

# 1. TOPOLOGICAL SPACES

## 1.1 TOPOLOGIES

DEFINITION 1.1 (Topology). For a set  $X$ , a *topology on  $X$*  is a collection  $\tau$  of subsets of  $X$  which satisfies,

1.  $\emptyset, X \in \tau$ .
2.  $U_1, \dots, U_n \in \tau \implies U_1 \cap \dots \cap U_n \in \tau$ .
3.  $(U_\alpha)_{\alpha \in A} \in \tau \implies \bigcup_{\alpha \in A} U_\alpha \in \tau$ .

The pair  $(X, \tau)$  is called a *topological space*.

DEFINITION 1.2 (Neighbourhood). Given a topological space  $X$ , a *neighbourhood* of a point  $p \in X$  is any open set  $U_p \in \tau$  containing  $p$ .

A *neighbourhood of the subset  $K \subseteq X$*  is an open set containing  $K$ .

PROPOSITION 1.3 (Metric topology). Let  $(M, d)$  be a metric space and let  $\tau$  be the collection of sets which are open in the sense of metric spaces. Then,  $(M, \tau)$  is a topological space.

*Proof.* Proof of the statement amounts to showing that each of the defining properties of a topological space are true in the metric space.

1.  $\emptyset$  is vacuously open in  $M$ , and  $M$  is clearly also open. Hence  $\emptyset, M \in \tau$ .
2. Take  $U_1, \dots, U_n \in \tau$ , and consider a point  $p \in U_1 \cap \dots \cap U_n$ . Since each of the sets  $U_i$  are open, there exist values  $r_i$  such that  $B(p; r_i) \subseteq U_i$  for all  $1 \leq i \leq n$ . Taking  $r = \min_{1 \leq i \leq n} \{r_i\}$ , we have that  $B(p; r) \subseteq U_1 \cap \dots \cap U_n$ , and hence this intersection is open.
3. For every point in the union, there is at least one open set  $U_\alpha$  containing the point, and an open ball around the point. Hence the union is open.

□

NOTATION. We have the following standard notations for common sets.

- The open unit  $n$ -ball:

$$\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$$

- The closed unit  $n$ -ball:

$$\overline{\mathbb{B}}^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

- The unit  $n$ -sphere:

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$

DEFINITION 1.4 (Metrisable). A topological space  $(X, \tau)$  is said to be *metrisable* if its topology is the same as that generated by some metric on  $X$ .

### 1.1.1 CLOSED SUBSETS

DEFINITION 1.5 (Closed sets). A subset  $F$  of a topological space  $X$  is said to be *closed* if its complement  $X \setminus F$  is open.

For the proceeding four definitions, let  $A$  be any subset of a topological space  $X$ .

DEFINITION 1.6 (Closure). The *closure* of  $A$  in  $X$  is defined to be,

$$\overline{A} = \bigcap \{B \subseteq X : B \supseteq A \text{ and } B \text{ is closed in } X\}$$

DEFINITION 1.7 (Interior). The *interior* of  $A$  is defined to be,

$$\text{Int } A = \bigcup \{C \subseteq X : C \subseteq A \text{ and } C \text{ is open in } X\}$$

DEFINITION 1.8 (Exterior). The *exterior* of  $A$  is defined to be,

$$\text{Ext } A = X \setminus \overline{A}$$

DEFINITION 1.9 (Boundary). The *boundary* of  $A$  is defined to be,

$$\partial A = X \setminus (\text{Int } A \cup \text{Ext } A)$$

PROPOSITION 1.10. Let  $X$  be a topological space and let  $A \subseteq X$  be any subset.

1. A point is in  $\text{Int } A$  if and only if it has a neighbourhood contained in  $A$ .
2. A point is in  $\text{Ext } A$  if and only if it has a neighbourhood contained in  $X \setminus A$ .
3. A point is in  $\partial A$  if and only if every neighbourhood of it contains both a point of  $A$  and a point of  $X \setminus A$ .
4. A point is in  $\overline{A}$  if and only if every neighbourhood of it contains a point of  $A$ .
5.  $\overline{A} = A \cup \partial A = \text{Int } A \cup \partial A$ .
6.  $\text{Int } A$  and  $\text{Ext } A$  are open in  $X$ , while  $\overline{A}$  and  $\partial A$  are closed in  $X$ .
7. The following are equivalent:
  - $A$  is open in  $X$ .
  - $A = \text{Int } A$ .
  - $A$  contains none of its boundary points.
  - Every point of  $A$  has a neighbourhood contained in  $A$ .

8. The following are equivalent:

- $A$  is closed in  $X$ .
- $A = \overline{A}$ .
- $A$  contains all of its boundary points.
- Every point of  $X \setminus A$  has a neighbourhood contained in  $X \setminus A$ .

*Proof.* We will work through each statement in turn.

1. Consider the following logical equivalence

$$\begin{aligned}
 x \in \text{Int } A & \\
 \iff & \\
 x \in \bigcup \{B \subseteq A : B \in \tau_X\} & \\
 \iff & \\
 \exists B \subseteq A \text{ s.t. } (x \in B \wedge B \in \tau_X). &
 \end{aligned}$$

This is exactly the statement that there exists an open neighbourhood of  $x \in \text{Int } A$  contained in  $\text{Int } A$ .

2. Assume first that  $x \in \text{Ext } A$ . Then, by definition, we have that  $x \in X \setminus \overline{A}$ . Since  $\overline{A}$  is an intersection of closed sets, it is closed, and hence  $X \setminus \overline{A}$  is open. Furthermore,  $A \subseteq \overline{A}$ , and equivalently,  $X \setminus \overline{A} \subseteq X \setminus A$ . Therefore,  $X \setminus \overline{A}$  is a neighbourhood of  $x$  satisfying the desired property.

For the reverse implication, assume that  $x$  has an open neighbourhood  $U \subseteq X \setminus A$ . Since  $U$  is open,  $X \setminus U$  is closed, and furthermore,  $X \setminus U \supseteq A$ . From the definition of  $\overline{A}$ , it must be the case that  $x \notin \overline{A}$ , hence  $x \in \text{Ext } A$ .

3. Given the previous two results, and the definition of  $\partial A$ , the statement is clear.

4. Given the second result, and the definition of  $\text{Ext } A$ , the statement is clear.

5. We first aim to show that  $A \cup \partial A \subseteq \overline{A}$ . Considering each of the LHS components in turn, we will consider neighbourhoods of points. Firstly, for  $x \in A$ , any neighbourhood of  $x$  will contain  $x$  itself, and hence  $x \in \overline{A}$ . Secondly, for  $x \in \partial A$ , we know by the third result that every neighbourhood contains a point of  $A$  (and  $X \setminus A$ ), therefore  $x \in \overline{A}$  also.

By the definition of  $\text{Int } A$  we also have that  $\text{Int } A \subseteq A$  – the interior is a union of open sets contained by  $A$ . Therefore, so far we have,

$$\text{Int } A \cup \partial A \subseteq A \cup \partial A \subseteq \overline{A}$$

We can also show by elementary set operations that,

$$\begin{aligned}
 \text{Int } A \cup \partial A &= \text{Int } A \cup (X \setminus (\text{Int } A \cup \text{Ext } A)) \\
 &= X \setminus \text{Ext } A \\
 &= \overline{A}
 \end{aligned}$$

And hence the chain of inclusions is a chain of equalities, as needed.

6. Clearly  $\text{Int } A$  is open, as a union of open sets. The same follows for the closedness of  $\overline{A}$  as an intersection of closed sets. As  $\text{Ext } A$  is the  $X$ -complement of  $\overline{A}$ , it is open. As  $\partial A$  is the  $X$ -complement of a union of open sets, it is closed.
7. We first assume that  $A$  is open. Then clearly, every point  $x \in A$  has an open neighbourhood contained in  $A$ , since  $A \subseteq A$ . From this we know that  $\partial A = \emptyset$  from the third result. From this we know that,  $A = \text{Int } A$  since,

$$\begin{aligned} (A \cup \partial A = \text{Int } A \cup \partial A) \wedge (A \cap \partial A = \emptyset) \\ \iff \\ A = \text{Int } A \end{aligned}$$

also relying on the disjointness of  $\text{Int } A$  and  $\partial A$ .

If  $A = \text{Int } A$ , then  $A$  is open, since  $\text{Int } A$  is open. From this we have a full circle of equivalences.

8. These equivalences are easily validated, in a similar way as to the previous result.

□

**DEFINITION 1.11** (Limit and isolated points). Given a topological space  $X$ , and a set  $A \subseteq X$ , we say that a point  $p \in X$  is a *limit point* of  $A$  if every neighbourhood of  $p$  contains a point of  $A$  other than  $p$ .

On the other hand, a point  $p \in A$  is called an *isolated point* of  $A$  if there exists a neighbourhood  $U$  of  $p$  such that  $U \cap A = \{p\}$ .

**PROBLEM 1.1.** Show that a subset is closed if and only if it contains all of its limit points.

**SOLUTION 1.1.** We can make a neat argument using the previous proposition. Since every neighbourhood of every limit point  $x$  of  $A$  contains a point of  $A$  other than the point  $x$  itself, the point  $x$  must be contained by the closure  $\overline{A}$ . Furthermore since,  $A$  is closed if and only if  $A = \overline{A}$ , every limit point is contained by  $A$  if and only if  $A = \overline{A}$  if and only if  $A$  is closed.

**DEFINITION 1.12** (Dense). A subset  $A$  of a topological space  $X$  is said to be *dense in  $X$*  if  $\overline{A} = X$ .

**PROBLEM 1.2.** Show that a subset  $A \subseteq X$  is dense in  $X$  if and only if every nonempty open subset of  $X$  contains a point in  $A$ .

**SOLUTION 1.2.**  $\overline{A} = X$  if and only if every neighbourhood of every point in  $X$  has nonempty intersection with  $A$ . In particular, the statement ‘every neighbourhood of every point in  $X$ ’ is exhaustive of nonempty open sets in the topology of  $X$ , and hence the statement follows.

## 1.2 CONVERGENCE AND CONTINUITY

**DEFINITION 1.13** (Sequence convergence). For a topological space  $X$  and a sequence  $(x_n)_{n=1}^{\infty}$  of points in  $X$ , we say that the *sequence converges* to  $x$  if for every neighbourhood  $U$  of  $x$ , there exists some  $N \in \mathbb{N}$  such that  $x_n \in U$  for all  $n \geq N$ .

PROBLEM 1.3. Show that in a metric space, the topological definition of convergence is equivalent to the metric space definition.

SOLUTION 1.3. To solve this exercise, we want to show that a sequence converges in a metric space if and only if the sequence converges in the respective topological space with the induced metric topology. Consider the following equivalences, starting with the definition of convergence in a metric space.

$$\begin{aligned}
& \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } d(x_n, x) < \epsilon \forall n \geq N \\
& \iff \\
& \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } x_n \in B^{(d)}(x; \epsilon) \forall n \geq N \\
& \iff \\
& \text{for every neighbourhood } U \text{ of } x \exists N \in \mathbb{N} \text{ s.t. } x_n \in U \forall n \geq N
\end{aligned}$$

PROBLEM 1.4. For a topological space  $X$ , a subset  $A$  and a sequence  $(x_i) \in A$ , show that  $x = \lim_{i \rightarrow \infty} x_i \in \overline{A}$ .

SOLUTION 1.4. Suppose for the sake of contradiction that this wasn't the case, and the limit  $x$  of a convergent sequence  $(x_i)$  was such that  $x \in \text{Ext } A$ . Then, by a previous result, there exists an open subset  $U \subseteq X \setminus A$  such that  $x \in U$ . Since  $x_i \in A$  for all  $i$ ,  $x_i \notin U$  for all  $i$ , and hence the sequence cannot be convergent; the desired contradiction.

DEFINITION 1.14 (Continuity). If  $X$  and  $Y$  are topological spaces, a map  $f : X \rightarrow Y$  is said to be *continuous* if for every open subset  $U \subseteq Y$ , its preimage  $f^{-1}(U)$  is open in  $X$ .

PROPOSITION 1.15. A map between topological spaces is continuous if and only if the preimage of every closed subset is closed.

*Proof.* Consider the following equivalences.

$$\begin{aligned}
& f \text{ is continuous} \\
& \iff \\
& f^{-1}(B) \text{ is open, for all } B \text{ open} \\
& \iff \\
& f^{-1}(Y \setminus A) \text{ is open, for all } A \text{ closed} \\
& \iff \\
& f^{-1}(Y) \setminus f^{-1}(A) \text{ is open, for all } A \text{ closed} \\
& \iff \\
& X \setminus f^{-1}(A) \text{ is open, for all } A \text{ closed} \\
& \iff \\
& f^{-1}(A) \text{ is closed, for all } A \text{ closed.}
\end{aligned}$$

□

PROPOSITION 1.16. Let  $X, Y$  and  $Z$  be topological spaces.

1. Every constant map  $f : X \rightarrow Y$  is continuous.
2. The identity map  $\text{Id}_X : X \rightarrow X$  is continuous.
3. If  $f : X \rightarrow Y$  is continuous, so is the restriction of  $f$  to any open subset of  $X$ .
4. If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are continuous, then so is their composition  $g \circ f : X \rightarrow Z$ .

*Proof.* We deal with each of the statements in turn.

1. Suppose that  $f : X \rightarrow Y : x \mapsto a \in Y$ . Then, an open set  $U \subseteq Y$  is either  $U \supseteq \{a\}$  or  $U \not\supseteq \{a\}$ . In the first case,  $f^{-1}(U) = X$  which is open, and in the second case  $f^{-1}(U) = \emptyset$ , which is also open.
2. Let  $U$  be an open set in  $X$ . Then  $f^{-1}(U) = U$ , open by hypothesis.
3. This is clear.
4. Consider an open set  $U \subseteq Z$ . Then,

$$\begin{aligned} (g \circ f)^{-1}(U) &= (f^{-1} \circ g^{-1})(U) \\ &= f^{-1}(g^{-1}(U)) \end{aligned}$$

where the continuity of each of these functions ensures that the composition is continuous.

□

PROPOSITION 1.17 (Local criterion for continuity). A map  $f : X \rightarrow Y$  is continuous if and only if each point of  $X$  has a neighbourhood on which the restriction of  $f$  is continuous.

*Proof.* If the function  $f$  is continuous, then we can simply take  $X$  to be the open neighbourhood of every point.

On the contrary, let  $U \subseteq Y$ , where we aim to show that  $f^{-1}(U)$  is open in  $X$ . Taking some  $x \in f^{-1}(U)$ , we know that there exists neighbourhood  $V_x$  of  $x$  such that  $f|_{V_x}$  is continuous. In particular,  $f|_{V_x}^{-1}(U)$  is open. Also,

$$f|_{V_x}^{-1}(U) = \{x \in V_x : f(x) \in U\} = V_x \cap f^{-1}(U)$$

and in particular,  $f|_{V_x}^{-1}(U) \subseteq f^{-1}(U)$ , is an open neighbourhood of  $x$ . The arbitrary choice of  $x \in f^{-1}(U)$  then determines that  $f^{-1}(U)$  is open. □

DEFINITION 1.18 (Homeomorphism). A *homeomorphism* from  $X$  to  $Y$  is a bijective map  $\phi : X \rightarrow Y$  such that  $\phi$  and  $\phi^{-1}$  are continuous.

If there exists a homeomorphism between  $X$  and  $Y$ , we say that  $X$  and  $Y$  are *topologically equivalent*, or *homeomorphic*. We denote this relation by  $X \approx Y$ .

PROBLEM 1.5. Show that homeomorphisms provide an equivalence relation on topological spaces.

SOLUTION 1.5. We aim to show that homeomorphisms are reflexive, transitive and symmetric.

1. Reflexive: taking  $\text{Id}_X : X \rightarrow X$ , we see that this map is bijective, continuous, and has continuous inverse. Hence  $X \approx X$ .
2. Transitive: consider the homeomorphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . Then,  $g \circ f : X \rightarrow Z$  is continuous, bijective from  $X$  to  $Z$ , and has continuous inverse  $f^{-1} \circ g^{-1} : Z \rightarrow X$ . Hence  $X \approx Z$ .
3. Symmetric: taking  $f^{-1}$  as the homeomorphism between  $Y$  and  $X$  is sufficient. Hence  $Y \approx X$ .

PROBLEM 1.6. Let  $(X_1, \tau_1)$  and  $(X_2, \tau_2)$  be topological spaces and let  $f : X_1 \rightarrow X_2$  be a bijective map. Show that  $f$  is a homeomorphism if and only if  $f(\tau_1) = \tau_2$  in the sense that  $U \in \tau_1$  if and only if  $f(U) \in \tau_2$ .

SOLUTION 1.6. First, assume that  $f(\tau_1) = \tau_2$ , in the sense described. Then  $V \in \tau_2$  if and only if there exists some  $U \in \tau_1$  such that  $f(U) = V$ . Since  $f$  is bijective by assumption,

$$f^{-1}(V) = f^{-1}(f(U)) = U$$

is open, and therefore  $f$  is continuous. A similar argument follows for the continuity of  $f^{-1}$ .

On the contrary, suppose that  $f$  is a homeomorphism. Then,  $f$  and  $f^{-1}$  are continuous, and,

$$\begin{aligned} U &\in \tau_1 \\ \implies \\ (f^{-1})^{-1}(U) &= f(U) \in \tau_2 \\ \implies \\ f^{-1}(f(U)) &= U \in \tau_1 \end{aligned}$$

which is the statement we wanted.

DEFINITION 1.19 (Finer and coarser). Given two topologies  $\tau_1, \tau_2$  on  $X$ , we say that  $\tau_1$  is *finer* than  $\tau_2$  if  $\tau_1 \supseteq \tau_2$  and *coarser* than  $\tau_2$  if  $\tau_1 \subseteq \tau_2$ .

PROBLEM 1.7. Show that the identity map of  $X$  is continuous as a map from  $(X, \tau_1)$  to  $(X, \tau_2)$  if and only if  $\tau_1$  is finer than  $\tau_2$ , and is a homeomorphism if and only if  $\tau_1 = \tau_2$ .

SOLUTION 1.7. Considering the map  $\text{Id}_X : (X, \tau_1) \rightarrow (X, \tau_2)$ , we have the following equivalences,

$$\begin{aligned} \tau_2 &\subseteq \tau_1 \\ \iff \\ U \in \tau_2 &\implies U \in \tau_1 \\ \iff \\ U \in \tau_2 &\implies \text{Id}_X^{-1}(U) \in \tau_1 \\ \iff \\ \text{Id}_X &\text{ is continuous .} \end{aligned}$$

The identity is bijective, so is a homeomorphism if and only if it is continuous with continuous

inverse. This is the case if and only if  $\tau_1 \subseteq \tau_2$  and  $\tau_2 \subseteq \tau_1$ , that is if and only if  $\tau_1 = \tau_2$ .

DEFINITION 1.20 (Open and closed maps). A function  $f : X \rightarrow Y$  is said to be an *open map* if it takes open subsets of  $X$  to open subsets of  $Y$ .

A function  $f : X \rightarrow Y$  is said to be a *closed map* if it takes closed subsets of  $X$  to closed subsets of  $Y$ .

PROBLEM 1.8. Suppose that  $f : X \rightarrow Y$  is a bijective continuous map. Show that the following are equivalent.

1.  $f$  is a homeomorphism.
2.  $f$  is an open map.
3.  $f$  is a closed map.

SOLUTION 1.8. Given the assumptions that  $f$  is continuous and bijective,

$$\begin{aligned}
 & f \text{ is a homeomorphism} \\
 & \iff \\
 & f^{-1} \text{ is continuous} \\
 & \iff \\
 & U \in \tau_X \implies f(U) \in \tau_Y \\
 & \iff \\
 & f \text{ is open.}
 \end{aligned}$$

PROPOSITION 1.21. Let  $f : X \rightarrow Y$  be a map of topological spaces.

1.  $f$  is continuous if and only if  $f(\overline{A}) \subseteq \overline{f(A)}$  for all  $A \subseteq X$ .
2.  $f$  is closed if and only if  $f(\overline{A}) \supseteq \overline{f(A)}$  for all  $A \subseteq X$ .
3.  $f$  is continuous if and only if  $f^{-1}(\text{Int } B) \subseteq \text{Int } f^{-1}(B)$  for all  $B \subseteq Y$ .
4.  $f$  is open if and only if  $f^{-1}(\text{Int } B) \supseteq \text{Int } f^{-1}(B)$  for all  $B \subseteq Y$ .

*Proof.* 1. If  $f$  is closed, then  $f(\overline{A})$  is closed. Since  $A \subseteq \overline{A}$ ,  $f(A) \subseteq f(\overline{A})$ . The closure of  $f(A)$  is defined as,

$$\overline{f(A)} = \bigcap_{B \supseteq f(A)} \{B : X \setminus B \in \tau_X\}.$$

We know that  $f(\overline{A})$  is a closed set containing  $f(A)$ , and the result follows. □

DEFINITION 1.22 (Local homeomorphism). A map  $f : X \rightarrow Y$  is called a *local homeomorphism* if every point  $x \in X$  has a neighbourhood  $U \subseteq X$  such that  $f(U)$  is an open subset of  $Y$  and  $f|_U : U \rightarrow f(U)$  is a homeomorphism.



PROPOSITION 1.23 (Properties of local homeomorphisms). We have the following properties,

1. Every homeomorphism is a local homeomorphism.
2. Every local homeomorphism is continuous and open.
3. Every bijective local homeomorphism is a homeomorphism.

*Proof.* We work through the statements in turn.

1. This is clear. Since  $X$  is open, and  $f(X) = Y$ ,  $f$  is a valid local homeomorphism for all points  $x \in X$ .
2. We know from the local criterion for continuity that a function  $f$  is continuous if and only if each point  $x \in X$  has a neighbourhood  $U$  such that  $f|_U$  is continuous. If we consider a local homeomorphism  $f$ , we know that this function is a homeomorphism on open neighbourhoods of every points  $x \in X$ , and therefore is continuous. We also know that every restriction of  $f$  is open, and therefore  $f$  is open also.
3. Using a previous result, we know that a continuous, open bijection is a homeomorphism.

□

### 1.3 HAUSDORFF SPACES

DEFINITION 1.24 (Hausdorff). A topological space  $X$  is said to be *Hausdorff* if given any two points  $p \neq q \in X$ , there exist neighbourhoods  $U, V$  of  $p$  and  $q$  respectively such that  $U \cap V = \emptyset$ .

PROBLEM 1.9. Suppose that for every  $p \in X$  there exists a continuous function  $f : X \rightarrow \mathbb{R}$  such that  $f^{-1}(0) = \{p\}$ . Show that  $X$  is Hausdorff.

SOLUTION 1.9. TODO

PROPOSITION 1.25. Let  $X$  be a Hausdorff space.

1. Every finite subset of  $X$  is closed.
2. If a sequence  $(p_i) \in X$  converges to a limit  $p \in X$ , the limit is unique.

*Proof.* TODO

□

PROBLEM 1.10. Show that the only Hausdorff topology on a finite set is the discrete topology.

SOLUTION 1.10. TODO

PROPOSITION 1.26. Suppose  $X$  is a Hausdorff space and  $A \subseteq X$ . If  $p \in X$  is a limit point of  $A$ , then every neighbourhood of  $p$  contains infinitely many points of  $A$ .

*Proof.* TODO

□

## 1.4 BASES AND COUNTABILITY

DEFINITION 1.27 (Basis). A collection  $\mathcal{B}$  of subsets of  $X$  is a *basis for the topology* of  $X$  if,

1. Every element of  $\mathcal{B}$  is an open set of  $X$ . That is  $\mathcal{B} \subseteq \tau$ .
2. Every open subset of  $X$  is the union of some collection of the elements of  $\mathcal{B}$ .

PROBLEM 1.11. Suppose that  $\mathcal{B}$  is a basis for  $X$ . Show that a subset  $U \subseteq X$  is open if and only if for each  $p \in U$ , there exists  $B \in \mathcal{B}$  such that  $p \in B \subseteq U$ .

SOLUTION 1.11. TODO

PROPOSITION 1.28. Let  $X$  and  $Y$  be topological spaces and  $\mathcal{B}$  a basis for  $Y$ . A map  $f : X \rightarrow Y$  is continuous if and only if for every basis subset  $B \in \mathcal{B}$ , the subset  $f^{-1}(B)$  is open in  $X$ .

*Proof.* TODO □

### 1.4.1 DEFINING A TOPOLOGY FROM A BASIS

PROPOSITION 1.29. A collection  $\mathcal{B}$  of open subsets of  $X$  is a basis if and only if the following two conditions hold,

1.  $\bigcup_{B \in \mathcal{B}} B = X$ .
2. If  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists an element  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

If this is the case, there is a unique topology on  $X$  for which  $\mathcal{B}$  is a basis, called the *generated topology* with respect to  $\mathcal{B}$ .

*Proof.* TODO □

### 1.4.2 COUNTABILITY PROPERTIES

DEFINITION 1.30 (Neighbourhood basis). If  $X$  is a topological space and  $p \in X$ , a collection  $\mathcal{B}_p$  of neighbourhoods of  $p$  is called a *neighbourhood basis* for  $X$  at  $p$  if every neighbourhood of  $p$  contains some  $B \in \mathcal{B}_p$ .

DEFINITION 1.31 (First countability). We say that a topological space  $X$  is *first countable* if there exists a countable neighbourhood basis at all points  $x \in X$ .

DEFINITION 1.32 (Nested neighbourhood basis). If  $X$  is a topological space and  $p \in X$ , a sequence  $(U_n)_{n \in \mathbb{N}}$  of neighbourhoods of  $p$  is called a *nested neighbourhood basis* if  $U_{n+1} \subseteq U_n$  for each  $n$ , and the sequence, when viewed as a collection is a neighbourhood basis of  $X$  at  $p$ .

LEMMA 1.33 (Nested neighbourhood basis lemma). Let  $X$  be a first countable space. Then there exists a nested neighbourhood basis at  $p$ , for every  $p \in X$ .

*Proof.* TODO □

DEFINITION 1.34 (Eventually in). If  $(x_i)_{i=1}^{\infty}$  is a sequence of points in the topological space  $X$  and  $A \subseteq X$ , we say that the sequence is *eventually in*  $A$  if there exists some  $n \in \mathbb{N}$  such that  $x_i \in A$  for all  $i \geq n$ .

LEMMA 1.35. Let  $X$  be first countable,  $A$  be any subset of  $X$ , and  $x \in X$ .

1.  $x \in \overline{A}$  if and only if  $x$  is a limit of a sequence of points in  $A$ .
2.  $x \in \text{Int } A$  if and only if every sequence in  $X$  converging to  $x$  is eventually in  $A$ .
3.  $A$  is closed in  $X$  if and only if  $A$  contains every limit of every convergent sequence of points in  $A$ .
4.  $A$  is open in  $X$  if and only if every sequence in  $X$  converging to a point of  $A$  is eventually in  $A$ .

*Proof.* TODO

□

EXAMPLE 1.36 (A non-first countable space). TODO

DEFINITION 1.37 (Second countability). A topological space is said to be *second countable* if it admits a countable basis for its topology.

DEFINITION 1.38 (Covers). A collection  $\mathcal{U}$  of subsets of  $X$  is called a *cover* of  $X$  if every points  $x \in X$  is contained by at least one  $U \in \mathcal{U}$ . The cover is called an *open cover* if every  $U \in \mathcal{U}$  is open, and a *closed cover* if every  $U \in \mathcal{U}$  is closed.

Given a cover  $\mathcal{U}$ , a *subcover* of  $\mathcal{U}$  is a subcollection  $\mathcal{U}' \subseteq \mathcal{U}$  which covers  $X$ .

DEFINITION 1.39 (Separable). A topological space is called *separable* if it contains a countable dense subset.

DEFINITION 1.40 (Lindelöf). A topological space,  $X$  is said to be a *Lindelöf space* if every open cover of  $X$  has a countable subcover.

THEOREM 1.41 (Properties of second countable spaces). Let  $X$  be a second countable space.

1.  $X$  is first countable.
2.  $X$  is separable.
3.  $X$  is Lindelöf.

*Proof.* TODO

□

## 1.5 MANIFOLDS

DEFINITION 1.42 (Locally Euclidean). A topological space  $M$  is called *locally Euclidean* of dimension  $n$  if every point of  $M$  has a neighbourhood in  $M$  that is homeomorphic to an open subset of  $\mathbb{R}^n$ .

LEMMA 1.43. A topological space  $M$  is locally Euclidean of dimension  $n$  if and only if either of the following properties hold:

1. Every points of  $M$  has a neighbourhood homeomorphic to an open ball in  $\mathbb{R}^n$ .
2. Every points of  $M$  has a neighbourhood homeomorphic to  $\mathbb{R}^n$ .

*Proof.* TODO

□

DEFINITION 1.44 (Topological manifold). An  $n$ -dimensional *topological manifold* is a second countable Hausdorff space which is locally Euclidean of dimension  $n$ .

PROPOSITION 1.45. Every open subset of an  $n$ -manifold is an  $n$ -manifold.

*Proof.* TODO

□

PROBLEM 1.12. Show that a topological space is a 0-manifold if and only if it is a countable discrete space.

SOLUTION 1.12. TODO

PROPOSITION 1.46. A separable metric space that is locally Euclidean of dimension  $n$  is an  $n$ -manifold.

### 1.5.1 MANIFOLDS WITH BOUNDARY

This content is basically the same, just slightly uglier. I will skip the sections on manifolds with boundary as they come up.

## 1.6 PROBLEMS