1 Assigned Problems

Problem 1.1. *For each integer n, if n is odd, then* $8|(n^2-1)$

Lemma 1: For every integer n, n^2+n is even
Proof: By theorem 16, we assert that an integer n is be either even or odd. Thus we have two cases.

Case 1: n is odd.
By definition, $n = 2k+1$, where $k \in Z$
Substituting, (n^2+n) becomes $((2k+1)^2+(2k+1))$
Expanding and simplifying, $4k^2+6k+2 = 2(2k^2+3k+1)$
By closure, $2k^2+3k+1$ is an integer.
By definition, $2(2k^2+3k+1)$ is even.

Case 2: n is even.
By definition, $n = 2k$, where $k \in Z$
Substituting, (n^2+n) becomes $((2k)^2+(2k))$
Expanding, $((2k)^2+(2k)) = (4k^2+2k)$
Simplifying, $(4k^2+2k) = 2(k^2+k)$
By definition, $2(k^2+k)$ is even.

Thus for every integer n, n^2+n is even.

Proof: Suppose n is odd.
By definition, $n = 2k+1$, where $k \in Z$
We can write $8|((2k+1)^2-1)$
By definition, $((2k+1)^2-1) = 8*b$, $b, k \in \mathbb{Z}$
Expanding, $4k^2+4k = 8*b$, $b, k \in \mathbb{Z}$
$4k^2+4k = 8b$, $b, k \in \mathbb{Z}$
By Lemma 1, $k^2+k = 2p$, $p, k \in \mathbb{Z}$
Rewriting, we have $4(2p) = 8b$, $p, b \in \mathbb{Z}$
Simplifying, $8p = 8b$, $p, b \in \mathbb{Z}$
By defn, p is divisible by 8.
Thus, we can write $((2k+1)^2-1) = 8*b$, $b, k \in \mathbb{Z}$
Rewritten, $(n^2-1) = 8*b$, $b \in \mathbb{Z}$
From this we can conclude that if n is odd, then $8|(n^2-1)$

□

Problem 1.2. *For each integer n, if n is odd, then* $8|(n^2 - 1)$

Proof: Suppose n is odd.

By definition, $n = 2L + 1$, where $L \in Z$

By trichotomy we have 3 possible cases for L.

1. $L > 0$
2. $L = 0$
3. $L < 0$

Because we are only considering integers,

1. $L \geq 1$
2. $L = 0$
3. $L \leq -1$

We proceed with proof by induction in the case where $L \geq 1$

Base Case: Suppose L = 1.
n = 2(1) + 1 = 3
$8|((3)^2 - 1) = 8|(9 - 1) = 8|8$
By Theorem 10, 8|8

Induction Step: Suppose $8|((2L + 1)^2 - 1)$
By definition, $((2L + 1)^2 - 1) = 8 * b$, $b \in \mathbb{Z}$
Expanding, $4L^2 + 4L = 8 * b$
$4L^2 + 4L = 8b$

Goal: Prove $8|((2(L + 1) + 1)^2 - 1)$
Simplifying $((2(L + 1) + 1)^2 - 1)$, we have $((2L + 3)^2 - 1)$
Expanding, $4L^2 + 12L + 8$
By additive identity, $4L^2 + 12L + 8 = 4L^2 + 12L + 8 + 0$
By additive inverse, $(4L - 4L) = 0$
$4L^2 + 12L + 8 + (4L - 4L) = 4L^2 + 4L + 8L + 8$
Substituting, $8b + 8L + 8$, $b, L \in \mathbb{Z}$
8(b + L + 1), $b, L \in \mathbb{Z}$
By integer closure, $(b + L + 1)$ is equal to some $k \in \mathbb{Z}$
Rewriting, we have $8|8k$
By definition, this presumes that $8k = 8b$ for $k, b \in \mathbb{Z}$
By theorem 1, k = b for $k, b \in \mathbb{Z}$
In any case where k = b, $8|8k$

By definition, $8|((2(L+1)+1)^2-1)$

We now evaluate the case where L = 0
By substitution, n = 2(0) + 1 = 1
Goal: Show $8|((1)^2-1)$
Simplifying, $(1*1)=1$ and $1-1=0$
Thus, we have $8|0$
Rewriting, $0=8*k$, $k \in \mathbb{Z}$
By E8, there exists $k=0$ such that 8 * 0 = 0
Therefore, $8|0$ and that when $L=0$, $n|0$

We now evaluate the case where $L \leq -1$ using induction

Base Case: Suppose $L=-1$.
$n=2(-1)+1=-2+1=-1$
Substuting, $8|((-1)^2-1)$
Simplifying, $8|0$
By definition, $0=8*k$
By E8, there exists $k=0$ such that 8 * 0 = 0
Therefore, $8|0$ and also $8|(n^2-1)$ when $L=-1$

Induction Step: Suppose $8|((2L+1)^2-1)$
By definition, $((2L+1)^2-1)=8*b$, $b \in \mathbb{Z}$
Expanding, $4L^2+4L=8*b$
$4L^2+4L=8b$

Goal: Prove $8|((2(L+1)+1)^2-1)$
Simplifying $((2(L+1)+1)^2-1)$, we have $((2L+3)^2-1)$
Expanding, $4L^2+12L+8$
By additive identity, $4L^2+12L+8=4L^2+12L+8+0$
By additive inverse, $(4L-4L)=0$
$4L^2+12L+8+(4L-4L)=4L^2+4L+8L+8$
Substituting, $8b+8L+8$, $b, L \in \mathbb{Z}$
8(b + L + 1), $b, L \in \mathbb{Z}$
By integer closure, $(b+L+1)$ is equal to some $k \in \mathbb{Z}$
Rewriting, we have $8|8k$
By definition, this presumes that $8k=8b$ for $k, b \in \mathbb{Z}$
By theorem 1, k = b for $k, b \in \mathbb{Z}$
In any case where k = b, $8|8k$
By definition, $8|((2(L+1)+1)^2-1)$

$\square$

Problem 1.3. *For every integer n,* $n^3 \equiv n \pmod 3$

Lemma 1: For integers $a, b \in \mathbb{Z}$ with $b > 1$, there exists a unique r such that $a \equiv r \pmod b$ where $0 \leq r < b$

Proof: By Lemma 1, we have the three following cases.
Case 1: $n \equiv 0 \pmod 3$
By Theorem 20, $n^3 \equiv 0^3 \pmod 3$
Simplifying, $n^3 \equiv 0 \pmod 3$
By Problem 5 Week 4, $n^3 \equiv n \pmod 3$
Case 2: $n \equiv 1 \pmod 3$
By Theorem 20, $n^3 \equiv 1^3 \pmod 3$
$n^3 \equiv 1 \pmod 3$
By Problem 5 Week 4, $n^3 \equiv n \pmod 3$
Case 3: $n \equiv 2 \pmod 3$
By Theorem 20, $n^3 \equiv 2^3 \pmod 3$
Simplifying, $n^3 \equiv 8 \pmod 3$
Goal: Establish that $8 \pmod 3 \equiv 2 \pmod 3$
Calculating both, we have $8 \pmod 3 = 2$, and $2 \pmod 3 = 2$
Since they are equal, $n^3 \equiv 2 \pmod 3$
By Problem 5 Week 4 Part, $n^3 \equiv n \pmod 3$

We can see that regardless of the congruence class (and individual cases), we have $n^3 \equiv n \pmod 3$ for every integer n. $\square$

Problem 1.4. *Let $N = \{0, 1, 2, ...\}$ be the set of natural numbers. Prove for $n \in N$, we must have 6 divides $(n^3 - n)$.*

Lemma 1: For integers $a, b \in \mathbb{Z}$ with $b > 1$, there exists a unique r such that $a \equiv r \pmod{b}$ where $0 \leq r < b$

Proof: By definition, $6|(n^3 - n)$ can be written as $n^3 \equiv n \pmod 6$
By Lemma 1, we have the following 6 cases

Case 1: $n \equiv 0 \pmod 6$
By Theorem 20, $n^3 \equiv 0^3 \pmod 6$
Simplifying, $n^3 \equiv 0 \pmod 6$
By Problem 5 Week 4, $n^3 \equiv n \pmod 6$
By definition, $6|(n^3 - n)$

Case 2: $n \equiv 1 \pmod 6$
By Theorem 20, $n^3 \equiv 1^3 \pmod 6$
Simplifying, $n^3 \equiv 1 \pmod 6$
By Problem 5 Week 4, $n^3 \equiv n \pmod 6$
By definition, $6|(n^3 - n)$

Case 3: $n \equiv 2 \pmod 6$
By Theorem 20, $n^3 \equiv 2^3 \pmod 6$
Simplifying, $n^3 \equiv 8 \pmod 6$
Calculating, we have $8 \pmod 6 = 2$, and $2 \pmod 6 = 2$
Because of this equality and problem 5 week 4 part 3, $n^3 \equiv n \pmod 6$
By definition, $6|(n^3 - n)$

Case 4: $n \equiv 3 \pmod 6$
By Theorem 20, $n^3 \equiv 3^3 \pmod 6$
Simplifying, $n^3 \equiv 27 \pmod 6$
Using the division theorem, we have $27 \pmod 6 = 3 \pmod 6$ because $27 = (3 * 6) + 3$
Because of this equality and problem 5 week 4 part 3, $n^3 \equiv n \pmod 6$
By definition, $6|(n^3 - n)$

Case 5: $n \equiv 4 \pmod 6$
By Theorem 20, $n^3 \equiv 4^3 \pmod 6$
Simplifying, $n^3 \equiv 64 \pmod 6$
Using the division theorem, we have $64 \pmod 6 = 4 \pmod 6$ because $64 = (10 * 6) + 4$

Because of this equality and problem 5 week 4 part 3, $n^3 \equiv n \pmod 6$
By definition, $6|(n^3 - n)$

Case 6: $n \equiv 5 \pmod 6$
By Theorem 20, $n^3 \equiv 5^3 \pmod 6$
Simplifying, $n^3 \equiv 125 \pmod 6$
Using the division theorem, we have $125 \pmod 6 = 5 \pmod 6$ because $125 = (20 * 6) + 5$
Because of this equality and problem 5 week 4 part 3, $n^3 \equiv n \pmod 6$
By definition, $6|(n^3 - n)$

We can see that regardless of the congruence class (and individual cases), we have 6 divides $(n^3 - n)$, when $n \in \mathbb{N}$. $\square$