Introduction to Topological Manifolds

Samuel Ireson

November 26, 2024

Contents

1. Topological Spaces

1.1 Topologies

Definition 1.1 (Topology). For a set X, a *topology on X* is a collection τ of subsets of X which satisfies,

1. $\varnothing, X \in \tau$.
2. $U_1, ..., U_n \in \tau \implies U_1 \cap \cdots \cap U_n \in \tau$.
3. $(U_\alpha)_{\alpha \in A} \in \tau \implies \bigcup_{\alpha \in A} U_\alpha \in \tau$.

The pair (X, τ) is called a *topological space.*

Definition 1.2 (Neighbourhood). Given a topological space X, a *neighbourhood* of a point $p \in X$ is any open set $U_p \in \tau$ containing p.

A *neighbourhood of the subset* $K \subseteq X$ is an open set containing K.

Proposition 1.3 (Metric topology). Let (M, d) be a metric space and let τ be the collection of sets which are open in the sense of metric spaces. Then, (M, τ) is a topological space.

Proof. Proof of the statement amounts to showing that each of the defining properties of a topological space are true in the metric space.

1. $\varnothing$ is vacuously open in M, and M is clearly also open. Hence $\varnothing, M \in \tau$.
2. Take $U_1, ..., U_n \in \tau$, and consider a point $p \in U_1 \cap$. Since each of the sets U_i are open, there exist values r_i such that $B(p; r_i) \subseteq U_i$ for all $1 \leq i \leq n$. Taking $r = \min_{1 \leq i \leq n}\{r_i\}$, we have that $B(p; r) \subseteq U_1 \cap \cdots \cap U_n$, and hence this intersection is open.
3. For every point in the union, there is at least one open set U_α containing the point, and an open ball around the point. Hence the union is open.

$\square$

Notation. We have the following standard notations for common sets.

- The open unit n-ball:

$$\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$$

- The closed unit n-ball:

$$\overline{\mathbb{B}}^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

- The unit n-sphere:

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$

DEFINITION 1.4 (Metrizable). A topological space (X, τ) is said to be *metrizable* if its topology is the same as that generated by some metric on X.

1.1.1 Closed subsets

DEFINITION 1.5 (Closed sets). A subset F of a topological space X is said to be *closed* if its complement $X \setminus F$ is open.

For the proceeding four definitions, let A be any subset of a topological space X.

DEFINITION 1.6 (Closure). The *closure* of A in X is defined to be,

$$\overline{A} = \bigcap \{B \subseteq X : B \supseteq A \text{ and } B \text{ is closed in } X\}$$

DEFINITION 1.7 (Interior). The *interior* of A is defined to be,

$$\operatorname{Int} A = \bigcup \{C \subseteq X : C \subseteq A \text{ and } C \text{ is open in } X\}$$

DEFINITION 1.8 (Exterior). The *exterior* of A is defined to be,

$$\operatorname{Ext} A = X \setminus \overline{A}$$

DEFINITION 1.9 (Boundary). The *boundary* of A is defined to be,

$$\partial A = X \setminus (\operatorname{Int} A \cup \operatorname{Ext} A)$$

PROPOSITION 1.10. Let X be a topological space and let $A \subseteq X$ be any subset.

1. A point is in $\operatorname{Int} A$ if and only if it has a neighbourhood contained in A.
2. A point is in $\operatorname{Ext} A$ if and only is it has a neighbourhood contained in $X \setminus A$.
3. A point is in ∂A if and only if every neighbourhood of it contains both a point of A and a point of $X \setminus A$.
4. A point is in $\overline{A}$ if and only if every neighourhood of it contains a point of A.
5. $\overline{A} = A \cup \partial A = \operatorname{Int} A \cup \partial A$.
6. $\operatorname{Int} A$ and $\operatorname{Ext} A$ are open in X, while $\overline{A}$ and ∂A are closed in X.
7. The following are equivalent:
   - A is open in X.
   - $A = \operatorname{Int} A$.
   - A contains none of its boundary points.
   - Every point of A has a neighbourhood contained in A.

8. The following are equivalent:
   - A is closed in X.
   - $A = \overline{A}$.
   - A contains all of its boundary points.
   - Every point of $X \setminus A$ has a neighbourhood contained in $X \setminus A$.

Proof. We will work through each statement in turn.

1. Consider the following logical equivalence

$$
\begin{gathered}
x \in \operatorname{Int} A \\
\iff \\
x \in \bigcup \{B \subseteq A : B \in \tau_X\} \\
\iff \\
\exists B \subseteq A \text{ s.t. } (x \in B \wedge B \in \tau_X).
\end{gathered}
$$

   This is exactly the statement that there exists an open neighbourhood of $x \in \operatorname{Int} A$ contained in $\operatorname{Int} A$.

2. Assume first that $x \in \operatorname{Ext} A$. Then, by definition, we have that $x \in X \setminus \overline{A}$. Since $\overline{A}$ is an intersection of closed sets, it is closed, and hence $X \setminus \overline{A}$ is open. Furthermore, $A \subseteq \overline{A}$, and equivalently, $X \setminus \overline{A} \subseteq X \setminus A$. Therefore, $X \setminus \overline{A}$ is a neighbourhood of x satisfying the desired property.

   For the reverse implication, assume that x has an open neighbourhood $U \subseteq X \setminus A$. Since U is open, $X \setminus U$ is closed, and furthermore, $X \setminus U \supseteq A$. From the definition of $\overline{A}$, it must be the case that $x \notin \overline{A}$, hence $x \in \operatorname{Ext} A$.

3. Given the previous two results, and the definition of ∂A, the statement is clear.

4. Given the second result, and the definition of $\operatorname{Ext} A$, the statement is clear.

5. We first aim to show that $A \cup \partial A \subseteq \overline{A}$. Considering each of the LHS components in turn, we will consider neighbourhoods of points. Firstly, for $x \in A$, any neighbourhood of x will contain x itself, and hence $x \in \overline{A}$. Secondly, for $x \in \partial A$, we know by the third result that every neighbourhood contains a point of A (and $X \setminus A$), therefore $x \in \overline{A}$ also.

   By the definition of $\operatorname{Int} A$ we also have that $\operatorname{Int} A \subseteq A$ – the interior is a union of open sets contained by A. Therefore, so far we have,

$$
\operatorname{Int} A \cup \partial A \subseteq A \cup \partial A \subseteq \overline{A}
$$

   We can also show by elementary set operations that,

$$
\begin{aligned}
\operatorname{Int} A \cup \partial A &= \operatorname{Int} A \cup (X \setminus (\operatorname{Int} A \cup \operatorname{Ext} A)) \\
&= X \setminus \operatorname{Ext} A \\
&= \overline{A}
\end{aligned}
$$

   And hence the chain of inclusions is a chain of equalities, as needed.

6. Clearly $\operatorname{Int} A$ is open, as a union of open sets. The same follows for the closedness of $\overline{A}$ as an intersection of closed sets. As $\operatorname{Ext} A$ is the X-complement of $\overline{A}$, it is open. As ∂A is the X-complement of a union of open sets, it is closed.
7. We first assume that A is open. Then clearly, every point $x \in A$ has an open neighbourhood contained in A, since $A \subseteq A$. From this we know that $\partial A = \varnothing$ from the third result. From this we know that, $A = \operatorname{Int} A$ since,

$$(A \cup \partial A = \operatorname{Int} A \cup \partial A) \wedge (A \cap \partial A = \varnothing)$$
$$\iff$$
$$A = \operatorname{Int} A$$

   also relying on the disjointness of $\operatorname{Int} A$ and ∂A.

   If $A = \operatorname{Int} A$, then A is open, since $\operatorname{Int} A$ is open. From this we have a full circle of equivalences.
8. These equivalences are easily validated, in a similar way as to the previous result.

$\square$

DEFINITION 1.11 (Limit and isolated points). Given a topological space X, and a set $A \subseteq X$, we say that a point $p \in X$ is a *limit point* of A if every neighbourhood of p contains a point of A other than p.

On the other hand, a point $p \in A$ is called an *isolated point* of A if there exists a neighbourhood U of p such that $U \cap A = \{p\}$.

PROBLEM 1.1. Show that a subset is closed if and only if it contains all of its limit points.

SOLUTION 1.1. We can make a neat argument using the previous proposition. Since every neighbourhood of every limit point x of A contains a point of A other than the point x itself, the point x must be contained by the closure $\overline{A}$. Furthermore since, A is closed if and only if $A = \overline{A}$, every limit point is contained by A if and only if $A = \overline{A}$ if and only if A is closed.

DEFINITION 1.12 (Dense). A subset A of a topological space X is said to be *dense in* X if $\overline{A} = X$.

PROBLEM 1.2. Show that a subset $A \subseteq X$ is dense in X if and only if every nonempty open subset of X contains a point in A.

SOLUTION 1.2. $\overline{A} = X$ if and only if every neighbourhood of every point in X has nonempty intersection with A. In particular, the statement 'every neighbourhood of every point in X' is exhaustive of nonempty open sets in the topology of X, and hence the statement follows.

1.2 Convergence and continuity

DEFINITION 1.13 (Sequence convergence). For a topological space X and a sequence $(x_n)_{n=1}^{\infty}$ of points in X, we say that the *sequence converges* to x if for every neighbourhood U of x, there exists some $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

PROBLEM 1.3. Show that in a metric space, the topological definition of convergence is equivalent to the metric space definition.

SOLUTION 1.3. To solve this exercise, we want to show that a sequence converges in a metric space if and only if the sequence converges in the respective topological space with the induced metric topology. Consider the following equivalences, starting with the definition of convergence in a metric space.

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } d(x_n, x) < \epsilon \ \forall n \geq N$$
$$\iff$$
$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } x_n \in B^{(d)}(x; \epsilon) \ \forall n \geq N$$
$$\iff$$
$$\text{for every neighbourhood } U \text{ of } x \ \exists N \in \mathbb{N} \text{ s.t. } x_n \in U \ \forall n \geq N$$

PROBLEM 1.4. For a topological space X, a subset A and a sequence $(x_i) \in A$, show that $x = \lim_{i\to\infty} x_i \in \overline{A}$.

SOLUTION 1.4. Suppose for the sake of contradiction that this wasn't the case, and the limit x of a convergent sequence (x_i) was such that $x \in \operatorname{Ext} A$. Then, by a previous result, there exists an open subset $U \subseteq X \setminus A$ such that $x \in U$. Since $x_i \in A$ for all i, $x_i \notin U$ for all i, and hence the sequence cannot be convergent; the desired contradiction.

DEFINITION 1.14 (Continuity). If X and Y are topological spaces, a map $f : X \to Y$ is said to be *continuous* if for every open subset $U \subseteq Y$, its preimage $f^{-1}(U)$ is open in X.

PROPOSITION 1.15. A map between topological spaces is continuous if and only if the preimage of every closed subset is closed.

Proof. Consider the following equivalences.

$$f \text{ is continuous}$$
$$\iff$$
$$f^{-1}(B) \text{ is open, for all } B \text{ open}$$
$$\iff$$
$$f^{-1}(Y \setminus A) \text{ is open, for all } A \text{ closed}$$
$$\iff$$
$$f^{-1}(Y) \setminus f^{-1}(A) \text{ is open, for all } A \text{ closed}$$
$$\iff$$
$$X \setminus f^{-1}(A) \text{ is open, for all } A \text{ closed}$$
$$\iff$$
$$f^{-1}(A) \text{ is closed, for all } A \text{ closed.}$$

□

Proposition 1.16. Let X, Y and Z be topological spaces.

1. Every constant map $f : X \to Y$ is continuous.
2. The identity map $\mathrm{Id}_X : X \to X$ is continuous.
3. If $f : X \to Y$ is continuous, so is the restriction of f to any open subset of X.
4. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is their composition $g \circ f : X \to Z$.

Proof. We deal with each of the statements in turn.

1. Suppose that $f : X \to Y : x \mapsto a \in Y$. Then, an open set $U \subseteq Y$ is either $U \supseteq \{a\}$ or $U \not\supseteq \{a\}$. In the first case, $f^{-1}(U) = X$ which is open, and in the second case $f^{-1}(U) = \varnothing$, which is also open.
2. Let U be an open set in X. Then $f^{-1}(U) = U$, open by hypothesis.
3. This is clear.
4. Consider an open set $U \subseteq Z$. Then,

$$\begin{aligned}(g \circ f)^{-1}(U) &= (f^{-1} \circ g^{-1})(U) \\ &= f^{-1}(g^{-1}(U))\end{aligned}$$

   where the continuity of each of these functions ensures that the composition is continuous.

□

Proposition 1.17 (Local criterion for continuity). A map $f : X \to Y$ is continuous if and only if each point of X has a neighbourhood on which the restriction of f is continuous.

Proof. If the function f is continuous, then we can simply take X to be the open neighbourhood of every point.

On the contrary, let $U \subseteq Y$, where we aim to show that $f^{-1}(U)$ is open in X. Taking some $x \in f^{-1}(U)$, we know that there exists neighbourhood V_x of x such that $f|_{V_x}$ is continuous. In particular, $f|_{V_x}^{-1}(U)$ is open. Also,

$$f|_{V_x}^{-1}(U) = \{x \in V_x : f(x) \in U\} = V_x \cap f^{-1}(U)$$

and in particular, $f|_{V_x}^{-1}(U) \subseteq f^{-1}(U)$, is an open neighbourhood of x. The arbitrary choice of $x \in f^{-1}(U)$ then determines that $f^{-1}(U)$ is open. □

Definition 1.18 (Homeomorphism). A *homeomorphism* from X to Y is a bijective map $\phi : X \to Y$ such that ϕ and ϕ^{-1} are continuous.

If there exists a homeomorphism between X and Y, we say that X and Y are *topologically equivalent*, or *homeomorphic*. We denote this relation by $X \approx Y$.

Problem 1.5. Show that homeomorphisms provide an equivalence relation on topological spaces.

SOLUTION 1.5. We aim to show that homeomorphisms are reflexive, transitive and symmetric.

1. Reflexive: taking $\mathrm{Id}_X : X \to X$, we see that this map is bijective, continuous, and has continuous inverse. Hence $X \approx X$.
2. Transitive: consider the homeomorphisms $f : X \to Y$ and $g : Y \to Z$. Then, $g \circ f : X \to Z$ is continuous, bijective from X to Z, and has continuous inverse $f^{-1} \circ g^{-1} : Z \to X$. Hence $X \approx Z$.
3. Symmetric: taking f^{-1} as the homeomorphism between Y and X is sufficient. Hence $Y \approx X$.

PROBLEM 1.6. Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and let $f : X_1 \to X_2$ be a bijective map. Show that f is a homeomorphism if and only if $f(\tau_1) = \tau_2$ in the sense that $U \in \tau_1$ if and only if $f(U) \in \tau_2$.

SOLUTION 1.6. First, assume that $f(\tau_1) = \tau_2$, in the sense described. Then $V \in \tau_2$ if and only if there exists some $U \in \tau_1$ such that $f(U) = V$. Since f is bijective by assumption,

$$f^{-1}(V) = f^{-1}(f(U)) = U$$

is open, and therefore f is continuous. A similar argument follows for the continuity of f^{-1}.

On the contrary, suppose that f is a homeomorphism. Then, f and f^{-1} are continuous, and,

$$\begin{gathered} U \in \tau_1 \\ \Longrightarrow \\ (f^{-1})^{-1}(U) = f(U) \in \tau_2 \\ \Longrightarrow \\ f^{-1}(f(U)) = U \in \tau_1 \end{gathered}$$

which is the statement we wanted.

DEFINITION 1.19 (Finer and coarser). Given two topologies τ_1, τ_2 on X, we say that τ_1 is *finer* than τ_2 if $\tau_1 \supseteq \tau_2$ and *coarser* than τ_2 if $\tau_1 \subseteq \tau_2$.

PROBLEM 1.7. Show that the identity map of X is continuous as a map from (X, τ_1) to (X, τ_2) if and only if τ_1 is finer than τ_2, and is a homeomorphism if and only if $\tau_1 = \tau_2$.

SOLUTION 1.7. Considering the map $\mathrm{Id}_X : (X, \tau_1) \to (X, \tau_2)$, we have the following equivalences,

$$\begin{gathered} \tau_2 \subseteq \tau_1 \\ \Longleftrightarrow \\ U \in \tau_2 \implies U \in \tau_1 \\ \Longleftrightarrow \\ U \in \tau_2 \implies \mathrm{Id}_X^{-1}(U) \in \tau_1 \\ \Longleftrightarrow \\ \mathrm{Id}_X \text{ is continuous }. \end{gathered}$$

The identity is bijective, so is a homeomorphism if and only if it is continuous with continuous inverse. This is the case if and only if $\tau_1 \subseteq \tau_2$ and $\tau_2 \subseteq \tau_1$, that is if and only if $\tau_1 = \tau_2$.

DEFINITION 1.20 (Open and closed maps). A function $f : X \to Y$ is said to be an *open map* if it takes open subsets of X to open subsets of Y.

A function $f : X \to Y$ is said to be a *closed map* if it takes closed subsets of X to closed subsets of Y.

PROBLEM 1.8. Suppose that $f : X \to Y$ is a bijective continuous map. Show that the following are equivalent.

1. f is a homeomorphism.
2. f is an open map.
3. f is a closed map.

SOLUTION 1.8. Given the assumptions that f is continuous and bijective,

$$\begin{gathered} f \text{ is a homeomorphism} \\ \Longleftrightarrow \\ f^{-1} \text{ is continuous} \\ \Longleftrightarrow \\ U \in \tau_X \implies f(U) \in \tau_Y \\ \Longleftrightarrow \\ f \text{ is open.} \end{gathered}$$

PROPOSITION 1.21. Let $f : X \to Y$ be a map of topological spaces.

1. f is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.
2. f is closed if and only if $f(\overline{A}) \supseteq \overline{f(A)}$ for all $A \subseteq X$.
3. f is continuous if and only if $f^{-1}(\operatorname{Int} B) \subseteq \operatorname{Int} f^{-1}(B)$ for all $B \subseteq Y$.
4. f is open if and only if $f^{-1}(\operatorname{Int} B) \supseteq \operatorname{Int} f^{-1}(B)$ for all $B \subseteq Y$.

Proof. 1. If f is closed, then $f(\overline{A})$ is closed. Since $A \subseteq \overline{A}$, $f(A) \subseteq f(\overline{A})$. The closure of $f(A)$ is defined as,

$$\overline{f(A)} = \bigcap_{B \supseteq f(A)} \{B : X \setminus B \in \tau_X\}.$$

We know that $f(\overline{A})$ is a closed set containing $f(A)$, and the result follows.

□

DEFINITION 1.22 (Local homeomorphism). A map $f : X \to Y$ is called a *local homeomorphism* if every point $x \in X$ has a neighbourhood $U \subseteq X$ such that $f(U)$ is an open subset of Y and $f|_U : U \to f(U)$ is a homeomorphism.

Proposition 1.23 (Properties of local homeomorphisms). We have the following properties,

1. Every homeomorphism is a local homeomorphism.
2. Every local homeomorphism is continuous and open.
3. Every bijective local homeomorphism is a homeomorphism.

Proof. We work through the statements in turn.

1. This is clear. Since X is open, and $f(X) = Y$, f is a valid local homeomorphism for all points $x \in X$.
2. We know from the local criterion for continuity that a function f is continuous if and only if each point $x \in X$ has a neighbourhood U such that $f|_U$ is continuous. If we consider a local homeomorphism f, we know that this function is a homeomorphism on open neighbourhoods of every points $x \in X$, and therefore is continuous. We also know that every restriction of f is open, and therefore f is open also.
3. Using a previous result, we know that a continuous, open bijection is a homeomorphism.

□

1.3 Hausdorff spaces

Definition 1.24 (Hausdorff). A topological space X is said to be *Hausdorff* if given any two points $p \neq q \in X$, there exist neighbourhoods U, V of p and q respectively such that $U \cap V = \varnothing$.

Problem 1.9. Suppose that for every $p \in X$ there exists a continuous function $f : X \to \mathbb{R}$ such that $f^{-1}(0) = \{p\}$. Show that X is Hausdorff.

Solution 1.9. TODO

Proposition 1.25. Let X be a Hausdorff space.

1. Every finite subset of X is closed.
2. If a sequence $(p_i) \in X$ converges to a limit $p \in X$, the limit is unique.

Proof. TODO □

Problem 1.10. Show that the only Hausdorff topology on a finite set is the discrete topology.

Solution 1.10. TODO

Proposition 1.26. Suppose X is a Hausdorff space and $A \subseteq X$. If $p \in X$ is a limit point of A, then every neighbourhood of p contains infinitely many points of A.

Proof. TODO □

1.4 Bases and countability

Definition 1.27 (Basis). A collection $\mathcal{B}$ of subsets of X is a *basis for the topology* of X if,

1. Every element of $\mathcal{B}$ is an open set of X. That is $\mathcal{B} \subseteq \tau$.
2. Every open subset of X is the union of some collection of the elements of $\mathcal{B}$.

Problem 1.11. Suppose that $\mathcal{B}$ is a basis for X. Show that a subset $U \subseteq X$ is open if and only if for each $p \in U$, there exists $B \in \mathcal{B}$ such that $p \in B \subseteq U$.

Solution 1.11. TODO

Proposition 1.28. Let X and Y be topological spaces and $\mathcal{B}$ a basis for Y. A map $f : X \to Y$ is continuous if and only if for every basis subset $B \in \mathcal{B}$, the subset $f^{-1}(B)$ is open in X.

Proof. TODO □

1.4.1 Defining a topology from a basis

Proposition 1.29. A collection $\mathcal{B}$ of open subsets of X is a basis if and only if the following two conditions hold,

1. $\bigcup_{B \in \mathcal{B}} B = X$.
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists an element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If this is the case, there is a unique topology on X for which $\mathcal{B}$ is a basis, called the *generated topology* with respect to $\mathcal{B}$.

Proof. TODO □

1.4.2 Countability properties

Definition 1.30 (Neighbourhood basis). If X is a topological space and $p \in X$, a collection $\mathcal{B}_p$ of neighbourhoods of p is called a *neighbourhood basis* for X at p if every neighbourhood of p contains some $B \in \mathcal{B}_p$.

Definition 1.31 (First countability). We say that a topological space X is *first countable* if there exists a countable neighbourhood basis at all points $x \in X$.

Definition 1.32 (Nested neighbourhood basis). If X is a topological space and $p \in X$, a sequence $(U_n)_{n \in \mathbb{N}}$ of neighbourhoods of p is called a *nested neighbourhood basis* if $U_{n+1} \subseteq U_n$ for each n, and the sequence, when viewed as a collection if a neighbourhood basis of X at p.

Lemma 1.33 (Nested neighbourhood basis lemma). Let X be a first countable space. Then there exists a nested neighbourhood basis at p, for every $p \in X$.

Proof. TODO □

Definition 1.34 (Eventually in). If $(x_i)_{i=1}^{\infty}$ is a sequence of points in the topological space X and $A \subseteq X$, we say that the sequence is *eventually in A* if there exists some $n \in \mathbb{N}$ such that $x_i \in A$ for all $i \geq n$.

Lemma 1.35. Let X be first countable, A be any subset of X, and $x \in X$.

1. $x \in \overline{A}$ if and only if x is a limit of a sequence of points in A.
2. $x \in \operatorname{Int} A$ if and only if every sequence in X converging to x is eventually in A.
3. A is closed in X if and only if A contains every limit of every convergent sequence of points in A.
4. A is open in X if and only if every sequence in X converging to a point of A is eventually in A.

Proof. TODO □

Example 1.36 (A non-first countable space). TODO

Definition 1.37 (Second countability). A topological space is said to be *second countable* if it admits a countable basis for its topology.

Definition 1.38 (Covers). A collection $\mathcal{U}$ of subsets of X is called a *cover* of X if every points $x \in X$ is contained by at least one $U \in \mathcal{U}$. The cover is called an *open cover* if every $U \in \mathcal{U}$ is open, and a *closed cover* if every $U \in \mathcal{U}$ is closed.

Given a cover $\mathcal{U}$, a *subcover* of $\mathcal{U}$ is a subcollection $\mathcal{U}' \subseteq \mathcal{U}$ which covers X.

Definition 1.39 (Separable). A topological space is called *separable* if it contains a countable dense subset.

Definition 1.40 (Lindelöf). A topological space, X is said to be a *Lindelöf space* if every open cover of X has a countable subcover.

Theorem 1.41 (Properties of second countable spaces). Let X be a second countable space.

1. X is first countable.
2. X is separable.
3. X is Lindelöf.

Proof. TODO □

1.5 Manifolds

Definition 1.42 (Locally Euclidean). A topological space M is called *locally Euclidean* of dimension n if every point of M has a neighbourhood in M that is homeomorphic to an open subset of $\mathbb{R}^n$.

Lemma 1.43. A topological space M is locally Euclidean of dimension n if and only if either of the following properties hold:

1. Every points of M has a neighbourhood homeomorphic to an open ball in $\mathbb{R}^n$.
2. Every points of M has a neighbourhood homeomorphic to $\mathbb{R}^n$.

Proof. TODO $\square$

Definition 1.44 (Topological manifold). An n-dimensional *topological manifold* is a secound countable Hausdorff space which is locally Euclidean of dimension n.

Proposition 1.45. Every open subset of an n-manifold is an n-manifold.

Proof. TODO $\square$

Problem 1.12. Show that a topological space is a 0-manifold if and only if it is a countable discrete space.

Solution 1.12. TODO

Proposition 1.46. A separable metric space that is locally Euclidean of dimension n is an n-manifold.

1.5.1 Manifolds with boundary

This content is basically the same, just slightly uglier. I will skip the sections on manifolds with boundary as they come up.

1.6 Problems

2. New spaces from old

2.1 Subspaces

Definition 2.1 (Subspace topology). Let X be a topological space and $S \subseteq X$ any subset. The *subset topology* on S is the collection,

$$\tau_S = \{U \subseteq S : U = S \cap V \text{ for some } V \in \tau_X\}$$

Problem 2.1. Verify that the subset topology is indeed a topology on S.

Solution 2.1. TODO

Problem 2.2. Suppose S is a subspace of X. Prove that $B \subseteq S$ is closed in S if and only if it is equal to the intersection of S with some closed subset of X.

Solution 2.2. TODO

Proposition 2.2. Suppose S is a subspace of the topological space X,

1. If $U \subseteq S \subseteq X$, U is open in S, and S is open in X, then U is open in X.
2. If $U \subseteq S \subseteq X$, U is closed in S, and S is closed in X, then U is closed in X.
3. If U is a subset of S that is either open or closed in X then it is also respectively open or closed in S.

Proof. TODO □

Problem 2.3. Suppose that $U \subseteq S \subseteq X$.

1. Show that the closure of U in S is equal to $\overline{U} \cap S$.
2. Show that the interior of U in S contains $\operatorname{Int} U \cap S$, but the oppositve inclusion is not necessarily true.

Solution 2.3. TODO

Theorem 2.3 (Characteristic property of the subspace topology). Suppose X is a topological space and $S \subseteq X$ is a subspace. For any topological space Y, a map $f : Y \to S$ is continuous if any only if the composition $\iota_S \circ f : Y \to X$ is continuous.

Proof. TODO □

Corollary 2.4. The inclusion map $\iota_S : S \hookrightarrow X$ is continuous.

Proof. TODO □

Corollary 2.5. Let $f : X \to Y$ be a continuous map between topological spaces. Then,

1. The restriction of f to any subspace $S \subseteq X$ is continuous.
2. If T is a subspace of Y that contains $f(X)$, then $f : X \to T$ is continuous.
3. If Y is a subspace of Z, then $f : X \to Z$ is continuous.

Proof. TODO □

Proposition 2.6 (Other subspace properties). Let S be a subspace of X.

1. If $R \subseteq S$ is a subspace of S, then R is a subspace of X.
2. If $\mathcal{B}$ is a basis for the topology of X, then
$$\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}\}$$
is a basis for the topology of S.
3. If (p_i) is a sequence of points in S, and $p \in S$, then $p_i \to p$ in S if and only if $p_i \to p$ in X.
4. Every subspace of a Hausdorff space is Hausdorff.
5. Every subspace of a first countable space is first countable.
6. Every subspace of a second countable space is second countable.

Proof. TODO □

2.1.1 Topological embeddings

Definition 2.7 (Topological embedding). An injective continuous map that is a homeomorphism onto its image is called a *topological embedding.*

Problem 2.4. Show that the inclusion map $\iota_S : S \hookrightarrow X$ is topological embedding.

Solution 2.4. TODO

Proposition 2.8. A continuous inhective map that is either open or closed is a topological embedding.

Proof. TODO □

Proposition 2.9. A surjective topological embedding is a homeomorphism

Proof. TODO □

DEFINITION 2.10. If $U \subseteq \mathbb{R}^n$ is an open subset and $f : U \to \mathbb{R}^k$ is any continuous map, the *graph of* f is the subset $\Gamma(f) \subseteq \mathbb{R}^{n+k}$ defined by,

$$\Gamma(f) = \{(x, y) = (x_1, ..., x_n, y_1, ..., y_k) : x \in U \wedge y = f(x)\}$$

PROBLEM 2.5. Show that $\Gamma(f)$ is a manifold.

SOLUTION 2.5. TODO

LEMMA 2.11 (Gluing lemma). Let X and Y be topological spaces, and let $\{A_i\}$ be either an arbitary open cover of X or a finite closed cover of X. Suppose that $f_i : A_i \to Y$ are continuous maps which coincide on intersections: $f_i|_{A_i \cap A_j} = f_j|_{A_i \cap A_j}$. Then, there exists a unique continuous map $f : X \to Y$ such that $f|_{A_i} = f_i$.

Proof. TODO □

THEOREM 2.12 (Uniqueness of the subspace topology). Suppose S is a subset of a topological space X. The subspace topology on S is the unique topology satisfying the characteristic property.

Proof. TODO □

2.2 PRODUCT SPACES

DEFINITION 2.13 (Product topology). Let $X_1, ..., X_n$ be topological spaces. The *product topology* is defined on their Cartesian product $X_1 \times \cdots \times X_n$ as the topology generated by the basis,

$$\mathcal{B} = \{U_1 \times \cdots \times U_n : U_i \in \tau_{X_i} \text{ for } i = 1, ..., n\}$$

PROBLEM 2.6. Prove that $\mathcal{B}$ is a basis for a topology.

SOLUTION 2.6. TODO

PROBLEM 2.7. Show that the product topology on $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}$ is the same as the metric topology induced by the Euclidean distance function.

SOLUTION 2.7. TODO

THEOREM 2.14 (Characteristic property of the product topology). Suppose that $X_1 \times \cdots \times X_n$ is a product space. For any topological space Y, a map $f : Y \to X_1 \times \cdot \times X_n$ is continuous if and only if each of its component functions $f_i = \pi_i \circ f$ is continuous where $\pi_i : X_1 \times \cdots \times X_n \to X_i$ is the canonical projection.

Proof. TODO □

Corollary 2.15. Each canonical projection π_i is continuous.

Proof. TODO □

Theorem 2.16 (Uniqueness of the product topology). The product topology on $X_1 \times \cdots \times X_n$ is the unique topology satisfying the characteristic property.

Proof. TODO □

Proposition 2.17 (Other product space properties). Let $X_1, ..., X_n$ be topological spaces.

1. The product topology is associative.
2. For any $i \in \{1, ..., n\}$ and any points $x_j \in X_j$, $j \neq i$, the map,

$$\begin{aligned} f &: X_i \to X_1 \times \cdots \times X_n \\ &: x \mapsto (x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n) \end{aligned}$$

   is a topological embedding of X_i into the product space.
3. Each canonical projection π_i is open.
4. If for each i, $\mathcal{B}_i$ is a basis for the topology of X_i, then the set

$$\{B_1 \times \cdots \times B_n : B_i \in \mathcal{B}_i\}$$

   is a basis for the product topology on $X_1 \times \cdots \times X_n$.
5. If S_i is a subspace of X_i for $i = 1, ..., n$, then the product topology and the subspace topology on $S_1 \times \cdots \times S_n \subseteq X_1 \times \cdots \times X_n$ are equal.
6. If each X_i is Hausdorff, so is $X_1 \times \cdots \times X_n$.
7. If each X_i is first countable, so is $X_1 \times \cdots \times X_n$.
8. If each X_i is second countable, so is $X_1 \times \cdots \times X_n$.

Proof. TODO □

Definition 2.18 (Product map). If $f_i : X_i \to Y_i$ are maps for $i = 1, ..., k$, their *product map* is defined as,

$$\begin{aligned} f_1 \times \cdots \times f_k &: X_1 \times \cdots \times X_k \to Y_1 \times \cdots Y_k \\ &: (x_1, ..., x_k) \mapsto (f_1(x_1), ..., f_k(x_k)) \end{aligned}$$

Proposition 2.19. A product of continuous maps is continuous, and a product of homeomorphisms is a homeomorphism.

Proof. TODO □

PROPOSITION 2.20. If $M_1, ..., M_k$ are manifolds of dimension $n_1, ..., n_k$ respectively, then product space $M_1 \times \cdots \times M_k$ is a manifold of dimension $n_1 + \cdots + n_k$.

Proof. TODO ☐

2.3 Disjoint union spaces

2.4 Quotient spaces

2.5 Adjunction spaces

2.6 Topological groups and group actions

2.7 Problems

3. Connectedness and compactness

3.1 Connectedness

3.2 Compactness

3.3 Local compactness

3.4 Paracompactness

3.5 Proper maps

3.6 Problems

4. Cell complexes

I'm excited to get to this part!

4.1 Cell complexes and CW complexes

4.2 Topological properties of CW complexes

4.3 Classification of 1D manifolds

4.4 Simplicial complexes

4.5 Problems

5. Compact surfaces

6. Homotopy and the fundamental group

7. The circle

7.1 Lifting properties of the circle

7.2 The fundamental group of the circle

7.3 Degree theory for the circle

7.4 Problems

8. Some group theory

8.1 Free products

8.2 Free groups

8.3 Presentations of groups

8.4 Free Abelian groups

8.5 Problems

9. The Seifer-Van Kampen theorem

9.1 Statement

9.2 Applications

9.3 Fundamental groups of compact surfaces

9.4 Proof

9.5 Problems

10. Covering maps

11. Group actions and covering maps

11.1 The automorphism group of a covering

11.2 Quotients by group actions

11.3 The classification theorem

11.4 Proper group actions

11.5 Problems

12. Homology