# 1 Introduction

**Theorem 1.1.**

*Proof:* We prove this by induction.
Base case: $n = 0$
By definition of the fibonacci sequence, $f_{5(0)} = f_0 = 0$
Goal: Define that 0 is a multiple of 5.
By definition, $5|0$ implies $0 = 5 * q$ where $q \in \mathbb{Z}$
By theorem E8, q can be 0.
Thus, $5|f_0$

Inductive Case: Suppose $f_{5k}$ is a multiple of 5
By definition, $f_{5k} = 5 * q$, where $q \in \mathbb{Z}$
Goal: Show that $f_{5(k+1)}$ is a multiple of 5
Working backwards, we can rewrite $f_{5(k+1)}$ as $f_{5k+5}$
Using the definition of the fibonacci sequence,
We can define, $f_{5k+5} = f_{5k+4} + f_{5k+3}$
$f_{5k+4} = f_{5k+3} + f_{5k+2}$
$f_{5k+3} = f_{5k+2} + f_{5k+1}$
$f_{5k+2} = f_{5k+1} + f_{5k}$
With some algebra, $f_{5k+5} = (f_{5k+3} + f_{5k+2}) + (f_{5k+2} + f_{5k+1})$
Simplifying, $(f_{5k+2} + f_{5k+1} + f_{5k+1} + f_{5k}) + (f_{5k+1} + f_{5k} + f_{5k+1})$
Further, $(f_{5k+1} + f_{5k} + f_{5k+1} + f_{5k+1} + f_{5k}) + (f_{5k+1} + f_{5k} + f_{5k+1})$
Finally, $5(f_{5k+1}) + 3(f_{5k})$
By our inductive hypothesis, we can rewrite $f_{5k}$ as $5 * q$, where $q \in \mathbb{Z}$
So we have, $5(f_{5k+1}) + (3)(5)(q)$, $q \in \mathbb{Z}$
Factoring, $5((f_{5k+1}) + (3)(q))$, $q \in \mathbb{Z}$
By definition, since 5 is a factor of our expression, it is a multiple of 5.

$\square$

**Theorem 1.2.**

*Proof:* We prove this by strong induction.

Lets evaluate our base cases:

**Case 1:** n = 0

Working backwards,

$f_0 = \frac{1}{\sqrt{5}}(\alpha^0 - \beta^0)$

By definition, the Fibonacci sequence defines $f_0 = 0$

Thus, $0 = \frac{1}{\sqrt{5}}(\alpha^0 - \beta^0)$

Rewriting, we have $0 = \frac{1}{\sqrt{5}}(1 - 1)$

Thus, $0 = \frac{1}{\sqrt{5}}(0)$

By E8, $0 = 0$

With context, we have shown that the first value of the fibonacci sequence $f_0$ is equivalent to $\frac{1}{\sqrt{5}}(\alpha^0 - \beta^0)$

**Case 2:** n = 1

Working backwards,

Substituting, $f_1 = \frac{1}{\sqrt{5}}(\alpha^1 - \beta^1)$

By definition, the Fibonacci sequence defines $f_1 = 1$

Thus, $1 = \frac{1}{\sqrt{5}}(\alpha^1 - \beta^1)$

Substituting, $1 = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2})$

Simplifying, $1 = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}-(1-\sqrt{5})}{2})$

$1 = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}-1+\sqrt{5}}{2})$

$1 = \frac{1}{\sqrt{5}}(\frac{2\sqrt{5}}{2})$

$1 = \frac{1}{\sqrt{5}}(\sqrt{5})$

$1 = 1$

Thus, we have shown that $f_1 = \frac{1}{\sqrt{5}}(\alpha^1 - \beta^1)$

Inductive Step: Assume that $f_k = \frac{1}{\sqrt{5}}(\alpha^k - \beta^k)$ holds for all $0 \leq k \leq n$, where $n \geq 1$.

Goal: Show that $f_{k+1} = \frac{1}{\sqrt{5}}(\alpha^{k+1} - \beta^{k+1})$

By the Fibonacci sequence, we can define $f_{k+1} = f_k + f_{k-1}$

Using our inductive hypothesis,

$f_{k+1} = \frac{1}{\sqrt{5}}(\alpha^k - \beta^k) + \frac{1}{\sqrt{5}}(\alpha^{k-1} - \beta^{k-1})$

$= \frac{(\alpha^k - \beta^k)}{\sqrt{5}} + \frac{(\alpha^{k-1} - \beta^{k-1})}{\sqrt{5}}$

$= \frac{(\alpha^k - \beta^k + \alpha^{k-1} - \beta^{k-1})}{\sqrt{5}}$

$= \frac{(\alpha^k + \alpha^{k-1} - \beta^k - \beta^{k-1})}{\sqrt{5}}$
$= \frac{\alpha^{k-1}(1+\alpha) - \beta^{k-1}(1+\beta)}{\sqrt{5}}$
Substituting from our definitions,
$= \frac{\alpha^{k-1}(\alpha^2) - \beta^{k-1}(\beta^2)}{\sqrt{5}}$
Thus, $f_{k+1} = \frac{\alpha^{k+1} - \beta^{k+1}}{\sqrt{5}}$
Finally, $f_{k+1} = \frac{1}{\sqrt{5}}(\alpha^{k+1} - \beta^{k+1})$ $\square$

**Theorem 1.3.**

*Proof:* We prove this by strong induction.

Lets evaluate our base cases:

**Case 1:** n = 1

$1 = (2k+1)2^L$, where $k, L \in \mathbb{N}$

Suppose we have $k = 0$, $L = 0$.

$1 = (2(0)+1)2^0$

Simplifying, $1 = 1$

Thus, we have shown there exist a $k, L \in \mathbb{N}$ that satisfy the statement.

**Inductive Step:**

Suppose for $1 \leq n \leq b$, b can be written in the form $(2k+1)2^L$, where $k, L \in \mathbb{N}$

Goal: Show that b + 1 can also be written as $(2k+1)2^L$, where $k, L \in \mathbb{N}$

By theorem 16, b + 1 can either be even or odd. Thus we have two cases.

**Case 1:** b + 1 is even

By definition, b + 1 = 2q, where $q \in \mathbb{Z}$

Because q is less than b. By our hypothesis, $q = (2k+1)2^L$, where $k, L \in \mathbb{N}$

Substituting we have our intended form, $b + 1 = 2((2k+1)2^L)$

Further, $b + 1 = (2k+1)2^{L+1}$, where $k, L \in \mathbb{N}$

Thus we have shown that when n = b + 1, n can be written as the product of an odd number and a power of 2.

**Case 1:** b + 1 is odd

By definition, b + 1 = 2q + 1, where $q \in \mathbb{Z}$

Because q is less than b. By our hypothesis, $q = (2k+1)2^L$, where $k, L \in \mathbb{N}$

Substituting, $b + 1 = 2((2k+1)2^L) + 1$

By closure, $(2k+1)2^L \in \mathbb{N}$

Rewriting, $b + 1 = 2(m) + 1$, where $m, b \in \mathbb{N}$

Because $2^0 = 1$, and $a * 1 = a$,

Further, $b + 1 = (2m+1)(2^0)$

Thus we have shown that when n = b + 1, n can be written as the product of an odd number and a power of 2.

$\square$

**Theorem 1.4.**

(a) $x \notin A \cap B$ if and only if $x \notin A$ or $x \notin B$
Rationale: $x \in A \cap B \equiv x \in A$ and $x \in B$
$\neg(x \in A$ and $x \in B) \equiv (x \notin A$ or $x \notin B)$

(b) $x \notin A \cup B$ if and only if $x \notin A$ and $x \notin B$
Rationale: $x \in A \cap B \equiv x \in A$ or $x \in B$
$\neg(x \in A$ or $x \in B) \equiv (x \notin A$ and $x \notin B)$

(c) $x \notin A \setminus B$ if and only if $x \notin A$ or $x \in B$
Rationale: $x \in A \setminus B \equiv x \in A$ and $x \notin B$
$\neg(x \in A$ and $x \notin B) \equiv (x \notin A$ or $x \in B)$