1 Introduction

Theorem 1.1. *A sequence is an infinite list of numbers that are indexed by* $\mathbb{N}$ *or a subset of* $\mathbb{N}$. *We can often write a sequence in the form* $a_1, a_2, ...a_n$

Example: $(A_n)_{n=0}^{\infty}$ □

Theorem 1.2. *We can define a sequence recursively (recursive definition). The Fibonacci sequence* f_n *is defined as follows:*

Example:

$$f_n = \begin{cases} f_0 = 1, f_1 = 1 \\ f_{n+2} = f_{n+1} + f_n \text{ if } n \geq 0. \end{cases}$$

□

Theorem 1.3. *For each $n \in \mathbb{N}$, the Fibonacci number f_{3n} is an even natural number.*

Proof: We prove this by induction.

Base case: when $n = 0, f_0$ is even.

Inductive step: Suppose f_{3k} is even for some $k \geq 0$

We want to prove $f_{3(k+1)}$ is even

Note $f_{3(k+1)} = f_{3k+3}$

By the recursive definition, $f_{3k+3} = f_{3k+2} + f_{3k+1}$

Further simplifying, $(f_{3k+1} + f_{3k}) + f_{3k+1}$

$2f_{3k+1} + f_{3k}$

Substituting, $2f_{3k+1} + 2L$

Thus, $2(f_{3k+1} + L)$ is even.

By the closure of the set of integers and the recursive definition, this is an integer.

By induction, this statement is true for any $\mathbb{N}$ $\square$

Theorem 1.4. *How many ways can you tile a 2 by n grid with dominoes?*

Illustrated: Working from a simpler case, suppose n = 1. There is only one way to fill the grid.

When n = 2, there are only two ways like such $\|$ and $=$

When n = 3, there are 3 ways in which you can tile the dominoes

When n = 4, there are 5 ways.

When n = 5, there are 8 ways. □

Illustrated: For any integer $n \geq 1$, the number of ways to tile a 2 by n grid with dominoes is the (n+1)th Fibonacci number, f_{n+1}

Recall,

$$f_n = \begin{cases} f_0 = 1, f_1 = 1 \\ f_{n+2} = f_{n+1} + f_n \text{ if } n \geq 0. \end{cases}$$

Proof using induction.

Base Case:

Suppose n = 1. There is 1 way to tile a 2 by 1 grid and $f_1 = 1$

Suppose n = 2. There are 2 ways to tile a 2 by 2 grid and $f_3 = 2$

Inductive Case:

Suppose the number of ways to tile an n by k grid is f_{k+1}

Suppose the number of ways to tile a 2 by (k + 1) grid is f_{k+2}

Goal: Find out the number of ways to tile a 2 by (k + 2) grid.

Consider the top left square of this 2 by (k + 2) grid.

There are only two ways in which it can be covered

Case 1:

This square is covered by a vertical d.

The remaining part is a 2 by (k + 1) grid.

By the inductive hypothesis, there are f_{k+2} to cover the grid.

Case 2:

The square is covered by a horizontal d.

The square underneath it must be covered by a horizontal domino.

The remaining grid is a 2 by k grid. Which has f_{k+1} ways to tile. □

Technique: Strong Mathematical Induction.

Base case: Prove $P(K_0)$ is true

Inductive Step: For every integer $k \geq k_0$, prove P(k + 1) is true under the assumption that it is true for all smaller cases, instead of assuming that it is true for one case.

Thus we are proving $P(K_0) \land P(K_0 + 1)$ ... $\land P(k) \implies P(k + 1)$

Theorem: Every positive integer $n \geq 2$ is either a prime number or is a product of prime numbers. $\square$