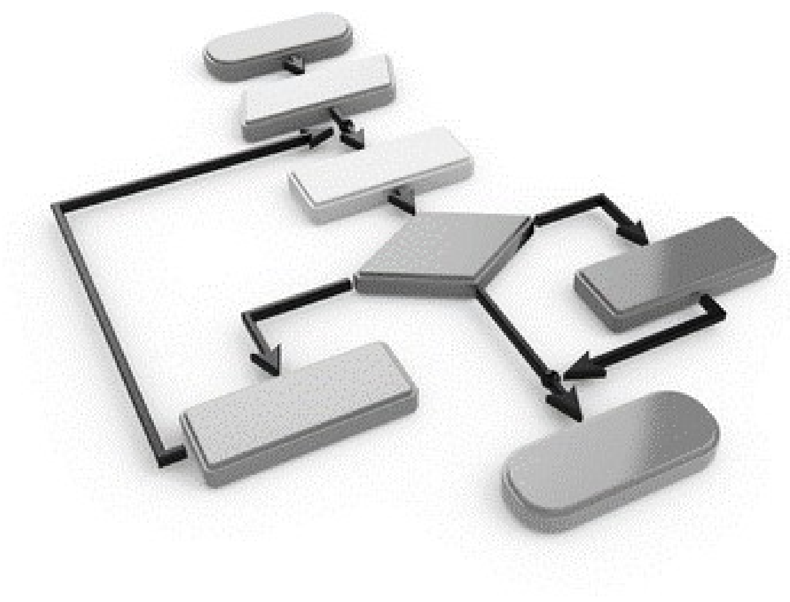


Dynamic Programming: MCM

CSE 301: Combinatorial Optimization

Matrix-chain Multiplication

- Input: a sequence (chain) $\langle A_1, A_2, \ldots, A_n \rangle$ of n matrices
- Aim: compute the product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$
- A product of matrices is fully parenthesized if
  - It is either a single matrix
  - Or, the product of two fully parenthesized matrix products surrounded by a pair of parentheses.

    $$\triangleright \left(A_i (A_{i+1} A_{i+2} \ldots A_j) \right)$$

    $$\triangleright \left((A_i A_{i+1} A_{i+2} \ldots A_{j-1}) A_j \right)$$

    $$\triangleright \left((A_i A_{i+1} A_{i+2} \ldots A_k)(A_{k+1} A_{k+2} \ldots A_j) \right) \qquad \text{for } i \le k < j$$
  - All parenthesizations yield the same product; matrix product is associative

Matrix-chain Multiplication

- Input: $\langle A_1, A_2, A_3, A_4 \rangle$
- 5 distinct ways of full parenthesization

$$(A_1(A_2(A_3A_4)))$$

$$(A_1((A_2A_3)A_4))$$

$$((A_1A_2)(A_3A_4))$$

$$((A_1(A_2A_3))A_4)$$

$$(((A_1A_2)A_3)A_4)$$

- The way we parenthesize a chain of matrices can have a dramatic effect on the cost of computing the product

Matrix-chain Multiplication

Cost of Multiplying two Matrices

Matrix has two attributes

- rows[A]: # of rows
- cols[A]: # of columns

# of scalar mult-adds in
$C \leftarrow AB$ is
rows[A]×cols[B]×cols[A]

$\left.\begin{array}{l} A: (p \times q) \\ B: (q \times r) \end{array}\right\}$ $C = A \cdot B$ is $p \times r$.

# of mult-adds is $p \times r \times q$

MATRIX-MULTIPLY(A, B)

```
if cols[A]≠rows[B] then
    error("incompatible dimensions")
for i ←1 to rows[A] do
    for j←1 to cols[B] do
        C[i,j] ← 0
        for k←1 to cols[A] do
            C[i,j]←
C[i,j]+A[i,k]·B[k,j]
return C
```

Matrix-chain Multiplication

Input: a chain $\langle A_1, A_2, \ldots, A_n \rangle$ of n matrices, A_i is a $p_{i-1} \times p_i$ matrix

Aim: fully parenthesize the product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$ such that the number of scalar mult-adds are minimized.

- Ex.: $\langle A_1, A_2, A_3 \rangle$ where A_1: 10×100; A_2: 100×5; A_3: 5×50

$$((\underbrace{A_1 A_2}_{10\times5}) \underbrace{A_3}_{5\times50}): \underbrace{10 \times 100 \times 5}_{A_1 A_2} + \underbrace{10 \times 5 \times 50}_{(A_1 A_2) A_3} = 7500$$

$$(\underbrace{A_1}_{10\times100} \underbrace{(A_2 A_3)}_{100\times50})): \underbrace{100\times5\times50}_{A_2 A_3} + \underbrace{10\times100\times50}_{A_1 (A_2 A_3)} = 75000$$

$\Rightarrow$ First parenthesization yields 10 times faster computation.

Number of Parenthesizations

- Brute force approach: exhaustively check all parenthesizations
- P(n): # of parenthesizations of a sequence of n matrices
- We can split sequence between kth and $(k+1)$st matrices for any $k=1, 2, \ldots, n-1$, then parenthesize the two resulting sequences independently, i.e.,

$$(A_1 A_2 A_3 \ldots A_k)(A_{k+1} A_{k+2} \ldots A_n)$$

- We obtain the recurrence

$$P(1) = 1 \text{ and } P(n) = \sum_{k=1}^{n-1} P(k)P(n-k)$$

Number of Parenthesizations

- The recurrence generates the sequence of Catalan Numbers
- Solution is $P(n) = C(n-1)$ where

$$C(n) = \frac{1}{n+1}\binom{2n}{n} = \Omega(4^n/n^{3/2})$$

- The number of solutions is exponential in n
- Therefore, brute force approach is a poor strategy

Establishing the Recurrence

Consider the subproblem of parenthesizing

$A_{i\ldots j} = A_i A_{i+1} \cdots A_j$ for $1 \le i \le j \le n$

$p_{i-1}p_kp_j$

m[i, k] $= A_{i\ldots k} A_{k+1\ldots j}$ m[k+1,j] for $i \le k < j$

Assume that the optimal parenthesization splits

the product $A_i A_{i+1} \cdots A_j$ at k ($i \le k < j$)

$$m[i, j] = \underbrace{m[i, k]}_{} + \underbrace{m[k+1, j]}_{} + \underbrace{p_{i-1}p_kp_j}_{}$$

min # of scalar multiplications to compute $A_{i\ldots k}$

min # of multiplications to compute $A_{k+1\ldots j}$

# of multiplications to compute $A_{i\ldots k}A_{k+1\ldots j}$

The Recurrence relation

$$m[i,j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\} & \text{if } i < j \end{cases}$$

Recursive Matrix-chain

Recursive matrix-chain order

RMC(p, i, j)

if $i = j$ **then**
  return 0
$m[i, j] \leftarrow \infty$
for $k \leftarrow i$ **to** $j-1$ **do**
  $q \leftarrow \mathrm{RMC}(p, i, k) + \mathrm{RMC}(p, k+1, j) + p_{i-1} p_k p_j$
  if $q < m[i, j]$ **then**
    $m[i, j] \leftarrow q$
return $m[i, j]$

Running Time of Recursive Matrix-chain

$$T(1) \geq 1$$

$$T(n) \geq 1 + \sum_{k=1}^{n-1} \left(T(k) + T(n-k) + 1 \right) \text{ for } n > 1$$

- For $i = 1, 2, \ldots, n$ each term $T(i)$ appears twice
  - Once as $T(k)$, and once as $T(n-k)$
- Collect $n-1$ 1's in the summation together with the front 1

$$\boxed{T(n) \geq 2 \sum_{i=1}^{n-1} T(i) + n}$$

- Prove via substitution that T(n) is $\Omega(2^n)$

Running Time of Recursive Matrix-chain

- Try to show that $T(n) \geq 2^{n-1}$ (by induction)

Base case: $T(1) \geq 1 = 2^0 = 2^{1-1}$ for $n = 1$

IH: $T(i) \geq 2^{i-1}$ for all $i = 1, 2, \ldots, n-1$ and $n \geq 2$

$$T(n) \geq 2\sum_{i=1}^{n-1} 2^{i-1} + n$$

$$= 2\sum_{i=0}^{n-2} 2^{i} + n = 2(2^{n-1} - 1) + n$$

$$= 2^{n-1} + (2^{n-1} - 2 + n)$$

$\Rightarrow T(n) \geq 2^{n-1}$ Q.E.D.

Elements of Dynamic Programming

When should we look for a DP solution to an optimization problem?

Two key ingredients for the problem

- Optimal substructure
- Overlapping subproblems

Elements of Dynamic Programming

Optimal Substructure

- A problem exhibits optimal substructure
  - if an optimal solution to a problem contains within it optimal solutions to subproblems
- Example: matrix-chain-multiplication

Optimal parenthesization of $A_1A_2\ldots A_n$ that splits the product between A_k and A_{k+1},

contains within it optimal soln's to the problems of parenthesizing $A_1A_2\ldots A_k$ and $A_{k+1}A_{k+2}\ldots A_n$

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

Elements of Dynamic Programming

Overlapping Subproblems

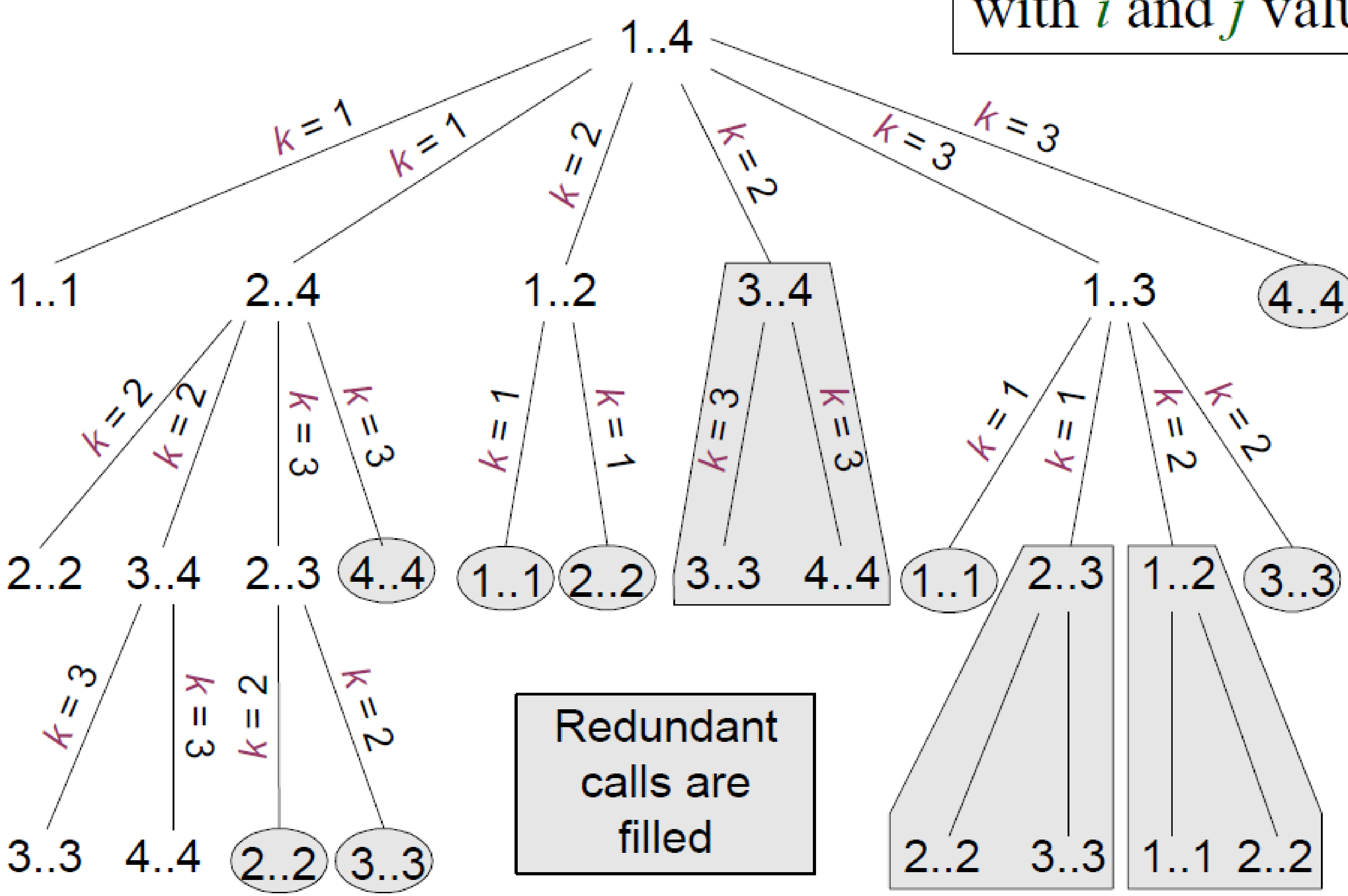
- Total number of distinct subproblems should be polynomial in the input size
- When a recursive algorithm revisits the same problem over and over again

  we say that the optimization problem has overlapping subproblems

Overlapping Subproblems in RMC Execution

Recursion tree for RMC(p,1,4)

Nodes are labeled with i and j values

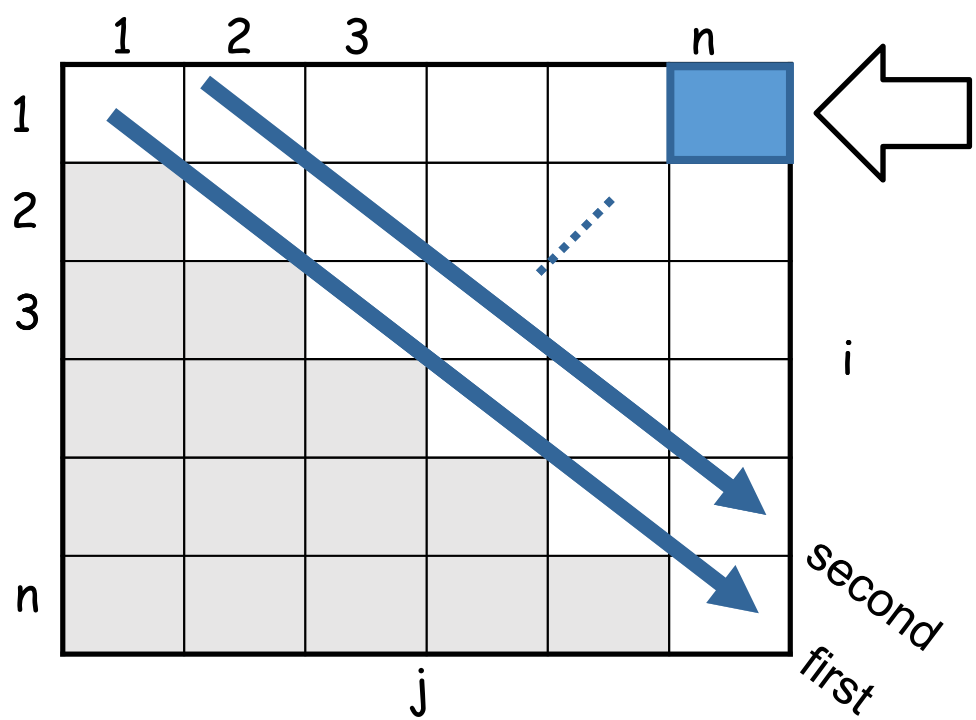


Redundant calls are filled

Using Dynamic Programming

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \leq k < j} \{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

- Length = 1: $i = j$, $i = 1, 2, \ldots, n$
- Length = 2: $j = i + 1$, $i = 1, 2, \ldots, n-1$



$m[1, n]$ gives the optimal solution to the problem

Compute rows from diagonal to top and from left to right

In a similar matrix s, we keep the optimal values of k

Matrix-Chain-Order

Alg.: MATRIX-CHAIN-ORDER(p)

```
n = p.length-1
let m[1..n, 1..n] and s[1..n-1, 2..n] be new tables
for i = 1 to n
    m[i, i] = 0
for l = 2 to n
    for i = 1 to n – l + 1
        j = i + l – 1
        m[i, j] =
        for k = i to j – 1
            q = m[i, k] + m[k+1, j] + p_{i-1} p_k p_j
            if q < m[i, j]
                m[i, j] = q
                s[i, j] = k
return m and s
```

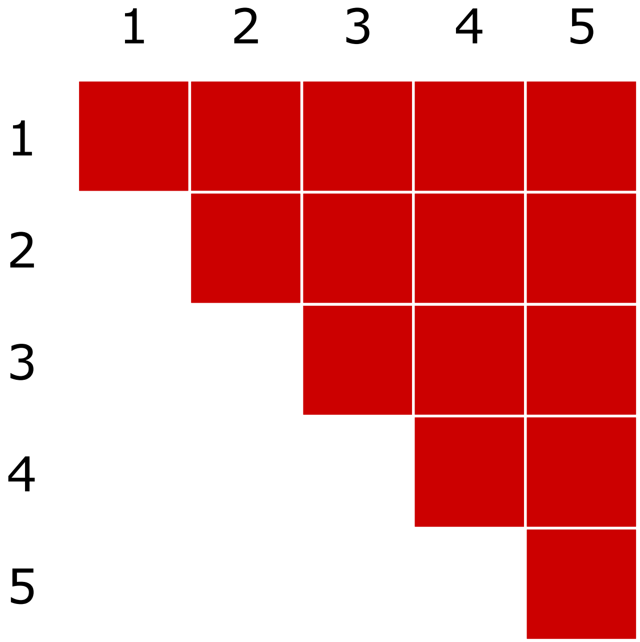
Analysis

- Our algorithm computes the minimum-cost table m and the split table s
- The optimal solution can be constructed from the split table s (shown later)
- Each entry s[i, j]=k shows where to split the product A_i A_{i+1} ... A_j for the minimum cost
- There are 3 nested loops and each can iterate at most n times, so the total running time is $\Theta(n^3)$.

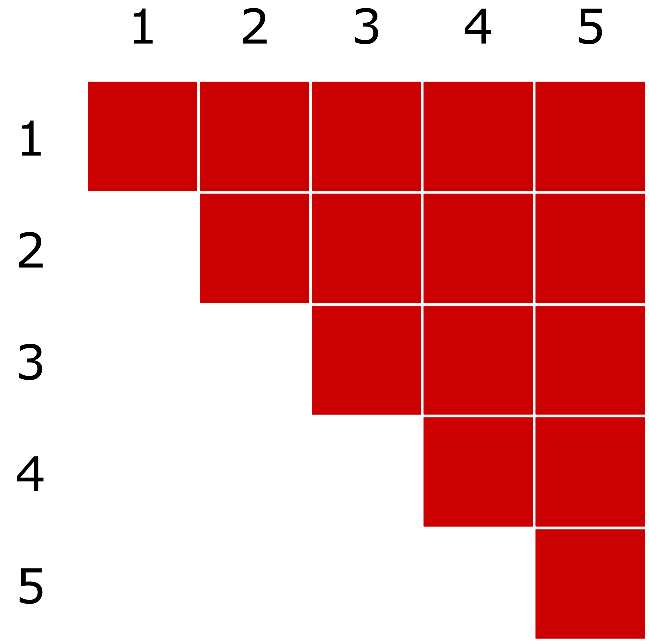
Example:

p = (10, 5, 1, 10, 2, 10)

[10×5]×[5×1]×[1×10]×[10×2]×[2×10]



m(i,j), i ≤ j



s(i,j), i ≤ j

Example:

p = (10, 5, 1, 10, 2, 10)

$[10 \times 5] \times [5 \times 1] \times [1 \times 10] \times [10 \times 2] \times [2 \times 10]$

m(i,i) = 0

	1	2	3	4	5
1	0				
2		0			
3			0		
4				0	
5					0

m(i,j), i ≤ j

	1	2	3	4	5
1					
2					
3					
4					
5					

s(i,j), i ≤ j

Example:

p = (10, 5, 1, 10, 2, 10)

[10×5]×[5×1]×[1×10]×[10×2]×[2×10]

$m(i,i+1) = p_{i-1}p_ip_{i+1}$ e.g., $m(2,3) = p_1p_2p_3 = 5*1*10 =$ 50, $m(1,2)=10*5*1 =$ 50, $m(3,4) = 1*10*2=20$, …

$s(i,i+1) = i$

	1	2	3	4	5
1	0	50			
2		0	50		
3			0	20	
4				0	200
5					0

m(i,j), i ≤ j

	1	2	3	4	5
1		1			
2			2		
3				3	
4					4
5					

s(i,j), i ≤ j

Example:

$p = (10, 5, 1, 10, 2, 10)$

$[10\times5]\times[5\times1]\times[1\times10]\times[10\times2]\times[2\times10]$

$m(i,i+2) = \min\{m(i,i) + m(i+1,i+2) + p_{i-1}p_ip_{i+2},$

$m(i,i+1) + m(i+2,i+2) + p_{i-1}p_{i+1}p_{i+2}\}$

	1	2	3	4	5
1	0	50			
2		0	50		
3			0	20	
4				0	200
5					0

$m(i,j),\ i \le j$

	1	2	3	4	5
1		1			
2			2		
3				3	
4					4
5					

$s(i,j),\ i \le j$

Example:

$p = (10, 5, 1, 10, 2, 10)$

$[10\times5]\times[5\times1]\times[1\times10]\times[10\times2]\times[2\times10]$

$m(2,4) = \min\{m(2,2) + m(3,4) + p_1p_2p_4,\ \ m(2,3) + m(4,4) + p_1p_3p_4\}$

$= \min\{0+20+5*1*2, 50+0+5*10*2\} = 30$

	1	2	3	4	5
1	0	50	150		
2		0	50	30	
3			0	20	40
4				0	200
5					0

$m(i,j),\ i \le j$

	1	2	3	4	5
1		1	2		
2			2	2	
3				3	3
4					4
5					

$s(i,j),\ i \le j$

Example:

[10×5]×[5×1]×[1×10]×[10×2]×[2×10]

$$m(i,i+3) = \min\{m(i,i) + m(i+1,i+3) + p_{i-1}p_ip_{i+3},$$
$$m(i,i+1) + m(i+2,i+3) + p_{i-1}p_{i+1}p_{i+3},$$
$$m(i,i+2) + m(i+3,i+3) + p_{i-1}p_{i+2}p_{i+3}\}$$

	1	2	3	4	5
1	0	50	150	90	
2		0	50	30	90
3			0	20	40
4				0	200
5					0

m(i,j), i ≤ j

	1	2	3	4	5
1		1	2	2	
2			2	2	2
3				3	3
4					4
5					

s(i,j), i ≤ j

Example:

$[10\times5]\times[5\times1]\times[1\times10]\times[10\times2]\times[2\times10]$

$m(i,i+4) = \min\{m(i,i) + m(i+1,i+4) + p_{i-1}p_ip_{i+4},$
$m(i,i+1) + m(i+2,i+4) + p_{i-1}p_{i+1}p_{i+4},$ $m(i,i+2) + m(i+3,i+4) + p_{i-1}p_{i+2}p_{i+4},$ $m(i,i+3) + m(i+4,i+4) + p_{i-1}p_{i+3}p_{i+4}\}$

	1	2	3	4	5
1	0	50	150	90	190
2		0	50	30	90
3			0	20	40
4				0	200
5					0

m(i,j), i ≤ j

	1	2	3	4	5
1		1	2	2	2
2			2	2	2
3				3	3
4					4
5					

s(i,j), i ≤ j

Print optimal parenthesis

Alg.: PRINT-OPTIMAL-PARENS(*s, i, j*)

```
if i == j
    print "Ai"
else print "("
    PRINT-OPTIMAL-PARENS(s, i, s[i, j])
    PRINT-OPTIMAL-PARENS(s, s[i, j] + 1, j)
    print ")"
```

Example:

Optimal multiplication sequence

$s(1,5) = 2$

- $A_{15} = A_{12} \times A_{35}$

	1	2	3	4	5
1	0	50	150	90	190
2		0	50	30	90
3			0	20	40
4				0	200
5					0

m(i,j), i ≤ j

	1	2	3	4	5
1		1	2	2	2
2			2	2	2
3				3	3
4					4
5					

s(i,j), i ≤ j

Example:

$A_{15} = A_{12} \times A_{35}$

$s(1,2) = 1$ ▶ $A_{12} = A_{11} \times A_{22}$

$\rightarrow A_{15} = (A_{11} \times A_{22}) \times A_{35}$

	1	2	3	4	5
1	0	50	150	90	190
2		0	50	30	90
3			0	20	40
4				0	200
5					0

$m(i,j),\ i \le j$

	1	2	3	4	5
1		1	2	2	2
2			2	2	2
3				3	4
4					4
5					

$s(i,j),\ i \le j$

Example:

$$A_{15} = (A_{11} \times A_{22}) \times A_{35}$$

$$s(3,5) = 4 \blacktriangleright A_{35} = A_{34} \times A_{55}$$

$$\rightarrow A_{15} = (A_{11} \times A_{22}) \times (A_{34} \times A_{55})$$

	1	2	3	4	5
1	0	50	150	90	190
2		0	50	30	90
3			0	20	40
4				0	200
5					0

$m(i,j),\ i \leq j$

	1	2	3	4	5
1		1	2	2	2
2			2	2	2
3				3	4
4					4
5					

$s(i,j),\ i \leq j$

Example:

$A_{15} = (A_{11} \times A_{22}) \times (A_{34} \times A_{55})$

$s(3,4) = 3$ ▶ $A_{34} = A_{33} \times A_{44}$

$\rightarrow A_{15} = (A_{11} \times A_{22}) \times ((A_{33} \times A_{44}) \times A_{55})$

	1	2	3	4	5
1	0	50	150	90	190
2		0	50	30	90
3			0	20	40
4				0	200
5					0

$m(i,j),\ i \leq j$

	1	2	3	4	5
1		1	2	2	2
2			2	2	2
3				3	4
4					4
5					

$s(i,j),\ i \leq j$

Memoization

- Offers the efficiency of the usual DP approach while maintaining top-down strategy
- Idea is to memoize the natural, but inefficient, recursive algorithm

Memoization

- Maintains an entry in a table for the soln to each subproblem
- Each table entry contains a special value to indicate that the entry has yet to be filled in
- When the subproblem is first encountered its solution is computed and then stored in the table
- Each subsequent time that the subproblem encountered the value stored in the table is simply looked up and returned

Memoized Matrix-Chain

Alg.: MEMOIZED-MATRIX-CHAIN(p)

```
n 🌎 length[p] – 1
for i 🌎 1 to n
        do for j 🌎 i to n
                do m[i, j] 🌎 👈
return LOOKUP-CHAIN(p, 1, n)
```

Initialize the m table with large values that indicate whether the values of m[i, j] have been computed

Top-down approach

Memoized Matrix-Chain

Alg.: LOOKUP-CHAIN(p, i, j)

```
if m[i, j] < ∞
        then return m[i, j]
if i = j
    then m[i, j] ← 0
    else for k ← i to j – 1
                do q ← LOOKUP-CHAIN(p, i, k) +
                     LOOKUP-CHAIN(p, k+1, j) + p_{i-1} p_k p_j
                   if q < m[i, j]
                      then m[i, j] ← q
return m[i, j]
```