1 Assigned Problems

Problem 1.1. *For each integer n, if n is odd, then* $8|(n^2 - 1)$

Proof: Suppose n is odd.
By definition, $n = 2L + 1$, where $L \in Z$
By trichotomy we have 3 possible cases for L.

1. $L > 0$
2. $L = 0$
3. $L < 0$

Because we are only considering integers,

1. $L \geq 1$
2. $L = 0$
3. $L \leq -1$

We proceed with proof by induction in the case where $L \geq 1$

Base Case: Suppose L = 1.
n = 2(1) + 1 = 3
$8|((3)^2 - 1) = 8|(9 - 1) = 8|8$
By Theorem 10, $8|8$

Induction Step: Suppose $8|((2L + 1)^2 - 1)$
By definition, $((2L + 1)^2 - 1) = 8 * b$, $b \in \mathbb{Z}$
Expanding, $4L^2 + 4L = 8 * b$
$4L^2 + 4L = 8b$

Goal: Prove $8|((2(L + 1) + 1)^2 - 1)$
Simplifying $((2(L + 1) + 1)^2 - 1)$, we have $((2L + 3)^2 - 1)$
Expanding, $4L^2 + 12L + 8$
By additive identity, $4L^2 + 12L + 8 = 4L^2 + 12L + 8 + 0$
By additive inverse, $(4L - 4L) = 0$
$4L^2 + 12L + 8 + (4L - 4L) = 4L^2 + 4L + 8L + 8$
Substituting, $8b + 8L + 8$, $b, L \in \mathbb{Z}$
8(b + L + 1), $b, L \in \mathbb{Z}$
By integer closure, $(b + L + 1)$ is equal to some $k \in \mathbb{Z}$
Rewriting, we have $8|8k$
By definition, this presumes that $8k = 8b$ for $k, b \in \mathbb{Z}$

By theorem 1, k = b for $k, b \in \mathbb{Z}$
In any case where k = b, $8|8k$
By definition, $8|((2(L+1)+1)^2 - 1)$

We now evaluate the case where L = 0
By substitution, n = 2(0) + 1 = 1
Goal: Show $8|((1)^2 - 1)$
Simplifying, $(1 * 1) = 1$ and $1 - 1 = 0$
Thus, we have $8|0$
Rewriting, $0 = 8 * k$, $k \in \mathbb{Z}$
By E8, there exists $k = 0$ such that 8 * 0 = 0
Therefore, $8|0$ and that when $L = 0$, $n|0$

We now evaluate the case where $L \leq -1$ using induction

Base Case: Suppose $L = -1$.
$n = 2(-1) + 1 = -2 + 1 = -1$
Substuting, $8|((-1)^2 - 1)$
Simplifying, $8|0$
By definition, $0 = 8 * k$
By E8, there exists $k = 0$ such that 8 * 0 = 0
Therefore, $8|0$ and also $8|(n^2 - 1)$ when $L = -1$

Induction Step: Suppose $8|((2L+1)^2 - 1)$
By definition, $((2L+1)^2 - 1) = 8 * b$, $b \in \mathbb{Z}$
Expanding, $4L^2 + 4L = 8 * b$
$4L^2 + 4L = 8b$

Goal: Prove $8|((2(L+1)+1)^2 - 1)$
Simplifying $((2(L+1)+1)^2 - 1)$, we have $((2L+3)^2 - 1)$
Expanding, $4L^2 + 12L + 8$
By additive identity, $4L^2 + 12L + 8 = 4L^2 + 12L + 8 + 0$
By additive inverse, $(4L - 4L) = 0$
$4L^2 + 12L + 8 + (4L - 4L) = 4L^2 + 4L + 8L + 8$
Substituting, $8b + 8L + 8$, $b, L \in \mathbb{Z}$
8(b + L + 1), $b, L \in \mathbb{Z}$
By integer closure, $(b + L + 1)$ is equal to some $k \in \mathbb{Z}$
Rewriting, we have $8|8k$
By definition, this presumes that $8k = 8b$ for $k, b \in \mathbb{Z}$
By theorem 1, k = b for $k, b \in \mathbb{Z}$
In any case where k = b, $8|8k$

By definition, $8|((2(L+1)+1)^2-1)$

$\square$

Problem 1.2. *For every integer n,* $n^3 \equiv n \pmod 3$

Lemma 1: For integers $a, b \in \mathbb{Z}$ with $b > 1$, there exists a unique r such that $a \equiv r \pmod b$ where $0 \leq r < b$

Proof: By Lemma 1, we must have one of the three following cases.
Case 1: $n \equiv 0 \pmod 3$
By Theorem 20, $n^3 \equiv 0^3 \pmod 3$
Simplifying, $n^3 \equiv 0 \pmod 3$
By Problem 5 Week 4, $n^3 \equiv n \pmod 3$
Case 1: $n \equiv 1 \pmod 3$
By Theorem 20, $n^3 \equiv 1^3 \pmod 3$
Similarly, $n^3 \equiv n \pmod 3$
Case 1: $n \equiv 2 \pmod 3$
By Theorem 20, $n^3 \equiv 2^3 \pmod 3$
Simplifying, $n^3 \equiv 8 \pmod 3$ □