1 Introduction

Theorem 1.1. *A sequence is an infinite list of numbers that are indexed by* $\mathbb{N}$ *or a subset of* $\mathbb{N}$. *We can often write a sequence in the form* $a_1, a_2, ...a_n$

Example: $(A_n)_{n=0}^{\infty}$ □

Theorem 1.2. *We can define a sequence recursively (recursive definition). The Fibonacci sequence* f_n *is defined as follows:*

Example:

$$f_n = \begin{cases} f_0 = 1, f_1 = 1 \\ f_{n+2} = f_{n+1} + f_n \text{ if } n \geq 0. \end{cases}$$

□

Theorem 1.3. *For each $n \in \mathbb{N}$, the Fibonacci number f_{3n} is an even natural number.*

Proof: We prove this by induction.

Base case: when $n = 0$, f_0 is even.

Inductive step: Suppose f_{3k} is even for some $k \geq 0$

We want to prove $f_{3(k+1)}$ is even

Note $f_{3(k+1)} = f_{3k+3}$

By the recursive definition, $f_{3k+3} = f_{3k+2} + f_{3k+1}$

Further simplifying, $(f_{3k+1} + f_{3k}) + f_{3k+1}$

$2f_{3k+1} + f_{3k}$

Substituting, $2f_{3k+1} + 2L$

Thus, $2(f_{3k+1} + L)$ is even.

By the closure of the set of integers and the recursive definition, this is an integer.

By induction, this statement is true for any $\mathbb{N}$ $\square$

Theorem 1.4. *How many ways can you tile a 2 by n grid with dominoes?*

Illustrated: Working from a simpler case, suppose n = 1. There is only one way to fill the grid.

When n = 2, there are only two ways like such $\|$ and $=$

When n = 3, there are 3 ways in which you can tile the dominoes

When n = 4, there are 5 ways.

When n = 5, there are 8 ways. □

Illustrated: For any integer $n \geq 1$, the number of ways to tile a 2 by n grid with dominoes is the (n+1)th Fibonacci number, f_{n+1}

Recall,

$$f_n = \begin{cases} f_0 = 1, f_1 = 1 \\ f_{n+2} = f_{n+1} + f_n \text{ if } n \geq 0. \end{cases}$$

Proof using induction.

Base Case:

Suppose n = 1. There is 1 way to tile a 2 by 1 grid and $f_1 = 1$

Suppose n = 2. There are 2 ways to tile a 2 by 2 grid and $f_3 = 2$

Inductive Case:

Suppose the number of ways to tile an n by k grid is f_{k+1}

Suppose the number of ways to tile a 2 by (k + 1) grid is f_{k+2}

Goal: Find out the number of ways to tile a 2 by (k + 2) grid.

Consider the top left square of this 2 by (k + 2) grid.

There are only two ways in which it can be covered

Case 1:

This square is covered by a vertical d.

The remaining part is a 2 by (k + 1) grid.

By the inductive hypothesis, there are f_{k+2} to cover the grid.

Case 2:

The square is covered by a horizontal d.

The square underneath it must be covered by a horizontal domino.

The remaining grid is a 2 by k grid. Which has f_{k+1} ways to tile. □

Technique: Strong Mathematical Induction.

Base case: Prove $P(K_0)$ is true

Inductive Step: For every integer $k \geq k_0$, prove P(k + 1) is true under the assumption that it is true for all smaller cases, instead of assuming that it is true for one case.

Thus we are proving $P(K_0) \wedge P(K_0 + 1) \text{ ... } \wedge P(k) \implies P(k+1)$

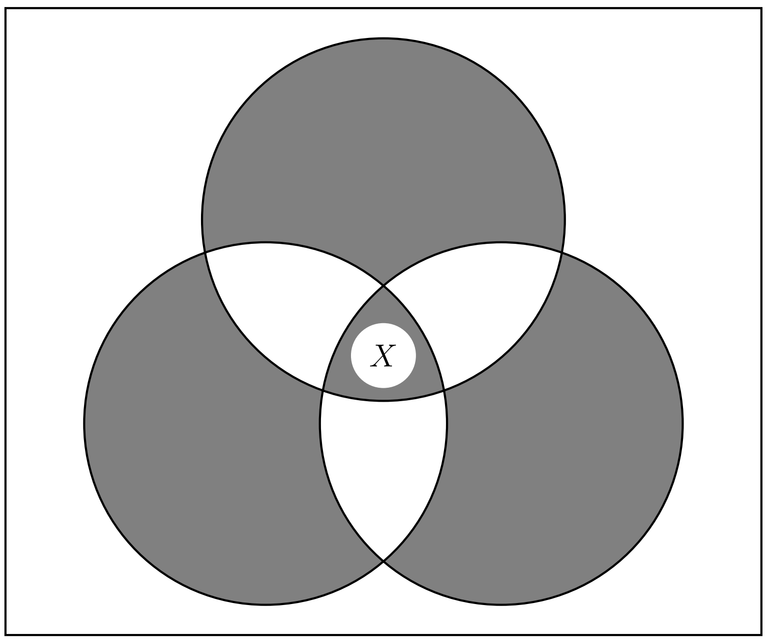
Theorem: Every positive integer $n \geq 2$ is either a prime number or is a product of prime numbers. $\square$

Let X, Y be two sets. The union of X and Y, denoted by $X \cup Y$ is $\{x \in U | x \in X \text{ or } x \in Y\}$

The intersection of X and Y, denote by $X \cap Y$, is $\{x \in U | x \in X$ and $x \in Y\}$

The set difference of X and Y (relative complement of Y w.r.t X), denoted by $X - Y$ (or $X \backslash Y$) is $\{x \in U | x \in Y$ and $x \notin X\}$

Set Difference Diagram



The complement of X^c is regarded as $U - X$ (The universal set minus the set X)

Example Problem:

Let X = $\{1, 2, 3, 4\}$

Let Y = $\{x \in \mathbb{R} | -1 < x \leq 3\}$

$X \cup Y$ = 1, 2, 3, 4

$X \cap Y$ = 0, 1, 2, 3

$X \backslash Y$ = 4

$Y \backslash X$ = 0

$Y^c = \{x \in \mathbb{U} | x \leq -1 \text{ or } x \geq 3\}$

Power Set: Let U be the universal set and let A be a subset of U. The power set of A, denoted by $\mathcal{P}(A)$, is the set of all subsets of A, that is $\mathcal{P}(A) = \{Y \subseteq U \mid Y \subseteq A\}$.

Example 1:
Let $U = \mathbb{R}$ and $A = \{1, 2\}$.
Subsets of A: $\emptyset$, $\{1\}$, $\{2\}$, $\{1, 2\}$.
$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

Example 2:
Let $U = \mathbb{R}$ and $A = \{-1, 4, 7\}$.
Subsets of A: $\emptyset$, $\{-1\}$, $\{4\}$, $\{7\}$, $\{-1, 4\}$, $\{-1, 7\}$, $\{4, 7\}$, $\{-1, 4, 7\}$.
$\mathcal{P}(A) = \{\emptyset, \{-1\}, \{4\}, \{7\}, \{-1, 4\}, \{-1, 7\}, \{4, 7\}, \{-1, 4, 7\}\}$.

Cardinality:
The cardinality of a finite set is the number of elements of the set.
Card(A), where $A = \{-1, 4, 7\}$ is equal to 3.

Question:
Suppose $\text{card}(A) = n$. What is $\text{card}(\mathcal{P}(A))$?

$$\text{card}(\mathcal{P}(A)) = 2^n$$

Theorem: If A is a finite set with $\text{card}(A) = n$, then $\text{card}(\mathcal{P}(A)) = 2^n$

Proof: We prove it by induction.

(1) Base Case: $n = 0$, so $A = \emptyset$.

$$\mathcal{P}(A) = \{\emptyset\}$$

Therefore,

$$\text{card}(\mathcal{P}(A)) = 1 = 2^0$$

(2) Inductive Step: Suppose any finite set A with $\text{card}(A) = k$ has a power set $\mathcal{P}(A)$ with cardinality 2^k.

Now consider a set with card $= k + 1$.

Here we focus on the $(k + 1)th$ element, A_{k+1}

For any subset of A, we only have 2 possiblities.

Case 1: This subset contains A_{k+1}. If we remove A_{k+1} from this subset, we get a subset of a set with card $= k$.

By the inductive hypothesis, we have 2^k possiblities for such subsets.

Case 2: This subset does not contain A_{k+1}. Then it is a subset of a set with card $= k$. By the inductive hypothesis, we have 2^k in this case. Thus, A has $2^k + 2^k = 2^{k+1}$ possibilities of subsets. $\square$

Technique (Element Chasing):
To prove $A \subseteq B$, for every $x \in A$, prove $x \in B$.
To disprove $A \subseteq B$, give an example of some $x \in A$ such that $x \notin B$.

Theorem 1.5. *Suppose A, B, C are subsets. Then $A \setminus (B \cup C) \subseteq (A \setminus B) \cup (A \setminus C)$.*

Proof: Let $x \in A \setminus (B \cup C)$, which means $x \in A$ and $x \notin (B \cup C)$.
Since $x \in A$ and $x \notin B$, then $x \in A \setminus B$.
Then by the definition of the union of two sets, $x \in (A \setminus B) \cup (A \setminus C)$.
We conclude $A \setminus (B \cup C) \subseteq (A \setminus B) \cup (A \setminus C)$. □

Question: Do we have $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cup C)$?
Counterexample:
$A = \{1, 2, 3\}$
$B = \{2\}$
$C = \{3\}$
$A \setminus B = \{1, 3\}$
$A \setminus C = \{1, 2\}$
$(A \setminus B) \cup (A \setminus C) = \{1, 2, 3\}$

Theorem 1.6. *Suppose A, B, C are subsets. Then A (B union C) = (A B) intersection (A C)*

Proof: We prove two inclusions.
(1) We prove A (B union C) subset (A B) union (A C)
(2) We prove ...

□