2. New spaces from old

2.1 Subspaces

Definition 2.1 (Subspace topology). Let X be a topological space and $S \subseteq X$ any subset. The *subset topology* on S is the collection,

$$\tau_S = \{U \subseteq S : U = S \cap V \text{ for some } V \in \tau_X\}$$

Problem 2.1. Verify that the subset topology is indeed a topology on S.

Solution 2.1. TODO

Problem 2.2. Suppose S is a subspace of X. Prove that $B \subseteq S$ is closed in S if and only if it is equal to the intersection of S with some closed subset of X.

Solution 2.2. TODO

Proposition 2.2. Suppose S is a subspace of the topological space X,

1. If $U \subseteq S \subseteq X$, U is open in S, and S is open in X, then U is open in X.
2. If $U \subseteq S \subseteq X$, U is closed in S, and S is closed in X, then U is closed in X.
3. If U is a subset of S that is either open or closed in X then it is also respectively open or closed in S.

Proof. TODO $\square$

Problem 2.3. Suppose that $U \subseteq S \subseteq X$.

1. Show that the closure of U in S is equal to $\overline{U} \cap S$.
2. Show that the interior of U in S contains $\operatorname{Int} U \cap S$, but the oppositve inclusion is not necessarily true.

Solution 2.3. TODO

Theorem 2.3 (Characteristic property of the subspace topology). Suppose X is a topological space and $S \subseteq X$ is a subspace. For any topological space Y, a map $f : Y \to S$ is continuous if any only if the composition $\iota_S \circ f : Y \to X$ is continuous.

Proof. TODO $\square$

COROLLARY 2.4. The inclusion map $\iota_S : S \hookrightarrow X$ is continuous.

Proof. TODO □

COROLLARY 2.5. Let $f : X \to Y$ be a continuous map between topological spaces. Then,

1. The restriction of f to any subspace $S \subseteq X$ is continuous.
2. If T is a subspace of Y that contains $f(X)$, then $f : X \to T$ is continuous.
3. If Y is a subspace of Z, then $f : X \to Z$ is continuous.

Proof. TODO □

PROPOSITION 2.6 (Other subspace properties). Let S be a subspace of X.

1. If $R \subseteq S$ is a subspace of S, then R is a subspace of X.
2. If $\mathcal{B}$ is a basis for the topology of X, then
$$\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}\}$$
is a basis for the topology of S.
3. If (p_i) is a sequence of points in S, and $p \in S$, then $p_i \to p$ in S if and only if $p_i \to p$ in X.
4. Every subspace of a Hausdorff space is Hausdorff.
5. Every subspace of a first countable space is first countable.
6. Every subspace of a second countable space is second countable.

Proof. TODO □

2.1.1 Topological embeddings

DEFINITION 2.7 (Topological embedding). An injective continuous map that is a homeomorphism onto its image is called a *topological embedding*.

PROBLEM 2.4. Show that the inclusion map $\iota_S : S \hookrightarrow X$ is topological embedding.

SOLUTION 2.4. TODO

PROPOSITION 2.8. A continuous inhective map that is either open or closed is a topological embedding.

Proof. TODO □

PROPOSITION 2.9. A surjective topological embedding is a homeomorphism

Proof. TODO □

DEFINITION 2.10. If $U \subseteq \mathbb{R}^n$ is an open subset and $f : U \to \mathbb{R}^k$ is any continuous map, the *graph of* f is the subset $\Gamma(f) \subseteq \mathbb{R}^{n+k}$ defined by,

$$\Gamma(f) = \{(x, y) = (x_1, ..., x_n, y_1, ..., y_k) : x \in U \wedge y = f(x)\}$$

PROBLEM 2.5. Show that $\Gamma(f)$ is a manifold.

SOLUTION 2.5. TODO

LEMMA 2.11 (Gluing lemma). Let X and Y be topological spaces, and let $\{A_i\}$ be either an arbitary open cover of X or a finite closed cover of X. Suppose that $f_i : A_i \to Y$ are continuous maps which coincide on intersections: $f_i|_{A_i \cap A_j} = f_j|_{A_i \cap A_j}$. Then, there exists a unique continuous map $f : X \to Y$ such that $f|_{A_i} = f_i$.

Proof. TODO □

THEOREM 2.12 (Uniqueness of the subspace topology). Suppose S is a subset of a topological space X. The subspace topology on S is the unique topology satisfying the characteristic property.

Proof. TODO □

2.2 PRODUCT SPACES

DEFINITION 2.13 (Product topology). Let $X_1, ..., X_n$ be topological spaces. The *product topology* is defined on their Cartesian product $X_1 \times \cdots \times X_n$ as the topology generated by the basis,

$$\mathcal{B} = \{U_1 \times \cdots \times U_n : U_i \in \tau_{X_i} \text{ for } i = 1, ..., n\}$$

PROBLEM 2.6. Prove that $\mathcal{B}$ is a basis for a topology.

SOLUTION 2.6. TODO

PROBLEM 2.7. Show that the product topology on $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}$ is the same as the metric topology induced by the Euclidean distance function.

SOLUTION 2.7. TODO

THEOREM 2.14 (Characteristic property of the product topology). Suppose that $X_1 \times \cdots \times X_n$ is a product space. For any topological space Y, a map $f : Y \to X_1 \times \cdot \times X_n$ is continuous if and only if each of its component functions $f_i = \pi_i \circ f$ is continuous where $\pi_i : X_1 \times \cdots \times X_n \to X_i$ is the canonical projection.

Proof. TODO □

COROLLARY 2.15. Each canonical projection π_i is continuous.

Proof. TODO □

THEOREM 2.16 (Uniqueness of the product topology). The product topology on $X_1 \times \cdots \times X_n$ is the unique topology satisfying the characteristic property.

Proof. TODO □

PROPOSITION 2.17 (Other product space properties). Let $X_1, ..., X_n$ be topological spaces.

1. The product topology is associative.
2. For any $i \in \{1, ..., n\}$ and any points $x_j \in X_j$, $j \neq i$, the map,
$$\begin{aligned} f &: X_i \to X_1 \times \cdots \times X_n \\ &: x \mapsto (x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n) \end{aligned}$$
is a topological embedding of X_i into the product space.
3. Each canonical projection π_i is open.
4. If for each i, $\mathcal{B}_i$ is a basis for the topology of X_i, then the set
$$\{B_1 \times \cdots \times B_n : B_i \in \mathcal{B}_i\}$$
is a basis for the product topology on $X_1 \times \cdots \times X_n$.
5. If S_i is a subspace of X_i for $i = 1, ..., n$, then the product topology and the subspace topology on $S_1 \times \cdots \times S_n \subseteq X_1 \times \cdots \times X_n$ are equal.
6. If each X_i is Hausdorff, so is $X_1 \times \cdots \times X_n$.
7. If each X_i is first countable, so is $X_1 \times \cdots \times X_n$.
8. If each X_i is second countable, so is $X_1 \times \cdots \times X_n$.

Proof. TODO □

DEFINITION 2.18 (Product map). If $f_i : X_i \to Y_i$ are maps for $i = 1, ..., k$, their *product map* is defined as,
$$\begin{aligned} f_1 \times \cdots \times f_k &: X_1 \times \cdots \times X_k \to Y_1 \times \cdots Y_k \\ &: (x_1, ..., x_k) \mapsto (f_1(x_1), ..., f_k(x_k)) \end{aligned}$$

PROPOSITION 2.19. A product of continuous maps is continuous, and a product of homeomorphisms is a homeomorphism.

Proof. TODO □

PROPOSITION 2.20. If $M_1, ..., M_k$ are manifolds of dimension $n_1, ..., n_k$ respectively, then product space $M_1 \times \cdots \times M_k$ is a manifold of dimension $n_1 + \cdots + n_k$.

Proof. TODO □

2.3 Disjoint union spaces

2.4 Quotient spaces

2.5 Adjunction spaces

2.6 Topological groups and group actions

2.7 Problems