1 Assigned Problems

Problem 1.1. *For every* $a, b, c \in \mathbb{R}$*, if* $a < b$ *and* $c > 0$*, then* $ac < bc$*.*

Proof: Suppose $a < b$ and $c > 0$.
By order axioms $b - a$ is positive, and c is positive by Theorem 6
By closure, $(b - a)(c) \in \mathbb{R}^+$
If $ac < bc$, then by order axioms we can write $bc - ac > 0$
By distributivity, $bc - ac = c(b - a)$
By commutativity, $c * (b - a) = (b - a) * c$
By above closure, $c * (b - a) > 0$ thus satisfying $ac < bc$

$\square$

Problem 1.2. *For every $a, b \in \mathbb{R}$, $ab > 0$ if and only if a and b are both positive or both negative.*

Proof: $\longrightarrow$

Suppose $ab > 0$

Goal: $a, b \in \mathbb{R}^+ \vee a, b \in \mathbb{R}^-$

By the trichotomy of real numbers, there exist 3 possibilities for both a, b

Thus, $3 * 3 = 9$ total combinations

Case #	a	b
1	$a > 0$	$b > 0$
2	$a > 0$	$b < 0$
3	$a > 0$	$b = 0$
4	$a < 0$	$b > 0$
5	$a < 0$	$b < 0$
6	$a < 0$	$b = 0$
7	$a = 0$	$b > 0$
8	$a = 0$	$b < 0$
9	$a = 0$	$b = 0$

Cases 3, 6, 7, 8 where either $(a = 0)$ or $(b = 0)$ by Theorem 4 equate to 0, because some $a * 0 = 0$ where $a \in \mathbb{R}$.

This contradicts $a * b > 0$ because $a * b = 0$ in all 4 cases.

Similarly, case 9 where both $a, b = 0$

By theorem 4, $a * 0 = 0$, thus $0 * 0 = 0$

By substitution, we can rewrite $0 * 0 = 0$ as $a * b = 0$

This contradicts $a * b > 0$ and the trichotomy principle.

Thus remains,

Case #	a	b
1	$a > 0$	$b > 0$
2	$a > 0$	$b < 0$
4	$a < 0$	$b > 0$
5	$a < 0$	$b < 0$

For case 1, suppose both $a, b > 0$
By theorem 6, both a, b are positive.
By positive closure, $a * b \in \mathbb{R}^+$, by definition $a * b$ is positive and $a * b > 0$
Our result is consistent with $ab > 0$
For case 2, suppose $a > 0$ and $b < 0$
By theorem 6, a is positive.
By order axioms and negativity, $b < 0 \equiv -b > 0$
By positive closure, $-b * a \in \mathbb{R}^+$.
By multiplication, $-b * a \equiv -ba$.
By order axioms and the definition of positivity, if $-ba \in \mathbb{R}^+$ then $-(-ba) \in \mathbb{R}^-$.
By E7, $-(-ba) = ba$
By commutativity, $ba = ab$
Thus, we derive $ab \in \mathbb{R}^-$, but this contradicts our hypothesis.
For case 4, suppose $a < 0$ and $b > 0$
By theorem 6, a is positive.
By order axioms, $a < 0 \equiv -a > 0$
By positive closure, $-a * b \in \mathbb{R}$
By multiplication, $-a * b = -ab$
By order axioms and the definition of positivity, if $-ab \in \mathbb{R}^+$ then $-(-ab) \in \mathbb{R}^-$.
By E7, $-(-ab) = ab$
Thus, we derive $ab \in \mathbb{R}^-$, but this contradicts our hypothesis.
The final case 5, in which $a, b < 0$
By positivity and order axioms $a < 0$ then $-a > 0$ and similarly $b < 0$ then $-b > 0$.
By positive closure, $-a * -b \in \mathbb{R}^+$.
By theorem e11, $-a * -b = ab$
Thus we have $ab \in \mathbb{R}^+$.
The only affirming cases here are 1, 5 where $a, b \in \mathbb{R}^+$ or $a, b \in \mathbb{R}^+$
Thus satisfying this statement.
$\longleftarrow$
Suppose $a, b \in \mathbb{R}^+ \vee a, b \in \mathbb{R}^-$
Case 1: $a, b \in \mathbb{R}^+$
By positive closure, $a * b \in \mathbb{R}^+$
By theorem 6 and positivity, $a * b > 0$

Case 2: $a, b \in \mathbb{R}^-$

By negativity and order axioms if $a < 0$ then $-a > 0$ and similarly if $b < 0$ then $-b > 0$.

By positive closure, $-a * -b \in \mathbb{R}^+$.

By theorem E11, $-a * -b = ab$

By positivity and theorem 6, $ab > 0$

$\square$