1 Assigned Problems

Problem 1.1. *For every* $a, b, c \in \mathbb{R}$*, if* $a < b$ *and* $c > 0$*, then* $ac < bc$*.*

Proof: Suppose $a < b$ and $c > 0$.
If $b - a$ is positive, and c is positive by Theorem 6
By positive closure, $(b - a)(c) \in \mathbb{R}$
If $ac < bc$, then $bc - ac > 0$
By distributivity, $bc - ac = c(b - a)$
By closure, $c(b - a) > 0$ thus satisfying $ac < bc$

$\square$

Problem 1.2. *For every $a, b \in \mathbb{R}$, $ab > 0$ if and only if a and b are both positive or both negative.*

Proof: $\longrightarrow$

Suppose $ab > 0$
Goal: $a, b \in \mathbb{R}^+ \vee a, b \in \mathbb{R}^-$
Case 1: $a, b \in \mathbb{R}^+$
By closure of positive real numbers, $a * b \in \mathbb{R}^+$
By theorem 6, such $(a * b) > 0$
Case 2: $a, b \in \mathbb{R}^-$
QUESTION: If we prove case 1, why prove case 2?
$\longleftarrow$
Suppose $a, b \in \mathbb{R}^+ \vee a, b \in \mathbb{R}^-$
By conditional laws,
$(A \vee B) \implies C$
$\equiv C \vee \neg(A \wedge B)$
$\equiv C \vee (\neg A \wedge \neg B)$
$\equiv (C \vee \neg A) \wedge (C \vee \neg B)$
$(A \implies C) \wedge (B \implies C)$

$\square$

Problem 1.3. *For every integer n, $n^3 \equiv n \pmod 3$.*

Proof: By Corollary 1, we only have one of the three cases:
Case 1: $n \equiv 0 \pmod 3$

$$n^3 \equiv n \pmod 3 \longrightarrow 3 \mid (n^3 - n)$$
$$3 \mid (n^3 - n) \longrightarrow (n)(n-1)(n-1)$$

By Theorem 20, we have $n^3 \equiv 0^3 \pmod 3$.
Case 2: $n \equiv 1 \pmod 3$
Similarly, $n = n^3 \pmod 3$.
Case 3: $n \equiv 2 \pmod 3$

$\square$

Problem 1.4. *Let $n \in \mathbb{Z}$. Then n is even if and only if n^2 is even. Similarly, n is odd if and only if n^2 is odd.*

Proof: $\longrightarrow$

Suppose n is even. Goal: n^2 is even.
Then $n = 2k$ for some $k \in \mathbb{Z}$.
So, $n^2 = (2k) \times n = 2(kn)$, where $kn \in \mathbb{Z}$ by the closure of $\mathbb{Z}$.
Hence, n^2 is even.

$\longleftarrow$

Need to prove: if n^2 is even, then n is even.
Contrapositive: If n is not even, then n^2 is not even.
By Theorem 16, if an integer is not even, then it is odd.
Suppose n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$.

$$\text{So, } n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Since $2k^2 + 2k \in \mathbb{Z}$, n^2 is odd. $\square$

2 Revisit

Let $x \in \mathbb{R}$. We say x is rational if there exist integers p and q with $q \neq 0$ such that $x = \frac{p}{q}$. We say x is irrational if it is not rational.

Problem 2.1. $\sqrt{2}$ *is irrational.*

Proof: There exist $p, q \in \mathbb{Z}$ such that $\sqrt{2} = \frac{p}{q}$, $q \neq 0$.

$2 = \frac{p^2}{q^2} \Rightarrow 2q^2 = p^2$
$\Rightarrow p^2$ is even
$\Rightarrow p$ is even ($p = 2k$, where $k \in \mathbb{Z}$)
$\Rightarrow 2q^2 = (2k)^2 = 4k^2$.
$q^2 = 2k^2$
$\Rightarrow q^2$ is even
$\Rightarrow q$ is even ($q = 2L$, where $L \in \mathbb{Z}$)
$\sqrt{2} = \frac{p}{q}$ p is negative and $-p$ is positive. □

Remark: We use proof by contradiction, then we prove that the well-ordering property is contradicted.