

## 1 Preliminaries

For population  $N(t)$ , and max supported pop  $N^*$ . Let  $n = \frac{N}{N^*}$ .

**Logistic model:**

$$\frac{dn}{dt} = rn(1-n) \quad \left| \begin{array}{l} \text{General} \\ \text{Solution} \end{array} \right. : n(t) = \frac{ce^{rt}}{1+ce^{rt}}.$$

**Logistic model (with predation):**  $\frac{dn}{dt} = rn(1 - \frac{n}{s}) - \frac{n^2}{1-n^2}$

**Lotka-Volterra predator-prey model:**  $n$  = prey population,  $p$  = predator population:

$$\frac{dn}{dt} = rn(1-p), \quad \frac{dp}{dt} = p(1-n)$$

**Converting non-autonomous systems to autonomous:**

For non-autonomous system of  $d$  equations:

$$\begin{aligned} \frac{dy_1}{dt} &= f_1(t, y_1, \dots, y_d), \\ \frac{dy_2}{dt} &= f_2(t, y_1, \dots, y_d), \\ &\vdots \end{aligned}$$

$$\frac{dy_d}{dt} = f_d(t, y_1, \dots, y_d).$$

We can make it into an autonomous one by introducing the variable  $y_{d+1} \equiv t$  in place of  $t$ , (thus creating an autonomous system of  $d+1$  variables) and introduce new indep var  $s$  such that  $ds/dt = 1$ :

$$\begin{aligned} \frac{dy_1}{ds} &= f_1(t, y_1, \dots, y_{d+1}), \\ \frac{dy_2}{ds} &= f_2(t, y_1, \dots, y_{d+1}), \\ &\vdots \end{aligned}$$

$$\begin{aligned} \frac{dy_d}{ds} &= f_d(t, y_1, \dots, y_{d+1}), \\ \frac{dy_{d+1}}{ds} &= 1. \end{aligned}$$

**Picard's Thm:** let the function  $f(\cdot, \cdot)$  be a continuous function of its arguments in a region of the plane containing the rectangle  $D = \{(t, y) : t_0 \leq t \leq T, |y - y_0| \leq K\}$ ,  $T, K > 0$ . Suppose  $\exists L$  (Lipschitz constant) such that  $|f(t, u) - f(t, v)| \leq L|u - v|$  whenever  $(t, u), (t, v) \in D$ . Since  $D$  closed & bounded,  $\exists M_f > 0$  such that  $M_f = \max\{|f(t, u)| : (t, u) \in D\}$ . Assume  $M_f(T - t_0) \leq K$ . Then there exists a unique continuously differentiable function  $t \mapsto y(t)$ , defined on  $[t_0, T]$ , such that  $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$ .

*Note:*  $\frac{\partial f}{\partial y}$  being continuous  $\Rightarrow$  existence of a Lipschitz constant,

We can take  $L = \sup_D \left[ \frac{\partial f}{\partial y} \right]$ .

**Thm 1.3.2 (Local Existence / Uniqueness of Solutions for ODE Systems):** Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  is continuous and has continuous partial derivatives w.r.t. all components of the dependant var  $y$  in a neighbourhood of the point  $(t_0, y_0)$ . Then there is an interval  $I = (t_0 - \delta, t_0 + \delta)$  containing a unique function  $y$  (continuously differentiable on  $I$ ) that satisfies  $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$ .

## 2 Euler's Method and Taylor Series Methods

**Euler's Method:** for finding approximate solutions to  $\frac{dy}{dt} = f(t, y)$  in time interval  $t \in [a, b]$ .

Let  $t_m = a + mh$ ,  $h \in \mathbb{R}$  small. Start with  $y_0 = y(t_0) = y(a)$  and use iteration function  $y_{n+1} = y_n + hf(t_n, y_n)$ .  $y_i \approx y(t_i)$ .

**Error:**  $e_n = |y_n - y(t_n)|$ .

**Lemma 2.6.1:** Suppose a given sequence of non-negative numbers  $(v_n)$  satisfies  $v_{n+1} \leq Av_n + B$ ,  $A > 1, B > 0$ . Then for

$$n = 0, 1, 2, \dots, \quad v_n \leq A^n v_0 + \frac{A^n - 1}{A - 1} B.$$

**Bound on error:**  $|e_n| \leq e^{(b-a)L} \frac{M}{2L} h = Dh$  where  $h$  is the timestep,  $M$  bounds  $|y''(t)|$ , and  $L$  bounds  $\frac{\partial y}{\partial t}$ .

**Thm 2.6.1** Consider the IVP  $\frac{dy}{dt} = f(t, y), y(a) = y_0$ . Suppose  $\exists$  unique, twice-differentiable function  $f$ , continuous everywhere with continuous, bounded partial derivative  $\left| \frac{\partial f}{\partial y} \right| < L$  with  $L > 0$ . Then for  $n = 0, 1, \dots, N$ , and some  $D > 0$ , the solution  $y_n$  given by Euler's method at  $t_n$  satisfies  $e_n = |y_n - y(t_n)| \leq Dh$  where  $h = (b - a)/N, t_n = a + hn$ .

**Big O (Landau) notation:** If method is  $O(h^p)$  then the error decays at least as quickly as  $h^p$  (for small  $h$ ).

$$(\exists h_0, C > 0) : (\forall 0 < h < h_0), |z| \leq Ch^p \Rightarrow z = O(h^p)$$

Say  $z$  is of order  $p$ .

**The Flow Map:** Consider IVP  $\frac{dy}{dt} = f(t, y), y(a) = y_0$  with unique solution in  $t \in [a, b]$ . Starting at arbitrary  $t_0 \in [a, b], y_0$  we may see where  $y(t)$  ends up, denoted  $y(t; t_0, y_0)$ .

Fix  $t_0$  and look at the **flow map**  $\Phi_{t_0, h}(y_0) = y(t_0 + h; t_0, y_0)$ . (actually a family of maps parameterised by  $h$ ).

**Numerical methods approximate flow maps:** Euler's method approximates flow map with  $\hat{\Phi}_{t, h}(y) = y + hf(t, y)$ .

**One-step methods:** approximate the solution through the iteration of an approximated flow map.

**Constructing Taylor series methods:**

Start with Taylor series:

$$y(t_0 + h) = y(t_0) + y'(t_0)h + \frac{1}{2}y''(t_0)h^2 + \frac{1}{6}y'''(t_0)h^3 + \dots$$

Also  $y'(t) = f(t, y) \Rightarrow$

$$y'' = \frac{d}{dt} f(t, y) = \frac{\partial}{\partial t} f(t, y) \frac{dt}{dt} + \frac{\partial}{\partial y} f(t, y) \frac{dy}{dt} = \frac{\partial}{\partial t} f(t, y) + \frac{\partial}{\partial y} f(t, y) y' = f_t + f_y f. \quad (\text{By chain rule})$$

$$\Phi_{t, h}(y) = y + hf(t, y) + \frac{1}{2}h^2(f_t(t, y) + f_y(t, y)f(t, y)) + \frac{1}{6}y'''h^3 + \dots$$

Which we can truncate to get the 2nd order Taylor series method

$$\hat{\Phi}_{t, h}(y) = y + hf(t, y) + \frac{1}{2}h^2(f_t(t, y) + f_y(t, y)f(t, y)).$$

## 3 Convergence of One-Step Methods

**Def 3.1.2 (Convergence):** A method is said to be **convergent** iff for any  $T$ ,

$$\lim_{h \rightarrow 0} \max_{n=0, 1, \dots, N} \|e_n\| = 0.$$

**Def 3.2.1 (Local Error):** The **local error** of a one-step method is the difference between the flow map  $\Phi_h$  and it's discrete approximation  $\Psi_h$

$$le(y, h) = \Psi_h(y) - \Phi_h(y).$$

It measures how much error is introduced in a single timestep of size  $h$ .

**Def 3.2.2 (Consistency):** Suppose the local error for our method satisfies

$$\|le(y, h)\| \leq Ch^{p+1}$$

where  $C$  is a constant that depends on  $y(t)$  and it's derivatives, and  $p \geq 1$ . Then the method is **consistent** at order  $p$ .

**Def 3.2.3 (Stability):** Suppose that a method satisfies an  $h$ -independent Lipschitz condition on  $D$  (spatial domain)

$$\|\Psi_h(u) - \Psi_h(v)\| \leq (1 + h\hat{L})\|u - v\| \quad \forall u, v \in D.$$

Then the method is **stable**. *Note  $\hat{L}$  need not be the same Lipschitz constant as for the vector field.*

**Thm 3.2.1 (Convergence of One-Step Methods):** Given a differential equation and a one-step method  $\Psi_h$  which is **consistent** and **stable**. Then the method is **convergent**.

**Interpolating Polynomials:** Given  $s$  distinct *abscissa points*  $c_0, \dots, c_s$  and *data points*  $g_0, \dots, g_s$ , there exists a unique interpolating polynomial  $P(x) \in \mathbb{P}_{s-1}$  passing through all points  $(c_i, g_i)$ .

**Lagrange Polynomials:** For a set of abscissae  $c_0, \dots, c_s$ , the Lagrange polynomials  $\ell_i$ ,  $i = 1, \dots, s$  are defined by

$$\ell_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^s \frac{x - c_j}{c_i - c_j}.$$

The Lagrange polynomial  $\ell_i$  is the interpolating polynomial through the data  $g_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$ .  $\{\ell_i\}$  form a basis for  $\mathbb{P}_{s-1}$ , and any polynomial  $Q(x)$  has the simple form

$$Q(x) = \sum_{i=1}^s Q(c_i) \ell_i(x) = \sum_{i=1}^s g_i \ell_i(x).$$

**Numerical Quadrature:** Given a smooth function  $g(x) : \mathbb{R} \rightarrow \mathbb{R}$ , and  $s$  **quadrature points**  $0 \leq c_1 < \dots < c_s \leq 1$  we can estimate  $\int_0^1 g(x) dx$  by integrating the corresponding interpolating polynomial  $P(x) \in \mathbb{P}_{s-1}$ . Define the weights

$$b_i = \int_0^1 \ell_i(x) dx.$$

Then our approximate integral is

$$\int_0^1 g(x) dx \approx \int P(x) dx = \sum_{i=1}^s g(c_i) \int_0^1 \ell_i(x) dx = \sum_{i=1}^s b_i g(c_i)$$

Therefore for interval  $[t_0, t_0 + h]$  we have

$$\int_{t_0}^{t_0+h} g(x) dx \approx \int_{t_0}^{t_0+h} P(x) dx = \sum_{i=1}^s b_i g(t_0 + hc_i).$$

A quadrature rule has **order**  $p$  if it integrates any polynomial  $\in \mathbb{P}_{p-1}$  exactly. We always have  $p \geq s$ , and for optimal choice of  $c_i$  we have  $p = 2s$ .

**One-Step Collocation:** Given an ODE we wish to construct the **collocation polynomial**  $u(t) \in \mathbb{R}^d$  that satisfies

$$u(t_0) = y_0 u'(t_0 + c_i h) = f(u(t_0 + c_i h)).$$

In particular, it agrees with our solut at  $t_0$ , and it's derivative matches that of the solution at each  $c_1, \dots, c_s$ . We can use such a polynomial to approximate a numerical solution to our ODE by decomposing it's derivative  $u'$  into Lagrange polynomial components, and then integrating over  $[t_0, t_0 + h]$  to get  $u(t_0 + h) = u(t_1)$ .

Let  $F_i = u'(t_0 + hc_i)$  be the value of the derivative of the polynomial at node  $c_i$ . Then

$$F_i = f(y_0 + h \sum_{j=1}^s a_{ij} F_j), \quad (\text{A})$$

$$y_{n+1} = y_n + h \sum_{i=1}^s b_i F_i. \quad (\text{B})$$

where

$$a_{ij} = \int_0^{c_i} \ell_j(x) dx,$$

$$b_i = \int_0^1 \ell_i(x) dx.$$

First solve the  $sd$ -dimensional system of non-linear equations given by (A), and plug into (B).

**Rem 3.6.1 (Continuous Approximations):** Collocation provides a continuous approximation  $u(t)$  of the solution  $y(t)$

on each interval  $[t_n, t_{n+1}]$ .

**Rem 3.6.2 (Optimal Node Placement):** For optimal order of accuracy, use **Gauss-Legendre** collocation methods. This means placing nodes at roots of shifted Legendre polynomials.

For  $s = 1, 2, 3$ , the optimal nodes are

$$\begin{aligned} c_1 &= \frac{1}{2}, \quad p = 2, \\ c_1 &= \frac{1}{2} - \frac{\sqrt{3}}{6}, \quad c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}, \quad p = 4, \\ c_1 &= \frac{1}{2} - \frac{\sqrt{15}}{10}, \quad c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{2} + \frac{\sqrt{15}}{10}, \quad p = 6. \end{aligned}$$

**Runge-Kutta Methods (Autonomous Case):** Generalisation of collocation methods, since they don't have coefficients that rely on integrals of Lagrange polynomials, they can have any coefficients. A **Runge-Kutta** method is any method of form

$$Y_i = y_n + h \sum_{j=1}^s a_{ij} f(Y_j), \quad i = 1, \dots, s,$$

$$y_{n+1} = y_n + h \sum_{i=1}^s b_i f(Y_i).$$

where  $s$  is the **number of stages**,  $b_i$  are the **weights**, and  $a_{ij}$  are the **internal coefficients**. Such a method generates the discrete flow-map

$$\Psi_h(y) = y + h \sum_{i=1}^s b_i f(Y_i(y, h)).$$

**Butcher Tables:** Store coefficients of Runge-Kutta methods in form

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

where  $c = (c_i)$ ,  $b = (b_j)$ ,  $A = (a_{ij})$ . If  $A$  is lower triangular (with 0s on diag) then the method is **explicit**, otherwise **implicit**.

**Butcher Table Examples:**

Euler Method (Explicit):

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}.$$

Trapezoidal (Implicit):

$$\begin{array}{c|cc} 0 & & \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}.$$

**Order Conditions for RK Methods:**

Consider an RK method with  $b = (b_j)$ ,  $A = (a_{ij})$ . For the order to be  $p$ , the following conditions must be satisfied (as well as any conditions for it to be  $< p$ ):

$$p = 1 \Rightarrow \sum_{i=1}^s b_i = 1,$$

$$p = 2 \Rightarrow \sum_{i=1}^s b_i c_i = \frac{1}{2}, \quad \text{That is } b^T c = \frac{1}{2},$$

$$p = 3 \Rightarrow \sum_{i=1}^s b_i c_i^2 = \frac{1}{3} \text{ And } \sum_{i=1}^s \sum_{j=1}^s b_i a_{ij} c_j = \frac{1}{6}.$$

There is no explicit RK method of order greater than 4. For best order, use Gauss-Legendre methods, which have order  $p = 2s$ .

**Def 4.1.1 (Fixed Point):** A point  $y^*$  is a fixed point of  $f(y)$  if  $f(y^*) = 0$ . Solution passing through a fixed point will be

$y(t) \equiv y^*$ , constant in time. Denote set of all fixed points of an ode system as  $\mathcal{F} = \{y \in \mathbb{R}^d : \Phi(y) = y\}$

**Def 4.2.1 (Fixed Point of Numerical Method):** Consider a one-step numerical method with described by the map  $\Psi_h(y)$ . Then a point  $y^*$  is a **fixed point** if  $\Psi_h(y^*) = y^*$ , and therefore produces the constant in time approximate solution  $y_n \equiv y^*$ . Denote the set of all fixed points of  $\Psi_h$  by  $\mathcal{F}_h = \{y \in \mathbb{R}^d : \Psi_h(y) = y\}$ .

For the Euler method  $\mathcal{F} = \mathcal{F}_h$ .

Fixed points in  $\mathcal{F}_h$  that are not fixed points of  $\mathcal{F}$  are called **spurious fixed points**.

**Thm 4.2.1:** For Runge-Kutta methods,  $\mathcal{F}_h \supseteq \mathcal{F}$ . Generally spurious fixed points of RK methods move depending on  $h$ , and usually tend to  $\infty$ .

**Def 4.3.1 (Stability and Asymptotic Stability):**  $y^*$  is: **stable** for the given ODE if  $\forall \varepsilon > 0$  (sufficiently small)  $\exists \delta > 0$  such that  $\forall t > 0$

$$\|y_0 - y^*\| < \delta \Rightarrow \|y(t; y_0) - y^*\| < \varepsilon;$$

**asymptotically stable** if it is **stable** and  $\exists \gamma > 0$  such that for any initial condition such that  $\|y - 0\| < \gamma$

$$\lim_{t \rightarrow \infty} \|y(t; y_0) - y^*\| = 0;$$

**unstable** if it is not **stable**.

**Thm 4.3.1 (Linearisation Thm):** Consider the equation in  $\mathbb{R}^d$

$$\frac{dy}{dt} = By + F(y)$$

subject to initial condition  $y(0) = y_0 \in \mathbb{R}^d$ . Assume  $B \in \text{Mat}(d, \mathbb{R})$  has all eigenvalues with negative real parts, and  $F(y) \in C^1$  in a neighbourhood of  $y = 0 \in \mathbb{R}^d$ , with  $F(0) = 0 \in \mathbb{R}^d$  and  $F'(0) = 0 \in \mathbb{R}^{d \times d}$ , where  $F'(y)$  is the jacobian of  $F$ . Then  $y = 0 \in \mathbb{R}^d$  is an **asymptotically stable** critical point. If  $B$  has any eigenvalues with positive real part, then  $y = 0$  is **unstable**.

**Thm 4.3.2 (Linearisation Thm II):** Suppose that our derivative  $f \in C^2$  has a fixed point  $y^*$ . If the eigenvalues of

$$J = f'(y)$$

lie strictly in the left half plane of  $\mathbb{C}$ , then  $y^*$  is **asymptotically stable**. If  $J$  has any eigenvalues in the right half plane of  $\mathbb{C}$ , then  $y^*$  is **unstable**.

**Def 4.4.1 (Stability and Asymptotic Stability of Maps):**

Consider a general map  $\Psi : \mathbb{R} \rightarrow \mathbb{R}$  and fixed point  $y^*$  of  $\Psi$  such that  $\Psi(y^*) = y^*$ . Define  $y^n(y_0)$  to be  $n$  applications of  $\Psi$  to  $y_0$ , so  $y^2(y_0) = \Psi(\Psi(y_0))$ . We say that  $y^*$  is:

**stable** for the given ODE if  $\forall \varepsilon > 0$  (sufficiently small)  $\exists \delta > 0$  such that  $\forall t > 0$

$$\|y_0 - y^*\| < \delta \Rightarrow \|y^n(y_0) - y^*\| < \varepsilon;$$

**asymptotically stable** if it is **stable** and  $\exists \gamma > 0$  such that for any initial condition such that  $\|y - 0\| < \gamma$

$$\lim_{t \rightarrow \infty} \|y^n(y_0) - y^*\| = 0;$$

**unstable** if it is not **stable**.

**spectral radius:** Let  $K$  be a matrix. Then  $\rho(K)$  denotes the **spectral radius** of  $K$ , the radius of the smallest circle centered at the origin enclosing all eigenvalues of  $K$ .

**Thm 4.4.1 (Spectral Radius and Stability):**

Let  $z_n = \|K^n y_0\|$ .

Then  $z_n \rightarrow 0$  as  $n \rightarrow \infty$  for all  $y_0$ , iff  $\rho(K) < 1$ .

Moreover,  $z_n \rightarrow \infty$  for some  $y_0$  iff  $\rho(K) > 1$ .

Finally, if  $\rho(K) = 1$  then  $z_n$  remains bounded as  $n \rightarrow \infty$ .

**Thm 4.4.2 (Stability and Asymptotic Stability of Iteration Maps):**

Let  $\Psi$  be a smooth ( $C^2$ ) map.

Then the fixed point  $y^*$  is **asymptotically stable** for the iteration  $y_{n+1} = \Psi(y_n)$  if

$$\rho(\Psi'(y^*)) < 1.$$

The fixed point  $y^*$  is **unstable** if  $\rho(\Psi'(y^*)) > 1$ .

The marginal case  $\rho(\Psi'(y^*)) = 1$  is delicate and must be considered on a case-by-case basis.

**Stability Function:** Consider an RK method with  $y_{n+1} = R(h\lambda)y_n$  (for scalar ODEs). Then  $R(h\lambda)$  is a rational function, and if the method is **explicit** then  $R(h\lambda)$  is a polynomial. Call  $R(h\lambda)$  the **stability function** of our method.

**Matrix Representation of RK Method (Scalar):**

An RK method for a scalar ODE  $y' = \mu y$  with internal stages

$$Y_i = y_n + \mu \sum_{j=1}^s a_{ij} Y_j, \quad i = 1, 2, \dots, s$$

can be written in matrix form as

$$Y = y_n \mathbf{1} + \mu A Y$$

$$Y = y_n (I - \mu A)^{-1} \mathbf{1}$$

where  $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^s$ . Then

$$y_{n+1} = y_n + \mu \sum_{j=1}^s b_j Y_j$$

$$y_{n+1} = y_n + \mu b^T Y$$

and so

$$y_{n+1} = R(\mu) y_n, \quad R(\mu) = 1 + \mu b^T (I - \mu A)^{-1} \mathbf{1}.$$

**Matrix Representation of RK Method (Vector):**

Consider an ODE  $y' = B y$ ,  $y \in \mathbb{R}^d$  and RK method define by

$$Y_i = y_n + h \sum_{j=1}^s a_{ij} B Y_j$$

$$y_{n+1} = y_n + h \sum_{j=1}^s b_i B Y_i.$$

Expand  $y_n, y_{n+1}, Y_i$  in the eigenbasis of  $B$ , and let  $U$  be the matrix with eigenvectors of  $B$  as columns. Write  $B = U \Lambda U^{-1}$  where  $\Lambda$  is diagonal matrix of eigenvalues. Define  $z_n, Z_i$  by

$$y_n = U z_n, \quad Y_i = U Z_i, \quad i = 1, \dots, s,$$

and then rewrite our RK method as

$$Z_i = z_n + h \sum_{j=1}^s a_{ij} \Lambda Z_j$$

$$z_{n+1} = z_n + h \sum_{j=1}^s b_i \Lambda Z_j.$$

and so, since  $\Lambda$  is diagonal, our system decouples into  $d$  independent scalar iterations, which we know how to deal with.

**Thm 4.5.1:** Given the ODE  $y' = By$ , where  $y \in \mathbb{R}^d$ , and  $B$  has a basis of eigenvectors and the eigenvalues  $\lambda_1, \dots, \lambda_d$ , consider applying a given RK method. The RK method has an (asymptotically) stable fixed point at the origin when applied to

$$\frac{dy}{dt} = By$$

iff the same method has an (asymptotically) stable fixed point at the origin when applied to each of the scalar differential equations

$$\frac{dy}{dt} = \lambda_i y.$$

**Cor 4.5.1:** Consider a linear ODE  $\frac{dy}{dt} = By$  with diagonalizable matrix  $B$ . Let an RK method be given with stability function  $R$ . The origin is stable for this RK method to  $\frac{dy}{dt} = By$  (at stepsize  $h$ ) iff

$$|R(h\lambda)| < 1$$

for all eigenvalues  $\lambda \in \sigma(B)$ . (The origin is unstable if  $|R(h\lambda)| > 1$  for any eigenvalue  $\lambda$ ).

**Stability Region:** The **stability region** of a numerical method is the set of all points such that  $\hat{R}(\mu) = |R(\mu)| < 1$ .

RK4 has the stability function

$$R(\mu) = 1 + \mu + \frac{1}{2}\mu^2 + \frac{1}{6}\mu^3 + \frac{1}{24}\mu^4.$$

The boundary of the stability region is the set of all values such that  $R(\mu) = e^{i\theta}$ , that is  $R(\mu)$  lies on the unit circle.

**A-Stability:** A numerical method is **A-stable** if its stability region contains the entire left half-plane. In this case, the method will be stable independent of  $h$ .