

1 Preliminaries

For population $N(t)$, and max supported pop N^* . Let $n = \frac{N}{N^*}$.

Logistic model:

$$\frac{dn}{dt} = rn(1-n) \quad \left| \begin{array}{l} \text{General} \\ \text{Solution} \end{array} \right| : n(t) = \frac{ce^{rt}}{1+ce^{rt}}.$$

Logistic model (with predation): $\frac{dn}{dt} = rn(1 - \frac{n}{s}) - \frac{n^2}{1-n^2}$

Lotka-Volterra predator-prey model: n = prey population, p = predator population:

$$\frac{dn}{dt} = rn(1-p), \quad \frac{dp}{dt} = p(1-n)$$

Converting non-autonomous systems to autonomous:

For non-autonomous system of d equations:

$$\frac{dy_1}{dt} = f_1(t, y_1, \dots, y_d),$$

$$\frac{dy_2}{dt} = f_2(t, y_1, \dots, y_d),$$

\vdots

$$\frac{dy_d}{dt} = f_d(t, y_1, \dots, y_d).$$

We can make it into an autonomous one by introducing the variable $y_{d+1} \equiv t$ in place of t , (thus creating an autonomous system of $d+1$ variables) and introduce new indep var s such that $ds/dt = 1$:

$$\frac{dy_1}{ds} = f_1(t, y_1, \dots, y_{d+1}),$$

$$\frac{dy_2}{ds} = f_2(t, y_1, \dots, y_{d+1}),$$

\vdots

$$\frac{dy_d}{ds} = f_d(t, y_1, \dots, y_{d+1}),$$

$$\frac{dy_{d+1}}{ds} = 1.$$

Picard's Thm: let the function $f(\cdot, \cdot)$ be a continuous function of its arguments in a region of the plane containing the rectangle $D = \{(t, y) : t_0 \leq t \leq T, |y - y_0| \leq K\}$, $T, K > 0$. Suppose $\exists L$ (Lipschitz constant) such that $|f(t, u) - f(t, v)| \leq L|u - v|$ whenever $(t, u), (t, v) \in D$. Since D closed & bounded, $\exists M_f > 0$ such that $M_f = \max\{|f(t, u)| : (t, u) \in D\}$. Assume $M_f(T - t_0) \leq K$. Then there exists a unique continuously differentiable function $t \mapsto y(t)$, defined on $[t_0, T]$, such that $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$.

Note: $\frac{\partial f}{\partial y}$ being continuous \Rightarrow existence of a Lipschitz constant,

We can take $L = \sup_D \left[\frac{\partial f}{\partial y} \right]$.

Thm 1.3.2 (Local Existence / Uniqueness of Solutions for ODE Systems): Suppose $f : \mathbb{R} \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ is continuous and has continuous partial derivatives w.r.t. all components of the dependant var y in a neighbourhood of the point (t_0, y_0) . Then there is an interval $I = (t_0 - \delta, t_0 + \delta)$ containing a unique function y (continuously differentiable on I) that satisfies $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$.

2 Euler's Method and Taylor Series Methods

Euler's Method: for finding approximate solutions to $\frac{dy}{dt} = f(t, y)$ in time interval $t \in [a, b]$.

Let $t_m = a + mh$, $h \in \mathbb{R}$ small. Start with $y_0 = y(t_0) = y(a)$ and use iteration function $y_{n+1} = y_n + hf(t_n, y_n)$. $y_i \approx y(t_i)$.

Error: $e_n = |y_n - y(t_n)|$.

Lemma 2.6.1: Suppose a given sequence of non-negative numbers (v_n) satisfies $v_{n+1} \leq Av_n + B$, $A > 1, B > 0$. Then for

$$n = 0, 1, 2, \dots, \quad v_n \leq A^n v_0 + \frac{A^n - 1}{A - 1} B.$$

Bound on error: $|e_n| \leq e^{(b-a)L} \frac{M}{2L} h = Dh$ where h is the timestep, M bounds $|y''(t)|$, and L bounds $\frac{\partial y}{\partial t}$.

Thm 2.6.1 Consider the IVP $\frac{dy}{dt} = f(t, y), y(a) = y_0$. Suppose \exists unique, twice-differentiable function f , continuous everywhere with continuous, bounded partial derivative $\left| \frac{\partial f}{\partial y} \right| < L$ with $L > 0$. Then for $n = 0, 1, \dots, N$, and some $D > 0$, the solution y_n given by Euler's method at t_n satisfies $e_n = |y_n - y(t_n)| \leq Dh$ where $h = (b - a)/N, t_n = a + hn$.

Big O (Landau) notation: If method is $O(h^p)$ then the error decays at least as quickly as h^p (for small h).

$$(\exists h_0, C > 0) : (\forall 0 < h < h_0), |z| \leq Ch^p \Rightarrow z = O(h^p)$$

Say z is of order p .

The Flow Map: Consider IVP $\frac{dy}{dt} = f(t, y), y(a) = y_0$ with unique solution in $t \in [a, b]$. Starting at arbitrary $t_0 \in [a, b], y_0$ we may see where $y(t)$ ends up, denoted $y(t; t_0, y_0)$.

Fix t_0 and look at the **flow map** $\Phi_{t_0, h}(y_0) = y(t_0 + h; t_0, y_0)$. (actually a family of maps parameterised by h).

Numerical methods approximate flow maps: Euler's method approximates flow map with $\hat{\Phi}_{t, h}(y) = y + hf(t, y)$.

One-step methods: approximate the solution through the iteration of an approximated flow map.

Constructing Taylor series methods:

Start with Taylor series:

$$y(t_0 + h) = y(t_0) + y'(t_0)h + \frac{1}{2}y''(t_0)h^2 + \frac{1}{6}y'''(t_0)h^3 + \dots$$

Also $y'(t) = f(t, y) \Rightarrow$

$$y'' = \frac{d}{dt} f(t, y) = \frac{\partial}{\partial t} f(t, y) \frac{dt}{dt} + \frac{\partial}{\partial y} f(t, y) \frac{dy}{dt}$$

$$= \frac{\partial}{\partial t} f(t, y) + \frac{\partial}{\partial y} f(t, y) y' = f_t + f_y f. \quad (\text{By chain rule})$$

$$\Phi_{t, h}(y) = y + hf(t, y) + \frac{1}{2}h^2(f_t(t, y) + f_y(t, y)f(t, y)) + \frac{1}{6}y'''h^3 + \dots$$

Which we can truncate to get the 2nd order Taylor series method

$$\hat{\Phi}_{t, h}(y) = y + hf(t, y) + \frac{1}{2}h^2(f_t(t, y) + f_y(t, y)f(t, y)).$$

3 Convergence of One-Step Methods

Def 3.1.2 (Convergence): A method is said to be **convergent** iff for any T ,

$$\lim_{h \rightarrow 0} \max_{n=0, 1, \dots, N} \|e_n\| = 0.$$

Def 3.2.1 (Local Error): The **local error** of a one-step method is the difference between the flow map Φ_h and it's discrete approximation Ψ_h

$$le(y, h) = \Psi_h(y) - \Phi_h(y).$$

It measures how much error is introduced in a single timestep of size h .

Def 3.2.2 (Consistency): Suppose the local error for our method satisfies

$$\|le(y, h)\| \leq Ch^{p+1}$$

where C is a constant that depends on $y(t)$ and it's derivatives, and $p \geq 1$. Then the method is **consistent** at order p .

Def 3.2.3 (Stability): Suppose that a method satisfies an h -independent Lipschitz condition on D (spatial domain)

$$\|\Psi_h(u) - \Psi_h(v)\| \leq (1 + h\hat{L})\|u - v\| \quad \forall u, v \in D.$$

Then the method is **stable**. *Note \hat{L} need not be the same Lipschitz constant as for the vector field.*

Thm 3.2.1 (Convergence of One-Step Methods): Given a differential equation and a one-step method Ψ_h which is **consistent** and **stable**. Then the method is **convergent**.

Interpolating Polynomials: Given s distinct *abscissa points* c_0, \dots, c_s and *data points* g_0, \dots, g_s , there exists a unique interpolating polynomial $P(x) \in \mathbb{P}_{s-1}$ passing through all points (c_i, g_i) .

Lagrange Polynomials: For a set of abscissae c_0, \dots, c_s , the Lagrange polynomials ℓ_i , $i = 1, \dots, s$ are defined by

$$\ell_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^s \frac{x - c_j}{c_i - c_j}.$$

The Lagrange polynomial ℓ_i is the interpolating polynomial through the data $g_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$. $\{\ell_i\}$ form a basis for \mathbb{P}_{s-1} ,

and any polynomial $Q(x)$ has the simple form

$$Q(x) = \sum_{i=1}^s Q(c_i) \ell_i(x) = \sum_{i=1}^s g_i \ell_i(x).$$

Numerical Quadrature: Given a smooth function $g(x) : \mathbb{R} \rightarrow \mathbb{R}$, and s **quadrature points** $0 \leq c_1 < \dots < c_s \leq 1$ we can estimate $\int_0^1 g(x) dx$ by integrating the corresponding interpolating polynomial $P(x) \in \mathbb{P}_{s-1}$. Define the weights

$$b_i = \int_0^1 \ell_i(x) dx.$$

Then our approximate integral is

$$\int_0^1 g(x) dx \approx \int P(x) dx = \sum_{i=1}^s g(c_i) \int_0^1 \ell_i(x) dx = \sum_{i=1}^s b_i g(c_i)$$

Therefore for interval $[t_0, t_0 + h]$ we have

$$\int_{t_0}^{t_0+h} g(x) dx \approx \int_{t_0}^{t_0+h} P(x) dx = \sum_{i=1}^s b_i g(t_0 + hc_i).$$

A quadrature rule has **order** p if it integrates any polynomial $\in \mathbb{P}_{p-1}$ exactly. We always have $p \geq s$, and for optimal choice of c_i we have $p = 2s$.