Numerical Linear Algebra

Iterative methods for eigenvalue problem[1]

[1] Steven J. Leon, *Linear Algebra with Applications*, 8th ed, Pearson Prentice Hall

Introduction:

Eigenvalues and eigenvectors: A non-zero column-vector $\mathbf{v}$ is called the eigenvector of a matrix $\mathbf{A}$ with the eigenvalue λ, if

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$

Real and symmetric matrix: An $n \times n$ matrix $\mathbf{A}$ is real and symmetric, if

$$A_{ij} \in \mathbb{R}, \quad \text{and} \quad \mathbf{A}^{\mathsf{T}} = \mathbf{A}$$

Orthogonal (orthonormal) matrix: An $n \times n$ matrix $\mathbf{Q}$ is orthogonal, if

$$\mathbf{Q}^{\mathsf{T}}\mathbf{Q} = \mathbf{Q}\mathbf{Q}^{\mathsf{T}} = \mathbf{1}$$

$$\Rightarrow \mathbf{Q}^{\mathsf{T}} = \mathbf{Q}^{-1}$$

Orthonormal eigenvectors: A set of eigenvectors $\mathbf{V} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n\}$ is called orthonormal set if it is an orthogonal set, that is

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \mathbf{v}_i^{\mathsf{T}} \mathbf{v}_j = \mathbf{v}_j^{\mathsf{T}} \mathbf{v}_i = 0, \quad \forall\, i \neq j$$

and the norm or length of each vector is 1, that is,

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \mathbf{v}_i^{\mathsf{T}} \mathbf{v}_j = \mathbf{v}_j^{\mathsf{T}} \mathbf{v}_i = 1, \quad \forall\, i = j$$

So a set of eigenvectors $\mathbf{V}$ is orthonormal if

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \mathbf{v}_i^{\mathsf{T}} \mathbf{v}_j = \mathbf{v}_j^{\mathsf{T}} \mathbf{v}_i = \begin{cases} 0, & \text{if} \quad i \neq j \\ 1, & \text{if} \quad i = j \end{cases} = \delta_{ij}$$

where δ_{ij} is Kronecker's delta.

Matrix diagonalization:

Matrix diagonalization means finding all eigenvalues and (optionally) eigenvectors of a matrix.

If an $n \times n$ matrix $\mathbf{A}$ is real and symmetric, then it has n real eigenvalues $\lambda_1, \cdots, \lambda_n$ and its (orthogonalized) eigenvectors $\mathbf{V} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ form a full basis, $\mathbf{V}^\intercal\mathbf{V} = \mathbf{V}\mathbf{V}^\intercal = \mathbf{1}$, in which the matrix is diagonal,

$$\mathbf{V}^\intercal\mathbf{A}\mathbf{V} = \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Orthogonal transformations,

$$\mathbf{A} \longrightarrow \mathbf{Q}^\mathsf{T}\mathbf{A}\mathbf{Q}$$

where $\mathbf{Q}^\mathsf{T}\mathbf{Q} = \mathbf{1}$, and, generally,

Similarity transformations,

$$\mathbf{A} \longrightarrow \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$$

preserve eigenvalues and eigenvectors. Therefore one of the strategies to diagonalize a matrix is to apply a sequence of similarity transformations (also called rotations) which (iteratively) turn the matrix into diagonal form.

Jacobi Eigenvalue Algorithm:

It is an iterative method for the calculation of the eigenvalues and eigenvectors of a real symmetric matrix (a process known as diagonalization). It is named after Carl Gustav Jacob Jacobi, who first proposed the method in 1846, but only became widely used in the 1950s with the advent of computers.(Ref: Wikipedia)

If $\mathbf{A}$ is a real symmetric matrix, then all the eigen values of $\mathbf{A}$ are real and there exists an orthogonal metrix $\mathbf{P}$ (consisting of the orthonormal eigen vectors of $\mathbf{A}$) such that $\mathbf{P}^{\intercal}\mathbf{A}\mathbf{P} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix.

This method uses a series of orthogonal similarity transformations using rotation matrices to arrive at the diagonal matrix $\mathbf{D}$ i.e. the basic idea is to choose special orthogonal matrices that zero out specified off-diagonal elements. *'Givens rotations'* are used in this method.

Givens rotation: A Givens rotation is represented by a matrix of the form

$$\mathbf{G}(p,q,\theta) = \begin{array}{cc} \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c & \cdots & -s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & s & \cdots & c & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} & \begin{array}{l} \\ \\ p \; row \\ \\ q \; row \\ \\ \\ \end{array} \\ \begin{array}{ccccccc} & & \begin{array}{c} p \\ col \end{array} & & \begin{array}{c} q \\ col \end{array} & & \end{array} & \end{array}$$

where $c \equiv cos(\theta)$ and $s \equiv sin(\theta)$