1 Preliminaries

For population $N(t)$, and max supported pop N^*. Let $n = \frac{N}{N^*}$.

Logistic model:

$$\frac{dn}{dt} = rn(1-n) \Bigg| \text{General Solution}: n(t) = \frac{ce^{rt}}{1+ce^{rt}}.$$

Logistic model (with predation): $\frac{dn}{dt} = rn(1-\frac{n}{s}) - \frac{n^2}{1-n^2}$

Lotka-Volterra predator-prey model: n = prey population, p = predator population:

$$\frac{dn}{dt} = rn(1-p), \quad \frac{dp}{dt} = p(1-n)$$

Converting non-autonomous systems to autonomous:

For non-autonomous system of d equations:

$\frac{dy_1}{dt} = f_1(t, y_1, ...y_d)$,
$\frac{dy_2}{dt} = f_2(t, y_1, ...y_d)$,
$\vdots$
$\frac{dy_d}{dt} = f_d(t, y_1, ...y_d)$.

We can make it into an autonomous one by introducing the variable $y_{d+1} \equiv t$ in place of t, (thus creating an autonomous system of $d+1$ variables) and introduce new indep var s such that $ds/dt = 1$:

$\frac{dy_1}{ds} = f_1(t, y_1, ...y_{d+1})$,
$\frac{dy_2}{ds} = f_2(t, y_1, ...y_{d+1})$,
$\vdots$
$\frac{dy_d}{ds} = f_d(t, y_1, ...y_{d+1})$,
$\frac{dy_{d+1}}{ds} = 1$.

Picard's Thm: let the function $f(\cdot,\cdot)$ be a continuous function of its arguments in a region of the plane containing the rectangle $D = \{(t,y) : t_0 \le t \le T, |y-y_0| \le K\}$, $T, K > 0$. Suppose $\exists L$ (Lipschitz constant) such that $|f(t,u) - f(t,v)| \le L|u-v|$ whenever $(t,u),(t,v) \in D$. Since D closed & bounded, $\exists M_f > 0$ such that $M_f = \max\{|f(t,u)| : (t,u) \in D\}$. Assume $M_f(T-t_0) \le K$. Then there exists a unique continuously differentiable function $t \mapsto y(t)$, defined on $[t_0, T]$, such that $\frac{dy}{dt} = f(t,y), y(t_0) = y_0$.

Note: $\frac{\partial f}{\partial y}$ being continuous $\Rightarrow$ existence of a Lipschitz constant, We can take $L = \sup_D \left[\frac{\partial f}{\partial y}\right]$.

Thm 1.3.2 (Local Existence / Uniqueness of Solutions for ODE Systems): Suppose $f : \mathbb{R} \to \mathbb{R}^d \times \mathbb{R}^d$ is continuous and has continuous partial derivatives w.r.t. all components of the dependant var y in a neighbourhood of the point (t_0, y_0). Then there is an interval $I = (t_0 - \delta, t_0 + \delta)$ containing a unique function y (continuously differentiable on I) that satisfies $\frac{dy}{dt} = f(t,y), y(t_0) = y_0$.

2 Euler's Method and Taylor Series Methods

Euler's Method: for finding approximate solutions to $\frac{dy}{dt} = f(t,y)$ in time interval $t \in [a,b]$.

Let $t_m = a + mh$, $h \in \mathbb{R}$ small. Start with $y_0 = y(t_0) = y(a)$ and use iteration function $\boxed{y_{n+1} = y_n + hf(t_n, y_n)}$. $y_i \approx y(t_i)$.

Error: $\boxed{e_n = |y_n - y(t_n)|}$.

Lemma 2.6.1: Suppose a given sequence of non-negative numbers (v_n) satisfies $v_{n+1} \le Av_n + B$, $A > 1, B > 0$. Then for $n = 0, 1, 2, ...,$ $\boxed{v_n \le A^n v_0 + \frac{A^n - 1}{A - 1}B}$.

Bound on error: $\boxed{|e_n| \le e^{(b-a)L}\frac{M}{2L}h = Dh}$ where h is the timestep, M bounds $|y''(t)|$, and L bounds $\frac{\partial y}{\partial t}$.

Thm 2.6.1 Consider the IVP $\frac{dy}{dt} = f(t,y), y(a) = y_0$. Suppose $\exists$ unique, twice-differentiable function f, continuous everywhere with continuous, bounded partial derivative $\left|\frac{\partial f}{\partial y}\right| < L$ with $L > 0$.Then for $n = 0, 1, ..., N$, and some $D > 0$, the solution y_n given by Euler's method at t_n satisfies $\boxed{e_n = |y_n - y(t_n)| \le Dh}$ where $h = (b-a)/N, t_n = a + hn$.

Big O (Landau) notation: If method is $O(h^p)$ then the error decays at least as quickly as h^p (for small h).

$\boxed{(\exists\, h_0, C > 0) : (\forall\, 0 < h < h_0), |z| \le Ch^p \Rightarrow z = O(h^p)}$

Say z is of order p.

The Flow Map: Consider IVP $\frac{dy}{dt} = f(t,y), y(a) = y_0$ with unique solution in $t \in [a,b]$. Starting at arbitrary $t_0 \in [a,b], y_0$ we may see where $y(t)$ ends up, denoted $y(t; t_0, y_0)$.

Fix t_0 and look at the **flow map** $\boxed{\Phi_{t_0,h}(y_0) = y(t_0 + h; t_0, y_0)}$.

(*actually a family of maps parameterised by* h).

Numerical methods approximate flow maps: Euler's method approximates flow map with $\hat{\Phi}_{t,h}(y) = y + hf(t,y)$.

One-step methods: approximate the solution through the iteration of an approximated flow map.

Constructing Taylor series methods:

Start with Taylor series:

$y(t_0 + h) = y(t_0) + y'(t_0)h + \frac{1}{2}y''(t_0)h^2 + \frac{1}{6}y'''(t_0)h^3 + ...$

Also $y'(t) = f(t,y) \Rightarrow$

$y'' = \frac{d}{dt}f(t,y) = \frac{\partial}{\partial t}f(t,y)\frac{dt}{dt} + \frac{\partial}{\partial y}f(t,y)\frac{dy}{dt}$

$= \frac{\partial}{\partial t}f(t,y) + \frac{\partial}{\partial y}f(t,y)y' = f_t + f_y f$. (By chain rule)

$\Phi_{t,h}(y) = y + hf(t,y) + \frac{1}{2}h^2(f_t(t,y) + f_y(t,y)f(t,y)) + \frac{1}{6}y'''h^3 + ...$

Which we can truncate to get the 2nd order Taylor series method

$\boxed{\hat{\Phi}_{t,h}(y) = y + hf(t,y) + \frac{1}{2}h^2(f_t(t,y) + f_y(t,y)f(t,y))}$.

3 Convergence of One-Step Methods

Def 3.1.2 (Convergence): A method is said to be **convergent** iff for any T,

$$\lim_{\substack{h \to 0 \\ h = T/N}} \max_{n=0,1,...,N} \|e_n\| = 0.$$

Def 3.2.1 (Local Error): The **local error** of a one-step method is the difference between the flow map Φ_h and it's descrete approximation Ψ_h

$$le(y,h) = \Psi_h(y) - \Phi_h(y).$$

It measures how much error is introduced in a single timestep of size h.

Def 3.2.2 (Consistency): Suppose the local error for our method satisfies

$$\|le(y,h)\| \le Ch^{p+1}$$

where C is a constant that depends on $y(t)$ and it's derivatives, and $p \ge 1$. Then the method is **consistent** at order p.

Def 3.2.3 (Stability): Suppose that a method satisfies an $h-$independant Lipschitz condition on D (spatial domain)

$$\|\Psi_h(u) - \Psi_h(v)\| \le (1 + h\hat{L})\|u - v\| \quad \forall\, u, v \in D.$$

Then the method is **stable**. *Note* $\hat{L}$ *need not be the same Lipschitz constant as for the vector field.*

Thm 3.2.1 (Convergence of One-Step Methods): Given a differential equation and a one-step method Ψ_h which is **consistent** and **stable**. Then the method is **convergent**.

Interpolating Polynomials: Given distinct *abscissa points* $c_1, ...c_s$ and *data points* $g_1, ..., g_s$, there exists a unique interpolating polynomial $P(x) \in \mathbb{P}_{s-1}$ passing through all points (c_i, g_i).

Lagrange Polynomials