Entanglement Entropy in Condensed-Matter Systems

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Abstract

This is a review to the application of entanglement entropy in condensed matter systems for general audience. The article is based on a series of presentations in "Student Seminar Theoretical Physics, 2016 Fall" at Utrecht University.

I. Introduction

When in quantum mechanics there is a system of particles that behave in a way such that the quantum state of each particle cannot be described individually, but rather we are forced to consider the quantum state of the system as a whole, we say that those particles are *entangled*. Entanglement is one of the most fascinating aspects of Quantum Mechanics and the center of numerous debates throughout history. Einstein, Podolski and Rosen strongly opposed the idea of entanglement by devising what is called the EPR paradox, which would not be resolved until 1964 when John Bell refuted one of the assumptions of the argument by proving that it was incompatible with the results derived from quantum mechanics. Despite Einstein's opposition entanglement is real and in 2015 it was experimentally observed by a team of physicists at TU Delft in a *loophole-free Bell test* which leaves little room for skeptics.

The concept of entanglement has sparked strong interest in many fields of physics. An example of such a field is condensed matter physics, in which the framework of quantum entanglement has reached a solid and mature status and is the subject of intensive research nowadays.

Because entanglement is such a relevant concept it is important to find a means of quantifying it. This object that we consider the measure of entanglement must obey certain properties, such as being null for systems in which the quantum state of the components can be described individually and replicate the behavior of the system as a whole, and being unchanged by transformations that only affect the quantum state of individual components. Such an object exists and is called Entanglement Entropy (EE). Furthermore EE is maximum when the state we are considering is maximally entangled, making it a competent way to quantify entanglement.

Another instrumental result in this field is the so called *Area Law* which states that if we divide a system in two subsystems then the EE of the system is proportional to the boundary between the subsystems. This law is only valid for the ground state of local Hamiltonians.

II. EE in 1D Systems

The area law is very important because it gives us an expectation of what the entanglement entropy might be. In one dimensional systems we expect the area law to translate into a constant entropy. This is intuitively explained by assuming a finite correlation length, that is, a maximum distance for which one particle can be entangled with another particle. Assume there is a finite range: if we break the system in two parts and look at the entanglement entropy of just one of them, we will always have the same entanglement breaking as we make it bigger, as long as this part has more particles than the correlation length. Hence we also get the same entanglement entropy! For a concrete example, if we have a correlation length of 2 and in a chain of 10 particles, having a partition of 3 particles centered in the middle of the chain breaks the same net entanglement as if we move it one position to the left or to the right: we always have two particles on the each side that are entangled in exactly the same way with the partition. If we make it a 4 particles partition, we are still breaking net entanglement only with two particles on each side, which means we still have the same entanglement entropy. Obviously this is only valid as long as we do not get the partition too close from the end of the chain. In this toy system we can see how the entropy is constant as expected.

However, we know this is only valid in a true ground state, which only exists if there is an energy cost to go from this to the first excited state of the system. If no such a cost exists we have a strongly correlated system, implying an infinite correlation length, a prediction of the violation of the area law: the entanglement entropy will not be constant anymore, because it matters where and how long the partition is in relation with the whole chain. In this case entanglement entropy will scale logarithmicaly with those parameters.

Seeing a change from a constant to a logarithmic entanglement entropy means that we have stumbled upon a phase transition, if it can be mapped, respectively, to infinite and finite correlation lengths, which is a defining characterisctic of such phenomena, and it can even happen at zero temperature, unlike ordinary phase transitions. This relation has been confirmed for systems in which we expect phase transitions in one dimension, theoretically, numerically and experimentally. For higher dimensions the relation between area law and correlation length/phase transitions is more involved and it is harder, if possible at all, to draw general conclusions.

III. 2D Triangular Lattices

At the lowest possible temperature (T=0) for many-body systems, although there is no thermal fluctuation, there are always still quantum fluctuations due to Heisenberg's uncertainty principle. These quantum fluctuations can drive the system into different phases based on other external physical parameters different from temperature such as magnetic field, pressure, chemical composition, etc. These interesting phase of matter are called quantum phases (e.g. Superconductor Insulator Transition, non-Fermi liquid (or "strange metal"), Quantum magnets, quantum Hall system, etc. (2)) Motivated by methods developed in quantum information theories (7), we are going to look at the entanglement properties in the ground state of our system. From there, we can detect different quantum phases and also quantum phase transitions based on the variation of these entanglement properties. In contrast with the well-studied 1-dimensional (1D) system (10), there is much less known about higher dimensional systems due to the difficulties in solving non-integrable large systems using calculation techniques such as Exact Diagonalization (ED), Quantum Monte Carlo (QMC) or Tensor Networks. Thanks to interesting features of geometric frustration systems (3), we will focus specifically on the 2D triangular spin lattice systems in two cases of spin-1/2 and spin-1 particles. The spin-1/2 case will be theoretically described by spin-1/2 XXZ Hamiltonian model while the Heisenberg Bilinear-Biquadratic (BB) Hamiltonian model is for the spin-1 case (5).

Based on the parameter θ described in the Hamiltonian models, there will be 5 different phases for spin-1/2 case. Meanwhile, there are four locally ordered phases for the spin-1 case as seen in Figure 1

In the next section, we will introduce, apply and compare 2 different computational techniques: Exact Diagonalization (ED) and a rather recently introduced technique called Cluster Mean Field (CMF) (12, 13, 6) to detect different quantum phase transitions manifested in our small (up to 12 sites) systems.

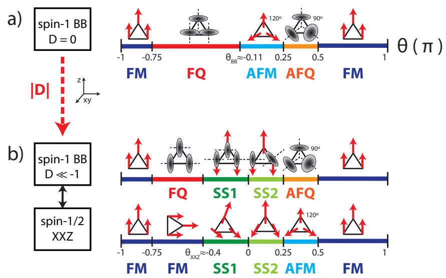

Figure 1: *Sketch of the phase diagram for the analyzed spin models (5).*

There is an important relationship between degeneracies of entanglement spectrum to the underlying symmetries of the system. However, in 2D system, the emergence or vanishing of these degeneracies strongly depends on the way the partition is made. In comparison with entanglement entropy (EE) as the usual choice of global entanglement measure in 1D system, we found that in 2D triangular spin lattice system, the geometry entanglement (GE) first introduced by (9) is more suitable to detect quantum phase transitions due to its insensitivity to the chosen cuts of the system.

IV. EE in Triangular Lattices

In higher dimensional systems, the calculations that leads to the before mentioned Entanglement Entropy cannot be made analytically. Hence, those problems have been dealt with using computational methods (2).

One of them is Exact Diagonalization (ED). It involves generating every matrix element of the Hamiltonian in a chosen basis and computing its eigenvalues and eigenvectors in order to determine the ground state for a given continuous parameter (14). It is then possible to measure entanglement using the definition of von Neunmann Entropy. The downside of this method is the way it scales with the size of the lattice (4^N), making it impractical for large systems.

The Cluster Mean Field (CMF) approach addresses that problem using by dividing the lattice in clusters, and treating the neighboring interactions as mean fields (12). The trade-off is the potential numerical instability due to the non-linearity of the equations.

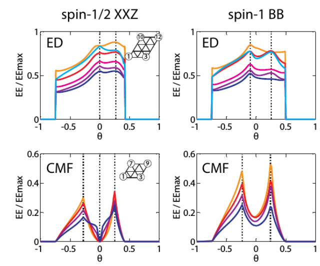

Figure 2: *Normalized EE for different partitions in a standard triangular lattice, using both outlined methods by (5)*

In results shown in Figure 2 one can observe some entanglement discontinuities and maxima. These are associated with the phase transitions outlined in Figure 1.

We wanted to perform a similar simulation using ED but with different boundary conditions, such that each site has 4 instead of 6 neighbors. The profile obtained in Figure 3 shows a qualitatively similar result but with an interesting local maximum only present in CMF computations thus far.

The fact that the normalized EE is changes with partition size reveals that the way the partition is chosen is relevant to the overall entanglement.

One can also check that the classical order parameter ($\langle S_z \rangle$, *i.e.* the magnetization) is no longer suitable to study phase transitions near $T = 0$, what is known as *quantum criticality*. This is where Entanglement Entropy provides useful insights.

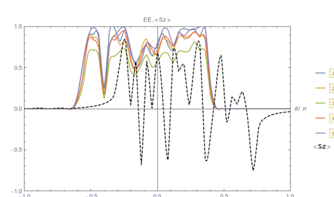

Figure 3: *EE calculation with different partition sizes for the triangular chain plotted along the magnetization*

V. Outlook

One of the ongoing efforts is classifying which kind of systems violate the area law and which kinds of systems obey the area law for the EE. Then it may be interesting to find what this system property can be used for in relation to the system's other properties. For example, it has been discovered that free fermions with a well defined Fermi surface violate the area law in more than one dimension (4), while fermions with an energy gap between their ground and first excited state do obey the area law.

Another promising application is the probing of a special kind of phase transition, a topological phase transition (1) (8). A current field of research is classifying the topological phase transitions into universality classes. Once the universality class of the transition is characterized by determining the relevant critical exponents, the universal behavior of the EE after the phase transition may be predicted. EE has also shown promise in the field of non-equilibrium quantum dynamics. It has been shown that in closed non-equilibrium systems, EE can grow in time (11). This leads to the interesting conclusion that there will be entropy in the system which is not created by heat. Further research is being done on how this impacts the dynamics of closed non-equilibrium systems.

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