Problem x.1: Power Iteration Method

Learning Objectives:

- Understand the algorithm of the Power Iteration Method.
- Learn to code this algorithm.
- Use Power Iteration Method for computing the largest eigenvalue of a matrix.

Power Iteration Method:

Let us consider a 3×3 matrix **A** and assume a initial approximation to eigenvector $\mathbf{x}^{(0)} = (1,0,0)^T$, The Power iteration method for approximating next eigenpair is

$$\boldsymbol{x}^{(1)} \leftarrow \boldsymbol{A}\boldsymbol{x}^{(0)}, \quad \boldsymbol{x}^{(1)} \leftarrow \frac{\boldsymbol{x}^{(1)}}{\|\boldsymbol{x}^{(1)}\|}, \quad \lambda^{(1)} \leftarrow \|\boldsymbol{x}^{(1)}\| \quad \text{or} \quad \frac{\boldsymbol{A}\boldsymbol{x}^{(0)} \cdot \boldsymbol{x}^{(1)}}{\boldsymbol{x}^{(1)} \cdot \boldsymbol{x}^{(1)}}$$

Follow the below procedure to get the largest eigenvalue and the corresponding eigenvector:

1. Get/initialize the matrix **A** and the initial approximation of eigenvector $\mathbf{x}^{(0)}$
2. Compute $\mathbf{x}^{(i)} \leftarrow \mathbf{A}\mathbf{x}^{(i-1)}$
3. Compute $\mathbf{x}^{(i)} \leftarrow \mathbf{x}^{(i)}/\| \mathbf{x}^{(i)} \|$
4. *Compute* $\lambda^{(i)} \leftarrow \| \mathbf{x}^{(i)} \|$
5. Repeat from step 2 until $| \lambda^{(i)} - \lambda^{(i-1)} | < \varepsilon$.

Sample Input/output:

```
user@host:~

user@host:~$ ./a.out

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Iter  |   ( x_1       x_2      x_3 )^T  |   Lambda
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  0   |  1.000000  0.000000  0.000000   |
  1   |  1.000000 -0.500000  0.000000   |  2.000000
  2   |  1.000000 -0.800000  0.200000   |  2.500000
  3   |  1.000000 -1.000000  0.428571   |  2.800000
  4   |  0.875000 -1.000000  0.541667   |  3.428571
  5   |  0.804878 -1.000000  0.609756   |  3.416667
  6   |  0.764286 -1.000000  0.650000   |  3.414634
  7   |  0.740586 -1.000000  0.673640   |  3.414286
  8   |  0.726716 -1.000000  0.687500   |  3.414226
  9   |  0.718593 -1.000000  0.695621   |  3.414216
 10   |  0.713835 -1.000000  0.700378   |  3.414214
 11   |  0.711048 -1.000000  0.703165   |  3.414214
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Approximate eigenvalue = 3.414214
Corresponding eigenvector = ( 0.711048, -1.000000,  0.703165)^T
```

Tasks:

1. Write a program to find the largest eigenvalue and the corresponding eigenvector of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$