{d^5y}{dt} + \frac{dy}{dt} = y \text{ 5th order}$$
$$u_{xx} + u_{yy} = 0 \text{ 2nd order}$$

Definition 1.1.6 An ODE is called **linear** if it is linear in y, i.e. it is of the form

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = g(t)$$

$\Diamond$

- ***Linear***:
  - $y' + 4y = 0$,
  - $t^2y'' + \cos ty = 1$,
  - and $\frac{y'}{t} - y = t^2$.
- ***Nonlinear***:
  - $\left(\frac{du}{dt}\right)^2 + y = 1$,
  - $yy' + y = 1$,
  - $y'' + 3e^y y$,
  - and $\frac{1}{y} - y' = 1$.

Nonlinear ODEs are some of the hardest equations to solve! In fact, most of the time, one won't be able to find an exact formula for the solution of a differential equation. Much of the study of differential equations comes down to qualitative analysis and approximate solutions. But one nice thing about studying ODEs is that we can always check if a function is really a solution to a differential equation or not. Futhermore, the equations can frequently be understood without an explicit solution at all.

1.2 Slope fields (direction fields)

In this section, we will learn about a *qualitative* techinque - that is, we will learn to interpret the behavior of the solutions of differential equations using the equations themselves.

1.2.1 Slope fields

Generally, there are three broad approaches to differential equations:

1. ***Analytically:*** This means that one actually finds a formula for the solutions of a differential equations, typically using integral techniques.
2. ***Numerically:*** But often, it is very difficult to find an actual formula for the solution, even though there may be a solution. Thus one can use computers and algorithms to numerically approximate the solution. Even in the numerical case it is important to understand limitations of the calculated solutions.
3. ***Qualitatively:*** Maybe we don't need the full solution of a differential equation. We can use our knowledge of ODEs to have an idea of how the solution behaves. For example, maybe the only thing you want to answer about the solution is what the following asymptotic limit is:

$$\lim_{t\to\infty} y(t).$$

When we have an equation of the form $\frac{dy}{dt} = f(t, y)$. We can always make a **slope field** for the ODE. A slope field contains minitangets at several points of a graph that describe the behavior of solutions through those points.

Example 1.2.1 Slope field for $y' = t - y$.

t	y	$f(t, y) = t - y$
-1	1	2
-1	0	1
-1	-1	0
0	1	1
0	0	0
0	-1	-1
1	1	0
1	0	-1
1	-1	-2

□

We can use mathematical software (the open source SageMath platform here) to generate slopefields that are accurate and easy to read.

```
t, y = var('t,y')
plot_slope_field(y - t, (t, -3, 3), (y, -3, 3))
```

- Qualitatively, given a starting point, we can predict the long-term behavior of a particular solution. One feature to notice in the picture above is that solutions seem to have very different asymptotic behavior depending on whether the initial point is above or below the line $y = t + 1$.
- Slope fields like the one above allow us to sketch what solution curves *might* look like, as curves through a given point must have matching slope there. (Indeed, before computers, working by hand and sketching solutions was a time-consuming and detail-oriented art.)
- You can find a link to `Dfield`, an applet that generates slope/direction fields, at this link.
- Slope fields can also be visualized in Desmos in various ways. A simple example can be found here.

1.2.2 Two important cases

There are two forms of first order equations that are relatively open to qualitative analysis, the cases where the function $f(t, y)$ on the RHS is a function of just one variable.

Type 1: $\frac{dy}{dt} = f(t)$.

- The slopes are always the same in each *vertical line*. Draw a picture.
- These are the simple ODEs corresponding to finding antiderivatives in Calculus II.
- For example, the slope field of $\frac{dy}{dt} = 2t$ is computed

```
g = Graphics()
t, y = var('t,␣y')
g += plot_slope_field(2*t, (t,-5,5),(y,-5,5))
g += plot(t^2 - 4, (t, -5,5), ymin = -5, ymax = 5)
g += plot(t^2 - 5, (t, -5,5), ymin = -5, ymax = 5)
g.show()
```

- Integrating both sides gives the general solution as the family of parabolas $y = t^2 + C$, which you should be able to see in the slope field.

Type 2: $\frac{dy}{dt} = f(y)$

- These are called **autonomous equations**.
- The slopes are always the same in each *horizontal line*.
- Autonomous equations have equilibrium solutions whenever $f(y) = 0$.
- For example, to do a qualitative analysis of the autonomous equation

$$\frac{dy}{dt} = 4y(1 - y)$$

first identify the equilibria (in this case $y = 0$ and $y = 1$).

Now check slopes between the equilibrium solutions -

$$y'(2) = (-)$$
$$y'(.5) = (+)$$
$$y'(-1) = (-)$$

Before we visualize, we can already see what the long term behavior should be. The equilibrium solution $y = 1$ attracts solutions (this is called a stable equilibrium) while the solution $y = 0$ pushes solutions away (which is called an unstable equilibrium).

```
t, y  = var('t,y')
plot_slope_field(4*y*(1-y), (t,-2,2),(y,-2,2))
```

When trying to match slope fields you should always follow these steps:

1. Factor!
2. Find the equilibrium solutions.
3. Test points between equilibrium solutions.

Chapter 2

First Order Differential Equations

2.1 Linear equations and integrating factors

Recall that a first order differential equation is an equation involving functions and their first derivatives. The general form of a first order equation is

$$\frac{dy}{dt} = f(t, y)$$

where t is the independent variable. This chapter will concern many standard and useful forms of first order equations along with methods of solution. A **linear first order differential equation** has the form

$$\frac{dy}{dt} = a(t)y + b(t).$$

This equation can be rewritten into **standard form** as

$$\frac{dy}{dt} + p(t)y = g(t)$$

where $p(t) = -a(t)$ and $g(t) = b(t)$.

Example 2.1.1 The first order linear differential equation

$$y' = t^2 y + \cos t$$

can be rewritten in standard form as

$$y' - t^2 y = \cos t.$$

$\square$

2.1.1 Integrating factors method

Consider a first order linear differential equation in standard form

$$y' + p(t)y = g(t). \tag{2.1.1}$$

Then notice that $\frac{dy}{dt} + p(t)y$ looks awfully like a *product rule* of some sort. (Remember that the product rule gives a method for computing a derivative of a product of functions by $(fg)' = f'g + fg'$.) In the product rule, there are two functions. In our case, clearly one function will be $y(t)$; what will the second function be? We need to adjust the equation so that we can undo a product rule amd simplify the equation. We denote by $\mu(t)$ the **integrating factor** that makes the LHS into a product rule. We can figure out what μ needs to be by computation. Let's multiply both sides by $\mu(t)$ and get

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t),$$

and if we want the LHS to be a product rule then

$$LHS = \frac{d\left[\mu(t)y(t)\right]}{dt} = \mu(t)\frac{dy}{dt} + \mu(t)p(t)y.$$

Let's just assume this works for now. (We will find out precisely what $\mu(t)$ needs to be in the following section). Setting the LHS to RHS we get

$$\frac{d\left[\mu(t)y(t)\right]}{dt} = \mu(t)g(t).$$

Then integrating we get

$$\int \frac{d\left[\mu(t)y(t)\right]}{dt}dt = \int \mu(t)g(t)dt.$$

But we know integrating cancels differentiation, and thus the LHS equals $\mu(t)y(t)$ so that

$$\mu(t)y(t) = \int \mu(t)g(t)dt + C.$$

Since y is the desired solution, we divide by $\mu(t)$ we get that

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right],$$

which is the **general solution** to the ODE in (2.1.1).

2.1.2 Finding the integrating factor

So recall that for the product rule to work we have

$$\frac{d\left[\mu(t)y(t)\right]}{dt} = \mu(t)\frac{dy}{dt} + \mu(t)p(t)y$$

but then this only happens if the derivative of $\mu(t)$ is $\mu(t)g(t)$ (by product rule!!!!). Thus,

$$\frac{d\left[\mu(t)\right]}{dt} = \mu(t)p(t).$$

Rewrite this as

$$\frac{d\mu}{dt} = \mu p$$

which is a separable equation (discussed at length in the next section). Thus, we can integrate both sides and get

$$\int \frac{d\mu}{\mu} = \int p(t)dt \iff \ln|\mu| = \int p(t)dt$$
$$\iff \mu = e^{\int p(t)dt}.$$

We then have a formula for the integrating factor μ:

An integrating factor μ for a first order linear ODE (as in (2.1.1)) is given by

$$\mu(t) = e^{\int p(t)\,dt}.$$

2.1.3 Examples

Example 2.1.2 (without formula). Find general solution of

$$\frac{dy}{dt} = \frac{3}{t}y + t^5.$$

- Step 1: Rewrite as

  $$\frac{dy}{dt} - \frac{3}{t}y = t^5$$

  so that $p(t) = -\frac{3}{t}$ and $g(t) = t^5$.

- Step 2: Find an integrating factor:

  $$\mu(t) = e^{\int -\frac{3}{t}dt} = e^{-3\ln t} = t^{-3} = \frac{1}{t^3}.$$

  Note we only need an integrating factor, not a general integrating factor. So we never need to have a $+C$ in this step! In the next step we will note that we also don't need the absolute value inside the natural log (why not?).

- Step 3: Multiply BOTH SIDES of the equation by $\mu(t)$ and get

  $$\frac{1}{t^3}\frac{dy}{dt} - \frac{3}{t^4}y = t^2$$

  and notice that

  $$\begin{gathered} \frac{1}{t^3}\frac{dy}{dt} - \frac{3}{t^4}y = t^2 \\ \Downarrow \\ \frac{d}{dt}\left[\frac{1}{t^3}y\right] = t^2 \end{gathered}$$

- Step 4: Integrate and solve for $y(t)$ (don't forget the constant C in this step, which is very important!)

  $$\begin{aligned} \int \frac{d\left[\frac{1}{t^3}y\right]}{dt} = \int t^2 dt + C \iff \qquad & \frac{1}{t^3}y = \frac{t^3}{3} + C \\ \iff \qquad & y(t) = \frac{t^6}{3} + Ct^3. \end{aligned}$$

□

Example 2.1.3 (using formula). Solve the IVP:

$$\frac{dy}{dt} = \frac{3}{t}y + t^5$$

with $y(1) = \frac{4}{3}$. In this example we'll skip the previous steps and go straight to using the formula.

- Step 1: Rewrite as
$$\frac{dy}{dt} - \frac{3}{t}y = t^5$$
so that $p(t) = -\frac{3}{t}$ and $g(t) = t^5$.

- Step 2: Find an integrating factor:
$$\mu(t) = e^{\int -\frac{3}{t}\,dt} = e^{-3\ln t} = t^{-3} = \frac{1}{t^3}.$$

- Step 3: I can just plug in the formula and get
$$\begin{aligned} y(t) = & \quad \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt. + C\right] \\ = & \quad t^3\left[\int \frac{1}{t^3}t^5 dt + C\right] \\ = & \quad t^3\left[\frac{t^3}{3} + C\right] = \frac{1}{3}t^6 + Ct^3. \end{aligned}$$

- Step 4: Since $y(1) = \frac{4}{3}$ then
$$\frac{4}{3} = \frac{1}{3} + C$$
so $C = 1$ so that
$$y(t) = \frac{1}{3}t^6 + t^3.$$

□

Example 2.1.4 (using formula). Find the general solution for
$$\frac{dy}{dt} = y + 9\cos t^2.$$

- Step 1: Rewrite as
$$\frac{dy}{dt} - y = 9\cos t^2$$
so that $p(t) = -1$ and $g(t) = 9\cos t^2$.

- Step 2: Find an integrating factor:
$$\mu(t) = e^{\int -1 dt} = e^{-t}.$$
Note we only need an integrating factor, not a general integrating factor. So we never need to have a $+C$ in this step!

- Step 3: I can go through the process again, or I can just plug in the formula and get
$$\begin{aligned} y(t) = & \quad \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt. + C\right] \\ = & \quad \frac{1}{e^{-t}}\left[\int e^{-t}9\cos t^2 dt + C\right] \\ = & \quad e^t\left[\int e^{-t}9\cos t^2 dt + C\right] \end{aligned}$$
Can't integrate, so we leave the answer in this form.

□

Example 2.1.5 (using formula). Find general solution of

$$t^3y' + 4t^2y = e^{-t}.$$

- Step 1: Rewrite as
$$y' + \frac{4}{t}y = \frac{e^{-t}}{t^3}$$
so that $p(t) = \frac{4}{t}$ and $g(t) = \frac{e^{-t}}{t^3}$.

- Step 2: Find an integrating factor:
$$\mu(t) = e^{\int \frac{4}{t}dt} = e^{4\ln|t|} = t^4.$$
Note we only need an integrating factor, not a general integrating factor. So we never need to have a $+C$ in this step!

- Step 3: I can go through the process again, or I can just plug in the formula and get

$$\begin{aligned}
y(t) = & \quad \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt. + C\right] \\
= & \quad \frac{1}{t^4}\left[\int t^4\frac{e^{-t}}{t^3}dt + C\right] \\
= & \quad \frac{1}{t^4}\left[\int te^{-t}dt + C\right] \\
= & \quad \frac{1}{t^4}\left[-te^{-t} - e^{-t} + C\right] \\
= & \quad -\frac{1}{t^3}e^{-t} - \frac{1}{t^4}e^{-t} + \frac{C}{t^4}.
\end{aligned}$$

where we used *integration by parts.* □

2.2 Separable equations

One of the easiest methods to solve a first order ODE is called **separation of variables** (presuming that one can integrate the result!). In the previous section, we focused on linear equations, which cover a good deal of first order ODEs. But we want to be able to solve at least some nonlinear equations if they fall into easily computable forms.

The technique we'll use in this section will only work if the first order ODE is **separable**. We say a first order ODE is separable if we can write it in the following form:

$$\frac{dy}{dt} = g(t)h(y). \tag{2.2.1}$$

If we can write it this way, then separate the variables to get (put all the y's on one side and all t's on the other side)

$$\frac{1}{h(y)}dy = g(t)dt,$$

and then integrate both side with respect to their respective variable. This is legal by a u-substitution argument. (This is informal algebra!)

Note: Sometimes you'll see a separable equation written in the following **differential** form:

$$M(x)dy + N(y)dy = 0.$$

Example 2.2.1 (Not separable). Notice that $\frac{dy}{dt} = y + t$ is not separable.

But we can solve this using the methods of the previous section. □

Example 2.2.2 Find the general solution of

$$\frac{dy}{dt} = \frac{t}{y^2}.$$

Separate variables in the equation, integrate and then solve for y:

$$\begin{aligned} \frac{dy}{dt} = \frac{t}{y^2} \iff & & y^2 dy = t\,dt \\ \iff & & \int y^2 dy = \int t\,dt \\ \iff & & \frac{y^3}{3} = \frac{t^2}{2} + c_1 \\ \iff & & y = \sqrt[3]{\frac{3t^2}{2} + 3c_1}. \end{aligned}$$

We can then rename $C = 3c_1$ and get the *general solution*

$$y(t) = \sqrt[3]{\frac{3t^2}{2} + C}.$$

If you are able to solve for y exactly, then this is called an **explicit** solution, because we can solve exactly with a formula. □

Example 2.2.3 (missing solution). Find the general solution for

$$\frac{dy}{dt} = y^2.$$

- First let's find the equilibrium solutions: $y(t) = 0$ is the only one.
- Then use the general separation of variables procedure

$$\begin{aligned} \frac{dy}{dt} = y^2 \iff & & \frac{1}{y^2} dy = dt \\ \iff & & \int \frac{1}{y^2} dy = \int dt \\ \iff & & -\frac{1}{y} = t + C. \end{aligned}$$

- But notice that

  $$y = -\frac{1}{t + C}$$

  does NOT solve the IVP with $y(0) = 0$. Thus we have to include the equilibrium solution $y(t) = 0$ to get the complete general solution. In this case we say the general solution is:

  $$y_g(t) = \begin{cases} 0 \\ -\frac{1}{t+C} \end{cases}$$

Moral of the story: Always find the equilibrium solutions first in case there are any missing solutions from separating variables!

```
t, y = var('t,y')
g = Graphics()
g += plot_slope_field(y^2, (t, -5,5),(y,-5,5))
g += plot(-1/(t + 4), (t, -5,5), ymin = -5, ymax = 5)
g += plot(0, (t, -5,5), ymin=-5, ymax=5)
g.show()
```

□

Example 2.2.4 (Clever quadratic formula trick). Solve the IVP:

$$\frac{dy}{dx} = \frac{2x+1}{y+1} \quad y(0) = 1$$

- Note there are no equilibrium solutions
- Use the general separation of variables procedure

$$\begin{aligned} \frac{dy}{dt} = \frac{2x+1}{y+1} \iff & \qquad \int (y+1)\,dy = \int (2x+1)\,dx \\ \iff & \qquad \frac{y^2}{2} + y = x^2 + x + C \\ \iff & \qquad \frac{y^2}{2} + y - x^2 - x + c = 0 \\ \iff & \qquad y^2 + 2y - 2x^2 - 2x + C = 0 \end{aligned}$$

- Then we can use the quadratic formula on

$$ay^2 + by + c = 0$$

where

$$\begin{aligned} a &= 1 \\ b &= 2 \\ c &= -2x^2 - 2x + C \end{aligned}$$

hence an explicit solution is given by

$$\begin{aligned} y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{4 - 4\left(-2x^2 - 2x + C\right)}}{2} \\ &= -1 \pm \sqrt{1 + 2x^2 + 2x + C} \\ &= -1 \pm \sqrt{2x^2 + 2x + C}. \end{aligned}$$

Thus the general solution is

$$y_g(x) = -1 \pm \sqrt{2x^2 + 2x + C}.$$

- Now to solve the IVP use the initial condition: $y(0) = 1$

$$1 = y(0) = -1 \pm \sqrt{C}$$

so that

$$2 = \pm\sqrt{c}$$

since the LHS is positive we choose the positive sign in the $\pm$ so that

$$2 = \sqrt{c}$$

hence

$$c = 4$$

so that we get the particular solution

$$y_p(x) = -1 + \sqrt{2x^2 + 2x + 4}.$$

```
x, y = var('x,y')
g = Graphics()
g += plot_slope_field((2*x+1)/(y+1), (x, -5,5),(y,-5,5))
g += plot(-1 + sqrt(2*x^2 + 2*x + 4), (x, -5,5), ymin = -5,
    ymax = 5)
g += points([(0,1)], size = 50)
g.show()
```

□

Example 2.2.5 Implicit solutions. Find the general solution for

$$\frac{dy}{dt} = \frac{y}{1+y^2}.$$

In this example, we would get

$$\ln|y| + \frac{y^2}{2} = t + C$$

and leave it that way as there is no nice way to solve this. But any function $y(t)$ that satisfies the equation above is a solution to our ODE. When we write solutions this way, we call this an **implicit solution**, as the equation implicitly defines y as a function of t. □

Example 2.2.6 Solve the IVP

$$\frac{dy}{dt} = t^4 y \quad y(0) = 1.$$

- Solution:
- Start with equilibrium solutions $y = 0$.
- Get $|y| = Ce^{t^5/5}$ but notice that by choise of C this shortens to $y = Ce^{t^5/5}$.
- Note that this includes the equilibrum solution $y = 0$ by setting $C = 0$.
- Thus then general solution is given by

  $$y_g(t) = Ce^{t^5/5}.$$

- To solve the IVP we use the initial condition

  $$1 = y(0) = Ce^0 = C$$

  thus $C = 1$, hence the particular solution to the IVP is

  $$y_p(t) = e^{t^5/5}.$$

```
t, y = var('t,y')
g = Graphics()
g += plot_slope_field(y*t^4, (t, -5,5),(y,-5,5))
g += plot(exp((t^5)/5), (t, -5,5), ymin = -5, ymax = 5)
g += points([(0,1)], size = 50)
g.show()
```

□

Example 2.2.7 Find the general solution:

$$\frac{dy}{dt} = (y+1)(y+5).$$

- Solution:
- Start with equilibrium solutions $y = -1, -5$.
- Use partial fractions to get $\frac{1}{(y+1)(y+5)} = \frac{1/4}{y+1} - \frac{1/4}{y+5}$.
- The solution is

$$\begin{aligned} \frac{1}{4}\ln|y+1| - \frac{1}{4}\ln|y+5| = t + C \iff& \qquad \ln\left|\frac{y+1}{y+5}\right| = 4t + C_1 \\ \iff& \qquad \left|\frac{y+1}{y+5}\right| = C_2 e^{4t} \\ \iff& \qquad \frac{y+1}{y+5} = C_3 e^{4t} \\ \iff& \qquad y = \frac{5ke^{4t} - 1}{1 - ke^{4t}}. \end{aligned}$$

- This yields all solutions but the equilibrium solution $y = -5$. Note that $y = -1$ can be found by taking $k = -1$. Thus

$$y_g(t) = \begin{cases} y(t) = \frac{5ke^{4t}-1}{1-ke^{4t}} \\ y(t) = -5. \end{cases}$$

□

2.3 Separable homogeneous equations and the substitution method

Consider an ODE

$$\frac{dy}{dx} = f(x, y)$$

and suppose we can rewrite it in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right). \tag{2.3.1}$$

An equation of the form (2.3.1) is called **homogeneous**.

- To solve this equation. We will define a new variable

$$v = \frac{y}{x} \quad \text{(Important)}$$

 and write everything in terms of only v and x!

- *Solve* for y: and get
$$y = xv.$$
- Implicitly differentiate both sides:
$$\frac{dy}{dx} = x\frac{dv}{dx} + 1 \cdot v. \quad \text{(Important)}$$

The two important equations we come up with are:

$$\text{General homogeneous substitution:} \begin{cases} v = \frac{y}{x} \\ \frac{dy}{dx} = x\frac{dv}{dx} + v \end{cases}$$

Example 2.3.1 Consider
$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}.$$

Part (a): Show that this ODE is homogeneous and rewrite the entire equation by only v and x.

To see this we divide the numerator and denominator by x^2 and get
$$\frac{dy}{dx} = \frac{1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2}{1}.$$

Then replacing $\frac{dy}{dx} = x\frac{dv}{dx} + v$ and $v = \frac{y}{x}$, we get a new equation
$$x\frac{dv}{dx} + v = 1 + v + v^2.$$

Part (b): Solve the ODE in terms of v and then return everything into terms of y, x.

We rewrite
$$\begin{aligned} x\frac{dv}{dx} = 1 + v^2 &\iff \int \frac{dv}{1+v^2} = \int \frac{1}{x}dx \\ &\iff \tan^{-1}(v) = \ln|x| + c \\ &\iff \tan^{-1}\left(\frac{y}{x}\right) = \ln|x| + c. \end{aligned}$$

Then
$$\frac{y}{x} = \tan(\ln|x| + C)$$
and the general solution is
$$y_g(x) = x\tan(\ln|x| + C)$$

□

Example 2.3.2 Find the solution to
$$y' = \frac{y}{x} + \frac{x}{y} \quad , x > 0.$$

Step 1: First check if you can apply any of the method of the previous sections (linear? separable?). The equation is neither linear, or separable. But notice that this is *homogeneous* for if we let $v = \frac{y}{x}$ then

$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{y/x} = v + \frac{1}{v}.$$

Step 2: Recall that $\frac{dy}{dx} = x\frac{dv}{dx} + v$, so plug this into the LHS, and get

$$\begin{aligned} x\frac{dv}{dx} + v = v + \frac{1}{v} &\iff \int v dv = \int \frac{1}{x} dx \\ &\iff \frac{v^2}{2} = \ln|x| + C \\ &\iff y^2 = 2x^2 \ln|x| + kx^2. \end{aligned}$$

and we get the general solution

$$y_g(x) = \pm\sqrt{2x^2 \ln|x| + Cx^2}$$

```
x, y = var('x,y')
g = Graphics()
g+= plot_slope_field(y/x + x/y, (x,-10,10),(y,-10,10))
g+= plot(sqrt(2*x^2*log(abs(x))+x^2),(x,1.1,10), ymin=-10,
    ymax=10)
g+= plot(-sqrt(2*x^2*log(abs(x))+3*x^2),(x,-10,
    -1.1),ymin=-10, ymax=10)
g.show()
```

□

Example 2.3.3 (not always the same substitution). Rewrite the equation

$$\frac{dy}{dx} = e^{9y-x}$$

in terms of only v, x by letting $v = 9y - x$.

Solution: Use the substitution $v = 9y - x$, then solve for y and get

$$y = \frac{1}{9}v + \frac{1}{9}x.$$

Then using implicit differentiation,

$$\frac{dy}{dx} = \frac{1}{9}\frac{dv}{dx} + \frac{1}{9}$$

and hence

$$\begin{aligned} \frac{dy}{dx} = e^{9y-x} &\iff \frac{1}{9}\frac{dv}{dx} + \frac{1}{9} = e^v \\ &\iff \frac{dv}{dx} = 9e^v - 1. \end{aligned}$$

and this can be easily solved by separating variables. □

2.4 Modeling with Differential Equations

What are Differential Equations used for? Predicting the future! That is, differential equations typically appear in the context of **mathematical models** for physical situations that can be dscribed using the language of change.

Models are used in many areas, including science, engineering, and finance. Once a model has been written down in terms of a differential equation, there are three broad approaches to understanding the system being studied.

- Analytic: explicit solutions
- Qualitative: Use geometry to see long term behaviour. For example, to check if the population is increasing or decreasing.
- Numerical: Approximations to actual solutions.

2.4.1 Model building

1. State Assumptions (science step, Newton's law of motions, etc, ...)
2. Describe variables, parameters: Independent variables (t, x), dependent variables (y, u), parameters (k, α) (do not change with time)
3. Create Equations:
   - Rate of change = slope = derivative.
   - the word "is" means equal.
   - A is proportional to B means $A = kB$.

Example 2.4.1 Population growth.

- ***Goal:*** Want to write a differential equation that models population growth of say zebras.
- ***Assumption:*** The rate of growth of the population is proportional to the size of the population.
- ***Problem:*** Write a differential equation that governs this. Let $P(t)$ be the population of zebras at time t. So for now we have

$$\frac{dP}{dt} = k \cdot P.$$

  Note here that k is a **parameter** that can be changed or selected for a specific situation once we know more information. For example if we know the proportion is $k = 2$, then

$$\frac{dP}{dt} = 2 \cdot P$$

  and we already saw earlier that $P(t) = Ce^{2t}$ is a solution to this.

□

Example 2.4.2 Mixing problem 1. ***Problem:*** A vat contains 60L of water with 5kg of salt water dissolved in it. A salt water solution that contains 2kg of salt per liter enters the vat at a rate of 3 L/min. Pure water is also flowing into the vat at a rate of 2 L/min. The solution in the vat is kept well mixed and is drained at a rate of 5 L/min, so that the rate in is the same as the rate out. Thus there is always 60L of salt water at any given time. How much salt is in the tank after 30 minutes? What is the long term behavior?

Solution:

- ***Step 1:*** Define variables.

Let $y(t)$ =amount of salt at time t. Let $y(0) = 5$ kg.

- ***Step 2:*** Find rate in/rate out

  Two equations from basic physics will help here. First,

$$\text{mass(kg)} \;=\; \text{density(kg/L)} \;\times\; \text{volume(L)}.$$

  Second,

$$\text{rate of mass (kg/min)} \;=\; \text{concentration (kg/L)} \times \text{rate of volume (L/min)}.$$

  Finally, when the volume of the mixture doesn't change, it must by that

$$\text{volume rate in} \;=\; \text{volume rate out}$$

  Using the information from the problem we have

$$\begin{aligned} \text{Rate in} &= \underbrace{\left(2\frac{\text{kg}}{\text{L}}\right)\left(3\frac{\text{L}}{\text{min}}\right)}_{\text{-salt water solution}} + \underbrace{\left(0\frac{\text{kg}}{L}\right)\left(2\frac{\text{L}}{\text{min}}\right)}_{\text{–pure water}} \\ &= 6\frac{\text{kg}}{\text{min}}. \end{aligned}$$

  and

$$\begin{aligned} \text{Rate out} &= \left(\begin{array}{c}\text{concentrarion}\\ \text{of stuff going out}\end{array}\right) \times \text{Rate} \\ &= \left(\frac{y(t)}{60}\frac{\text{kg}}{\text{L}}\right) \times 5\frac{\text{L}}{\text{min}}. \\ &= \frac{y(t)}{12}\frac{\text{kg}}{\text{min}}. \end{aligned}$$

- ***Step 3:*** Write the IVP

  Always recall that for mixing problems we have

$$\begin{aligned} \frac{dy}{dt} &= \text{Rate in} - \text{Rate out} \\ &= 6 - \frac{y}{12}. \end{aligned}$$

  and the initial condition is

$$y(0) = 5.$$

- ***Step 4:*** Find the common denominator and solve using separation of variables.

  Write

$$\frac{dy}{dt} = 6 - \frac{y}{12} = \frac{72 - y}{12}$$

  and using separation of variables we get

$$\begin{aligned} \frac{dy}{dt} = \frac{72 - y}{12} &\iff \frac{dy}{72 - y} = \frac{dt}{12} \\ &\iff -\ln|72 - y| = \frac{t}{12} + C_1 \\ &\iff \ln|72 - y| = \frac{-t}{12} + C_2 \end{aligned}$$

$$
\begin{aligned}
\iff & \quad |72 - y| = C_3 e^{-\frac{t}{12}} \\
\iff & \quad 72 - y = k e^{-\frac{t}{12}} \\
\iff & \quad y = 72 - k e^{-\frac{t}{12}}.
\end{aligned}
$$

Solving the IVP by using $y(0) = 5$, we get

$$
\begin{aligned}
y(0) = 5 \iff & \quad 72 - ke^0 = 5 \\
\iff & \quad k = 72 - 5 = 67
\end{aligned}
$$

so the final solution is

$$y(t) = 72 - 67e^{-\frac{t}{12}}.$$

- ***Step 5:***

  After 30 minutes there is

  $$y(30) = 72 - 67e^{-\frac{30}{12}} = 66.5 \text{ kg}.$$

  The long term behavior is simply the limit:

  $$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} 72 - 67e^{-\frac{30}{12}} = 72 - 0 = 72.$$

$\square$

Example 2.4.3 Mixing problem II. The difference here is that now we allow the total volume of fluid to vary, when before it was kept fixed.

Problem: A 400-gallon tank initially contains 200 gallons of water containing 3 pounds of sugar per gallon. Suppose water containing 5 pounds per gallon flows into the the top of the tank at a rate of 6 gallons per minute. The water in the tank is kept well mixed, and 4 gallons per minute are removed from the bottom of the tank. How much sugar is in the tank when the tank is full?

Solution:

- ***Step 1:*** Define variables

  Let $y(t)$ = amount of sugar at time t, which is in minutes. Let $y(0) = 3 \times 200 = 600$ pounds.

- ***Step 2:*** Find Rate in/ Rate out

  Note that for anything that comes in you can always find the rate in as

  $$\text{Rate in} = \begin{pmatrix} \text{concentration} \\ \text{of sugar coming in} \end{pmatrix} \times \text{Rate}.$$

  Similarly you can always find the Rate out as

  $$\text{Rate out} = \begin{pmatrix} \text{concentration} \\ \text{of sugar coming out} \end{pmatrix} \times \text{Rate}.$$

  We have

  $$
  \begin{aligned}
  \text{Rate in} = & \quad \underbrace{\left(5\frac{\text{pounds}}{\text{gallon}}\right)\left(6\frac{\text{gallons}}{\text{min}}\right)}_{\text{-sugar water solution}} \\
  = & \quad 30\frac{\text{pounds}}{\text{gallon}}.
  \end{aligned}
  $$

To find the concentration of sugar coming out we have know the amount of water at time t.

$$\begin{aligned}\text{Water at time } t = & \quad 200 \text{ gallons} + \left(6\frac{\text{gallons}}{\text{min}} - 4\frac{\text{gallons}}{\text{min}}\right) t \\ = & \quad 200 + 2t,\end{aligned}$$

So

$$\begin{aligned}\text{Rate out} = & \quad \left(\begin{array}{c}\text{concentration} \\ \text{of stuff going out}\end{array}\right) \times \text{Rate} \\ = & \quad \left(\frac{y(t)}{200+2t}\frac{\text{pounds}}{\text{gallon}}\right) \times 4\frac{\text{gallons}}{\text{min}}. \\ = & \quad 4\frac{y(t)}{200+2t}\frac{\text{pound}}{\text{min}}.\end{aligned}$$

- ***Step 3:*** Write the IVP

  Always recall that for mixing problems we have

$$\begin{aligned}\frac{dy}{dt} = & \quad \text{Rate in} - \text{Rate out} \\ = & \quad 30 - \frac{4}{200+2t}y.\end{aligned}$$

  and the initial condition

$$y(0) = 600.$$

- ***Step 4:*** Solve using the Method of integrating factors:

  Write

$$\frac{dy}{dt} + \frac{4}{200+2t}y = 30$$

  so that $g(t) = \frac{4}{200+2t}$ and $b(t) = 30$. Thus the integrating factor is

$$\mu(t) = e^{4\int \frac{dt}{200+2t}} = e^{2\int \frac{dt}{100+t}} = e^{2\ln(100+t)} = (100+t)^2.$$

  Thus using the formula, we have that

$$\begin{aligned}y(t) = & \quad \frac{1}{\mu(t)}\left[\int \mu(t)b(t)dt. + C\right] \\ = & \quad \frac{1}{(100+t)^2}\left[30\int (100+t)^2 dt. + C\right] \\ = & \quad \frac{1}{(100+t)^2}\left[30\frac{(100+t)^3}{3}. + C\right] \\ = & \quad \frac{1}{(100+t)^2}\left[10(100+t)^3 + C\right]\end{aligned}$$

  and using $y(0) = 600$ we get that

$$600 = \frac{1}{100^2}\left[10 \cdot 100^3 + C\right]$$

  so that

$$C = -4,000,000$$

  and thus

$$y(t) = \frac{10(100+t)^3 - 4,000,000}{(100+t)^2}.$$

- ***Step5:*** Answer the question

  Since the amount of water in the tank is $200 + 2t$ then it fills up when

  $$200 + 2t = 400$$

  so that $t = 100$. Thus the amount of sugar is

  $$\begin{aligned} y(100) &= \frac{10\,(200)^3 - 4,000,000}{(200)^2} \\ &= 1,900 \text{ pounds.} \end{aligned}$$

$\square$

2.5 Modeling with differential equations - more problems

2.5.1 Newton's Law of Cooling

Newton's Law of Cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and its surroundings.

That is, let $T(t)$ be the temperature of the object, while T_s is the surrounding temperature. Then by Newton's Law of cooling there is some constant of proportionality k, such that

$$\frac{dT}{dt} = k\,(T - T_s)\,.$$

Example 2.5.1 Newton's Law of Cooling. Suppose there was a murder in a room that is 70° F. Assume the victim had a temperature of 98.6° when murdered. Let t_c be the time it took for someone to finally discover the corpse since its death. As a detective, your goal is to find out how long ago the body died. Here is the given information

- Fact 1: At the time time someone discovered the body, the temperature of the corpse was 72.5.
- Fact 2: One hour after the body was discovered, the temperature of the corpse was 72.
- Question: Find the critical value of t_c

Solution: One needs to solve the following IVP: Let $T(t)$ be the temperature of the victim, then

$$\frac{dT}{dt} = k\,(T - 70)\,, \quad T\,(0) = 98.6$$

and need to use the information

$$\begin{aligned} T(t_c) &= 72.5, \\ T(t_c + 1) &= 72. \end{aligned}$$

to solve for k.

First solving for $T(t)$ we get

$$T(t) = 70 + (98.6 - 70)\,e^{kt}$$

$$= 70 + 28.6e^{kt}.$$

Then using

$$72.5 = 70 + 28.6e^{kt_c},$$
$$72 = 70 + 28.6e^{k(t_c+1)}$$

Solving the first equation for k we get

$$k = \frac{1}{t_c} \ln \frac{2.5}{28.6}$$

and plugging this into second equation we get

$$72 = 70 + 28.6e^{\frac{1}{t_c} \ln \frac{2.5}{28.6}(t_c+1)}$$

and hence

$$t_c \approx 10.92 \text{ hours.}$$

and $k = -0.223$. □

2.5.2 Free fall with friction

We will consider problems involving **free-fall** with and without initial velocities. We will also consider when there is some **air resistance** of magnitude $R(v)$ directed opposite to the velocity v.

Setting up an equation:

Assume that the positive direction is up (that is, gravity is negatively oriented).

Since we know that $F = \text{mass} \times \text{acceleration} = m\frac{dv}{dt}$. This will always be the LHS of our equation.

The RHS depends on the problem given (e.g. free fall, throwing object up? is there resistance?)

Thus our equations in free fall will be in the form

$$m\frac{dv}{dt} = \pm R(v) - mg.$$

The plus/minus symbol is present as friction/resistance is always in the opposite direction of motion.

- We'll have $-R(v)$: If object is going up, i.e. $v > 0$. (Since air resistance $R(v)$ is directed opposite to the velocity v)
- We'll have $+R(v)$ if the object is going down, i.e. $v < 0$ (Since air resistance $R(v)$ is directed opposite to the velocity v)

Example 2.5.2 Suppose an object with mass 10 kg is launched upward with initial velocity 20 m/s from a platform that is 3 meters high. Suppose there is a force due to air resistance of magnitude $|v|$ directed opposite to the velocity, where the velocity v is measured in m/s. We neglect the variation of the earth's gravitational fields with distance. (Since it's not going very high anyways)

- ***Part (a):*** Find the maximum height above the ground that the object reaches.

  Solution: Suppose we consider when the object is going up in the air before it has reached the maximum height. Let $R(v) = |v|$ be the resistance, then using what've discussed above we have

$$m\frac{dv}{dt} = -R(v) - mg,$$

and we have $-R(v)$ since the object is still going up. Since the object is going up then $v > 0$. Recall that

$$|v| = \begin{cases} v & v > 0 \\ -v & v < 0 \end{cases}$$

then

$$m\frac{dv}{dt} = -|v| - mg = -v - mg.$$

Hence

$$m\frac{dv}{dt} = -v - mg.$$

Solving this we have that

$$\begin{aligned} \int \frac{dv}{v+mg} = \frac{\int -dt}{m} &\iff \ln|v+mg| = -\frac{t}{m} + C \\ &\iff |v+mg| = Ce^{-t/m} \\ &\iff v + mg = Ce^{-t/m} \\ \\ &\iff v = Ce^{-t/m} - mg. \end{aligned}$$

Since $v(0) = 20$ Then we can solve for C and obtain (using $g = 9.8\text{m/s}^2$

$$\begin{aligned} v(t) &= (20 + mg)\, e^{-t/m} - mg, \\ &= 118e^{-t/10} - 98. \end{aligned}$$

and this equation is valid only when the object is going up.
The maximum happens when velocity is equal to zero. Thus set $v(t_1) = 0$ and we get that

$$\begin{aligned} 0 = 118e^{-t/10} - 98 &\iff t_1 = -10\ln\left(\frac{98}{118}\right) \\ &\iff t_1 \approx 1.86. \end{aligned}$$

Solve for position: We get

$$\begin{aligned} x(t) &= \int v(t)dt + C \\ &= -1180e^{-t/10} - 98t + C. \end{aligned}$$

Since $x(0) = 3$, then

$$\begin{aligned} 3 = -1180e^0 - 98 \cdot 0 + C &\iff 3 = -1180 + C \\ &\iff C = 1183. \end{aligned}$$

Thus

$$x(t) = -1180e^{-t/10} - 98t + 1183.$$

Then

$$\begin{aligned} \text{maximum height} &= x(1.86) \\ &\approx 21. \end{aligned}$$

- ***Part (b):*** Find the time that the object hits the ground, assuming it missed the platform.

  Solution: We need to find the equation of when the object is falling down.When the object is falling down we thus have the following equation:

  $$m\frac{dv}{dt} = R(v) - mg,$$

  and we have $R(v)$ since the object is going down. Thus since the object is going *down* then $v < 0$. Recall that

  $$|v| = \begin{cases} v & v > 0 \\ -v & v < 0 \end{cases}$$

  then $|v| = -v$ so that

  $$m\frac{dv}{dt} = |v| - mg = -v - mg.$$

  hence

  $$m\frac{dv}{dt} = -v - mg.$$

  Solving this we have that $v_2(t) = Ce^{-t/m} - mg$ with initial condition $v_2(0) = 0$. Thus

  Then

  $$\begin{aligned} x_2(t) &= \int v_2(t)dt + C \\ &= -980e^{-t/10} - 98t + C \end{aligned}$$

  since

  $$x_2(0) = \text{maximum height} = 21$$

  then solving for C we have

  $$x_2(t) = -980e^{-t/10} - 98t + 1001.$$

  To find out when $x_2(t)$ hits the ground we need to find t_2 such that $x_2(t_2) = 0$ thus (using a calculator)

  $$0 = -980e^{-t_2/10} - 98t_2 + 1001 \iff t_2 \approx 2.14.$$

  Thus the ball hits the ground by adding the time it takes to reach its maximum plus the time after that:

  $$t_0 = t_1 + t_2 = 1.86 + 2.14 = 4 \text{ seconds}.$$

$\square$

Example 2.5.3 Consider the same scenario as before. A object with mass 10 kg is launched upward with initial velocity 20 m/s from a platform that is 3 meters high. Except, there is a force due to air resistance of magnitude $v^2/5$ directed opposite to the velocity, where the velocity v is measured in m/s.

- ***Part (a):*** Write the differential equation for velocity, when the object is still going up.

  Solution: Let $R(v) = v^2/5$ be the resistance, then

$$m\frac{dv}{dt} = -R(v) - mg,$$

  and we have $-R(v)$ since the object is still going up. Thus

$$m\frac{dv}{dt} = -\frac{v^2}{5} - mg \iff m\frac{dv}{dt} = -\frac{v^2}{5} - 98$$

- ***Part (b):*** Write the differential equation for velocity, when the object has already reched maximum and is already going down..

  Solution: Let $R(v) = v^2/5$ be the resistance, then using the above we have

$$m\frac{dv}{dt} = R(v) - mg,$$

  and we have $R(v)$ since the object is going down. Thus

$$m\frac{dv}{dt} = \frac{v^2}{5} - 98.$$

$\square$

Example 2.5.4 Suppose we fly a plane at an altitude of 5000 ft and drop a watermelon that weighs 64 pounds vertically downward. Assume that the force of air resistance, which is directed opposite to the velocity, is of magnitude $|v|/128$. (Use $g = 32$ ft/sec^2)

Question: Find how long it takes for the watermelon to hit the ground.

Solution: Since the watermelon is falling down, $v < 0$. Hence

$$m\frac{dv}{dt} = R(v) - mg,$$

where $R(v)$ is positive. Now recall that

$$\text{weight} = mg$$

then $m = \frac{64}{32} = 2$.

Then $R(v) = |v|/128 = v/128$, and so

$$\begin{aligned} m\frac{dv}{dt} = \frac{v}{128} - mg &\iff 2\frac{dv}{dt} = \frac{v}{128} - 64 \\ &\iff \frac{dv}{dt} = \frac{v}{256} - 32 \\ &\iff \int \frac{dv}{v - 256 \cdot 32} = \int \frac{1}{256}dt \\ &\iff v(t) = Ce^{t/256} + 256 \cdot 32. \end{aligned}$$

and since $v(0) = 0$ then

$$v(t) = -256 \cdot 32e^{t/256} + 256 \cdot 32.$$

Solving for the distance traveled $x(t)$ from the ground we have

$$x(t) = -(256)^2 \cdot 32e^{t/256} + 32 \cdot (256)\, t + C$$

and letting $x(0) = 0$, then

$$x(t) = -(256)^2 \cdot 32e^{t/256} + 32 \cdot (256)\, t + (256)^2 \cdot 32$$

Then, noting that since we chose the plane to be altitude 0 and the melon falls 5000ft down,

$$x(t_0) = -5000 \iff t_0 \approx 17.88 \text{ seconds.}$$

□

2.6 Existence and Uniqueness of Solutions

2.6.1 Existence theorems

We want to know if solutions even exist to a given ODE.

- If this models a physical phenomona and no solutions exists, then there is something seriously wrong about your model.
- Why spend time trying to find a solution, and doing all the things in previous sections if no solutions exist.

By way of analogy, consider the polynomial equation

$$2x^5 - 10x + 3 = 0.$$

Plugging $x = \pm 1$ into $f(x) = 2x^5 - 10x + 5$ we get $f(1) = -5$ and $f(-1) = 11$. What can we conclude about the solution set?

- We draw a continuous sketch of this graph, and show it must cross the $x-$axis.
- By the intermediate value theorem we know that at least one solution exists, since somewhere in between $x = -1$ and $x = 1$ the function $f(x)$ must have crossed the $x-$ axis.
- There could be more than one, we'd like to know if we should stop searching for more solutions.
- This is a difficult questions. There is no "quadratic formula" for 5th degree polynomials.

On the other hand, no real solutions exist for $x^2 + 1 = 0$.

In the context of differential equations, there are theorems that tell us conditions for when a solution must **exist** for a linear first order ODE and when we know that solution is **unique**.

Theorem 2.6.1 Linear 1st order ODE Existence and Uniqueness Theorem. *If the function p and g are continuous on an open interval $I = (a, b)$ containing the point $t = t_0$, then there **exists** a **unique** function $y = \phi(t)$ that satisties the IVP*

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

for each t in I and where y_0 is an arbitrary initial value.

This theorem guarantees the existence and uniqueness of solutions under the assumption of the theorem. *This is only for IVP, nothing to do with separate solutions to ODE's* (which we already know there are many).

This theorem allows you to know the domain before even solving for the solution. That is, we know where the solution is valid, which is an important consideration in modeling problems.

Example 2.6.2 ***Part (a):*** Without solving the problem, what is the largest interval in which the solution of the given IVP is certain to exist by the Existence and Uniqueness Theorem?

$$(t-1)y' + \cos t y = \frac{e^t}{t-6} \quad y(3) = -4$$

Solution: We rewrite as

$$y' + \frac{\cos t}{(t-1)} y = \frac{e^t}{(t-6)(t-1)}$$

Since $\frac{\cos t}{(t-1)}$ and $\frac{e^t}{(t-6)(t-1)}$ are only both continuous for every $t \neq 1, 6$. The intervals are:

$$(-\infty, 1) \cup (1, 6) \cup (6, \infty).$$

But since the interval $I = (1, 6)$ is the only one that contains the initial point $t_0 = 03$ is in I. Then we know there exists a unique solution $y = \phi(t)$ on the interval $(1, 6)$.

Part (b): What if I change the initial condition to

$$y(8) = 7,$$

then what is the interval I?

Solution: Then $I = (6, \infty)$. □

Example 2.6.3 ***Part (a):*** Without solving the problem, what is the largest interval in which the solution of the given IVP is certain to exist by the Existence and Uniqueness Theorem?

$$t^2 y' + \frac{\ln(t-1)}{e^{t-2}} y = \frac{t-5}{\sin(t-4)} \quad y(3) = \pi$$

Solution: We rewrite as

$$y' + \frac{\ln(t-1)}{t^2 e^{t-2}} y = \frac{t-5}{t^2 \sin(t-4)}$$

\item The function $\frac{\ln|t-1|}{t^2 e^{t-2}}$ is continuous when $t \neq 0$ and $t-1 > 0$. So continuous on $(1, \infty)$.

The function $\frac{t-5}{t^2 \sin(t-4)}$ is continuous when $t \neq 0$ and when $t - 4 \neq n\pi \implies t \neq 4 + n\pi$. So the problem points are $t = 0$ and $t = \ldots, 4-2\pi, 4-\pi, 4, 4+\pi, 4+2\pi$. Note that $4 + \pi \approx 7.14$ hence both functions are simultaneously continuous on

$$(1, 4) \cup (4, 4+\pi) \cup (4+\pi, 4+2\pi) \cup \cdots$$

since $t_0 = 3$ falls inside $(1, 4)$ then the solution to this IVP must have domain

$$I = (1, 4).$$

Part (b): What if I change the initial condition to

$$y(8) = 10,$$

then what is I?

Then $I = (4 + \pi, 4 + 2\pi)$. □

For general first order equations, the situation is more complicated. The content of the following theorem is that if the differential equation is nice enough, we can find some small circle where the solutions exist and are unique.

Theorem 2.6.4 General 1st Order ODE existence and uniqueness theorem. *Suppose $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous functions in a rectangle of the form*

$$\{(t, y) \mid a < t < b, c < y < d\}$$

*in the $ty-$plane. If (t_0, y_0) is a point inside the rectangle. then there exists a unique $\epsilon > 0$ and a **unique function** $y(t) = \phi(t)$ defined for $(t_0 - \epsilon, t_0 + \epsilon)$ that solves the initial value problem*

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0.$$

Warning: Unlike Theorem 2.6.1, the statement of Theorem 2.6.4 does not tell you what domain the solution will be valid for. In this case, you really do have to explicitly find the solution to figure out the domain of the function.

Corollary 2.6.5 *Moreover assuming the same conditions as Theorem 1, if (t_0, y_0) is a point in this rectangle and if $y_1(t)$ and $y_2(t)$ are two functions that solve the IVP*

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$, then

$$y_1(t) = y_2(t)$$

for $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

Restatement of Uniqueness Theorem: If two solutions y_1, y_2 to an ODE that satisfies the condition of the uniqueness theorem, then if they are in the same place at the same time, then they must be the same function! That is, the uniqueness condition says that if y_1, y_2 are two solutions to some ODE and y_1 and y_2 are equal at some point t_0, then $y_1(t) = y_2(t)$ for all t in some interval. *It's either all or nothing.*

Example 2.6.6 Consider

$$\frac{dy}{dt} = (y-1)^{1/2} \qquad y(0) = 1.$$

Part (a): Is this a linear or nonlinear equation? Can you use Theorem 2.6.1?

Solution: This is a nonlinear equation, due to the expression $(y-1)^{1/2}$. Since Theorem 2.6.1 only applies to linear equations, we can't use it for this IVP.

Part (b): Using Theorem 2.6.4 (the general theorem), can you guarantee that there is a unique solution to this IVP? Why?

Solution: To apply Theorem 2.6.4, we need the right hand side equation

$$f(t, y) = (y-1)^{1/2}$$

to be continuous and we need

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y-1}}$$

to be continous around the point $(t_0, y_0) = (0, 1)$. But since $\frac{1}{2\sqrt{y-1}}$ is not continuous when $y_0 = 1$, then we cannot guarantee uniqueness of the solution. (That is, this is a poor model at this point.)

Moral: There could be multiple solutions to this IVP. If a solution exists, it may not be unique! □

In summary:

- Theorem 2.6.1: Allows to check if there exists a unique solution for Linear Equations. Also tells us what the possible domain is.
- Theorem 2.6.4: Allows to check if there exists a unique solution for general first order equations. Does not tell us about possible domains.

2.6.2 More examples. Domains of solutions.

A **partial derivative** is a derivative of a function in more than one variable with respect to just one of the variables, considering the other variables as constants. For a function $f = f(t, y)$, we denote the partial derivative with respect to t by $\frac{\partial}{\partial t}f$ or by the compact notation f_t. For example, take the partials of $y^2 + t^2, yt$ and y^2t.

$$\frac{\partial}{\partial y}[y^2 + t^2] = 2y + 0 = 2y$$

and

$$\frac{\partial}{\partial t}[y^2 + t^2] = 0 + 2t = 2t.$$

For the second function,

$$\frac{\partial}{\partial y}y^2t = 2yt$$

and

$$\frac{\partial}{\partial t}y^2t = y^2.$$

Notice that Theorem 2.6.4 only gives you a function $y(t)$ defined for some interval $(t_0 - \epsilon, t_0 + \epsilon)$. We need to keep in mind that the number $\epsilon > 0$ may be super small, so the solution may not be valid for big t. So this affects how we can apply this theorem real world solutions.

Example 2.6.7 Finding the domain of a solution. Consider

$$\frac{dy}{dt} = 1 + y^2 \qquad y(0) = 0.$$

Part (a): Find where in the $t - y$ plane the hypothesis of Theorem 2.6.4 is satisfied:

Solution: Note that $f(t, y) = 1 + y^2$ and $\frac{\partial f}{\partial y} = 2y$ are always continuous, thus satisfied in all of $\mathbb{R}^2$.

Part (b): Find the actual interval in which the IVP exists uniquely. We ***must*** keep in mind that the solution could blow up at an asymptote.

Solve using separable equations and get $y(t) = \tan(t + c)$ and with initial condition you get $y(t) = \tan(t)$. But this solution is only valid for $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

```
t,y = var('t,y')
g = Graphics()
g += plot_slope_field(1 + y^2, (t,-5,5), (y,-5,5))
g += plot(tan(t), (t,-5,5), ymax=5, ymin=-5)
g.show()
```

Moral: Unlike Theorem 2.6.1, the conclusion of Theorem 2.6.4 does not say that the solution needs to exist in the entire rectangle, it just says that there exists some (possibly mysterious) interval in which it exists. □

Example 2.6.8 Lack of uniqueness. Consider the problem $\frac{dy}{dt} = 3y^{2/3}$ and $y(0) = 0$.

Question 1: Show that $y_1(t) = 0$ and $y_2(t) = t^3$ are two solutions to this IVP. Why does this not contradict the uniqueness assertion of Theorem 2.6.4?

Solution: We know the equilibrium $y_1(t) = 0$ which solves the IVP is one solution. Use separation of variables to get $y(t) = (t + c)^3$ so that $y_2(t) = t^3$.

That is, the solution through $(0, 0)$ is ***NOT UNIQUE!***

This doesn't contradict Theorem 2.6.4 because we can compute $\frac{\partial f}{\partial y} = 2y^{-\frac{1}{3}} = \frac{2}{y^{1/3}}$. This is not continuous at $(t_0, y_0) = (0, 0)$, and so the theorem does not apply.

Question 2: Consider the problem $\frac{dy}{dt} = 3y^{2/3}$ and $y(1) = 1$. Find where in the $t - y$ plane solutions exist uniquely.

Solution: Solutions exist uniquely in any rectangle not containing $(0, 0)$.

□

Example 2.6.9 Applications of uniqueness: comparing solutions. Consider the equation $\frac{dy}{dt} = \frac{(1+t)^2}{(1+y)^2}$.

We can easily check that $y_1(t) = t$ is a solution. Now say that $y_2(t)$ is the unique solution to the IVP

$$\frac{dy}{dt} = \frac{(1+t)^2}{(1+y)^2} \quad y(0) = -.1.$$

It is very hard to solve this equation. But we can still say something about the behavior of y_2. In particular, $y_2(t)$ can't cross the other solution $y_1(t) = t$. Then we can say that

$$y_2(t) < t$$

for all t, which is a useful and striking bound on the behavior of y_2.

```
t,y = var('t,y')
g = Graphics()
g += plot_slope_field((1+t)^2/(1+y)^2, (t,-5,5), (y,-5,5))
g += plot(t, (t,-5,5), ymin=-5, ymax = 5)
Y =function('Y')(t)
ode = diff(Y,t) == (1+t)^2/(1+Y)^2
g += desolve_rk4(ode, Y, ivar=t, ics=[0,-.1], step=.1,
    end_points=[-5,5], output = 'plot', xmin=-5, xmax = 5,
    color='red')
g.show()
```

□

2.6.3 Summary

- We must check continuity conditions to have uniqueness and existence.
- Uniqueness implies that solutions can't cross each other.
- Specifically, uniqueness implies that solutions can't cross equilibrium solutions.
- We can use uniqueness to say that solutions are between other solutions, which can give strong qualitative understanding.

2.7 Autonomous Equations and Population dynamics

Definition 2.7.1 An **autonomous differential equation** is of the form

$$\frac{dy}{dt} = f(y).$$

$\Diamond$

We will only deal with autonomous equations in this section. Autonomous equations are preferable for some physical models are autonomous (self-govering). For example a compressed spring has the same force at 4:00am and at 10:00pm.

2.7.1 Examples of autonomous systems

2.7.1.1 Population growth/decay

Assumption: The rate of growth of the population is proportional to the size of the population. Thus if k is the proportionality constant (growth rate) we have

$$\frac{dP}{dt} = kP.$$

But here P is the dependent variable, t is time, which is the independent variable. Thus $P = P(t)$ is actually a function! This is a ODE. We can also write it $P' = kP$, or the physics way, $\dot{P} = kP$.

2.7.1.2 Logistic Growth:

Assumption: If population is small, then rate of growth is proportional to its size. If population is too large to be supported by its resources and environment, then the population will decrease, that $\frac{dP}{dt} < 0$. We can restate the assumptions as

1. $\frac{dP}{dt} \approx kP$ if P is small.
2. If $P > N$ then $\frac{dP}{dt} < 0$.

In this case, we have the **logistic growth model**

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P$$

2.7.2 Phase lines

Suppose $\frac{dy}{dt} = y(1-y)$, which has the slope field

```
t, y = var('t,y')
plot_slope_field(y*(1-y), (t,-3,3), (y, -3, 3))
```

Since the slopes are the same at each horizontal direction we can compress this information to something *easier* to draw, the **phase line** for the autonomous equation.

Rope Metaphor: We can reduce the entire 2d picture into a “rope” that the function climbs up and down.

1. Start with IVP $\frac{dy}{dt} = f(y)$ and $y(0) = y_0$.
2. Draw a rope at start at y_0.
3. At each y write $f(y)$ on this rope to indicate the slope at that y.
4. If $f(y) = 0$ stay put. If $f(y) > 0$ then climb up the rope, if $f(y) < 0$ then climb down the rope.
5. Bigger values for $f(y)$ means climb faster as t moves through time.
6. If you let $y(t)$ your location on the rope, then $y(t)$ is a solution to the IVP.

The rope in Figure 2.7.2 is the Phase line, but instead of numbers we use arrows to represent the slope.

The phase line for this equation has two points representing the equilibrium solutions, and arrows indicating the sign of the slopes given by $f(y)$ between the equilibira.

Figure 2.7.2 Phase line for $\frac{dy}{dt} = y(1-y)$

For example, the phase line shows that as y is close to $y = 1$ from below, then the function keeps increasing, and thus must approach asymptotically to the equilibrium solution.

A sketch of some possible solutions looks like:

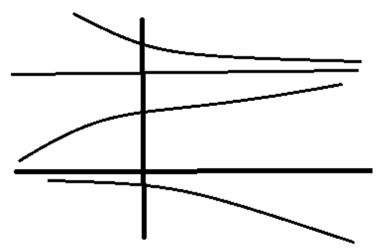


Figure 2.7.3 Solution sketch for $\frac{dy}{dt} = y(1-y)$

From our first sketch we can always notice the following things about sketching curves:

1. If $f(y(0)) = 0$ then $y(0)$ is an equilibirum solution and $y(t) = y(0)$ for all t.
2. If $f(y(0)) > 0$ then $y(t)$ is *increasing* for all t and either $y(y) \to \infty$ as $t \to \infty$ or $y(t)$ tends to first equilibirum point *larger* than $y(0)$.
3. If $f(y(0)) < 0$ then then $y(t)$ is *decreasing* for all t and either $y(y) \to -\infty$ as $t \to \infty$ or $y(t)$ tends to first equilibirum point *smaller* than $y(0)$.

Example 2.7.4 Curve Sketching. We let

$$\frac{dy}{dt} = (2 - y)\sin y.$$

1. Find equilibrium points $y = 2$ and $y = n\pi$ (so infinite amount)
2. Plug points and get that the phase line is :

3. Talk about what happens when things are getting close to the equilibrium solutions.

4. Sketch curves:

$\square$

Example 2.7.5 We don't know how quickly things jump. Show that the graph $\frac{dP}{dt} = (1 - \frac{P}{20})^3(\frac{P}{5} - 1)P^7$ has Phase line $[\ominus 20 \oplus 5 \ominus 0 \oplus]$ $\begin{array}{c}\vee\\ 20\\ \wedge\\ 5\\ \vee\\ 0\\ \wedge\end{array}$ but 5 jumps to 20 very quickly (like 0.00001 quick.

```
t,y=var('t,y')
plot_slope_field((1 - y/20)^3 * (y/5 - 1) *y^7, (t,-5,5),
    (y,0, 20))
```

$\square$

Example 2.7.6 Not all solutions exist for all t. Consider the equation $\frac{dy}{dt} = (1 + y)^2$.

The phase line is $[\ominus - 1\oplus]$ $\begin{array}{c}\wedge\\ -1\\ \wedge\end{array}$ Sketch a curve.

These increasing/decreasing behaviors could be asymptotes. (Phase LINE DOES NOT TELL US THIS INFO)

ACTUAL SOL: $y(t) = -1 - \frac{1}{t+c}$. Asymptote at $t = c$.

If $y(0) > -1$ then draw possible curve.

```
t,y=var('t,y')
g = Graphics()
g+= plot_slope_field((1+y)^2, (t,-5,5),(y,-5,5))
g+= plot(-1-1/(t + 2), (t,-5,5), ymax = 5, ymin = -5)
g.show()
```

$\square$

Example 2.7.7 Cusps. Consider the equation $\frac{dy}{dt} = \frac{1}{1-y}$.

The phase line would be:

We can sketch "cusp-like" curves. Once a curve has fallen into a hole once it reaches the dotted line.

```
t,y=var('t,y')
g = Graphics()
g+= plot_slope_field(1/(1-y), (t, -5, 5), (y, -5, 5))
g+= plot(1 + sqrt(-2*t + 4), (t, -5, 2))
g+= plot(1 - sqrt(-2*t -2), (t, -5, -1), color="red")
g.show()
```

$\square$

Role of Equilibrium points:

The solutions to autonomous equations either

1. Tend to $\pm\infty$
2. Tend to the equilibrium solutions.
3. Stay consistently increasing/decreasing within equilibrium solutions.

2.7.3 Classification of Equilibrium Solutions

Recall what **asymptotic** means: say that f is asymptotic to the line $y = c$ if

$$\lim_{t\to\infty} f(t) = c.$$

We can classify the equilibrium solutions to an autonomous equation by looking at the behavior of "nearby" solutions. Solutions fall into one of three categories.

1. ***Asymptotically stable (sink)***
   (a) y_0 is an **asymptotically stable** equilibrium if any solution with initial condition sufficiently close to y_0 is asymptotic to y_0 as t increases.
   (b) Phase Line looks like this: $[\ominus y_0 \oplus]$ $\begin{matrix}\vee\\ y_0\\ \wedge\end{matrix}$
   (c) Graph looks like: (reminds you that it is falling into something)
   (d) In a graph of $f(y)$ vs. y, we have $f'(y_0) < 0$.
2. ***Asymptotically unstable (source):***
   (a) y_0 is an **asymptotically unstable** equilibrium if any solution with initial condition sufficiently close to y_0 tends torward y_0 as t decreases.
   (b) The phase line looks like this: $[\oplus y_0 \ominus]$ $\begin{matrix}\wedge\\ y_0\\ \vee\end{matrix}$
   (c) Graph looks like: (reminds you that it is coming from one place)
   (d) In $f(y)$ vs. y graph, we have $f'(y_0) > 0$.
3. ***Semistable:***
   (a) y_0 is an ***asymptotically semistable*** equilibrium if it doesn't fit the category of a sink or source \item Phase Line looks like this: $[\oplus y_0 \oplus]$ $\begin{matrix}\wedge\\ y_0\\ \wedge\end{matrix}$ or $[\ominus y_0 \ominus]$ $\begin{matrix}\vee\\ y_0\\ \vee\end{matrix}$
   (b) Graph looks like:

Example 2.7.8 Drawing solution from the $f(y)$ vs. y graph). Consider the equation $\frac{dy}{dt} = y^2 + y - 6 = (y + 3)(y - 2)$.

The phase line is $[\oplus 2 \ominus -3 \oplus]$ $\begin{array}{c} \wedge \\ 2 \\ \vee \\ -3 \\ \wedge \end{array}$

How can these be classified? □

Example 2.7.9 (Using $f(y)$). We can figure out classification directly from the graph of $f(y)$.

Here, node means semistable, sink means stable, and source means unstable. □

Example 2.7.10 Suppose we only know the graph of $f(y)$ not the actual formula.

Then draw phase line : $[\ominus c \oplus b \ominus a \oplus]$ $\begin{array}{c} \vee \\ c \\ \wedge \\ b \\ \vee \\ a \\ \wedge \end{array}$ Now sketch some solution curves.

□

2.8 Exact equations

This section introduces a family of equations that arise naturally in physical contexts. For example, suppose that we had a detailed temperature map of a hot metal sheet. Can we predict how the heat will flow? The answer to this question is provided by the gradient, studied in multi-variable calculus. But what about the opposite scenario? Given a heat flow map, can we reconstruct the original temperature distribution? To answer this question, we'll consider what are known as **exact equations**. First, we recall some multivariable calculus.

2.8.1 Partial derivatives and the gradient

Suppose that $z = f(x, y)$ is a function of two independent variables. Just as in one variable, we want to understand how the graph of f changes at a point, but now we have lots of different directions to look at. To compute the rate of change in the x and y directions, we can use the **partial derivative** in those directions. When we take a partial derivative with respect to a variable, we treat all other variables as constant so that we're isolating our view to change in that specific direction.

Definition 2.8.1 Let $z = f(x, y)$. The partial derivative of f with respect to x is

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

The partial derivative of f with respect to y is

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

◊

In practice, we can use the single variable differentiation rules, treating the other variable like it is a fixed number. For example,

$$\frac{\partial}{\partial x}x^2 + x\cos y + y^2 = 2x + \cos y + 0.$$

Frequently, when given a function $z = f(x, y)$, we wish to know the direction of greatest slope or rate of change. For example, if $z = f(x, y)$ represents the height of a mountain, the direction that water flows downhill will be in the direction of sleepest descent. We can use the partial derivatives to define a **vector field** that at each point (x, y) gives the direction of greatest slope of the graph of f.

Definition 2.8.2 The **gradient** of $z = f(x, y)$ is the vector-valued function

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle.$$

This is written shorthand as $\nabla f = \langle f_x, f_y \rangle$. ◊

You should think of the gradient in this context as a slope field - at every point in the $x - y$ plane, the gradient attaches a vector that indicates the direction of greatest slope.

If we think of $z = f(x, y)$ as a voltage map or a temperature map or a height map, a standard visualization is to use curves to represent points that have equal heights. Such curves are called **isotherms** or **equipotential lines** or altitude lines.

Example 2.8.3 Equipotentials and surfaces. Let $f(x, y) = -x^2 + 3y - y^2$. The following code will plot some equipotentials of f. In particular, we will plot the curves corresponding to $z = 0, 1, 2$.

```
x,y = var('x,y')
g = Graphics()
g+= implicit_plot(-x^2  + 3*y - y^2, (x, -2,2), (y, 0,3),
    color = "red" )
g+=implicit_plot(-x^2  + 3*y - y^2 - 1, (x, -2,2), (y, 0,3),
    color="yellow" )
g+= implicit_plot(-x^2  + 3*y - y^2-2, (x, -2,2), (y, 0,3),
    color="green" )
g.show()
```

Now compare to the surface itself:

```
var('x␣y␣z')
f(x, y) = -x^2  + 3*y - y^2
P = implicit_plot3d(f-z, (x ,-1, 3), (y, 0, 3), (z, -2, 3))
Q = plot3d(0, (-1,3), (0,3), color="red", opacity=".4")
R = plot3d(1, (-1,3), (0,3), color="yellow", opacity = ".4")
S = plot3d(2, (-1,3), (0,3), color ="green", opacity = ".4")
P + Q +R +S
```

□

Example 2.8.4 Relationship between equipotentials and gradients. One of the most important geometric facts about the relationship between the **potential function** f and the gradient field ∇f is that *equipotentials are perpendicular to gradients.* Using our previous example:

```
x,y = var('x,y')
g = Graphics()
g+= implicit_plot(-x^2  + 3*y - y^2, (x, -2,2), (y, 0,3),
    color = "red" )
g+=implicit_plot(-x^2  + 3*y - y^2 - 1, (x, -2,2), (y, 0,3),
    color="yellow" )
g+= implicit_plot(-x^2  + 3*y - y^2-2, (x, -2,2), (y, 0,3),
    color="green" )
g+= plot_vector_field((-2*x, 3 - 2*y), (-2,2), (0,3))
g.show()
```

□

In summary, given a **potential function** $z = f(x, y)$,

1. We can find the equipotential lines (lines of constant height)
$$C = f(x, y);$$
2. we can find the gradient field
$$\nabla f = \langle f_x, f_y \rangle;$$
3. and we know that at a given point, the equipotential and the gradient line are perpendicular.

2.8.2 From gradient field to potential function

How do we know when we can go the other direction? That is, as we asked at the top of the section, when given a vector field $F(x, y) = (M(x, y), N(x, y))$, how can we recover a potential function f? Essentially, this is asking us to find a function so that $\nabla f = F$ (the "antiderivative of F is f"). Like integration problems, this may not always exist.

Definition 2.8.5 A vector field F is called **conservative** if there exists a potential function f so that $\nabla f = F$. ♢

One marker of nice functions in two variables is the conclusion of **Clairaut's theorem**, which states that the **mixed partial derivatives** of f are equal - that is, for a nice f,
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$
Suppose for the moment that a vector field
$$F = \langle M, N \rangle = \nabla f = \langle f_x, f_y \rangle$$
and that the derivatives $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ all exist and are continuous. Clairaut's theorem will force
$$\frac{\partial M}{\partial y} = f_{xy} = f_{yx} = \frac{\partial N}{\partial x}.$$
It turns out to be the case that this condition, $M_y = N_x$, is not only necessary but sufficient on nice enough domains like rectangles.

Theorem 2.8.6 *Let $F = \langle M(x, y), N(x, y) \rangle$ be a vector field so that the partial derivatives of M and N exist and are continuous on a rectangle $a \leq x \leq b, c \leq y \leq d$. Then there exists a potential function f so that $\nabla f = F$ if*
$$M_y = N_x.$$

2.8.3 Differential equations and equipotentials

The **total derivative** of a function $z = f(x, y)$ is given by the expression

$$df = f_x\, dx + f_y\, dy.$$

Notice that the functions that appear as components in df are the components of the gradient ∇f. If we were given the equation of an equipotential for f, say

$$f(x, y) = C,$$

then the total derivative of the equation is

$$\begin{aligned} df &= dC \\ \Rightarrow f_x\, dx + f_y\, dy &= 0 \\ \Rightarrow M\, dx + N\, dy &= 0. \end{aligned}$$

If we have further that f has continuous second partial derivatives on a rectangle $a \le x \le b, c \le y \le d$, then Clairaut's theorem gives

$$M_y = f_{xy} = f_{yx} = N_x.$$

The upshot of all of this is that we can view an equation of the form

$$M\, dx + N\, dy = 0$$

as a differential equation that seeks to find the equipotentials $f(x, y) = C$ for some unknown function f with $\nabla f = \langle M, N\rangle$.

2.8.4 Exact equations

Consider an equation $M(x, y)dx + N(x, y)dy = 0$. We say this equation is **exact** if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; that is, as discussed in the previous section, the equation represents a differential equation that seeks to find equipotentials of a function f so that $\nabla f = \langle M, N\rangle$.

Example 2.8.7 Suppose $\frac{dy}{dx} = \frac{-2x-y^2}{2xy}$. We can rewrite this as $\left(2x + y^2\right) dx + 2xydy = 0$ then $M = 2x + y^2$ and $N = 2xy$. Computing the partial derivatives,

$$\begin{aligned} M_y &= 2y \\ N_x &= 2y \end{aligned}$$

are $M_y = N_x$. Thus this equation is exact. □

Theorem 2.8.8 *If M, N, M_y, N_x are all continuous on a rectangle $[a, b] \times [c, d]$ and*

$$M\, dx + N\, dy = 0$$

is exact then there exists a function ψ such that

$$\psi_x(x, y) = M(x, y) \text{ and } \psi_y(x, y) = N(x, y)$$

and such that $\psi(x, y) = C$ gives an implicit solution to the ODE.

Proof. If ψ satisfies $\psi_x = M$ and $\psi_y = N$ such that $\psi(x, y) = C$ then ψ defines a function $y = \phi(x)$ implicitly. Then we show $\phi(x)$ solves the ODE. Note that $0 = M(x, y) + N(x, y)y' = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\left(\psi\left(x, \phi(x)\right)\right)$ by the multivariable chain rule. Thus if we integrate both sides

$$\begin{aligned} 0 &= \frac{d}{dx}\left(\psi\left(x, \phi(x)\right)\right) \\ \iff \int 0 dx &= \int \frac{d}{dx}\left(\psi\left(x, \phi(x)\right)\right) dx \\ \iff c &= \psi\left(x, \phi(x)\right), \end{aligned}$$

as needed. ■

Solving exact equations: If $Mdx + Ndy = 0$ is exact then

$$\begin{array}{ll} \psi_x = M(x,y) & \Longrightarrow \psi = \int M(x,y)dx + h(y) \\ & \quad\Downarrow \\ \psi_y = N(x,y) & \quad \psi_y = \frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + h'(y) \end{array}$$

and then solve for $h(y)$.

Another way: One may also solve it by starting with the second equation:

$$\begin{array}{ll} \psi_x = M(x,y) & \quad \psi_x = \frac{\partial}{\partial x}\left(\int N(x,y)dx\right) + g'(x) \\ & \quad\Uparrow \\ \psi_y = N(x,y) \Longrightarrow & \quad \psi = \int N(x,y)dy + g(x). \end{array}$$

Example 2.8.9 We know $\left(2x + y^2\right) dx + 2xydy = 0$ is exact.

1. Show it's exact(done earlier) and follow the arrows until you close the diagram:

$$\begin{array}{ll} \text{Start here:} \psi_x = 2x + y^2 & \Longrightarrow \psi = \int \left(2x + y^2\right) dx + h(y) \\ & \quad \boldsymbol{\psi = x^2 + y^2 x + h(y)} \\ & \quad \Downarrow \\ \psi_y = 2xy & \Longleftarrow \psi_y = 2xy + h'(y) \end{array}$$

2. Solve for $h(y)$ by noting that since

$$\begin{aligned} 2xy = 2xy + h'(y) &\Longrightarrow h'(y) = 0 \\ &\Longrightarrow h(y) = C. \end{aligned}$$

3. Put it all together and get $\psi(x, y) = x^2 + y^2 x + C$ and hence the *implicit solution* is $x^2 + y^2 x = C$.

□

Example 2.8.10 Solve $\left(y \cos x + 2xe^y\right) + \left(\sin x + x^2 e^y - y^2\right) y' = 0$.

1. To show it's exact note that $\left(y \cos x + 2xe^y\right) dx + \left(\sin x + x^2 e^y - y^2\right) dy = 0$, and not hard to see that

$$\begin{aligned} M_y &= \cos x + 2xe^y \\ N_x &= \cos x + 2xe^y \end{aligned}$$

and they are equal, thus this ODE is exact. Follow the arrows until close the diagram:

$$\begin{array}{lll}
\text{Start here:}\psi_x = y\cos x + 2xe^y & \implies \psi = \int (y\cos x + 2xe^y)\,dx + h(y) \\
 & \boldsymbol{\psi = y\sin x + x^2e^y + h(y)} \\
 & \Downarrow \\
\psi_y = \sin x + x^2e^y - y^2 & \Longleftarrow \psi_y = \sin x + x^2e^y + h'(y)
\end{array}$$

2. Solve for $h(y)$ by noting that since

$$\begin{aligned}
\sin x + x^2e^y - y^2 = \sin x + x^2e^y + h'(y) &\implies h'(y) = -y^2 \\
&\implies h(y) = -\frac{y^3}{3}
\end{aligned}$$

3. Put it all together and get $\psi(x,y) = y\sin x + x^2e^y - y$ and hence the *implicit solution* is $y\sin x + x^2e^y - \frac{y^3}{3} = C$.

```
var('x,y')
P = plot_vector_field((y*cos(x) + 2*x*e^y, sin(x) + x^2*e^y
    - y^2), (-3,3),(-3,3))
Q = implicit_plot(y *sin(x) + x^2*e^y - y^3/3 -
    5,(-3,3),(-3,3), color="red")
R = implicit_plot(y *sin(x) + x^2*e^y - y^3/3 -
    3,(-3,3),(-3,3), color ="orange")
S = implicit_plot(y *sin(x) + x^2*e^y - y^3/3,(-3,3),(-3,3),
    color ="blue")
P + Q +R +S
```

$\square$

Example 2.8.11 Find the value of b for which the given equation is exact, and then solve it using that b: $\left(xy^2 + bx^2y\right)dx + (x+y)\,x^2dy$

1. If this equation is exact then $M_y = N_x$,

$$\begin{aligned}
M_y &= 2xy + bx^2 \\
N_x &= 3x^2 + 2yx
\end{aligned}$$

and are only equal when $b = 3$. Follow the arrows until close the diagram:

$$\begin{array}{ll}
\text{Start here:}\psi_x = xy^2 + 3x^2y & \implies \psi = \int \left(xy^2 + 3x^2y\right)dx + h(y) \\
 & \boldsymbol{\psi = \frac{1}{2}x^2y^2 + x^3y + h(y)} \\
 & \Downarrow \\
\psi_y = x^3 + x^2y & \Longleftarrow \psi_y = x^2y + x^3 + h'(y)
\end{array}$$

2. Solve for $h(y)$ by noting that since

$$\begin{aligned}
x^3 + x^2y = x^2y + x^3 + h'(y) &\implies h'(y) = 0 \\
&\implies h(y) = C
\end{aligned}$$

3. Put it all together and get $\psi(x,y) = \frac{1}{2}x^2y^2 + x^3y + C$ and hence the *implicit solution* is $\frac{1}{2}x^2y^2 + x^3y = C$.

$\square$

Example 2.8.12 Solve $(x\cos x + e^y)\,dx + xe^y dy$

1. If this equation is exact then $M_y = N_x$, and
$$M_y = e^y$$
$$N_x = e^y$$
Now note that it is actually easier to integrate N with respect to y: Thus we can start the diagram in the other direction
$$\begin{array}{lcl} \psi_x = x\cos x + e^y & \Longleftarrow = \psi_x = e^y + g'(x) \\ & \boldsymbol{\psi = xe^y + g(x)} \\ & \Uparrow \\ \text{Start here:} \psi_y = xe^y & \Longrightarrow \psi_y = \int (xe^y)\,dy + g(x) \end{array}$$

2. Solve for $g(x)$ by noting that since
$$x\cos x + e^y = e^y + g'(x) \implies g'(x) = x\cos x$$
but at the end of the day we can't avoid the harder integration, as we still need to integration by parts to $g(x) = x\sin x + \cos x$

3. Put it all together and get $\psi(x, y) = xe^y + x\sin x + \cos x$ and hence the *implicit solution* is $xe^y + x\sin x + \cos x = C$.

```
var('x,y')
P = plot_vector_field((x*cos(x) + e^y, x*e^y), (-3,3),(0,5))
Q = implicit_plot(x*e^y + x*sin(x) + cos(x) -
    120,(-3,3),(0,5), color="red")
R = implicit_plot(x*e^y + x*sin(x) + cos(x) -
    50,(-3,3),(0,5), color ="orange")
S = implicit_plot(x*e^y + x*sin(x) + cos(x) -
    10,(-3,3),(0,5), color ="blue")
P + Q +R +S
```

□

2.9 Euler's method

In practice, many if not most differential equations do not have explicit solutions. If an equation does happen to fall into a form that we have a solution method for, there is no guarantee that we can integrate the result. Thus, it is important to have approaches that can sketch curves and approximate solutions in the absence of explicit formulas.

One of the most straightforward approaches to first order equations of the form
$$\frac{dy}{dt} = f(t, y)$$
is **Euler's method**, which approximates a solution to an initial value problem with small pieces of tangent line.

Suppose we are given an initial value problem
$$\frac{dy}{dt} = f(t, y) \qquad y(t_0) = y_0.$$

- Let $h =$ step size. These are our $t-$axis increments.
- Let $t_0 =$ our starting point. Then our next point will be $t_1 = t_0 + h$, then $t_2 = t_1 + h$. Notice that this means

$$t_{k+1} - t_k = h.$$

For example suppose $t_0 = 1$ and $h = .5$, then $t_0 = 1, t_1 = 1.5, t_2 = 2, \ldots$. So how do we find the explicit values for y_k other than just guessing?

Observation 2.9.1 For small step size h, the slope of the tangent line at (t_0, y_0) is a reasonable approximation for the secant line connecting $(t_0, y(t_0))$ to the point $(t_1), y(t_1)$. That is,

$$\frac{y(t_{k+1}) - y(t_k)}{t_{k+1} - t_k} = \frac{y(t_{k+1}) - y(t_k)}{h} \approx f(t_k, y(t_k)).$$

Let $y_0 = y(t_0)$. Now, denote by y_1 the *approximation* of $y(t_1)$ given by

$$y(t_1) \approx y_1 := y_0 + f(t_0, y_0)h.$$

Iteration of this idea to produce a larger approximate graph of a solution is the key idea of Euler's method.

2.9.1 Euler's method

Definition 2.9.2 Given an initial condition $y(t_0) = y_0$ and step size h, compute (t_{k+1}, y_{k+1}) from the preceding point (t_k, y_k) as follows:

$$\begin{aligned} t_{k+1} &= t_k + h \\ y_{k+1} &= y_k + f(t_k, y_k)\, h. \end{aligned}$$

$\diamondsuit$

Example 2.9.3 Suppose we have the autonomous equation

$$\frac{dy}{dt} = 2y - 1 \qquad , y(0) = 1,$$

with $h = 0.1$ and $0 \leq t \leq 1$.

1. Our first point is $(t_0, y_0) = (0, 1)$.
2. We can compute the formula for this and get $t_{k+1} = t_k + .1$ and notice that $f(t, y) = 2y - 1$.

$$y_{k+1} = y_k + f(t_k, y_k)\, h = y_k + (2y_k - 1)(.1).$$

3. Make a table:

k	t_k	$y_k = y_{k-1} + f(t_{k-1}, y_{k-1})\, h$	$f(t_k, y_k) = 2y_k - 1$
0	0	1	1
1	0.1	$y_1 = 1 + 1 \cdot (.1) = \mathbf{1.1}$	$f(t_1, y_1) = 2(1.1) - 1 = \mathbf{1.20}$
2	0.2	$y_2 = 1.1 + (1.20) \cdot (.1) = \mathbf{1.22}$	$f(t_2, y_2) = 2(1.22) - 1 = \mathbf{1.44}$
3	0.3	$y_3 = 1.22 + (1.20) \cdot (.1) = \mathbf{1.364}$	$f(t_3, y_3) = 2(1.22) - 1 = \mathbf{1.73}$
4	0.4	1.537	2.07
	.5	1.744	2.49
	.6	1.993	2.98
	.7	2.292	3.58
	.8	2.65	4.3
	0.9	3.080	5.16
	1.0	3.596	3.596

Notice that actual value is $y(1) = \frac{e^2+1}{2} = 4.195$ and our approximation is $y(1) \approx 3.596$, which is a little short, but it makes sense all the slopes are always below the graph. □

Example 2.9.4 Our previous example didn't have any ts to plug in. So suppose we have

$$\frac{dy}{dt} = -2ty^2, \quad y(0) = 1, \quad h = \frac{1}{2}$$

1. Our first point is $(t_0, y_0) = (0, 1)$.

2. We can compute the formula for this and get $t_{k+1} = t_k + .5$ and notice that $f(t, y) = -2ty^2$.

$$y_{k+1} = y_k + f(t_k, y_k)\, h = y_k + \left(-2t_k y_k^2\right)\left(\frac{1}{2}\right).$$

3. Make a table:

k	t_k	$y_k = y_{k-1} + f(t_{k-1}, y_{k-1})\, h$	$f(t_k, y_k) = -2t_k y_k^2$
0	0	1	0
1	$\frac{1}{2}$	$y_1 = 1 + 0 \cdot (\frac{1}{2}) = \mathbf{1}$	$f(t_1, y_1) = -2\frac{1}{2}1^1 = \mathbf{-1}$
2	1	$y_2 = 1 + (-1) \cdot (\frac{1}{2}) = \mathbf{\frac{1}{2}}$	$f(t_2, y_2) = -2(1)(\frac{1}{2})^2 = -\mathbf{\frac{1}{2}}$
3	$1.5 = \frac{3}{2}$	$y_3 = \frac{1}{2} + (-\frac{1}{2}) \cdot (\frac{1}{2}) = \frac{1}{4}$	$f(t_3, y_3) = -2(\frac{3}{2})(\frac{1}{4})^2 = -\mathbf{\frac{3}{16}}$
4	2	$\frac{1}{4} + \left(-\frac{3}{16}\right) \cdot \left(\frac{1}{2}\right) = .15625$	

A plot of our approximate solution is given below:

In code, this might look like

```
var('t,y')
f(t, y) = -2*t*y^2
t0 = 0
y0 = 1
A = plot_slope_field(f, (0,3), (-1,3))

#approximate solution by Euler's method
h = .5
time = [t0 + n*h for n in range(5)]
yk = [y0]
def ynext(n):
    return yk[n-1] + f(time[n-1], yk[n-1])*h
for i in range(1,5):
    yk.append(ynext(i))
L = [[time[i], yk[i]] for i in range(5)]
B = line(L)

#actual solution
g(t) = 1/(t^2 + 1)
C = plot(g, (0,2), color = "red")
A + B + C
```

□

Chapter 3

Second Order Linear Equations

3.1 Motivation - mass-spring systems.

This chapter is concerned with **second order differential equations**, and in particular those with **constant coefficients**. That is, we're going to be spending quite a bit of time thinking about equations of the form

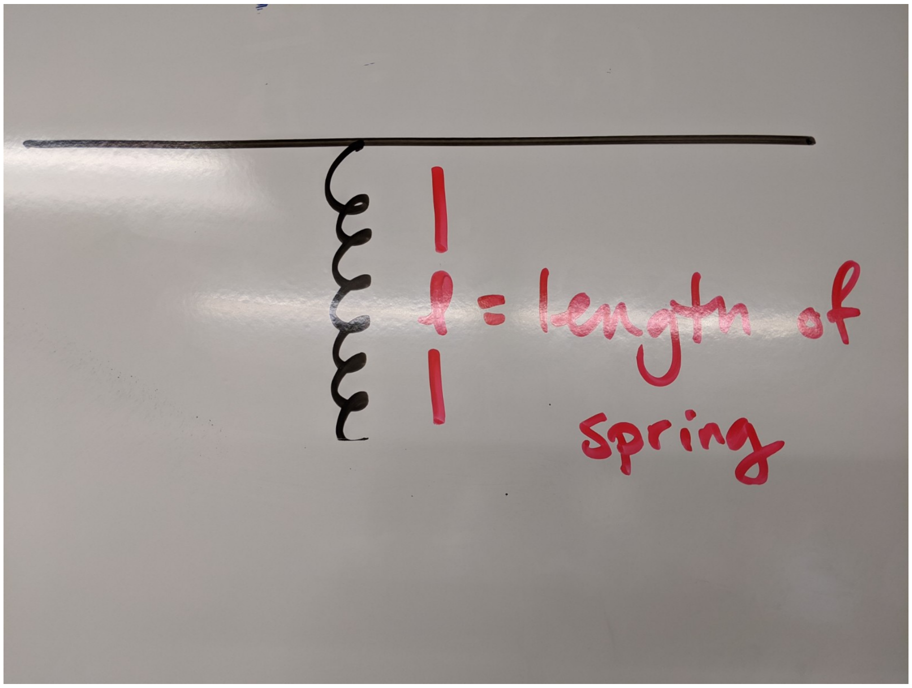
$$ay'' + by' + cy = 0$$

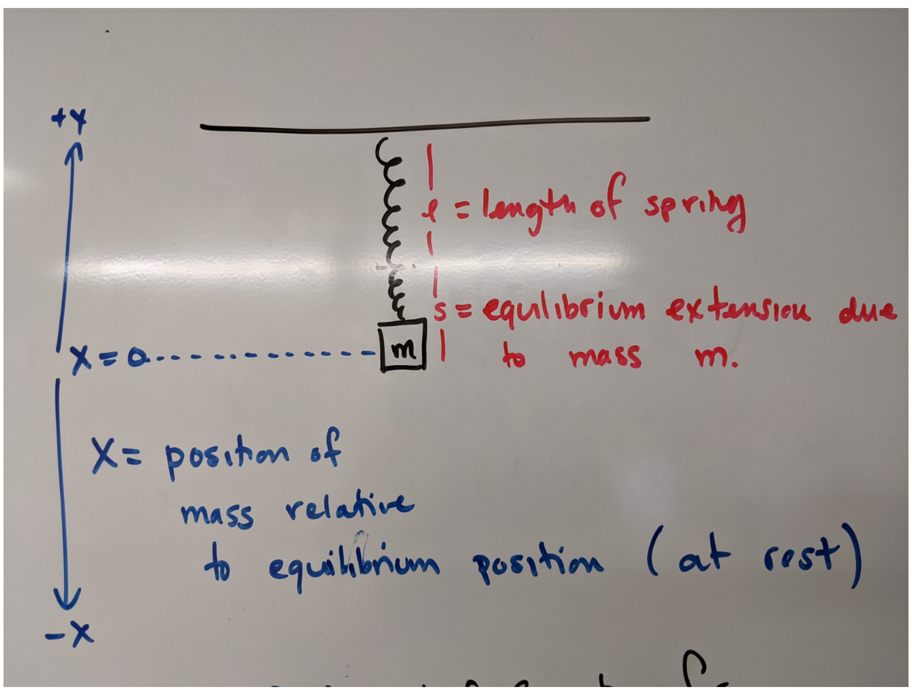
and

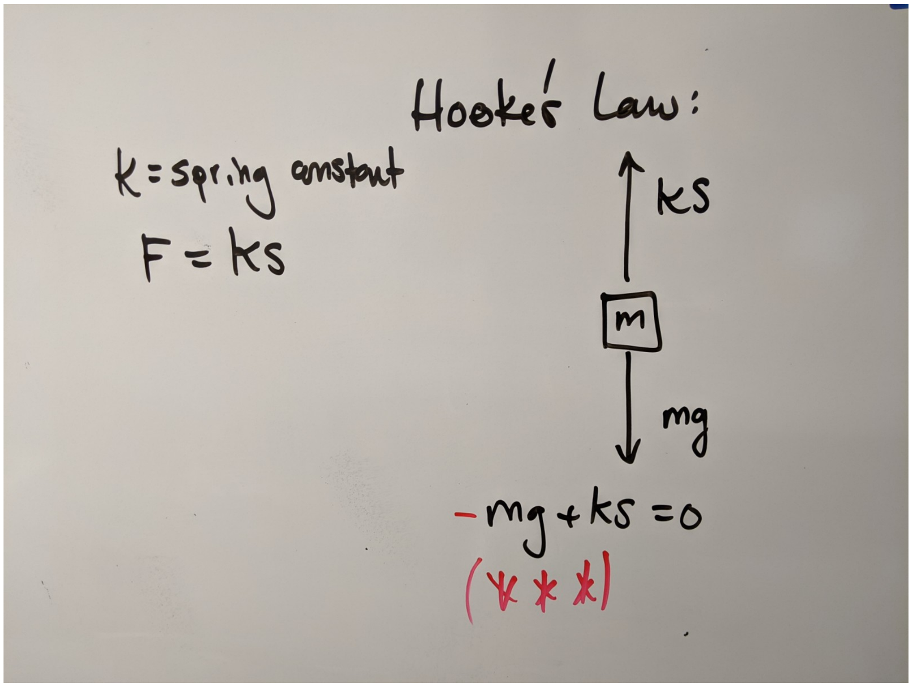
$$ay'' + by' + cy = f(t).$$

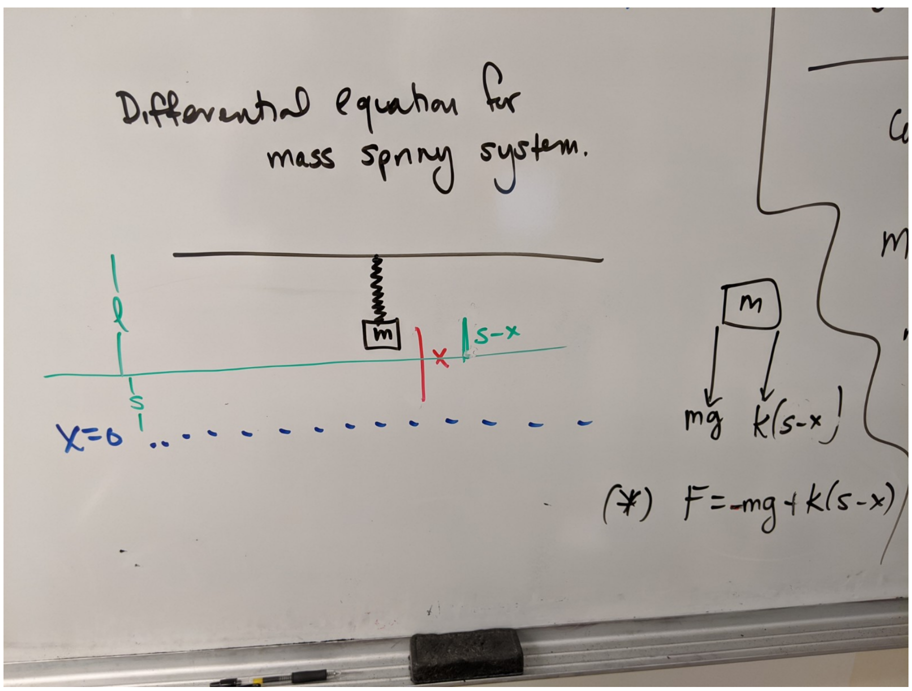
At first glance, these equations seem artificially simple in structure. However, some of the most useful differential equations in the physical sciences and mathematics have this form, which motivates our close attention to second order linear equations with constant coefficients.

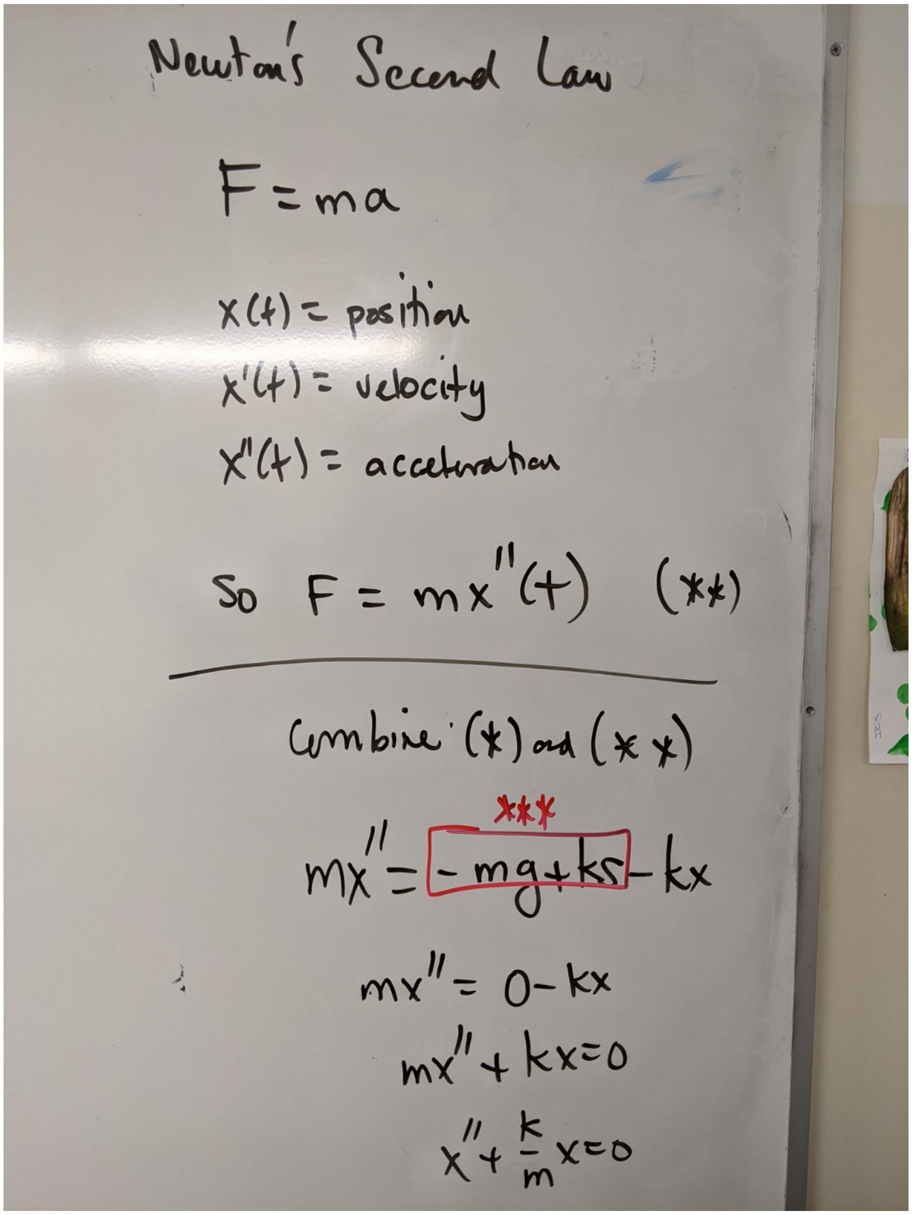
3.1.1 Undamped mass-spring systems






Hooke's Law:
k = spring constant
F = ks
ks
m
mg
−mg + ks = 0
(***)


Differential equation for mass spring system.
ℓ
m
x
s−x
s
x=0
m
mg
k(s−x)
(*) F = −mg + k(s−x)



The pictures above are a derivation of the differential equation for a simple **mass-spring system**. Notice that the resulting equation has a second derivative x'' in it - thus, this is a second order equation where the position of the mass $x(t)$ relative to the equilibrium position is a function of time t:

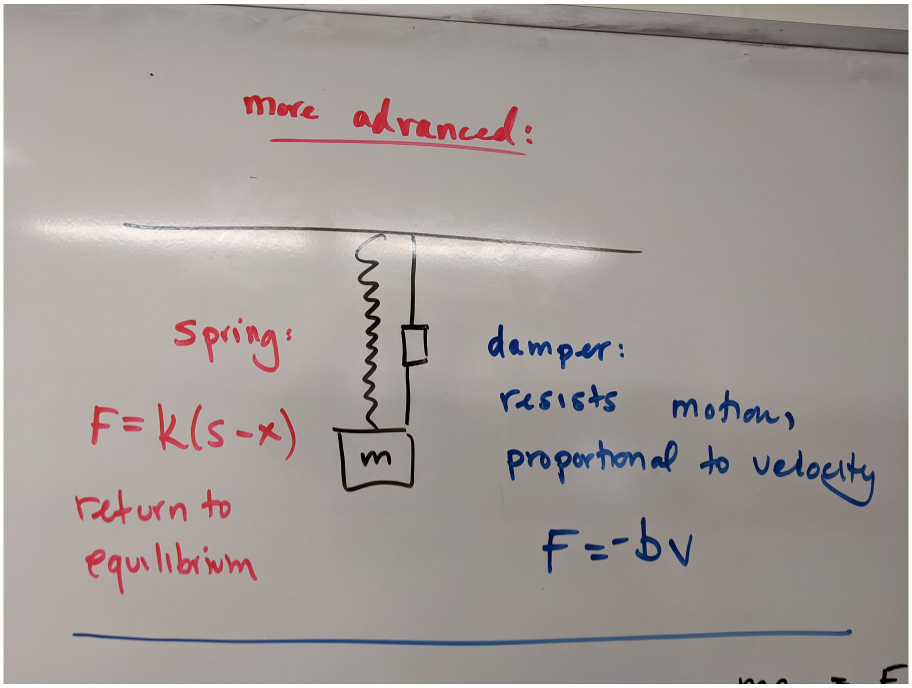
$$x'' + \frac{k}{m}x = 0.$$

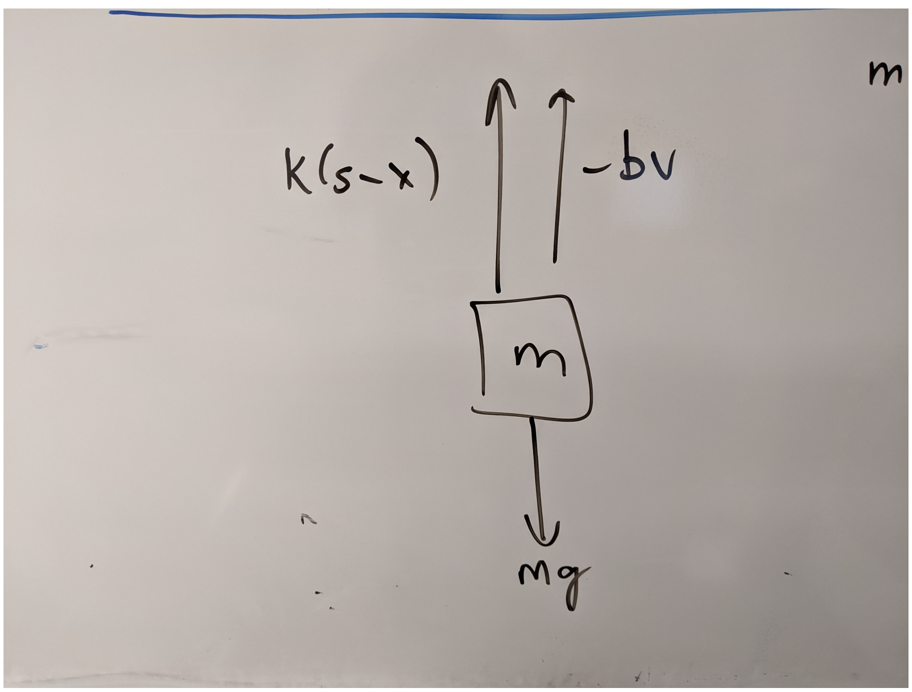
Also, note the important fact that the equation is linear and has constant coefficients. If we want to describe how the solutions to this equation behave, we should study second order equations with constant coefficients.

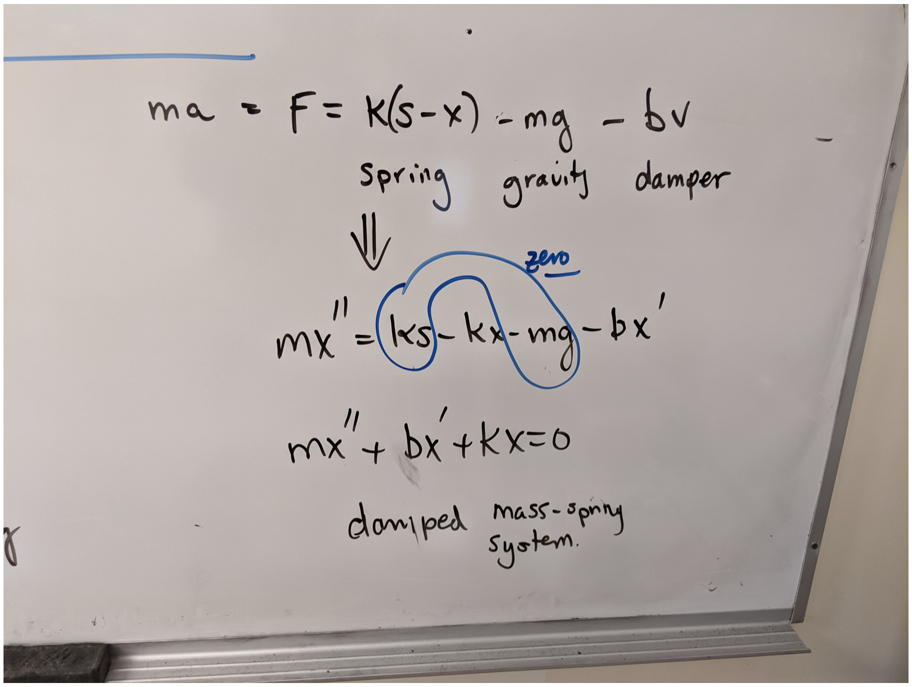
3.1.2 Damped mass-spring systems

One way to model more complicated situations with a mass-spring system is to include a **damper** that applies force against the direction of motion. The spring/shock absorber system in a car wheel is an example of a **damped**

mass-spring system. The pictures below derive the equation for this in the case that we assume that the damper exerts a force proportional to and in the opposite direction from the velocity of the mass.







Thus, the equation for a damped mass-spring system is

$$x'' + \frac{b}{m}x' + \frac{k}{m}x = 0$$

which is also a second order linear equation with constant coefficients.

The mass-spring system is one of the most useful models in all of science. For example, RLC circuits (resistor/inductor/capacitor) are typically modeled as mass-spring systems.

This motivates the study of second order linear differential equations with constant coefficients, even though that might seem like an extremely restricted family of problems to think about. In later sections, we will extend our ideas to consider what happens if an external *driving* or *forcing function* is applied to the system.

We should already have an idea about the sorts of functions that solve these systems.

1. Physical intuition tells us that when we release a mass attached to a spring from a non-equilibrium starting position, the mass oscillates up and down around the equilibrium position. This suggests that sine or cosine waves might be involved in the solution.

2. Mathematical intuition tells us that functions that are related to their second derivatives by constant factors are also sines and cosines. That is, $(\sin x)'' = -\sin x$.

We will see that, indeed, this intuition is correct for undamped systems with no external forces.

3.2 Second order linear equations

A **general second order ODE** is of the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right).$$

As is the case with first order equations, we can describe a more structured family of equations. A 2^{nd} **order linear ODE** is of the form

$$a(t)y'' + b(t)y' + c(t)y = d(t)$$

which can be rewritten as

$$y'' + p(t)y' + q(t)y = g(t).$$

A 2nd order ODE is called **homogeneous** if

$$a(t)y'' + b(t)y' + c(t)y = 0$$

and **nonhomogeneous** if

$$a(t)y'' + b(t)y' + c(t)y = d(t)$$

for some $d(t)$ that is NOT identically zero.

An intial value problem for a second order ODE needs to have ***two*** initial conditions:

$$\begin{aligned} y(t_0) &= y_0, \\ y'(t_0) &= y_0'. \end{aligned}$$

3.2.1 Second order linear homogeneous ODEs with constant coefficients (characteristic equation)

We will begin with the **2nd order linear homogeneous ODEs with constant coefficients** that we introduced and motivated in the previous section:

$$ay'' + by' + cy = 0$$

where a, b, c are real constants.

Consider $y'' - y = 0$ or

$$y'' = y.$$

Can we think of a solution to this ODE from Calculus 1? A function where its second derivative is equal to itself?

A moment of thought should lead us to conclude that there are two obvious solutions: $y_1(t) = e^t$ and $y_2(t) = e^{-t}$. But it not hard to check that the larger family of functions $y = c_1e^t$ and $y = c_2e^{-t}$ are also solutions for any constants c_1, c_2.

Now consider the general ODE

$$ay'' + by' + cy = 0.$$

Guided by our observation about the solutions to the previous equation, let us assume solutions are of the form $y(t) = e^{rt}$. (If this is an unsatisfying assumption, another appoach will be explained shortly). Then

$$\begin{aligned} y(t) &= e^{rt} \\ y'(t) &= re^{rt} \\ y''(t) &= r^2e^{rt}, \end{aligned}$$

and plugging this into the ODE we have

$$LHS = ar^2e^{rt} + bre^{rt} + ce^{rt} \stackrel{?}{=} 0$$

$$e^{rt}\left(ar^2+br+c\right) \stackrel{?}{=} 0.$$

and since $e^{rt} \neq 0$ then

$$ar^2+br+c=0.$$

Using standard methods for quadratic equations, we can solve for the roots $r = r_1, r_2$.

This is called the **characteristic equation** of this ODE. If the roots r_1, r_2 are *real and distinct*, then the **general solution to the homogeneous equation** is of the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

The justification for gluing together the solutions will be presented in the next section as the "principle of superposition".

Example 3.2.1 Let's find the general solution of

$$y'' + 5y' + 6y = 0.$$

1. We'll guess that the solution to a solution is $y(t) = e^{rt}$ for some r. Then get
$$\left(r^2+5r+6\right)e^{rt} = 0$$
so that we must have $r^2+5r+6 = (r+2)(r+3) = 0$ so that $r = -2, -3$.
2. So $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$ are solutions and
$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$
is the general solution.

□

Example 3.2.2 Let's find the solution to the following IVP

$$y'' + 5y' + 6y = 0 \quad y(0) = 2, y'(0) = -1.$$

Solving for the particular solution. We have $y(0) = 2$ and $y'(0) = -1$. Differentiating $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$ we get $y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$ and set up the following system:

$$\begin{aligned} c_1 + c_2 &= 2 \\ -2c_1 - 3c_2 &= -1 \end{aligned}$$

and get $c_1 = 5, c_2 = -3$. So the particular solution is

$$y(t) = 5e^{-2t} - 3e^{-3t}.$$

□

Example 3.2.3 Let's find the general solution of

$$2\frac{d^2y}{dt^2} + 7\frac{dy}{dt} - 4y = 0.$$

1. We'll guess that the solution to a solution is $y(t) = e^{rt}$ for some r. Then get
$$\left(2r^2+7r-4\right)e^{rt} = 0$$
so that we must have $2r^2+7r-4 = (2r-1)(r+4) = 0$ so that $r = \frac{1}{2}, -4$.

2. So $y_1(t) = e^{t/2}$ and $y_2(t) = e^{-4t}$ are solutions and

$$y(t) = c_1 e^{t/2} + c_2 e^{-4t}$$

is the general solution.

□

3.2.2 Derivatives as linear operators and the characteristic equation

It might seem unsatisfying to assume the form of an answer and then show that our guess worked. Is there a mathematical justification for making this guess? The answer is yes, if we're willing to push our understanding of differentiation slightly.

Define the **differential operator** D to be the operation of applying $\frac{d}{dx}$ to a function $y(x)$. The operator D can be thought of as a function on functions:

$$D(\text{function}) = \text{ derivative}.$$

Any order of derivative can be thought of this way. For example,

$$y''' = \frac{d^3y}{dx^3} = \frac{d^3}{dx^3}y = D^3y.$$

We need a notion of D^0 - what should that mean? It makes sense to set $D^0(y) = y$ - that is, take no derivatives of y. We will use the symbol 1 to represent this function, the **identity function** that leaves y unchanged.

In Calculus, we learn that derivatives follow certain rules. Two of the most important are

$$D(f+g) = Df + Dg$$

and

$$D(cf) = cD(f).$$

These two properties together show that D is a **linear function**.

Using differential operator notation allows us to transform an ODE and use algebraic techniques to solve it using first order methods. Consider the equation

$$y'' + 3y' + 2y = 0.$$

This can be written

$$D^2y + 3Dy + 2y = 0$$

using operator notation. But since each operator is applied to the same input, y, we can use function notation to write

$$(D^2 + 3D + 2)(y) = 0.$$

The operator expression $D^2 + 3D + 2$ can be factored as symbols into $(D+2)(D+1)$ (that this is equal to the original equation applied to y is a consequence of linearity). Now, using the fact that the derivative of 0 is 0, we assert that the solutions to the equation

$$(D+1)(D+2)y = (D+2)(D+1)y = 0$$

are the solutions to $(D+1)(y) = 0$ and $(D+2)y = 0$, which are just the first order equations

$$y' + y = 0$$

$$y' + 2y = 0$$

These are separable, with solutions $y_1 = c_1 e^{-x}$ and $y_2 = c_2 e^{-2x}$. So the general solution (justified by the principle of superposition, covered in the next section) to the ODE is

$$y = c_1 e^{-x} + c_2 e^{-2x}.$$

Note that the operator polynomial $D^2 + 3D + 2$ is precisely equivalent to the characteristic polynomial $r^2 + 3r + 2$.

3.3 Solutions to Linear Equations; the Wronskian

In this section, we will consider equations of the form

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

where a, b, c are constants. This is a second order, linear, homogeneous equation. Our goal is to find the general solution of these equations.

Theorem 3.3.1 (Existence and Uniqueness for 2nd order linear ODES). *Consider the IVP*

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

where p, q, g are continuous on an open interval I that contains t_0. Then there exists a unique solution $y = \phi(t)$, and the solution exists throughout all of I.

Recall that this theorem implies that a solution to this IVP

1. exists,
2. is unique
3. and the solution ϕ is defined throughout all of I.

In fact it says more, namely that ϕ is at least twice differentiable on I.

Example 3.3.2 Find the longest interval in which the solution to the IVP is certain to exist by Theorem 3.3.1:

$$\left(t^4 - 4t^2\right) y'' + \cos t y' - e^t y = 0, \quad y(1) = 2, \ y'(1) = 1.$$

Solution: Rewrite the equation as

$$y'' + \frac{\cos t}{t^2\left(t^2 - 4\right)} y' - \frac{e^t}{t^2\left(t^2 - 4\right)} y = 0.$$

so that $p(t) = \frac{\cos t}{t^2(t^2-4)}$ and $q(t) = -\frac{e^t}{t^2(t^2-4)}$ which are both continuous on $(-\infty, -2) \cup (-2, 0) \cup (0, 2) \cup (2, \infty)$. Since $t_0 = 1 \in (0, 2)$ then $I = (0, 2)$ is the longest interval where $p(t)$ and $q(t)$ are both continuous that contains t_0. □

3.3.1 Principle of Superposition

We now give a name to idea that linear combinations of solutions to a linear homogeneous differential equation remain solutions. (Again, the underlying principle is the linearity of the differential operator.)

Theorem 3.3.3 Superposition of solutions to linear homogeneous

ODE. *If y_1 and y_2 are two solutions to an ODE*

$$y'' + p(t)y' + q(t)y = 0,$$

then the linear combination $y(t) = c_1y_1(t) + c_2y_2(t)$ is also a solution for any values c_1, c_2.

Warning: The principle of superposition holds only if the equation is *linear* and *homogeneous*.

Example 3.3.4 Suppose $y_1(t) = e^{-t}$ and $y_2(t) = e^t$ are two solutions to $y'' - y = 0$. Since this is a linear homogeneous ODE then the principle of superposition says that the function

$$y(t) = 2e^{-t} + 3e^t$$

is also a solution. □

Example 3.3.5 It is not hard to check that $y_1(t) = 1$ and $y_2(t) = t^{\frac{1}{2}}$ are solutions to

$$yy'' + (y')^2 = 0, \;\; t > 0.$$

Part (a): Show $y(t) = 1 + 2t^{\frac{1}{2}}$ is not a solution to this ODE:
Solution: First compute

$$\begin{aligned} y(t) &= 1 + 2t^{\frac{1}{2}} \\ y'(t) &= t^{-\frac{1}{2}} \\ y''(t) &= -\frac{1}{2}t^{-\frac{3}{2}} \end{aligned}$$

To show this simply check if the LHS equal to 0:

$$\begin{aligned} LHS = yy'' + (y')^2 &= \left(1 + 2t^{\frac{1}{2}}\right)\left(-\frac{1}{2t^{3/2}}\right) + \left(\frac{1}{t^{\frac{1}{2}}}\right)^2 \\ &= -\frac{1}{2t^{3/2}} - \frac{1}{t} + \frac{1}{t} = -\frac{1}{2t^{3/2}} \neq 0, \end{aligned}$$

Part (b): Why does this not contradict the Principle of Superposition?
Solution: To apply the principle, the equation needs to be linear. The term $(y')^2$ in the ODE makes this nonlinear, hence we can't even use the principle in the first place. □

3.3.2 Fundamental sets of solutions

Suppose that $y_1(t)$ and $y_2(t)$ are two solutions to a second order linear homogeneous equation. When do we know that

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

is the **general solution** to the ODE? That is, when do we know that we can obtain *every single solution* to an IVP? To answer that we need to define some machinery.

Definition 3.3.6 The determinant of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

◊

Definition 3.3.7 The **Wronskian** of the solutions $y_1(t)$ and $y_2(t)$ to a second order linear homogeneous ODE is the function

$$W = W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

$\Diamond$

The Wronskian computes a function that can be used to check of a solution set is sufficient to construct every possible solution.

Theorem 3.3.8 General solution theorem. *Suppose y_1 and y_2 are two solutions to the ODE*

$$y'' + p(t)y' + q(t)y = 0$$

in some interval I where p, q are continuous. Then the family of solutions

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for arbtitrary c_1, c_2 is the general solution if and only if the Wronskian $W(y_1, y_2)$ is not zero for at least one point t_0 in I.

Example 3.3.9 Find the general solution to

$$y'' + 4y' - 5y = 0.$$

Solution: In the last section we showed that to find solutions to this ODE we simply need to solve the characteristic equation

$$r^2 + 4r - 5 = (r-1)(r+5) = 0$$

and get $r = 1, -5$ so that

$$y(t) = c_1 e^t + c_2 e^{-5t}$$

gives other solutions to the ODE. To show this gives all of them, we simply need to show the Wronksian is not always zero:

$$\begin{aligned} W\left(e^t, e^{-5t}\right) &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^t & e^{-5t} \\ e^t & -5e^{-5t} \end{vmatrix} \\ &= -5e^{-4t} - e^{-4t} \\ &= -6e^{-4t} \\ &\neq 0. \end{aligned}$$

$\square$

To find the general solution of $y'' + p(t)y' + q(t)y = g(t)$, we only need to find two (y_1, y_2) solutions whose Wronskian is nonzero:

1. First find two solutions y_1, y_2.
2. Then check $W(y_1, y_2) \neq 0$ for at least one point in the interval.

Definition 3.3.10 The solutions y_1 and y_2 are said to form a **fundamental set of solutions** to

$$y'' + p(t)y' + q(t)y = 0$$

if $W(y_1, y_2) \neq 0$. $\Diamond$

Example 3.3.11 Verify that $y_1(t) = t^{\frac{1}{2}}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions of

$$2t^2y'' + 3ty' - y = 0, \quad t > 0.$$

Solution:

Part (a): First we verify these are indeed solutions by plugging them into the LHS and checking that they equal zero. First computer some derivatives

$$\begin{aligned} y_1(t) &= t^{\frac{1}{2}} & y_2(t) &= t^{-1} \\ y_2'(t) &= \tfrac{1}{2}t^{-\frac{1}{2}} & y_2'(t) &= -t^{-2} \\ y_3'(t) &= -\tfrac{1}{4}t^{-\frac{3}{2}} & y_2''(t) &= -t^{-2}. \end{aligned}$$

Plugging y_1 into LHS we get

$$\begin{aligned} LHS &= 2t^2y_1'' + 3ty_1' - y_1 \\ &= 2t^2\left(-\frac{1}{4}t^{-\frac{3}{2}}\right) + 3t\left(\frac{1}{2}t^{-\frac{1}{2}}\right) - \left(t^{\frac{1}{2}}\right) \\ &= -\frac{1}{2}t^{\frac{1}{2}} + \frac{3}{2}t^{\frac{1}{2}} - t^{\frac{1}{2}} \\ &= 0. \end{aligned}$$

Thus y_1 is a solution. It is very similar to show y_2 is a solution.

Part (b): To show y_1y_2 form a fundamental set of solutions, we simply need to show that $W(y_1, y_2)$ is nonzero:

$$W(y_1, y_2) = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-2} \end{vmatrix} = -\frac{3}{2}t^{-3/2} \neq 0$$

which is nonzero for $t > 0$. □

3.4 Complex roots of the characteristic equation

3.4.1 Complex numbers:

Complex numbers are of the form $z = a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. Thus $i^2 = -1$. Complex numbers have a **polar representation** $z = re^{i\theta}$, where $r = \sqrt{a^2 + b^2}$ and $\theta = \arctan \frac{y}{x}$. In the polar representation, we should think about r as a radius and $e^{i\theta}$ as a point on the unit circle.

We should think about the exponential part as representing motion on a circle. The unit circle from trigonometry gives $(x, y) = (\cos\theta, \sin\theta)$ for an angle θ on the unit circle. This connection is made explicit in what is known as **Euler's formula:**

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

So

$$e^{a+ib} = e^a e^{ib} = e^a\left(\cos b + i\sin b\right) = e^a\cos b + ie^a\sin b.$$

3.4.2 Complex roots to the characteristic equation

Suppose we are solving the constant coefficient 2nd order linear differential equation

$$ay'' + by' + cy = 0.$$

and that solving the characteristic equation

$$ar^2 + br + c = 0$$

gives that the roots are

$$r_1 = a + ib \text{ and } r_2 = a - ib.$$

(Remember that complex roots of real polynomials always come in conjugate pairs.)

These roots are distinct, so we can apply the form of solutions we developed in the previous section. So for the root $r_1 = a + ib$, the solution is of the form

$$\begin{aligned} y(t) &= e^{r_1 t} = e^{(a+ib)t} = e^{at}e^{ibt} \\ &= e^{at}\left(\cos(bt) + i\sin(bt)\right) \\ &= e^{at}\cos(bt) + ie^{at}\sin(bt) \\ &= u(t) + iv(t) \end{aligned}$$

where $u(t) = e^{at}\cos(bt)$ is the real part and $v(t) = e^{at}\sin(bt)$ is the imaginary part.

The complex form of the solution is frequently useful for computational purposes, but in practice (and for non-mathematicians) we prefer real solutions. Because the differential operator is linear, we have the following theorem:

Theorem 3.4.1 *If $y(t) = u(t) + iv(t)$ is a complex solution to an ODE of the form $ay'' + by' + cy = 0$, then so are $u(t)$ and $v(t)$.*

Therefore since $u(t) = e^{at}\cos(bt)$ and $v(t) = e^{at}\sin(bt)$ are solutions we can compute (after some tedious work) that the Wronskian of u and v are:

$$W(u, v)(t) = \mu e^{2at} \neq 0 \text{ as long as } b \neq 0.$$

Hence by the Theorem 3.3.8, because the Wronskian is not zero we have that $u(t)$ and $v(t)$ form a fundamental set of solutions. In other words, the general solution to $ay'' + by' + cy = 0$ is

$$y_g = e^{at}(c_1\cos(bt) + c_2\sin(bt)).$$

3.4.3 Examples

So far, we can solve two cases of second order linear constant coefficient homogeneous differential equations. For

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

and roots r_1, r_2 of

$$ar^2 + br + c = 0,$$

the general solutions are the following:

Roots	General solution	Example
$r_1, r_2 =$ real, distinct	$y(t) = c_1e^{r_1t} + c_2e^{r_2t}$	$(r+1)(r-1) = r^2 - 1 = 0$
$r = a \pm ib$, imaginary	$y(t) = c_1e^{at}\cos bt + c_2e^{at}\sin bt$	$r^2 + 1 = 0$

Example 3.4.2 Let's find the general solution of

$$\frac{d^2y}{dt} + 4\frac{dy}{dt} + 13y = 0$$

Step 1: We can jump straight to the characteristic equation:

$$r^2 + 4r + 13 = 0,$$

which we can solve this using the quadratic formula:

$$r = \frac{-4 \pm \sqrt{16 - 4 \cdot 13}}{2} = -2 \pm \frac{1}{2}\sqrt{4(4-13)} = -2 \pm \sqrt{-9} = -2 \pm 3i.$$

(Or you can use a completing the square trick
Step 2: The general solution is

$$y(t) = c_1 e^{-2t} \cos 3t + c_2 e^{-2t} \sin 3t.$$

□

Example 3.4.3 Let's find the particular solution to the IVP:

$$y'' + 9y = 0, \quad y(0) = -2, \;\; y'(0) = 1$$

Step 1: We can jump straight to the characteristic equation:

$$r^2 + 9 = 0$$

and get $r = \pm 3i$.
Step 2: The general solution is

$$\begin{aligned} y(t) &= c_1 e^{0t} \cos 3t + c_2 e^{0t} \sin 3t. \\ &= c_1 \cos 3t + c_2 \sin 3t. \end{aligned}$$

Step 3: Using the initial conditions $y(0) = -2, \;\; y'(0) = 1$ we need to first take a derivative

$$\begin{aligned} y(t) &= c_1 \cos 3t + c_2 \sin 3t \\ y'(t) &= -3c_1 \sin 3t + 3c_2 \cos 3t \end{aligned}$$

hence

$$\begin{aligned} -2 &= y(0) = c_1 + 0 \\ 1 &= y'(0) = 0 + 3c_2 \end{aligned}$$

so that

$$c_1 = -2, c_2 = \frac{1}{3}.$$

Thus the solution is

$$y(t) = -2\cos 3t + \frac{1}{3}\sin 3t.$$

□

Example 3.4.4 Suppose we get that the general solution comes out to

$$y(t) = c_1 e^{3t} \cos t + c_2 e^{3t} \sin t.$$

Then just remember when finding constants corresponding to initial conditions that we need to use product rule to find the derivative of $y(t)$:

$$y'(t) = 3c_1 e^{3t} \cos t - c_1 e^{3t} \sin t + 3c_2 e^{3t} \sin t + c_2 e^{3t} \cos t.$$

□

3.5 Repeated roots; Reduction of order

3.5.1 Repeated roots

Suppose that we are given the differential equation

$$ay'' + by' + cy = 0$$

and that $b^2 - 4ac = 0$. Then the characteristic equation must have the form

$$\begin{aligned} ar^2 + br + c &= ar^2 + br + \frac{b^2}{4a} \\ &= a(r + \frac{b}{2a})^2 \end{aligned}$$

That is, the equation has just one root, $r = -\frac{b}{2a}$. The results of the previous section say that the function $y = e^{-\frac{b}{2a}t}$ is a solution to the equation. But this is just one solution. We can't use another copy of the function as a second solution because it isn't linearly independent from the first - we will not have a fundamental set, nor will we know the general equation.

So the question is how to get a second solution from only our knowledge of the first. Because the roots are repeated, we may well guess that some kind of modification of the first solution $y_1 = e^{-\frac{b}{2a}t}$ will give us what we want, but we need to use more than a constant or again we'll fail to have a fundamental set. It seems reasonable to guess that the second solution should have the form

$$y_2 = u(t)y_1(t)$$

for some unknown (but non-constant) function u.

So suppose that $y_2 = uy_1$ solves the equation. We'll need to compute expressions for y_2'' and y_2' so that we can plug our "solution" into the differential equation.

$$\begin{aligned} y_2 &= ue^{-\frac{b}{2a}t} \\ y_2' &= u'e^{-\frac{b}{2a}t} - \frac{b}{2a}ue^{-\frac{b}{2a}t} \\ y_2'' &= u''e^{-\frac{b}{2a}t} - \frac{b}{2a}u'e^{-\frac{b}{2a}t} - \frac{b}{2a}u'e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}ue^{-\frac{b}{2a}t} \end{aligned}$$

Then

$$\begin{aligned} 0 &= ay_2'' + by_2' + cy_2 \\ &= ay_2'' + by_2' + \frac{b^2}{4a}y \\ &= a(u''e^{-\frac{b}{2a}t} - \frac{b}{a}u'e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}ue^{-\frac{b}{2a}t}) \\ &\qquad + b(u'e^{-\frac{b}{2a}t} - \frac{b}{2a}ue^{-\frac{b}{2a}t}) + \frac{b^2}{4a}(ue^{-\frac{b}{2a}t}) \\ &= au''e^{-\frac{b}{2a}t}. \end{aligned}$$

Since $a \neq 0$ and $e^{-\frac{b}{2a}t} \neq 0$, then it must be that

$$au''e^{-\frac{b}{2a}t} = 0 \Rightarrow u'' = 0.$$

Integrating twice gets us that $u = Ct + D$ for arbitrary constants C, D. Then our second solution is $y_2 = uy_1 = (Ct + D)y_1$. Since we're working with linear differential equations, the principle of superposition allows us to set $D = 0$ (since we already know that scalar multiples of y_1 are solutions) and $C = 1$ (since any scalar multiple of a solution is a solution). Thus, $y_2 = ty_1$ is our proposed second solution.

To check that we have a fundamental set, we can compute the Wronskian $W(y_1, ty_1)$:

$$W(y_1, ty_1) = \begin{vmatrix} y_1 & ty_1 \\ y_1' & y_1 + ty_1' \end{vmatrix} = y_1^2 = e^{-\frac{b}{a}t} \neq 0 \text{ for any } t.$$

We conclude that the set $y_1 = e^{-\frac{b}{2a}t}, y_2 = te^{-\frac{b}{2a}t}$ is a fundemental set of solutions, and so the general solution for an equation with repeated roots is

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 te^{-\frac{b}{2a}t}.$$

Example 3.5.1 Consider the equation

$$y'' + 6y' + 9y = 0.$$

The characteristic equation is

$$r^2 + 6r + 9 = (r + 3)^2$$

and so we have the repeated root $r = -3$. Our first solution is $y_1 = e^{-3t}$. By the argument above, our second linearly independent solution is $y_2 = te^{-3t}$ and the general solution is

$$y = c_1 e^{-3t} + c_2 te^{-3}.$$

□

Example 3.5.2 Find the general solution of

$$y'' - 10y' + 25y = 0$$

The characteristic equation is

$$r^2 - 10r + 25 = (r - 5)^2 = 0,$$

and so we have the repeated root $r = 5$. Then the general solution is

$$y = c_1 e^{5t} + c_2 te^{5t}.$$

□

The table below summarizes the solutions for homogeneous linear second order differential equations with constant coefficients.

roots :	general solution	example
$r_1, r_2 =$ real, distinct	$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$	$(r+1)(r-1) = 0$
$r = a \pm ib$, imaginary	$y(t) = c_1 e^{at} \cos bt + c_2 e^{at} \sin bt$	$r^2 + 1 = 0$
$r = r_1$, real, repeated	$y(t) = c_1 e^{r_1 t} + c_2 te^{r_1 t}$.	$(r-2)^2 = 0$

3.5.2 Reduction of order

In the previous examples, when we had only one solution y_1, we found a second solution $y_2 = ty_1$ by multiplying by t. We did this by making the guess that

$y_2 = uy_1$. This idea works for general second order homogenous linear equations, not just those with constant coefficients.

For example, suppose we know that $y_1(t) = t$ is a solution to

$$t^2y'' + 2ty' - 2y = 0, \quad t > 0.$$

To find the second solution $y_2(t)$ of this ODE, we guess

$$y_2(t) = v(t)y_1(t) = v(t)t.$$

First, take derivatives:

$$\begin{aligned}
y_2(t) &= v(t)t \\
y_2'(t) &= v'(t)t + v(t) \\
y_2''(t) &= v''(t)t + v'(t) + v'(t) \\
&= v''(t)t + 2v'(t).
\end{aligned}$$

Then plug y_2 and its derivatives into the LHS of the ODE:

$$\begin{aligned}
LHS &= t^2y_2'' + 2ty_2' - 2y_2 \\
&= t^2\left(v''(t)t + 2v'(t)\right) + 2t\left(v'(t)t + v(t)\right) - 2\left(v(t)t\right) \\
&= t^3v''(t) + 2t^2v'(t) + 2t^2v'(t) + 2tv(t) - 2tv(t) \\
&= t^3v'' + 4t^2v'.
\end{aligned}$$

Setting the LHS equal to zero means

$$v''t + 4v' = 0.$$

Now we notice that

$$t^3v'' + 4t^2v' = 0$$

is really a first order equation is disguise, using the substitution $w = v'$.

> The equation
>
> $$a(t)v'' + b(t)v' = 0$$
>
> using the substitution $w = v'$ becomes
>
> $$a(t)w' + b(t)w = 0,$$
>
> which is separable and first order.

Then we need to solve

$$w' + \frac{4}{t}w = 0.$$

This equation is separable, and gives

$$w = \frac{k_1}{t^4}$$

Now we need to reverse the substitution to find v.

$$v' = w = k_1t^{-4}$$

hence

$$v = k_1t^{-3} + k_2.$$

To finish we have that $y_2 = v \cdot t = \left(k_1 t^{-3} + k_2\right) t = k_1 t^{-2} + k_2 t$. Set $k_2 = 0$ and $k_1 = 1$. Thus

$$y_2(t) = t^{-2}$$

is a linearly independent solution. Hence the general solution is given by

$$\begin{aligned} y(t) &= c_1 y_1(t) + c_2 y_2(t) \\ &= c_1 t + c_2 t^{-2}. \end{aligned}$$

This example illustrates the central ideas of reduction of order.

Given a second order linear differential equation

$$a(t)y'' + b(t)y' + c(t)y = r(t)$$

with a known solution y_1, a second solution is given by

$$y_2 = v(t)y_1.$$

This substitution will reduce the order of the equation from second to first in terms of $w = v'$.

3.6 Non-homogeneous equations - Undetermined coefficients

3.6.1 Nonhomogeneous equations

An equation is called **nonhomogeneous** if there is a **forcing term** - that is, a function of just the independent variable on the RHS of the equation. For example, consider the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

where p, q, g are (continuous) functions on some open interval I. The function $g(t)$ is the forcing or driving function.

Consider the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0, .$$

whose general solution we'll call y_h. Now suppose that the **particular solution** $y_p(t)$ solves the nonhomogeneous equation. Then $y_p + y_h$ also solves the equation, since

$$\begin{aligned} &(y_p + y_h)'' + p(y_p + y_h)' + q(y_p + y_h) \\ &= y_p'' + py_p' + qy_p + y_h'' + py_h' + qy_h \\ &= g(t) + 0 \\ &= g(t). \end{aligned}$$

We record this result in the following theorem.

Theorem 3.6.1 *The general solution of*

$$y'' + p(t)y' + q(t)y = g(t)$$

is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$$

where y_1, y_2 are a fundamental set of solutions of the corresponding homogeneous equation, and $y_p(t)$ is a particular solution to the nonhomogeneous equation.

Steps to solving $y'' + p(t)y' + q(t)y = g(t)$:

1. We already know how to find the fundamental set of solutions y_1, y_2 for the homogeneous equation. We have that $y_h = c_1 y_1 + c_2 y_2$ is the general solution to the corresponding homogeneous equation.

2. Find a particular solution y_p using the **method of undetermined coefficients**.

3. The pieces can be summed into the general solution: $y(t) = y_h + y_p = c_1 y_1 + c_2 y_2 + y_p$.

3.6.2 Method of undetermined coefficients

The idea of the method of undetermined coefficients is to guess what the particular solution y_p should be, based on what $g(t)$ looks like. If we think about the LHS of a differential equation as a machine that takes as input some function y and produces as output a function $y'' + py' + qy$, we can guess the sort of input that will be required to produce the output $g(t)$ as a sum of derivatives of y.

Our guess of y_p will always be the general form of $g(t)$, for nice function. For example, we will guess that polynomial inputs produce polynomial outputs, that exponential inputs produce exponential outputs, and that trigonometric inputs produce trigonometric outputs.

If $g(t)$ looks like	Then $y_p(t)$ is
$P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$	$t^s \left[A_m t^m + A_{m-1} t^{m-1} + \cdots + A_0\right]$
$e^{\alpha t} P_m(t)$	$t^s e^{\alpha t} \left[A_m t^m + A_{m-1} t^{m-1} + \cdots + A_0\right]$
$P_m(t) e^{\alpha t} \cos \beta t$ or $P_m(t) e^{\alpha t} \sin \beta t$	$t^s \left[\left(A_m t^m + \cdots + A_0\right) e^{\alpha t} \cos \beta t \right.$ $\left. + \left(B_m t^m + \cdots + B_0\right) e^{\alpha t} \sin \beta t\right]$

This looks more complicated that it actually is because there's a possibility that $g(t)$ is actually a solution to the homogeneous equation. So $s =$ the smallest nonnegative integer ($s = 0, 1$,or 2) such that no term of y_p is a solution to the corresponding homogeneous equation. (For example, given the equation $x'' + 4x' + 4x = e^{-2t}$, we can't guess $y_p = Ae^{-2t}$ because this solves the homogeneous equation.)

Example 3.6.2 Find the solution to the following IVP:

$$y'' + 5y' + 6y = e^{-t}. \qquad y(0) = 1, y'(0) = \frac{1}{2}$$

Step 1: Find $y_h(t)$, which is simply the general solution of

$$y'' + 5y' + 6y = 0.$$

Solving the charateristic polynomial $r^2 + 5r + 6 = (r + 2)(r + 3) = 0$, we get $r = -2, -3$ so that the solution is

$$y_h(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

Step 2: We find $y_p(t)$ by making our guess and to find the undertermined coefficient. So we let $y_p(t) = Ae^{-t}$ and plug y_p into the LHS:

$$y_p'' + 5y_p' + 6y_p = \qquad Ae^{-t} - 5Ae^{-t} + 6Ae^{-t}$$

$$= \qquad 2Ae^{-t}$$

Step 3: Set the LHS equal to the RHS and solve for A to get

$$2Ae^{-t} = e^{-t}$$

so that $A = \frac{1}{2}$.

Step 4: Plug A back in and get $y_h(t) = \frac{1}{2}e^{-t}$ and a general solution of

$$y(t) = c_1e^{-2t} + c_2e^{-3t} + \frac{1}{2}e^{-t}.$$

Final IVP step: Now we need to find c_1 and c_2 using $y(0) = 1$ and $y'(0) = \frac{1}{2}$ and set up the following system of equations:

$$\begin{aligned} c_1 + c_2 + \frac{1}{2} &= 1 \\ -2c_1 - 3c_2 - \frac{1}{2} &= \frac{1}{2} \end{aligned}$$

which comes from $y(t) = c_1e^{-2t} + c_2e^{-3t} + \frac{1}{2}e^{-t}$ and $y'(t) = -2k_1e^{-2t} - 3k_2e^{-3t} - \frac{1}{2}e^{-t}$. Solving this we get $c_1 = \frac{5}{2}$ and $c_2 = -2$ thus the solution to the IVP is

$$y(t) = \frac{5}{2}e^{-2t} - 2e^{-3t} + \frac{1}{2}e^{-t}.$$

$\square$

Example 3.6.3 Find the general solution of

$$\frac{d^2y}{dt} - 5\frac{dy}{dt} + 4y = e^{4t}.$$

Step 1: Find $y_h(t)$. We solve $r^2 - 5r + 4 = (r-1)(r-4) = 0$ and get $r = 1, 4$ so that the solution is

$$y_h(t) = c_1e^t + c_2e^{4t}.$$

Step 2: $y_p(t) = Ae^{4t}$ is the wrong guess because

$$\frac{d^2y_p}{dt} - 5\frac{dy_p}{dt} + 4y_p = 16Ae^{4t} - 20Ae^{4t} + 4Ae^{4t} = 0.$$

But we should have known that this wouldn't work. The term e^{4t} is part of the homogeneous solution, so plugging it into the LHS will give 0. We can modify our guess by multiplying by t. Our second guess should be $y_p(t) = Ate^{4t}$. Find y_p' and $y_p''(t)$ on the side and plug into LHS and get

$$\begin{aligned} \frac{d^2y_p}{dt} - 5\frac{dy_p}{dt} + 4y_p = &\quad \left(8Ae^{4t} + 16Ate^{4t}\right) - 5\left(Ae^{4t} + 4Ate^{4t}\right) + 4Ate^{4t} \\ = &\quad 3Ae^{4t} \end{aligned}$$

Now set LHS equal to RHS and get $3Ae^{4t} = e^{4t}$ so that $A = \frac{1}{3}$.

Step 3: Plug A back in and get $y_p(t) = \frac{1}{3}e^{4t}$ and a general solution of

$$y(t) = c_1e^t + c_2e^{4t} + \frac{1}{3}te^{4t}.$$

$\square$

Example 3.6.4 Find the general solution of

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 4\cos 2t$$

Step 1: Find y_h which is the general solution to the unforced equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 0$$

which since $r^2 + 2r + 10 = 0$ gives $r = -1 \pm 3i$ must be

$$y_h(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t.$$

Step 2: Now as long as the RHS $g(t)$ is not part of y_c then we can use that as our guess. So we let $y_p(t) = A\cos 2t + B\sin 2t$.

Step 3: Plug into the LHS and set equal to RHS

$$\begin{aligned}\frac{d^2y_p}{dt^2} + 2\frac{dy_p}{dt} + 10y = {} & [-4A\cos 2t - 4B\sin 2t] \\ & + 2[-2A\sin 2t + 2B\cos 2t] + 10[A\cos 2t + B\sin 2t]\end{aligned}$$

which gives us

$$[-4A + 4B + 10A]\cos 2t + [-4B - 4A + 10B]\sin 2t = 4\cos 2t$$

so that

$$\begin{aligned} 6A + 4B &= 4 \\ -4A + 6B &= 0 \end{aligned}$$

gives us $A = \frac{6}{13}, B = \frac{4}{13}$.

Step 4: Plug into general solution of $y(t) = y_h(t) + y_p(t)$ and get

$$y(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t + \frac{6}{13}\cos 2t + \frac{4}{13}\sin 2t.$$

□

Example 3.6.5 Find the general form of a particular solution of

$$y'' - 2y' - 3y = 5te^{-t}.$$

Step 1: Find y_h which is $y_h = c_1 e^{-t} + c_2 e^{3t}$.

Step 2: Using our table our first guess will be

$$y_p = (At + B)e^{-t}$$

since $At + B$ is the general form of a one degree polynomial. But this doesn't work because Be^{-t} is included in the y_c as $c_1 e^{-t}$

Step 2 (Second guess): Now guess

$$y_p = t(At + B)e^{-t}$$

so that both At^2e^{-t} and Bte^{-t} are different than the terms in y_c. □

Example 3.6.6 Find the general form of a particular solution of

$$y'' + 6y' + 9y = -7te^{-3t} + t^3$$

Step 1: The characteristic equation is $r^2 + 6r + 9 = (r+3)^2 = 0$, which gives the repeated root of $r_1 = r_2 = -3$. Hence

$$y_h(t) = c_1 e^{-3t} + c_2 t e^{-3t}$$

Using our table we make our first guess as

$$y_p = (At + B)\, e^{-3t} + Ct^3 + Dt^2 + Et + F$$

but this is wrong, since $(At + B)\, e^{-3t}$ is included in the y_h. So our second guess is to multiply *only that part* by t, and get

$$y_p = t\,(At + B)\, e^{-3t} + Ct^3 + Dt^2 + Et + F.$$

But this still doesn't work since Bte^{-3t} is included in the y_c as $c_2 te^{-3t}$. Our third guess is to multiply again only that part by t and get

$$y_p = t^2\,(At + B)\, e^{-3t} + Ct^3 + Dt^2 + Et + F,$$

which works since none of the terms in the y_p are included in the homogeneous solution y_h. □

Example 3.6.7 Find the general form of a particular solution of

$$y'' + y = t + t\sin t$$

Step 1: As in Example 3 we know $y_h(t) = c_1 \cos t + c_2 \sin t$.

Our first guess would normally be $y_p = At+B+[(Dt+E)\cos t + (Ft+G)\sin t]$ but notice that since $E\cos t$ and $G \sin t$ is included in the y_h, we need to multiply by t and get our final guess of

$$y_p = At + B + t\,[(Dt + E)\cos t + (Ft + G)\sin t]$$

□

Example 3.6.8 Find the general form of a particular solution of

$$y'' + 2y' + 10y = 4e^{-t}\cos 3t + 17$$

Step 1: As in Example 3 we know $y_h(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t$.
Since $e^{-t}\cos 3t$ is already inside our y_c we need to multiply by t .

$$y_p = t\left(Ae^{-t}\cos 3t + Be^{-t}\sin 3t\right) + C.$$

Note that 17 is a zero degree polynomial , which is why we have the C in the y_p. □

3.7 Variation of Parameters

Consider the equation

$$y'' + 4y = \frac{3}{\sin t}$$

MOUC doesn't work with quotients, only products. We will learn a (more complicated) general formula to solving more general linear non-homoheneous 2nd order ODEs.

Theorem 3.7.1 Variation of Parameters. *If p, q, and g are continuous on an open interval I, and if the functions $\{y_1, y_2\}$ form a fundamental set of* *solutions to the corresponding homogeneous equation*

$$y'' + p(t)y' + q(t)y = 0,$$

then a particular solution to

$$y'' + p(t)y' + q(t)y = g(t)$$

is given by

$$\begin{aligned} y_p(t) &= -y_1(t) \int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)} ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)} ds \\ &= -y_1(t) \left[\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} dt \right] + y_2(t) \left[\int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} dt \right] \end{aligned}$$

if the antiderivates exist, where t_0 is any value in I. Then the general solution to the non-homogeneous solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t).$$

Proof. The proof can be found in any differential equations text, or online. The idea is this: Suppose

$$y_h(t) = c_1 y_1(t) + c_2 y_1(t)$$

is the general solution to

$$y'' + p(t)y' + q(t)y = 0.$$

Then the idea is to use the following guess:

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

for the non-homogeneous equation, and also make the extra assumption that

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0. \quad (\star)$$

The validity of this assumption is difficult to justify without higher level mathematics, but one can at least take comfort in that we have an extra constraint to play with, so for computational convenience we select condition $(\star)$.

Then take derivatives, simplify and put them back into the differential equation. This will always reduce to

$$\begin{aligned} LHS &= y_p'' + p(t)y_p' + q(t)y_p \\ &= \text{work} \\ &= u_1' y_1'(t) + u_2'(t) y_2'(t) \end{aligned}$$

and set LHS to RHS which is $g(t)$ hence we get

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t). \quad (\star\star)$$

Putting $(\star)$ and $(\star\star)$ together we have the two equations:

$$\begin{cases} u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \\ u_1' y_1'(t) + u_2'(t)y_2'(t) = g(t) \end{cases}$$

which boils to solving for $u_1'(t)$ and $u_2'(t)$ and getting

$$\begin{cases} u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)} \\ u_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} \end{cases}$$

which by integrating we have

$$\begin{cases} u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt \\ u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt \end{cases}$$

■

Example 3.7.2 Find a particular solution to

$$y'' + 4y = \frac{1}{\cos(2t)}.$$

Step1: First find y_h if possible. In this case $y_h$will be given by solving $r^2 + 4 = 0$ so that $r = \pm 2i$ hence

$$y_h(t) = c_1 \cos(2t) + c_2 \sin(2t).$$

Thus $y_1(t) = \cos(2t)$ and $y_2(t) = \sin(2t)$.

Step2: Find the Wronskian:

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} \\ &= 2\cos^2(2t) + 2\sin^2(2t) \\ &= 2\left[\cos^2(2t) + \sin^2(2t)\right] \\ &= 2 \cdot 1 = 2. \end{aligned}$$

Step3: Use our formula with $g(t) = \frac{1}{\cos(2t)}$ and get

$$\begin{aligned} y_p(t) &= -y_1(t)\left[\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt\right] + y_2(t)\left[\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt\right] \\ &= -\cos(2t)\left[\int \frac{1}{2}\frac{\sin(2t)}{\cos(2t)}dt\right] + \sin(2t)\left[\int \frac{\cos(2t)}{2}\frac{1}{\cos(2t)}dt\right] \\ &= -\cos(2t)\left[\frac{1}{2}\int \frac{\sin(2t)}{\cos(2t)}dt\right] + \frac{t}{2}\sin(2t) \end{aligned}$$

Now you can remember the antiderivative of $\int \tan(2t)dt$ or use u-substitution with $u = \cos(2t)$ and get $du = -2\sin(2t)dt$ so that

$$\int \frac{\sin(2t)}{\cos(2t)}dt = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| = -\frac{1}{2}\ln|\cos(2t)|$$

hence

$$y_p(t) = \frac{1}{4}\cos(2t)\ln|\cos(2t)| + \frac{t}{2}\sin(2t).$$

□

Example 3.7.3 Find the general solution to

$$t^2y'' + 2ty' - 2y = 6t$$

given that

$$y_1(t) = t, \;\; y_2(t) = t^{-2}$$

forms a fundamental set of solution for the corresponding homogeneous differential equation.

Step 1: Since $y_1(t) = t, \;\; y_2(t) = t^{-2}$ forms a fundamental set of solution, this means that the general solution for the homogeneous equation is

$$y_h = c_1 t + c_2 t^{-2}.$$

Step 2: Find the Wronskian:

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} \\ &= -2t^{-2} - t^{-2} = -3t^{-2} \neq 0, \end{aligned}$$

Step 3: Rewrite the equation in the form $y'' + p(t)y' + q(t)y = g(t)$ and hence

$$y'' + \frac{2}{t}y' - \frac{2}{t^2}y = \frac{6}{t}.$$

Use our formula with $g(t) = \frac{6}{t}$ and get

$$\begin{aligned} y_p(t) &= -y_1(t)\left[\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt\right] + y_2(t)\left[\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt\right] \\ &= -t\left[\int \frac{t^{-2}}{-3t^{-2}}\frac{6}{t}dt\right] + t^{-2}\left[\int \frac{t}{-3t^{-2})}\frac{6}{t}dt\right] \\ &= -t\left[\int \frac{2}{t}dt\right] + t^{-2}\left[\int -2t^2 dt\right] \\ &= -t\left[2\ln t\right] + t^{-2}\left[-\frac{2}{3}t^3\right] \\ &= -2t\ln t - \frac{2}{3}t. \end{aligned}$$

Hence, the general solution is

$$\begin{aligned} y(t) &= y_h + y_p \\ &= c_1 t + c_2 t^{-2} - 2t\ln t - \frac{2}{3}t. \end{aligned}$$

□

Example 3.7.4 Find the general solution to

$$t^2 y'' - 3ty' + 3y = 8t^3, \quad t > 0$$

given that

$$y_1(t) = t, \;\; y_2(t) = t^3$$

forms a fundamental set of solution for the corresponding homogeneous differential equation.

Step 1: Since $y_1(t) = t, \;\; y_2(t) = t^3$ forms a fundamental set of solution, this means that the general solution for the homogeneous equation is

$$y_h = c_1 t + c_2 t^3.$$

Step 2: Find the Wronskian:

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} \\ &= 3t^3 - t^3 = 2t^3 \neq 0, \end{aligned}$$

Step 3: Rewrite the equation in the form $y'' + p(t)y' + q(t)y = g(t)$ and hence

$$y'' - \frac{3}{t}y' + \frac{3}{t^2}y = 8t, .$$

Use our formula with $g(t) = 8t$ and get

$$\begin{aligned}
y_p(t) &= -y_1(t)\left[\int \frac{y_2(t)g(t)}{W\left(y_1, y_2\right)(t)}dt\right] + y_2(t)\left[\int \frac{y_1(t)g(t)}{W\left(y_1, y_2\right)(t)}dt\right] \\
&= -t\left[\int \frac{t^3}{2t^3}8tdt\right] + t^3\left[\int \frac{t}{2t^3}8tdt\right] \\
&= -t\left[\int 4tdt\right] + t^3\left[\int \frac{4}{t}dt\right] \\
&= -t\left[2t^2\right] + t^3\left[4\ln t\right] \\
&= -2t^3 + 4t^3\ln t
\end{aligned}$$

hence the general solution is

$$\begin{aligned}
y(t) &= y_h + y_p \\
&= c_1 t + c_2 t^3 - 2t^3 + 4t^3 \ln t.
\end{aligned}$$

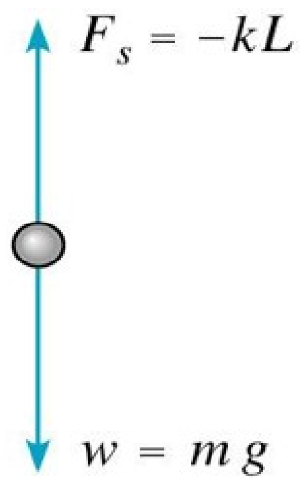
□

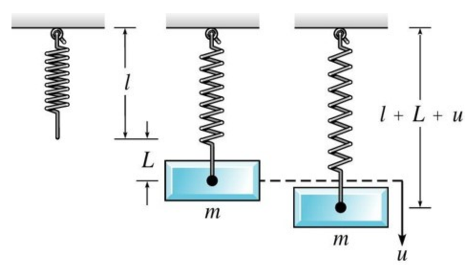
3.8 Harmonic oscillations

We'll now take a closer look at the model that motivated our study of second order equations with constant coefficients - harmonic oscillators. The simplest example of a harmonic oscillator are the motions of a spring-mass system.

3.8.1 Mass-spring systems

Suppose a mass m hangs from a vertical spring of original length l.





We will study the motion of a mass when it is acted on by an external force (forcing function) and/or is initially displaced. Let $u(t)$ =displacement of the mass from its equilibrium position at time t. The motion of the mass $u(t)$ is modeled by the following:

$$mu''(t) + \gamma u'(t) + ku(t) = F(t) \quad u(0) = u_0, u'(0) = v_0.$$

where m, γ, k are positive.

The specific constants depend on the measurement system in use. m is found from $w = mg$. γ is given in units of $\frac{\text{weight unit}\cdot s}{\text{distance unit}}$. k is found using Hooke's Law, $mg = kL$.

Example 3.8.1 A 4 lb mass stretches a spring 2 inches. The mass is displaced an additional 6 in. and then released; and is in a medium with a damping coefficients $\gamma = 2\frac{\text{lb sec}}{\text{ft}}$. Formulate the IVP that governs the motion of this mass.

Solution:

Find m: $w = mg$ which implies

$$m = \frac{w}{g} = \frac{4 \text{ lb}}{32 \text{ ft/s}^2} = \frac{1}{8}\frac{\text{lb}s^2}{ft}$$

Find γ: Given

$$\gamma = 2\frac{\text{lb sec}}{\text{ft}}.$$

Find k: (Hooke's Law)

$$k = \frac{mg}{L} = \frac{4 \text{ lb}}{2 \text{ in}} = \frac{4 \text{ lb}}{(1/6) \text{ ft}} = 24\frac{\text{lb}}{ft}.$$

Thus

$$\frac{1}{8}u'' + 2u' + 24u = 0$$

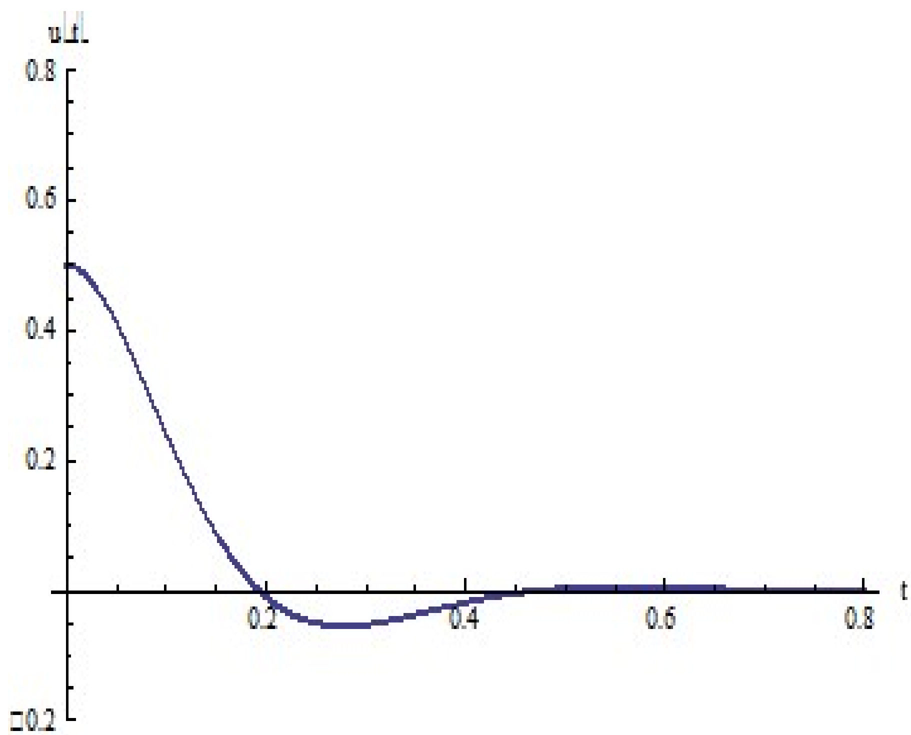
hence

$$u'' + 16u' + 192u = 0, \quad u(0) = \frac{1}{2}, \quad u'(0) = 0$$

since $u(0) = 6\text{in}\ \frac{1\text{ft}}{12\text{in}} = \frac{1}{2}$.

Solving this

$$u(t) = \frac{1}{4}e^{-8t}\left(2\cos\left(8\sqrt{2}t\right) + \sqrt{2}\sin\left(8\sqrt{2}t\right)\right).$$



□

Definition 3.8.2 When

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t = R\cos(\omega_0 t - \delta)$$

then $\omega_0 =$ is the **natural frequency** of the system. $\Diamond$

3.8.2 Undamped harmonic oscillator

When the damping coefficient $\gamma = 0$ (nothing stopping it from oscillating forever), we have

$$mu'' + ku = 0$$

so that $mr^2 + k = 0$ gives $r = \pm i\sqrt{\frac{k}{m}}$.This is a special number, so we'll denote it $\omega_0 = \sqrt{\frac{k}{m}}$. We get

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

with period $\frac{2\pi}{\omega}$ and the **natural frequency** of the system is ω_0.

3.8.3 Damped harmonic oscillator:

A **damper** is an applied force that resists velocity (that is, the damping force is always opposite the direction of motion). A common way to model a damping force is as proportional to the velocity $u'(t)$. When damped, the model equation becomes

$$mu''(t) + \gamma u'(t) + ku(t) = 0.$$

In general we'll have the following characteristic equation

$$mr^2 + \gamma r + k = 0,$$

and solving for the roots we get

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}.$$

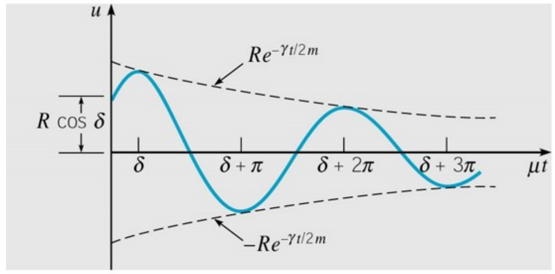
3.8.4 Types of oscillating systems

Different types of behavior are possible depending on the value of $b^2 - 4mk$. We'll classify the possible cases in the following way:

- If $\gamma = 0$,
  - the oscillator is **undamped**.
  - Mass oscillates forever
  - The natural period is $2\pi\sqrt{\frac{m}{k}}$.
- If $\gamma > 0$ and $\gamma^2 - 4km < 0$ (which happens when the roots are $r = \alpha \pm \beta i$)
  - The oscillator is **underdamped**. The mass oscillates back and forth as it tends to its rest position. The solutions are

    $$u = Re^{-\gamma t/(2m)}\cos(\mu t - \delta)$$

    and u is bounded between $\pm Re^{-\gamma t/(2m)}$.

- 

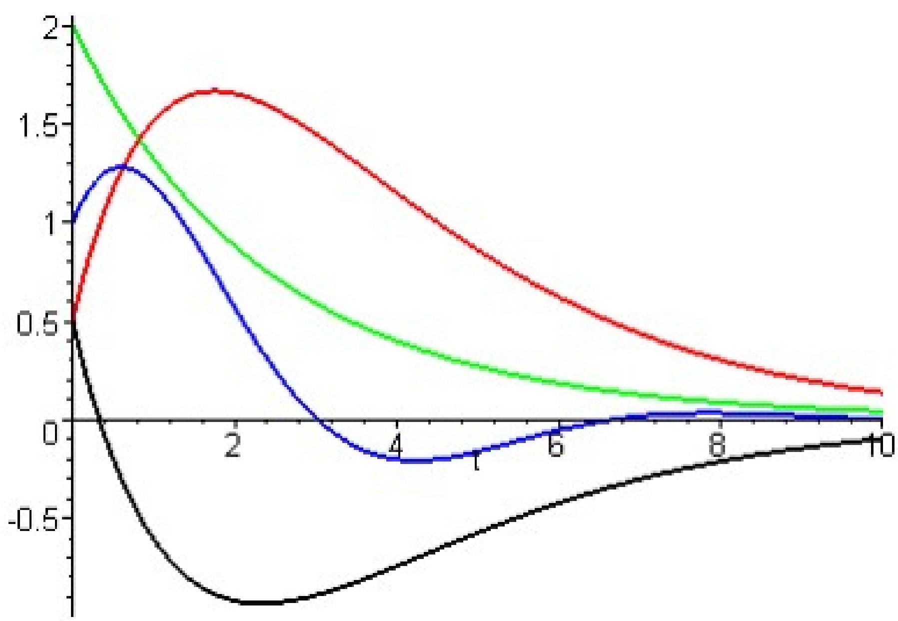
- If $\gamma > 0$ and $\gamma^2 - 4km > 0$ (which happens when there are two distinct r_1, r_2):
  - The oscillator is **overdamped**. The mass tends to its rest position but does not oscillate.
  - The solutions are

$$u = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \;\; r_1, r_2 < 0$$

- If $\gamma > 0$ and $\gamma^2 - 4km = 0$ (which happens when there is one negative r):
  - The oscillator is **critically damped.** The mass tends to its rest position but does not oscillate.
  - Solutions tend to the origin tangent to the unique line of eigenvectors.
  - The solutions are

$$u = c_1 e^{-\gamma t/(2m)} + c_2 t e^{-\gamma t/(2m)}$$

The image below illustrates the behavior possible in different cases. Underdamped is in blue, overdamped in green, and critically damped in red and black.

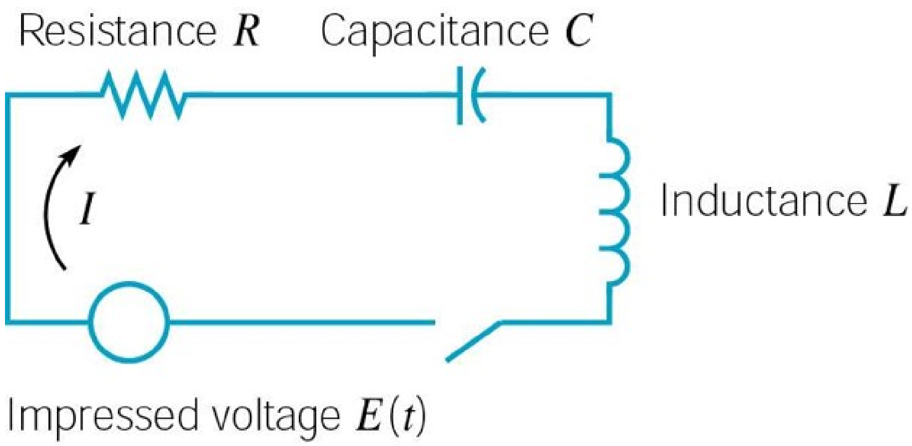


3.8.5 Electric Circuts

The flow of electric charge in certain basic electrical circuits (**RLC** for resistor (R), inductor (L), capacitor (C)) is modeled by second order linear ODEs with constant coefficients:

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t), \quad Q(0) = Q_0, \;\; Q'(0) = Q_0'$$

where Q = charge (coulombs).



3.9 Forced oscillations

We consider equations of the form

$$mu'' + \gamma u' + ku = F(t)$$

where $m > 0, \gamma > 0, k > 0$ are mass, damping coefficients, and spring constant. Here $F(t)$ represents **external force** done to the mass-spring system. (e.g wind or cars driving on a bridge) While these forcing functions come in many forms, one of the most important cases is where the forcing function is itself a harmonic oscillation (you can think of this as corresponding to an ambient vibration).

3.9.1 Harmonic forcing functions

Suppose that the forcing function is $F(t) = F_0 \cos \omega t$, so that the governing equation for the system is

$$mu'' + \gamma u' + ku = F(t).$$

Since this is a linear equation in u, recall that we can write the solution as

$$\begin{aligned} u(t) &= c_1 u_1(t) + c_2 u_2(t) + A \cos \omega t + B \sin \omega t \\ &= u_h + u_p, \end{aligned}$$

and it turns out that $\lim_{t\to\infty} u_h(t) = 0$. (See examples above)

We call $u_h(t)$ a **transient solution**, as its contribution to the system disappears as time progresses. The particular solution $u_p(t) = A \cos \omega t + B \sin \omega t$ is called the **steady-state solution**, as it is the long-term, limiting behavior of the system.

Example 3.9.1 Consider a undamped harmonic oscillator with model equation

$$u'' + 2u = \cos(\omega t), \quad \omega \neq \sqrt{2}.$$

Find the general solution $u(t)$. What is the natural frequency of the system?

Solution:

Step 1: Recall that $r^2 + 2 = 0$ so $r = \pm\sqrt{2}i$, so that

$$u_h(t) = c_1 \cos\left(\sqrt{2}t\right) + c_2 \sin\left(\sqrt{2}t\right).$$

Step 2: We make our first guess

$$u_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

and there are no repeats with u_h as long as $\omega \neq 0$, hence we have the correct guess. Thus

$$\begin{aligned} u_p'(t) &= -A\omega\sin(\omega t) + B\omega\cos(\omega t) \\ u_p''(t) &= -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t). \end{aligned}$$

Plugging this into the LHS, we have

$$\begin{aligned} LHS = u_p'' + 2u_p &= \left[-A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)\right] \\ &\quad + 2A\cos(\omega t) + 2B\sin(\omega t) \\ &= A\left(2-\omega^2\right)\cos(\omega t) + B\left(2-\omega^2\right)\sin(\omega t) \end{aligned}$$

Then, setting $LHS = RHS = 1 \cdot \cos(\omega t) + 0\sin(\omega t)$ we have

$$\begin{aligned} A\left(2-\omega^2\right) &= 1, \quad B\left(2-\omega^2\right) = 0 \\ A &= \frac{1}{2-\omega^2}, \quad B = 0 \end{aligned}$$

so that

$$u(t) = c_1\cos\left(\sqrt{2}t\right) + c_2\sin\left(\sqrt{2}t\right) + \frac{1}{2-\omega^2}\cos(\omega t).$$

□

3.9.2 Resonance (undamped systems)

When the forcing function of an undamped system is far from the natural frequency of a undamped system, forcing function and the natural behavior of the system interact in a combination of constructive and destructive interference.

```
var('x')
y = function('y')(x)
de = diff(y, x, 2) + 2*y == cos(5*x)
f = desolve(de, y, [0,0,0])
plot(f, [0,10*pi], title = f)
```

However, when the frequency of the forcing function approaches the natural frequency of the system, constructive and destructive interference tend to cluster into locally strong or weak oscillations called **beats**.

$$\frac{\omega_0}{2\pi} \approx \frac{\omega}{2\pi} \iff \omega_0 \approx \omega.$$

Solutions in this case typically look something like this:

When the natural and forcing frequencies are equal, the result is **harmonic resonance**:

$$\omega_0 = \omega.$$

We need a new u_p to solve for when resonance happens, since we can't plug $\omega = \sqrt{2}$, into $\frac{1}{2-\omega^2}\cos(\omega t)$. We will see that solution looks like this:

That is, the energy of the forcing function is stored in the system with no destructive interference, which results in increasingly wide oscillations.

Example 3.9.2 Solve the following undamped harmonic oscillator:

$$u'' + 2u = \cos\left(\sqrt{2}t\right), \quad u(0) = 0,\ u'(0) = 0.$$

What is the natural frequency? What is the frequency for the external force? What kind of behavior of the solution will you get get?

Solution:

Step1: Recall that $r^2 + 2 = 0$ so $r = \pm\sqrt{2}i$, so that

$$u_h(t) = c_1 \cos\left(\sqrt{2}t\right) + c_2 \sin\left(\sqrt{2}t\right).$$

The natural frequency of the system is $\omega_0 = \sqrt{2}$. The external frequency is $\omega = \sqrt{2}$. Since they match, then we will get resonance!

Step 2: We make our first guess

$$u_p(t) = A\cos\left(\sqrt{2}t\right) + B\sin\left(\sqrt{2}t\right)$$

but we know that this overlaps with the homogeneous solution, so we choose instead our second guess (by multiplying old guess by t)

$$u_p(t) = At\cos\left(\sqrt{2}t\right) + Bt\sin\left(\sqrt{2}t\right).$$

Then

$$\begin{aligned}
u_p'(t) &= A\cos\sqrt{2}t - A\sqrt{2}t\sin\sqrt{2}t \\
&\quad + B\sin\sqrt{2}t + B\sqrt{2}t\cos\sqrt{2}t \\
u_p''(t) &= -\sqrt{2}A\sin\sqrt{2}t - A\sqrt{2}\sin\sqrt{2}t \\
&\quad - A2t\cos\sqrt{2}t \\
&\quad B\sqrt{2}\cos\sqrt{2}t + B\sqrt{2}\cos\sqrt{2}t \\
&\quad - B2t\sin\sqrt{2}t
\end{aligned}$$

Plugging this into the LHS we have

$$\begin{aligned}
LHS = u_p'' + 2u_p &= \text{simplify} \\
&= 2\sqrt{2}B\cos\left(\sqrt{2}t\right) - 2\sqrt{2}A\sin\left(\sqrt{2}t\right).
\end{aligned}$$

Now, setting $LHS = RHS = 1 \cdot \cos\left(\sqrt{2}t\right) + 0\sin\left(\sqrt{2}t\right)$, we have

$$\begin{aligned}
2\sqrt{2}B = 1, &\quad -2\sqrt{2}A = 0 \\
B = \frac{1}{2\sqrt{2}}, &\quad A = 0
\end{aligned}$$

so that

$$u(t) = c_1 \cos\left(\sqrt{2}t\right) + c_2 \sin\left(\sqrt{2}t\right) + \frac{1}{2\sqrt{2}} t \sin\left(\sqrt{2}t\right).$$

Using initial condition we have $c_1 = 0, c_2 = 0$.

$$u(t) = \frac{1}{2\sqrt{2}} t \sin\left(\sqrt{2}t\right).$$

Hence we get the picture similar to the one above since

$$\frac{1}{2\sqrt{2}} t \sin\left(\sqrt{2}t\right) \approx \pm\frac{t}{2\sqrt{2}} \text{ when } t \text{ is large.}$$

```
var('x')
y = function('y')(x)
de = diff(y, x, 2) + 2*y == cos(sqrt(2)*x)
f = desolve(de, y, [0,0,0])
plot(f, [0,10*pi], title = f)
```

□

Chapter 4

Higher order differential equations

4.1 Linear equations

4.1.1 General linear equations

Everything we did in Chapter 3 for second order linear equations can be extended to higher order systems. Suppose we have the nth order linear equation

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = g(t).$$

We assume that $a_n(t), \ldots, a_0(t)$ are continuous functions on an interval I, and that $a_n(t) \neq 0$ inside the interval: so that we can write it in standand form as

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t). \quad (\star)$$

with initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ \cdots, y^{(n-1)}(t_0) = y_0^{(n-1)}. \quad (\star).$$

Theorem 4.1.1 Existence/uniqueness. *Let a linear differential equation be given in form $(\star)$. If $p_{n-1}(t), \ldots, p_0(t)$ are continuous functions on an open interval I (containing t_0), then there exists a unique solution $y = \phi(t)$ throughout all of I to the IVP in $(\star)$.*

Example 4.1.2 Consider the ODE

$$(t-2)y^{(4)} + \sin t y''' + \ln t y = \sqrt{t+5}.$$

Find the intervals where you are guaranteed a unique solution to this ODE by the Uniqueness and Existence Theorem.

Rewriting, we have

$$y^{(4)} + \frac{\sin t}{(t-2)}y''' + \frac{\ln t}{(t-2)}y = \frac{\sqrt{t+5}}{(t-2)}$$

and

- $\frac{\sin t}{(t-2)}$ is comtinuous when $t \neq 2$
- $\frac{\ln t}{(t-2)}$ is continuous when $t > 0$ and $t \neq 2$, and
- $\frac{\sqrt{t+5}}{(t-2)}$ is continuous when $t \geq -5$ and $t \neq 2$.

Making a number line we see that all three functions are continuous when either on the interval $(0,2)$ or $(2,\infty)$. □

4.1.2 Constant coefficients

Now consider the homogeneous nth order linear equation with constant coefficients.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

As we did in the 2nd order case, the first thing we do is guess that the solution will look like $y = e^{rt}$ and

$$\begin{aligned} y &= e^{rt} \\ y' &= re^{rt}, \\ &\vdots \\ y^{(n)} &= r^n e^{rt}. \end{aligned}$$

Plugging into the LHS and setting equal to zero we have

$$\begin{aligned} LHS &= a_n\left(r^n e^{rt}\right) + \cdots + a_1\left(re^{rt}\right) + a_0\left(e^{rt}\right) \\ &= e^{rt}\left(a_n r^n + \cdots + a_0\right) \\ &= RHS = 0 \end{aligned}$$

hence

$$e^{rt}\left(a_n r^n + \cdots + a_0\right) = 0.$$

But since $e^{rt} \neq 0$ then

$$a_n r^n + \cdots + a_1 r + a_0 = 0.$$

As before the **characteristic equation** is given by:

$$\underbrace{a_n r^n + \cdots + a_0}_{Z(r)} = 0,$$

where we call $Z(r)$ the characteristic polynomial. How do we solve $n-$degree polynomials? By factoring! The fundamental theorem of algebra guarantees that an nth degree polynomial factors into n linear terms (assuming that we allow complex roots).

$$Z(t) = a_n\left(r - r_1\right)\left(r - r_2\right)\cdots\left(r - r_n\right).$$

HOWEVER, there is no general approach to factoring polynomials of degree greater than 4. Numerical techniques are necessary in these cases.

Solutions to the ODE are built exactly like in the 2nd degree case. If there are any repeat solutions, then keep multiplying by t until you don't have any more repeat solutions.

Example 4.1.3 Find general solution and the particular solution to the IVP

$$y''' - 2y'' - y' + 2y = 0. \quad y(0) = 0, y'(0) = 1, y'(0) = 2.$$

(Hint: Suppose you know $r^3 - 2r^2 - r + 2 = (r-2)(r+1)(r-1)$)

The characteristic equation is $2r^3 - 4r^2 - 2r + 4 = 0$ and by the hint we have

$$(r-2)(r+1)(r-1) = 0,$$

hence the general solution is $y(t) = c_1 e^{2t} + c_2 e^{-t} + c_3 e^t$. To find the particular solution to the IVP we start by:

$$\begin{aligned} y(t) &= c_1 e^{2t} + c_2 e^{-t} + c_3 e^t \\ y'(t) &= 2c_1 e^{2t} - c_2 e^{-t} + c_3 e^t \\ y''(t) &= 4c_1 e^{2t} + c_2 e^{-t} + c_2 e^t. \end{aligned}$$

Then we have to solve the following system of equations:

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 1 &= 2c_1 - c_2 + c_3 \\ 2 &= 4c_1 + c_2 + c_3 \end{aligned}$$

and get $c_1 = \frac{2}{3}$, $c_2 = -\frac{1}{6}$ and $c_3 = -\frac{1}{2}$, hence

$$y(t) = \frac{2}{3}e^{2t} - \frac{1}{6}e^{-t} - \frac{1}{2}e^t.$$

$\square$

Example 4.1.4 Find general solution of

$$y^{(4)} + 8y''' + 16y'' = 0.$$

(Hint: $r^4 + 8r^3 + 16r^2 = r^2 (r+4)^2$)

The characteristic polynomial is

$$r^4 + 8r^3 + 16r^2 = 0$$

which by the hint we know factors as

$$r^2 (r+4)^2 = 0.$$

Note that since this a 4th degree polynomial we need to have 4 roots: $0, 0, -4, -4$. So we use the same method we do when we have repeats and get

$$\begin{aligned} y(t) &= c_1 e^{0t} + c_2 t e^{0t} + c_3 e^{-4t} + c_4 t e^{-4t} \\ &= c_1 + c_2 t + c_3 e^{-4t} + c_4 t e^{-4t}. \end{aligned}$$

$\square$

Example 4.1.5 Solve

$$y^{(4)} + y''' - 5y'' + y' - 6y = 0.$$

(Hint: Suppose $(r-2)(r+3)\left(r^2+1\right)$)

The characteristic equation is given by

$$r^4 + r^3 - 5r^2 + r - 6 = 0$$

and by the hint

$$Z(r) = (r-2)(r+3)\left(r^2+1\right) = 0$$

which gives

$$r = 2, -3, \pm i$$

hence

$$y(t) = c_1 e^{2t} + c_2 e^{-3t} + c_3 \cos t + c_3 \sin t.$$

$\square$

Example 4.1.6 Solve

$$y''' - 3y'' + 3y' - y = 0.$$

(Hint: $r^3 - 3r^2 + 3r - 1 = (r-1)(r-1)^2$)

The characteristic polynomial is $r^3 - 3r^2 + 3r - 1 = 0$ and by the hint,

$$(r-1)^3 = 0$$

\item So that $r = 1, 1, 1$

$$y(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t.$$

□

Example 4.1.7 Solve

$$y^{(4)} + 8y'' - 9y = 0$$

(Hint: $r^4 + 8r^2 - 9 = \left(r^2 - 1\right)\left(r^2 + 9\right)$)

By the hint

$$\begin{aligned} r^4 + 8r^2 - 9 &= \left(r^2 - 1\right)\left(r^2 + 9\right) \\ &= (r-1)(r+1)(r-3i)(r+3i). \end{aligned}$$

then

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos(3t) + c_4 \sin(3t).$$

□

Example 4.1.8 Suppose the roots of the characteristic equation are

$$2, 3, 3, 3, 2 \pm 3i, 2 \pm 3i$$

then the general solution is

$$\begin{aligned} y(t) = c_1 e^{2t} &+ c_2 e^{3t} + c_3 t e^{3t} + c_4 t^2 e^{3t} \\ &+ c_5 e^{2t} \cos(3t) + c_6 e^{2t} \sin(3t) \\ &+ c_5 t e^{2t} \cos(3t) + c_6 t e^{2t} \sin(3t). \end{aligned}$$

□

4.2 The Method of Undetermined Coefficients

We consider

$$y^{(n)} + p_{n-1}(t) y^{(n-1)} + \cdots + p_1(t) y' + p_0(t) y = g(t)$$

where $g(t)$ can be a polynomial, sin, cos, exp or products of these. Recall the general solution is of the form: $y = y_h + y_p$ where y_h is the general solution of the corresponding homogeneous equation and y_p is a particular solution to the non-homogeneous equation.

Example 4.2.1 Find the general solution of

$$y''' - y'' - y' + y = 2e^{-t} + 3.$$

(Hint: $r^3 - r^2 - r + 1 = (r-1)(r-1)(r+1)$)

Step 1: We find y_h: Solve $r^3 - r^2 - r + 1 = 0$, but by the hint

$$(r-1)(r-1)(r+1) = 0$$

so that $y_h = c_1 e^t + c_2 t e^t + c_3 e^{-t}$.

Step 2: Find y_p: We first guess $y_p = Ae^{-t} + B$, but there are repeats with y_h hence we get a second guess of

$$\begin{aligned}
y_p &= Ate^{-t} + B \\
y_p' &= Ae^{-t} - Ate^{-t} \\
y_p'' &= -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t} \\
y_p''' &= 2Ae^{-t} + Ae^{-t} - Ate^{-t} = 3Ae^{-t} - Ate^{-t}
\end{aligned}$$

Hence

$$\begin{aligned}
LHS = {} & 3Ae^{-t} - Ate^{-t} \\
& + 2Ae^{-t} - Ate^{-t} \\
& - Ae^{-t} + Ate^{-t} \\
& + Ate^{-t} + B \\
& = 4Ae^{-t} + B
\end{aligned}$$

Step 3: Set LHS=RHS so that

$$LHS = 4Ae^{-t} + B = 2e^{-t} + 3 = RHS$$

hence

$$\begin{aligned}
4A &= 2, B = 3 \\
A &= \frac{1}{2}
\end{aligned}$$

hence

$$y_p = \frac{1}{2}te^{-t} + 3$$

so that the General Solution is

$$y = c_1e^t + c_2te^t + c_3e^{-t} + \frac{1}{2}te^{-t} + 3.$$

$\square$

Example 4.2.2 Consider

$$y''' + 4y' = t + \sin(4t).$$

Find the general form of y_p.

Step 1: We find y_h: Solve

$$\begin{aligned}
r^3 + 4r &= 0 \\
r\left(r^2 + 4\right) &= 0
\end{aligned}$$

so that $y_h = c_1 + c_2\cos 2t + c_3\sin 2t$.

Step 2: Find y_p: We first guess $y_p = At + B + C\cos(4t) + D\sin(4t)$. But B is already in y_c as c_1. So instead make the second guess $y_p = t\,(At + B) + C\cos(4t) + D\sin(4t)$ which is correct. $\square$

Example 4.2.3 Consider

$$y^{(4)} - 2y'' + y = e^t + te^{-t}.$$

Find the general form of y_p. (Hint: $r^4 - 2r^2 + 1 = \left(r^2 - 1\right)^2$)

Step 1: We find y_h: Solve

$$r^4 - 2r^2 + 1 = 0$$
$$\left(r^2 - 1\right)^2 = 0$$

so that $y_h = c_1 e^t + c_2 t e^t + c_3 e^{-t} + c_4 t e^{-t}$.

Step 2: Find y_p:

1. First guess: $y_p = Ae^t + (Bt + C)\, e^{-t}$.
2. Second Guess: $y_p = Ate^t + \left(Bt^2 + Ct\right) e^{-t}$.
3. Third Guess: $y_p = At^2 e^t + \left(Bt^3 + Ct^2\right) e^{-t}$.

□

Example 4.2.4 Suppose

$$y^{(5)} = t^3,$$

find the general form for y_p.

We find y_h: The roots to $r^5 = 0$ are

$$r = 0, 0, 0, 0, 0$$

so that

$$y_c = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_4 t^4$$

Step 2: Find y_p:

1. First Guess: $y_p = At^3 + Bt^2 + Ct + D$
2. Final Guess:}} $y_p = t^5 \left(At^3 + Bt^2 + Ct + D\right)$

□

Chapter 5

Series Solutions

5.1 Power Series

Suppose that we're given the problem

$$\frac{dy}{dx} = e^{x^2}.$$

This is pretty obviously a separable differential equation, and after resorting we get

$$dy = e^{x^2}\,dx,$$

and so

$$y = \int e^{x^2}\,dx.$$

Unfortunately, the function e^{x^2} (an enormously useful function that shows up regularly in probability and statistics in the form of the Gaussian or normal distribution), despite being a very nice function, *has no closed form antiderivative.* That is, there is no function that I can write down for y. Can we really not solve a problem like $\int_0^1 e^{x^2}\,dx$?

In calculus, we learn that one approach to a large class of problems like this is to use **power series**, which are expressions of the form

$$\sum_{k=0}^{\infty} a_k(x-a)^k = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots$$

where $a_0, a_1, \dots$ are constants. (We'd really like to think of a power series like giant polynomial, but this is isn't always justified...)

A power series **converges at** x_0 if

$$\lim_{n\to\infty} a_0 + a_1(x_0-a) + \ldots + a_n(x_0-a)^n = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_k(x_0-a)^k = \lim_{n\to\infty} S_n \text{ exists.}$$

The function $S_n(x)$ is called the n**th partial sum**. Often times, we'll use the shorthand notation

$$\sum_{k=0}^{\infty} a_k(x-a)^k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k(x-a)^k$$

to represent the limit of the partial sums.

The constant a is called the center of the power series. A power series centered at a always converges at $x = a$ (since all but one of the terms is 0). If there is *any* other point for the series converges, we call it a **convergent power series**. If a series fails to converge anywhere except at $x = a$, we call it a **divergent power series**. A series is called **absolutely convergent** at x_0 if

$$\sum_{k=0}^{\infty} |a_k| \, |x_0 - a|^k \text{ exists.}$$

If a powre series converges absolutely for some $x = x_0$, then it also converges for every value of x closer to the center a than x_0 - that is, the series converges for $|x - a| < |x_0 - a|$. The proof of this follows from the squeeze theorem and properties of positive series.

$$\begin{aligned}
&|x - a| \leq |x_0 - a| \\
&\Leftrightarrow \sum_{k=0}^{n} |a_k| \, |x - a|^k \leq \sum_{k=0}^{m} |a_k| \, |x_0 - a|^k \\
&\Leftrightarrow \lim_{n\to\infty} \sum_{k=0}^{n} |a_k| \, |x - a|^k \leq \lim_{n\to\infty} \sum_{k=0}^{m} |a_k| \, |x_0 - a|^k
\end{aligned}$$

and since the larger sum exists, so too must the smaller. This leads to a question of some importance with power series generally - for a given center a, what is the largest set around a on which the series converges? absolutely? The largest number ρ for which a power series centered at a converges for every $x \in (a - \rho, a + \rho)$ (alteratively all x so that $|x - a| < \rho$) is called the **radius of convergence**. How can we find it?

Two of the most powerful tests for convergence we encounter in calculus are the root and ratio tests, both of which work very nicely on power series because power series contain terms of the form $(x - a)^n$.

Theorem 5.1.1 Ratio test for convergence. *Let $\sum_{k=0}^{\infty} c_k$ be an infinite series, and suppose that the limit*

$$L = \lim_{n\to\infty} \left| \frac{c_{k+1}}{c_k} \right| \textit{ exists.}$$

Then the series converges if $L < 1$ and diverges if $L > 1$. If $L = 1$, the test is inconclusive.

Now let's apply the ratio test to a power series to see if we can discover for which values of x the series converges. Consider the series $S = \sum_{k=0}^{\infty} a_x (x-a)^k$. We can apply the ratio test by computing the limit (if it exists)

$$\begin{aligned}
L &= \lim_{k\to\infty} \left| a_{k+1}(x-a)^{k+1} \right| a_k (x-a)^k \\
&= \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| |x - a| \, .
\end{aligned}$$

Presuming that L exists, the ratio test says that the series will converge if $L < 1$. Let $A = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$. Then $L < 1$ is equivalent to

$$\begin{aligned}
L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| |x - a| &< 1 \\
A \, |x - a| &< 1 \\
|x - a| &< \frac{1}{A}
\end{aligned}$$

Theorem 5.1.2 *Let $\sum a_k(x-a)^k$ be a power series centered at a so that $A = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ exists. If $A = 0$, then the radius of convergence for the series is $\rho = \infty$ (that is, the series converges for every x). Otherwise, the radius of converge is $\rho = \frac{1}{A}$. (that is, the series converges for $x \in (a-\rho, a+\rho)$).*

A convergent power series defines a very special kind of function on its interval of convergence. Let

$$f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$$

with domain $\mathcal{D} = (a-\rho, a+\rho)$. The function f defined this way is called an **analytic function**. Such functions are extremely nice - they are smooth, they have derivatives to all orders, they can be integrated as many times as one requires. (n.b.: Functions formed from power series are the central object of study in complex analysis, which consequently is a beautiful subject.)

In calculus, we learn a method for deriving power series of a function called **Taylor series**.

Definition 5.1.3 If f is analytic on an open interval I centered at a, then for all $x \in I$,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k;$$

that is, f has a power series centered at a with $a_k = \frac{f^{(k)}(a)}{k!}$. $\diamondsuit$

Finite Taylor series, called **Taylor polynomials**, are typically excellent approximations of functions near $x = a$.

So why care about this? We can essentially treat Taylor series like "infinitely long polynomials" where they converge - convergent power series can be treated algebraically like functions and are easily integrated and differentiated:

$$\begin{aligned}
\frac{d}{dx}f(x) &= \frac{d}{dx}\left[\sum_{k=0}^{\infty} a_k(x-a)^k\right] \\
&= \sum_{k=0}^{\infty} \frac{d}{dx} a_k(x-a)^k \\
&= \sum_{k=0}^{\infty} k a_k(x-a)^{k-1} \\
&= \sum_{k=1}^{\infty} k a_k(x-a)^{k-1} \quad \text{(first term was 0)} \\
&= \sum_{k=0}^{\infty} (k+1)a_{k+1}(x-a)^k \quad \text{(re-index)}
\end{aligned}$$

Taking another derivative gets us an expression for the second derivative of a power series:

$$\frac{d^2}{dx^2}f(x) = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}(x-a)^k.$$

If $f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$, then we have

$$f'(x) = \sum_{k=0}^{\infty}(k+1)a_{k+1}(x-a)^k$$
$$f''(x) = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}(x-a)^k$$

We should generally have the following basic power series of common functions memorized:

$f(x)$	a	$\sum a_k(x-a)^k$	ρ	IOC
e^x	0	$\sum \frac{x^k}{k!}$	∞	$(-\infty, \infty)$
$\cos x$	0	$\sum (-1)^k \frac{x^{2k}}{(2k)!}$	∞	$(-\infty, \infty)$
$\sin x$	0	$\sum (-1)^k \frac{x^{2k+1}}{(2k+1)!}$	∞	$(-\infty, \infty)$
$\ln(1+x)$	0	$\sum (-1)^{k+1} \frac{x^k}{k}$	1	$(-1, 1)$
$\frac{1}{1-x}$	0	$\sum x^k$	1	$(-1, 1)$

Example 5.1.4 Integrate e^{x^2}. We can now answer the question posed at the beginning of this discussion: since $e^x = \sum \frac{x^k}{k!}$, we get

$$e^{x^2} = \sum \frac{(x^2)^k}{k!} = \sum \frac{x^{2k}}{k!}.$$

Then

$$\begin{aligned} \int e^{x^2}\,dx &= \int \sum \frac{x^{2k}}{k!}\,dx \\ &= \sum \int \frac{x^{2k}}{k!}\,dx \\ &= \left[\sum \frac{x^{2k+1}}{(2k+1)k!}\right] + C \end{aligned}$$

This function has no closed form, but the result can be worked with easily (for example, in approximation). □

Example 5.1.5 Applying the ratio test. Find the radius and interval of convergence of the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{10^k}(x-5)^k.$$

To apply the ratio test, we need the terms

$$c_k = \frac{(-1)^k}{10^k}(x-5)^k$$

and

$$c_{k+1} = \frac{(-1)^{k+1}}{10^{k+1}}(x-5)^{k+1}.$$

Then

$$\begin{aligned} \lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right| &= \lim_{k\to\infty}\left|\frac{(-1)^{k+1}(x-5)^{k+1}}{10^{k+1}} \cdot \frac{10^k}{(-1)^k}(-1)^k(x-5)^k\right| \\ &= \lim_{k\to\infty}\left|\frac{1}{10}(x-5)\right| = \frac{1}{10}\,|x-5| \end{aligned}$$

which by the ratio test will converge if

$$\frac{1}{10}|x-5| < 1$$
$$\text{or } |x-5| < 10.$$

Then the radius of convergence $\rho = 10$, and the interval of converge, which has endpoints given by $a \pm \rho$, is $(-5, 15)$. □

5.2 Method of series solutions

The idea of this section is to use series to construct solutions to linear differential equations that don't have obvious closed form solutions (which is most of them). First, we need to know that the differential equation is itself sufficiently well-behaved to have analytic solutions.

Definition 5.2.1 A point $x = x_0$ is called an **ordinary point** for a homogeneous second order differential equation of the form

$$y'' + P(x)y' + Q(x)y = 0$$

if P and Q are analytic at x_0. (Typical functions for P, Q are trig functions, exponentials, polynomials, and rational functions, which are analytic away from asymptotes). ◊

Example 5.2.2 Every point $x = x_0$ is ordinary for the differential equation

$$y'' + e^x y = 0.$$

□

Example 5.2.3 Every positive x_0 is ordinary for the equation

$$y'' + (\ln x)y = 0.$$

However, $\ln x$ has an asymptote at $x = 0$, so this is a **singular point** for the equation. (Singular points are important to consider but quite complicated to analyze, and addressed in further advanced courses in ODE). □

The big idea of power series solutions is that if

$$y'' + Py' + Qy = 0$$

then at any ordinary point $x = x_0$ we can find two linearly independent power series centered at x_0 that solve the differential equation of the form

$$y = \sum_{k=0}^{\infty} c_k (x - x_0)^k.$$

The challenge is to find the coefficients c_k.

Example 5.2.4 First order example. To illustrate the method, we'll begin with the power series approach to a first order differential equation

$$y' - 2y = 0.$$

Of course, we already know that the solution to this equation should be $y = ke^{2x}$ by the method of characteristic equations, so we should expect our series answer to recover that.

Since the equation is regular at every point, (2 is analytic), for convenience, we will work at the regular point $x = 0$. We will assume that the solution $y = f(x)$ is a power series, which gives the expressions

$$
\begin{aligned}
y &= \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \dots; \\
y' &= \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k = c_1 + 2c_2 x + 3c_3 x^2 + \dots.
\end{aligned}
$$

Plugging into the differential equation, we get

$$
\begin{aligned}
(c_1 + 2c_2 x + 3c_3 x^2 + \dots) - 2(c_0 + c_1 x + c_2 x^2 + \dots) &= 0 \\
(c_1 - 2c_0) + (2c_2 - 2c_1)x + (3c_3 - 2c_2)x^2 + \dots &= 0.
\end{aligned}
$$

What we get when we set all the coefficients equal to 0 is called a **recursion**: if we know c_0, we can get c_1, which lets us get c_2 and so on. We use substitution to get all the values in terms of the first value c_0.

$$
\begin{array}{rcc}
c_1 - 2c_0 = 0 & \Rightarrow & c_1 = 2c_0 \\
2c_2 - 2c_1 = 0 & \Rightarrow & c_2 = c_1 = 2c_0 \\
3c_3 - 2c_2 = 0 & \Rightarrow & c_3 = \frac{2}{3}c_2 = \frac{4}{3}c_0 \\
4c_4 - 2c_3 = 0 & \Rightarrow & c_4 = \frac{2}{4}c_3 = \frac{2}{3}c_0 \\
\vdots & \vdots & \vdots
\end{array}
$$

Now we can substitute these expressions into our assumed solution y:

$$
\begin{aligned}
y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \\
&= c_0 + (2c_0)x + (2c_0)x^2 + (\frac{4}{3}c_0)x^3 + \dots \\
&= c_0 \underbrace{\left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots\right)}_{\text{homogeneous solution}}
\end{aligned}
$$

The quantity c_0 comes from an intital condition, and the series in this case turns out to be the power series of the function e^{2x} (check if you like!). □

Example 5.2.5 Airy's equation. The next example first appeared in work on optics in 1838. It is a very simple second order linear equation, yet does *not have any closed form solutions.* That is, the only approach to finding solutions is to use series methods. The solutions turn out to have applications in quantum physics as well as in optics. The equation is the straightforward looking

$$y'' - xy = 0.$$

Since $P = 0$ and $Q = x$, both of which are analytic functions, series solutions exist everywhere, so we assume a center point of $x_0 = 0$. Then we have the expressions

$$
\begin{aligned}
y &= \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots \\
y' &= \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots \\
y'' &= \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \dots
\end{aligned}
$$

Plugging into Airy's equation, we get

$$\begin{aligned}
&(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \ldots) \\
&\qquad\qquad - x(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots) = 0 \\
&(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \ldots) \\
&\qquad\qquad - (c_0x + c_1x^2 + c_2x^3 + c_3x^4 + c_4x^5 + \ldots) = 0
\end{aligned}$$

which resorts into

$$\begin{aligned}
&2c_2 + (6c_3 - c_0)x + (12c_4 - c_1)x^2 + (20c_5 - c_2)x^3 \\
&\qquad\qquad + (30c_6 - c_3)x^4 + (42c_7 - c_4)x^5 + \ldots = 0
\end{aligned}$$

Now we set the coefficients of each term equal to 0.

First, notice that $2c_2 = 0$ means that $c_2 = 0$. But $20c_5 - c_2 = 0$, and so $c_5 = 0$ as well. Since, c_8 is given in terms of c_5, c_8 is also 0, and so on for the family of coefficients $c_2, c_5, c_8, c_{11}, \ldots$. The is one family of coefficients.

The second family is in terms of c_0:

$$\begin{array}{rcr}
6c_3 - c_0 = 0 & \Rightarrow & c_3 = \frac{1}{6}c_0 \\
30c_6 - c_3 = 0 & \Rightarrow & c_6 = \frac{1}{6\cdot 5}c_3 = \frac{1}{6\cdot 5\cdot 3\cdot 2}c_0 \\
72c_9 - c_6 = 0 & \Rightarrow & c_9 = \frac{1}{9\cdot 8}c_6 = \frac{1}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}c_0 \\
\vdots & \vdots & \vdots
\end{array}$$

which gives expressions for $c_3, c_6, c_9, c_12, \ldots$ in terms of c_0 (this will correspond to the first linearly independent solution).

The third family of solutions is given by c_1:

$$\begin{array}{rcr}
12c_4 - c_1 = 0 & \Rightarrow & c_4 = \frac{1}{4\cdot 3}c_1 \\
42c_7 - c_3 = 0 & \Rightarrow & c_7 = \frac{1}{7\cdot 6}c_3 = \frac{1}{7\cdot 6\cdot 4\cdot 3}c_1 \\
90c_{10} - c_7 & \Rightarrow & c_{10} = \frac{1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}c_1 \\
\vdots & \vdots & \vdots
\end{array}$$

which gives expressions for $c_4, c_7, c_{10}, c_{13}, \ldots$ in terms of c_1 (this represents the second linearly independent solution).

Then we are ready to solve the equation. Starting with our assumed solution of the form $y = \sum c_k x^k$ and sorting into the three families we've identified, we get

$$\begin{aligned}
y &= c_0 + c_1x + c_2x^2 + c_3x^3 \ldots \\
&= c_0 + c_3x^3 + c_6x^6 + c_9x^9 + \ldots \qquad c_0 \text{ family} \\
&\quad + c_1x + c_4x^4 + c_7x^7 + c_{10}x^{10} + \ldots \qquad c_1 \text{ family} \\
&\quad + c_2x^2 + c_5x^5 + c_8x^8 + \ldots \qquad c_2 \text{ family} \\
&= c_0\left(1 + \frac{1}{3\cdot 2}x^3 + \frac{1}{6\cdot 5\cdot 3\cdot 2}x^6 + \frac{1}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}x^9 + \ldots\right) \\
&\quad + c_1\left(x + \frac{1}{4\cdot 3}x^4 + \frac{1}{7\cdot 6\cdot 4\cdot 3}x^7 + \frac{1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}x^{10} + \ldots\right) \\
&\quad + 0 + 0 + 0 + 0 + \ldots
\end{aligned}$$

Then the two linearly independent solutions to Airy's equation are

$$y_1 = 1 + \frac{1}{3\cdot 2}x^3 + \frac{1}{6\cdot 5\cdot 3\cdot 2}x^6 + \frac{1}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}x^9 + \ldots$$

and

$$y_2 = x + \frac{1}{4 \cdot 3}x^4 + \frac{1}{7 \cdot 6 \cdot 4 \cdot 3}x^7 + \frac{1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3}x^{10} + \ldots$$

where neither function has a closed form. (The complicated behavior of the solutions is part of why there isn't much better than a series or improper integral form, as can be seen from their plots. The plot below is a slight rearrangment of the two solutions into a different fundamental set, but preserves the same behavior - namely, the functions are oscillating and then become exponential.)

```
y =  Graphics();
y += plot(airy_ai(x), (x,-10,5), ymin = -1, ymax = 1, color
    = 'red');
y += plot(airy_bi(x), (x,-10,5), ymin = -1, ymax = 1);
y.show()
```

□

Example 5.2.6 Bessel equations. Another important example is the so callled **Bessel Differential Equation**:

$$x^2y'' + xy' + \left(x^2 - \nu^2\right) y = 0, \quad x > 0$$

where ν is some constant. For sake of simplicity, let us pick $\nu = 0$, so that

$$x^2y'' + xy' + x^2y = 0 \quad x > 0$$

and we can rewrite this equation by dividing by x^2 to get

$$y'' + \frac{1}{x}y' + y = 0, \quad x > 0.$$

Much like Airy's equation, this seems like such a simple equation, but it turns out there is no nice solution in terms of elementary functions. In turns out that one way to solve this Bessel ODE is to use power series. (Since $x > 0$, the function $\frac{1}{x}$ is analytic.) One can find out that

$$y_1(x) = J_0(x),$$
$$y_2(x) = Y_0(x)$$

where

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2\, 2^{2n}}x^{2n}.$$

$J_0(x)$ is called the Bessel function of first kind of order $\nu = 0$. $Y_0(x)$ is called the Bessel function of second kind of order $\nu = 0$. $Y_0(x)$ can also be represented by a series, but is more complicated. Another way to write Y_0 is as an integral,

$$Y_0(x) = -\frac{2}{\pi}\int_1^{\infty} \frac{\cos(tx)}{\sqrt{t^2-1}}dt, \; x > 0$$

Thus the general solution to Bessel Equation

$$y'' + \frac{1}{x}y' + y = 0, \;\; x > 0$$

is given by

$$\begin{aligned} y(x) &= c_1J_0(x) + c_2Y_0(x) \\ &= c_1\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2\, 2^{2n}}x^{2n} - c_2\int_1^{\infty} \frac{2}{\pi}\frac{\cos(tx)}{\sqrt{t^2-1}}dt. \end{aligned}$$

□

Chapter 6

The Laplace Transform

6.1 The definition of the Laplace transform

In this section, we define $\mathcal{L}$, the Laplace transform, which is one of the major mathematical tools of engineering and the mathematics of physical systems. Before defining the Laplace Transform we review **improper integrals**, since its definition depends on it.

An improper integral with an infinite limit of integration is shorthand for the limit

$$\int_a^\infty f(t)dt = \lim_{B\to\infty}\int_a^B f(t)dt.$$

If the limit converges then the improper integral converges. If the limit diverges, then the improper integral diverges.

We also need to be able to integrate through simple discontinuities in functions that change definition.

Definition 6.1.1 A function f is **piecewise continuous** on $\alpha \le t \le \beta$ if it is continuous there except for a finite number of jump (or removable) discontinuities. ◊

Example 6.1.2 Are the following functions piecewise continutions?

$$f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ 1 & 1 < t \le 2 \\ 4-t & 2 < t \le 3 \end{cases}$$

and

$$g(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ (t-1)^{-1} & 1 < t \le 2 \\ 1 & 2 < t \le 3. \end{cases}$$

Solution: Sketch the graphs $f(t)$ is piecewise continuous since it only has a jump discontinuity. $g(t)$ is *not* since it has a discontinuity that is not jump or removable. □

Example 6.1.3 Integrating piecewise functions. Consider

Then
$$\int_\alpha^\beta f(t)dt = \int_\alpha^{t_1} f(t)dt + \int_{t_1}^{t_2} f(t)dt + \int_{t_2}^{\beta} f(t)dt.$$

□

The goal of using the Laplace transform is to change the differential equations, which have techniques of solution built on calculus, into the language of algebraic equations, which have techniques of solution built on high school algebra. The hope is that once we discover a solution to the algebraic equation, we can transform it back into a solution for the original ODE.

$$\begin{array}{ccc} \text{ODE Equation} & \overset{\mathcal{L}}{\Longrightarrow} & \text{Algebraic Equation} \\ & & \Downarrow \\ \text{Turn it into an ODE Solution} & \underset{\mathcal{L}^{-1}}{\Longleftarrow} & \text{Solve the Algebrac EQ} \end{array}$$

Thus the Laplace transforms a function $f(t)$ into a function $F(s)$, which is representied symbolically as

$$f(t) \xrightarrow{\mathcal{L}} F(s).$$

Generally, a **transform** of a function $f(t)$ turns $f(t)$m> into a different function with potentially more tractible properties. We will transform functions $f(t)$ of t in the **time domain** into functions $F(s)$ of s in the **frequency domain**. The use of these terms is deliberate and will become clear on further study.

We are now ready to define the transform of greatest use in the solution of linear differential equations.

Definition 6.1.4 The **Laplace transform** of f is given by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt.$$

We assume s is real (though in general it can be complex). ◊

In general, we should be suspicious of integrals involving infinity as a limit of integration. Why should we believe that such an integral is likely to converge? The key observation is that the exponential function e^{-st} tends to 0 *very quickly*. In fact exponentials dominate all polynomials, which is going to mean that the functions most commonly used in differential equations (polynomials, trig functions, and exponentials with domain restrictions) are all reasonably controlled by the exponential function e^{-st}.

Existence of $\mathcal{L}\{f(t)\}$.

If f is piecewise continuous for $[0, a]$ for all a and $|f(t)| \leq Ke^{ct}$ for large t, then $\mathcal{L}[f(t)] = F(s)$ exists.

The following examples may seem cumbersome, but rest easy! After using the defintion to get a feel for how the Laplace transform works, we'll develop a table of known and useful transforms and our study will be more algebraic.

Example 6.1.5 Find the Laplace transform of $f(t) = e^{9t}$, $t \geq 0$.

Solution: We compute

$$\mathcal{L}\left\{e^{9t}\right\} = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{9t}e^{-st}dt$$

$$
\begin{aligned}
&= \int_0^\infty e^{(9-s)t} dt \\
&= \frac{1}{9-s} \left[e^{(9-s)t} \right]_{t=0}^{t=\infty} \\
&= \frac{1}{9-s} \left[\lim_{b\to\infty} e^{(9-s)b} - e^0 \right]
\end{aligned}
$$

but since

$$\lim_{b\to\infty} e^{(9-s)b} = \begin{cases} \infty & a-s>0 \\ 0 & a-s<0 \end{cases}$$

then

$$\mathcal{L}\left\{e^{9t}\right\} = \begin{cases} \frac{1}{s-9} & s>9 \\ \text{not defined} & s<9 \end{cases}.$$

□

Example 6.1.6 Find the Laplace transform of $f(t) = e^{at}$, $t \geq 0$.

We can use the same computation as in Example 1, but change every 9 to an a and get

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a} \quad s > a.$$

□

Example 6.1.7 Find the Laplace transform of $f(t) = 1$, $t \geq 0$.

Solution: Using $a = 0$ above we have that $f(t) = e^{0\cdot t} = 1$ hence we can use the formula above to get

$$\mathcal{L}\{1\} = \frac{1}{s} \quad s > 0.$$

□

Eventually, we'll make a table where we collect all of the Laplace transforms that we have computed, so that we don't have to redo the work everytime.

Example 6.1.8 Find the Laplace transform of $f(t) = \sin(at)$.

Solution: We compute

$$
\begin{aligned}
\mathcal{L}\{\sin at\} = F(s) &= \int_0^\infty e^{-st}\sin(at)dt \\
&= \lim_{B\to\infty} \int_0^B e^{-st}\sin(at)dt.
\end{aligned}
$$

Using integration by parts, we get

$$
\begin{aligned}
u &= \sin(at) & dv &= e^{-st}dt \\
du &= a\cos(at)dt & v &= -\frac{e^{-st}}{s}
\end{aligned}
$$

so we have

$$
\begin{aligned}
F(s) &= \lim_{B\to\infty} \left[-\frac{e^{-st}\sin(at)}{s}\bigg|_{t=0}^{t=B} + \int_0^B \frac{e^{-st}}{s} a\cos(at)dt \right] \\
&= \lim_{B\to\infty} \left[-\frac{e^{-sB}\sin(aB)}{s} + 0 + \int_0^B \frac{e^{-st}}{s} a\cos(at)dt \right] \\
&= 0 + \frac{a}{s}\int_0^\infty e^{-st}\cos(at)dt. \qquad (\star)
\end{aligned}
$$

Integrating $\int_0^\infty e^{-st}\cos(at)dt$ again we get

$$\begin{aligned} u &= \cos(at) \qquad dv = e^{-st}dt \\ du &= -a\sin(at)dt \qquad v = -\frac{e^{-st}}{s} \end{aligned}$$

$$\begin{aligned}
\int_0^\infty e^{-st}\cos(at)dt &= \lim_{B\to\infty}\left[-\frac{e^{-st}\cos(at)}{s}\Big|_{t=0}^{t=B} - \int_0^B \frac{e^{-st}}{s}a\sin(at)dt\right] \\
&= \lim_{B\to\infty}\left[-\frac{e^{-sB}\cos(aB)}{s} + \frac{e^{-st}}{s} - \frac{a}{s}\int_0^B \frac{e^{-st}}{s}a\sin(at)dt\right] \\
&= \left[0 + \frac{t}{s} - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)dt\right].
\end{aligned}$$

Plugging this back into ($\star$) we have

$$\begin{aligned}
F(t) &= \frac{a}{s}\left[\frac{1}{s} - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)dt\right] \\
&= \frac{a}{s}\left[\frac{1}{s} - \frac{a}{s}F(s)\right]
\end{aligned}$$

Then we can solve this equation using algebra for $F(s)$ and get

$$F(s) = \frac{a}{s^2+a^2}, \quad s > 0.$$

$\square$

Properties of the Laplace Transform: Linearity.

If f, g are two function where $\mathcal{L}$ exists for $s > a_1$ and $s > a_2$, respectively, Then

$$\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t))\} \pm \mathcal{L}\{g(t)\}, \quad s > \max\{a_1, a_2\},$$

and We have for $c \in \mathbb{R}$,

$$\mathcal{L}\{cf(t))\} = c\mathcal{L}\{f(t))\}.$$

Example 6.1.9 Find the Laplace transform of $f(t) = 7 - e^{2t} + 4\sin(3t)$.
Solution: Using what we have computed we get

$$\begin{aligned}
\mathcal{L}\left\{7 - e^{-5t} + 4\sin(3t)\right\} &= \mathcal{L}\{7\} - \mathcal{L}\left\{e^{(-5)t}\right\} + 4\mathcal{L}\{\sin(3t)\} \\
&= \frac{7}{s} - \frac{1}{s-(-5)} + 4\cdot\frac{3}{s^2+9} \\
&= \frac{7}{s} - \frac{1}{s+5} + \frac{12}{s^2+9}. \quad s > 0
\end{aligned}$$

$\square$

6.2 The Laplace Transform and Initial Value Problems

In this section we will show the connection between ODEs with given initial value conditions and Laplace Transforms. Recall that our previous methods for approaching IVPs involve solving first a homogeneous equation and then using another method, such as undertermined coefficients, to find a particular solution. Using the Laplace transform, we will be able to do this all at once.

First, we need to look at how the Laplace transform acts on derivatives.

Theorem 6.2.1 Laplace transform of $\frac{df}{dt}$. *Suppose f has a Laplace transform $\mathcal{L}\{f\} = F(s)$ and is controlled by some exponential Ke^at. Then the Laplace transform of f' is given by*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Proof. Let

$$\mathcal{L}\{f'(t)\} = \lim_{B\to\infty}\int_0^B f'(t)e^{-st}dt$$

and we use integration by parts

we have

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \lim_{B\to\infty}\left[f(t)e^{-st}\right]_{t=0}^{t=B} + \int_0^B f(t)se^{-st}dt\\ &= [0 - f(0)] + s\int_0^\infty f(t)e^{-st}dt\\ &= -f(0) + s\mathcal{L}\{f(t)\},\end{aligned}$$

here we use the condition from Theorem 6.2.1 that says $|f(t)| \leq Ke^{at}$ for $t \geq M$ which implies that $\lim_{B\to\infty} f(B)e^{-sB} = 0$ when $s > a$. Rearranging gives us the desired result. ■

Corollary 6.2.2 *Suppose $f, f', \ldots, f^{(n)}$ are nice functions that have Laplace transforms, then*

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0)\cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Example 6.2.3 $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$. $\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2f(0) - sf'(0) - f''(0)$. □

6.2.1 Inverse Laplace Transforms

The Inverse Laplace transform $\mathcal{L}^{-1}$ is the function that satisfies $\mathcal{L}^{-1}\{\mathcal{L}[f]\} = f$. In other words,

$$\mathcal{L}^{-1}\{F\} = f \iff \mathcal{L}\{f\} = F.$$

Some example inverse transforms.

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^{t}$$

$$\mathcal{L}^{-1}\left\{\frac{10}{s+1}\right\} = 10\mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\} = 10e^{-t}.$$

$$\mathcal{L}^{-1}\left\{\frac{6}{s^2+7}\right\} = 6\mathcal{L}^{-1}\left\{\frac{1}{s^2+\left(\sqrt{7}\right)^2}\right\} = \frac{6}{\sqrt{7}}\mathcal{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+\left(\sqrt{7}\right)^2}\right\} = \frac{6}{\sqrt{7}}\sin\left(\sqrt{7}t\right).$$

In general,

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}.$$

We'll typically have rational functions for which we need to find inverse transforms. As an example, let's compute $\mathcal{L}^{-1}\left\{\frac{4}{(s-1)(s+1)}\right\}$. Whenever we have linear or irreducible quadratic factors in the denominator, we need to use partial fractions:

$$\frac{4}{(s-1)(s+1)} = \frac{A}{(s-1)} + \frac{B}{(s+1)},$$

hence

$$\begin{aligned} 4 &= A(s+1) + B(s-1), \\ 0\cdot s + 4 &= (A+B)s + (A-B) \end{aligned}$$

so that

$$A + B = 0$$

$$A - B = 4$$

and get $A = 2, B - 2$. Thus

$$\frac{4}{(s-1)(s+1)} = \frac{2}{s-1} - \frac{2}{s+1}.$$

Therefore:

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{4}{(s-1)(s+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{4}{(s-1)(s+1)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{2}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} \\ &= 2e^{t} - 2e^{-t}. \end{aligned}$$

Example 6.2.4 Find $\mathcal{L}^{-1}\left\{\frac{6}{s(s+4)}\right\}$.

First

$$\frac{6}{s(s+4)} = \frac{A}{s} + \frac{B}{(s+4)}$$

so that

$$6 = A(s+4) + Bs$$

or

$$0s + 6 = (A+B)s + 4A$$

and get

so that $A = -\frac{3}{2}$ and $B = \frac{3}{2}$ hence

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{6}{s(s+4)}\right\} &= \mathcal{L}^{-1}\left\{\frac{3/2}{s} + \frac{-3/2}{(s+4)}\right\} \\
&= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{3}{2}\mathcal{L}\left\{\frac{1}{(s+4)}\right\} \\
&= \frac{3}{2}\cdot 1 - \frac{3}{2}\mathcal{L}\left\{\frac{1}{s-(-4)}\right\} \\
&= \frac{3}{2}\cdot 1 - \frac{3}{2}e^{-4t}
\end{aligned}$$

□

6.2.2 Higher order IVPs

We'll now combine the partial fraction technique with the Laplace transform to solve higher order examples.

Example 6.2.5 Solve

$$y' = y - 4e^{-t}, \ \ y(0) = 1$$

using Laplace transforms.

Step 1: Find the Laplace Transform of the ODE (The going forwards part):

$$\begin{aligned}
\mathcal{L}\{y'\} = \mathcal{L}\{y\} - 4\mathcal{L}\left\{e^{-t}\right\} &\iff s\mathcal{L}\{y\} - y(0) = \mathcal{L}\{y\} - 4\frac{1}{s+1} \\
&\iff s\mathcal{L}\{y\} - 1 = \mathcal{L}\{y\} - 4\frac{1}{s+1}.
\end{aligned}$$

Step 2: Solve for $\mathcal{L}\{y\}$ using algebra: and get

$$\mathcal{L}\{y\} = \frac{1}{s-1} - \frac{4}{(s-1)(s+1)}.$$

Step 3: We want to go backwards and invert this. But first let's do partial fractions:

$$\frac{4}{(s-1)(s+1)} = \frac{A}{(s-1)} + \frac{B}{(s+1)},$$

hence

$$\begin{aligned}
4 &= A(s+1) + B(s-1), \\
0\cdot s + 4 &= (A+B)s + (A-B)
\end{aligned}$$

so that

$$\begin{aligned}
A + B &= 0 \\
A - B &= 4
\end{aligned}$$

and get $A = 2, B = -2$. Thus

$$\frac{4}{(s-1)(s+1)} = \frac{2}{s-1} - \frac{2}{s+1}.$$

Step 4: Use the inverse Laplace transform to get

$$
\begin{aligned}
y = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{y\right\}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{4}{(s-1)(s+1)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \left(\mathcal{L}^{-1}\left\{\frac{2}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\}\right) \\
&= e^t - \mathcal{L}^{-1}\left\{\frac{2}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} \\
&= e^t - 2e^t + 2e^{-t} \\
&= -e^t + 2e^{-t}.
\end{aligned}
$$

$\square$

Example 6.2.6 Solve

$$y' + 4y = 6, \;\; y'(0) = 0$$

using Laplace transforms.

Step 1: Find the Laplace Transform of the ODE (The going forwards part):

$$\mathcal{L}\left\{y'\right\} + 4\mathcal{L}\left\{y\right\} = \mathcal{L}\left\{6\right\} \iff s\mathcal{L}\left\{y\right\} - y(0) + 4\mathcal{L}\left\{y\right\} = \frac{6}{s}$$

Step 2: Solve for $\mathcal{L}\left\{y\right\}$ using algebra: and get

$$\mathcal{L}\left\{y\right\} = \frac{6}{s\,(s+4)}.$$

Step 3: Partial Fractions (We did this already)

$$\frac{6}{s\,(s+4)} = \frac{3/2}{s} + \frac{-3/2}{(s+4)}$$

Step 4: Use the inverse Laplace transform to get

$$
\begin{aligned}
y = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{y\right\}\right\} &= \mathcal{L}^{-1}\left\{\frac{3/2}{s} + \frac{-3/2}{(s+4)}\right\} \\
&= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{3}{2}\mathcal{L}\left\{\frac{1}{(s+4)}\right\} \\
&= \frac{3}{2}\cdot 1 - \frac{3}{2}\mathcal{L}\left\{\frac{1}{s-(-4)}\right\} \\
&= \frac{3}{2}\cdot 1 - \frac{3}{2}e^{-4t}
\end{aligned}
$$

$\square$

6.3 Solutions to higher order IVPs

Recall the folloing consequence of the Laplace transform of a derivative:

Corollary 6.3.1 *Suppose $f, f', \dots, f^{(n)}$ are nice functions that have Laplace transforms, then*

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\left\{f(t)\right\} - s^{n-1}f(0)\cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

For example,

1. $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$.
2. $\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$.

Typically, we use a table to compute Laplace transforms and inverse transforms. The examples below refer to the transforms listed on the following table.

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. $t^n, n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}, n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$

6.3.1 Partial fractions

The essential step in computing an inverse Laplace transform is separating the function F into pieces that we can apply the inverse transform to. This is typically done via partial fractions. Make sure you first factor the denominator as much as possible

1. The correct form of the partial fractions is

$$\frac{5s}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

2. The correct form of the partial fractions is

$$\frac{6s+1}{(s-1)^3(s^2+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{Ds+E}{s^2+3}$$

3. The correct form of the partial fractions is

$$\frac{9s-1}{s(s^2+9)(s-5)} = \frac{A}{s} + \frac{Bs+C}{s^2+9} + \frac{D}{s-5}$$

4. The correct form of the partial fractions is

$$\frac{s^2+s-1}{(s^2+1)^3(s-1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{(s^2+1)^2} + \frac{Es+F}{(s^2+1)^3} + \frac{G}{s-1}.$$

5. The correct form of the partial fractions is

$$\frac{9s+1}{(s^4+1)(s^2+2s+10)s^2} = \frac{As^3+Bs^2+Cs+D}{s^4+1} + \frac{Es+F}{s^2+2s+10}.$$
$$+\frac{G}{s}+\frac{H}{s^2}.$$

6.3.2 Inverse Laplace Transforms

Example 6.3.2 Find the inverse transform of $F(s) = \frac{1}{s^4+s^2}$

First let's do partial fractions:

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

hence

$$1 = As\left(s^2+1\right) + B\left(s^2+1\right) + (Cs+D)s^2$$

so that

$$0s^3 + 0s^2 + 0s + 1 = (A+C)s^3 + (B+D)s^2 + As + B$$

and get the equations

and get $B = 1, A = 0, C = 0, D = -1$. Thus

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}.$$

Using Formulas 3 and 5 in the Laplace Trasform table:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}},\ \ \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}.$$

Use get the inverse Laplace transform:

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\\ &= t - \sin t,\end{aligned}$$

□

Example 6.3.3 (Harder) Find the inverse transform of $F(s) = \frac{1-2s}{s^2+4s+5}$. Note that we can't factor s^2+4s+5 with real roots, thus we will complete the square.

Completing the Square: Suppose we have s^2+bs+c, then the trick is to ADD/SUBTRACT $\left(\frac{b}{2}\right)^2$, and the polynomials will become $s^2+bs+c = \left(s+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$. To complete the square for s^2+4s+5: Then $b = 4$ hence we add/subtract $\left(\frac{b}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 4$. Thus

$$\begin{aligned}s^2+4s+5 &= s^2+4s+4+(-4+5)\\ &= (s+2)^2+1\end{aligned}$$

Going back to the problem of find the Laplace Transform we have that

$$F(s) = \frac{1-2s}{s^2+4s+5} = \frac{1-2s}{(s+2)^2+1}$$

and looking at Formulas 9 and 10 from the Laplace trasnform table:

$$\mathcal{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2} \text{ and } \mathcal{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2}.$$

We can apply these by separating $F(s)$ into pieces like this:=

$$\begin{aligned}\mathcal{L}^{-1}\left\{F(s)\right\} &= \mathcal{L}^{-1}\left\{\frac{1-2s}{(s+2)^2+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{-2(s+2)}{(s+2)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{\mathbf{+4}+1}{(s+2)^2+1}\right\} \\ &= -2\mathcal{L}^{-1}\left\{\frac{(s-(-2))}{(s-(-2))^2+1}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} \\ &= -2e^{-2t}\cos t + 5e^{-2t}\sin t.\end{aligned}$$

□

Example 6.3.4 Find the inverse transform of $F(s) = \frac{2s-8}{s^2-4s+5}$.

Note that we can't factor s^2-4s+5 with real roots, thus we will complete the square.

Completing the square: Suppose we have s^2+bs+c, then the trick is to ADD/SUBTRACT $\left(\frac{b}{2}\right)^2$, and the polynomials will become $s^2+bs+c = \left(s+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$. To complete the square for s^2-4s+5: Then $b=-4$ hence we add/subtract $\left(\frac{b}{2}\right)^2 = \left(\frac{-4}{2}\right)^2 = 4$. Thus

$$\begin{aligned}s^2-4s+5 &= s^2-4s+4+1 \\ &= (s-2)^2+1\end{aligned}$$

Going back to the problem of find the Laplace Transform we have that

$$F(s) = \frac{2s-8}{(s-2)^2+1} = \frac{2(s-2)}{(s-2)^2+1} - 4\frac{1}{(s-2)^2+1}$$

We can apply these by separating $F(s)$ into pieces like this:=

$$\begin{aligned}\mathcal{L}^{-1}\left\{F(s)\right\} &= \mathcal{L}^{-1}\left\{\frac{2s-8}{(s-2)^2+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{2(s-2)}{(s-2)^2+1}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1}\right\} \\ &= 2e^{2t}\cos t - 4e^{2t}\sin t\end{aligned}$$

□

Example 6.3.5 Find the inverse transform of $F(s) = \frac{2s-3}{s^2-4}$.

Notice that this one looks like Formulas 7 and 8 from the Table of Laplace Transforms:

$$\mathcal{L}\left\{\sinh(at)\right\} = \frac{a}{s^2-a^2} \text{ and } \mathcal{L}\left\{\cosh(at)\right\} = \frac{s}{s^2-a^2} \quad s > |a|.$$

Hence we can separate $F(s)$ into pieces so that we can make it look like the formulas above:

$$\begin{aligned}\mathcal{L}^{-1}\left\{F(s)\right\} &= \mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-4}\right\} \\ &= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2-2^2}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2-2^2}\right\} \\ &= 2\cosh(2t) - \frac{3}{2}\sinh(2t).\end{aligned}$$

□

Example 6.3.6 Find the inverse transform of $F(s) = \frac{3s}{s^2-s-6}$.

We want to use partial fractions

$$\frac{3s}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

and multiply both sides by the denominator of the LHS we get

$$3s = A(s+2) + B(s-3)$$

and rewriting we get

$$3s + 0 = (A+B)s + (2A - 3B)$$

so that

$$3 = A + B \text{ and } = 2A - 3B$$

and solving for A, B gets us

$$A = \frac{9}{5}, \;\; B = \frac{6}{5}.$$

So that using our table we have that

$$\begin{aligned}\mathcal{L}^{-1}\left\{F(s)\right\} &= \mathcal{L}^{-1}\left\{\frac{9/5}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{6/5}{s-(-2)}\right\} \\ &= \frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}.\end{aligned}$$

□

6.3.3 Using Laplace Transforms to solve IVPs

Example 6.3.7 Use Laplace Transforms to solve:

$$y''' + y' = 1, \;\; y(0) = y'(0) = y''(0) = 0.$$

Step 1: Find the Laplace Transform of the ODE (The going forwards part). Recall the formulas $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ and $\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$. Applying $\mathcal{L}$ to both sides we get

$$\begin{aligned}\mathcal{L}\left\{y''' + y'\right\} &= \mathcal{L}\left\{1\right\}, \iff \\ \left[s^3\mathcal{L}\left\{y\right\} - s^2 y(0) - sy'(0) - y''(0)\right] + \left[s\mathcal{L}\left\{y\right\} - y(0)\right] &= \frac{1}{s}, \iff \\ \left[s^3\mathcal{L}\left\{y\right\} - s^2\cdot 0 - s\cdot 0 - 0\right] + \left[s\mathcal{L}\left\{y\right\} - 0\right] &= \frac{1}{s}, \iff \\ \mathcal{L}\left\{y\right\}\left(s^3 + s\right) &= \frac{1}{s}, \iff\end{aligned}$$

Step 2: Solve for $\mathcal{L}\{y\}$ using algebra: and get

$$\mathcal{L}\{y\} = \frac{1}{s^2\left(s^2+1\right)}.$$

Step 3: We want to go backwards. But first let's do partial fractions: we did this in Example 1 of the Laplace transforms and got

$$\frac{1}{s^2\left(s^2+1\right)} = \frac{1}{s^2} - \frac{1}{s^2+1}.$$

Step 4: Formulas 3 and 5 in the Laplace Transform table:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \ \ \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}.$$

Use get the inverse Laplace transform:

$$\begin{aligned} y = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= t - \sin t, \end{aligned}$$

$\square$

Example 6.3.8 Use Laplace Transforms to solve:

$$y'' - 4y' + 5y = 2e^t, \ \ y(0) = 3, y'(0) = 1.$$

Step 1: Find the Laplace Transform of the ODE (The going forwards part). Recall the formulas $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ and $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$. Applying $\mathcal{L}$ to both sides we get

$$\begin{aligned} \mathcal{L}\{y'' - 4y' + 5y\} &= \mathcal{L}\left\{2e^t\right\}, \iff \\ \left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right] - 4\left[s\mathcal{L}\{y\} - y(0)\right] + 5\mathcal{L}\{y\} &= \frac{2}{s-1}, \iff \\ s^2\mathcal{L}\{y\} - 3s - 1 - 4s\mathcal{L}\{y\} + 12 + 5\mathcal{L}\{y\} &= \frac{2}{s-1}, \iff \\ \mathcal{L}\{y\}\left(s^2 - 4s + 5\right) &= \frac{2}{s-1} + 3s - 11, \iff \end{aligned}$$

Step 2: Solve for $\mathcal{L}\{y\}$ using algebra: and get

$$\mathcal{L}\{y\} = \frac{2}{(s-1)\left(s^2-4s+5\right)} + \frac{3s-11}{s^2-4s+5}.$$

Step 3: Do Partial Fractions and complete the square:

$$\frac{2}{(s-1)\left(s^2-4s+5\right)} = \frac{A}{s-1} + \frac{Bs+C}{s^2-4s+5}$$

and get $A = 1, B = -1, C = 3$ so that

$$\frac{2}{(s-1)\left(s^2-4s+5\right)} = \frac{1}{s-1} + \frac{-s+3}{s^2-4s+5}$$

Step 4: The inverse Laplace transform:

$$\begin{aligned} y &= \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{-s+3}{s^2-4s+5} + \frac{3s-11}{s^2-4s+5}\right\} \end{aligned}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{2s-8}{s^2-4s+5}\right\}$$

and recall that from an above example

$$\mathcal{L}^{-1}\left\{\frac{2s-8}{s^2-4s+5}\right\} = 2e^{2t}\cos t - 4e^{2t}\sin t$$

hence

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{2s-8}{s^2-4s+5}\right\} \\ &= e^t + 2e^{2t}\cos t - 4e^{2t}\sin t. \end{aligned}$$

□

Example 6.3.9 Take the Laplace transform of the following equation:

$$y'' + 4y = 3\cos t \qquad y(0) = y'(0) = 0.$$

Step 1: Find the Laplace Transform of the ODE (The going forwards part). Recall the formulas $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ and $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$. Applying $\mathcal{L}$ to both sides we get

$$\begin{aligned} \mathcal{L}\{y'' + 4y\} &= \mathcal{L}\{3\cos t\}, \iff \\ \left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right] + 4\mathcal{L}\{y\} &= \frac{3s}{s^2+1}, \iff \\ \mathcal{L}\{y\}\left(s^2+4\right) &= \frac{3s}{s^2+1}, \iff \end{aligned}$$

Step 2: Solve for $\mathcal{L}\{y\}$ using algebra: and get

$$\mathcal{L}\{y\} = \frac{3s}{(s^2+4)(s^2+1)}.$$

Step 3: Do Partial Fractions and complete the square:

$$\frac{3s}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+C}{s^2+1}$$

and get $A = -1, B = 0, C = 1, D = 0$ so that

$$\frac{3s}{(s^2+4)(s^2+1)} = \frac{-s}{s^2+4} + \frac{s}{s^2+1}$$

Step 4: The inverse Laplace transform:

$$\begin{aligned} y &= \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} \\ &= \mathcal{L}^{-1}\left\{\frac{-s}{s^2+4} + \frac{s}{s^2+1}\right\} \\ &= -\cos(2t) + \cos t. \end{aligned}$$

□

6.4 Step functions

Step functions are often used in problems involving the flow of electric circuts, and discontinuous impulsive forcing, such as in vibrations of mechanical systems.

Definition 6.4.1 The **Heaviside function**, or **unit step function** is defined by

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}.$$

$\Diamond$

Though it really doesn't matter, we will assume $c > 0$. The step function looks like

, which you can think of an "on-switch" that turns on at $t = c$. Note that $1 - u_c(t)$ is the corresponding "off-switch" and looks like:

Example 6.4.2 Sketch the following function and describe it as a piecewise function:

$$f(t) = 2tu_2(t) - (t-1)u_4(t).$$

We look at the critical points which are $t = 2, 4$ and consider different cases: $t < 2$, $f(t) = 0 + 0 = 0$ $2 \leq t < 4$, $f(t) = 2t \cdot 1 + 0 = 2t$, $4 \leq t$, $f(t) = 2t \cdot 1 - (t-1) \cdot 1 = t + 1$, hence

$$f(t) = \begin{cases} 0 & t < 2 \\ 2t & 2 \leq t < 4 \\ t+1 & t \geq 4. \end{cases}$$

$\square$

Example 6.4.3 Write $f(t)$ in terms of step functions:

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ t-1 & 1 \leq t < 2 \\ t-2 & 2 \leq t < 3 \\ 0 & 3 \leq t. \end{cases}$$

Solution: The discontinuity points are $t = 0, 1, 2, 3$. When $0 \leq t < 1$, the function will be $f(t) = tu_0(t) + \cdots$. Our goal is to figure out the rest. When $1 \leq t < 2$, the function will be $f(t) = tu_0(t) + \mathbf{?} \cdot u_1(t) + \cdots = t - 1$, hence

$$t + ? = t - 1 \quad \Longrightarrow ? = -1.$$

Hence $f(t) = tu_0(t) - 1 \cdot u_1(t) + \cdots$ When $2 \leq < 3$, the function will be $f(t) = tu_0(t) - 1 \cdot u_1(t) + \mathbf{?}u_2(t) + \cdots = t - 2$, hence

$$t - 1 + ? = t - 2 \quad \Longrightarrow ? = -1.$$

Hence $f(t) = tu_0(t) - 1 \cdot u_1(t) - 1u_2(t) + \cdots$ When $t \geq 3$, the function will be $f(t) = tu_0(t) - 1 \cdot u_1(t) - 1u_2(t) + \mathbf{?}u_3(t) \cdots = 0$, hence

$$t - 1 - 1 + ? = 0 \quad \Longrightarrow ? = 2 - t$$

Thus

$$f(t) = tu_0(t) - u_1(t) - u_2(t) + (2-t)u_3(t).$$

$\square$

We can compute the Laplace transform of $u_c(t)$:

$$\begin{aligned}\mathcal{L}\{u_c(t)\} &= \int_0^\infty u_c(t)e^{-st}dt = \int_c^\infty e^{-st}dt \\ &= \left[-\frac{e^{-st}}{s}\right]_{t=c}^{t=\infty} \\ &= \frac{e^{-cs}}{s}.\end{aligned}$$

Example 6.4.4 Find the Laplace Transform of

$$f(t) = \begin{cases} 2 & t < 3 \\ -3 & t \geq 3 \end{cases}.$$

Solution: First use the technique from the first two examples two write $f(t)$ in terms of u_c, and get

$$f(t) = 2 - 5u_3(t),$$

hence

$$F(s) = \mathcal{L}\{f(t)\} = \frac{2}{s} - 5\frac{e^{-3s}}{s}.$$

□

The theorems that come next are called the **translation theorems**, because they represent the effect of translation on both functions and their transforms.

Theorem 6.4.5 First translation theorem. *If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$ and $c > 0$, then*

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s),$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\}.$$

Remark: Note that $u_c(t)f(t-c)$ translates a function to the right by c, and leaves everything to the left as zero.

Theorem 6.4.6 Second translation theorem. *If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$ and $c > 0$, then*

$$\mathcal{L}\left\{e^{ct}f(t)\right\} = F(s-c), \quad s > a + c.$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s-c)\}$, then

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}.$$

The translation theorems are included in the table formulas:

1. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, n positive integer.
2. $\mathcal{L}\{t\} = \frac{1}{s^2}$, $\mathcal{L}\left\{t^2\right\} = \frac{2}{s^3}$, and $\mathcal{L}\left\{t^3\right\} = \frac{3!}{s^4}$.
3. $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$
4. $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$

5. $\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2 - a^2}$
6. $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2 - a^2}$
7. $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$
8. $\mathcal{L}\left\{e^{ct}f(t)\right\} = F(s-c)$

Example 6.4.7 Find the Laplace transform of

$$f(t) = \begin{cases} 0 & t < 2 \\ t^2 - 4t + 5 & t \geq 2 \end{cases}$$

Solution: First we complete the square by adding/subtracting $\left(\frac{b}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 4$ and get

$$t^2 - 4t + 5 = t^2 - 4t + 4 - 4 + 5 = (t-2)^2 - 4 + 5 = (t-2)^2 + 1$$

so that

$$\begin{aligned} f(t) &= \begin{cases} 0 & t < 2 \\ (t-2)^2 + 1 & t \geq 2 \end{cases} \\ &= u_2(t)\left[(t-2)^2 + 1\right] \\ &= u_2(t)(t-2)^2 + u_2(t), \end{aligned}$$

hence using formulas $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$ and $\mathcal{L}\left\{t^2\right\} = \frac{2}{s^3}$ and $c = 2$,

$$\begin{aligned} \mathcal{L}\{f(t)\} &= e^{-2s}F(s) + \frac{e^{-2s}}{s}, \text{ where } f(t-2) = (t-2)^2, f(t) = t^2 \\ &= e^{-2s}\frac{2}{s^3} + \frac{e^{-2s}}{s}. \end{aligned}$$

□

Example 6.4.8 Take the Inverse Laplace Transform of:$F(s) = \dfrac{e^{-2s}}{s^2 + s - 2}$

SolutionL In this example we can actually factor

$$\begin{aligned} \frac{e^{-2s}}{s^2 + s - 2} &= \frac{e^{-2s}}{(s+2)(s-1)} \\ &= e^{-2s}\left(\frac{-1/3}{s+2} + \frac{1/3}{s-1}\right), \text{ by partial fractions} \end{aligned}$$

and use $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$ (use this whenever use see an e^{-cs} when taking \textbf{inverses}!)

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)\} &= -\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-(-2)}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-1}\right\} \\ &= -\frac{1}{3}u_2(t)f_1(t-2) + \frac{1}{3}u_2(t)f_2(t-2). \end{aligned}$$

Use the fact that $\mathcal{L}\{f_1\} = \mathcal{L}\left\{e^{-2t}\right\} = \frac{1}{s+2}$ and $\mathcal{L}\{f_2\} = \mathcal{L}\left\{e^{t}\right\} = \frac{1}{s-1}$ hence

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{3}u_2(t)e^{-2(t-2)} + \frac{1}{3}u_2(t)e^{(t-2)}.$$

□

Example 6.4.9 Take the Inverse Laplace Transform of:$F(s) = \dfrac{9\,(s-3)\,e^{-5s}}{s^2-6s+13}$

Solution: In this example we can only complete the square since we can't factor and get

$$\frac{9\,(s-3)\,e^{-5s}}{s^2-6s+13} = \frac{9\,(s-3)\,e^{-5s}}{(s-3)^2+2^2}$$

Now note that by $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$ and $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$ we have

$$\mathcal{L}\{\cos(2t)\} = \frac{s}{s^2+2^2} \implies \mathcal{L}\left\{e^{3t}\cos(2t)\right\} = \frac{(s-3)}{(s-3)^2+2^2}$$

(Need to take care of the e^{-5s}) Now use $\mathcal{L}\{u_c(t)f_1(t-c)\} = e^{-cs}F_1(s)$ with $f_1(t) = e^{3t}\cos(2t)$ and $c=5$ so that $f_1(t-5) = e^{3(t-5)}\cos(2(t-5))$ hence

$$\mathcal{L}\left\{u_5(t)e^{3(t-5)}\cos(2(t-5))\right\} = e^{-5s}\frac{(s-3)}{(s-3)^2+2^2}$$

Thus multiplying both sides by 9

$$\mathcal{L}^{-1}\{F(s)\} = 9u_5(t)e^{3(t-5)}\cos(2(t-5)).$$

□

Example 6.4.10 Take the Inverse Laplace Transform of:$F(s) = \dfrac{e^{-7s}}{s^2-4}$.

Solution: We note $\mathcal{L}\{\sinh 2t\} = \frac{2}{s^2-4}$ and use $\mathcal{L}\{u_c(t)f_1(t-c)\} = e^{-cs}F_1(s)$ where $f_1(t) = \sinh 2t \implies f_1(t-7) = \sinh 2(t-7)$ to get that

$$\begin{aligned}\mathcal{L}\{u_7(t)f_1(t-7)\} = e^{-7s}F_1(s), &\implies \mathcal{L}\{u_7(t)\sinh 2(t-7)\} = \frac{2e^{-7s}}{s^2-4}\\ &\implies \frac{1}{2}\mathcal{L}\{u_7(t)\sinh 2(t-7)\} = \frac{e^{-7s}}{s^2-4}\end{aligned}$$

hence

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}u_7(t)\sinh 2(t-7).$$

□

Example 6.4.11 Take the Inverse Laplace Transform of:$F(s) = \dfrac{1}{s^2} + \dfrac{e^{-6s}}{(s-2)^3}$.

We know $\mathcal{L}\{t\} = \frac{1}{s^2}$, $\mathcal{L}\left\{t^2\right\} = \frac{2}{s^3}$ and $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$ and $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$ hence

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= t + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2e^{-6s}}{(s-2)^3}\right\}\\ &= t + \frac{1}{2}u_6(t)e^{2(t-6)}(t-6)^2.\end{aligned}$$

□

Example 6.4.12 Take the Inverse Laplace Transform of: $F(s) = \dfrac{1}{s^2 - 10s + 26}$.

Solution: (practice with using $\mathcal{L}\{e^{ct}g(t)\} = G(s-c)$). First we complete the square and get

$$\frac{1}{s^2 - 10s + 26} = \frac{1}{(s-5)^2 + 1}$$

and use $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$ with $a = 1$ so that $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$ Then use the fact that $\mathcal{L}\{e^{ct}g(t)\} = G(s-c)$ with $c = 5$ and $g(t) = \sin t$, thus we know that

$$\mathcal{L}\left\{e^{5t}\sin t\right\} = \frac{1}{(s-5)^2 + 1}$$

hence

$$\mathcal{L}^{-1}\{F(s)\} = e^{5t}\sin t.$$

$\square$

6.5 ODEs with discontinuous forcing functions

We will now do some examples involving intial value problems where the forcing function have pieces that switch on and off. This will require the use of step functions to turn parts of the forcing function on and off as the definition requires. You can think of this as a way to rewrite a multi-part definition into just one definition for a piecewise continuous function.

Example 6.5.1 Solve using Laplace Transforms:

$$y' = -y + u_3(t), \quad y(0) = 2.$$

Step 1: Take $\mathcal{L}$ of both sides and solve for $\mathcal{L}$

$$\mathcal{L}\{y'\} = \mathcal{L}\{-y\} + \mathcal{L}\{u_3(t)\}$$

so that

$$s\mathcal{L}\{y\} - y(0) = -\mathcal{L}\{y\} + \frac{e^{-3s}}{s}.$$

Step 2: Solve for $\mathcal{L}\{y\}$,

$$\mathcal{L}\{y\} = \frac{2}{s+1} + \frac{e^{-3s}}{s(s+1)}$$

Step 3: We do partial fractions on

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

Step 4: Take the inverse Laplace transform: Using $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}F(s)$, and get

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{\frac{2}{s+1} + e^{-3s}\frac{1}{s} - e^{-3s}\frac{1}{s+1}\right\} \\ &= 2e^{-t} + u_3(t) - u_3(t)e^{-(t-3)}. \end{aligned}$$

$\square$

Example 6.5.2 Solve using Laplace Transforms:

$$y' = -3y + 6u_4(t)e^{-(t-4)}, \quad y(0) = 5.$$

Step 1: Take $\mathcal{L}$ of both sides

$$\mathcal{L}\{y'\} = -3\mathcal{L}\{y\} + 6\mathcal{L}\left\{u_4(t)e^{-(t-4)}\right\}$$

and get

$$s\mathcal{L}\{y\} - y(0) = -3\mathcal{L}\{y\} + 6\mathcal{L}\left\{u_4(t)e^{-(t-4)}\right\}$$

Step 2: Solve for $\mathcal{L}\{y\}$ and get

$$\mathcal{L}\{y\} = \frac{5}{s+3} + \frac{6e^{-4s}}{(s+3)(s+1)}.$$

Step 3: We do partial fractions

$$\frac{6}{(s+3)(s+1)} = \frac{-3}{(s+3)} + \frac{3}{(s+1)}.$$

Step 3: Take the inverse Laplace transform: Using $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}F(s)$, and

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{\frac{5}{s+3} + \frac{6e^{-4s}}{(s+3)(s+1)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{5}{s+3}\right\} + \mathcal{L}^{-1}\left\{-3e^{-4s}\frac{1}{(s+3)} + 3e^{-4s}\frac{1}{(s+1)}\right\} \\ &= 5e^{-3t} - 3u_4(t)e^{-3(t-4)} + 3u_4(t)e^{-(t-4)}. \end{aligned}$$

□

Example 6.5.3 Solve using Laplace Transforms:

$$y'' + 4y = 3u_5(t)\sin(t-5), \quad y(0) = 1, y'(0) = 0.$$

Step 1: Take $\mathcal{L}$ of both sides and solve for $\mathcal{L}$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = 3\mathcal{L}\{u_5(t)\sin(t-5)\}$$

and recall $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}F(s)$, hence $a = 4$, $f(t-5) = \sin(t-5)$ hence $f(t) = \sin t$ and $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$ hence

$$\begin{aligned} s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} &= 3\frac{e^{-5s}}{s^2+1}, \implies \\ (s^2+4)\mathcal{L}\{y\} - s &= 3\frac{e^{-5s}}{s^2+1}, \implies \\ \mathcal{L}\{y\} &= 3\frac{e^{-5s}}{(s^2+4)(s^2+1)} + \frac{s}{s^2+4} \end{aligned}$$

Step 2: We do partial fractions on

$$\frac{3}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

hence

$$3 = (As+B)(s^2+1) + (Cs+D)(s^2+4), \implies$$

$$0 \cdot s^3 + 0 \cdot s^2 + 0 \cdot s + 3 = (A + C)\, s^3 + (B + D)\, s^2 + (A + 4C)\, s + (B + 4D)$$

hence

$$\begin{aligned} A + C &= 0 \\ B + D &= 0 \\ A + 4C &= 0 \\ B + 4D &= 3 \end{aligned}$$

and get

$$A = 0 \quad B = -1, \quad C = 0, \quad D = 1$$

hence

$$\frac{3}{(s^2+4)(s^2+1)} = \frac{-1}{s^2+4} + \frac{1}{s^2+1}$$

Step 3: Take the inverse Laplace transform: Using $\mathcal{L}\left[u_a(t) f(t-a)\right] = e^{-as}F(s)$, and $\mathcal{L}\left\{\sin(at)\right\} = \frac{a}{s^2+a^2}$ and $\mathcal{L}\left\{\cos(at)\right\} = \frac{a}{s^2+a^2}$ we have

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{\frac{-e^{-5s}}{s^2+4} + \frac{e^{-5s}}{s^2+1} + \frac{s}{s^2+4}\right\} \\ &= -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2e^{-5s}}{s^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} \\ &= -\frac{1}{2}u_5(t)\sin\left(2\,(t-5)\right) + u_5(t)\sin\left(t-5\right) + \cos\left(2t\right) \end{aligned}$$

□

Example 6.5.4 Solve using Laplace Transforms:

$$y^{(4)} - y = u_1(t) - u_2(t), \quad y(0) = 0, y'(0) = 0, y''(0) = 0., y'''(0) = 0..$$

Step 1: Take $\mathcal{L}$ of both sides and solve for $\mathcal{L}$

$$\mathcal{L}\left\{y^{(4)}\right\} - \mathcal{L}\left\{y\right\} = \mathcal{L}\left\{u_1(t) - u_2(t)\right\}$$

hence

$$\begin{aligned} & s^4\mathcal{L}\left\{y\right\} - s^3y(0) - s^2y'(0) - sy''(0) - sy''' - y'(0) - \mathcal{L}\left\{y\right\} \\ &= \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \implies \\ \left(s^4-1\right)\mathcal{L}\left\{y\right\} &= \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}, \implies \\ \mathcal{L}\left\{y\right\} &= \frac{e^{-s}}{s\left(s^4-1\right)} - \frac{e^{-2s}}{s\left(s^4-1\right)} \end{aligned}$$

Step 2: We do partial fractions on

$$\frac{1}{s\left(s^4-1\right)} = \frac{1}{s\left(s^2-1\right)\left(s^2+1\right)} = \frac{1}{s\left(s+1\right)\left(s-1\right)\left(s^2+1\right)} =$$

hence

$$\frac{1}{s\left(s^4-1\right)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1} + \frac{Ds+E}{s^2+1}$$

after doing the work to get the partial fractions you get

$$\frac{1}{s\left(s^4-1\right)} = -\frac{1}{s} + \frac{1}{4}\frac{1}{s+1} + \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{s}{s^2+1}$$

putting it back we need to take the inverse of

$$\begin{aligned} & e^{-s}\left[-\frac{1}{s}+\frac{1}{4}\frac{1}{s+1}+\frac{1}{4}\frac{1}{s-1}+\frac{1}{2}\frac{s}{s^2+1}\right] \\ & -e^{-2s}\left[-\frac{1}{s}+\frac{1}{4}\frac{1}{s+1}+\frac{1}{4}\frac{1}{s-1}+\frac{1}{2}\frac{s}{s^2+1}\right] \end{aligned}$$

Step 3: Take the inverse Laplace transform: Using $\mathcal{L}\left[u_a(t)f(t-a)\right] = e^{-as}F(s)$, and $\mathcal{L}\left\{\cos(at)\right\} = \frac{a}{s^2+a^2}$ and $\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$ we have

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{e^{-s}\left[-\frac{1}{s}+\frac{1}{4}\frac{1}{s+1}+\frac{1}{4}\frac{1}{s-1}+\frac{1}{2}\frac{s}{s^2+1}\right]\right\} \\ &\quad -\mathcal{L}^{-1}\left\{e^{-2s}\left[-\frac{1}{s}+\frac{1}{4}\frac{1}{s+1}+\frac{1}{4}\frac{1}{s-1}+\frac{1}{2}\frac{s}{s^2+1}\right]\right\} \\ &= -u_1(t)+u_1(t)\left[\frac{1}{4}e^{-1(t-1)}+\frac{1}{4}e^{1(t-1)}+\frac{1}{2}\cos\left(t-1\right)\right] \\ &\quad +u_2(t)-u_2(t)\left\{\frac{1}{4}e^{-1(t-2)}+\frac{1}{4}e^{1(t-2)}+\frac{1}{2}\cos\left(t-2\right)\right\} \end{aligned}$$

□

Example 6.5.5 Find the Laplace transform of

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 3t & 1 \le t < \infty \end{cases}.$$

Step 1: First let us rewrite this in terms of unit step functions. When $0 \le t < 1$: the function is $f(t) = t$ When $1 \le t < \infty$: then function is $f(t) = t + ? \cdot u_1(t) = 3t$ hence $? = 2t$ so that

$$f(t) = t + 2tu_1(t).$$

Step 2: Before we can take a Laplace transform, we notice that our fomula involves $\mathcal{L}\left\{u_c(t)g\left(t-c\right)\right\} = e^{-ct}\mathcal{L}\left\{g(t)\right\}$. Thus we will need to turn $2tu_1(t)$ into this form:

$$\begin{aligned} f(t) &= t + 2tu_1(t) \\ &= t + 2\left(t-\mathbf{1}\right)u_1(t) + \boldsymbol{2u_1(t)} \end{aligned}$$

hence

$$\begin{aligned} \mathcal{L}\left\{f(t)\right\} &= \mathcal{L}\left\{t\right\} + 2\mathcal{L}\left\{(t-1)u_1(t)\right\} + 2\mathcal{L}\left\{u_1(t)\right\} \\ &= \frac{1}{s^2} + 2e^{-s}\frac{1}{s^2} + 2e^{-s}\frac{1}{s}. \end{aligned}$$

□

We now introduce the following useful formula that is not included in the table, which records the translation of a step function:

Fact 6.5.6 *We have*

$$\mathcal{L}\left\{u_c(t)h(t)\right\} = e^{-cs}\mathcal{L}\left\{h\left(t+c\right)\right\}.$$

Example 6.5.7 Take the Laplace transform of $f(t) = u_1(t)te^t$.

Notice that we cannot use the formula $\mathcal{L}\left\{u_c(t)g\left(t-c\right)\right\} = e^{-ct}\mathcal{L}\left\{g(t)\right\}$ directly since te^t is not written as a function of $(t-1)$. Hence we'll need to use

$\mathcal{L}\{u_c(t)h(t)\} = e^{-cs}\mathcal{L}\{h(t+c)\}$ with $h(t) = te^t$ and $c = 1$. Thus

$$h(t+1) = (t+1)e^{t+1}$$

and we get

$$\begin{aligned}
\mathcal{L}\left\{u_1(t)te^t\right\} &= e^{-cs}\mathcal{L}\{h(t+c)\} \\
&= e^{-s}\mathcal{L}\left\{(t+1)e^{t+1}\right\} \\
&= e^{-s}\mathcal{L}\left\{te^te + e^te\right\} \\
&= e^{-s}\left(e\mathcal{L}\left\{te^t\right\} + e\mathcal{L}\left\{e^t\right\}\right) \\
&= e^{1-s}\left(\frac{1}{(s-1)^2} + \frac{1}{(s-1)}\right)
\end{aligned}$$

where we used formula 2 and 11 in the table. □

6.6 Impulse functions

One of the most useful capabilities of the Laplace transform is to deal with forcing functions that aren't even functions, but that occur with great regularity in the modeling of physical systems. The guiding question here that we should keep in mind is "what happens to an oscillating mass when it gets struck by an outside blow?".

We consider

$$ay'' + by' + cy = f(t),$$

where

$$g(t) = \begin{cases} \text{large} & t_0 - \tau < t < t_0 + \tau. \\ 0 & \text{elsewhere} \end{cases}$$

Here $g(t)$ is a **force** and

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)dt = \int_{-\infty}^{\infty} g(t)dt$$

is the **impulse** of the force, or the amount of force in a short time period about t_0.

If y =current in an electric circuit, $g(t) =$ is the time derivative of the voltage, then $I(\tau)$ is the total voltage impressed on circuit in the time interval $I = (t_0 - \tau, t_0 + \tau)$. We will use the following particular example of a force with $\tau = 0$ (to simplify things):

$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau, \\ 0 & \text{elsewhere} \end{cases},$$

where $\tau > 0$ is small.

We'll first look at some nice properties of $d_\tau(t)$:

1. $\lim\limits_{\tau\to 0^+} d_\tau(t) = 0$, whenever $t \neq 0$, and $\lim_{\tau\to 0^+} d_\tau(0) = \infty$.

2. $I(\tau) = \int_{-\tau}^{\tau} \frac{1}{2\tau} dt = \left[\frac{1}{2\tau}t\right]_{-\tau}^{\tau} = 1$ for every τ. Hence $\lim\limits_{\tau\to 0^+} I(\tau) = 1$,

We thus want to define a **unit impulse function** δ, with the properties

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(t) dt = 1.$$

This object isn't actually a function, but there is a mathematically rigorous way to define objects called *generalized functions* which includes δ. We call δ the **Dirac delta function**. We can think of δ as a limit of the $d_\tau(t)$ functions:

$$\delta(t) = \lim_{\tau \to 0^+} d_\tau(t).$$

We can consider a unit impulse at an arbitrary point $t = t_0$, meaning $\delta(t - t_0)$, hence

$$\delta(t - t_0) = 0, \; t \neq t_0,$$
$$\int_{-\infty}^{\infty} \delta(t - t_0) dt = 1.$$

We will now compute the Laplace Transform of $\delta(t - t_0)$, using the properties of the integral definition:

$$\begin{aligned}
\mathcal{L}\{\delta(t - t_0)\} &= \lim_{\tau \to 0^+} \mathcal{L}\{d_\tau(t - t_0)\} \\
&= \lim_{\tau \to 0^+} \int_{t_0 - \tau}^{t_0 + \tau} e^{-st} d_\tau(t - t_0)\, dt \\
&= \lim_{\tau \to 0^+} \frac{1}{2\tau} \int_{t_0 - \tau}^{t_0 + \tau} e^{-st} dt = \lim_{\tau \to 0^+} \frac{1}{2\tau} \left[\frac{e^{-st}}{t} \right]_{t = t_0 - \tau}^{t = t_0 + \tau} \\
&= \frac{e^{-st_0}}{s} \lim_{\tau \to 0^+} \frac{e^{s\tau} - e^{-s\tau}}{2\tau}, \text{ by algebra} \\
&= e^{-st_0} \lim_{\tau \to 0^+} \frac{\sinh(s\tau)}{s\tau}, \text{ by formula below} \\
&= e^{-st_0} \lim_{\tau \to 0^+} \frac{s\cosh(s\tau)}{s}, \text{ by L'Hopitals rule} \\
&= e^{-st_0}.
\end{aligned}$$

where we used the fact that

$$\sinh(s\tau) = \frac{e^{s\tau} - s^{-s\tau}}{2}.$$

In summary,

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}, \;\; t_0 > 0.$$

We now compute some simple examples:

- If $t_0 = 0$, then
$$\mathcal{L}\{\delta(t)\} = e^{-s \cdot 0} = 1.$$

- If $t_0 = 9$ then
$$\mathcal{L}\{\delta(t - 9)\} = e^{-9s}.$$

We'll note the following important property of δ functions, which is usually called "point evaluation". Suppose f is a continuous function. Then

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0).$$

In the next example we show how the delta function is connected to the Heaviside function.

Example 6.6.1 Solve the IVP

$$y' = \delta(t-c), \quad y(0) = y_0.$$

Take $\mathcal{L}$ of both sides and

$$\begin{aligned} \mathcal{L}\{y'\} &= \mathcal{L}\{\delta(t-c)\} \implies \\ s\mathcal{L}\{y\} - y(0) &= e^{-cs} \implies \\ \mathcal{L}\{y\} &= \frac{y_0}{s} + \frac{e^{-cs}}{s} \end{aligned}$$

hence

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{\frac{y_0}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-cs}}{s}\right\} \\ &= y_0 + u_c(t). \end{aligned}$$

□

The example shows that the derivative of the Heaviside function is the delta function! (To be totally clear, we should note that to define derivatives of discontinuous functions requires that we understand differentiation in the context of generalized functions. For now, we should accept the intuition that the "slope" of a jump dicontinuity is the "infinite spike" of the delta function.

Fact 6.6.2

$$\frac{d}{dt}[u_c(t)] = \delta(t-c).$$

Example 6.6.3 Solve the IVP

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \quad y(0) = 0, y'(0) = 0.$$

Step 1: Take $\mathcal{L}$ of both sides and solve for $\mathcal{L}\{y\}$:

$$\begin{aligned} \mathcal{L}\{y''\} + 4\mathcal{L}\{y\} &= \mathcal{L}\{\delta(t-\pi)\} - \mathcal{L}\{\delta(t-2\pi)\} \implies \\ s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} &= e^{-\pi s} - e^{-2\pi s} \implies \\ (s^2+4)\mathcal{L}\{y\} &= e^{-\pi s} - e^{-2\pi s} \implies \\ \mathcal{L}\{y\} &= \frac{e^{-\pi s}}{s^2+4} - \frac{e^{-2\pi s}}{s^2+4}. \end{aligned}$$

Step 2: Notice that we don't need to do partial fractions or complete the square here since s^2+4 is already a sum of two squares.

Step3: Take an inverse Laplace transform: Using $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}F(s)$ and $\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$ we get

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+4}\right\} \\ &= \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{2}{s^2+2^2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{2}{s^2+2^2}\right\} \end{aligned}$$

$$= \frac{1}{2}u_\pi(t)f_1\left(t-\pi\right) - \frac{1}{2}u_{2\pi}(t)f_2\left(t-2\pi\right)$$
$$= \frac{1}{2}u_\pi(t)\sin\left(2\left(t-\pi\right)\right) - \frac{1}{2}u_{2\pi}(t)\sin\left(2\left(t-2\pi\right)\right)$$

where $f_1, f_2 = \sin(2t)$. Now it turns out, that

$$\sin\left(2\left(t-\pi\right)\right) = \sin\left(2t-2\pi\right) = \sin(2t)$$

and

$$\sin\left(2\left(t-2\pi\right)\right) = \sin\left(2t-4\pi\right) = \sin(2t).$$

or in general

$$\sin\left(x\right) = \sin\left(x+2\pi\right).$$

Hence a possible multiple choice answer could be:

$$y = \frac{1}{2}u_\pi(t)\sin\left(2t\right) - \frac{1}{2}u_{2\pi}(t)\sin\left(2t\right).$$

□

Example 6.6.4 Solve the IVP

$$y'' + 2y' + 3y = \sin t + \delta\left(t-3\pi\right), \quad y(0)=0, y'(0)=0.$$

Take $\mathcal{L}$ of both sides and solve for $\mathcal{L}\left\{y\right\}$:

$$\begin{aligned}
\mathcal{L}\left\{y''\right\} + 2\mathcal{L}\left\{y'\right\} + 3\mathcal{L}\left\{y\right\} &= \frac{1}{s^2+1} + e^{-3\pi s} \implies \\
\left[s^2\mathcal{L}\left\{y\right\} - sy(0) - y'(0)\right] + 2\left[s\mathcal{L}\left\{y\right\} - y(0)\right] + 3\mathcal{L}\left\{y\right\} &= \frac{1}{s^2+1} + e^{-3\pi s} \implies \\
s^2\mathcal{L}\left\{y\right\} + 2s\mathcal{L}\left\{y\right\} + 3\mathcal{L}\left\{y\right\} &= \frac{1}{s^2+1} + e^{-3\pi s} \implies \\
\left(s^2+2s+3\right)\mathcal{L}\left\{y\right\} &= \frac{1}{s^2+1} + e^{-3\pi s} \implies \\
\mathcal{L}\left\{y\right\} = \frac{1}{\left(s^2+2s+3\right)\left(s^2+1\right)} &+ \frac{e^{-3\pi s}}{s^2+2s+3}
\end{aligned}$$

Step 2: First we do partial fractions:

$$\frac{1}{\left(s^2+2s+3\right)\left(s^2+1\right)} = \frac{As+B}{\left(s^2+2s+3\right)} + \frac{Cs+D}{\left(s^2+1\right)}$$

and do the algebra to get

$$A = \frac{1}{4}, B = \frac{1}{4}, C = -\frac{1}{4}, D = \frac{1}{4}$$

Also we need to complete the square:

$$s^2+2s+3 = \left(s+1\right)^2 + 2.$$

so that

$$\frac{1}{\left(s^2+2s+3\right)\left(s^2+1\right)} = \frac{1}{4}\left(\frac{s+1}{\left(s+1\right)^2+2} + \frac{-s+1}{\left(s^2+1\right)}\right)$$

Step 3: Take an inverse Laplace transform: Using $\mathcal{L}\left[u_a(t)f(t-a)\right] = e^{-as}F(s)$, $\mathcal{L}\left\{\cos(at)\right\} = \frac{s}{s^2+a^2}$ $\mathcal{L}\left\{\sin(at)\right\} = \frac{a}{s^2+a^2}$, Also using $\mathcal{L}\left\{e^{at}\cos(bt)\right\} = \frac{s-a}{(s-a)^2+b^2}$ $\mathcal{L}\left\{e^{at}\sin(bt)\right\} = \frac{b}{(s-a)^2+b^2}$ we get

$$y = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s+1}{\left(s+1\right)^2+2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{-s+1}{\left(s^2+1\right)}\right\}$$

$$
\begin{aligned}
&+\mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{(s+1)^2+2}\right\}\\
&=\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+\left(\sqrt{2}\right)^2}\right\}-\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}\\
&+\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}+\frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{e^{-3\pi s}\frac{\sqrt{2}}{(s+1)^2+\left(\sqrt{2}\right)^2}\right\}\\
&=\frac{1}{4}\left(e^{-t}\cos\left(\sqrt{2}t\right)-\cos t+\sin t\right)\\
&+\frac{1}{\sqrt{2}}u_{3\pi}(t)\,f_1(t-3\pi)\\
&=\frac{1}{4}\left(e^{-t}\cos\left(\sqrt{2}t\right)-\cos t+\sin t\right)\\
&+\frac{1}{\sqrt{2}}u_{3\pi}(t)\,e^{-1(t-3\pi)}\sin\left(\sqrt{2}\,(t-3\pi)\right)
\end{aligned}
$$

where $f_1=\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+1)^2+\left(\sqrt{2}\right)^2}\right\}=e^{-t}\cos\sqrt{2}t.$ $\square$

6.7 The convolution integral

Suppose we want to take the inverse Laplace transform of a product:
Is it true that

$$\mathcal{L}^{-1}\{F(s)G(s)\}\stackrel{?}{=}\mathcal{L}^{-1}\{F(s)\}\,\mathcal{L}^{-1}\{G(s)\}?$$

The answer is a resounding *NO!*

In order to take the inverse of a product, we need to define the **convolution integral**: Let $f(t), g(t)$ be two nice functions, then

$$(f\star g)(t)=\int_0^t f(t-\tau)\,g(\tau)\,d\tau=\int_0^t f(\tau)g(t-\tau)\,d\tau.$$

The function $h=f\star g$ is called the \textbf{convolution} of f and g. We can think of convolution as somehow "mixing" or "averaging" the two functions that are being convolved. In practice, convolutions are often used to take jagged, unsmooth functions as input and return smoothed functions as output by convoluting the rough function with an appropriate smooth one.

Theorem 6.7.1 *The Laplace transform of the convolution is*

$$\mathcal{L}\{(f\star g)(t)\}=\mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}=F(s)G(s);$$

that is

$$\mathcal{L}^{-1}\{F(s)G(s)\}=(f\star g)(t)=\int_0^t f(t-\tau)\,g(\tau)\,d\tau.$$

Convolutions have nice properties: We can treat $\star$ almost like multiplication of functions:

- $f\star g=g\star f$ (commutative)
- $f\star(g_1+g_2)=f\star g_1+f\star g_2$ (distributive)
- $(f\star g)\star h=f\star(g\star h)$ (associative)

- $f \star 0 = 0 \star f = 0$.

However it doesn't have all the properties of ordinary multiplication. In particular, $(f \star 1) \neq 1 \star f$.

Example 6.7.2 Find the Laplace transform of

$$h(t) = \int_0^t \sin(t-\tau)\cos\tau d\tau$$

Use $f = \sin t$ and $g = \cos t$ and we know that by the Theorem

$$\begin{aligned}\mathcal{L}\left\{\int_0^t \sin(t-\tau)\cos\tau d\tau\right\} &= \mathcal{L}\{\sin t\}\mathcal{L}\{\cos t\} \\ &= \frac{1}{s^2+1}\cdot\frac{s}{s^2+1} \\ &= \frac{s}{(s^2+1)^2}.\end{aligned}$$

□

Example 6.7.3 Find the Laplace transform of

$$e^t\int_0^t \sin\tau\cos(t-\tau)\,d\tau.$$

This question is testing if you know how to use formulas

$$\mathcal{L}\left\{e^{ct}f(t)\right\} = F(s-c)$$

hence we need to first take the Laplace transform of

$$\mathcal{L}\left\{\int_0^t \sin\tau\cos(t-\tau)\,d\tau\right\} = \mathcal{L}\left\{\int_0^t \sin(t-\tau)\cos(\tau)\,d\tau\right\} = \frac{s}{(s^2+1)^2}$$

from Example1. Hence using the formula above we have

$$\mathcal{L}\left\{e^t\int_0^t \sin\tau\cos(t-\tau)\,d\tau\right\} = \frac{s-1}{\left((s-1)^2+1\right)^2}$$

□

Example 6.7.4 Find the inverse Laplace transform of

$$H(s) = \frac{30}{(s-3)^3(s^2+25)}$$

Split up $H(S) = F(s)G(s)$ where $F(s) = \frac{2}{(s-3)^3}$ and $G(s) = \frac{5}{s^2+25}$ so that

$$H(s) = 3\cdot\frac{2!}{(s-3)^{2+1}}\cdot\frac{5}{s^2+5^2}$$

and since

$$\begin{aligned}\mathcal{L}^{-1}\{F\} &= \mathcal{L}^{-1}\left\{\frac{2!}{(s-3)^{2+1}}\right\} = t^2e^{3t}, \\ \mathcal{L}^{-1}\{G\} &= \mathcal{L}^{-1}\left\{\frac{5}{s^2+5^2}\right\} = \sin(5t)\end{aligned}$$

so that

$$\begin{aligned}\mathcal{L}^{-1}\{H(s)\} &= 3\int_0^t f(t-\tau)g(\tau)d\tau \\ &= 3\int_0^t (t-\tau)^2 e^{3(t-\tau)}\sin(5\tau)\,d\tau\end{aligned}$$

but you also need to be prepared that one of the possible solutions is

$$\begin{aligned}\mathcal{L}^{-1}\{H(s)\} &= 3\int_0^t f(\tau)g(t-\tau)d\tau \\ &= 3\int_0^t \tau^2 e^{3\tau}\sin(5(t-\tau))\,d\tau.\end{aligned}$$

$\square$

Example 6.7.5 Solve the IVP in terms of the convolution integrals:

$$4y'' + 4y' + 17y = g(t), \quad y(0) = 0, y'(0) = 0.$$

Step 1: Take $\mathcal{L}$ of both sides and solve $\mathcal{L}\{y\}$:

$$4\left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) + 4\left(s\mathcal{L}\{y\} - y(0)\right) + 17\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

and plugging in the initial conditions we have

$$\mathcal{L}\{y\}\left(4s^2 + 4s + 17\right) = \mathcal{L}\{g(t)\}$$

so that

$$\mathcal{L}\{y\} = \frac{\mathcal{L}\{g(t)\}}{4s^2 + 4s + 17}$$

Step 2: Instead of doing partial fractions we will use the convolution integral. But first let us complete the square by first wrtiting

$$4s^2 + 4s + 17 = 4\left(s^2 + s + \frac{17}{4}\right)$$

hence we want add/subtract $\left(\frac{b}{2}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$hence

$$\begin{aligned}4\left(s^2 + s + \frac{17}{4}\right) &= 4\left(s^2 + s + \frac{1}{4} - \frac{1}{4} + \frac{17}{4}\right) \\ &= 4\left(\left(s + \frac{1}{2}\right)^2 + \frac{16}{4}\right) \\ &= 4\left(\left(s + \frac{1}{2}\right)^2 + 4\right)\end{aligned}$$

hence

$$\frac{\mathcal{L}\{g(t)\}}{4s^2 + 4s + 17} = \frac{1}{4}\frac{1}{\left(\left(s + \frac{1}{2}\right)^2 + 4\right)}\mathcal{L}\{g(t)\}$$

Step 3: We take the inverse Laplace transform of

$$\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{\left(\left(s + \frac{1}{2}\right)^2 + 4\right)}\mathcal{L}\{g(t)\}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}\right\}$$

hence we need to take the inverse of

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{\left(\left(s+\frac{1}{2}\right)^2+4\right)}\right\} \\ &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{\left(\left(s+\frac{1}{2}\right)^2+4\right)}\right\} \\ &= \frac{1}{2}e^{-\frac{1}{2}t}\sin\left(2t\right). \end{aligned}$$

Thus using the formula $\mathcal{L}^{-1}\left\{F(s)G(s)\right\} = \left(f \star g\right)(t) = \int_0^t f\left(t-\tau\right)g\left(\tau\right)d\tau$ we have

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{\frac{1}{4}\mathcal{L}\left\{f(t)\right\}\mathcal{L}\left\{g(t)\right\}\right\} \\ &= \frac{1}{4}\int_0^t f\left(t-\tau\right)g\left(\tau\right)d\tau \\ &= \frac{1}{4}\int_0^t \frac{1}{2}e^{-\frac{1}{2}(t-\tau)}\sin\left(2\left(t-\tau\right)\right)g\left(\tau\right)d\tau \\ &= \frac{1}{8}\int_0^t e^{-\frac{1}{2}(t-\tau)}\sin\left(2\left(t-\tau\right)\right)g\left(\tau\right)d\tau. \end{aligned}$$

□

Example 6.7.6 Compute the following integral

$$\int_0^5 e^{-x}\sin x dx$$

using only Laplace transforms.

Solution:: First we want to write this as a convolution:

$$\int_0^5 e^{-x}\sin x dx = e^{-5}\int_0^5 e^{5-x}\sin x dx.$$

and let

$$h(t) = \int_0^t e^{t-\tau}\sin\tau d\tau.$$

The Laplace trasnform of this is

$$\begin{aligned} \mathcal{L}\left\{h(t)\right\} &= \mathcal{L}\left\{\int_0^t e^{t-\tau}\sin\tau d\tau\right\} \\ &= \mathcal{L}\left\{\int_0^t f(t-\tau)g(\tau)d\tau\right\} \\ &= \mathcal{L}\left\{f(t)\right\}\mathcal{L}\left\{g(t)\right\} \\ &= \mathcal{L}\left\{e^t\right\}\mathcal{L}\left\{\sin t\right\} \\ &= \frac{1}{\left(s-1\right)\left(s^2+1\right)} \end{aligned}$$

Now do partiial fractions on this and get

$$\frac{1}{\left(s-1\right)\left(s^2+1\right)} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{s+1}{s^2+1}\right)$$

Hence we can now take the inverse Laplace transform of this:

$$\begin{aligned} h(t) &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\sin t. \end{aligned}$$

Thus we computed that

$$h(t) = \int_0^t e^{t-\tau}\sin\tau d\tau = \frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\sin t$$

Thus

$$\begin{aligned} e^{-5}\int_0^5 e^{5-x}\sin x dx &= e^{-5}h(5) \\ &= e^{-5}\left(\frac{1}{2}e^5 - \frac{1}{2}\cos 5 - \frac{1}{2}\sin 5\right). \end{aligned}$$

□

Appendix A

A fast quadratic method

Consider the quadratic equation

$$ax^2 + bx + c = 0.$$

Students typically learn to solve this equation with the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or with the method of completing the square.

A recently introduced method for solving quadratic equations makes this process much easier, particularly when complex numbers are involved. We'll first look at the theory behind the method, and then we'll give some examples that show how easy it is to do. Suppose that $a = 1$. What we are looking for is two numbers, the **roots** R and S so that we can factor the equation as

$$(x - R)(x - S) = 0.$$

Multiplying out shows that

$$x^2 + bx + c = x^2 - (R + S)x + RS,$$

so that $-b = (R + S)$ and $c = RS$. Here is the key step: $R + S$ will equal $-b$ when the average of R and S is $-b/2$. Because parabolas are symmetric about the vertex, the number $-b/2$ is precisely halfway in between R and S. So we get

$$R = -b/2 + z \text{ and } S = -b/2 - z.$$

Since $RS = c$, we can compute

$$c = RS = (-b/2 + z)(-b/2 - z) = \frac{b^2}{4} - z^2$$

and so

$$z = \pm\sqrt{\frac{B^2}{4} - C},$$

which is the quadratic formula when $a = 1$.

This might seem a bit complicated to parse, but the idea is very easy to use in practice. Consider the equation

$$x^2 + 5x - 6 = 0.$$

Then we have that the roots are $5/2 \pm z$. Since the roots must multiply to c, we get

$$\begin{aligned}(5/2+z)(5/2-z) &= -6 \\ \Rightarrow 25/4 - z^2 &= -6 \\ \Rightarrow z^2 &= 49/4 \\ \Rightarrow z &= \pm 7/2\end{aligned}$$

Then the roots are $-5/2 \pm 7/2$, so $R = -6$ and $S = 1$, which means the equation factors as $(x+6)(x-1) = 0$.

This works even better for complex roots. Consider the equation

$$x^2 + 2x + 10 = 0$$

so that the roots have the form $-2/2 \pm z = -1 \pm z$. Then

$$(-1-z)(-1+z) = 10$$

and so

$$z^2 - 1 = -10$$

and

$$z = \pm 3i.$$

Then the solutions to the equation are $-1 \pm 3i$.

Appendix B

Exercises for Chapter 1

B.1 Section 1.1

Exercises

1.

2. $y(t) = t + 1$

$$\frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t}.$$

3. $y(x) = x + x \ln x$

$$x\frac{dy}{dx} = x + y, \quad y(1) = 3.$$

4.

$$\frac{dy}{dt} = y^2 + 2y$$

5.

$$\frac{dy}{dt} = y^4 t - 3y^3 t + 2y^2 t.$$

6.

(a) $\dfrac{dy}{dt} = 2yt$

(b) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ *Fun Fact:* This particular PDE is a very famous PDE and is called the *heat equation*. It models the flow of heat in a medium over time.

(c) $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ *Fun Fact:* This particular PDE is a very famous PDE and is called the *wave equation*. This PDE along with boundary conditions, describes the amplitude and phase of the wave.

(d) $x\dfrac{d^2y}{dx^2} = y\dfrac{dy}{dx} + x^2 y$

(e) $2y'' - y' + y = 0$

7.

(a) $\dfrac{dy}{dt} = 2yt$

(b) $y\dfrac{d^2y}{dt^2} = \cos t$

(c) $ty''' - y'' - 2y = 0$

(d) $\frac{dy^6}{dt^6} - 2\frac{dy}{dt} + y = t^2$

(e) $\cos y + y' = t$

(f) $6y''' - y^2 = y^{(5)}$

(g) $\frac{d^2y}{dt^2} = \frac{y}{y+t}$

B.2 Section 1.2

Exercises

1.

(a) $\frac{dy}{dt} = \sin t$

(b) $\frac{dy}{dt} = t - y$

(c) $\frac{dy}{dt} = 2 - y$

(d) $\frac{dy}{dt} = t$

2.

$$\frac{dy}{dt} = y^2 - t$$

(a) Suppose $y(t)$ is a solution to this ODE and also you know that $y(-1) = 1$. Then based on the slope field, what is your prediction for the long term behavior of $y(t)$. That is, what is your prediction of

$$\lim_{t\to\infty} y(t)?$$

(b) Suppose $y(t)$ is a solution to this ODE and also you know that $y(1) = 0$. Then based on the slope field, what is your prediction for the long term behavior of $y(t)$, that is, what is your prediction of

$$\lim_{t\to\infty} y(t) =?$$

3. $P(t)P(t)$

$$\frac{dP}{dt} = P(P-100)(P+100)/100000$$

(a) Suppose that the population of the Phan fish is 80 at time $t = 0$. What is the long term behavior for the population of the Phan fish? Will it keep increasing/decreasing, stabilize to a certain number, or go extinct?

Appendix C

Exercises for Chapter 2

C.1 Section 2.1

Exercises

1. Use a computer app to draw the direction field for the given differential equations. Use the direction field to describe the long term behavior of the solution for large t. (Meaning use the direction field to predict $\lim_{t\to\infty} y(t)$ for different starting points). Find the general solution of the given differential equations, and use it to determine how solutions behave as $t \to \infty$.

(a) $y' + 3y = t + e^{-2t}$

(b) $y' + y = te^{-t} + 1$

(c) $ty' - y = t^2 e^{-t}$

(d) $2y' + y = 3t$

2. Find the particular solution to given initial value problem.

(a) $y' - y = 2te^{2t}$, $y(0) = 1$

(b) $ty' + 2y = \sin t$, $y(\pi/2) = 1$, $t > 0$

3. Consider the following initial value problem:

$$ty' + (t+1)\,y = 2te^{-t}, \quad y(1) = a, \quad t > 0$$

where a is any real number. Find the particular solution that solves this IVP.

C.2 Section 2.2

Exercises

1. Find the general solutions for the following differential equations. Find the *explicit* solutions if you can. If you can't solve for y exactly, then leave it as an \emph{implicit} solution:

(a) $y' = ky$ where k is a parameter.

(b) $y' = \dfrac{x^2}{y}$

(c) $\dfrac{dy}{dx} = \dfrac{3x^2 - 1}{3 + 2y}$

(d) $xy' = \dfrac{\left(1 - y^2\right)^{1/2}}{y}$

(e) $\dfrac{dy}{dx} = \dfrac{x^2}{1 + y^2}$

(f) $\dfrac{dy}{dx} = \dfrac{x}{\cos\left(y^2\right) y}$

2. Consider the ODE

$$\frac{dy}{dt} = \frac{4y}{t}.$$

(a) What kind of differential equation is this? Is it linear? Is it separable?

(b) If the ODE is both separable and Linear. Then use both methods to solve this equation. And check to make sure you get the same answer.

3. Find the general solution to the following differential equation:

$$\frac{dy}{dt} = (y + 1)(y - 2).$$

(Hint: Use partial fractions!)

C.3 Section 2.3

Exercises

1. First check each if the following differential equations are homogeneous. Then find the general solutions for the following differential equations.

(a) $x^2 \dfrac{dy}{dx} = -\left(y^2 - yx\right).$

(b) $\dfrac{dy}{dx} = \dfrac{x + 3y + 2\frac{y^2}{x}}{3x + y}.$

(c) $\dfrac{dy}{dx} = \dfrac{y}{x} - \dfrac{x^2 - y^2}{2xy}.$

2. Consider the following homogeneous equation:

$$\frac{dy}{dx} = \frac{y - x}{y + x}.$$

(a) Use the substitution $v = \frac{y}{x}$ to rewrite the equation only in terms of v and x.

(b) Solve for the general solution.

3.

$$\frac{dy}{dx} = \frac{-y^2 - yx}{x^2}.$$

(a) Use the substitution $v = \frac{y}{x}$ to rewrite the equation only in terms of v and x.

(b) Solve for the general solution.

4. Using the given substitution, solve the differential equation:

(a) Rewrite $\frac{dy}{dx} + xy = x^2y^2$ using the substitution $u = \frac{1}{y}$, only in terms of u, x.

(b) Rewrite $\frac{dy}{dx} + y = \frac{x}{y^2}$ using the substitution $u = y^3$, only in terms of u, x.

C.4 Section 2.4

Exercises

1. Initially, a tank contains 100 L of water with 10 kg of sugar in solution. Water containing sugar flows into the tank at the rate of 2 L/min, and the well-stirred mixture in the tank flows out at the rate of 5 L/min. The concentration $c(t)$ of sugar in the incoming water varies as $c(t) = 2+\cos(3t)$ kg/L. Let $Q(t)$ be the amount of sugar (in kilograms) in the tank at time t (in minutes). Write the Initial Value Problem that $Q(t)$ satisfies.

2. Initially, a tank contains 500 L (liters) of pure water. Water containing 0.3kg of salt per liter is entering at a rate of 2 L/min, and the mixture is allowed to flow out of the tank at a rate of 1 L/min. Let $Q(t)$ be the amount of salt at time t measured in kilograms (kg). What is the IVP that $Q(t)$ satisfies?

3. Initially, a tank contains 400 L of water with 10 kg of salt in solution. Water containing 0.1 kg of salt per liter (L) is entering at a rate of 1 L/min, and the mixture is allowed to flow out of the tank at a rate of 2 L/min. Let $Q(t)$ be the amount of salt at time t measured in kilograms. What is the IVP that $Q(t)$ satisfies?

4. Consider a pond that initially constains 10 million gal of pure water. Water containing a polluted chemical flows into the pond at the rate of 6 million gal/year, and the mixture in the pond flows out at the rate of 5 million gal/year. The concentration $\gamma(t)$ of chemical in the incoming water varies as $\gamma(t) = 2 + \sin 2t$ grams/gal. Let $Q(t)$ be the amount of chemical at time t measured by millions of grams. What is the IVP that $Q(t)$ satisfies?

5. A tank contains 200 gal of liquid. Initially, the tank contains pure water. At time $t = 0$, brine containing 3 lb/gal of salt begins to pour into the tank at a rate of 2 gal/min, and the well-stirred mixture is allowed to drain away at the same rate. How many minutes must elapse before there are 100 lb of salt in the tank?

6. A huge tank initially contains 10 gallons (gal) of water with 6 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the well-stirred mixture is allowed to flow out of the tank at a rate of 2 gal/min. What is the amount of the salt in the tank after 10 min?

7. Initially a tank holds 40 gallons of water with 10 lb of salt in solution. A salt solution containing $\frac{1}{2}$b of salt per gallon runs into the tank at the rate of 4 gallons per minute. The well mixed solution runs out of the tank at a rate of 2 gallons per minute. Let $y(t)$ be the amount of salt in the tank after t minutes. Then what is $y(20)$.

C.5 Section 2.5

Exercises

1. A detective is called to the scene of a crime where a dead body has just been found.

(a) She arrives on the scene at 10:23 pm and begins her investigation. Immediately, the temperature of the body is taken and is found to be 80°F. The detective checks the programmable thermostat and finds that the room has been kept at a constant 68°F for the past 3 days.\

(b) After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be 78.5° F. This last temperature reading was taken exactly one hour after the first one.

(c) The next day the detective is asked by another investigator, "What time did our victim die?" Assuming that the victim's body temperature was normal (98.6°) prior to death, what is her answer to this question? Newton's Law of Cooling can be used to determine a victim's time of death.

C.6 Section 2.6

Exercises

1. What is the largest open interval in which the solution to the IVPs in part (a) and part (b) are guaranteed to exist by the existence and uniqueness theorems?

(a) The IVP given by:

$$\begin{cases} \left(t^2+t-2\right)y'+e^t y=\frac{(t-4)}{(t-6)} \\ y(-3)=-1. \end{cases}$$

(b) The IVP given by:

$$\begin{cases} \left(t^2+t-2\right)y'+e^t y=\frac{(t-4)}{(t-6)} \\ y(5)=47. \end{cases}$$

2. What is the largest open interval in which the solution of the initial value problem

$$\begin{cases} (t-3)\,y'+y=\frac{(t-3)\cdot\ln(t-1)}{t-10} \\ y(6)=-7. \end{cases}$$

is guaranteed to exist by the existence and uniqueness Theorem?

3. What is the largest open interval in which the solution of the initial value problem

$$\begin{cases} (t-1)\, y' + \sqrt{t+2} y = \frac{3}{t-3} \\ y(2) = -5. \end{cases}$$

is guaranteed to exist by the Existence and Uniqueness Theorem?

4. What is the largest open interval in which the solution of the initial value problem

$$\begin{cases} t^2 y' + \ln|t-4|\, y = \frac{t-1}{\sin t} \\ y(5) = 9. \end{cases}$$

is guaranteed to exist by the existence and uniqueness Theorem?

5. Consider the IVP below

$$\frac{dy}{dt} = y^{1/5}, \quad y(0) = 0.$$

(a) Is this a linear or nonlinear equation? Can you use Theorem 2.6.1?

(b) Using Theorem 2.6.4 (the general theorem), can you guarantee that there is a unique solution to this IVP? Why?

C.7 Section 2.7

Exercises

1. Consider the following differential equation:

$$\frac{dy}{dt} = (y+2)\,(y-1)\,(y+5)$$

(a) Draw a phase line. Classify the equilibrium solutions. Draw all possible sketch of solutions of this differential equation.

(b) Consider the IVP

$$\frac{dy}{dt} = (y+2)\,(y-1)\,(y+5)\,, \quad y(0) = 3.$$

Let $y(t)$ be the unique solution that solves this IVP. Draw a sketch of $y(t)$ and use it to find $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

2. Consider the following differential equation:

$$\frac{dy}{dt} = y\,(y-3)^2\,(y+4)$$

(a) Draw a phase line. Classify the equilibrium solutions.

(b) Draw all possible sketch of solutions of this differential equation.

(c) Consider the IVP

$$\frac{dy}{dt} = y\,(y-3)^2\,(y+4)\,, \quad y(0) = -5.$$

Let $y(t)$ be the unique solution that solves this IVP. Draw a sketch of $y(t)$ and use it to find $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

(d) Consider the IVP

$$\frac{dy}{dt} = y\,(y-3)^2\,(y+4)\,, \quad y(0) = 1.$$

Let $y(t)$ be the unique solution that solves this IVP. Draw a sketch of $y(t)$ and use it to find $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

3. Let $y(t)$ be the unique solution to the IVP given by

$$\frac{dy}{dt} = y^2 \sin y, \quad y(0) = 1.$$

Draw a phase line for the ODE to find out $\lim_{t\to\infty} y(t)$ for the unique solution of the IVP above.

4. Consider the differential equation

$$\frac{dy}{dt} = f\,(y)$$

where $f(y)$ is given by the following graph (in y versus $f(y)$):

Draw the phase line and classify the equilibrium solutions.

C.8 Section 2.8

Exercises

1.

(a) $(2x+3) + (2y-2)\,y' = 0$

(b) $(2x+4y) + (2x-2y)\,y' = 0$

(c) $\left(3x^2 - 2xy + 2\right) dx + \left(6y^2 - x^2 + 3\right) dy = 0$

(d) $\left(2xy^2 + 2y\right) + \left(2x^2 y + 2x\right) y' = 0$

(e) $\dfrac{dy}{dx} = -\dfrac{ax+by}{bx+cy}$

(f) $(e^x \sin y + 3y)\,dx - (3x - e^x \sin y)\,dy = 0$

(g) $\left(\frac{y}{x} + 6x\right) dx + (\ln x - 2)\,dy = 0,x > 0$

2.

$$\left(9x^2 + y - 1\right) dx - (4y - x)\,dy = 0, \quad y(1) = 0.$$

3. b

$$\left(ye^{2xy} + x\right) dx + bxe^{2xy} dy = 0.$$

C.9 Section 2.9

Exercises

1. Find the approximate values of the solution of the given initial value problem at $t = 0.1, 0.2, 0.3$ and 0.4 using Euler's Method with $h = 0.1$.

$$\frac{dy}{dt} = t + y, \quad y(0) = 1.$$

2. Find the approximate values of the solution of the given initial value problem at $t = 0.1, 0.2, 0.3$ and 0.4 using Euler's Method with $h = 0.05$.

$$\frac{dy}{dt} = t + y^2, \quad y(0) = 1.$$

3. Find the approximate value of $y\,(2)$ using Euler's Method with $h = 0.5$ for the solution of the following IVP

$$\frac{dy}{dt} = y\,(3 - ty)\,, \quad y(0) = 0.5.$$

4. Consider the solution $y(t)$ to the IVP:

$$\frac{dy}{dt} = y\,(t + y)\,/10, \quad y(0) = 1.$$

Use the slope field below with Euler's Method (using $h = .5$) to estimate the value of $y(3)$:

Appendix D

Exercises for Chapter 3

D.1 Section 3.2

Exercises

1. Check if the following function are solutions to the given EQ.

(a) Check directly if $y_1 = 2e^{5t}$ is a solution or not to $y'' - 6y' + 5y = 0$.

(b) Check directly if $y_2 = 2e^{t}$ is a solution or not to $y'' - 6y' + 5y = t$.

2. Recall that if $y(t) = e^{rt}$ is a solution to the ODE given by

$$ay'' + by' + cy = 0$$

for constant a, b, c where $a \neq 0$, then the exponent r in front the t must be a solution to the *characteristic equation* $ar^2 + br + c = 0$.

By yourself, rederive that if $y(t) = Ae^{rt}$ is a solution to the equation above, then the number r must satisfy the characteristic equation $ar^2 + br + c = 0$ or $A = 0$.

3. Use the method given in Section 3.1 to find the general solution to

$$y'' + 5y' - 6y = 0$$

4. Use the method given in Section 3.1 to find the general solution to

$$y'' - 7y' = 0$$

5. Use the method given in Section 3.1 to find the particular solution to the IVP

$$y'' + y' - 20y = 0, \quad y(0) = 18, y'(0) = 9$$

D.2 Section 3.3

Exercises

1. What is the largest open interval in which the solution of the initial value problem

$$(t-3)\,y'' + \sin t y' + y = \frac{\ln(t-1)}{t-10}, \qquad y(15) = -7, y'(15) = 10$$

is guaranteed to exist by Theorem 3.3.1?

2. What is the largest open interval in which the solution of the initial value problem

$$t^2 y'' + e^t y' + (t-1)\,y = \sqrt{t+2}, \quad y(-1) = 1, y'(-1) = 5$$

is guaranteed to exist by Theorem 3.3.1?

3.

$$y'' + p(t)y' + q(t)y = 0,$$

$p, q I$Theorem 3.3.8

4. Consider the equation

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0.$$

(a) Is the function $y_1(t) = t^{\frac{1}{2}}$ a solution to this ODE?

(b) Is the function $y_2(t) = t^{-1}$ a solution to this ODE?

(c) Use Theorem 3.3.8 to show that

$$y(t) = c_1 t^{\frac{1}{2}} + c_2 t^{-1}$$

gives the general solution to the ODE above.

D.3 Section 3.4

Exercises

1.

(a) $y'' + 16y = 0$

(b) $y'' - 4y' + 9y = 0$

(c) $y'' - 4y' + 29y = 0$

2.

$$y'' - 8y' + 17y = 0, \quad y(0) = -4, y'(0) = -1.$$

D.4 Section 3.5

D.4.1 Repeated roots

Exercises

1.

(a) $y'' + 14y' + 49y = 0$

(b) $y'' - 18y' + 81y = 0$

2.

$$y'' - 4y' + 4y = 0, \quad y(0) = 12, y'(0) = -3.$$

D.4.2 Reduction of order

Exercises

1. $y_1(t) = t$

$$t^2y'' - 3ty' + 3y = 0, \quad t > 0.$$

$y_2(t)y = c_1y_1 + c_2y_2$

2. $y_1(t) = t^{-1}$

$$2t^2y'' + ty' - 3y = 0, \quad t > 0.$$

$y_2(t)y = c_1y_1 + c_2y_2$

3. $y_1(t) = t$

$$t^2y'' + 2ty' - 2y = 0, \quad t > 0.$$

$y_2(t)y = c_1y_1 + c_2y_2$

4. $y_1(t) = t^2$

$$t^2y'' - 3ty' + 4y = 0, \quad t > 0.$$

$y_2(t)y = c_1y_1 + c_2y_2$

D.5 Section 3.6

Exercises

1.

$$y'' + y' - 2y = e^{3t}.$$

(a) Find the general solution.

(b) Find the particular solution to the IVP:

$$y'' + y' - 2y = e^{3t}, \quad y(0) = \frac{1}{10}, y'(0) = \frac{13}{10}.$$

2.

$$y'' - 2y' + 2y = e^{2t}.$$

3. Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' - 4y' + 3y = 4e^{3t}.$$

4. Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' - 2y' + y = e^{t}.$$

5. Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' + y' - 6y = 52\cos(2t).$$

6. Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' + 2y' + 3y = \sin(t).$$

7. For the following ODEs. Use the method of undertermined coefficients (MOUC) to make the correct guess for the y_p. You DO NOT have to solve for the coefficients, $A, B, C \ldots$. Simply make the correct guess for the y_p.

(a) $y'' - 2y' + y = te^t$,

(b) $y'' + y' - 2y = t^2 e^t$,

(c) $y'' + y' = t^2 + \cos t$,

(d) $y'' + y' - 6y = e^{5t} + \sin(3t)$,

(e) $y'' + y' - 2y = te^t + t^2$,

D.6 Section 3.7

Exercises

1. Consider the following ODE

$$y'' + 16y = \frac{1}{\sin(4t)}.$$

(a) Find a particular solution to the ODE above using the method of variation of parameters.

(b) What is the general solution to the ODE above?

2. Find the general solution to

$$t^2 y'' - 4ty' + 6y = t^3, \quad t > 0$$

given that

$$y_1(t) = t^2, \;\; y_2(t) = t^3$$

forms a fundamental set of solution for the corresponding homogeneous differential equation.

3. Find the general solution to

$$t^2 y'' - 3ty' + 3y = 8t^3, \quad t > 0$$

given that

$$y_1(t) = t, \;\; y_2(t) = t^3$$

forms a fundamental set of solution for the corresponding homogeneous differential equation.

4. Find the general solution to

$$2t^2 y'' + ty' - 3y = 2t^{5/2}, \quad t > 0$$

given that

$$y_1(t) = t^{-1}, \;\; y_2(t) = t^{3/2}$$

forms a fundamental set of solution for the corresponding homogeneous differential equation.

D.7 Section 3.8/Section 3.9

Exercises

1. A 64 lb mass stretches a spring 4 feet. The mass is displaced an additional 5 feet. and then released; and is in a medium with a damping coefficients $\gamma = 7\frac{\text{lb sec}}{\text{ft}}$. Suppose there is no external forcing. Formulate the IVP that governs the motion of this mass.

2. A 32 lb mass stretches a spring 4 feet. The mass is displaced an additional 6 feet. and then released with an initial velocity of 3 $\frac{\text{ft}}{\text{sec}}$; and is in a medium with a damping coefficients $\gamma = 2\frac{\text{lb sec}}{\text{ft}}$. Suppose there is an external forcing due to wind given by $F(t) = 3\cos(3t)$. Formulate the IVP that governs the motion of this mass.

3. Consider the following undamped harmonic oscillator with forcing:

$$u'' + 5u = \sin(3t), \quad u(0) = 0,\ u'(0) = 0.$$

What is the natural frequency? What is the frequency for the external force? Will you get resonance? What is your guess for u_p?

4. Consider the following undamped harmonic oscillator with forcing:

$$u'' + 16u = 7\cos(4t), \quad u(0) = 0,\ u'(0) = 0.$$

What is the natural frequency? What is the frequency for the external force? Will you get resonance? What is your guess for u_p?

Appendix E

Exercises for Chapter 4

E.1 Section 4.1

Exercises

1. What is the largest interval for which there exists a unique solution by the Existence and Uniqueness Theorem for the following IVP:

$$\begin{cases} (t-5)\, y^{(4)} - \frac{\ln(t+7)}{t} y'' + e^t y = \frac{t^2+1}{(t-1)} \\ y(2) = -1 \\ y'(2) = 1 \\ y''(2) = 2 \\ y'''(2) = 5. \end{cases}$$

2. Find general solution of

$$y''' + 10y'' + 7y' - 18y = 0.$$

(Hint: $r^3 + 10r^2 + 7r - 18 = (r-1)(r+2)(r+9)$)

3. Find general solution of

$$y^{(4)} - 10y''' + 36y'' - 54y' + 27y = 0.$$

(Hint: $r^4 - 10r^3 + 36r^2 - 54r + 27 = (r-1)(r-3)^3$)

4. Find general solution of

$$y^{(5)} - 4y^{(4)} + 13y''' - 36y'' + 36y = 0.$$

(Hint: $r^5 - 4r^4 + 13r^3 - 36r^2 + 36r = r(r-2)^2\left(r^2+9\right)$)

5. Find general solution of

$$y^{(4)} + 11y'' + 18y = 0.$$

(Hint: $r^4 + 11r^2 + 18 = \left(r^2+2\right)\left(r^2+9\right)$)

6. Find general solution of

$$y^{(6)} + 32y^{(4)} + 256y'' = 0.$$

(Hint: $r^6 + 32r^4 + 256r^2 = r^2\left(r^2+16\right)^2$)

E.2 Section 4.2

Exercises

1. Consider
$$y^{(4)} + 8y''' + 16y'' = 4e^{-3t} + \cos t.$$
Find the general form of y_p. (Hint: $r^3 - 4r^2 - 11r + 30 = (r+3)(r-2)(r-5)$)

2. Consider
$$y^{(4)} + 8y''' + 16y'' = t + e^t.$$
Find the general form of y_p(Hint: $r^4 + 8r^3 + 16r^2 = r^2 (r+4)^2$)

3. Consider
$$y^{(4)} - 10y''' + 36y'' - 54y' + 27y = 2te^t + \cos(3t),$$
and suppose you know that $y_h = c_1 e^t + c_2 e^{-3t} + c_3 t e^{-3t} + c_4 t^2 e^{-3t}$. Find the general form of y_p.

4. Consider
$$y^{(4)} - 2y''' = 2t + 1.$$
Find the general form of y_p. (Hint: $r^4 - 2r^3 = r^3 (r-2)$)

Appendix F

Exercises for Chapter 6

F.1 Section 6.1

Exercises

1. Use the definition of Laplace transform to find the Laplace transform of $f(t) = 1$. That is, find $\mathcal{L}\{1\}$.

2. Use the definition of Laplace transform to find the Laplace transform of $f(t) = t$. That is, find $\mathcal{L}\{t\}$.

3. Use the definition of Laplace transform to find the Laplace transform of $f(t) = t^2$. That is, find $\mathcal{L}\{t^2\}$.

4. Use the properties of Laplace transform and the following facts

$$\begin{aligned}
\mathcal{L}\{1\} &= \frac{1}{s}, s > 0 \\
\mathcal{L}\{e^{at}\} &= \frac{1}{s-a}, s > a, \\
\mathcal{L}\{t\} &= \frac{1}{s^2}, s > 0, \\
\mathcal{L}\{t^2\} &= \frac{2}{s^3}, s > 0, \\
\mathcal{L}\{\sin(at)\} &= \frac{a}{s^2+a^2}, s > 0, \\
\mathcal{L}\{\cos(at)\} &= \frac{s}{s^2+a^2}, s > 0,
\end{aligned}$$

to compute the Laplace transforms of the following functions.

(a) $\mathcal{L}\left\{2e^{5t} + 7\cos(3t) + 2t\right\} =$

(b) $\mathcal{L}\left\{-7e^{-9t} - 5t^2 - 5\sin(3t)\right\} =$

(c) $\mathcal{L}\left\{-5\sin(\sqrt{7}t) + 2 + 5t\right\} =$

(d) $\mathcal{L}\left\{4e^{-t} - 6e^{3t} + \cos(3t)\right\} =$

F.2 Section 6.2

Exercises

1. Use the table of Laplace Transforms to help you compute the following inverse Laplace transforms.

(a) $\mathcal{L}^{-1}\left\{\dfrac{5}{s-6}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{5}{7-s}+\dfrac{1}{s+3}\right\}$

(c) $\mathcal{L}^{-1}\left\{\dfrac{3}{s+9}-\dfrac{10}{s^2}\right\}$

(d) $\mathcal{L}^{-1}\left\{\dfrac{3}{s^2+7}+\dfrac{2}{(s-5)^3}\right\}$

(e) $\mathcal{L}^{-1}\left\{\dfrac{s-3}{(s-3)^2+36}\right\}$

(f) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+9}+\dfrac{2}{s}-\dfrac{s-1}{(s-1)^2+25}\right\}$

2. Solve the following IVP using Laplace Transforms:

$$y' + 4y = e^{-t}, \quad y(0) = 0$$

3. Solve the following IVP using Laplace Transforms:

$$y' + y = e^{-2t}, \quad y(0) = 2$$

4. Solve the following IVP using Laplace Transforms:

$$y' + 7y = 1, \quad y(0) = 3.$$

F.3 Section 6.3

Exercises

1. What is the correct form of the partial fractions?

(a) $\dfrac{5s-1}{(s-3)(s^2+2s+5)} =$

(b) $\dfrac{s-2}{(s-2)^2(s+5)} =$

(c) $\dfrac{s+1}{(s^2+9)(s^3+2)} =$

(d) $\dfrac{s}{(s+1)(s^2+10)s^3} =$

2. Take the inverse Laplace Transforms of the following:

(a) $F(s) = \dfrac{1}{s^2-8s+7}$

(b) $F(s) = \dfrac{s+7}{s^2+6s+13}$

(c) $F(s) = \dfrac{2s-1}{s^2-8s+18}$

3. Solve the following IVP using Laplace Transforms:

$$y'' + 4y = 8, \quad y(0) = 11, y'(0) = 5.$$

4. Solve the following IVP using Laplace Transforms:

$$y'' - 4y' + 5y = 2e^t, \quad y(0) = 3, y'(0) = 1.$$

F.4 Section 6.4

Exercises

1. Take the Laplace transforms of the following functions

(a) $f(t) = u_7(t)e^{6(t-7)}$

(b) $f(t) = u_2(t)e^{-9(t-2)}$

(c) $f(t) = u_2(t)\left(t-2\right)^3$

(d) $f(t) = u_6(t)\sin\left(3\left(t-6\right)\right)$

(e) $f(t) = u_1(t)\cos\left(7\left(t-1\right)\right)$

2. Take the inverse Laplace transforms of the following functions

(a) $F(s) = \dfrac{e^{-3s}}{s+1}$

(b) $F(s) = \dfrac{e^{-5s}}{s-7}$

(c) $F(s) = \dfrac{2e^{-2s}}{s^2+4}$

(d) $F(s) = \dfrac{se^{-9s}}{s^2+7}$

(e) $F(s) = \dfrac{(s+2)\,e^{-3s}}{(s+2)^2+16}$

3. Take the inverse Laplace transforms of

$$F(s) = \frac{e^{-3s}}{s^2-3s+2}.$$

4. Take the inverse Laplace transforms of

$$F(s) = \frac{se^{-9s}}{s^2+6s+11}.$$

F.5 Section 6.5

Exercises

1.

$$y' + 9y = u_5(t), \quad y(0) = -2.$$

2.

$$y' + y = u_7(t)e^{-2(t-7)}, \quad y(0) = 1.$$

3.

$$y'' + 9y = u_3(t)\sin\left(2\left(t-3\right)\right), \quad y(0) = 2.$$