Problem 12

a.) From 9.3, Special Case 2, we know that if $A(x,t) = \bar{A}(x) \neq 0$ (if area does not change with time), then the equation can be written as

$$\frac{\partial c(x,t)}{\partial t} = -\frac{1}{\bar{A}(x)}\frac{\partial}{\partial x}[J(x,t)\bar{A}(x)] \pm \sigma(x,t) \tag{1}$$

We now need to find an equation for $\bar{A}(x)$. Since arc length equals radius times angle, we get

$$\bar{A}(r) = \theta r h \tag{2}$$

where θ is the angle of the arc, r is radial distance, and h is height of the section. Therefore we get the equation

$$\frac{\partial c(r,t)}{\partial t} = -\frac{1}{\theta r h}\frac{\partial}{\partial r}[J(r,t)\theta r h] \pm \sigma(r,t) \tag{3}$$

Since θ and h are constants, we can factor them out to get

$$\frac{\partial c(r,t)}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}[J(r,t)r] \pm \sigma(r,t) \tag{4}$$

b) Extending the principle applied in part (a), we first need to find $\bar{A}(R)$. Since θ is small, we can approximate cross sectional area by taking horizontal arc length times vertical arc length. Therefore we get the equation

$$\bar{A}(R) = \theta_1\theta_2 R^2 \tag{5}$$

where θ_1 is the horizontal angle of the arc, θ_2 is the vertical angle of the arc, and R is radial distance. Combining this with equation (1) from above, we get

$$\frac{\partial c(R,t)}{\partial t} = -\frac{1}{\theta_1\theta_2 R^2}\frac{\partial}{\partial R}[J(R,t)\theta_1\theta_2 R^2] \pm \sigma(R,t) \tag{6}$$

Since θ_1 and θ_2 are constants, we can simplify the equation as such:

$$\frac{\partial c(R,t)}{\partial t} = -\frac{1}{R^2}\frac{\partial}{\partial R}[J(R,t)R^2] \pm \sigma(R,t) \tag{7}$$

c) Part A: In order to obtain the equations in 9.5, we apply Fick's law:

$$J = -\mathcal{D}\nabla c \tag{8}$$

In this case, we use the one-dimensional version:

$$J = -\mathcal{D}\frac{\partial c}{\partial x} \tag{9}$$

Applying this to equation (4), we get

$$\frac{\partial c(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}[\mathcal{D}\frac{\partial c}{\partial r}r] \pm \sigma(r,t) \tag{10}$$

Simplifying, and assuming that no particles are created or eliminated at the source, we get:

$$\frac{\partial c(r,t)}{\partial t} = \frac{\mathcal{D}}{r}\frac{\partial}{\partial r}(\frac{\partial c}{\partial r}r) \tag{11}$$

Part B: Once again, we apply Fick's law in one dimension to get

$$\frac{\partial c(R,t)}{\partial t} = \frac{1}{R^2}\frac{\partial}{\partial R}[\mathcal{D}\frac{\partial c}{\partial R}R^2] \pm \sigma(R,t) \tag{12}$$

We again simplify, assuming no particles are created or destroyed at the source.

$$\frac{\partial c(R,t)}{\partial t} = \frac{\mathcal{D}}{R^2}\frac{\partial}{\partial R}(\frac{\partial c}{\partial R}R^2) \tag{13}$$