Numerical Differential Equations

Template from Leon Lee @leon0241 on GitHub.

1 Preliminaries

Definition 1.1: Studying Differential Equations

Analytically - to find a solution in terms of y; **qualitatively** - study the properties based on analytical and/or graphical techniques; **computationally** - find an approximation solution numerically.

Recall 1.2: ODE Terminology

Ordinary has one independent variable. **Autonomous** does not depend on the independent variable. Linear if in form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a(t)y + b(t)$$

where $a(t)$ and $b(t)$ do not need to be linear in t.

Example 1.3: Examples of (Non)-Linear Equations

$$(1)\ \frac{\mathrm{d}y}{\mathrm{d}x} = kx^2y - cx^3, \quad (2)\ x' = e^{-t\beta}x, \quad (3)\ \dot{u} = \cos(\omega t)u - \sin(\omega t).$$

All the above are linear equations with y, x and u being the dependent variables and parameters k, c, β, ω. Below is a *linear system*

$$\dot{x} = u, \quad \dot{u} = -\omega^2 x \qquad \text{for an unspecified independent var.}$$

The below are *non-linear* equations

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u^2, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = kx^2t - cx^3$$

or (Duffing's oscillator) a *non-autonomous, non-linear* system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u, \quad \frac{\mathrm{d}u}{\mathrm{d}t} = -\beta x - \alpha x^3 - \delta u - \gamma\cos(\omega t).$$

Recall 1.4: Lipschitz Continuity

A function f is Lipschitz continuous if there exists some constant $L > 0$ (the Lipschitz constant) such that

$$|f(t,u) - f(t,v)| \le L|u - v|, \qquad \text{for all } t \text{ and } u, v \in D.$$

Remark 1.5: Lipschitz Continuity: Easier

If f is continuous in t and continuously differentiable (that is, the derivative exists and is itself continuous) in y then

$$\frac{\partial f}{\partial y} \text{ is continuous} \implies L = \sup_D \left|\frac{\partial f}{\partial y}\right|.$$

Proof; by the mean value theorem. MVT states that if f is continuous and differentiable over some $[a, b]$ then there exists c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \implies f(b) - f(a) = f'(c)(b - a).$$

It immediately follows how we can find L.

Theorem 1.6: Picard's Theorem (Existence/Uniqueness)

Let f be a cts func of its arguments in a region of the plane containing the rectangle

$$D = \{(t, y) \mid t_0 \le t \le T,\ |y - y_0| \le K\}$$

where T, K are some constants. Further suppose f is Lipschitz cts. Domain D is closed and bounded so there exists some

$$M_f = \max\{|f(t, u)| : (t, u) \in D.$$

If $M_f(T - t_0) \le K$ then there exists a unique, continuously differentiable function $y(t)$ defined on $[t_0, T]$ such that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \quad y(t_0) = y_0.$$

Note. This can be condensed and/or rewritten, do it.

Theorem 1.7: Local Existence and Uniqueness

Suppose $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is cts and has cts partial derivatives in w.r.t all variables in some neighbourhood of (t_0, y_0). Then there some interval $I = (t_0 - \delta, t_0 + \delta)$ such that some y satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \quad y(t_0) = y_0.$$

Recall 1.8: Slope Fields

Consider the ODE $y' = f(t, y)$. Any point $f(t_0, y_0)$ gives the slope of y; do this for all points of the plane and draw, typically with arrow.

Recall 1.9: Taylor Series Expansion

The standard definition of a Taylor series for some function f centered about a is given

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Of course this requires f to be infinitely differentiable. If f is a most $N + 1$ times differentiable on some interval $[a, b]$, we have the Taylor remainder theorem (remark $x_0 \in (a, b)$ and τ between x_0 and x)

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(N+1)}(\tau)}{(N + 1)!}(x - x_0)^{N+1}.$$

Example 1.10: Non-Autonomous Equations

2 Euler's Method and Taylor Series Methods

Definition 2.1: Euler's Method

$$y_{n+1} = y_n + hf(t_n, y_n), \quad \text{where } y_i \approx y(t_i).$$

The algorithm takes in h, f, (t_0, y_0) and N. Remark N steps gives $N+1$ time points $t_0, t_1, \ldots, t_N = t_0 + Nh$. Remark $h = (b-a)/N$.

Example 2.2: From Where?

Euler's method is exactly a *first order* Taylor series method. Expand the Taylor series (with remainder, first order) about t_0 at t_1,

$$y(t_1) = y(t_0) + y'(t_0)(t_1 - t_0) + \frac{1}{2}y''(\tau)(t_1 - t_0)^2.$$

where τ is between t_1 and t_0. We know nada of τ, but we approx the sol by last term die $(h^2 \to 0)$. With $t_1 = t_0 + h$ we are done.

Lemma 2.3: An Inequality

Sequence (v_n) of non-neg, for $A > 1$ and $B > 0$ constants.

$$v_{n+1} \leq Av_n + B \implies v_n \leq A^n v_0 + \frac{A^n - 1}{A - 1}B.$$

Theorem 2.4: Convergence of Euler's Method

Consider $y'(t) = f(t, y)$ with $y(a) = y_0$. Suppose unique, twice diff. sol $y(t)$ on $[a, b]$. Further suppose f is cts everywhere with cts, bounded partial derivatives

$$\left|\frac{\partial f}{\partial y}\right| \leq L, \quad L > 0.$$

Then for some constant D, solution y_n at t_n satisfies

$$|e_n| = |y_n - y(t_n)| \leq Dh \quad \text{where } D = e^{(b-a)L}\frac{M}{2L} : |y''(t)| \leq M.$$

It is stressed that D does not depend on N or h, but only on features of the problem and the solution. If h is not small enough, the error compounds and can become very large (recall examples).

Definition 2.5: The Big O (Landau) Notation

A quantity is (of order) $\mathcal{O}(h^p)$ if it decays at least as quickly as h^p when h is small enough. More formally, we write $z = \mathcal{O}(h^p)$ if there exist $h_0, C \in \mathbb{R}_+$ such that $|z| \leq Ch^p$ for all $0 < h \leq h_0$.

We are concerned with p. Say ... is of order p. Conventionally

$$z = \mathcal{O}(h^{p+1}) \implies z = \mathcal{O}(h^p).$$

Definition 2.6: Flow Maps

Idea "we look at a set of initial point and describe the journey to their endpoints".

$$\Phi_{t_0,h}(y_0) = y(t_0 + h;\ t_0, y_0)$$

that we read "given t_0, h". Determining the flow map is equivalent to solving the ODE; idea being to replace the map by an approx. i.e

$$\dot{\tilde{\Phi}}_{t,h}(y) = y + hf(t, y) \qquad \text{is Euler's method as a flow map approx.}$$

We have, from outside the course (at this moment),

$$\text{linear ODE } y' = Ay \implies \Phi_t(y) = e^{tA}y.$$

Recall 2.7: How to Differentiate

Suppose $z = f(x, y)$ and $x = g(t)$, $y = h(t)$. Then, from SVCDE,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

So if $y' = f(t, y)$ then

$$y'' = \frac{d(f(t,y))}{dt} = \frac{\partial f}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = f_t + f_y \cdot f.$$

Example 2.8: From Where? Remastered

(Second order Taylor series method.) Recall

$$y(t_0 + h) = y(t_0) + y'(t_0)h + \frac{1}{2}y''(t_0)h^2 + \frac{1}{6}y'''(t_0)h^3 + \mathcal{H.O.T}.$$

Then a *second order method* instead drops the y''' term (consider Taylor remainder if wanted) and we know $y' = f$ and $y'' = f_t + f_y \cdot f$. So, by considering a typical ODE, we have

$$y_{n+1} \approx y_n + hf(t_n, y_n) + \frac{1}{2}h^2(f_t(t_n, y_n) + f_y(t_n, y_n) \cdot f(t_n, y_n)).$$

Higher order Taylor series methods can of course be used. General case is a pain to express, but in practice is it simple (implicit diff).

Definition 2.9: One-Step Methods

Methods that approximate the solution through iteration of an approximate flow map (i.e Euler, Taylor) are *one-step* methods.

3 Convergence of One-Step Methods

Definition 3.1: Global Error

The *global error* after n steps is the diff. between approx and exact solution; $e_n := y_n - y(t_n)$. A good approx. will produce a global error that is small in norm at each step; i.e

$$\max_{n=0,\dots,N} ||e_n|| \leq \delta, \qquad \text{(user specified typically } \delta = 10^{-4}\text{)}.$$

Definition 3.2: Convergent Methods

For any given T, method *convergent* if

$$\lim_{h\to 0} \max_{n=0\dots,N} ||e_n|| = 0.$$

Remark this is **worst case error.** As $h \to 0$ we have $N \to \infty$.

Remark 3.3: Notational Remarks

We use $y(t+h) = \Phi_h(y(t))$ i.e let's look at the flow map of y from t to $t+h$. Determining the flow map is equivalent in many ways to solving the ODE. We are given a *one step numerical method* that we describe with $\Psi_h(y)$; an approximation of the flow map.

Definition 3.4: Local Error

The *local error* of a one-step num. method is the difference between the flow map and its approximation ie.

$$le(y,h) = \Psi_h(y) - \Phi_h(y).$$

Definition 3.5: Consistent (at Order p) Methods

Suppose some method satisfies

$$||le(y,h)|| \leq Ch^{p+1}$$

for $p \geq 1$ and some constant C depending on y and its derivatives. Then the method is *consistent* at order p with $p \geq 1$.

Definition 3.6: Stable Methods

Suppose method satisfies an h-dependent Lipschitz condition on D (below). Then the method is *stable*.

$$||\Psi_h(u) - \Psi_v)|| \leq (1 + h\hat{L})||u - v||, \quad \forall u, v \in D.$$

Theorem 3.7: Convergence of One-Step Methods

Consider some ODE and some one-step numerical method Ψ_h that is both *stable* and *consistent* c Then the global error satisfies

$$\max_{n=0,\dots,N} ||e_n|| = \mathcal{O}(h^p)$$

i.e any stable and consistent one step method will converge.

Construction of More One-Step Methods

Remark 3.8: The Idea

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y), \quad y(t_0) = y_0;$$

$$\int_t^{t+h} \frac{\mathrm{d}y}{\mathrm{d}t} = \int_t^{t+h} f(y) \implies y(t+h) - y(t) = \int_t^{t+h} f(y).$$

How do we approximate the RHS?

Example 3.9: Euler's Method

$$\int_t^{t+h} f(y) \approx hf(y(t)) \implies y(t+h) - y(t) \approx hf(y(t));$$

i.e Euler's method; $y_{n+1} - y_n = hf(y_n)$, an explicit method.

Example 3.10: Trapezoidal Rule

$$\int_t^{t+h} f(y(\tau))\mathrm{d}\tau \approx \frac{h}{2}\Big(f(y(t)) + f(y(t+h))\Big)$$

$$\implies y_{n+1} = y_n + \frac{h}{2}(f(y_n) + f(y_{n+1})), \quad \text{an implicit method.}$$

Definition 3.11: Polynomial Interpolation

Let $\mathbb{P}_s$ denote the space of real polynomials degree $\leq s$. Set of s *abscissa* c_i, correspond. data g_i, there exists a unique polynomial $P(x) \in \mathbb{P}_{s-1}$ through them. The *Lagrange* interpolating polynomials

$$\ell_i(x) = \prod_{j=1, j\neq i}^{s} \frac{x - c_j}{c_i - c_j}, \quad P(x) = \sum_{i=1}^{s} g_i \ell_i(x).$$

Definition 3.12: Quadrature Rules

A method to approx. a definite integral of one independent variable is a *numerical quadrature rule*. A quadrature rule has order p if it exactly integrates any polynomial in $\mathbb{P}_{p-1}$. We have $p \geq s$ always.

Theorem 3.13: Quadrature Formula

$$b_i = \int_0^1 \ell_i(x)\mathrm{d}x,$$

$$\int_{t_0}^{t_0+h} g(t)\mathrm{d}t = \int_0^1 h \cdot g(t_0 + hx)\mathrm{d}x \approx h\sum_{i=1}^{s} b_i g(t_0 + hc_i).$$

Definition 3.14: Collocation Methods

We construct a polynomial that passes through y_0 and agrees with the ODE at s nodes; then the numerical solution is the value of the polynomial at y_1. Below are necessary conditions;

$$u(t_0) = y_0,$$
$$u'(t_0 + c_i h) = f(u(t_0 + c_i h)), \quad i = 1, \dots, s.$$

Example 3.15: Construction of Collocation Methods

We construct u'. Consider F_i to be the values of the (as yet undetermined) polynomial at the nodes i.e $F_i := u'(t_0 + c_i h)$. Then

$$u'(t) = \sum_{i=1}^{s} F_i \ell_i \left(\frac{t - t_0}{h}\right)$$

and by integrating over $[t_0, t_0 + c_i h]$ with a change of variables $x = (t - t_0)/h$ we obtain

$$u(t_0 + c_i h) = y_0 + h\sum_{j=1}^{s} F_j \int_0^{c_i} \ell_j(x)\mathrm{d}x, \quad i = 1, \dots, s.$$

By defining a_{ij} and b_i, we substitute this result into our required collocation conditions and achieve the coupled non-linear system (remark - it is explicit).

$$a_{ij} := \int_0^{c_i} \ell_j \mathrm{d}x, \qquad b_i := \int_0^1 \ell_i(x)\mathrm{d}x, \qquad i, j = 1, \dots, s;$$

$$F_i = f\Big(y_n + h\sum_{j=1}^{s} a_{ij}F_j\Big), \quad y_{n+1} = y_n + h\sum_{i=1}^{s} b_i F_i.$$

Lemma 3.16: Collocation: A Continuous Approximation

We obtain a cts approx. of the sol $u(t)$ on each interval $[t_n, t_{n+1}]$.

Remark 3.17: Best Accuracy?

The Gauss-Legendre collocation methods uses nodes c_1 for

$$c_1 = \frac{1}{2}$$

$$c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}, c_2 = \frac{1}{2} + \frac{\sqrt{36}}{6},$$

$$c_1 = \frac{1}{2} - \frac{\sqrt{15}}{10}, c_2 = \frac{1}{2}, c_3 = \frac{1}{2} + \frac{\sqrt{15}}{10}.$$

4 Runge-Kutta Methods

Definition 4.1: Runge-Kutta Methods

$$Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j), \quad i = 1, \dots, s$$

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f(Y_i).$$

Here s num. stages, b_i weights, a_{ij} internal coefficients. This the *same formula* as collocation method but w/o restriction on $a/b/c$. It generates a discrete flow-map approximation

$$\Psi_h(y) = y + h\sum_{i=1}^{s} b_i f(Y_i(y,h)).$$

Remark 4.2: Internal Consistency

We have $c_i = \sum_{j=1}^{s} a_{ij}$ (assumed in the notes; not necessarily true).

Definition 4.3: Butcher Tables

$$\begin{array}{c|c} C & A \\ \hline & b^T \end{array} \quad \text{i.e} \quad \begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

If the matrix $A = (a_{ij})$ is strictly lower triangular then the method is *explicit*. Otherwise it is implicit.

Example 4.4: Examples Using the Butcher Table

The implicit midpoint rule

$$Y_1 = y_n + \frac{h}{2} f(Y_1), \quad y_{n+1} = y_n + hf(Y_1),$$

$$\text{i.e} \quad y_{n+1} = y_n + hf\left(\frac{y_{n+1}+y_n}{2}\right).$$

So in this case we have $s = 1$, $a_{11} = \frac{1}{2}$ and $b_i = 1$ and also $c_i = \frac{1}{2}$

$$\begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array} \quad \text{clearly implicit (see formula).}$$

The **four-stage** Runge-Kutta Method has

$$\begin{cases} Y_1 = y_n, \\ Y_2 = y_n + \frac{h}{2} f(Y_1), \\ Y_3 = y_n + \frac{h}{2} f(Y_2), \\ Y_4 = y_n + hf(Y_3), \end{cases} : y_{n+1} = h\left(\frac{1}{6}f(Y_1)+\frac{1}{3}f(Y_2)+\frac{1}{3}f(Y_3)+\frac{1}{6}f(Y_4)\right).$$

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array} \quad \text{consider } Y_i = y(t_n + c_i h).$$

Recall 4.5: Continuously Differentiable and Semisimple

Notation $C^{(n)}$ means function has a cts n-th derivative. Any eigenvalue λ is semisimple if the size of its largest Jordan block is 1.

Definition 4.6: The Jacobian ... And More

$$\text{the Jacobian } f' = \left(\frac{\partial f_i}{\partial y_i}\right), \quad i \le i, j \le d, \quad \text{notation } M = (m_{ij}).$$

$$f'' = \left(\frac{\partial^2 f_i}{\partial y_i \partial y_k}\right) : \text{operator} \sum_{j=1}^{d}\sum_{k=1}^{d} \frac{\partial^2 f_i}{\partial y_j \partial y_k} a_j b_k \text{ maps } a, b \in \mathbb{R}^d \to \mathbb{R}.$$

There is a long spiel but we achieve, in summary,

$$\begin{aligned} y' &= f, \\ y'' &= f'f, \\ y''' &= f''(f,f) + f'f'f; \qquad y(t),\ f(y). \end{aligned}$$

Recall Taylor series expansion

$$y(t+h) = y(t) + hy'(t) + \frac{1}{2}h^2 y''(t) + \frac{1}{6}h^3 y'''(t) + \mathcal{O}(h^4)$$

and the corresponding flow map

$$\Phi_h(y) = y + hf + \frac{h^2}{2}f'f + \frac{h^3}{6}\left(f''(f,f) + f'f'f\right) + \mathcal{O}(h^4).$$

The notes gives how to calculate the local error for;

$$\text{Euler's method:} \quad le(y,h) = \frac{h^2}{2}f'f + \mathcal{O}(h^3),$$

$$\text{Trapezium rule:} \quad le(y,h) = -\frac{1}{12}(f''(f,f) + f'f'f)h^3 + \mathcal{O}(h^4).$$

Theorem 4.7: Convergence of RK Methods

Respectively for $p = 1, 2, 3$ we require the following conditions;

$$\sum_{i=1}^{s} b_i = 1,$$

$$\sum_{i=1}^{s} b_i = 1, \quad \sum_{i=1}^{s} b_i c_i = \frac{1}{2},$$

$$\sum_{i=1}^{s} b_i = 1, \quad \sum_{i=1}^{s} b_i c_i = \frac{1}{2}, \sum_{i=1}^{s} b_i c_i^2 = \frac{1}{3}, \quad \sum_{i=1}^{s}\sum_{j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}.$$

Stability of Runge-Kutta Methods

Remark 4.8: The Idea

We look at the local sol. behaviour in the neighbourhood of fixed points; assume ODE autonomous, often about origin. Key takeaway; any linear ODE system and its' RK approx. can be decomposed into scalar linear problems involving the eigenvalues as coeffs.

Definition 4.9: Fixed Points of an ODE

Point y^* if $f(y^*) = 0$; that is, $y(t) \equiv y^*$ when $y(0) = y^*$.

Definition 4.10: (Spurious) Fixed Points

Fixed points of Φ_t identical to ODE; $\mathcal{F}$. We have fixed points of num. method $\mathcal{F}_h$ depend on h; $\mathcal{F}_h = \{y \in \mathbb{R}^d : \Psi_h(y) = y\}$. In general (for RK methods) $\mathcal{F} \subseteq \mathcal{F}_h$, for Euler, implicit mp we have $\mathcal{F} = \mathcal{F}_h$.

Spurious point: $y^* \in \mathcal{F}_h$ but $y^* \notin \mathcal{F}$. How do we identity? As $h \to 0$, all spurious tend to infinity; so test multiple h.

Definition 4.11: Formal Definitions for Stability

(i) y^* stable if for all $\epsilon > 0$, exists $\delta > 0$ s.t

$$||y_0 - y^*|| < \delta \implies ||y(t;\ y_0) - y^*|| < \epsilon, \quad t > 0;$$

(ii) asymptotically stable if stable and exists $\gamma > 0$ s.t

$$\lim_{t\to\infty} ||y(t;\ y_0) - -y^*|| = 0.$$

This can be analogously translated for a general map $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ by replacing y_0 with $y^n(y_0)$ where a fixed point is $y^* = \Psi(y^*)$.

Theorem 4.12: The Linearisation Theorem

Consider $\mathbb{R}^d$: $\frac{dy}{dt} = By + F(y)$ where $y(0) = y_0$; B is a constant matrix F is C^1 in a neighbourhood of y_0 with $F(0) = F'(0) = 0$ where F' is the Jacobian of F. If B's evals all negative real part then $y = 0$ is an asymptotically stable fixed point. If any positive, y^* unstable.

Lemma 4.13: Linearisation: The Sequel

Suppose $\frac{dy}{dt} = f(y)$ and f is C^2. If evals of $J = f'(y^*)$ (where f' is Jacobian of f) all strictly lie in the left complex half-plane then y^* is asymptotically stable. If any one in the right, y^* is unstable.

Definition 4.14: Spectral Radius

Denoted $\rho(K)$ of matrix K; "the smallest circle centered at origin enclosing all eigenvalues of K". In practice; max $||\lambda_i||$ for $\lambda_i \in \sigma(K)$.

Theorem 4.15: Linear Iteration

Consider $z_n = ||K^n y_0||$. Then

(i) $z_n \to 0$ for all y_0 iff $\rho(K) < 1$;

(ii) $z_n \to \infty$ for some y_0 iff $\rho(K) > 1$;

(iii) if $\rho(K) = 1$ and all eigenvalues on the unit circle are semisimple then $\{z_n\}$ remains bounded as $n \to \infty$.

Theorem 4.16: General (Smooth) Maps

Given a smooth (C^2) map Ψ, we consider the fixed point y^* for the iteration $y_{n+1} = \Psi(y_n)$. It

(i) is stable if $\rho(\Psi'(y^*)) < 1$,

(ii) is unstable if $\rho(\Psi'(y^*)) > 1$,

(iii) and we cannot say if we have inequality with 1.

5 Linear Stability of Numerical Methods

Remark 5.1: The Idea

We consider linearised $\frac{\mathrm{d}y}{\mathrm{d}t} = By$ where B is a matrix with a basis of eigenvectors (i.e can be written as $diag(\lambda_i) = \Lambda$.

Theorem 5.2: Stability Function of RK Methods

All Runge-Kutta methods applied to $y' = \lambda y$ can be written as $y_{n+1} = R(h\lambda)y_n$ where $R(\mu = h\lambda) = P(\mu)/Q(\mu)$, $R :=$ "stab func".

Theorem 5.3: Stability of The Origin

Consider $y' = By$ where B is $d \times d$ and has a basis of eigenvectors. Consider applying some RK method. Then the RK method has a stable (asymptotically stable) fixed point at the origin if and only if the same method has a stable (asymptotically) fixed point at the origin to $\frac{\mathrm{d}y}{\mathrm{d}t} = \lambda_i y$ for all λ_i.

Lemma 5.4: Diagonalisable with Stability Function

Consider $\frac{\mathrm{d}y}{\mathrm{d}t} = By$ with diagonalisable B. Consider an RK method with stab. func R. The origin is stable for this RK method at stepsize h if and only if $\hat{R} = |R(\mu)| < 1$ for all $u = h\lambda_i$. Unstable if > 1.

Example 5.5: Examples of Stability Functions

Recall we are requiring $\hat{R}(\mu) = |R(\mu)| < 1$ where $\mu = \lambda h$ (?).

(i) Euler's method $\hat{R}(\mu) = |1 + \mu|$; i.e a disc of radius 1 centered at the point -1.

(ii) Trapezium rule;

$$\hat{R}(\mu) = \left|\frac{1 + \mu/2}{1 - \mu/2}\right| < 1$$

i.e when $\mu/2$ is closer to -1 than 1 i.e the entire left complex half-plane (and thus is A-stable).

(iii) The implicit Euler $z_{n+1} = z_n + h\lambda z_{n+1}$ has $\hat{R}(\mu) = |1 - \mu|^{-1}$ i.e the exterior of the disk of radius 1 centered at 1 (so included entire left half-plane and is thus A-stable).

What about RK4? We have

$$R(\mu) = 1 + \mu + \frac{1}{2}\mu^2 + \frac{1}{6}\mu^3 + \frac{1}{24}\mu^4$$

and agrees with the Taylor series expansion of $\exp(h\lambda)$ through to fourth order. It is tricky to solve.

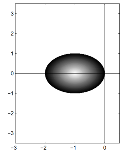

Figure 1: RK1

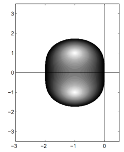

Figure 2: RK2

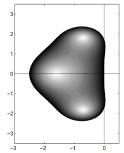

Figure 3: RK3

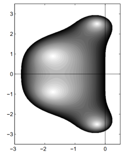

Figure 4: RK4

Definition 5.6: A-Stable Methods

If the stability region includes the entire left hand plane the method is *A-stable*. On linear systems, if the origin is stable for the ODE then it is also stable for the numerical method, regardless of stepsize.