Linear Algebra

Least Squares Problems[1] [2]

[1] Steven J. Leon, *Linear Algebra with Applications*, 8th ed, Pearson Prentice Hall

[2] Yaroslav Vorobets, http://www.math.tamu.edu/ yvorobet/MATH304/

Application 1. Hooke's law states that the force applied to a spring is proportional to the distance that the spring is stretched. Thus, if F is the force applied and x is the distance that the spring has been stretched, then $F = kx$. The proportionality constant k is called the *spring constant*.
Some physics students want to determine the spring constant for a given spring. They apply forces of 3, 5, and 8 pounds, which have the effect of stretching the spring 4, 7, and 11 inches, respectively. Using Hooke's law, they derive the following system of equations:

$$4k = 3$$
$$7k = 5$$
$$11k = 8$$

The system is clearly inconsistent, since each equation yields a different value of k. Rather than use any one of these values, the students decide to compute the least squares solution of the system:

$$(4,7,11)\begin{pmatrix}4\\7\\11\end{pmatrix}(k) = (4,7,11)\begin{pmatrix}3\\5\\8\end{pmatrix}$$

$$186k = 135$$

$$k \approx 0.726$$

Problem 1. Find the least squares solution of the system

$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 + 3x_2 &= 1 \\ 2x_1 - x_2 &= 2 \end{aligned}$$

Sol:

$$\begin{pmatrix} 1 & -2 & 2 \\ 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 \\ 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 9 & -7 \\ -7 & 11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

The least squares solution of the system is $\left(\frac{83}{50}, \frac{71}{50}\right)^T$

Problem 2. Find the constant function that is the least squares fit to the following data

x	0	1	2	3
$f(x)$	1	0	1	2

$$f(x) = c \Longrightarrow \begin{cases} \text{c=1} \\ \text{c=0} \\ \text{c=1} \\ \text{c=2} \end{cases} \Longrightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (c) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$(1,1,1,1) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (c) = (1,1,1,1) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$c = \frac{1}{4}(1+0+1+2) = 1$ (mean arithmetic value)

Problem 3. Find the linear polynomial that is the least squares fit to the following data

x	0	1	2	3
$f(x)$	1	0	1	2

$$f(x) = c_1 + c_2 x \Longrightarrow \begin{cases} c_1 = 1 \\ c_1 + c_2 = 0 \\ c_1 + 2c_2 = 1 \\ c_1 + 3c_2 = 2 \end{cases} \Longrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} \Longleftrightarrow \begin{cases} c_1 = 0.4 \\ c_2 = 0.4 \end{cases}$$

Problem 4. Find the quadratic polynomial that is the least squares fit to the following data

x	0	1	2	3
$f(x)$	1	0	1	2

$$f(x) = c_1 + c_2 x + c_3 x^2$$

$$\Longrightarrow \begin{cases} c_1 = 1 \\ c_1 + c_2 + c_3 = 0 \\ c_1 + 2c_2 + 4c_3 = 1 \\ c_1 + 3c_2 + 9c_3 = 2 \end{cases} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 22 \end{pmatrix} \Longleftrightarrow \begin{cases} c_1 = 0.9 \\ c_2 = -1.1 \\ c_3 = 0.5 \end{cases}$$