Numerical Differential Equations

Template from Leon Lee @leon0241 on GitHub.

1 Preliminaries

Definition 1.1: Studying Differential Equations

Analytically - to find a solution in terms of y; **qualitatively** - study the properties based on analytical and/or graphical techniques; **computationally** - find an approximation solution numerically.

Recall 1.2: ODE Terminology

Ordinary has one independent variable. **Autonomous** does not depend on the independent variable. Linear if in form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a(t)y + b(t)$$

where $a(t)$ and $b(t)$ do not need to be linear in t.

Example 1.3: Examples of (Non)-Linear Equations

$$(1)\ \frac{\mathrm{d}y}{\mathrm{d}x} = kx^2y - cx^3, \quad (2)\ x' = e^{-t\beta}x, \quad (3)\ \dot{u} = \cos(\omega t)u - \sin(\omega t).$$

All the above are linear equations with y, x and u being the dependent variables and parameters k, c, β, ω. Below is a *linear system*

$$\dot{x} = u, \quad \dot{u} = -\omega^2 x \qquad \text{for an unspecified independent var.}$$

The below are *non-linear* equations

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u^2, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = kx^2t - cx^3$$

or (Duffing's oscillator) a *non-autonomous, non-linear* system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u, \quad \frac{\mathrm{d}u}{\mathrm{d}t} = -\beta x - \alpha x^3 - \delta u - \gamma\cos(\omega t).$$

Recall 1.4: Lipschitz Continuity

A function f is Lipschitz continuous if there exists some constant $L > 0$ (the Lipschitz constant) such that

$$|f(t,u) - f(t,v)| \le L|u - v|, \qquad \text{for all } t \text{ and } u, v \in D.$$

Remark 1.5: Lipschitz Continuity: Easier

If f is continuous in t and continuously differentiable (that is, the derivative exists and is itself continuous) in y then

$$\frac{\partial f}{\partial y} \text{ is continuous} \implies L = \sup_D \left|\frac{\partial f}{\partial y}\right|.$$

Proof; by the mean value theorem. MVT states that if f is continuous and differentiable over some $[a, b]$ then there exists c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \implies f(b) - f(a) = f'(c)(b - a).$$

It immediately follows how we can find L.

Theorem 1.6: Picard's Theorem (Existence/Uniqueness)

Let f be a cts func of its arguments in a region of the plane containing the rectangle

$$D = \{(t, y) \mid t_0 \le t \le T,\ |y - y_0| \le K\}$$

where T, K are some constants. Further suppose f is Lipschitz cts. Domain D is closed and bounded so there exists some

$$M_f = \max\{|f(t, u)| : (t, u) \in D.$$

If $M_f(T - t_0) \le K$ then there exists a unique, continuously differentiable function $y(t)$ defined on $[t_0, T]$ such that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \quad y(t_0) = y_0.$$

Note. This can be condensed and/or rewritten, do it.

Theorem 1.7: Local Existence and Uniqueness

Suppose $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is cts and has cts partial derivatives in w.r.t all variables in some neighbourhood of (t_0, y_0). Then there some interval $I = (t_0 - \delta, t_0 + \delta)$ such that some y satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \quad y(t_0) = y_0.$$

Recall 1.8: Slope Fields

Consider the ODE $y' = f(t, y)$. Any point $f(t_0, y_0)$ gives the slope of y; do this for all points of the plane and draw, typically with arrow.

Recall 1.9: Taylor Series Expansion

The standard definition of a Taylor series for some function f centered about a is given

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Of course this requires f to be infinitely differentiable. If f is a most $N + 1$ times differentiable on some interval $[a, b]$, we have the Taylor remainder theorem (remark $x_0 \in (a, b)$ and τ between x_0 and x)

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(N+1)}(\tau)}{(N+1)!}(x - x_0)^{N+1}.$$

2 Euler's Method and Taylor Series Methods

Definition 2.1: Euler's Method

$$y_{n+1} = y_n + hf(t_n, y_n), \quad \text{where } y_i \approx y(t_i).$$

The algorithm takes in h, f, (t_0, y_0) and N. Remark N steps gives $N+1$ time points $t_0, t_1, \dots, t_N = t_0 + Nh$. Remark $h = (b-a)/N$.

Example 2.2: From Where?

Euler's method is exactly a *first order* Taylor series method. Expand the Taylor series (with remainder, first order) about t_0 at t_1,

$$y(t_1) = y(t_0) + y'(t_0)(t_1 - t_0) + \frac{1}{2}y''(\tau)(t_1 - t_0)^2.$$

where τ is between t_1 and t_0. We know nada of τ, but we approx the sol by last time die ($h^2 \to 0$). With $t_1 = t_0 + h$ we are done.

Lemma 2.3: An Inequality

Sequence (v_n) of non-neg, for $A > 1$ and $B > 0$ constants.

$$v_{n+1} \leq Av_n + B \implies v_n \leq A^n v_0 + \frac{A^n - 1}{A - 1}B.$$

Theorem 2.4: Convergence of Euler's Method

Consider $y'(t) = f(t, y)$ with $y(a) = y_0$. Suppose unique, twice diff. sol $y(t)$ on $[a, b]$. Further suppose f is cts everywhere with cts, bounded partial derivatives

$$\left|\frac{\partial f}{\partial y}\right| \leq L, \quad L > 0.$$

Then for some constant D, solution y_n at t_n satisfies

$$|e_n| = |y_n - y(t_n)| \leq Dh \quad \text{where } D = e^{(b-a)L}\frac{M}{2L} : |y''(t)| \leq M.$$

It is stressed that D does not depend on N or h, but only on features of the problem and the solution. If h is not small enough, the error compounds and can become very large (recall examples).

Definition 2.5: The Big O (Landau) Notation

A quantity is (of order) $\mathcal{O}(h^p)$ if it decays at least as quickly as h^p when h is small enough. More formally, we write $z = \mathcal{O}(h^p)$ if there exist $h_0, C \in \mathbb{R}_+$ such that $|z| \leq Ch^p$ for all $0 < h \leq h_0$.

We are concerned with p. Say ... is of order p. Conventionally

$$z = \mathcal{O}(h^{p+1}) \implies z = \mathcal{O}(h^p).$$

Definition 2.6: Flow Maps

Idea "we look at a set of initial point and describe the journey to their endpoints".

$$\Phi_{t_0,h}(y_0) = y(t_0 + h;\ t_0, y_0)$$

that we read "given t_0, h". Determining the flow map is equivalent to solving the ODE; idea being to replace the map by an approx. i.e

$$\dot{\tilde{\Phi}}_{t,h}(y) = y + hf(t, y) \qquad \text{is Euler's method as a flow map approx.}$$

We have, from outside the course (at this moment),

$$\text{linear ODE } y' = Ay \implies \Phi_t(y) = e^{tA}y.$$

Recall 2.7: How to Differentiate

Suppose $z = f(x, y)$ and $x = g(t)$, $y = h(t)$. Then, from SVCDE,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

So if $y' = f(t, y)$ then

$$y'' = \frac{d(f(t,y))}{dt} = \frac{\partial f}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = f_t + f_y \cdot f.$$

Example 2.8: From Where? Remastered

(Second order Taylor series method.) Recall

$$y(t_0 + h) = y(t_0) = y'(t_0)h + \frac{1}{2}y''(t_0)h^2 + \frac{1}{6}y'''(t_0)h^3 + \mathcal{H.O.T}.$$

Then a *second order method* instead drops the y''' term (consider Taylor remainder if wanted) and we know $y' = f$ and $y'' = f_t + f_y \cdot f$. So, by considering a typical ODE, we have

$$y_{n+1} \approx y_n + hf(t_n, y_n) + \frac{1}{2}h^2(f_t(t_n, y_n) + f_y(t_n, y_n) \cdot f(t_n, y_n)).$$

Higher order Taylor series methods can of course be used. General case is a pain to express, but in practice is it simple (implicit diff).